Notes on the Fundamental Theorem of Integral Calculus

I. Introduction. These notes supplement the discussion of line integrals presented in §1.6 of our text. Recall the Fundamental Theorem of Integral Calculus, as you learned it in Calculus I:

> *Suppose F is a real-valued function that is differentiable on an interval $[a,b]$ of the real line, and suppose F' is continuous on $[a,b]$. Then $\int_a^b F'(t)\,dt = F(b) - F(a)$.*

For example, if you want to integrate x^2 over $[0,1]$, the Fundamental Theorem says that it's only necessary to find a function F on $[0,1]$ with derivative x^2; say $F(x) = x^3/3$; then $\int_0^1 x^2\,dx = \int_0^1 F'(x)\,dx = F(1) - F(0) = 1/3$.

Exercise 1. Extend the version of the Fundamental Theorem stated above to *complex-valued* functions. (Just apply "real-valued" version to the real and imaginary parts of the complex-valued integral). Use the resulting theorem to find $\int_0^{i\pi/4} e^{it}\,dt$.

The goal of these notes is to prove the:

Fundamental Theorem of Integral Calculus for Line Integrals *Suppose G is an open subset of the plane with p and q (not necessarily distinct) points of G. Suppose γ is a smooth curve in G from p to q.*[1] *Then for any function F analytic on G,*

$$\int_\gamma F'(z)\,dz = F(q) - F(p).$$

Example 1. If γ is any smooth curve in $\mathbb{C}$ from p to q then $\int_\gamma z^2\,dz = (q^3 - p^3)/3$.

Proof. Use Fundamental Theorem for Line Integrals, with $F(z) = z^3/3$, just as we did above for the real case.

Example 2 (cf. Example 7, §1.5 of the text). Suppose γ is the line segment from $p = -\pi/2 + i$ to $q = \pi + i$. Find $\int_\gamma \cos z\,dz$.

Solution. Use the Fundamental Theorem with $F(z) = \sin z$ to obtain:

$$\int_\gamma \cos z\,dz = F(q) - F(p) = \sin(\pi + i) - \sin(-\pi/2 + i)....$$

Example 3 (the most important integral of all!). Suppose γ is a circle centered about the origin, oriented counter-clockwise. Then $\int_\gamma z^{-1}\,dz = 2\pi i$.

Proof. (slight modification of the argument done in class Wednesday, 2/26). Divide γ into two semicircles, say γ_1 from $-i$ to i in the closed right halfplane, and γ_2 from i to $-i$ in the

[1] This means that $\gamma : [a,b] \to G$ is differentiable on $[a,b]$, its derivative is continuous on $[a,b]$, and $\gamma(a) = p$, $\gamma(b) = q$

closed left halfplane[2]. Then the integral over the entire circle γ is the sum of the integrals over these semicircles.

For the integral over γ_1, let F be the principal branch of the complex logarithm, and take $G - \mathbb{C}\backslash\{(-\infty, 0]\}$ (make a sketch of G so that you can see its relationship with γ_1). Then F is analytic in G and γ_1 lies entirely in G, so by our Fundamental Theorem for Line Integrals:

$$\int_{\gamma_1} \frac{1}{z}\,dz = F(i) - F(-i) = \frac{\pi}{2}i - \frac{-\pi}{2}i = \pi i. \tag{1}$$

For the integral over γ_2, let $F(z) = \ln r + i\theta$ for $z = re^{i\theta}$ with $0 < \theta < 2\pi$. Now take $G = \mathbb{C}\backslash\{[0, \infty)\}$, so that (our new) F is analytic on (our new) G, γ_2 lies entirely in G (again, make the relevant sketches so that you understand this), and $F'(z) = 1/z$ at every point of G. By the Fundamental Theorem again:

$$\int_{\gamma_2} \frac{1}{z}\,dz = F(-i) - F(i) = \frac{3\pi}{2}i - \frac{\pi}{2}i = \pi i. \tag{2}$$

Thus by (1) and (2):

$$\int_{\gamma} \frac{1}{z}\,dz = \int_{\gamma_1} \frac{1}{z}\,dz + \int_{\gamma_2} \frac{1}{z}\,dz = \pi i + \pi i = 2\pi i,$$

as promised. □

Remark. We computed $\int_\gamma z^{-1}\,dz$ in class using nothing more than the definition, and it wasn't difficult—in fact it was *less difficult* than the above calculation. However, consider the following exercises:

Exercise 4. Compute $\int_\gamma z^{-1}\,dz$, where γ is a regular hexagon with center at the origin.

Exercise 5. Same problem, except now integrate z^n over γ, where n is any integer $\neq -1$.

Exercise 6. Generalize the previous two exercises to other simple closed curves in the plane, some of which may *not* surround the origin.

Some of the above exercises illustrate the following important

Corollary of Fund'l Thm. *If F is analytic on some open set Ω, then for every closed curve γ in Ω: $\int_\gamma F'(z)\,dz = 0$.*

Proof. It's trivial! γ begins and ends at the same point, so just apply the Fundamental Theorem with $p = q$. □

The key to proving the Fundamental Theorem for line integrals is the following version of:

[2] Be sure to sketch these curves!

The Chain Rule.[3] *Suppose F is analytic on an open subset G of the complex plane, and $\gamma : [a,b] \to G$ is a smooth curve. Then for any $t \in [a,b]$, the composite function $F \circ \gamma : [a,b] \to \mathbb{C}$ is differentiable at t, and*

$$(F \circ \gamma)'(t) = F'(\gamma(t))\gamma'(t) \qquad \textit{for all } t \in [a,b].$$

We'll prove the Chain Rule in a moment. Right now, let's note a few things:

(a) The "formula part" of the chain rule shows that $F \circ \gamma$ has *continuous* derivative on $[a,b]$, so $F \circ \gamma$ defines a smooth curve in the plane (draw a picture to illustrate this).

(b) This is the *fourth* version of the Chain Rule you've seen in your career in mathematics. The other three are:

 (i) The one you learned in Calculus I:

 If $\gamma : [a,b] \to [c,d]$ is differentiable on $[a,b]$ and $f : [c,d] \to \mathbb{R}$ is differentiable on $[c,d]$, then $f \circ \gamma : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$, and $(f \circ \gamma)'(t) = f'(\gamma(t))\gamma'(t)$ for each $t \in [a,b]$.

 (ii) The one we learned in §2.1 Example 4 of our text (without proving it):

 If g is analytic on an open subset Ω of the plane and f is analytic on an open set that contains $g(\Omega)$, then $f \circ g$ is analytic on Ω, and $(f \circ g)'(z) = f'(g(z))g'(z)$ for every $z \in \Omega$.

 (iii) The Chain Rule of Calc. III, which we'll state here, mostly without hypotheses, for functions of two variables:

 Suppose w depends differentiably on x and y, while x and y depend differentiably on t. Then w depends differentiably on t, and

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{dy}\frac{dy}{dt}.$$

Before setting out to prove our version of the Chain Rule, let's see how it gets used to prove the Fundamental Theorem for Line Integrals.

Chain Rule ⇒ Fundamental Theorem for Line Integrals. Recall that we're given F analytic on an open set G, and $\gamma : [a,b] \to G$ a smooth curve with $\gamma(a) = p$ and $\gamma(b) = q$. We want to show that $\int_\gamma F'(z)\,dz = F(q) - F(p)$. For this we begin with the *definition* of line integral:

$$\begin{aligned}
\int_\gamma F'(z)\,dz &\stackrel{\text{def}}{=} \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\
&= \int_a^b (F \circ \gamma)'(t)\,dt \qquad \text{(by the Chain Rule)} \\
&= (F \circ \gamma)(b) - (F \circ \gamma)(a) \quad (\textit{original} \text{ Fund'l Thm. Int. Calc.}) \\
&= F(q) - F(p)
\end{aligned}$$

□

[3] Be sure to draw diagrams to illustrate the composition of functions that's being talked about in each version of the chain rule discussed here!

Note that in the third line of the last calculation we used the *complex-valued* version of the "original" Fundamental Theorem of Integral Calculus (see Exercise 1, page 1 of these notes).

Proof of Chain Rule. Recall that we're given F analytic on the open set G of the plane and $\gamma : [a,b] \to G$ a smooth curve. Our goal is to show, for any $t \in [a,b]$, that $(F \circ \gamma)'(t) = F(\gamma(t))\gamma'(t)$.

The idea is to reduce everything to the real two-variable chain rule (the third one in the list on the previous page). For this write $u \stackrel{\text{def}}{=} \operatorname{Re} f$ and $v \stackrel{\text{def}}{=} \operatorname{Im} f$, so u and v are real-valued functions on G with partial derivatives existing at each point of G. Also let $\alpha \stackrel{\text{def}}{=} \operatorname{Re} \gamma$ and $\beta \stackrel{\text{def}}{=} \operatorname{Im} \gamma$, so α and β are real-valued differentiable functions on $[a,b]$.

As in Calc III, we'll think of $w \stackrel{\text{def}}{=} u \circ \gamma$ like this: $w = u(x,y)$ where $x = \alpha(t)$ and $y = \beta(t)$. So w is a function of t, hence repeating version (iii) of the Chain Rule:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{dy}\frac{dy}{dt},$$

that is, for each $t \in [a,b]$:

$$(u \circ \gamma)'(t) = \frac{\partial u}{\partial x}\,\alpha'(t) + \frac{\partial u}{\partial y}\,\beta'(t). \tag{3}$$

Upon replacing u by v in (3) we obtain:

$$(v \circ \gamma)'(t) = \frac{\partial v}{\partial x}\,\alpha'(t) + \frac{\partial v}{\partial y}\,\beta'(t). \tag{4}$$

Upon "adding equation (3) to i times equation (4)" we obtain:

$$(f \circ \gamma)'(t) \stackrel{\text{def}}{=} (u \circ \gamma)'(t) + i(v \circ \gamma)' = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\alpha'(t) + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\beta'(t). \tag{5}$$

The first term in large round brackets on the right-hand side of (5) is, from our work on the Cauchy-Riemann equations, just f'. Now use the Cauchy-Riemann equations on the second term in large round brackets:

$$\left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right) = \left(-\frac{\partial v}{\partial x} + i\frac{\partial u}{\partial x}\right) = i\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right) = if'$$

This, along with (5), and the (until now suppressed) fact that all partial derivatives are actually being evaluated at the point $\gamma(t)$, yields:

$$(f \circ \gamma)'(t) = f'(\gamma(t))(\alpha'(t) + i\beta'(t)) = f'(\gamma(t))\gamma'(t)$$

for each $t \in [a,b]$. $\square$