Ec 121a
Theory of Value

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Fall 2020

Lecture 1: Introduction to Comparative Statics

The models we study in this class are largely **qualitative**. Here is a simple example of what I mean by a qualitative model.

1.1 Supply and Demand in a Single Market

You may recall from Ec 11 that a **demand curve** expresses a relation between quantities and prices for buyers. This being economics, there are at least two interpretations of the demand curve. The **Walrasian demand curve** [?] gives the quantity that buyers are willing to buy as a function of the price they pay. As a function it maps prices to quantities. The **Marshallian demand curve** [?] gives for each quantity the price at which buyers are willing to buy that quantity. This is not to be confused with the **all-or-nothing demand curve**.

Since the time of Marshall it has been traditional to draw demand and supply curves with quantity as the abscissa, but I usually do not. Instead, I usually put price on the horizontal axis, since the theories we shall discuss treat price as the independent variable from the point of view of buyers and sellers. Fortunately, the sign of the slope is the same either way. See Figures **??** and **??**.

- The definition of a market and the Law of One Price.

1.2 Comparative Statics of Supply and Demand

The model has these pieces:

- The (Walrasian) **demand curve** gives the quantity that buyers are willing to buy as a function of the price they pay:

$$D(p) \text{ is the quantity demanded at price } p.$$

- Behavioral assumption: The demand curve is downward sloping. To simplify things, let's assume the demand curve is smooth and that

$$D'(p) < 0.$$

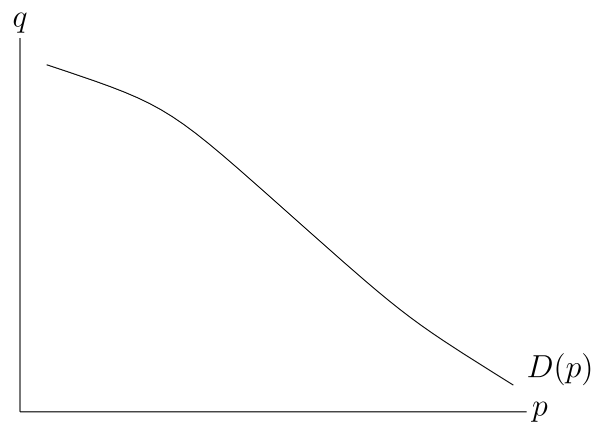

Figure 1.1. A (Walrasian) demand curve.

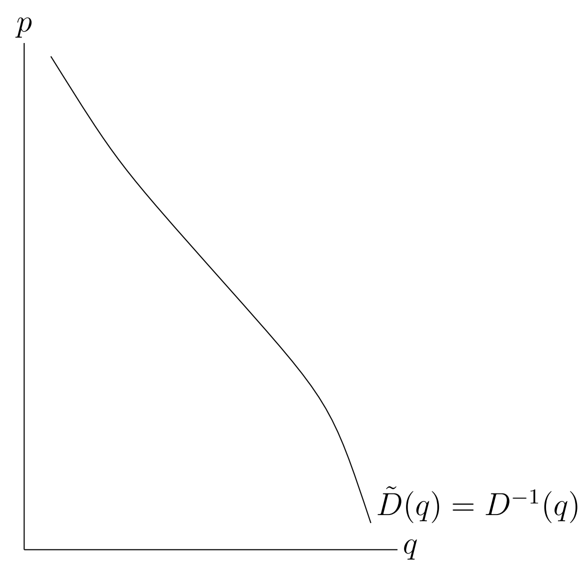

Figure 1.2. The same curve as a Marshallian demand.

• The (Walrasian) **supply curve** give the quantity that sellers are willing to sell as a function of the price they receive.

$$S(p) \text{ is the quantity supplied at price } p.$$

• Behavioral assumption: The supply curve is upward sloping:

$$S'(p) > 0.$$

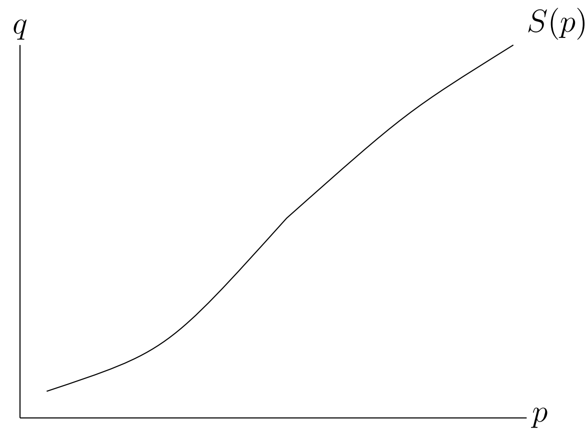

Figure 1.3. A (Walrasian) supply curve.

• The **market equilibrium price** p^* and **equilibrium quantity** q^* are determined by "market forces" so that the quantity demanded is equal to the quantity supplied, or the market "**clears**." That is,

$$D(p^*) - S(p^*) = 0.$$
$$q^* = D(p^*) = S(p^*).$$

• *By itself this tells us nothing about what the price and quantity might be—it is consistent with any values of p^* and q^*. Nevertheless the model is not without content.*

• The testable content does not come from characterizing the price-quantity pairs that can be market equilibria, but rather from the way the equilibrium changes in response to *interventions* or exogenous changes to the environment. The identification of how the static equilibrium changes in response to changes in outside factors is called **comparative statics**.

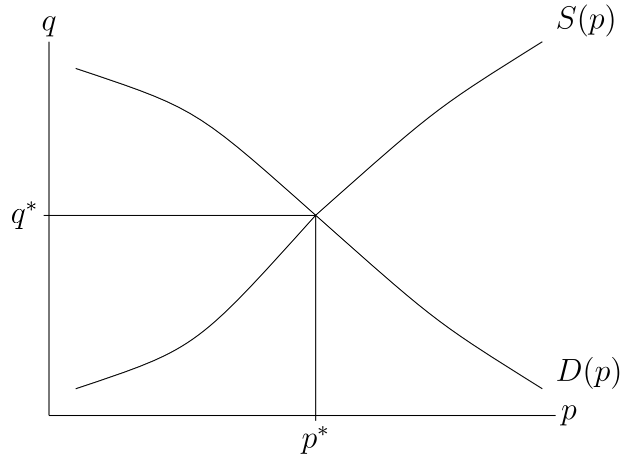

Figure 1.4. Market-clearing.

1.2.1 Testable implications

Suppose an *ad rem* excise tax of t per unit is imposed. What happens to the market price and quantity?

To answer this we need to be careful about which price we are speaking of. Let p_b denote the price the buyer pays and p_s the price the seller receives. Then

$$p_b - p_s = t.$$

So now we must modify the last part of the model to this:

- The market prices $\hat{p}_b(t)$ and $\hat{p}_s(t)$, which depend on the size of the tax, are determined by "market forces" so that the quantity demanded is equal to the quantity supplied.

$$\begin{gathered} D\Big(\hat{p}_b(t)\Big) - S\Big(\hat{p}_s(t)\Big) = 0. \\ \hat{q}(t) = D\Big(\hat{p}_b(t)\Big) = S\Big(\hat{p}_s(t)\Big). \\ \hat{p}_b(t) = \hat{p}_s(t) + t. \end{gathered} \tag{1}$$

We are interested in how $\hat{p}_b$ and $\hat{p}_s$ vary with the tax t. In other words, we want to know what we can about $\hat{p}'_s$, $\hat{p}'_b$, and $\hat{q}'$.

- Well, for starters, if there are market clearing prices, we know that for all t

$$D\Big(\hat{p}_s(t) + t\Big) - S\Big(\hat{p}_s(t)\Big) = 0. \tag{2}$$

Differentiate both sides to get

$$D'\Big(\hat{p}_s(t) + t\Big)\Big(\hat{p}'_s(t) + 1\Big) - S'\Big(\hat{p}_s(t)\Big)\hat{p}'_s(t) = 0.$$

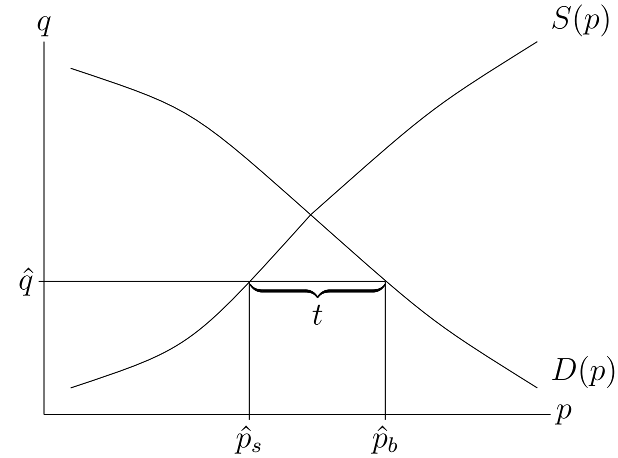

Figure 1.5. Market-clearing with an *ad rem* tax.

Solve to get

$$\hat{p}_s'(t) = -\frac{D'\big(\hat{p}_s(t)+t\big)}{D'\big(\hat{p}_s(t)+t\big) - S'\big(\hat{p}_s(t)\big)}. \tag{3}$$

Since $D' < 0$ and $S' > 0$, it follows that

$$-1 < \hat{p}_s'(t) < 0.$$

- Next try it with

$$D\big(\hat{p}_b(t)\big) - S\big(\hat{p}_b(t) - t\big) = 0.$$

Differentiating with respect to t yields

$$D'\big(\hat{p}_b(t)\big)p_b'(t) - S'\big(\hat{p}_b(t) - t\big)\big(\hat{p}_b'(t) - 1\big) = 0,$$

so

$$\hat{p}_b'(t) = -\frac{S'\big(\hat{p}_b(t)-t\big)}{D'\big(\hat{p}_b(t)\big) - S'\big(\hat{p}_b(t)-t\big)}. \tag{4}$$

This implies

$$0 < \hat{p}_b'(t) < 1.$$

- Now observe that since $t = \hat{p}_b(t) - \hat{p}_s(t)$, from (**??**)–(**??**) we have

$$\begin{aligned} t' = \hat{p}_b'(t) - \hat{p}_s'(t) &= \frac{-S'\big(\hat{p}_b(t)-t\big)}{D'\big(\hat{p}_b(t)\big) - S'\big(\hat{p}_b(t)-t\big)} - \frac{-D'\big(\hat{p}_s(t)+t\big)}{D'\big(\hat{p}_s(t)+t\big) - S'\big(\hat{p}_s(t)\big)} \\ &= \frac{D'\big(\hat{p}_s(t)+t\big) - S'\big(\hat{p}_s(t)\big)}{D'\big(\hat{p}_s(t)+t\big) - S'\big(\hat{p}_s(t)\big)} = \frac{D'\big(\hat{p}_b(t)\big) - S'\big(\hat{p}_b(t)-t\big)}{D'\big(\hat{p}_b(t)\big) - S'\big(\hat{p}_b(t)-t\big)} \\ &= 1. \end{aligned}$$

This is comforting.

• How do we know that market clearing prices will exist? That is, how can we be sure that the supply and demand curves cross? Here is a sufficient set of conditions that will guarantee it (assuming smoothness, which implies no jumps).

∘ If the price is low enough, then demand exceeds supply.

$$\lim_{p \to 0} D(p) > \lim_{p \to 0} S(p).$$

∘ If the price is high enough, then supply exceeds demand.

$$\lim_{p \to \infty} S(p) > \lim_{p \to \infty} D(p).$$

• The mathematical sticklers among you may object to the previous line of argument. I differentiated $\hat{p}_b$ and $\hat{p}_s$ with respect to t, but how did I know that these functions were differentiable? The answer is given by the Implicit Function Theorem, which is stated as Theorem **??** below. In our case, (**??**) states that $D(p+t) - S(p) = 0$, which is an equation of the form $f(p,t) = 0$. This *implicitly defines* p as a function of t. The Implicit Function Theorem states that if f is (continuously) differentiable, and if $\frac{\partial f}{\partial x} \neq 0$, then at least locally this equation uniquely defines p as a (continuously) differentiable function of t. For more details, see Section **??** below, or for more even details and extensions, see my online notes [**?**].

1.3 Market equilibria as maximizers

For a smooth function f, if its derivative is positive to the left of x^* and negative to the right, then it achieves a maximum at x^*. In our simple market, demand is greater than supply to the left of equilibrium and less than supply to the right. Thus excess demand (demand − supply) acts like the derivative of a function that has a maximum at the market equilibrium.

Define $A(p)$ by

$$A(p) = \int_0^p D(x) - S(x)\, dx,$$

which is the (signed) area under the Walrasian demand curve and above the Walrasian supply curve up to p. (There is a mathematical chance that this area might be infinite, but that could not be the case for a demand curve, as it would imply that the revenue obtainable by lowering the price and increasing the quantity would be unbounded, which cannot happen in a real economy.) Then by the First Fundamental Theorem of Calculus (see, e.g., Apostol [**?**, Theorem 5.1, p. 202]),

$$A'(p) = D(p) - S(p), \quad \text{and} \quad A''(p) = D'(p) - S'(p) < 0,$$

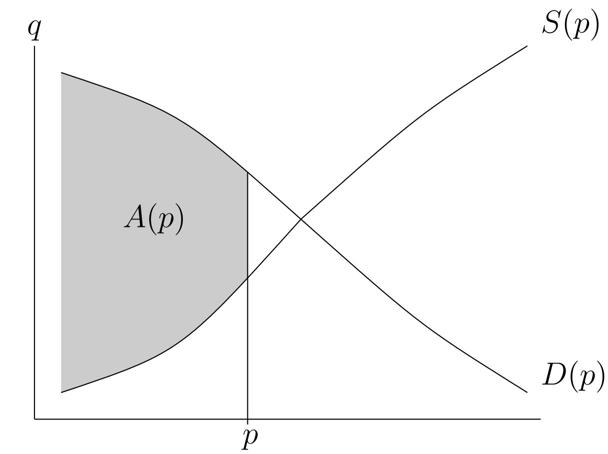

Figure 1.6. Market equilibrium maximizes $A(p)$.

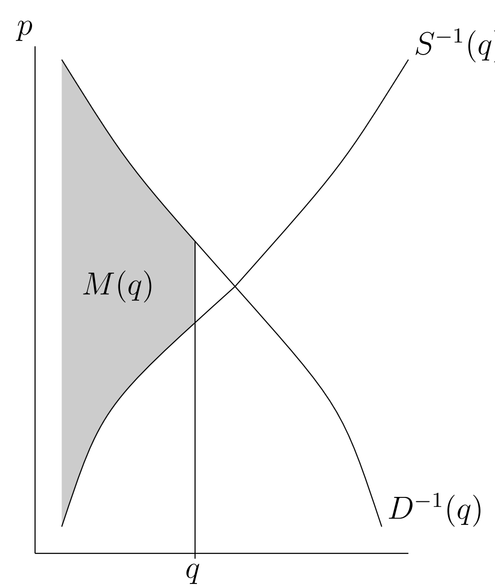

Figure 1.7. Market equilibrium maximizes Marshallian surplus $M(q)$.

so A is strictly concave. Its maximum occurs at the market clearing price p^*.

Thus, at least in this simple case, *finding the market clearing price is equivalent to maximizing an appropriate function.* This is a theme to which we shall return later. But now let us recast the above argument in a Marshallian framework.

The Marshallian demand curve is the inverse of the Walrasian demand curve and ditto for the supply curves. Marshallian **consumers' surplus** is the area under the Marshallian demand up to some quantity q. You may recall from Ec 11 (we shall derive it later) that the inverse supply curve of a price-taking seller is the marginal cost. Thus the area under the inverse supply curve up to q is the total variable cost of producing q. Thus the (signed) area $M(q)$ between the inverse demand curve and the inverse supply curve is equal to consumers' surplus minus (variable) cost. Once again, the market equilibrium quantity q^* maximizes the **Marshallian surplus** $M(q)$. See Figure **??**.

1.4 Appendix: The Implicit Function Theorem

An equation of the form

$$f(x,p) = y \tag{5}$$

implicitly defines x as a function of $p$on a domain P if there is a function ξ on P for which

$$f(\xi(p), p) = y$$

for all $p \in P$. It is traditional to assume that $y = 0$, but not essential. (We can always convert y to zero by defining $\hat{f}(x,p) = f(x,p) - y$. Then $f(x,p) = y$ if and only if $\hat{f}(x,p) = 0$.)

1.4.1 Classical Implicit Function Theorem *Let $X \times P$ be an open subset of $\boldsymbol{R}^{\mathrm{n}} \times \boldsymbol{R}^{\mathrm{m}}$, and let $f \colon X \times P \to \boldsymbol{R}^{\mathrm{n}}$ be C^k, for $k \geqslant 1$. Assume that $D_x f(\bar{x}, \bar{p})$ is invertible.*

*Then there are neighborhoods $U \subset X$ and $W \subset P$ of $\bar{x}$ and $\bar{p}$ on which (**??**) uniquely defines x as a function of p. That is, there is a function $\xi \colon W \to U$ such that:*

1. *$f(\xi(p); p) = f(\bar{x}, \bar{p})$ for all $p \in W$.*

2. *For each $p \in W$, $\xi(p)$ is the unique solution to (**??**) lying in U. In particular, then*

$$\xi(\bar{p}) = \bar{x}.$$

3. *ξ is C^k on W, and*

$$\begin{bmatrix} \dfrac{\partial \xi_1}{\partial p_1} & \cdots & \dfrac{\partial \xi_1}{\partial p_m} \\ \vdots & & \vdots \\ \dfrac{\partial \xi_n}{\partial p_1} & \cdots & \dfrac{\partial \xi_n}{\partial p_m} \end{bmatrix} = - \begin{bmatrix} \dfrac{\partial f_1}{\partial x_1} & \cdots & \dfrac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial x_1} & \cdots & \dfrac{\partial f_n}{\partial x_n} \end{bmatrix}^{-1} \begin{bmatrix} \dfrac{\partial f_1}{\partial p_1} & \cdots & \dfrac{\partial f_1}{\partial p_m} \\ \vdots & & \vdots \\ \dfrac{\partial f_n}{\partial p_1} & \cdots & \dfrac{\partial f_n}{\partial p_m} \end{bmatrix}.$$

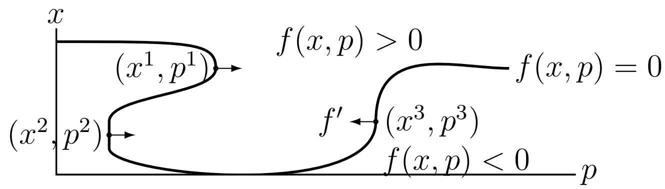

Figure 1.8. Looking for implicit functions.

1.4.2 Theorem (Implicit Function Theorem 0) *Let X be a subset of $\boldsymbol{R}^{\mathrm{n}}$, let P be a metric space, and let $f\colon X \times P \to \boldsymbol{R}^{\mathrm{n}}$ be continuous. Suppose the derivative $D_x f$ of f with respect to x exists at a point $(\bar{x}, \bar{p})$ in the interior of $X \times P$ and that $D_x f(\bar{x}, \bar{p})$ is invertible.*

Then for any neighborhood U of $\bar{x}$, there is a neighborhood W of $\bar{p}$ and a function $\xi\colon W \to U$ such that:

1. $\xi(\bar{p}) = \bar{x}$.
2. $f\big(\xi(p), p\big) = f(\bar{x}, \bar{p})$ *for all* $p \in W$.
3. ξ *is continuous at the point* $\bar{p}$.

However, it may be that ξ is neither continuous nor uniquely defined on any neighborhood of $\bar{p}$.

1.4.1 Examples

Figure **??** illustrates the Implicit Function Theorem for the special case $n = m = 1$, which is the only one I can draw. The figure is drawn sideways since we are looking for x as a function of p. In this case, the requirement that the differential with respect to x be invertible reduces to $\frac{\partial f}{\partial x} \neq 0$. That is, in the diagram the gradient of f may not be horizontal. In the figure, you can see that the points, (x^1, p^1), (x^2, p^2), and (x^3, p^3), the differentials $D_x f$ are zero. At (x^1, p^1) and (x^2, p^2) there is no way to define x as a continuous function of p locally. (Note however, that if we allowed a discontinuous function, we could define x as a function of p in a neighborhood of p^1 or p^2, but not uniquely.) At the point (x^3, p^3), we can uniquely define x as a function of p near p^3, but this function is not differentiable.

Another example of the failure of the conclusion of the Classical Implicit Function Theorem is provided by the following function.

1.4.3 Example (Differential not invertible) Define $f\colon \boldsymbol{R} \times \boldsymbol{R} \to \boldsymbol{R}$ by

$$f(x, p) = (x - p^2)(x - 2p^2).$$

Consider the function implicitly defined by $f(x, p) = 0$. The function f is zero along the parabolas $x = p^2$ and $x = 2p^2$, and in particular $f(0, 0) = 0$. See

Figure **??**. The hypothesis of the Implicit Function Theorem is not satisfied since $\frac{\partial f(0,0)}{\partial x} = 0$. The conclusion also fails. The problem here is not that a smooth implicit function through $(x, p) = (0, 0)$ fails to exist. The problem is that it is not unique. There are four distinct continuously differentiable implicitly defined functions. □

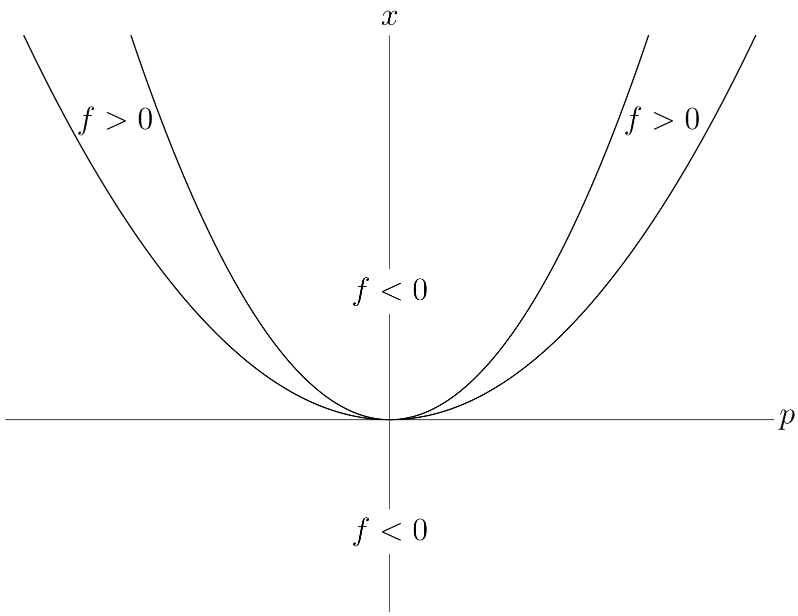

Figure 1.9. $f(x, p) = (p - x^2)(p - 2x^2)$.

1.4.4 Example (Lack of continuous differentiability) Consider the function $h(x) = x + 2x^2 \sin \frac{1}{x^2}$, see Figure **??**. This function is differentiable everywhere, but not continuously differentiable at zero. Furthermore, $h(0) = 0$, $h'(0) = 1$, but h is not monotone on any neighborhood of zero. Now consider the function $f(x, p) = h(x) - p$. It satisfies $f(0, 0) = 0$ and $\frac{\partial f(0,0)}{\partial x} \neq 0$, but there is no unique implicitly defined function on any neighborhood, nor is there any continuous implicitly defined function.

To see this, note that $f(x, p) = 0$ if and only if $h(x) = p$. So a unique implicitly defined function exists only if h is invertible on some neighborhood of zero. But this is not so, for given any $\varepsilon > 0$, there is some $0 < p < \frac{\varepsilon}{2}$ for which there are $0 < x < x' < \varepsilon$ satisfying $h(x) = h(x') = p$. It is also easy to see that no continuous function satisfies $h\big(\xi(p)\big) = p$ either. □

If x is more than one-dimensional there are subtler ways in which $D_x f$ may fail to be continuous. The next example is taken from Dieudonné [**?**, Problem 10.2.2, p. 273].

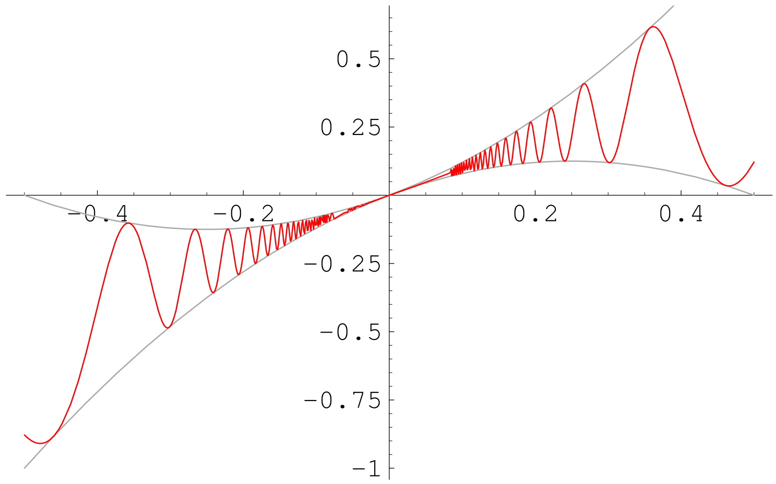

Figure 1.10. The function $x + 2x^2 \sin \frac{1}{x^2}$.

1.4.5 Example Define $f\colon \boldsymbol{R}^2 \to \boldsymbol{R}^2$ by

$$f_1(x, y) = x$$

and

$$f_2(x, y) = \begin{cases} y - x^2 & 0 \leqslant x^2 \leqslant y \\ \dfrac{y^2 - x^2 y}{x^2} & 0 \leqslant y < x^2 \\ -f_2(x, -y) & y < 0. \end{cases}$$

Dieudonné claims that f is everywhere differentiable on $\boldsymbol{R}^2$, and $Df(0,0)$ is the identity mapping, but Df is not continuous at the origin. I'll let you ponder that.

Furthermore in every neighborhood of the origin there are distinct points (x, y) and (x', y') with $f(x, y) = f(x', y')$. To find such a pair, pick a (small) $x' = x > 0$ and set $y = x^2$ and $y' = -x^2$. Then $f_1(x, y) = f_1(x', y') = x$, and $f_2(x, y) = f_2(x', y') = 0$.

This implies f has no local inverse, so the equation $f(x, y) - p = f(0, 0) = 0$ does not uniquely define (x, y) as a function of $p = (p_1, p_2)$ near the origin. □

References

[1] T. M. Apostol. 1967. *Calculus*, 2d. ed., volume 1. Waltham, Massachusetts: Blaisdell.

[2] K. C. Border. Notes on the implicit function theorem. http://www.its.caltech.edu/~kcborder/Notes/IFT.pdf

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