A Collection of Problems in Differential Calculus

Problems from Calculus I Final Examinations, 2000-2020 Department of Mathematics, Simon Fraser University

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Problems from Calculus I Final Examinations, 2000-2020 Department of Mathematics, Simon Fraser University

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To my sons, my best teachers. - Veselin Jungic

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Preface

The purpose of this collection of problems is to serve as a supplementary learning resource for students who are taking a differential calculus course at Simon Fraser University, Burnaby, BC, Canada. The collection contains problems posed at Math 151 - Calculus I and Math 150 - Calculus I With Review final examinations from 2000 to 2020. The problems are sorted by topic and most of them are accompanied with hints or solutions. Both courses, Math 150 and Math 151, are designed for students enrolled in one of Simon Fraser University's science (mathematics, physics, statistics, chemistry, earth sciences) or applied science (engineering, computing science) programs.

We welcome students from other institutions to use this collection as an additional resource in their quest to master differential calculus.

We hope that other calculus instructors will find this collection useful if only as a way to compare with their own practices.

Although all of the problems contained in this collection have been used, some of them multiple times, as final examination questions, their levels of difficulty may vary significantly from each other. The process of composing a final examination is never simple. The process is largely determined by institutionally prescribed course outcomes. The process also reflects individual instructors' goals of holding a fair and balanced examination that can serve as an objective tool for determining the level of each student's knowledge.

The last chapter in this collection contains a detailed list of recommendations to all students who are thinking about their well-being, learning, goals, and who want to be successful academically.

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No project such as this can be free from errors and incompleteness. The authors would be grateful to everyone who points out any typos, errors, or sends any other suggestion on how to improve this manuscript.

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Contents

Chapter 1

Limits and Continuity

1.1 Introduction

Use the following definitions, theorems, and properties to solve the problems contained in this Chapter.

Limit We write $\lim_{x\to a} f(x) = L$ and say "the limit of $f(x)$, as x approaches a, equals L" if it is possible to make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to a.

Limit - ε, δ Definition Let f be a function defined on some open interval that contains a, except possibly at a itself. Then we say that the limit of $f(x)$ as x approaches a is L, and we write $\lim_{x\to a} f(x) = L$ if for every number $\varepsilon > 0$ there is a $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < |x - a| < \delta$.

Limit and Right-hand and Left-hand Limits $\lim_{x\to a} f(x) = L \Leftrightarrow (\lim_{x\to a^-} f(x) = L$ and $\lim_{x\to a^+} f(x) = L)$

Infinite Limit Let f be a function defined on a neighbourhood of a, except possibly at a itself. Then $\lim_{x\to a} f(x) = \infty$ means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close to a, but not equal to a.

Vertical Asymptote The line $x = a$ is called a vertical asymptote of the curve $y = f(x)$ if at least one of the following statements is true:

$$\begin{array}{lll} \lim_{x\to a} f(x) = \infty & \lim_{x\to a^-} f(x) = \infty & \lim_{x\to a^+} f(x) = \infty \\ \lim_{x\to a} f(x) = -\infty & \lim_{x\to a^-} f(x) = -\infty & \lim_{x\to a^+} f(x) = -\infty \end{array}$$

Limit At Infinity Let f be a function defined on (a, ∞). Then $\lim_{x\to\infty} f(x) = L$ means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

Horizontal Asymptote The line $y = a$ is called a horizontal asymptote of the curve $y = f(x)$ if if at least one of the following statements is true:

$$\lim_{x\to\infty} f(x) = a \text{ or } \lim_{x\to-\infty} f(x) = a.$$

Limit Laws Let c be a constant and let the limits $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ exist. Then

1. $\lim_{x\to a}(f(x) \pm g(x)) = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x)$
2. $\lim_{x\to a}(c \cdot f(x)) = c \cdot \lim_{x\to a} f(x)$
3. $\lim_{x\to a}(f(x) \cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$
4. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ if $\lim_{x\to a} g(x) \neq 0$.

Squeeze Law If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

Trigonometric Limits $\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1$ and $\lim_{\theta\to 0} \frac{\cos\theta - 1}{\theta} = 0$.

The Number e $\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$ and $\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e$.

L'Hospital's Rule Suppose that f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a.) Suppose that $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$ or that $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$ if the limit on the right side exists (or is ∞ or $-\infty$).

Continuity We say that a function f is continuous at a number a if $\lim_{x\to a} f(x) = f(a)$.

Continuity and Limit If f is continuous at b and $\lim_{x\to a} g(x) = b$ then $\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)$.

Intermediate Value Theorem Let f be continuous on the closed interval $[a, b]$ and let $f(a) \neq f(b)$. For any number M between $f(a)$ and $f(b)$ there exists a number c in (a, b) such that $f(c) = M$.

1.2 Limits

Evaluate the following limits. Use limit theorems, not ε - δ techniques. If any of them fail to exist, say so and say why.

Evaluate the following limits.

1. $\lim_{x\to 10} \frac{x^2 - 100}{x - 10}$

Answer. 20.

2. $\lim_{x\to 10} \frac{x^2 - 99}{x - 10}$

Answer. Does not exist.

3. $\lim_{x\to 10} \frac{x^2 - 100}{x - 9}$

Answer. 0.

4. $\lim_{x\to 10} f(x)$, where $f(x) = x^2$ for all $x \neq 10$, but $f(10) = 99$.

Answer. 100.

5. $\lim_{x\to 10} \sqrt{-x^2 + 20x - 100}$

Answer. Does not exist. Consider the domain of $g(x) = \sqrt{-x^2 + 20x - 100} = \sqrt{-(x - 10)^2}$.

Evaluate the following limits.

6. $\displaystyle\lim_{x\to-4}\frac{x^2-16}{x+4}\ln|x|$

Answer. $-8\ln 4$.

7. $\displaystyle\lim_{x\to\infty}\frac{x^2}{e^{4x}-1-4x}$

Answer. 0. Note the exponential function in the denominator.

8. $\displaystyle\lim_{x\to-\infty}\frac{3x^6-7x^5+x}{5x^6+4x^5-3}$

Answer. $\dfrac{3}{5}$. Divide the numerator and denominator by the highest power.

9. $\displaystyle\lim_{x\to-\infty}\frac{5x^7-7x^5+1}{2x^7+6x^6-3}$

Answer. $\dfrac{5}{2}$.

10. $\displaystyle\lim_{x\to-\infty}\frac{2x+3x^3}{x^3+2x-1}$

Answer. 3.

11. $\displaystyle\lim_{x\to-\infty}\frac{5x+2x^3}{x^3+x-7}$

Answer. 2.

12. $\displaystyle\lim_{x\to\infty}\frac{ax^{17}+bx}{cx^{17}-dx^3},\ a,b,c,d\neq 0$

Answer. $\dfrac{a}{c}$.

13. $\displaystyle\lim_{x\to\infty}\frac{3x+|1-3x|}{1-5x}$

Hint. What is the value of $3x+|1-3x|$ if $x<\frac{1}{3}$?

Answer. 0.

14. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^6-3}}{\sqrt{x^6+5}}$

Answer. 1.

15. $\displaystyle\lim_{u\to\infty}\frac{u}{\sqrt{u^2+1}}$

Answer. 1.

16. $\displaystyle\lim_{x\to\infty}\frac{1+3x}{\sqrt{2x^2+x}}$

Answer. $\dfrac{3}{\sqrt{2}}$.

17. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{4x^2+3x}-7}{7-3x}$

Answer. $-\dfrac{2}{3}$.

18. $\displaystyle\lim_{x\to-\infty}\frac{\sqrt{x^2-9}}{2x-1}$

Answer. $-\dfrac{1}{2}$.

19. $\displaystyle\lim_{x\to1^+}\frac{\sqrt{x-1}}{x^2-1}$

Hint. Note that $x^2-1=(x-1)(x+1)$.

Answer. ∞.

20. Let $f(x)=\begin{cases}\frac{x^2-1}{|x-1|} & \text{if} \quad x\neq 1,\\ 4 & \text{if} \quad x=1.\end{cases}$ Find $\displaystyle\lim_{x\to1^-}f(x)$.

Hint. Which statement is true for $x<1$: $|x-1|=x-1$ or $|x-1|=1-x$?

Answer. -2.

21. Let $F(x)=\frac{2x^2-3x}{|2x-3|}$.

(a) Find $\displaystyle\lim_{x\to1.5^+}F(x)$.

(b) Find $\displaystyle\lim_{x\to1.5^-}F(x)$.

(c) Does $\displaystyle\lim_{x\to1.5}F(x)$ exist? Provide a reason.

Answer.

(a) 1.5.

(b) -1.5.

(c) No. The left-hand limit and the right-hand limit are not equal.

Evaluate the following limits. If any of them fail to exist, say so and say why.

22. $\displaystyle\lim_{x\to -2} \frac{2-|x|}{2+x}$

Answer. 1.

23. $\displaystyle\lim_{x\to 2^-} \frac{|x^2-4|}{10-5x}$

Answer. $\dfrac{4}{5}$.

24. $\displaystyle\lim_{x\to 4^-} \frac{|x-4|}{(x-4)^2}$

Answer. ∞.

25. $\displaystyle\lim_{x\to 8} \frac{(x-8)(x+2)}{|x-8|}$

Answer. Does not exist.

26. $\displaystyle\lim_{x\to 2} \left(\frac{1}{x^2+5x+6} - \frac{1}{x-2}\right)$

Answer. Does not exist.

27. $\displaystyle\lim_{x\to -1} \frac{x^2-x-2}{3x^2-x-1}$

Answer. 0.

28. $\displaystyle\lim_{x\to 16} \frac{\sqrt{x}-4}{x-16}$

Hint. Rationalize the numerator.

Answer. $\dfrac{1}{8}$.

29. $\displaystyle\lim_{x\to 8} \frac{\sqrt[3]{x}-2}{x-8}$

Hint. Note that $x-8 = (\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)$.

Answer. $\dfrac{1}{12}$.

30. $\displaystyle\lim_{x\to 4} \frac{2-\sqrt{x}}{4x-x^2}$

Answer. $\dfrac{1}{16}$

31. $\displaystyle\lim_{x\to 0} \frac{\sqrt{1+2x}-\sqrt{1-4x}}{x}$

Answer. 3.

32. Find constants a and b such that $\displaystyle\lim_{x\to 0} \frac{\sqrt{ax+b}-2}{x} = 1$.

Hint. Rationalize the numerator. Choose the value of b so that x becomes a factor in the numerator.

Answer. $a = b = 4$.

Evaluate the following limits. If any of them fail to exist, say so and say why.

33. $\displaystyle\lim_{x\to 5} e^{\frac{x-5}{\sqrt{x-1}-2}}$

Answer. e^4.

34. $\displaystyle\lim_{x\to 7} e^{\frac{\sqrt{x+2}-3}{x-7}}$

Answer. $e^{1/6}$.

35. $\displaystyle\lim_{t\to 0} \frac{\sqrt{\sin t+1}-1}{t}$

Answer. $\dfrac{1}{2}$.

36. $\displaystyle\lim_{x\to 8} \frac{x^{1/3}-2}{x-8}$

Hint. Note that $x-8 = (\sqrt[3]{x}-2)(\sqrt[3]{x^2}+2\sqrt[3]{x}+4)$.

Answer. $\dfrac{1}{12}$.

37. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+x}-x\right)$

Hint. Rationalize the numerator.

Answer. $\dfrac{1}{2}$.

38. $\displaystyle\lim_{x\to -\infty} \left(\sqrt{x^2+5x}-\sqrt{x^2+2x}\right)$

Hint. Rationalize the numerator. Note that $x\to -\infty$ and use the fact that if $x<0$ then $x = -\sqrt{x^2}$.

Answer. $-\dfrac{3}{2}$.

39. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2-x+1}-\sqrt{x^2+1}\right)$

Answer. $-\dfrac{1}{2}$.

40. $\displaystyle\lim_{x\to\infty} \left(\sqrt{x^2+3x-2}-x\right)$

Answer. $\dfrac{3}{2}$.

41. Is there a number b such that $\lim_{x\to -2} \dfrac{bx^2+15x+15+b}{x^2+x-2}$ exists? If so, find the value of b and the value of the limit.

Answer. $b = 3$.

Solution. Since the denominator approaches 0 as $x \to -2$, the necessary condition for this limit to exist is that the numerator approaches 0 as $x \to -2$. Thus we solve $4b - 30 + 15 + b = 0$ to obtain $b = 3$. $\lim_{x\to -2} \dfrac{3x^2+15x+18}{x^2+x-2} = -1$.

42. Determine the value of a so that $f(x) = \dfrac{x^2+ax+5}{x+1}$ has a slant asymptote $y = x + 3$.

Hint. Write $f(x) = x + \dfrac{(a-1)x+5}{x+1}$.

Answer. $a = 4$

43. Prove that $f(x) = \frac{\ln x}{x}$ has a horizontal asymptote $y = 0$.

Answer. $\lim_{x\to\infty} \dfrac{\ln x}{x} = 0$.

44. Let I be an open interval such that $4 \in I$ and let a function f be defined on a set $D = I\backslash\{4\}$. Evaluate $\lim_{x\to 4} f(x)$, where $x + 2 \le f(x) \le x^2 - 10$ for all $x \in D$.

Answer. 6.

Solution. From $\lim_{x\to 4}(x + 2) = 6$ and $\lim_{x\to 4}(x^2 - 10) = 6$, by the Squeeze Theorem, it follows that $\lim_{x\to 4} f(x) = 6$.

45. Evaluate $\lim_{x\to 1} f(x)$, where $2x - 1 \le f(x) \le x^2$ for all x in the interval $(0, 2)$.

Answer. 1.

Use the squeeze theorem to show that

46. $\lim_{x\to 0} x^4 \sin\left(\dfrac{1}{x}\right) = 0$.

Solution. Use the fact $-x^4 \le x^4 \sin\left(\dfrac{1}{x}\right) \le x^4$, $x \ne 0$.

47. $\lim_{x\to 0^+} \left(\sqrt{x}e^{\sin(1/x)}\right) = 0$.

Solution. From the fact that $|\sin(1/x)| \le 1$ for all $x \ne 0$ and the fact that the function $y = e^x$ is increasing conclude that $e^{-1} \le e^{\sin(1/x)} \le e$ for all $x \ne 0$. Thus $e^{-1} \cdot \sqrt{x} \le \sqrt{x}e^{\sin(1/x)} \le e \cdot \sqrt{x}$ for all $x > 0$. By the Squeeze Theorem, $\lim_{x\to 0^+} \left(\sqrt{x}e^{\sin(1/x)}\right) = 0$.

Evaluate the following limits. If any of them fail to exist, say so and say why.

48. $\lim_{x\to 0^+} \left[(x^2+x)^{1/3} \sin\left(\dfrac{1}{x^2}\right)\right]$

Hint. Squeeze Theorem.

Answer. 0.

49. $\lim_{x\to 0} x \sin\left(\dfrac{e}{x}\right)$

Hint. Squeeze Theorem.

Answer. 0.

50. $\lim_{x\to 0} x\sin\left(\frac{1}{x^2}\right)$

Hint. Squeeze Theorem.

Answer. 0.

51. $\lim_{x\to 0} \sqrt{x^2+x}\cdot\sin\left(\frac{\pi}{x}\right)$

Hint. Squeeze Theorem

Answer. 0.

52. $\lim_{x\to 0} x\cos^2\left(\frac{1}{x^2}\right)$

Hint. Squeeze Theorem.

Answer. 0.

53. $\lim_{x\to\pi/2^+} \frac{x}{\cot x}$

Answer. $-\infty$.

54. $\lim_{x\to 0} \frac{1-e^{-x}}{1-x}$

Answer. 0.

55. $\lim_{x\to 0} \frac{e^{2x}-1-2x}{x^2}$

Answer. 2.

56. $\lim_{x\to 2} \frac{e^x-e^2}{\cos\left(\frac{\pi x}{2}\right)+1}$

Answer. Does not exist.

57. $\lim_{x\to 1} \frac{x^2-1}{e^{1-x^7}-1}$

Answer. $-\frac{2}{7}$.

58. $\lim_{x\to 0} \frac{e^{-x^2}\cos(x^2)}{x^2}$

Answer. ∞.

59. $\lim_{x\to 1} \frac{x^{76}-1}{x^{45}-1}$

Hint. This is the case "0/0". Apply L'L'Hôpital's rule.

Answer. $\frac{76}{45}$.

60. $\lim_{x\to 1} \frac{x^a-1}{x^b-1}$, $a, b \neq 0$

Answer. $\frac{a}{b}$.

61. $\lim_{x\to 0} \frac{(\sin x)^{100}}{x^{99}\sin 2x}$

Hint. Write $\frac{1}{2}\cdot\left(\frac{\sin x}{x}\right)^{100}\cdot\frac{2x}{\sin 2x}$.

Answer. $\frac{1}{2}$.

62. $\lim_{x\to 0} \frac{x^{100}\sin 7x}{(\sin x)^{99}}$

Hint. Write $7\cdot\left(\frac{x}{\sin x}\right)^{101}\cdot\frac{\sin 7x}{7x}$.

Answer. 7.

63. $\lim_{x\to 0} \frac{x^{100}\sin 7x}{(\sin x)^{101}}$

Answer. 7.

64. $\lim_{x\to 0} \frac{\arcsin 3x}{\arcsin 5x}$

Hint. This is the case "0/0". Apply L'L'Hôpital's rule.

Answer. $\frac{3}{5}$.

65. $\lim_{x\to 0} \frac{\sin 3x}{\sin 5x}$

Answer. $\frac{3}{5}$.

66. $\lim_{x\to 0} \frac{x^3\sin\left(\frac{1}{x^2}\right)}{\sin x}$

Hint. Write $x^2\cdot\frac{x}{\sin x}\cdot\sin\left(\frac{1}{x^2}\right)$.

Answer. 0.

67. $\lim_{x\to 0} \frac{\sin x}{\sqrt{x\sin 4x}}$

Hint. $\frac{\sin x}{2|x|}\cdot\frac{1}{\sqrt{\frac{\sin 4x}{4x}}}$.

Answer. Does not exist.

68. $\lim\limits_{x\to 0} \dfrac{1-\cos x}{x\sin x}$

Hint. Write $\dfrac{1-\cos x}{x^2}\cdot\dfrac{x}{\sin x}$.

Answer. $\dfrac{1}{2}$.

69. $\lim\limits_{\theta\to \frac{3\pi}{2}} \dfrac{\cos\theta+1}{\sin\theta}$

Answer. -1.

70. $\lim\limits_{x\to \frac{\pi}{2}} \left(x-\dfrac{\pi}{2}\right)\tan x$

Answer. -1.

71. $\lim\limits_{x\to\infty} x\tan(1/x)$

Hint. Substitute $t=\dfrac{1}{x}$.

Answer. 1.

72. $\lim\limits_{x\to 0} \left(\dfrac{1}{\sin x}-\dfrac{1}{x}\right)$

Hint. This is the case "$\infty-\infty$". Write $\dfrac{x-\sin x}{x\sin x}$ and apply L'L'Hôpital's rule.

Answer. 0.

73. $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3}$

Answer. $\dfrac{1}{6}$.

74. $\lim\limits_{x\to 0}(\csc x-\cot x)$

Answer. 0.

75. $\lim\limits_{x\to 0^+}(\sin x)(\ln\sin x)$

Hint. This is the case "$0\cdot\infty$". Write $\dfrac{\ln\sin x}{\frac{1}{\sin x}}$ and apply L'L'Hôpital's rule.

Answer. 0.

76. $\lim\limits_{x\to\infty}\left(x\cdot\ln\dfrac{x-1}{x+1}\right)$

Answer. -2.

77. $\lim\limits_{x\to\infty}\dfrac{e^{\frac{x}{10}}}{x^3}$

Answer. ∞.

78. $\lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}}$

Hint. This is the case "∞/∞". Apply L'L'Hôpital's rule.

Answer. 0.

79. $\lim\limits_{x\to\infty}\dfrac{\ln 3x}{x^2}$

Answer. 0.

80. $\lim\limits_{x\to\infty}\dfrac{(\ln x)^2}{x}$

Answer. 0.

81. $\lim\limits_{x\to 1}\dfrac{\ln x}{x}$

Answer. 0.

82. $\lim\limits_{x\to 0}\dfrac{\ln(2+2x)-\ln 2}{x}$

Hint. This is the case "$0/0$". Write $\dfrac{\ln(1+x)}{x}$ and apply L'L'Hôpital's rule.

Answer. 1.

83. $\lim\limits_{x\to\infty}\dfrac{\ln((2x)^{1/2})}{\ln((3x)^{1/3})}$

Hint. Use properties of logarithms first.

Answer. $\dfrac{3}{2}$.

84. $\lim\limits_{x\to 0}\dfrac{\ln(1+3x)}{2x}$

Answer. $\dfrac{3}{2}$.

85. $\lim\limits_{x\to 1}\dfrac{\ln(1+3x)}{2x}$

Hint. The denominator approaches 2.

Answer. $\ln 2$.

86. $\lim\limits_{\theta \to \frac{\pi}{2}^+} \dfrac{\ln(\sin\theta)}{\cos\theta}$

Hint. This is the case “0/0”. Apply L’Hospital’s rule.

Answer. 0.

87. $\lim\limits_{x \to 1} \dfrac{1 - x + \ln x}{1 + \cos(\pi x)}$

Hint. Apply L’Hospital’s rule twice.

Answer. $-\dfrac{1}{\pi^2}$.

88. $\lim\limits_{x \to 0} \left(\dfrac{1}{x^2} - \dfrac{1}{\tan x}\right)$

Hint. This is the case "$\infty - \infty$". Write $\dfrac{\sin x - x^2\cos x}{x^2 \sin x}$ and apply L’Hospital’s rule.

Answer. ∞.

89. $\lim\limits_{x \to 0^+} \left(\dfrac{1}{x} - \dfrac{1}{e^x - 1}\right)$

Answer. $\dfrac{1}{2}$.

90. $\lim\limits_{x \to 0} (\cosh x)^{\frac{1}{x^2}}$

Hint. This is the case "1^∞". Write $e^{\frac{\ln\cosh x}{x^2}}$. Apply L’Hospital’s rule and use the fact that the exponential function $f(x) = e^x$ is continuous.

Answer. $e^{\frac{1}{2}}$.

91. $\lim\limits_{x \to 0^+} (\cos x)^{\frac{1}{x}}$

Answer. 1.

92. $\lim\limits_{x \to 0^+} (\cos x)^{\frac{1}{x^2}}$

Answer. $e^{-1/2}$.

93. $\lim\limits_{x \to 0^+} x^x$

Hint. This is the case "0^0". Write $x^x = e^{x\ln x} = e^{\frac{\ln x}{x^{-1}}}$. Apply L’Hospital’s rule and use the fact that the exponential function $f(x) = e^x$ is continuous.

Answer. 1.

94. $\lim\limits_{x \to 0^+} x^{\sqrt{x}}$

Answer. 1.

95. $\lim\limits_{x \to 0^+} x^{\tan x}$

Answer. 1.

96. $\lim\limits_{x \to 0^+} (\sin x)^{\tan x}$

Answer. 1.

97. $\lim\limits_{x \to 0} (1 + \sin x)^{\frac{1}{x}}$

Answer. e.

98. $\lim\limits_{x \to \infty} (x + \sin x)^{\frac{1}{x}}$

Hint. This is the case "∞^0".

Answer. 1.

99. $\lim\limits_{x \to \infty} x^{\frac{1}{x}}$

Answer. 1.

100. $\lim\limits_{x \to \infty} \left(1 + \dfrac{1}{x}\right)^{2x}$

Answer. e^2.

101. $\lim\limits_{x \to \infty} \left(1 + \sin\dfrac{3}{x}\right)^{x}$

Answer. e^3.

102. $\lim\limits_{x \to 0^+} (x + \sin x)^{\frac{1}{x}}$

Answer. 0.

103. $\lim\limits_{x \to 0^+} \left(\dfrac{x}{x+1}\right)^{x}$

Hint. Write $e^{x\ln\frac{x}{x+1}} = e^{x\ln x} \cdot e^{-x\ln(x+1)}$ and make your conclusion.

Answer. 1.

104. $\lim_{x\to e^+} (\ln x)^{\frac{1}{x-e}}$

Answer. $e^{\frac{1}{e}}$.

105. $\lim_{x\to e^+} (\ln x)^{\frac{1}{x}}$

Answer. 1.

106. $\lim_{x\to 0} e^{x\sin(1/x)}$

Hint. Use the Squeeze Theorem.

Answer. 1.

107. $\lim_{x\to 0} (1-2x)^{1/x}$

Hint. Write $\left((1-2x)^{-\frac{1}{2x}}\right)^{-2}$.

Answer. e^{-2}.

108. $\lim_{x\to 0^+} (1+7x)^{1/5x}$

Hint. Write $\left((1+7x)^{\frac{1}{7x}}\right)^{\frac{7}{5}}$.

Answer. $e^{\frac{7}{5}}$.

109. $\lim_{x\to 0^+} (1+3x)^{1/8x}$

Hint. Write $\left((1+3x)^{\frac{1}{3x}}\right)^{\frac{3}{8}}$.

Answer. $e^{\frac{3}{8}}$.

110. $\lim_{x\to 0} \left(1+\frac{x}{2}\right)^{3/x}$

Hint. Write $\left(\left(1+\frac{x}{2}\right)^{\frac{2}{x}}\right)^{\frac{3}{2}}$.

Answer. $e^{\frac{3}{2}}$.

111. Let $x_1 = 100$, and for $n \geq 1$, let $x_{n+1} = \frac{1}{2}\left(x_n + \frac{100}{x_n}\right)$. Assume that $L = \lim_{n\to\infty} x_n$ exists. Calculate L.

Hint. Use the fact that $L = \lim_{n\to\infty} x_n$ to conclude $L^2 = 100$. Can L be negative?

Answer. 10.

Compute the following limits, or show that they do not exist.

112. $\lim_{x\to 0} \frac{1-\cos x}{x^2}$

Hint. Write $\frac{2\sin^2 \frac{x}{2}}{x^2}$, or use L'L'Hôpital's rule.

Answer. $\frac{1}{2}$.

113. $\lim_{x\to 2\pi} \frac{1-\cos x}{x^2}$.

Answer. 0.

114. $\lim_{x\to -1} \arcsin x$.

Answer. Does not exist. Note that the domain of $f(x) = \arcsin x$ is the interval $[-1, 1]$.

Compute the following limits or state why they do not exist:

115. $\lim_{h\to 0} \frac{\sqrt[4]{16+h}-2}{2h}$

Answer. $\frac{1}{64}$.

116. $\lim_{x\to 1} \frac{\ln x}{\sin(\pi x)}$

Hint. Use L'L'Hôpital's rule.

Answer. $-\frac{1}{\pi}$.

117. $\lim_{u\to\infty} \frac{u}{\sqrt{u^2+1}}$

Hint. Divide the numerator and denominator by u.

Answer. 1.

118. $\lim_{x\to 0} (1-2x)^{1/x}$

Answer. e^{-2}.

119. $\lim_{x\to 0} \frac{(\sin x)^{100}}{x^{99}\sin(2x)}$

Answer. $\frac{1}{2}$

120. $\lim_{x\to\infty} \frac{1.01^x}{x^{100}}$

Hint. Think, exponential vs. polynomial.

Answer. ∞.

Find the following limits. If a limit does not exist, write 'DNE'. No justification necessary.

121. $\lim_{x\to 0} \frac{(2+x)^{2016}-2^{2016}}{x}$

Answer. $2016\cdot 2^{2015}$.

122. $\lim_{x\to\infty} (\sqrt{x^2+x}-x)$

Answer. $\frac{1}{2}$.

123. $\lim_{x\to 0} \cot(3x)\sin(7x)$

Answer. $\frac{7}{3}$.

124. $\lim_{x\to 0^+} x^x$

Answer. 1.

125. $\lim_{x\to\infty} \frac{x^2}{e^x}$

Answer. 0.

126. $\lim_{x\to 3} \frac{\sin x - x}{x^3}$

Answer. $\frac{\sin 3 - 3}{27}$.

Evaluate the following limits, if they exist.

127. $\lim_{x\to 0} \frac{f(x)}{|x|}$ given that $\lim_{x\to 0} xf(x) = 3$.

Hint. Consider $\lim_{x\to 0} \frac{xf(x)}{x|x|}$.

Answer. Does not exist.

128. $\lim_{x\to 1} \frac{\sin(x-1)}{x^2+x-2}$

Answer. $\frac{1}{3}$.

129. $\lim_{x\to -\infty} \frac{\sqrt{x^2+4x}}{4x+1}$

Hint. Note that $x<0$.

Answer. $-\frac{1}{4}$.

130. $\lim_{x\to\infty} \frac{\sqrt{x^4+2}}{x^4-4}$

Answer. 0

131. $\lim_{x\to\infty} (e^x+x)^{1/x}$

Answer. e.

Evaluate the following limits, if they exist.

132. $\lim_{x\to 4} \left[\frac{1}{\sqrt{x}-2} - \frac{4}{x-4}\right]$

Answer. $\frac{1}{4}$.

133. $\lim_{x\to 1} \frac{x^2-1}{e^{1-x^2}-1}$

Answer. -1.

134. $\lim_{x\to 0} (\sin x)(\ln x)$

Answer. 0.

Evaluate the following limits. Use "∞" or "$-\infty$" where appropriate.

135. $\lim_{x\to 1^-} \frac{x+1}{x^2-1}$

Answer. $-\infty$.

136. $\lim_{x\to 0} \frac{\sin 6x}{2x}$

Answer. 3.

137. $\lim_{x\to 0} \frac{\sinh 2x}{xe^x}$

Answer. 2.

138. $\lim_{x\to 0^+} (x^{0.01}\ln x)$

Answer. 0.

139. Use the $\varepsilon - -\delta$ definition of limits to prove that

$$\lim_{x\to 0} x^3 = 0.$$

Solution. Let $\varepsilon > 0$ be given. We need to find $\delta = \delta(\varepsilon) > 0$ such that $|x - 0| < \delta \Rightarrow |x^3 - 0| < \varepsilon$, which is the same as $|x| < \delta \Rightarrow |x^3| < \varepsilon$. Clearly, we can take $\delta = \sqrt[3]{\varepsilon}$. Indeed, for any $\varepsilon > 0$ we have that $|x| < \sqrt[3]{\varepsilon} \Rightarrow |x|^3 = |x^3| < \varepsilon$ and, by definition, $\lim\limits_{x\to 0} x^3 = 0$.

140.

(a) Sketch an approximate graph of $f(x) = 2x^2$ on $[0, 2]$. Next, draw the points $P(1, 0)$ and $Q(0, 2)$. When using the precise definition of $\lim_{x\to 1} f(x) = 2$, a number δ and another number ε are used. Show points on the graph which these values determine. (Recall that the interval determined by δ must not be greater than a particular interval determined by ε.)

(b) Use the graph to find a positive number δ so that whenever $|x-1| < \delta$ it is always true that $|2x^2 - 2| < \frac{1}{4}$.

(c) State exactly what has to be proved to establish this limit property of the function f.

Answer. For any $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon) > 0$ such that $|x - 1| < \delta \Rightarrow |2x^2 - 2| < \varepsilon$.

141. Give an example of a function $F = f + g$ such that the limits of f and g at a do not exist and that the limit of F at a exists.

Answer. Take, for example, $f(x) = \text{sign}(x)$, $g(x) = -\text{sign}(x)$, and $a = 0$.

142. If $\lim\limits_{x\to a}[f(x) + g(x)] = 2$ and $\lim\limits_{x\to a}[f(x) - g(x)] = 1$ find $\lim\limits_{x\to a}[f(x) \cdot g(x)]$.

Answer. $\dfrac{2}{3}$.

143. If f' is continuous, use L'Hospital's rule to show that

$$\lim_{h\to 0} \frac{f(x+h) - f(x-h)}{2h} = f'(x).$$

Explain the meaning of this equation with the aid of a diagram.

Answer. $\lim\limits_{h\to 0} \dfrac{f(x+h) - f(x-h)}{2h} = \lim\limits_{h\to 0} \dfrac{f'(x+h) + f'(x-h)}{2}$ and, since f' is continuous, $\lim\limits_{h\to 0} f'(x+h) = \lim\limits_{h\to 0} f'(x-h) = f'(x)$.

1.3 Continuity

Recall that a function f is continuous at a number a if $\lim\limits_{x\to a} f(x) = f(a)$. Alternatively, a function f is continuous at a number a if

(a) The function f is defined at the number a;

(b) The limit $\lim\limits_{x\to a} f(x)$ exists;

(c) $\lim\limits_{x\to a} f(x) = f(a)$.

1. Given the function

$$f(x) = \begin{cases} c - x & \text{if} \quad x \le \pi \\ c \sin x & \text{if} \quad x > \pi \end{cases}$$

(a) Find the value of the constant c so that the function $f(x)$ is continuous.

(b) For the value of c found in part (a), verify whether the 3 conditions for continuity are satisfied.

(c) Draw a graph of $f(x)$ from $x = -\pi$ to $x = 3\pi$ indicating the scaling used.

Hint. To find c, solve $\lim\limits_{x\to\pi^-} f(x) = \lim\limits_{x\to\pi^-} f(x)$ for c.

Answer.

(a) $c = \pi$.

(b) For all $x \le \pi$, $f = \pi - x$ is a linear function and so is continuous. For all $x \ge \pi$, $f = \pi \sin x$ is a sine function and so is continuous. When $x = \pi$, we have:

i. $f(\pi) = 0$.

ii. $\lim\limits_{x\to\pi^-} (\pi - x) = 0$, and $\lim\limits_{x\to\pi^-} \pi \sin(x) = 0$. Therefore, $\lim\limits_{x\to\pi} f(x)$ exists, and is equal to 0.

iii. $\lim\limits_{x\to\pi} f(x) = 0 = f(\pi)$.

(c)

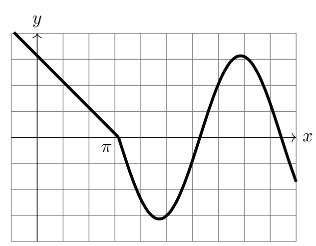


Figure 1.1 $c = \pi$

2.

(a) Use the Intermediate Value Theorem to show that $2^x = \dfrac{10}{x}$ for some $x > 0$.

(b) Show that the equation $2^x = \dfrac{10}{x}$ has no solution for $x < 0$.

Solution.

(a) Let $f(x) = 2^x - \dfrac{10}{x}$. Note that the domain of f is the set $\mathbb{R}\backslash\{0\}$ and that on its domain, as a sum of two continuous function, f is continuous.

(b) For all $x \in (-\infty, 0)$ we have that $\dfrac{10}{x} < 0$ which implies that for all $x \in (-\infty, 0)$ we have that all $f(x) > 0$.

3. Sketch a graph of the function

$$f(x) = \begin{cases} 2 - x^2 & \text{if } 0 \le x < 1 \\ \frac{5}{2} & \text{if } x = 1 \\ |2 - x| & \text{if } 1 < x \le 3 \\ \frac{1}{x-3} & \text{if } 3 < x \le 5 \\ 2 + \sin(2\pi x) & \text{if } 5 < x \le 6 \\ 2 & \text{if } x > 6 \end{cases}$$

Answer the following questions by YES or NO:

(a) Is f continuous at:

i. $x = 1$.

ii. $x = 6$.

(b) Do the following limits exist?

i. $\lim_{x\to 1} f(x)$.

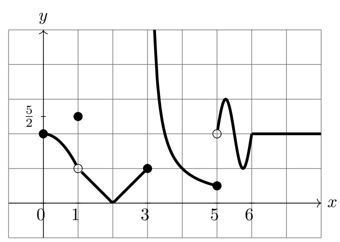
ii. $\lim_{x\to 3^-} f(x)$.

(c) Is f differentiable

i. at $x = 1$?

ii. on $(1, 3)$?

Answer.



(a) i. No.

ii. Yes.

(b) i. Yes.

ii. Yes.

(c) i. No.

ii. No.

4. Assume that

$$f(x) = \begin{cases} 2 + \sqrt{x} & \text{if } x \ge 1 \\ \frac{x}{2} + \frac{5}{2} & \text{if } x < 1 \end{cases}$$

(a) Determine whether or not f is continuous at $x = 1$. Justify your answer and state your conclusion.

(b) Using the definition of the derivative, determine $f'(1)$.

Answer.

(a) Check that $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$.

(b) $\frac{1}{2}$. Note $\lim\limits_{x\to 1^-} \frac{\frac{5+x}{2} - 3}{x-1} = \frac{1}{2}$ and $\lim\limits_{x\to 1^+} \frac{(2+\sqrt{x}) - 3}{x-1} = \frac{1}{2}$.

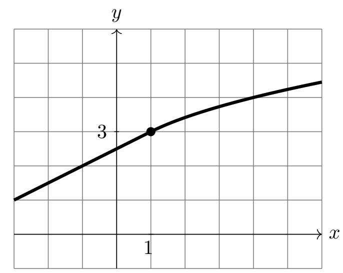


Figure 1.2 A continuous function

5. Find the value of b so that the function

$$f(x) = \begin{cases} x^3 + bx - 7 & \text{if } x \le 2 \\ be^{x-2} & \text{if } x > 2 \end{cases}$$

is continuous everywhere. Justify your answer.

Answer. -1.

6. Find the value of $a \in \mathbb{R}$ so that the function

$$f(x) = \begin{cases} ax^2 + 9 & \text{if } x > 0 \\ x^3 + a^3 & \text{if } x \ge 0 \end{cases}$$

is continuous.

Answer. $\sqrt[3]{9}$.

7. Find the value of $a \in \mathbb{R}$ so that the function

$$f(x) = \begin{cases} ax^2 + 2x & \text{if } x < 2 \\ x^3 - ax & \text{if } x \ge 2 \end{cases}$$

is continuous.

Answer. $\frac{2}{3}$.

8. Determine $a \in \mathbb{R}$ such that the function

$$f(x) \begin{cases} \frac{\cos x}{2x-\pi} & \text{if } x > \frac{\pi}{2} \\ ax & \text{if } x \le \frac{\pi}{2} \end{cases}$$

is continuous everywhere. Justify your answer.

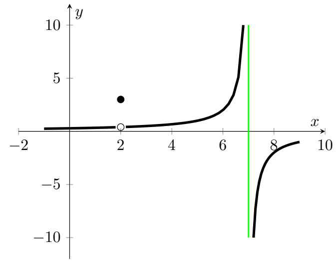
Answer. $-\frac{1}{\pi}$.

9. Give one example of a function $f(x)$ that is continuous for all values of x except $x = 3$, where it has a removable discontinuity. Explain how you know that f is discontinuous at $x = 3$, and how you know that the discontinuity is removable.

Answer. $f(x) = \frac{x^2 - 9}{x - 3}$ if $x \ne 3$ and $f(3) = 0$.

10. Sketch the graph of a function that has a removable discontinuity at $x = 2$ and an infinite discontinuity at $x = 7$, but is continuous everywhere else.

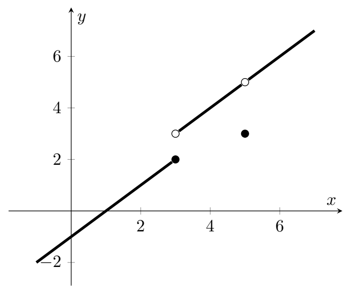
Answer.



Removable and infinite

11. Sketch the graph of a function that has a jump discontinuity at $x = 3$ and a removable discontinuity at $x = 5$, but is continuous everywhere else.

Answer.



Removable and jump

12. A function $h : I \to I$ is said to have a fixed point at $x = c \in I$ if $h(c) = c$. Suppose that the domain and range of a function $f(x)$ are both the interval $[0, 1]$ and that f is continuous on its domain with $f(0) \neq 0$ and $f(1) \neq 1$. Prove that f has at least one fixed point, i.e. prove that $f(c) = c$ for some $c \in (0, 1)$.

Hint. Consider the function $g(x) = f(x) - x$.

1.4 Miscellaneous

Solve the following equations:

1. $\pi^{x+1} = e$.

Answer. $x = \dfrac{1 - \ln \pi}{\ln \pi}$.

2. $2^{3^x} = 10$.

Answer. $x = -\dfrac{\log \log 2}{\log 3}$.

3. Find the domain of the function $f(x) = \dfrac{\ln(\ln(\ln x))}{x - 3} + \sin x$.

Answer. $(e, 3) \cup (3, \infty)$.

Solve the following problems:

4. What is meant by saying that L is the *limit* of $f(x)$ as x approaches a?

Answer. Give a definition of the limit.

5. What is meant by saying that the function $f(x)$ is *continuous* at $x = a$?

Answer. Give a definition of a function continuous at a point.

6. State two properties that a continuous function $f(x)$ can have, either of which guarantees the function is not differentiable at $x = a$. Draw an example for each.

Answer. A corner or a vertical tangent; $y = |x|$; $y = x^{\frac{1}{3}}$.

Chapter 2

Differentiation Rules

2.1 Introduction

Use the following definitions, techniques, properties, and algorithms to solve the problems contained in this Chapter.

Derivative The derivative of a function f at a number a is $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ if this limit exists.

Tangent Line An equation of the tangent line to $y = f(x)$ at $(a, f(a))$ is given by $y - f(a) = f'(a)(x - a)$.

Product and Quotient Rules If f and g are both differentiable, then $(fg)' = f \cdot g' + g \cdot f'$ and $\left(\dfrac{f}{g}\right)' = \dfrac{g \cdot f' - f \cdot g'}{g^2}$, with $g(x) \neq 0$.

Chain Rule If f and g are both differentiable and $F = f \circ g$ is the composite function defined by $F(x) = f(g(x))$, then F is differentiable and F' is given by $F'(x) = f'(g(x)) \cdot g'(x)$.

Implicit Differentiation Let a function $y = y(x)$ be implicitly defined by $F(x, y) = G(x, y)$. To find the derivative y' do the following:

1. Use the chain rule to differentiate both sides of the given equation, thinking of x as the independent variable.
2. Solve the resulting equation for $\dfrac{dy}{dx}$.

The Method of Related Rates If two variables are related by an equation and both are functions of a third variable (such as time), we can find a relation between their rates of change. We say the rates are related, and we can compute one if we know the other. We proceed as follows:

1. Identify the independent variable on which the other quantities depend and assign it a symbol, such as t. Also, assign symbols to the variable quantities that depend on t.
2. Find an equation that relates the dependent variables.
3. Differentiate both sides of the equation with respect to t (using the chain rule if necessary).
4. Substitute the given information into the related rates equation and solve for the unknown rate.

2.2 Derivatives

Recall that if $f'(a)$ exists then

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.$$

1.

(a) Assume that $f(x)$ is a real-valued function defined for all real numbers x on an open interval whose centre is a certain real number a. What does it mean to say that $f(x)$ has a derivative $f'(a)$ at $x=a$, and what is the value of $f'(a)$? (Give the definition of $f'(a)$.)

(b) Use the definition of $f'(a)$ you have just given in part (a) to show that if $f(x) = \dfrac{1}{2x-1}$ then $f'(3) = -0.08$.

(c) Find $\displaystyle\lim_{h\to 0} \frac{\sin^7\left(\frac{\pi}{6}+\frac{h}{2}\right) - \left(\frac{1}{2}\right)^7}{h}$.

Answer.

(a) $\displaystyle f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$.

(b) True.

(c) $frac7\sqrt{3}256$.

Solution.

(a) We state the definition of the derivative:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}..$$

(b) We have:

$$f'(3) = \lim_{h\to 0} \frac{\frac{1}{2(3+h)-1} - \frac{1}{5}}{h} = \lim_{h\to 0} \frac{-2}{5(5+2h)} = -0.08.$$

(c) We have:

$$\begin{aligned}\lim_{h\to 0} \frac{\sin^7\left(\frac{\pi}{6}+\frac{h}{2}\right) - \left(\frac{1}{2}\right)^7}{h} &= \frac{d}{dx}\left(\sin^7 \frac{x}{2}\right)\Big|_{x=\frac{\pi}{3}} \\ &= \frac{7}{2}\cdot \sin^6 \frac{\pi}{6} \cdot \cos\frac{\pi}{6} \\ &= \frac{7\sqrt{3}}{256}.\end{aligned}$$

2. Explain why the function

$$f(x) = \begin{cases} x^2+2x+1, & \text{if } x \le 0 \\ 1+\sin x, & \text{if } x > 0 \end{cases}$$

is continuous but not differentiable on the interval $(-1,1)$.

Answer. $\displaystyle\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = 1, \ \lim_{x\to 0^-} \frac{f(x)-f(0)}{x} = 2.$

3. Let I be a bounded function on $\mathbb{R}$ and define f by $f(x) = x^2 I(x)$. Show that f is differentiable at $x = 0$.

Solution. Let $|I(x)| \le M$ for all $x \in \mathbb{R}$. Then for any $h \ne 0$, $\left|\dfrac{h^2 I(h)}{h}\right| = |hI(h)|$. Use the Squeeze Theorem to conclude that f is differentiable at $x = 0$.

4. Use the definition of the derivative to find $f'(2)$ for $f(x) = x + \frac{1}{x}$.

Answer. $\frac{3}{4}$.

Solution. $f'(2) = \displaystyle\lim_{x\to 2} \frac{x + \frac{1}{x} - \frac{5}{2}}{x-2} = \frac{3}{4}$.

5. Use the definition of the derivative to find $f'(1)$ for $f(x) = 3x^2 - 4x + 1$.

Answer. 2.

6. Use the definition of the derivative to find the derivative of $f(x) = \sqrt{x}$. Do not use L'Hopital's rule.

Answer. $\dfrac{1}{2\sqrt{x}}$.

7. If g is continuous (but not differentiable) at $x = 0$, $g(0) = 8$, and $f(x) = xg(x)$, find $f'(0)$.

Solution. Since g is not differentiable we cannot use the product rule. $f'(0) = \displaystyle\lim_{h\to 0} \frac{hg(h)}{h} = 8$.

8. Using the definition of the derivative of $f(x)$ at $x = 4$, find the value of $f'(4)$ if $f(x) = \sqrt{5-x}$.

Answer. -0.5.

Solution. $f'(4) = \displaystyle\lim_{h\to 0} \frac{\sqrt{5-(x+h)} - 1}{h} = -0.5$.

9. Let f be a function that is continuous everywhere and let

$$F(x) = \begin{cases} \frac{f(x)\sin^2 x}{x} & \text{if} \quad x \ne 0, \\ 0 & \text{if} \quad x = 0. \end{cases}$$

Use the definition of derivatives to evaluate $F'(0)$. Your answer should be in terms of f.

Answer. $F'(0) = f(0)$.

Solution. $F'(0) = \displaystyle\lim_{h\to 0} \frac{\frac{f(h)\sin^2 h}{h}}{h} = \lim_{h\to 0} \frac{f(h)\sin^2 h}{h^2} = f(0)$.

10. The function

$$f(x) = \begin{cases} e^x & \text{if} \quad x \le 1 \\ mx + b & \text{if} \quad x > 1 \end{cases}$$

is continuous and differentiable at $x = 1$. Find the values for the constants m and b.

Solution. $m = e, b = 0$. Solve $\displaystyle\lim_{x\to 1^-} e^x = \lim_{x\to 1^+} (mx+b)$ and $\displaystyle\lim_{x\to 1^-} \frac{e^x - e}{x-1} = \lim_{x\to 1^+} \frac{mx + b - (m+b)}{x-1}$ for m and b.

11. Suppose the functions $F(x)$ and $G(x)$ satisfy the following properties:

$$\begin{array}{lll} F(3) = 2, & G(3) = 4, & G(0) = 3 \\ F'(3) = -1, & G'(3) = 0, & G'(0) = 0 \end{array}$$

(a) If $S(x) = \dfrac{F(x)}{G(x)}$, find $S'(3)$. Simplify your answer.

(b) If $T(x) = F(G(x))$, find $T'(0)$. Simplify your answer.

(c) If $U(x) = \ln(F(x))$, find $U'(3)$. Simplify your answer.

Answer.

(a) $S'(3) = \dfrac{F'(3)G(3) - F(3)G'(3)}{[G(3)]^2} = -\dfrac{1}{4}$.

(b) $T'(0) = F'(G(0)) \cdot G'(0) = 0$.

(c) $U'(3) = \dfrac{F'(3)}{F(3)} = -\dfrac{1}{2}$.

12. Suppose the functions $f(x)$ and $g(x)$ satisfy the following properties:

$$\begin{array}{lll} f(2) = 3, & g(2) = 4, & g(0) = 2 \\ f'(2) = -1, & g'(2) = 0, & g'(0) = 3 \end{array}$$

(a) Find an equation of the tangent line to the graph of the function f at the point $(2, f(2))$.

(b) If $h(x) = 2f(x) - 3g(x)$, find $h'(2)$.

(c) If $k(x) = \dfrac{f(x)}{g(x)}$, find $k'(2)$.

(d) If $p(x) = f(g(x))$, find $p'(0)$.

(e) If $r(x) = f(x) \cdot g(x)$, find $r'(2)$.

Answer.

(a) $y = 5 - x$.

(b) -2.

(c) $-\dfrac{1}{4}$.

(d) -3.

(e) -4.

13. Suppose that $f(x)$ and $g(x)$ are differentiable functions and that $h(x) = f(x)g(x)$. You are given the following table of values:

$h(1)$	24
$g(1)$	6
$f'(1)$	-2
$h'(1)$	20

Using the table, find $g'(1)$.

Solution. From $h(1) = f(1)g(1)$ and $h'(1) = f'(1)g(1) + f(1)g'(1)$ it follows that $g'(1) = 9$.

14. Given $F(x) = f^2(g(x))$, $g(1) = 2$, $g'(1) = 3$, $f(2) = 4$, and $f'(2) = 5$, find $F'(1)$.

Answer. $2f(g(1)) \cdot f'(g(1)) \cdot g'(1) = 120.$

15. Compute the derivative of $f(x) = \dfrac{x}{x-2}$ by

(a) using the limit definition of the derivative;

(b) using the quotient rule.

Answer. $f'(x) = -\dfrac{2}{(x-2)^2}$.

Solution.

(a) We compute:

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \lim_{h\to 0} \left[\frac{\left(\frac{x+h}{x+h-2}\right) - \left(\frac{x}{x-2}\right)}{h} \right] \\ &= \lim_{h\to 0} \left[\frac{(x+h)(x-2) - x(x+h-2)}{(x+h-2)(x-2)} \cdot \frac{1}{h} \right] \\ &= \lim_{h\to 0} \left[\frac{-2h}{(x+h-2)(x-2)} \cdot \frac{1}{h} \right] \\ &= \lim_{h\to 0} \left[\frac{-2}{(x+h-2)(x-2)} \right] \\ &= \frac{-2}{(x-2)^2}. \end{aligned}$$

(b) We compute:

$$\begin{aligned} f'(x) &= \frac{\left(\frac{d}{dx}x\right)(x-2) - x\left(\frac{d}{dx}(x-2)\right)}{(x-2)^2} \\ &= \frac{(x-2)-x}{(x-2)^2} \\ &= \frac{-2}{(x-2)^2}. \end{aligned}$$

16.

(a) Write down the formula for the derivative of $f(x) = \tan x$. State how you could use formulas for derivatives of the sine and cosine functions to derive this formula. (DO NOT do this derivation.)

(b) Use the formula given in part (a) to derive the formula for the derivative of the arctangent function.

(c) Use formulas indicated in parts (a) and (b) to evaluate and simplify the derivative of $g(x) = \tan(x^2) + \arctan(x^2)$ at $x = \dfrac{\sqrt{\pi}}{2}$. That is, you want to compute a simplified expression for $g'\left(\dfrac{\sqrt{\pi}}{2}\right)$.

Answer.

(a) $\sec^2 x$.

(b) $\frac{1}{1+x^2}$.

(c) $2\sqrt{\pi} + \frac{16\sqrt{\pi}}{16+\pi^2}$.

Solution.

(a) $f'(x) = \sec^2 x$. This follows from $\tan x = \dfrac{\sin x}{\cos x}$ by using the quotient rule.

(b) From $g(x) = \arctan x$, $x \in \mathbb{R}$, and $f'(g(x)) \cdot g'(x) = 1$, we conclude that $g'(x) = \cos^2(g(x))$. Next, suppose that $x > 0$ and consider the right triangle with the hypotenuse of the length 1 and with one angle measured $g(x)$ radians. Then $\tan g(x) = \tan(\arctan x) = x = \dfrac{\sin g(x)}{\cos g(x)} = \sqrt{\dfrac{1-g'(x)}{g'(x)}}$ which implies that $x^2 = \dfrac{1-g'(x)}{g'(x)}$. Thus $g'(x) = \dfrac{1}{1+x^2}$.

(c) From $g'(x) = 2x\sec x^2 + \dfrac{2x}{1+x^4}$ it follows that $g'\left(\dfrac{\sqrt{\pi}}{2}\right) = 2\sqrt{\pi} + \dfrac{16\sqrt{\pi}}{16+\pi^2}$.

17. Show that $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

Hint. Recall that if $f(x) = e^x$ then $f^{-1}(x) = \ln x$.

18. Show that $\dfrac{d}{dx}\sin^{-1} x = \dfrac{1}{\sqrt{1-x^2}}$.

Hint. Recall that if $f(x) = \sin^{-1} x$ is the inverse function of $g(x) = \sin x$, $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

19. If $g(x) = 2x^3 + \ln x$ is the derivative of $f(x)$, find

$$\lim_{x\to 0} \frac{f(1+x) - f(1)}{x}.$$

Answer. $f'(1) = g(1) = 2$.

20. Find

$$\lim_{x\to 0} \frac{\sqrt{1+x} + (1+x)^7 - 2}{x}.$$

Answer. $\dfrac{d}{dx}\left(\sqrt{x} + x^7\right)\Big|_{x=1} = \dfrac{15}{2}$.

21. Find a function f and a number a such that $\lim\limits_{h\to 0} \dfrac{(2+h)^6 - 64}{h} = f'(a)$.

Answer. $f(x) = x^6$, $a = 2$.

22. If $g(x)$ is differentiable and $f(x) = (\cos x)e^{g(x)}$, what is $f'(x)$?

Answer. $f'(x) = (-\sin x + g'(x) \cdot \cos x) \cdot e^{g(x)}$.

23. If $g(x)$ is differentiable and $f(x) = (\sin x)\ln g(x)$, what is $f'(x)$?

Answer. $f'(x) = (\cos x)\ln(g(x)) + \dfrac{g'(x) \cdot \sin x}{g(x)}$.

24. Let $f(x) = x^2 \sin\left(\dfrac{1}{x}\right)$ if $x \neq 0$, and $f(0) = 0$. Find $f'(0)$ (or say why it doesn't exist.)

Hint. Note that, for $h \neq 0$, $\left|\dfrac{h^2 \sin \frac{1}{h}}{h}\right| = \left|h \sin \dfrac{1}{h}\right| \leq |h|$. Use the Squeeze Theorem.

Answer. $f'(0) = 0$.

25. Let $f(x) = 2x + \cos x$. Say why $f(x)$ is an increasing function for all x. If $g(x) = f^{-1}(x)$, calculate $g'(0)$.

Answer. $f'(x) = 2 - \sin x > 0$ for all $x \in \mathbb{R}$. Let $g(0) = \alpha$. Then $g'(0) = \dfrac{1}{f'(g(0))} = \dfrac{1}{2 - \sin \alpha}$.

26. Show that $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$.

Solution. Let $f(x) = \sin x$, $x \in \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$. Then, for $x \in (-1, 1)$, $(f^{-1})'(x) = \dfrac{1}{\cos(f^{-1}(x))}$. Suppose that $x \in (0, 1)$ and let $\alpha = f^{-1}(x)$. Consider the right triangle with the hypothenuse of the length 1 and an angle measured α radians. The length of the leg opposite to the angle α equals $\sin \alpha = x$ which implies $\dfrac{d}{dx}(\sin^{-1} x) = \dfrac{1}{\sqrt{1 - x^2}}$.

27. Suppose that f is a differentiable function such that $f(g(x)) = x$ and $f'(x) = 1 + (f(x))^2$. Show that $g'(x) = \dfrac{1}{1 + x^2}$.

Hint. Use the chain rule and the given property of $f'(x)$ to get $(1 + (f(g(x)))^2) \cdot g'(x) = 1$.

28. If $y = \dfrac{\sqrt{x^2 + 1} - \sqrt{x^2 - 1}}{\sqrt{x^2 + 1} + \sqrt{x^2 - 1}}$, show that $\dfrac{dy}{dx} = 2x - \dfrac{2x^3}{\sqrt{x^4 - 1}}$.

Hint. Write $y = \dfrac{1}{2} \cdot (2x^2 - 2\sqrt{x^4 - 1})$.

29. Let f be a function differentiable on $\mathbb{R}$ and such that for all $x \neq 2$, $f(x) = \dfrac{x^4 - 16}{x - 2}$. Find $f^{(4)}(2)$.

Hint. Note $f(x) = (x + 2)(x^2 + 4)$.

Answer. $f^{(4)}(2) = 0$.

30. Given $y = \dfrac{1}{x} + \cos 2x$, find $\dfrac{d^5 y}{dx^5}$. Simplify your answer.

Answer. $y' = -\dfrac{5!}{x^6} - 2^5 \sin 2x$.

31. Find the values of A and B that make

$$f(x) = \begin{cases} x^2 + 1 & \text{if} \quad x \geq 0 \\ A \sin x + B \cos x & \text{if} \quad x < 0 \end{cases}$$

differentiable at $x = 0$.

Answer. $A = 0$, $B = 1$.

32. Find the values of A and B that make

$$f(x) = \begin{cases} x^2 + 1 & \text{if} \quad x < 0 \\ Ax + B & \text{if} \quad x \geq 0 \end{cases}$$

differentiable at $x = 0$.

Answer. $A = 0$, $B = 1$.

33. If f and g are two functions for which $f' = g$ and $g' = f$ for all x, then prove that $f^2 - g^2$ must be a constant.

Hint. Use the chain rule to differentiate $f^2 - g^2$.

34. Show that if f and g are twice differentiable functions (i.e. both have continuous second derivatives) then $(fg)'' = f''g + 2f'g' + fg''$.

Hint. Use the product rule twice.

35. Find y' when $y = \dfrac{(x+2)^{3\ln x}}{(x^2+1)^{1/2}}$.

Hint. Use the logarithmic differentiation.

Answer. $y' = \left(\dfrac{3\ln(x+2)}{x} + \dfrac{3\ln x}{x+2} - \dfrac{x}{x^2+1}\right) \cdot \dfrac{(x+2)^{3\ln x}}{(x^2+1)^{1/2}}$.

36. Find y' when $y = e^{4\cosh\sqrt{x}}$.

Answer. $y = \dfrac{2}{\sqrt{x}} \sinh\sqrt{x} \cdot e^{4\cosh\sqrt{x}}$.

37. Find $f'(0)$ for the function $f(x) = \sin^{-1}(x^2 + x) + 5^x$.

Answer. From $f'(x) = \dfrac{2x+1}{\sqrt{1-(x^2+x)^2}} + 5^x \ln 5$ it follows that $f'(0) = 1 + \ln 5$.

38. Let $f(x) = \log_a(3x^2 - 2)$. For what value of a is $f'(1) = 3$?

Answer. e^2.

39. Let $f(x) = e^{a(x^2-1)}$. For what value of a is $f'(1) = 4$?

Answer. 2.

40. Let $f(x) = \ln((x^2+1)^a)$. For what value of a is $f'(2) = 2$?

Answer. $\dfrac{5}{2}$.

41. Given

$$y = \frac{\sqrt{1+2x}\,\sqrt[4]{1+4x}\,\sqrt[6]{1+6x}\ldots\sqrt[100]{1+100x}}{\sqrt[3]{1+3x}\,\sqrt[5]{1+5x}\,\sqrt[7]{1+7x}\ldots\sqrt[101]{1+101x}},$$

find y' at $x = 0$.

Hint. Write as a product.

Answer. 0.

42. Evaluate $D_t \cos^{-1}(\cosh(e^{-3t}))$, without simplifying your answer.

Answer. $\dfrac{3e^{-3t}\sinh(e^{-3t})}{\sqrt{1-\cosh^2(e^{-3t})}}$

43. Use logarithmic differentiation to find $y'(u)$ as a function of u alone, where

$$y(u) = \left(\frac{(u+1)(u+2)}{(u^2+1)(u^2+2)}\right)^{1/3},$$

without simplifying your answer.

Answer. $y'(u) = \dfrac{1}{3}\left(\dfrac{1}{u+1} + \dfrac{1}{u+2} - \dfrac{2u}{u^2+1} - \dfrac{2u}{u^2+2}\right) \cdot \left(\dfrac{(u+1)(u+2)}{(u^2+1)(u^2+2)}\right)^{1/3}$.

44. Given $y = \tan(\cos^{-1}(e^{4x}))$, find $\dfrac{dy}{dx}$. Do not simplify your answer.

Answer. $y' = -\dfrac{4e^{-4x}}{\sqrt{1-e^{8x}}}$.

Find the derivatives of the following functions:

45. $y = \cosh(\arcsin(x^2 \ln x))$.

Answer. $y' = \dfrac{x \ln(ex^2) \sinh(\arcsin(x^2 \ln x))}{\sqrt{1 - x^4 \ln^2 x}}$.

46. $y = \ln(\tan(7^{1-5x}))$.

Answer. $y' = -\dfrac{10 \cdot 7^{1-5x} \ln 7}{\sin(2 \cdot 7^{1-5x})}$.

Find the derivatives of the following functions:

47. $y = e^{\cos x^2}$

Answer. $y' = -2xe^{\cos x^2} \sin x^2$.

48. $y = x^{20} \arctan x$

Answer. $y' = x^{19} \left(20 \arctan x + \dfrac{x}{1 + x^2}\right)$.

49. $y = x^{\ln x}$

Answer. $y' = 2x^{\ln x - 1} \ln x$.

Find the derivatives of the following functions:

50. $y = e^{3 \ln(2x+1)}$

Answer. $y' = \dfrac{6e^{3 \ln(2x+1)}}{2x + 1}$.

51. $y = x^{2x}$

Answer. $y' = 2x^{2x}(\ln x + 1)$.

52. $y = \dfrac{e^{2x}}{(x^2 + 1)^3 (1 + \sin x)^5}$

Answer. $y' = \dfrac{e^{2x}}{(x^2 + 1)^3 (1 + \sin x)^5} \cdot \left(2 - \dfrac{6x}{x^2 + 1} - \dfrac{5 \cos x}{1 + \sin x}\right)$.

53. $x^2 + 2xy^2 = 3y + 4$

Answer. $y' = \dfrac{2x + 2y^2}{3 - 4xy}$.

Find the derivatives of the following functions:

54. $y = x^{\sinh x}$

Answer. $y' = (x \cosh x + \sinh x) x^{\sinh x}$.

55. $\ln(x + y) = xy - y^3$

Answer. $y' = \dfrac{xy + y^2 - 1}{3xy^2 + 3y^3 - x^2 - xy - 1}$.

Find the derivatives of the following functions:

56. $y = \sec(\sinh x)$

Answer. $y' = \sec(\sinh x) \tan(\sinh x) \cosh x$.

57. $e^x + e^y = x^e + y^e + e^3$

Answer. $y' = \dfrac{ex^{e-1} - e^x}{e^y - ey^{e-1}}$.

Find the derivatives of the following functions:

58. $f(x) = \dfrac{3x^2+1}{e^x}$

Answer. $f'(x) = (6x - 3x^2 - 1)e^{-x}$.

59. $g(z) = \sin\sqrt{z^2+1}$

Answer. $g'(z) = \dfrac{z\cos\sqrt{z^2+1}}{\sqrt{z^2+1}}$.

60. $h(y) = \sqrt{\dfrac{\cos y}{y}}$

Answer. $h'(y) = -\dfrac{y\tan y + 1}{2y}\sqrt{\dfrac{\cos y}{y}}$.

Find the derivatives of the following functions:

61. $f(x) = \dfrac{1}{x + \frac{1}{x}}$

Answer. $f'(x) = \dfrac{1-x^2}{(x^2+1)^2}$.

62. $g(x) = \ln(\sqrt{x^2+1}\sin^4 x)$

Answer. $g'(x) = \dfrac{x}{x^2+1} + 4\cot x$.

Find the derivatives of the following functions:

63. $f(x) = \arctan(\sqrt{x})$

Answer. $f'(x) = \dfrac{1}{2\sqrt{x}(1+x)}$.

64. $f(x) = \cosh(5\ln x)$

Answer. $f'(x) = \dfrac{5\sinh(5\ln x)}{x}$.

Find the derivatives of the following functions:

65. $f(x) = 10^{3x}$

Answer. $f'(x) = 3 \cdot 10^{3x} \cdot \ln 10$.

66. $f(x) = x^{10}\tanh x$

Answer. $f'(x) = x^9(10\tanh x + x\ \mathrm{sech}^2 x)$.

67. $f(x) = x^{\cos x}$

Answer. $f'(x) = \left(\dfrac{\cos x}{x} - \sin x \ln x\right) x^{\cos x}$.

Find the derivatives of the following functions:

68. $y = \dfrac{e^{x^2+1}}{x\sin x}$

Answer. $y' = \dfrac{(2x^3-1)\sin x - x\cos x}{x^2\sin^2 x} \cdot e^{x^2+1}$.

69. $f(x) = x^{x^2}$

Answer. $f'(x) = (2\ln x + 1)x^{x^2+1}$.

70. $f(x) = \ln(\cos 3x)$

Answer. $f'(x) = -3\tan 3x$.

Find the derivatives of the following functions:

71. $f(x) = \dfrac{(x-1)^2}{(x+1)^3}$

Answer. $f'(x) = \dfrac{(5-x)(x-1)}{(x+1)^4}$.

72. $f(x) = 2^{2x} - (x^2+1)^{2/3}$

Answer. $f'(x) = 2^{2x+1}\ln 2 - \frac{4x}{3\sqrt[3]{x^2+1}}$.

73. $f(x) = \tan^2(x^2)$

Answer. $f'(x) = 4x\tan(x^2)\cdot\sec^2 x^2$.

74. $f(x) = x^{\arctan x}$

Answer. $f'(x) = \left(\dfrac{\ln x}{1+x^2} + \dfrac{\arctan x}{x}\right)\cdot x^{\arctan x}$.

75. Compute $f'''(x)$ where $f(x) = \sinh(2x)$.

Answer. $f'''(x) = 8\cosh(2x)$.

Find the derivatives of the following functions:

76. $f(x) = 5x + x^5 + 5^x + \sqrt[5]{x} + \ln 5$

Answer. $f'(x) = 5 + 5x^4 + 5^x\ln 5 + \dfrac{1}{5\sqrt[5]{x^4}}$.

77. $y = x^{10}\tanh x$

Answer. $y' = x^9(10\tanh x + x\ \mathrm{sech}^2 x)$.

78. $y = (\ln x)^{\cos x}$

Answer. $y' = \left(\dfrac{\cos x}{x\ln x} - \sin x\ln\ln x\right)\cdot(\ln x)^{\cos x}$.

Find the derivatives of the following functions:

79. $f(x) = \ln(\sinh x)$

Answer. $f'(x) = \coth x$.

80. $f(x) = e^{x\cos x}$

Answer. $f'(x) = (\cos x - x\sin x)e^{x\cos x}$.

81. $f(x) = \dfrac{\sin x}{1+\cos x}$

Answer. $f'(x) = \dfrac{1+\cos x}{(1+\cos x)^2}$.

82. $f(x) = x^x$

Answer. $f'(x) = (\ln x + 1)x^x$.

Find the derivatives of the following functions:

83. $f(x) = g(x^3)$, if $g(x) = \frac{1}{x^2}$

Answer. $f'(x) = \dfrac{3}{x^4}$.

84. $f(x) = x^2\sin^2(2x^2)$

Answer. $f'(x) = 2x\sin^2(2x^2) + 8x^3\sin(2x^2)\cos(x^2)$.

85. $f(x) = (x+2)^x$

Answer. $f'(x) = \left(\ln(x+2) + \dfrac{x}{x+2}\right)(x+2)^x$.

Find the derivatives of the following functions:

86. $y = \sec\sqrt{x^2+1}$

Answer. $y' = \dfrac{x\sec\sqrt{x^2+1}\tan\sqrt{x^2+1}}{\sqrt{x^2+1}}$.

87. $y = x^{e^x}$

Answer. $y' = \left(e^x\ln x + \dfrac{e^x}{x}\right)x^{e^x}$.

Find the derivatives of the following functions:

88. $y = x^3 + 3^x + x^{3x}$

Answer. $y' = 3x^2 + 3^x\ln 3 + 3(\ln x + 1)x^{3x}$.

89. $y = e^{-5x}\cosh 3x$

Answer. $y' = -(e^{-2x} + 4e^{-8x})$.

90. $y = \arctan\left(\sqrt{x^2-1}\right)$

Answer. $y' = \dfrac{1}{x\sqrt{x^2-1}}$.

91. $y = \dfrac{x^5 e^{x^3}\sqrt[3]{x^2+1}}{(x+1)^4}$

Answer. $y' = \left(\dfrac{5}{x} + 3x^2 + \dfrac{2x}{3(x^2+1)} - \dfrac{4}{x+1}\right)\cdot\dfrac{x^5 e^{x^3}\sqrt[3]{x^2+1}}{(x+1)^4}$.

Find the derivatives of the following functions:

92. $f(x) = \dfrac{\ln(x^2-3x+8)}{\sec(x^2+7x)}$

Answer. $f'(x) = \dfrac{1}{\sec(x^2+7x)}\cdot\left(\dfrac{2x-3}{x^2-3x+8} - (2x+7)\cdot\ln(x^2-3x+8)\tan(x^2+7x)\right)$.

93. $f(x) = \arctan(\cosh(2x-3))$

Answer. $f'(x) = \dfrac{2\sinh(2x-3)}{1+\cosh^2(2x-3)}$

94. $f(x) = \cos(e^{3x-4})$

Answer. $f'(x) = -3e^{3x-4}\sin(e^{3x-4})$.

95. $f(x) = (\tan x)^{\ln x + x^2}$

Answer. $f'(x) = \left((x^{-1}+2x)\ln\tan x + \dfrac{\ln x + x^2}{\sin x\cos x}\right)\cdot(\tan x)^{\ln x + x^2}$.

96. $f(x) = (\sec^2 x - \tan^2 x)^{45}$

Answer. $f'(x) = 0$.

Find the derivatives of the following functions:

97. $h(t) = e^{-\tan\left(\frac{t}{3}\right)}$

Answer. $h'(t) = -\dfrac{1}{3}\sec^2\left(\dfrac{t}{3}\right)\cdot e^{-\tan\left(\frac{t}{3}\right)}$

98. $2y^{2/3} = 4y^2\ln x$

Hint. Note that $y = \left(\dfrac{1}{2\ln x}\right)^{3/4}$.

Answer. $y' = -\dfrac{3}{8x\ln^2 x}\left(\dfrac{1}{2\ln x}\right)^{-1/4}$.

99. $f(y) = 3^{\log_7(\arcsin y)}$

Answer. $f'(y) = \dfrac{\ln 3}{\ln 7\cdot\sqrt{1-y^2}\cdot\arcsin y}\cdot 3^{\log_7(\arcsin y)}$

Find the derivatives of the following functions:

100. $f(x) = \sin^{-1}(x^2+x)+5^x$

Answer. $f'(x) = \dfrac{2x+1}{\sqrt{1-(x^2+x)^2}} + 5^x \ln 5.$

101. $g(x) = \cosh\left(\frac{\sqrt{x+1}}{x^2-3}\right)$

Answer. $g'(x) = -\dfrac{3x^2+4x+3}{2\sqrt{x+1}(x^2-3)^2} \sinh\left(\dfrac{\sqrt{x+1}}{x^2-3}\right).$

102. $f(x) = \dfrac{3^{\cos x}}{e^{2x}}$

Answer. $f'(c) = -3^{\cos x} \cdot (\sin x \cdot \ln 3 + 2)\, e^{-2x}.$

Find the derivatives of the following functions:

103. $f(x) = \dfrac{\sinh^{-1}(2^x)}{e^{4x}+a},\ a \in \mathbb{R}$

Answer. $f'(x) = \dfrac{2^x(e^{4x}+a)\ln 2 - 4e^{4x}\sinh^{-1}(2^x)\sqrt{2^{2x}+1}}{(e^{4x}+a)^2\sqrt{2^{2x}+1}}.$

104. $g(x) = \dfrac{(2+\cos(3x^2))e^{\pi x}}{3\sqrt{x}}$

Hint. Use logarithmic differentiation.

Answer. $g'(x) = g(x) \cdot \left(-\dfrac{6x\sin(3x^2)}{2+\cos(3x^2)} + \pi - \dfrac{3}{2}\right).$

105. $f(x) = \dfrac{5^{\cos x}}{\sin x}$

Answer. $f'(x) = -5^{\cos x} \cdot \left(\ln 5 + \cos x \cdot \csc^2 x\right).$

106. $y = x^{\arcsin x}$

Answer. $y' = x^{\arcsin x} \cdot \left(\dfrac{\ln x}{\sqrt{1-x^2}} + \dfrac{\arcsin x}{x}\right).$

Find the derivatives of the following functions:

107. $f(x) = \dfrac{xe^x}{\cos(x^2)}$

Answer. $f'(x) = \dfrac{(1+x)\cos(x^2)+2x^2\sin(x^2)}{\cos^2(x^2)} e^x.$

108. Find $\dfrac{d^2y}{dx^2}$ if $y = \arctan(x^2)$.

Answer. $\dfrac{d^2y}{dx^2} = -\dfrac{4x^3}{(1+x^4)^2}$

109. $y = x^{\sqrt{x}}$

Answer. $y' = \dfrac{1}{2}x^{\sqrt{x}-\frac{1}{2}}\ln(e^2x)$

Find the derivatives of the following functions:

110. $f(x) = \dfrac{x\ln x}{\sin(2x+3)}$

Answer. $f'(x) = \dfrac{(1+\ln x)\sin(2x+3) - 2x\ln x\cos(2x+3)}{\sin^2(2x+3)}.$

111. $f(x) = \dfrac{e^{\cos x}}{x^2+x}$

Answer. $f'(x) = \dfrac{(x^2+x)\sin x + 2x - 1}{(x^2+x)^2} \cdot e^{\cos x}.$

112. $f(x) = \dfrac{(x^4+4x+5)^{10}}{\sqrt{x^4-x^2+2}} \cdot \dfrac{1}{(x^3+x-6)^3}$

Answer. $f'(x) = \dfrac{(x^4+4x+5)^{10}}{\sqrt{x^4-x^2+2}\cdot(x^3+x-6)^3} \cdot \left(\dfrac{40(x^3+1)}{x^4+4x+5} - \dfrac{x(2x^2-1)}{x^4-x^2+2} - \dfrac{3(3x^2+1)}{x^3+x-6}\right).$

113. Find y'' if $y = e^{e^x}$.

Answer. $y'' = e^{e^x+x} \cdot (e^x+1).$

Find the derivatives of the following functions:

114. $f(x) = 3x^4 + \ln x$, find $f'(2)$

Answer. 96.5.

115. $f(x) = x^{\sqrt{x}+1}$

Answer. $f'(x) = x^{\sqrt{x}}\left(\dfrac{\sqrt{x}\ln(e^2x)}{2} + 1\right).$

116. $g(\theta) = \cos\left(\frac{\theta}{2}\right)$, find $g^{(11)}(\theta)$

Answer. $g^{(11)}(\theta) = \dfrac{1}{2^{11}} \cdot \sin\left(\dfrac{\theta}{2}\right).$

Find the derivatives of the following functions:

117. $f(x) = \dfrac{5^{\log_2(\pi)} e^{\cos(x)}}{\cos(x)}$

Answer. $f'(x) = \dfrac{5^{\log_2 \pi} e^{\cos x}(1 - \cos x)\sin x}{\cos^2 x}$.

118. $y = \cos^x(x)$

Answer. $y'(x) = \cos^x(x) \cdot (\ln(\cos x) - x \tan x)$.

119. $g(t) = \sqrt{4t^2+3}$, find $g''(t)$

Answer. $g''(t) = 12(4t^2+3)^{-3/2}$.

Find the derivatives of the following functions:

120. $f(x) = g\left(x^3\right)$, if $g(x) = \dfrac{1}{x^2}$

Answer. $f'(x) = -6x^{-7}$.

121. $f(x) = x^2 \sin^2(2x^2)$

Answer. $f'(x) = 2x\sin(2x^2) + 4x^3\cos(2x^2)$.

122. $f(x) = (x+2)^x$

Answer. $f'(x) = (x+2)^x\left(\dfrac{x}{x+2} + \ln(x+2)\right)$.

Find the derivatives of the following functions:

123. $y = \dfrac{-4}{x+2}$, find y''

Answer. $y'' = -8(x+2)^{-3}$.

124. $f(x) = x^6 e^x + 5e^{2x}$

Answer. $f'(x) = ((x+6)x^5 + 10e^x)e^x$.

125. $f(x) = \cos(\sin(x^3))$

Answer. $f'(x) = -3x^2\sin(\sin(x^3))\cos(x^3)$.

Find the derivatives of the following functions:

126. $f(x) = 1 - 3^x + x^2 + \dfrac{x}{\sqrt{1-x}}$

Answer. $f'(x) = 2x - 3^x \ln 3 + \dfrac{2-x}{2(1-x)^{3/2}}$.

127. $g(z) = \sqrt{\log(|2z+1|)}$

Answer. $g'(z) = \dfrac{1}{(2z+1)\ln 10\sqrt{\log|2z+1|}}$

128. $y = \dfrac{x\sec x}{5\ln(x^2)}$

Answer. $y' = \dfrac{((1+x\tan x)\ln x - 1)\sec x}{10\ln^2 x}$

129. $y = \sinh(7^{2x} - \sqrt{x})$

Answer. $y' = \left(2\cdot 7^{2x}\ln 7 - \dfrac{1}{2\sqrt{x}}\right)\cosh(7^{2x} - \sqrt{x})$.

2.3 Related Rates

To solve a related rates problem you need to do the following:

(a) Identify the independent variable on which the other quantities depend and assign it a symbol, such as t. Also, assign symbols to the variable quantities that depend on t.

(b) Find an equation that relates the dependent variables.

(c) Differentiate both sides of the equation with respect to t (using the chain rule if necessary).

(d) Substitute the given information into the related rates equation and solve for the unknown rate.

Solve the following related rates problems:

1. A bug is walking on the parabola $y = x^2$. At what point on the parabola are the $x-$ and $y-$ coordinates changing at the same rate?

Hint. Solve for x in the system of equations: $\dfrac{dy}{dt} = 2x\dfrac{dx}{dt}$, $\dfrac{dy}{dt} = \dfrac{dx}{dt}$.

Answer. $x = \dfrac{1}{2}$, $y = \dfrac{1}{4}$.

2. A particle is moving along the parabola $y = x^2 - 4x + 8$. Its $x-$coordinate as a function of time t (in seconds) is $x(t) = -2t^3 + 5$ metres. Let l be the line joining the origin $(0, 0)$ to the particle. Determine how quickly the angle between the x-axis and the line l is changing when $x = 3$.

Answer. $-\frac{3}{17}$ rad/s.

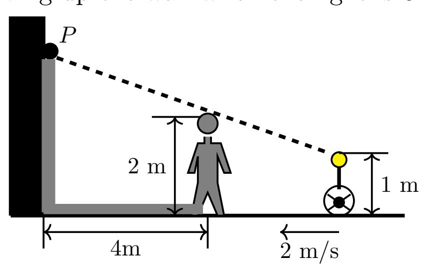
Solution. Let α be the angle between the x-axis and the line l. Then $\tan\alpha = x - 4 + \dfrac{8}{x}$ and $\sec^2\alpha \cdot \dfrac{d\alpha}{dt} = \left(1 - \dfrac{8}{x^2}\right) \cdot \dfrac{dx}{dt}$. Therefore, $\left.\dfrac{d\alpha}{dt}\right|_{x=3} = -\dfrac{3}{17}$ rad/s.

3. A child 1.5 m tall walks towards a lamppost on the level ground at the rate 0.25 m/sec. The lamppost is 10 m high. How fast is the length of the child's shadow decreasing when the child is 4 m from the post?

Answer. -0.44m/s.

Solution. Let $x = x(t)$ be the distance between the wall and the child. Let $s = s(t)$ be the length of the child's shadow. Using similar triangles, it is found that $\dfrac{10}{1.5} = \dfrac{x + s}{s}$. It follows that $\dfrac{ds}{dt} = \dfrac{3}{17} \cdot \dfrac{dx}{dt}$. $\left.\dfrac{ds}{dt}\right|_{x=4} = -0.44$m/s.

4. A light moving at 2 m/sec approaches a 2-m tall man standing 4 m from a wall. The light is 1 m above the ground. How fast is the tip P of the man's shadow moving up the wall when the light is 8 m from the wall?)



Answer. 0.5m/s.

Solution. Let $x = x(t)$ be the distance between the light and the wall in metres. Let $p = p(t)$ be the height of the man's shadow in metres. Using similar triangles it is found that $\dfrac{p - 1}{1} = \dfrac{x}{x - 4}$. It follows that $\dfrac{dp}{dt} = -\dfrac{4}{(x-4)^2} \cdot \dfrac{dx}{dt}$. $\left.\dfrac{dp}{dt}\right|_{x=8} = 0.5$m/s.

5. A ladder 15 ft long rests against a vertical wall. Its top slides down the wall while its bottom moves away along the level ground at a speed of 2 ft/s. How fast is the angle between the top of the ladder and the wall changing when the angle is $\pi/3$ radians?

Answer. $\frac{4}{15}$ ft/s.

Solution. Let $x = x(t)$ be the distance between the bottom of the ladder and the wall. It is given that, at any time t, $\dfrac{dx}{dt} = 2$ ft/s. Let $\theta = \theta(t)$ be the angle between the top of the ladder and the wall. Then $\sin\theta = \dfrac{x(t)}{15}$. It follows that $\cos\theta \cdot \dfrac{d\theta}{dt} = \dfrac{1}{15}\dfrac{dx}{dt}$. Thus when $\theta = \dfrac{\pi}{3}$ the rate of change of θ is given by $\dfrac{d\theta}{dt} = \dfrac{4}{15}$ ft/s.

6. A ladder 12 metres long leans against a wall. The foot of the ladder is pulled away from the wall at the rate $\frac{1}{2}$ m/min. At what rate is the top of the ladder falling when the foot of the ladder is 4 metres from the wall?

Answer. $-\frac{\sqrt{2}}{8}$ m/min.

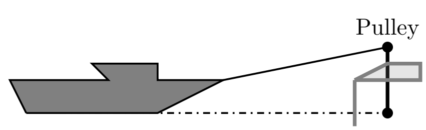
Solution. Let $x = x(t)$ be the distance between the foot of the ladder and the wall and let $y = y(t)$ be the distance between the top of the ladder and the ground. It is given that, at any time t, $\dfrac{dx}{dt} = \dfrac{1}{2}$ m/min. From $x^2 + y^2 = 144$ it follows that $x \cdot \dfrac{dx}{dt} + y\dfrac{dy}{dt} = 0$. Thus when $x(t) = 4$ we have that $y(t) = 8\sqrt{2}$ and $4 \cdot \dfrac{1}{2} + 8\sqrt{2}\dfrac{dy}{dt} = 0$. The top of the ladder is falling at the rate $\dfrac{dy}{dt} = -\dfrac{\sqrt{2}}{8}$ m/min.

7. A rocket R is launched vertically and it is tracked from a radar station S which is 4 miles away from the launch site at the same height above sea level. At a certain instant after launch, R is 5 miles away from S and the distance from R to S is increasing at a rate of 3600 miles per hour. Compute the vertical speed v of the rocket at this instant.

Answer. 6000 mi/h.

Solution. Let $x = x(t)$ be the height of the rocket at time t and let $y = y(t)$ be the distance between the rocket and radar station. It is given that, at any time t, $x^2 = y^2 - 16$. Thus, at any time t, $x \cdot \dfrac{dx}{dt} = y\dfrac{dy}{dt}$. At the instant when $y = 5$ miles and $\dfrac{dy}{dt} = 3600$ mi/h we have that $x = 3$ miles and we conclude that, at that instant, $3\dfrac{dx}{dt} = 5 \cdot 3600$. Thus the vertical speed of the rocket is $v = \dfrac{dx}{dt} = 6000$ mi/h.

8. A boat is pulled into a dock by means of a rope attached to a pulley on the dock. The rope is attached to the bow of the boat at a point 1 m below the pulley. If the rope is pulled through the pulley at a rate of 1 m/sec, at what rate will the boat be approaching the dock when 10 m of rope is out?



Answer. $\dfrac{10}{\sqrt{99}}$ m/sec.

Solution. Let $x = x(t)$ be the distance between the dock and the bow of the boat at time t and let $y = y(t)$ be the length of the rope between the pulley and the bow at time t. It is given that $\dfrac{dy}{dt} = 1$ m/sec. From

$x^2+1=y^2$ it follows that $\dfrac{dx}{dt}=\dfrac{y}{x}$ m/sec. Since $y=10$ implies $x=\sqrt{99}$ we conclude that when 10 m of rope is out then the boat is approaching the dock at the rate of $\dfrac{10}{\sqrt{99}}$ m/sec.

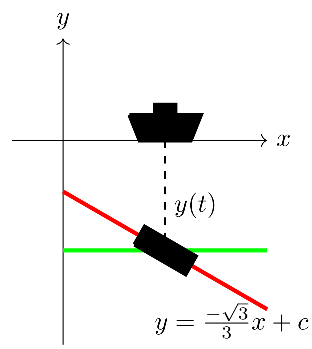
9. A ship is moving on the surface of the ocean in a straight line at 10 km/hr. At the same time, an enemy submarine maintains a position directly below the ship while diving at an angle of 30 degrees below the horizontal.

(a) provide a diagram of this problem situation. Use x as the line of movement of the ship in kilometres and y as the depth of the submarine in kilometres.

(b) How fast is the submarine depth increasing?

Answer.

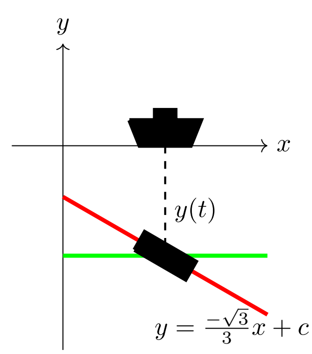
(a) let $c \leq 0$.



(b) $-\frac{10\sqrt{3}}{3}$ km/hr.

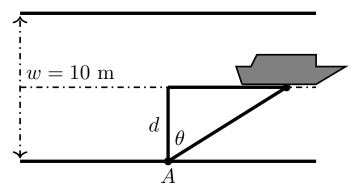
Solution.

(a) Let the positive direction of the x-axis be the line of the movement of the ship. If $(x(t),0)$ is the position of the ship at time t then the position of the submarine is given by $(x(t),y(t))$ with $y(t)= -\dfrac{\sqrt{3}}{3}\cdot x(t)+c$, for some negative constant c.



(b) We have that the position of the submarine is given by $(x(t),y(t))$ with $y(t)=-\dfrac{\sqrt{3}}{3}\cdot x(t)+c$, for some negative constant c. Therefore, $\dfrac{dy}{dt}=-\dfrac{\sqrt{3}}{3}\dfrac{dx}{dt}==-\dfrac{10\sqrt{3}}{3}$ km/hr.

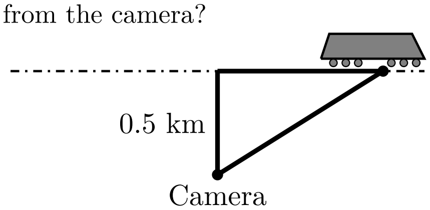
10. A person (A) situated at the edge of the river observes the passage of a speed boat going downstream. The boat travels exactly through the middle of the river (at the distance d from the riverbank.) The river is 10 m wide. When the boat is at $\theta = 60^0$ the observer measures the rate of change of the angle θ to be 2 radians/second. What is the speed, v, of the speed boat at that instant?



Answer. 40 m/s.

Solution. From $y = 5\tan\theta$ we get that, at any time t, $\dfrac{dy}{dt} = 5\sec^2\theta\dfrac{d\theta}{dt}$. At the instant when $\theta = \dfrac{\pi}{3}$ radians we have that $v = \dfrac{dy}{dt} = 5\cdot\sec^2\dfrac{\pi}{3}\cdot 2 = 40$ m/s.

11. A high speed train is traveling at 3 km/min along a straight track. The train is moving away from a movie camera which is located 0.5 km from the track. The camera keeps turning so as to always point at the front of the train. How fast (in radians per minute) is the camera rotating when the train is 1 km from the camera?



Answer. 1.5 rad/min.

Solution. Let $x = x(t)$ be the horizontal distance (in km) of the train with respect to the camera. Let $D = D(t)$ be the distance (in km) between the camera and the train. From the relationship $\tan\theta = \dfrac{x}{0.5}$ it follows that $\dfrac{d\theta}{dt} = 2\cos^2\theta\cdot\dfrac{dx}{dt}$. When $D = 1$, $\cos\theta = 2$, and therefore $\left.\dfrac{d\theta}{dt}\right|_{D=1} = 1.5$ rad/min.

12. An airplane is flying horizontally at an altitude of $y = 3$ km and at a speed of 480 km/h passes directly above an observer on the ground. How fast is the distance D from the observer to the airplane increasing 30 seconds later?

Answer. 384 km/h.

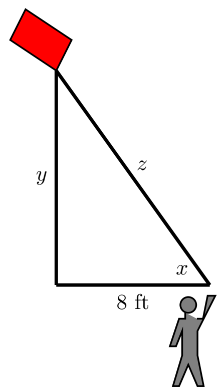
Solution. After time t (in hours) the plane is $480t$ km away from the point directly above the observer. Thus, at time t, the distance between the observer and the plane is $D = \sqrt{3^2 + (480t)^2}$. We differentiate $D^2 = 9 + 230,400t^2$ with respect to t to get $2D\dfrac{dD}{dt} = 460,800t$. Since 30 sec $= \dfrac{1}{120}$ hours it follows that the distance between the observer and the plane after 30 seconds equals $D = 5$ km. Thus, 30 seconds later the distance D from the observer to the airplane is increasing at the rate of $\left.\dfrac{dD}{dt}\right|_{t=\frac{1}{120}} =$ 384 km/h.

13. An airplane flying horizontally at a constant height of 1000 m above a fixed radar station. At a certain instant the angle of elevation θ from the station is $\frac{\pi}{4}$ radians and decreasing at a rate of 0.1 rad/sec. What is the speed of the aircraft at this moment.

Answer. $100\sqrt{2}$ m/sec.

Solution. Let y be the distance between the airplane and the radar station. Then, as the hypothenuse in a right angle triangle with the angle θ and the opposite leg of length 1000 m, $y = \frac{1000}{\sin\theta}$. Since it is given that $\frac{d\theta}{dt} = -0.1$ rad/sec, it follows that $\frac{dy}{dt} = -\frac{1000\cos\theta}{\sin^2\theta} \cdot \frac{d\theta}{dt} = \frac{100\cos\theta}{\sin^2\theta}$ m/sec. Hence if $\theta = \frac{\pi}{4}$, the speed of the plane is given by $\left.\frac{dy}{dt}\right|_{t=\frac{\pi}{4}} = 100\sqrt{2}$ m/sec.

14. A kite is rising vertically at a constant speed of 2 m/s from a location at ground level which is 8 m away from the person handling the string of the kite.



(a) Let z be the distance from the kite to the person. Find the rate of change of z with respect to time t when $z = 10$.

(b) Let x be the angle the string makes with the horizontal. Find the rate of change of x with respect to time t when the kite is $y = 6$ m above ground.

Answer.

(a) 1.2 m/s.

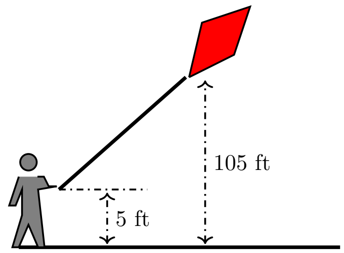
(b) $\frac{4}{25}$ m/s.

Solution.

(a) From $z^2 = 64 + 4t^2$ it follows that $2zz' = 8t$. If $z = 10$ then $t = 3$ and at that instant $z' = 1.2$ m/s.

(b) Since the height of the kite after t seconds is $2t$ metres, it follows that $\tan x = \frac{2t}{8}$. Thus $\frac{x'}{\cos^2 x} = \frac{1}{4}$. If $y = 6$ then $t = 3$ and $\tan x = \frac{3}{4}$. It follows that $\cos x = \frac{4}{5}$ and at that instant the rate of change of x is given by $x' = x'(3) = \frac{4}{25}$ m/s.

15. A girl flying a kite holds the string 5 feet above the ground level and lets the string out at a rate of 2 ft/sec as the kite moves horizontally at an altitude of 105 feet. Assuming there is no sag in the string, find the rate at which the kite is moving when 125 feet of string has been let out.



Answer. 3.33 ft/s.

Solution. Let $D = D(t)$ be the distance between the girl's hand and the kite in feet. From the relationship $D^2 = x^2 + y^2$, it follows that $D\dfrac{dD}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$. When $D = 125$ft, $y = 100$ft and $x = 75$ft, and $\left.\dfrac{dx}{dt}\right|_{D=125} = 3.33$ ft/s.

16. A balloon is rising at a constant speed 4m/sec. A boy is cycling along a straight road at a speed of 8m/sec. When he passes under the balloon, it is 36 metres above him. How fast is the distance between the boy and balloon increasing 3 seconds later.

Answer. $\frac{16}{\sqrt{5}}$ m/sec.

Solution. Let $x = x(t)$ be the distance (in metres) between the boy and the balloon at time t. Then $[x(t)]^2 = (8t)^2 + (36 + 4t)^2$. From $2x(t)x'(t) = 128t + 8(36 + 4t)$. From $x(3) = 24\sqrt{5}$ m, it follows that $x'(3) = \dfrac{16}{\sqrt{5}}$ m/sec.

17. A boy is standing on a road holding a balloon and a girl is running towards him at 4 m/sec. At $t = 0$ the girl is 10 m away and the boy releases the ballon which rises vertically at a speed of 2 m/sec. How fast is the distance from the girl to the ballon changing 2 seconds later?

Answer. $\frac{2y(0)}{\sqrt{4+(y(0)+4)^2}}$m/sec.

Solution. Let $D = D(t)$ be the distance (in metres) between the girl and the balloon at time t. Let $x = x(t)$ and $y = y(t)$ be the horizontal and vertical distances to the balloon respectively. Observe that, for $t > 0$, $\frac{dx}{dt} = -4$ m/sec and $\frac{dy}{dt} = 2$ m/sec. From the relationship $D^2 = x^2 + y^2$, it follows that $D\dfrac{dD}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$. At time $t = 2$, $x = 2$, $y = y(0) + 4$, and $D = \sqrt{4 + (y(0) + 4)^2}$. Therefore, $\left.\dfrac{dD}{dt}\right|_{t=2} = \dfrac{2y(0)}{\sqrt{4 + (y(0) + 4)^2}}$m/sec.

18. A helicopter takes off from a point 80 m away from an observer located on the ground, and rises vertically at 2 m/s. At what rate is the elevation angle of the observer's line of sight to the helicopter changing when the helicopter is 60 m above the ground.

Answer. $\dfrac{1}{50}$ m/sec.

Solution. Let $\theta = \theta(t)$ be the elevation angle. From $\tan\theta = \dfrac{2t}{80}$ it follows that $\frac{d\theta}{dt} = \frac{\cos^2\theta}{40}$. When $t = 30$ we have $\tan\theta = \frac{3}{4}$ and $\cos\theta = \frac{4}{5}$. Thus when the helicopter is 60 m above the ground the elevation angle of the observer's line of sight to the helicopter is changing at the rate $\frac{1}{50}$ m/sec.

19. An oil slick on a lake is surrounded by a floating circular containment boom. As the boom is pulled in, the circular containment area shrinks (all the while maintaining the shape of a circle.) If the boom is pulled in at the rate of 5 m/min, at what rate is the containment area shrinking when it has a diameter of 100m?

Answer. 500π m^2/min.

Solution. Let r denotes the radius of the circular containment area. It is given that $\frac{dr}{dt} = -5$ m/min. From the fact that the area at time t is given by $A = r^2\pi$, where $r = r(t)$, it follows that $\frac{dA}{dt} = 2r\pi\frac{dr}{dt} = -10r\pi$ m^2/min. Hence when $r = 50$m then the area shrinks at the rate of $10 \cdot 50 \cdot \pi = 500\pi$ m^2/min.

20. A rectangle is inscribed in the unit circle so that its sides are parallel to the coordinate axis. Let $\theta \in \left(0, \frac{\pi}{2}\right)$ be the angle between the positive x-axis and the ray with the initial point at the origin and passing through the top-right vertex P of the rectangle. Suppose that the angle θ is increasing at the rate of 2 radians/second and suppose that all lengths are measured in centimetres. At which rate is the area of the rectangle changing when $\theta = \frac{\pi}{3}$? Is the area increasing or decreasing at that moment? Why? Show all your work. Do not forget to use the appropriate units. Clearly explain your reasoning.

Answer. -4 cm^2/sec.

Solution. The area $A = A(t)$ of the inscribed rectangle can be described as $A = 4xy$, where x and y are the coordinates of point P. Recall that $x = \cos\theta$ and $y = \sin\theta$. It follows that $A(t) = 2\sin(2\theta)$ and $\frac{dA}{dt} = 4\cos(2\theta) \cdot \frac{d\theta}{dt}$. Therefore, $\left.\frac{dA}{dt}\right|_{\theta=\frac{\pi}{3}} = -4$ cm^2/sec.

21. Consider a cube of variable size. (The edge length is increasing.) Assume that the volume of the cube is increasing at the rate of 10 cm^3/minute. How fast is the surface area increasing when the edge length is 8 cm?

Answer. 5 cm^2/min.

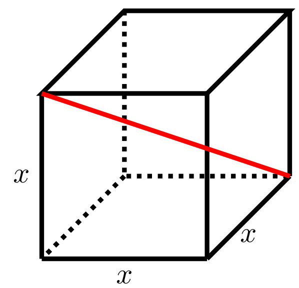
Solution. Let $x = x(t)$ be the edge length. Then the volume is given by $V = x^3$ and the surface area is given by $S = 6x^2$. It is given that $\frac{dV}{dt} = 10$. This implies that $3x^2\frac{dx}{dt} = 10$ at any time t and we conclude that at the instant when $x = 8$ the edge is increasing at the rate $\frac{5}{96}$cm/min. This fact together with $\frac{dS}{dt} = 12x\frac{dx}{dt}$ implies that at the instant when $x = 8$ the surface area is increasing at the rate $12 \cdot 8 \cdot \frac{5}{96} = 5$ cm^2/min.

22. Consider a cube of variable size. (The edge length is increasing.) Assume that the surface area of the cube is increasing at the rate of 6 cm^2/minute. How fast is the volume increasing when the edge length is 5 cm?

Answer. 7.5 cm^3/min.

Solution. Let $x = x(t)$ be the edge length of the cube in cm. Given that the volume, $V = x^3$ it follows that $\frac{dV}{dt} = 3x^2\frac{dx}{dt}$. The surface area $S = S(t)$ is described by $S = 6x^2$ leading to the relationship $\frac{dx}{dt} = \frac{1}{12x} \cdot \frac{dS}{dt}$. Therefore, $\frac{dV}{dt} = \frac{x}{4} \cdot \frac{dS}{dt}$ and $\left.\frac{dV}{dt}\right|_{x=4} = 7.5$ cm^3/min.

23. At what rate is the diagonal of a cube changing if its edges are decreasing at a rate of 3 cm/s?



Answer. $-3\sqrt{3}$ cm/sec.

Solution. Let $x = x(t)$ be the edge length of the cube in cm. The length of the diagonal, $z = z(x)$ can be described as $z = \sqrt{3}x$. It follows that $\frac{dz}{dt} = \sqrt{3}\frac{dx}{dt} = -3\sqrt{3}$ cm/sec.

24. The volume of an ice cube is decreasing at a rate of 5 m^3/s. What is the rate of change of the side length at the instant when the side lengths are 2 m?

Answer. -2.5m/sec

Solution. Let $x = x(t)$ be the side length of the ice cube. Since the volume of the cube can be expressed as $V = x^3$, it follows that $\frac{dx}{dt} = \frac{1}{3x^2}\frac{dV}{dt}$. $\left.\frac{dx}{dt}\right|_{x=2} = -2.5$m/sec

25. The height of a rectangular box is increasing at a rate of 2 metres per second while the volume is decreasing at a rate of 5 cubic metres per second. If the base of the box is a square, at what rate is one of the sides of the base decreasing, at the moment when the base area is 64 square metres and the height is 8 metres?

Answer. $\frac{133}{128}$ m/sec.

Solution. Let $H = H(t)$ be the height of the box, let $x = x(t)$ be the length of a side of the base, and $V = V(t) = Hx^2$. It is given that $\frac{dH}{dt} = 2$ m/sec and $\frac{dV}{dt} = 2x\frac{dx}{dt}H + x^2\frac{dH}{dt} = -5$ m^3/sec. The question is to find the value of $\frac{dx}{dt}$ at the instant when $x^2 = 64$ m^2 and $H = 8$ m. Thus, at that instant, one of the sides of the base is decreasing at the rate of $\frac{133}{128}$ m/sec.

26. A coffee filter has the shape of an inverted cone with a fixed top radius R and height H. Water drains out of the filter at a rate of 10 cm^3/min.

When the depth h of the water is 8 cm, the depth is decreasing at a rate of 2 cm/min.

(a) Express the volume of the cone as a function of the depth of the water only.

(b) What is the ratio R/H?

Answer.

(a) $V = \dfrac{\pi}{3}\dfrac{R^2h^3}{H^2}$.

(b) $\dfrac{1}{8}\sqrt{\dfrac{5}{\pi}}$.

Solution.

(a) The volume of the water is expressed as $V = \dfrac{1}{3}\pi r^2 h$. The radius r can be represented as $r = \dfrac{Rh}{H}$ using similar triangles. Overall, $V = \dfrac{\pi}{3}\dfrac{R^2h^3}{H^2}$.

(b) From $\dfrac{dV}{dt} = \pi h^2 \left(\dfrac{R}{H}\right)^2 \dfrac{dh}{dt}$ it follows $\dfrac{R}{H} = \dfrac{1}{8}\sqrt{\dfrac{5}{\pi}}$.

27. Sand is pouring out of a tube at 1 cubic metre per second. It forms a pile which has the shape of a cone. The height of the cone is equal to the radius of the circle at its base. How fast is the sandpile rising when it is 2 metres high?

Answer. $\dfrac{1}{4\pi}$ m/sec.

Solution. Let $H = H(t)$ be the height of the pile, let $r = r(t)$ be the radius of the base, and let $V = V(t)$ be the volume of the cone. It is given that $H = r$ (which implies that $V = \dfrac{H^3\pi}{3}$) and that $\dfrac{dV}{dt} = H^2\pi\dfrac{dH}{dt} = 1$ m^3/sec. The question is to find the value of $\dfrac{dH}{dt}$ at the instant when $H = 2$. Thus at that instant the sandpile is rising at the rate of $\dfrac{1}{4\pi}$ m/sec.

28. Gravel is being dumped from a conveyer belt at a rate of 1 cubic metre per second. It makes a pile in the the shape of a cone whose base diameter and height are always equal. How fast is the height of the pile changing when it is 5 metres high?

Answer. $\frac{4}{25\pi}$ m/sec.

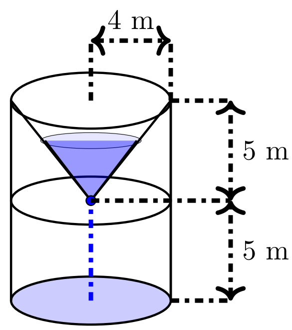
Solution. Let $x = x(t)$ be the height and diameter of the gravel pile. The volume of this pile can be expressed as $V = \dfrac{\pi x^2}{12}$. Given $\left.\dfrac{dV}{dt}\right|_{x=5} = 1$ m^3/sec it follows that $\left.\dfrac{dx}{dt}\right|_{x=5} = \dfrac{4}{25\pi}$ m/sec.

29. A water tank is in the shape of a cone with its vertex pointing downwards. The tank has a radius of 3 m and is 5 m high. At first the tank is full of water, but at time $t = 0$ (in seconds), a small hole at the vertex is opened and the water begins to drain. When the height of water in the tank has dropped to 3 m, the water is flowing out at 2 m^3/s. At what rate, in metres per second, is the water level dropping then?

Answer. $\dfrac{50}{81\pi}$ m/sec.

Solution. Let $H = H(t)$ be the height of water, let $r = r(t)$ be the radius of the surface of water, and let $V = V(t)$ be the volume of water in the cone at time t. It is given that $r = \dfrac{3H}{5}$ which implies that $V = \dfrac{3H^3\pi}{25}$. The question is to find the value of $\dfrac{dH}{dt}$ at the instant when $H = 3$ and $\dfrac{dV}{dt} = -2$ m^3/sec . Thus at that instant the water level dropping at the rate of $\dfrac{50}{81\pi}$ m/sec.

30. A conical tank with an upper radius of 4 m and a height of 5 m drains into a cylindrical tank with radius of 4 m and a height of 5 m. The water level in the conical tank is dropping at a rate of 0.5 m/min when the water level of the conical tank is 3m. At what rate is the water level in the cylindrical tank rising at that point?



Answer. $\frac{9}{50}$ m/min.

Solution. Let $V = V(t)$ be the volume of water in the cone at time t, $h = h(t)$ be the height of water in the cone, $w = w(t)$ be the volume of water in the cylinder and $H = H(t)$ be the height of water in the cylinder. The volume of water in the cone can be expressed as $V = \dfrac{16}{75}\pi h^3$ leading to $\dfrac{dV}{dt} = -\dfrac{dw}{dt} = \dfrac{48}{75}\pi h^2 \dfrac{dh}{dt} = -\dfrac{72}{25\pi}$ m^3/min. The volume of water in the cylinder can be expressed as $w = \pi r^2 h$ leading to $\dfrac{dh}{dt} = \dfrac{1}{16\pi}\dfrac{dw}{dt} = \dfrac{9}{50}$ m/min.

31. A boy starts walking north at a speed of 1.5 m/s, and a girl starts walking west from the same point P at the same time at a speed of 2 m/s. At what rate is the distance between the boy and the girl increasing 6 seconds later?

Answer. 2.5 m/s.

Solution. The distance between the boy and the girl is given by $z = \sqrt{x^2 + y^2}$ where $x = x(t)$ and $y = y(t)$ are the distances covered by the boy and the girl in time t, respectively. The question is to find $z'(6)$. We differentiate $z^2 = x^2 + y^2$ to get $zz' = xx' - yy'$. From $x(6) = 9$, $y(6) = 12$, $z(6) = 15$, $x'(t) = 1.5$, and $y'(t) = 2$ it follows that $z'(6) = 2.5$ m/s.

32. At noon of a certain day, ship A is 60 miles due north of ship B. If ship A sails east at speed of 15 miles per hour and B sails north at speed of 12.25 miles per hour, determine how rapidly the distance between them is changing 4 hours later?

Answer. 12.54 miles/hour.

Solution. The distance between the two ships is given by $z = \sqrt{x^2 + (60 - y)^2}$ where $x = x(t)$ and $y = y(t)$ are the distances covered by the ship A and the ship B in time t, respectively. The question is to find $z'(4)$. We differentiate $z^2 = x^2 + (60 - y)^2$ to get $zz' = xx' - (60 - y)y'$. From $x(4) = 60$, $y(4) = 49$, $z(4) = 61$, $x'(t) = 15$, and $y'(t) = 12.25$ it follows that $z'(4) = \dfrac{765.25}{61} \approx 12.54$ miles/hour.

33. A police car, approaching a right-angled intersection from the north, is chasing a speeding SUV that has turned the corner and is now moving straight east. When the police car is 0.6 km north of the intersection and the SUV is 0.8 km east of the intersection, the police determine with radar that the distance between them and the SUV is increasing at 20 km/hr. If the police car is moving at 60 km/hr at the instant of measurement, what is the speed of the SUV?

Answer. 70 km/hr.

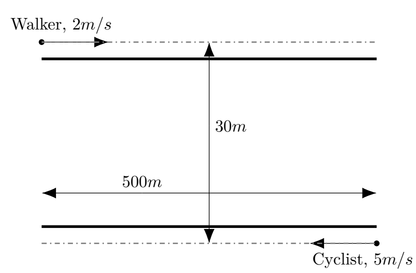
Solution. Let $x = x(t)$ be the distance between the police car and the intersection and let $y = y(t)$ be the distance between the SUV and the intersection. The distance between the two cars is given by $z = \sqrt{x^2 + y^2}$. The question is to find the value of $\dfrac{dy}{dt}$ at the instant when $x = 0.6$ km, $y = 0.8$ km, $\dfrac{dz}{dt} = 20$ km/hr, and $\dfrac{dx}{dt} = -60$ km/hr. We differentiate $z^2 = x^2 + y^2$ to get $z\dfrac{dz}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$. Since, at the given instance, $z = 1$, we have that $\dfrac{dy}{dt} = 70$ km/hr.

34. A ball lands at a point A. As soon as the ball lands, Puppy #1 starts off 15m North of A running at 3 m/s in the direction of A, and Puppy #2 starts off 12m East of A running at 2 m/s in the direction of ??. At what rate is the distance between the puppies changing when they are 5m apart. In answering this question, sketch a diagram and define your terms clearly.

Answer. $\frac{17}{5}$ m/sec.

Solution. Let $z = z(t)$ be the distance between the two puppies at time t. From $z^2 = x^2 + y^2$ it follows that $z\dfrac{dz}{dt} = x\dfrac{dx}{dt} + y\dfrac{dy}{dt}$ and $\left.\dfrac{dz}{dt}\right|_{z=5} = \dfrac{17}{5}$ m/sec.

35. A person is walking east along a river bank path at a speed of $2m/s$. A cyclist is on a path on the opposite bank cycling west at a speed of $5m/s$. The cyclist is initially $500m$ east of the walker. If the paths are $30m$ apart how fast is the distance between the walker and cyclist changing after one minute?



Answer. $-\frac{56}{\sqrt{73}}$ m/sec.

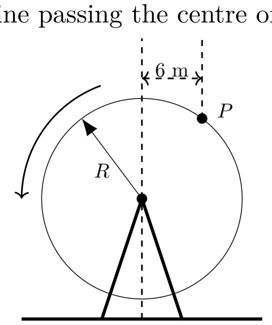
Solution. Let $z = z(t)$ be the distance between the runner and cyclist in metres at time t. Let $x = x(t)$ be the horizontal distance between the two individuals. Then $z(t)$ can be expressed as $z^2 = x^2 + 900$ which leads to $z\dfrac{dz}{dt} = x\dfrac{dx}{dt}$. The reduction in horizontal distance between the two individuals is given by $\dfrac{dx}{dt} = -5-2 = -7$ m/sec. Observe that when $t = 1$ minute then $x = 80$ m and $z = 10\sqrt{73}$ m. $\left.\dfrac{dz}{dt}\right|_{t=1\text{ min}} = -\dfrac{56}{\sqrt{73}}$ m/sec.

36. A lighthouse is located on a small island 3 km off-shore from the nearest point P on a straight shoreline. Its light makes 4 revolutions per minute. How fast is the light beam moving along the shoreline when it is shining on a point 1 km along the shoreline from P?

Answer. $\dfrac{80\pi}{3}$ km/min.

Solution. Let the point L represents the lighthouse, let at time t the light beam shines on the point $A = A(t)$ on the shoreline, and let $x = x(t)$ be the distance between A and P. Let $\theta = \theta(t)$ be the measure in radians of $\angle PLA$. It is given that $x = 3\tan\theta$ and $\dfrac{d\theta}{dt} = 8\pi$ radians/minute. The question is to find $\dfrac{dx}{dt}$ at the instant when $x = 1$. First we note that $\dfrac{dx}{dt} = 3\sec^2\theta\dfrac{d\theta}{dt}$. Secondly, at the instant when $x = 1$ we have that $\tan\theta = \dfrac{1}{3}$ which implies that $\cos\theta = \dfrac{3}{\sqrt{10}}$. Hence, when shining on a point one kilometre away from P, the light beam moving along the shoreline at the rate of $\dfrac{80\pi}{3}$ km/min.

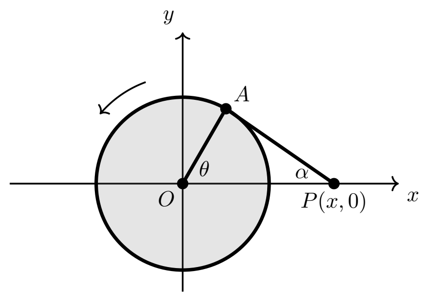
37. You are riding on a ferris wheel of diameter 20 metres. The wheel is rotating at 1 revolution per minute. How fast are you rising when you are at the point P in the Figure below, that is you are 6 metres horizontally away from the vertical line passing the centre of the wheel?



Answer. 12 m/sec.

Solution. Let $x = x(t)$ be the horizontal and let $y = y(t)$ be the vertical distance of the rider from the centre of the Ferris wheel (in metres). From the relationship $y = 10\cos\theta$ it follows that $\dfrac{dy}{dt} = -10\sin\theta \cdot \dfrac{d\theta}{dt}$. When $x = 6$ m then $\sin\theta = \dfrac{3}{5}$. Therefore, $\left.\dfrac{dy}{dt}\right|_{x=6} = 12$ m/sec.

38. The Figure below shows a rotating wheel with radius 40 cm and a connecting rod AP with length 1.2 m. The pin P slides back and forth along the x-axis as the wheel rotates counterclockwise at a rate of 360 revolutions per minute.



(a) Find the angular velocity of the connecting rod, $\frac{d\alpha}{dt}$, in radians per second, when $\theta = \frac{\pi}{3}$.

(b) Express the distance $x = |OP|$ in terms of θ.

(c) Find an expression for the velocity of the pin P in terms of θ.

Answer.

(a) $\frac{12\pi}{\sqrt{33}}$ rad/sec.

(b) $x = 40(\cos\theta + \sqrt{8 + \cos\theta})$.

(c) $-40\left(1 + \dfrac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \cdot \sin\theta$ rad/sec.

Solution.

(a) Observe that $\dfrac{d\theta}{dt} = 12\pi$ rad/sec. Using the Law of Sines we find that $3\sin\alpha = \sin\theta$. Hence $3\cos\alpha \cdot \dfrac{d\alpha}{dt} = \cos\theta \cdot \dfrac{d\theta}{dt}$. If $\theta = \dfrac{\pi}{3}$ then $\sin\alpha = \dfrac{\sqrt{3}}{6}$ and $\left.\dfrac{d\alpha}{dt}\right|_{\theta=\frac{\pi}{3}} = \dfrac{12\pi}{\sqrt{33}}$ rad/sec.

(b) By the Law of Cosines $120^2 = x^2 + 40^2 - 80x\cos\theta$. It follows that $x = 40(\cos\theta + \sqrt{8 + \cos\theta})$.

(c) $\dfrac{dx}{dt} = -40\left(1 + \dfrac{\cos\theta}{\sqrt{8 + \cos^2\theta}}\right) \cdot \sin\theta$ rad/sec.

2.4 Tangent Lines and Implicit Differentiation

Solve the following problems.

1.

(a) Find $\displaystyle\lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$ where $f(x) = \dfrac{3x+1}{x-2}$.

(b) What does the result in (a) tell you about the tangent line to the graph of $y = f(x)$ at $x = 1$?

(c) Find the equation of the tangent line to $y = f(x)$ at $x = 1$.

Answer.

(a) $f'(1) = -7$.

(b) The slope is equal to -7.

(c) $y = -7x + 3$.

2. Consider a tangent of the curve $y = 17 - 2x^2$ that goes through the point $(3, 1)$.

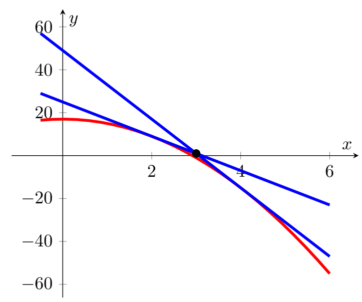
(a) Provide a diagram of this situation. Can you draw two tangent lines?

(b) Find the slopes of those tangent lines.

(c) Find the equation of those tangent lines.

Answer.

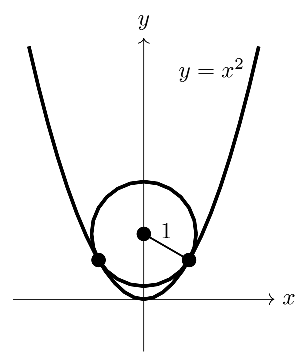
(a)



(b) Observe that all lines through $(3, 1)$ are given by $y = 1 + k(x - 3)$ and then find k so that the equation $17 - 2x^2 = 1 + k(x - 3)$ has a unique solution. $k = -16$ or $k = -8$.

(c) $y = -16x + 49$ and $y = -8x + 25$.

3. The Figure below shows a circle with the radius 1 inscribed in the parabola $y = x^2$. Find the centre of the circle.



Hint. The question is to find $a \in \mathbb{R}^+$ such that the circle $x^2 + (y-a)^2 = 1$ and the parabola $y = x^2$ have the same tangent lines at their intersection points.

Answer. $a = \dfrac{5}{4}$.

Given that $\sinh x = \dfrac{e^x - e^{-x}}{2}$ and $\cosh x = \dfrac{e^x + e^{-x}}{2}$:

4. Prove that $\cosh(x + y) = \cosh x \cosh y + \sinh x \sinh y$.

5. Find $\lim\limits_{x \to \infty} \tanh x$.

Answer. 1.

6. Find the equation of the tangent line to the curve $y = 4 + 3x + \cosh x$ at the point $(0, 5)$.

   Answer. $y = 3x + 5$.

7. At what point on the curve $y = \sinh x$ does the tangent line have a slope of 1?

   Hint. Solve $y' = \cosh x = 1$.

   Answer. The point is $(0, 0)$.

8. Find the point(s) on the curve $y = x^3$ where the line through the point $(4, 0)$ is tangent to the curve.

   Hint. Solutions of $-a^3 = 3a^2(4 - a)$ are $a = 0$ and $a = 6$.

   Answer. The points are $(0, 0)$ and $(6, 216)$.

9. Find the equation of the tangent line to $y = x^2 - 1$ that has slope equal to -4.

   Answer. $y = -4x - 5$.

10. Find the equation of the tangent line to $y = x^2$ passing through the point $(0, -3)$.

    Answer. $y = 2\sqrt{3}x \pm 3$.

11.

    (a) Find $\arcsin\left(-\frac{1}{\sqrt{2}}\right)$

    (b) Find the equation of the tangent line to the graph of $y = \arcsin x$ when $x = -\frac{1}{\sqrt{2}}$.

    Answer.

    (a) $-\frac{\pi}{4}$.

    (b) $y = \sqrt{2}x + 1 - \frac{\pi}{4}$.

12. Find the equation of the line that is tangent to the graph of $y = \sqrt{x} - \frac{1}{\sqrt{x}}$ at $x = 1$.

    Answer. $y = x - 1$.

13. Find the equation of the line that is tangent to the graph of $y = e^{x^2}$ at $x = 2$.

    Answer. $y = e^4(4x - 7)$.

14. Find the values of c such that the line $y = \frac{3x}{2} + 6$ is tangent to the curve $y = c\sqrt{x}$.

    Answer. $c = \pm 6$.

15. Find the values of a such that the tangent line to $f(x) = \frac{2x^2}{1 + x^2}$ at the point $(a, f(a))$ is parallel to the tangent of $g(x) = \tan^{-1}(x)$ at $(a, g(a))$.

    Answer. $a = 2 \pm \sqrt{3}$.

16. Consider the function $y = 3x^2 + 2x - 10$ on the interval $[1, 5]$. Does this function have a tangent line with a slope of 20 anywhere on this interval? Explain.

    Hint. Use the Intermediate Value Theorem for the function $y'(x)$. Alternatively, solve $y' = 20$.

Answer. Yes.

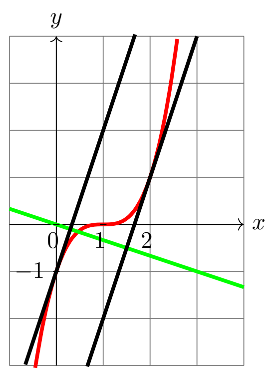
17. Let C be the curve $y = (x-1)^3$ and let L be the line $3y + x = 0$.

(a) Find the equation of all lines that are tangent to C and are also perpendicular to L.

(b) Draw a labeled diagram showing the curve C, the line L, and the line(s) of your solution to part (a). For each line of your solution, mark on the diagram the point where it is tangent to C and (without necessarily calculating the coordinates) the point where it is perpendicular to L.

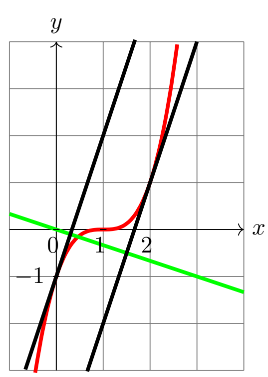
Answer.

(a) The lines are $y = 3x - 1$ and $y = 3x - 5$.

(b) 

Solution.

(a) We note that $y' = 3(x-1)^2$. Two lines, none of them horizontal, are perpendicular to each other if the product of their slopes equals -1. Thus to find all points on the curve C with the property that the tangent line is perpendicular to the line L we solve the equation $-\frac{1}{3} \cdot 3(x-1)^2 = -1$. Hence $x = 0$ or $x = 2$. The lines are $y = 3x - 1$ and $y = 3x - 5$.

(b) 

18. Find the value of $h'(0)$ if $h(x) + x\cos(h(x)) = x^2 + 3x$.

Answer. $h'(0) = 2$.

19. Find the derivative of y with respect to x for the curve $e^y \ln(x+y)+1 = \cos(xy)$ at the point $(1,0)$.

Answer. -1.

Solution. From $e^y \cdot \left(\dfrac{dy}{dx}\cdot \ln(x+y) + \dfrac{1+\frac{dy}{dx}}{x+y}\right) = -\left(y+\dfrac{dy}{dx}\right)\cdot \sin(xy)$ it follows that $\left.\dfrac{dy}{dx}\right|_{x=1} = -1$.

Find $\dfrac{dy}{dx}$ if:

20. $x^5 + y^5 = 5xy$

Answer. $\dfrac{dy}{dx} = \dfrac{y-x^4}{y^4-x}$.

21. $x^2 + x^3y - xy^2 = 1$

Answer. $y' = \dfrac{-3x^2y-2x+y^2}{x^3-2xy}$.

22. $x^y = y^x$

Answer. $\dfrac{dy}{dx} = \dfrac{y(x\ln y - y)}{x(y\ln x - x)}$.

Solution.

$$\begin{aligned} y\ln x &= x \ln y \\ \frac{dy}{dx}\cdot \ln x + \frac{y}{x} &= \ln y + \frac{x}{y}\cdot\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{y(x\ln y - y)}{x(y\ln x - x)}. \end{aligned}$$

23. $e^y - 3^x = x\sinh y$

Answer. $\dfrac{dy}{dx} = \dfrac{3^x \ln 3 + \sinh y}{e^y - x\cosh y}$.

24. $y - 2ye^{xy} = 0$

Answer. $y' = \dfrac{2y^2e^xy}{1-2xye^xy-2e^xy}$.

25. $\sinh x - \cos y = x^2y$

Answer. $\dfrac{dy}{dx} = \dfrac{\cosh x - 2xy}{x^2 - \sin y}$.

26. $\ln(x-y) = xy + y^3$

Answer. $\dfrac{dy}{dx} = \dfrac{1-y(x-y)}{1+(x-y)(x+3y^2)}$.

27. $\ln y + x = x^2 + x\cos y$

Answer. $y' = \dfrac{y(2x+\cos(y)-1}{xy\sin(y)+1}$.

28. $x\sin y + y\cos x = 1$

Answer. $y' = \dfrac{y\sin(x)-\sin(y)}{x\cos(y)+\cos(x)}$.

29. $\sin y + x = x^2 + x\cos y$

Answer. $y' = \dfrac{2x+\cos(y)-1}{x\sin(y)+\cos(y)}$.

30. $\cos x + e^y = x^2 + \tan^{-1} y$

Answer. $y' = \dfrac{(y^2+1)(2x+\sin?(x))}{e^yy^2+e^y-1}$.

31. $y\cos(x^2) = x\sin(y^2)$

Answer. $y' = \dfrac{2xy\sin(x^2)+\sin(y^2)}{\cos(x^2)-2xy\cos(y^2)}$.

32. $\tan y + \ln x = x^2 + \tan^{-1} y$

Answer. $y' = \dfrac{(2x^2-1)(y^2+1)}{x(y^2\sec^2(y)+\tan^2(y))}$.

33. $(xy+1)^2 = \tan(x+y^3)$

Answer. $y' = \dfrac{\sec^2(x+y^3)-2y(2xy^+1)}{3y^2\sec^2(x+y^3)-2x(xy+1)}$.

34. Find the slope of the tangent line to the curve $y + x\ln y - 2x = 0$ at the point $(1/2, 1)$.

Answer. $\frac{4}{3}$.

Use implicit differentiation to answer the following:

35. Find the tangent line to the graph of $\sin(x+y) = y^2\cos x$ at $(0,0)$.

Answer. $x + y = 0$.

36. Show that the tangent lines to the graph of $x^2 - xy + y^2 = 3$, at the points where the graph crosses the x-axis, are parallel to each other.

Solution. The graph crosses the x-axis at the points $(\pm\sqrt{3}, 0)$. The claim follows from the fact that $2x - y - xy' + 2yy' = 0$ implies that if $x = \pm\sqrt{3}$ and $y = 0$ then $y' = 2$.

37. The curve implicitly defined by

$$x\sin y + y\sin x = \pi$$

passes through the point $P = \left(\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(a) Find the slope of the tangent line through P

(b) Write the tangent line through P.

Answer.

(a) -1

(b) $y = \pi - x$

38. Write the equation of the line tangent to the curve $\sin(x+y) = xe^{x+y}$ at the origin $(0,0)$.

Answer. $y = 0$.

39. Write the equation of the line tangent to the curve $\sin(x+y) = 2x - 2y$ at the point (π, π).

Answer. $y - \pi = \frac{1}{3}(x - \pi)$.

40. Find the slope of the tangent line to the curve $xy = 6e^{2x-3y}$ at the point $(3,2)$.

Answer. $\frac{10}{21}$.

41. Find the x-coordinates of points on the curve $x^3 + y^3 = 6xy$ where the tangent line is vertical.

Answer. $x = 0$, $x = 2\sqrt{3}4$.

42.

(a) Find $\frac{dy}{dx}$ for the function defined implicitly by $x^2y + ay^2 = b$, where a and b are fixed constants.

(b) For the function defined in part (a) find the values of the constants a and b if the point $(1,1)$ is on the graph and the tangent line at $(1,1)$ is $4x + 3y = 7$.

Answer.

(a) $-\frac{2xy}{x^2 + 2ay}$.

(b) $a = \dfrac{1}{4}$ and $b = \dfrac{5}{4}$.

Solution.

(a) Let a, b be constants. Then we have

$$\begin{aligned} \frac{d}{dx}\left(x^2y + ay^2\right) &= \frac{d}{dx}(b) \\ \frac{d}{dx}\left(x^2y\right) + \frac{d}{dx}\left(ay^2\right) &= 0 \\ 2xy + x^2\frac{dy}{dx} + 2ay\frac{dy}{dx} &= 0 \\ \frac{dy}{dx}\left(x^2 + 2ay\right) &= -2xy \\ \frac{dy}{dx} &= \frac{-2xy}{x^2 + 2ay}. \end{aligned}$$

(b) We solve the system of equations:

$$\begin{aligned} 1 + a =& b \\ -\frac{2}{1+2a} =& -\frac{4}{3} \end{aligned}$$

to get $a = \dfrac{1}{4}$ and $b = \dfrac{5}{4}$.

43. Let l be any tangent to the curve $\sqrt{x} + \sqrt{y} = \sqrt{k}$, $k > 0$. Show that the sum of the x-intercept and the y-intercept of l is k.

Answer. $(a + \sqrt{ab}) + (b + \sqrt{ab}) = (\sqrt{a} + \sqrt{b})^2 = k$.

Solution. From $\dfrac{dy}{dx} = -\sqrt{\dfrac{y}{x}}$ we get that the tangent line l to the curve at any of its points (a, b) is given by $y - b = -\sqrt{\frac{b}{a}}(x - a)$. The sum of the x-intercept and the y-intercept of l is given by $(a + \sqrt{ab}) + (b + \sqrt{ab}) = (\sqrt{a} + \sqrt{b})^2 = k$.

44. Show that the length of the portion of any tangent line to the curve

$$x^{2/3} + y^{2/3} = 9,$$

cut off by the coordinate axis is constant. What is this length?

Answer. $\sqrt{9^3} = 27$.

Solution. From $\dfrac{2}{3\sqrt[3]{x}} + \dfrac{2y'}{3\sqrt[3]{y}} = 0$ we conclude that $y' = -\sqrt[3]{\dfrac{y}{x}}$. Thus the tangent line through the point (a, b) on the curve is given by $y - b = -\sqrt[3]{\dfrac{b}{a}}(x - a)$. Its x and y intercepts are $\left(a + \sqrt[3]{ab^2}, 0\right)$ and $\left(0, b + \sqrt[3]{a^2b}\right)$. Thus the square of the portion of the tangent line cut off by the coordinate axis is

$$\begin{aligned} \left(a + \sqrt[3]{ab^2}\right)^2 + \left(b + \sqrt[3]{a^2b}\right)^2 &= a^2 + 2a\sqrt[3]{ab^2} + b\sqrt[3]{a^2b} + b^2 + 2b\sqrt[3]{a^2b} + a\sqrt[3]{ab^2} \\ &= \left(\sqrt[3]{a^2} + \sqrt[3]{b^2}\right)^3 \\ &= 9^3 \end{aligned} .$$

The length of the portion is $\sqrt{9^3} = 27$.

45. Let C denote the circle whose equation is $(x-5)^2+y^2=25$. Notice that the point $(8,-4)$ lies on the circle C. Find the equation of the line that is tangent to C at the point $(8,-4)$.

Answer. $y+4=\dfrac{3}{4}(x-8)$.

46. The so called *devil's curve* is described by the equation

$$y^2(y^2-4)=x^2(x^2-5).$$

(a) Compute the y-intercepts of the curve.

(b) Use implicit differentiation to find an expression for $\dfrac{dy}{dx}$ at the point (x,y).

(c) Give an equation for the tangent line to curve at $(\sqrt{5},0)$.

Answer.

(a) $(0,0)$, $(0,\pm 2)$.

(b) $y'=\dfrac{x(2x^2-5)}{2y(y^2-2)}$.

(c) $x=\sqrt{5}$.

47. The equation $e^y+y(x-2)=x^2-8$ defines y implicitly as a function of x near the point $(3,0)$.

(a) Determine the value of y' at this point.

(b) Use the linear approximation to estimate the value of y when $x=2.98$.

Answer.

(a) $y'(3)=3$.

(b) $y(2.98)\approx 3\cdot 2.98-9=-0.06$

48. The equation $e^y+y(x-3)=x^2-15$ defines y implicitly as a function of x near the point $A(4,0)$.

(a) Determine the values of y' and y'' at this point.

(b) Use the tangent line approximation to estimate the value of y when $x=3.95$.

(c) Is the true value of y greater or less than the approximation in part (b)? Make a sketch showing how the curve relates to the tangent line near the point $A(4,0)$.

Answer.

(a) $y'(4)=4$ and $y”(4)=-11$

(b) $y(3.95)\approx -0.2$.

(c) Since the curve is concave down, the tangent line is above the curve and the approximation is an overestimate.

Chapter 3

Applications of Differentiation

3.1 Introduction

Use the following definitions, theorems, and properties to solve the problems contained in this Chapter.

Absolute Maximum and Minimum A function f has an absolute maximum at c if $f(c) \geq f(x)$ for all $x \in D$, the domain of f. The number $f(c)$ is called the maximum value of f on D. A function f has an absolute minimum at c if $f(c) \leq f(x)$ for all $x \in D$, the domain of f. The number $f(c)$ is called the minimum value of f on D.

Local Maximum and Minimum A function f has a local maximum at c if $f(c) \geq f(x)$ for all x in an open interval containing c. A function f has a local minimum at c if $f(c) \leq f(x)$ for all x in an open interval containing c.

Extreme Value Theorem If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.

Fermat's Theorem If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

Critical Number A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

Closed Interval Method To find the absolute maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

1. Find the values of f at the critical numbers of f in (a, b).
2. Find the values of f at the endpoints of the interval.
3. The largest of the values from Step 1 and Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

Rolle's Theorem Let f be a function that satisfies the following three hypotheses:

1. f is continuous on the closed interval $[a, b]$.

2. f is differentiable on the open interval (a, b).
3. $f(a) = f(b)$.

Then there is a number c in (a, b) such that $f'(c) = 0$.

The Mean Value Theorem Let f be a function that satisfies the following hypotheses:

1. f is continuous on the closed interval $[a, b]$.
2. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

Increasing/Decreasing Test 1. If $f'(x) > 0$ on an interval, then f is increasing on that interval.

2. If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

The First Derivative Test Suppose that c is a critical number of a continuous function f.

1. If f' changes from positive to negative at c, then f has a local maximum at c.
2. If f' changes from negative to positive at c, then f has a local minimum at c.
3. If f' does not change sign at c, then f has no local minimum or maximum at c.

Concavity If the graph of f lies above all of its tangent lines on an interval I, then it is called concave upward on I. If the graph of f lies below all of its tangents on I, it is called concave downward on I.

Concavity Test. 1. If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.

2. If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

Inflection Point A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

The Second Derivative Test Suppose f'' is continuous near c.

1. If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.
2. If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

Linear Approximation The linear function $L(x) = f(a) + f'(a)(x - a)$ is called the linearization of f at a. For x close to a we have that $f(x) \approx L(x) = f(a) + f'(a)(x - a)$ and this approximation is called the linear approximation of f at a.

Differential Let f be a function differentiable at $x \in \mathbb{R}$. Let $\Delta x = dx$ be a (small) given number. The differential dy is defined as $dy = f'(x)\Delta x$.

Newton's Method To estimate a solution, say $x = r$, to the equation $f(x) = 0$:

1. Begin with an initial guess x_1.
2. Calculate $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$.
3. If x_n is known then $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$.
4. If x_n and x_{n+1} agree to k decimal places then x_n approximates the root r up to k decimal places and $f(x_n) \approx 0$.

Antiderivative A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all $x \in I$.

Natural Growth/Decay Equation The natural growth/decay is modelled by the initial-value problem

$$\frac{dy}{dt} = ky, \quad y(0) = y_0, \quad k \in \mathbb{R}\backslash\{0\}.$$

Newton's Law of Cooling and Heating is given as

$$\frac{dT}{dt} = k(T - T_s)$$

where k is a constant, $T = T(t)$ is the temperature of the object at time t and T_s is the temperature of surroundings.

3.2 Curve Sketching

Solve the following problems.

1. Give an example of a function with one critical point which is also an inflection point. You must provide the equation of your function.

Answer. $f(x) = x^3$.

2. Give an example of a function that satisfies $f(-1) = f(1) = 0$ and $f'(x) > 0$ for all x in the domain of f'.

Answer. $f(x) = x - \text{sign}(x)$, $x \neq 0$.

3. Determine the value of a so that

$$f(x) = \frac{x^2 + ax + 5}{x + 1}$$

has a slant asymptote $y = x + 3$.

Answer. 4.

4. For what values of the constants a and b does the function $f(x) = \ln a + bx^2 - \ln x$ have an extremum value $f(2) = 1$?

Answer. $a = 2\sqrt{e}$, $b = 0.125$.

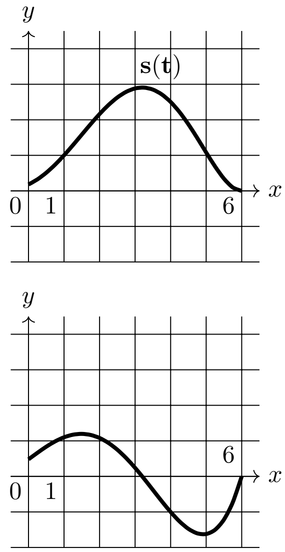
5. The function $f(x) = ax^3 + bx$ has a local extreme value of 2 at $x = 1$. Determine whether this extremum is a local maximum or local minimum.

Answer. Local maximum.

6. Prove that the function $f(x) = x^{151} + x^{37} + x + 3$ has neither a local maximum nor a local minimum.

Answer. Observe that $f'(x) > 0$ for all $x \in \mathbb{R}$.

7. A particle moves along a line with a position function $s(t)$ where s is measured in metres and t in seconds. Four graphs are shown below: one corresponds to the function $s(t)$, one to the velocity $v(t)$ of the particle, one to its acceleration $a(t)$ and one is unrelated.



(a) Observe that the position function s is already labeled. Identify the graphs of $v(t)$ and $a(t)$.

(b) Find all time intervals when the particle is slowing down and when it is speeding up.

(c) Estimate the total distance travelled by the particle over the interval $[1, 6]$.

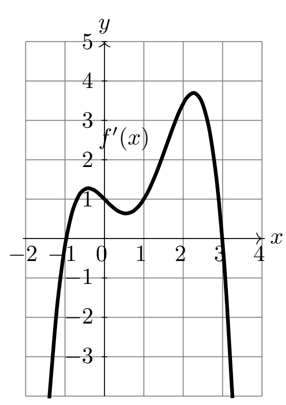
Answer.

(a) Velocity — bottom left; acceleration — bottom right.

(b) Speeding up on $(0, 1.5)$ and $(4, 5)$.

(c) Approximately 5.8 units.

8. Function f is differentiable everywhere. The graph of f' is depicted in the Figure below. f' is negative and concave down at all points not shown in this graph.



(a) Does the function f have a local maximum? If so, determine the approximate coordinate(s) of the local maximum point(s).

(b) Does the function f have a local minimum? If so, determine the approximate coordinate(s) of the local minimum point(s).

(c) Does the function f have any inflection points? If so, determine the approximate coordinate(s) of the inflection point(s).

(d) Determine the interval(s) on which the function f is decreasing.

(e) Determine the interval(s) on which f'' is decreasing.

(f) If f is a polynomial function, what is the least possible degree of f?

Answer.

(a) $(3, f(3))$.

(b) $(-1, f(-1))$.

(c) $(-0.5, f(-0.5))$, $(0.5, f(0.5))$, $(2.3, f(2.3))$.

(d) $(-\infty, -1)$ and $(3, \infty)$.

(e) $(-\infty, 0)$ and $(1.5, \infty)$.

(f) 5. Observe that the second derivative has three zeros.

9. Sketch the graph of $f(x) = 3x^4 - 8x^3 + 10$, after answering the following questions.

(a) Where is the graph increasing, and where is decreasing?

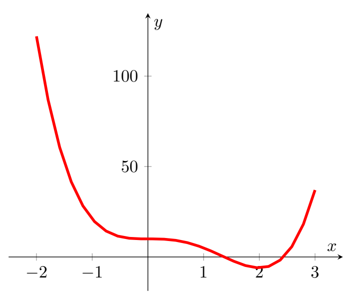
(b) Where is the graph concave upward, and where is it concave downward?

(c) Where are the local minima and local maxima? Establish conclusively that they *are* local minima and maxima.

(d) Where are the inflection points?

(e) What happens to $f(x)$ as $x \to \infty$ and as $x \to -\infty$.

Answer.



Solution.

(a) From $f'(x) = 12x^2(x-2)$ we conclude that $f'(x) > 0$ for $x > 2$ and $f'(x) < 0$ for $x < 2$. So f is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(b) From $f"(x) = 12x(3x-4)$ it follows that $f"(x) > 0$ for $x < 0$ or $x > \frac{4}{3}$ and $f"(x) < 0$ for $x \in \left(0, \frac{4}{3}\right)$. Also $f"(x) = 0$ for $x = 0$ and $x = \frac{4}{3}$. Thus f is concave upward on $(-\infty, 0)$ and on

(c) Critical numbers are $x = 0$ and $x = 2$. Since $f'(x)$ does not change sign at $x = 0$ there is no local maximum or minimum there. (Note also that $f"(0) = 0$ and that the second derivative test is inconclusive.) Since $f'(x)$ changes from negative to positive at $x = 2$ there is a local minimum at $x = 2$. (Note also that $f"(2) > 0$, so second derivative test says there is a local minimum.)

(d) Inflection points are $(0, 10)$ and $\left(\frac{4}{3}, f\left(\frac{4}{3}\right)\right)$.

(e) $\lim_{x\to\pm\infty} f(x) = \infty$.

10. In this question we consider the function $f(x) = \dfrac{x-3}{\sqrt{x^2-9}}$.

(a) Find the domain of f.

(b) Find the coordinates of all x- and y-intercepts, if any.

(c) Find all horizontal and vertical asymptotes, if any.

(d) Find all critical numbers, if any.

(e) Find all intervals on which f is increasing and those on which f is decreasing.

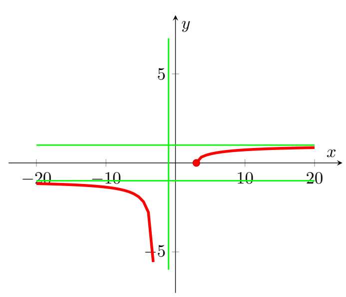
(f) Find the (x, y) coordinates of all maximum and minimum points, if any.

(g) Find all intervals on which f is concave up and those on which f is concave down.

(h) Find the (x, y) coordinates of all inflection points, if any.

(i) Sketch the graph of $y = f(x)$ using all of the above information. All relevant points must be labeled.

Answer.



$f(x) = \dfrac{x-3}{\sqrt{x^2-9}}$

Solution.

(a) From $x^2 - 9 > 0$ it follows that the domain of the function f is the set $(-\infty, -3) \cup (3, \infty)$.

(b) The function is not defined at $x = 0$, so there is no the y-intercept. Note that $f(x) \neq 0$ for all x in the domain of f.

(c) From $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = -1$ we conclude that there are two horizontal asymptotes, $y = 1$ (when $x \to \infty$) and $y = -1$ (when $x \to -\infty$). From $\lim_{x\to 3^+} f(x) = 0$ and $\lim_{x\to -3^-} f(x) = -\infty$ it follows that there is a vertical asymptote at $x = -3$.

(d) Since, for all x in the domain of f, $f'(x) = \dfrac{3(x-3)}{(x^2-9)^{3/2}} \neq 0$ we conclude that there is no critical number for the function f.

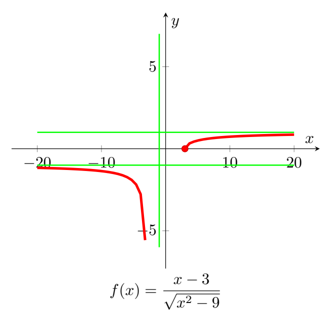
(e) Note that $f'(x) > 0$ for $x > 3$ and $f'(x) < 0$ for $x < -3$. Thus f increasing on $(3, \infty)$ and decreasing on $(-\infty, -3)$.

(f) Since the domain of f is the union of two open intervals and since the function is monotone on each of those intervals, it follows that the function f has neither (local or absolute) a maximum nor a minimum.

(g) From $f''(x) = -\dfrac{6(x-3)(x-\frac{3}{2})}{(x^2-9)^{5/2}}$ it follows that $f''(x) < 0$ for all x in the domain of f. Therefore $f(x)$ is concave downwards on its domain.

(h) None.

(i)



11. Given $f(x) = \dfrac{x^2-1}{x}$:

(a) Find the domain and x-intercepts.

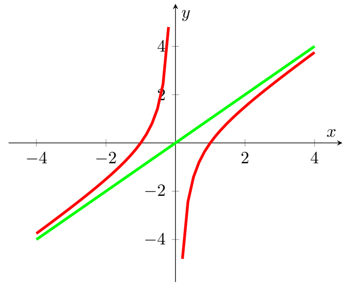
(b) Find all asymptotes.

(c) Determine the intervals on which the function is increasing or decreasing. Find the local maximum and minimum, if they exist.

(d) Determine the intervals on which the function is concave upward or downward. Find the inflection points, if they exist.

(e) Sketch the graph.

Answer.



$f(x) = \dfrac{x^2-1}{x}$

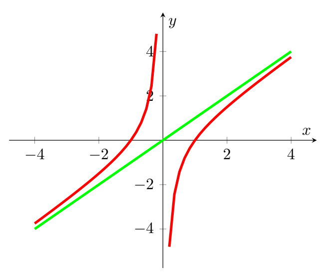
Solution.

(a) The domain of the function f is the set $\mathbb{R}\backslash\{0\}$. The x-intercepts are ± 1. Since 0 not in domain of f there is no y-intercept.

(b) From $\lim\limits_{x\to 0^-} f(x) = -\infty$ and $\lim\limits_{x\to 0^+} f(x) = \infty$ it follows that the vertical asymptote is the line $x = 0$. Since $\lim\limits_{x\to\pm\infty} f(x) = \lim\limits_{x\to\pm\infty}\left(x - \dfrac{1}{x}\right) = \pm\infty$ we conclude that there is no horizontal asymptote. Finally, the fact $f(x) = x - \dfrac{1}{x}$ implies that f has the slant (oblique) asymptote $y = x$.

(c) For all $x \in \mathbb{R}\backslash\{0\}$, $f'(x) = \dfrac{x^2+1}{x^2} > 0$ so the function f is increasing on $(-\infty, 0)$ and on $(0, \infty)$. The function f has no critical numbers and thus cannot have a local maximum or minimum.

(d) Since $f''(x) = -\dfrac{2}{x^3}$ it follows that $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. Therefore f is concave upward on $(-\infty, 0)$ and concave downward on $(0, \infty)$. There are no points of inflection.

(e) 

$f(x) = \dfrac{x^2-1}{x}$

12. Given $f(x) = \dfrac{x^2}{x^2-1}$:

(a) Find the domain of f.

(b) Is f an even function, odd function, or neither?

(c) Find all the x- and y- intercepts of the graph of f.

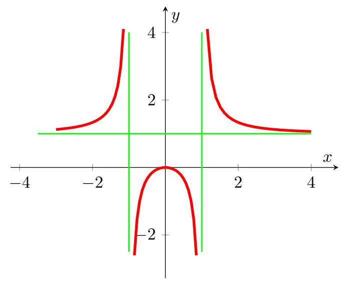
(d) Find all horizontal, vertical, and slant asymptotes of the graph of f. If asymptote(s) of a certain kind are absent, explain why.

(e) Find the intervals where the graph of f is increasing or decreasing, and locations and values of the local maxima and minima.

(f) Find the intervals where the the graph of f is concave upward or downward and the inflection points.

(g) Sketch the graph and clearly indicate the features found in parts (a)–(f).

Answer.



$f(x) = \dfrac{x^2}{x^2 - 1}$

Solution.

(a) All $x \in \mathbb{R}$ such that $x \neq \pm 1$.

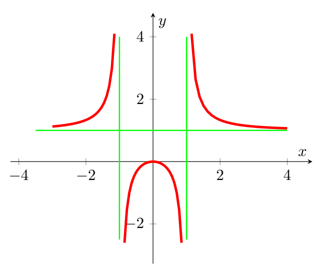
(b) Since $f(-x) = f(x)$, f is an even function.

(c) $(0, 0)$.

(d) Vertical asymptotes when $x = \pm 1$. Horizontal asymptote at $y = 1$.

(e) Increasing on $(0, -1) \cup (-1, -\infty)$. Decreasing on $(0, 1) \cup (1, \infty)$.

(f) CCU on $(-\infty, -1) \cup (1, \infty)$. CCD on $(-1, 1)$. No inflection points.

(g) 

$f(x) = \dfrac{x^2}{x^2 - 1}$

13. Given $f(x) = \dfrac{x^2}{x^2 + 9}$:

(a) Find the domain of f.

(b) Is f an even function, odd function, or neither?

(c) Find all the x- and y- intercepts of the graph of f.

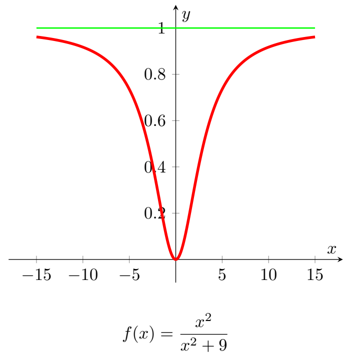
(d) Find all horizontal, vertical, and slant asymptotes of the graph of f. If asymptote of a certain kind are absent, explain why.

(e) Find the intervals where the graph of f is increasing or decreasing, and locations and values of the local maxima and minima.

(f) Find the intervals where the the graph of f is concave upward or downward and the inflection points.

(g) Sketch the graph and clearly indicate the features found in parts (a)–(f).

Answer.



Solution.

(a) All $x \in \mathbb{R}$.

(b) Since $f(-x) = f(x)$, f is an even function.

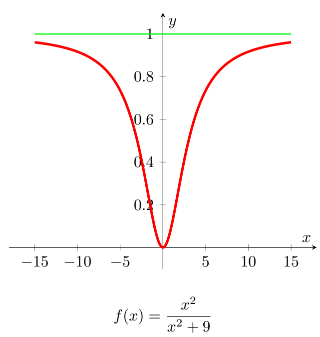
(c) $(0, 0)$.

(d) Horizontal asymptote at $y = 1$.

(e) Increasing on $(0, \infty)$. Decreasing on $(-\infty, 0)$.

(f) CCU on $(-\sqrt{3}, \sqrt{3})$. CCD on $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$. Inflection points at $(-\sqrt{3}, 1/4)$ and $(\sqrt{3}, 1/4)$.

(g)



14. Suppose

$$f(x) = \frac{x}{x^2+4},\ f'(x) = \frac{4-x^2}{(x^2+4)^2},\ f''(x) = \frac{2x^3-24x}{(x^2+4)^3}.$$

(a) Determine all vertical and horizontal asymptotes of f.

(b) Determine the intervals on which the function is increasing or decreasing.

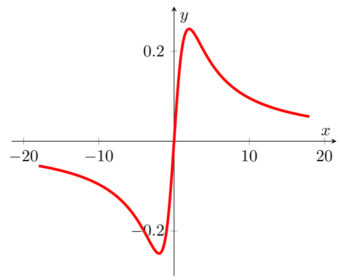
(c) Determine all local maxima and minima of f.

(d) Determine where f is concave upward and where it is concave downward.

(e) Find all inflection points.

(f) Sketch the graph.

Answer.



$f(x) = \dfrac{x}{x^2+4}$

Solution.

(a) All $x \in \mathbb{R}$.

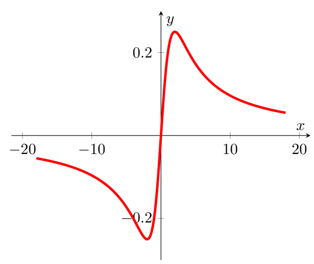
(b) Increasing on $(-2, 2)$. Decreasing on $(-\infty, -2) \cup (2, \infty)$.

(c) Local max at $(2, 1/4)$. Local min at $(-2, -1/4)$.

(d) CCU on $(-2\sqrt{3}, 0) \cup (2\sqrt{3}, \infty)$. CCD on $(-\infty, -2\sqrt{3}) \cup (0, 2\sqrt{3})$.

(e) Inflection points at $(-2\sqrt{3}, -\sqrt{3}/8), (0, 0), (2\sqrt{3}, \sqrt{3}, 8)$.

(f)



$f(x) = \dfrac{x}{x^2+4}$

15. Suppose

$$f(x) = \frac{x^2}{x-1},\ f'(x) = \frac{x(x-2)}{(x-1)^2},\ f''(x) = \frac{2}{(x-1)^3}.$$

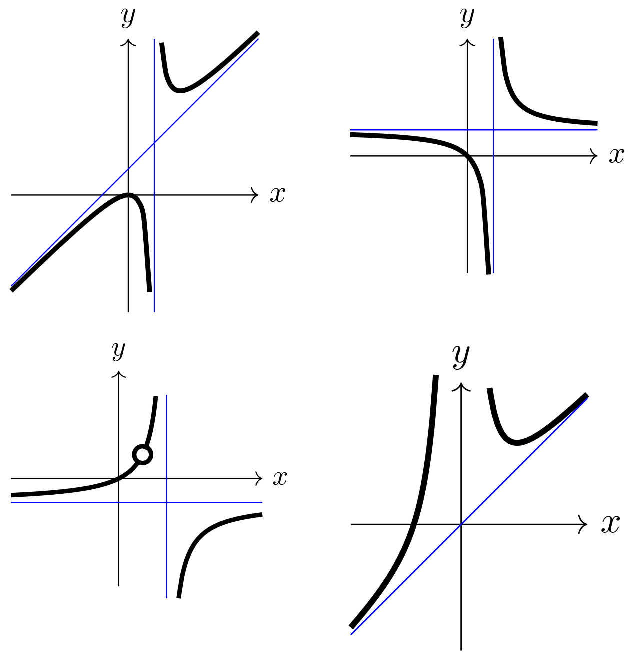
(a) Determine the critical points of f.

(b) Determine the intervals on which the function is increasing or decreasing. Classify the critical point(s) as either local maxima, local minima, or neither.

(c) Determine where f is concave upward and where it is concave downward. Identify any inflection points.

(d) What is the end behaviour of f (i.e. what is happening for large positive and negative x values)? For instance, does f have any horizontal or slant asymptotes?

(e) Indicate which of the following graphs is the graph of $y = f(x)$ by circling the graph of your choice.



Answer. Top Left.

16. Consider the function $f(x) = x^3 - 2x^2 - x + 1$ on the interval $[-1, 3]$.

(a) The derivative of f is:

(b) The critical points for f are:

(c) The second derivative of f is:

(d) The points of inflection of f are:

(e) The intervals on which f is increasing are:

(f) The intervals on which f is concave up are:

(g) The intervals on which f is concave down are:

(h) f has an absolute maximum at:

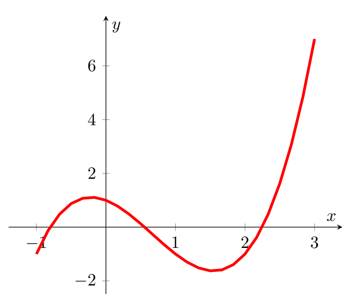
(i) f has an absolute minimum at:

(j) f has a local but not absolute maximum at:

(k) f has a local but not absolute minimum at:

(l) Sketch the graph of $y = f(x)$ using all of the above information. All relevant points must be labeled.

Answer.



$f(x) = x^3 - 2x^2 - x + 1$

17. The aim of this problem is to sketch the graph of $f(x)$ where

$$f(x) = \frac{x^2-2}{x^4},\; f'(x) = -2\frac{x^2-4}{x^5},\; f"(x) = 2\frac{3x^2-20}{x^6}.$$

(a) Find the following derivatives and say what, if anything, they tell us about asymptotes of the graph of $y = f(x)$.

i. $\lim_{x\to 0^-} f(x)$

ii. $\lim_{x\to 0^+} f(x)$

iii. $\lim_{x\to -\infty} f(x)$

iv. $\lim_{x\to \infty} f(x)$

(b) Find the intervals on which f is increasing and decreasing.

(c) Find the intervals on which f is concave up and concave down.

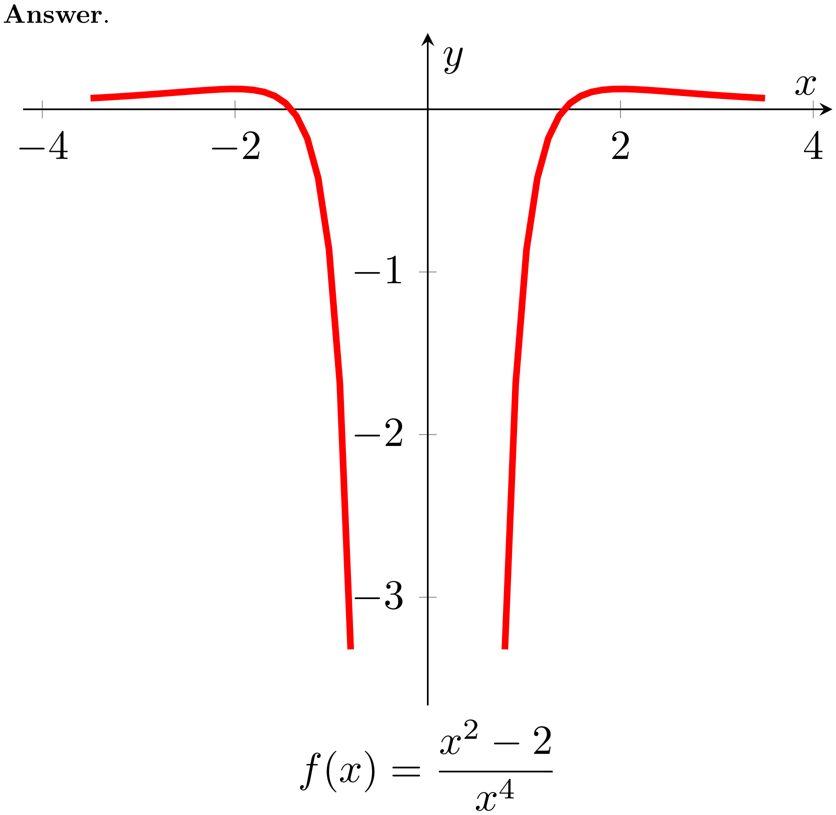
(d) Find the x-intercepts.

(e) Find the coordinates of all inflection points.

(f) Indicate the coordinates of all points where the graph of f has horizontal tangents. Are they local minima or maxima. Justify.

(g) Sketch the graph of $y = f(x)$ using all of the above information. All relevant points must be labeled. You may need the numerical values: $\sqrt{2} \approx 1.414$, $\sqrt{\frac{20}{3}} \approx 2.582$, and $\frac{21}{200} = 0.105$.

Answer.



18. The goal of this exercise is to sketch the plot of $f(x) = \dfrac{1}{(1+e^x)^2}$.

(a) Find the domain of f.

(b) Prove that the derivative of f is given by $f'(x) = -\dfrac{2e^x}{(1+e^x)^3}$.

(c) Prove that the second derivative of f is given by $f''(x) = \dfrac{2e^x(2e^x-1)}{(1+e^x)^4}$.

(d) Find the equations of the two horizontal asymptotes by finding limits of $f(x)$ at $x \to +\infty$ and $-\infty$.

(e) Find any x- and y-intercepts.

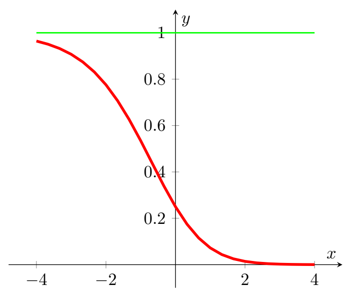
(f) Prove there are no critical points.

(g) Prove that $\left(-\ln 2, \dfrac{4}{9}\right)$ is the only inflection point.

(h) Find the intervals on which f is concave up and those on which f is concave down.

(i) Sketch the graph of $y = f(x)$ using all of the above information. (You may need $\ln 2 \approx 0.7$, $\frac{4}{9} \approx 0.44$.)

Answer.



$f(x) = \dfrac{1}{(1+e^x)^2}$

19. Let

$$f(x) = \frac{x^2-4x}{(x+4)^2},\ f'(x) = \frac{4(3x-4)}{(x+4)^3},\ f''(x) = -\frac{24(x-4)}{(x+4)^4}.$$

(a) Find any x- and y-intercepts.

(b) Determine any horizontal asymptotes of $f(x)$ by taking appropriate limits.

(c) Determine any vertical asymptotes of $f(x)$ by taking appropriate limits.

(d) Fill in the sign of each factor of $f'(x)$ on the indicated intervals and thereby determine the sign of $f'(x)$. Use this to determine where

$f(x)$ is increasing/decreasing.

	$(-\infty,-4)$	$(-4,-4/3)$	$(4/3,\infty)$
$(x+4)^3$			
$3x-4$			
$f'(x)$			
$f(x)$			

(e) Determine the location of any local extrema of $f(x)$ and indicate whether they are minima or maxima.

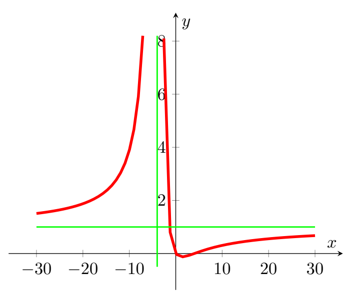
(f) Fill in the sign of each factor of f"(x) on the indicated intervals and thereby determine the sign of f"(x). Use this to determine where $f(x)$ is concave up/down.

	$(-\infty,-4)$	$(-4,-4)$	$(4,\infty)$
-24^3			
$(x+4)^4$			
$x-4$			
$f''(x)$			
$f(x)$			

(g) Determine the locations of any inflection points of $f(x)$.

(h) Sketch the graph of $y=f(x)$ using all of the above information.

Answer.



$f(x) = \dfrac{x^2-4x}{(x+4)^2}$

20. Let

$$f(x) = \frac{4-4x}{x^2}, \quad f'(x) = \frac{4(x-2)}{x^3}, \quad f''(x) = \frac{8(3-x)}{x^4}.$$

Determine the following. Show all your work.

(a) The domain of f.

(b) The x- and y-coordinates of all intercepts.

(c) All asymptotes.

(d) The intervals on which f increases and the intervals on which f decreases.

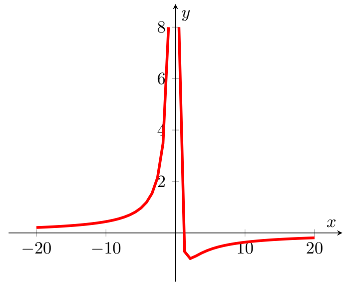
(e) The x- and y-coordinates of all critical points, each classified as a local maximum, minimum or neither.

(f) The intervals on which f is concave up and the intervals on which f is concave down.

(g) The x- and y-coordinates of all inflection points.

(h) Sketch the graph of f using all of the above information and label all pertinent points and lines.

Answer.



$f(x) = \frac{4-4x}{x^2}$

Solution.

(a) All $x \in \mathbb{R}$ such that $x \neq 0$.

(b) $(1, 0)$.

(c) Vertical asymptote at $x = 0$. Horizontal asymptote at $y = 0$.

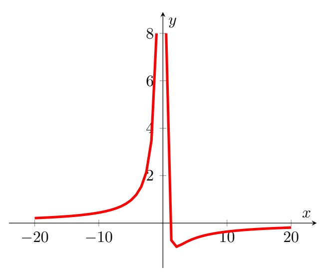
(d) Increasing on $(-\infty, 0) \cup (2, \infty)$. Decreasing on $(0, 2)$.

(e) One critical point at $(2, -1)$. Local min.

(f) CCU on $(-\infty, 0) \cup (0, 3)$. CCD on $(3, \infty)$.

(g) One inflection point at $(3, -8/9)$.

(h) **Answer**.



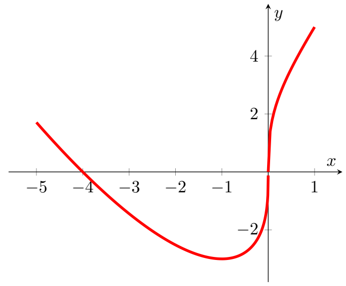
$f(x) = \frac{4-4x}{x^2}$

21. Sketch the graph of

$$y = 4x^{1/3} + x^{4/3}.$$

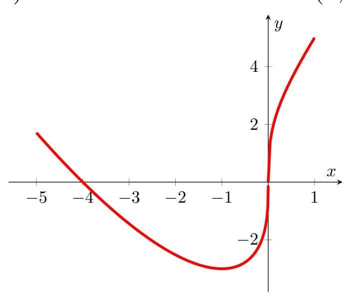
On your graph *clearly* indicate and label all intercepts, local extrema, and inflection points.

Answer.



$y = 4x^{1/3} + x^{4/3}$

Solution. Note that the domain of the given function is the set of all real numbers. The y-intercept is the point $(0,0)$ and the x-intercepts are $(-4,0)$ and $(0,0)$. From $y' = \dfrac{4}{3}x^{1/3}\left(\dfrac{1}{x}+1\right)$ we conclude that y' is not defined at $x = 0$ and that $y' = 0$ if $x = -1$. Thus the critical numbers are $x = -1$ and $x = 0$. Also $y' < 0$ on $(-\infty,-1)$ and $y' > 0$ on $(-1,0)\cup(0,\infty)$. Hence the function has a local minimum at $x = -1$. Note that the y-axis is a vertical asymptote to the graph of the given function. From $y'' = \dfrac{4}{9}x^{-5/3}(x-2)$ it follows that $y''(x) > 0$ on $(-\infty,0)\cup(2,\infty$ and $y''(x) < 0$ on $(0,2)$. Points of inflection are $(0,0)$ and $(2, 6\cdot 2^{1/3})$.



$y = 4x^{1/3} + x^{4/3}$

22. Consider the function $f(x) = x^{2/3}\left(\dfrac{5}{2} - x\right)$.

(a) Explain why f is continuous for $x \in (-\infty,\infty)$.

(b) Determine the behavior of f as $x \to \pm\infty$.

(c) Given that $f'(x) = \dfrac{5}{3x^{1/3}}(1-x)$, determine the regions of increase and decrease of f.

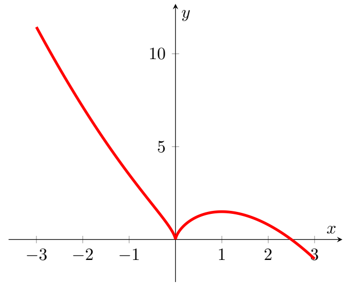
(d) Identify the locations of the relative extrema of f and classify them as maxima or minima.

(e) Given that $f''(x) = -\dfrac{5}{9x^{4/3}}(1+2x)$, determine the concavity of f.

(f) Identify the inflection points of f.

(g) Sketch the graph of f.

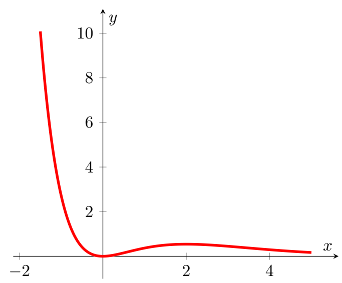
Answer.



$y = x^{2/3}\left(\frac{5}{2} - x\right)$

23. Graph $y = x^x$.

Answer.



$y = x^x$

24. Let

$$f(x) = \frac{x^2+2}{x^2-4},\ f'(x) = \frac{-12x}{(x^2-4)^2},\ f''(x) = \frac{12(3x^2+4)}{(x^2-4)^3}.$$

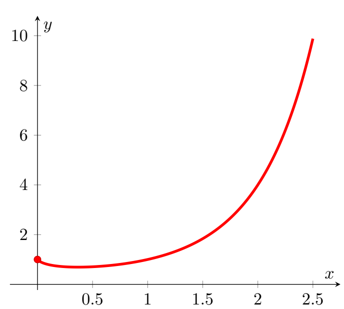
(a) Find the horizontal and vertical asymptotes of the given function (if any).

(b) Find the intervals where the function is increasing or decreasing, and local maximum and local minimum values of the function (if any).

(c) Find the intervals where the function is concave upward or downward and the inflection points.

(d) Sketch a graph of the function.

Answer.



$f(x) = \frac{x^2+2}{x^2-4}$

25. Let

$$f(x) = x^2e^{-x},\ f'(x) = (2x - x^2)e^{-x},\ f''(x) = (2 - 4x + x^2)e^{-x}.$$

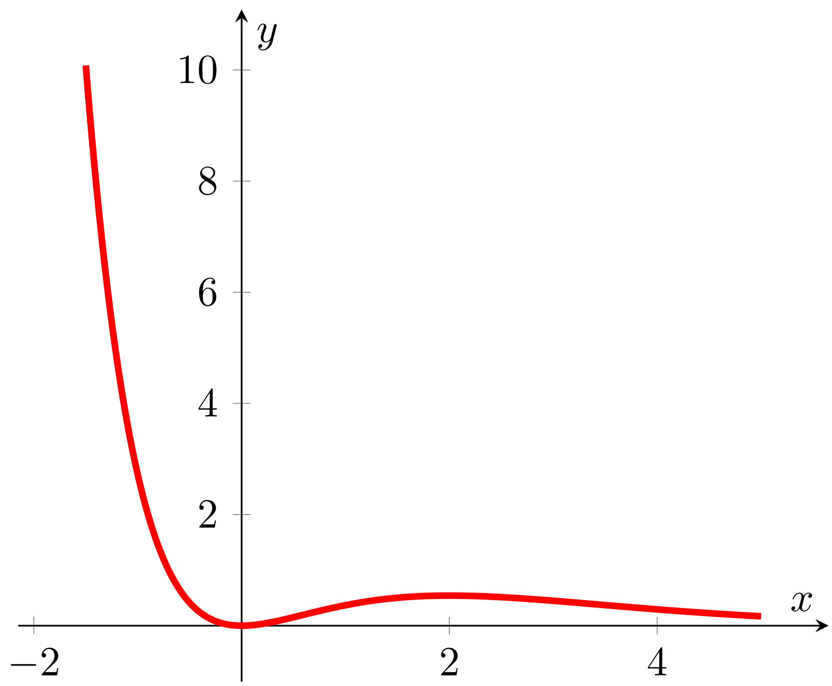
(a) Does the graph f have any vertical or horizontal asymptotes. If so, what are they?

(b) Determine the intervals of increase and decrease of this function. Also determine the extreme values.

(c) Determine the intervals of upwards and downwards concavity of this function. Also determine any points of inflection.

(d) Sketch a graph of this function. Clearly label the local extrema and points of inflection.

Answer.



$$f(x) = x^2 e^{-x}$$

26. Let

$$f(x) = e^{1/x},\ f'(x) = -\frac{e^{1/x}}{x^2},\ f"(x) = \frac{e^{1/x}(2x+1)}{x^4}.$$

(a) What is the domain of f.

(b) Determine any points of intersection of the graph of f with the x and y axes.

(c) Determine any horizontal asymptotes of f.

(d) Determine any vertical asymptotes of f.

(e) For each interval in the table below, indicate whether f is increasing or decreasing.

	$(-\infty, 0)$	$(0, \infty)$
$f(x)$		

(f) Determine the x coordinates of any local extrema of f.

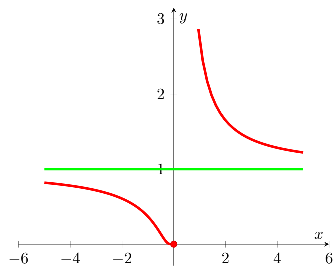
(g) For each interval in the table below, indicate whether f is concave up or concave down.

	$(-\infty,-1/2)$	$(-1/2,0)$	$(0,\infty)$
$f(x)$			

(h) Determine the x coordinates of any inflection points on the graph of f.

(i) Sketch the graph of $y = f(x)$ using all of the above information. All relevant points must be labeled.

Answer.



$f(x) = e^{1/x}$

27. Let $f(x) = e^{-2x^2}$.

(a) Find any horizontal and vertical asymptotes of $f(x)$.

(b) Find the intervals where $f(x)$ is increasing and the intervals where $f(x)$ is decreasing.

(c) Find the x values at which any local maxima or local minima occur and state whether a local maximum or a local minimum occurs.

(d) Give the interval where $f(x)$ is concave up and the intervals where $f(x)$ is concave down.

(e) Find the inflection points of $f(x)$.

Answer.

(a) $y = 0$.

(b) Increasing on $(-\infty, 0)$.

(c) Local maximum at $x = 0$.

(d) Concave down on $(-2, 2)$.

(e) Inflection points at $x = \pm 2$.

28. Consider the function $f(x) = x^3e^{-x+5}$. The first and second derivatives are:

$$f'(x) = x^2(3-x)e^{-x+5}, \;\; f''^{(x)} = x(x^2-6x+6)e^{-x+5}.$$

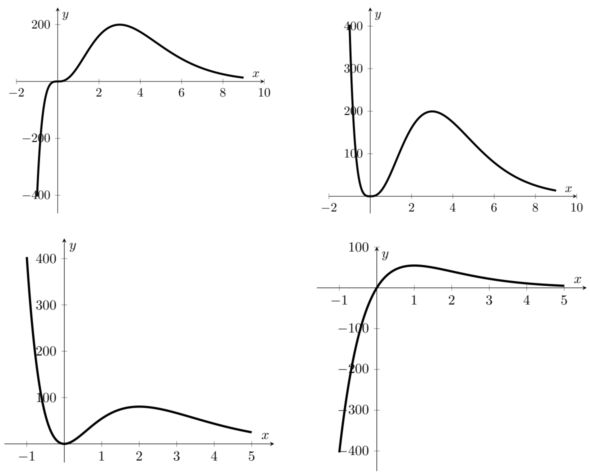
(a) Determine the critical points of f.

(b) Determine the intervals on which the function is increasing, and those on which the function is decreasing. Classify the critical point(s) as either local maxima, local minima, or neither.

(c) Determine where f is concave up and where it is concave down. Identify any inflection points.

(d) What is the end behaviour of f (i.e. what is happening as $x \to \infty$ and $x \to -\infty$)?

(e) Indicate which of graphs in the Figure below is the graph of $y = f(x)$. Also, identify the critical points and inflection points on the graph you've chosen and write in the x-coordinates of these points.



Answer.

(a) $(0, 0)$, $(3, 9e^2)$.

(b) Increasing on $(-\infty, 3)$ and decreasing on $(3, \infty)$. A local (global) maximum at $(3, 9e^2)$. The other critical point is neither a local maximum nor a local minimum.

(c) Note that $x^2 - 6x + 6 = (x - (3 - \sqrt{3}))(x - (3 + \sqrt{3}))$. The function is concave up on $(0, 3 - \sqrt{3})$ and $(3 + \sqrt{3}, \infty)$ and concave down on $(-\infty, 0)$ and $(3 - \sqrt{3}, 3 + \sqrt{3})$. The inflection points are $(0, 0)$, $(3 - \sqrt{3}, (3 - \sqrt{3})^3 e^{2+\sqrt{3}})$, and $(3 + \sqrt{3}, (3 + \sqrt{3})^3 e^{2-\sqrt{3}})$.

(d) $\lim_{x \to -\infty} f(x) = -\infty$, $\lim_{x \to \infty} f(x) = 0$.

(e) Top Left.

29. Let $f(x) = \dfrac{18(x-1)}{x^2}$. Then $f'(x) = \dfrac{18(2-x)}{x^3}$ and $f''(x) = \dfrac{36(x-3)}{x^4}$. Give the following:

(a) The domain of f.

(b) The x and y coordinates of all intercepts of f.

(c) The equations of asymptotes.

(d) The intervals on which f is increasing and those on which f is decreasing.

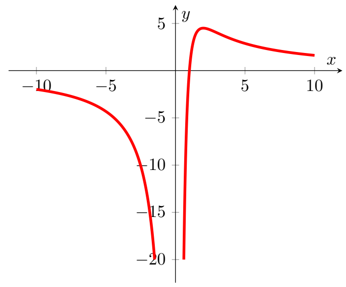
(e) The x and y coordinates of all critical points, each classified as a max, min, or neither.

(f) The intervals on which f is concave up and those on which f is concave down.

(g) The x and y coordinates of all inflection points of f.

(h) Sketch the graph of $y = f(x)$ using all of the above information. All relevant points must be labeled.

Answer.



$f(x) = \dfrac{18(x-1)}{x^2}$

30. Let $f(x) = \dfrac{x+3}{\sqrt{x^2+1}}$. Then $f'(x) = \dfrac{1-3x}{(x^2+1)^{3/2}}$ and $f"(x) = \dfrac{6x^2-3x-3}{(x^2+1)^{5/2}}$. Give the following:

(a) The domain of f.

(b) The x and y coordinates of all intercepts of f.

(c) The equations of asymptotes, if any.

(d) All critical points, if any.

(e) The intervals on which f is increasing and those on which f is decreasing.

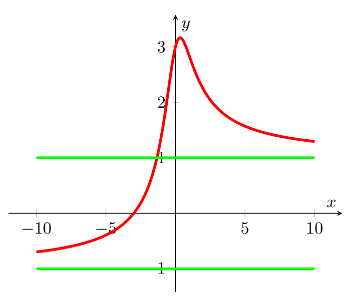
(f) The classification of each critical point, if any, as a minimum or maximum, local or global, or not an extremum.

(g) The intervals on which f is concave up and those on which f is concave down.

(h) The x and y coordinates of all inflection points of f, if any.

(i) Sketch the graph of $y = f(x)$ using all of the above information. All relevant points must be labeled.

Answer.



$f(x) = \dfrac{x+3}{\sqrt{x^2+1}}$

31. Consider the function

$$f(x) = (5-2x)x^{2/3}.$$

For your convenience, the first and second derivative of $f(x)$ are

$$f'(x) = \frac{10(1-x)}{3x^{1/3}}, \; f''(x) = -\frac{10(1+2x)}{9x^{4/3}}.$$

(a) Locate any intercepts of f.

(b) Locate any asymptotes of f.

(c) Determine where f is increasing and decreasing.

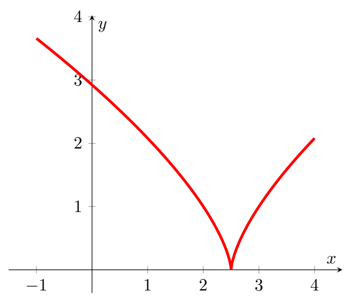
(d) Locate the local extrema of f.

(e) Determine where f is concave upward or concave downward.

(f) Locate all inflection points of f.

(g) Sketch a graph of the function.

Answer.



$f(x) = (5-2x)x^{\frac{2}{3}}$

32. Consider the function

$$f(x) = \frac{x}{x^2-1}.$$

For your convenience, the first and second derivative of $f(x)$ are

$$f'(x) = -\frac{x^2+1}{(x^2-1)^2}, \text{ and } f''(x) = \frac{2x(x^2+3)}{(x^2-1)^3}.$$

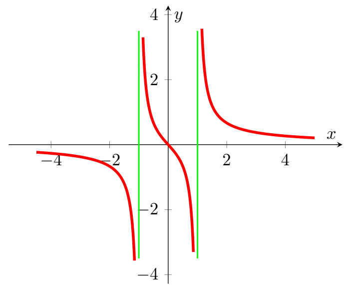
(a) Determine any horizontal and vertical asymptotes of f.

(b) Determine the open intervals on which f is increasing as well those on which f is decreasing.

(c) Determine the open intervals on which f is concave upward as well those on which f is concave downward.

(d) Based on the information found in Parts (a) to (c), sketch a graph of f. Indicate any relative extremum, inflection points and intercepts on your graph.

Answer.



$f(x) = \frac{x}{x^2-1}$

33. Given is the function

$$f(x) = \frac{x^3 - 2x}{3x^2 - 9}.$$

(a) Give the domain of f.

(b) Is f an even function, odd function, or neither?

(c) Find all the x and y-intercepts of the graph f.

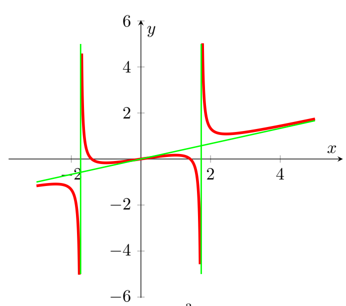
(d) Find all horizontal, vertical, and slant asymptotes of the graph of f. If asymptotes of a certain kind are absent, explain why.

(e) Find the intervals where the graph of f is increasing or decreasing and the locations and values of the local maxima and minima.

(f) Find the intervals where the graph of f is concave upward and concave downward and the infection points.

(g) Sketch the graph and clearly indicate the features found in parts (a)-(f).

Answer.



$f(x) = \frac{x^3-2x}{3x^2-9}$

34. Suppose f is a function satisfying the following conditions:

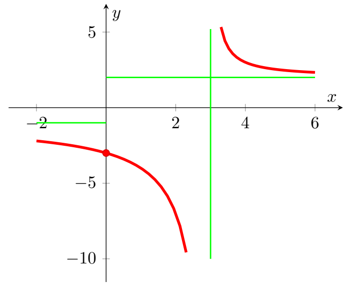
$$\lim_{x\to 3^+} f(x) = +\infty, \ \lim_{x\to 3^-} f(x) = -\infty, \ \lim_{x\to\infty} f(x) = 2, \ \lim_{x\to -\infty} f(x) = -1,$$

$f(0) = -3$, and

$$f'(x) < 0 \text{ for all } x \neq 3, \ f"(x) > 0 \text{ for all } x > 3, \ f"(x) < 0 \text{ for all } x < 3.$$

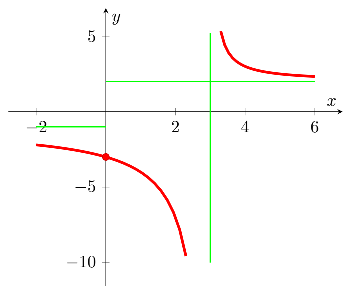
Draw a graph of the function f with all asymptotes and intercepts clearly labeled.

Answer.



Three asymptotes

Solution. It is given that the y-intercept is the point $(0, -3)$. Note that the given function has a vertical asymptote $x = 3$ and two horizontal asymptotes, $y = -1$, when $x \to -\infty$, and $y = 2$, when $x \to \infty$. Also, the function f is decreasing on $(-\infty, 3)$ and $(3, \infty)$. Finally, f is concave upwards on $(3, \infty)$ and concave downwards on $(-\infty, 3)$.



Three asymptotes

35.

(a) Plot the graph of a function f which has only one point of discontinuity on its domain $[-4, \infty)$ and that satisfies:

$$\begin{array}{lll}
\lim_{x\to\infty} f(x) = -2 & f''(x) < 0 & \text{if} \quad -4 < x < -1 \\
\lim_{x\to 0^-} f(x) = \infty & f''(x) > 0 & \text{if} \quad -1 < x < 0 \\
f(0) = 2 & f''(x) < 0 & \text{if} \quad 0 < x < 4 \\
 & f''(x) > 0 & \text{if} \quad 4 < x < \infty \\
 & f'(x) < 0 & \text{if} \quad -4 < x < -1 \\
 & f'(x) > 0 & \text{if} \quad -1 < x < 0 \\
 & f'(x) > 0 & \text{if} \quad 0 < x < 2 \\
 & f'(x) < 0 & \text{if} \quad 2 < x < \infty
\end{array}$$

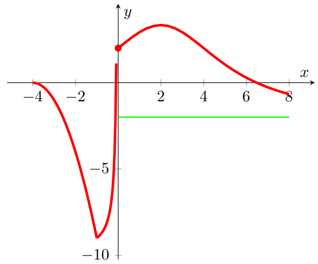
(b) Find all points of inflection for this graph, and for each point of inflection, determine if it is possible that $f''(x) = 0$.

Solution.

(a) The graph has a vertical asymptote $y = 0$ and a horizontal asymptote $x = -2$. The following table summarizes the rest of the given information.

Interval	$(-4, -1)$	$(-1, 0)$	$(0, 2)$	$(2, 4)$	$(4, \infty)$
Monotonity	Decreasing	Increasing	Increasing	Decreasing	Decreasing
Concavity	Downwards	Upwards	Downwards	Downwards	Upwards

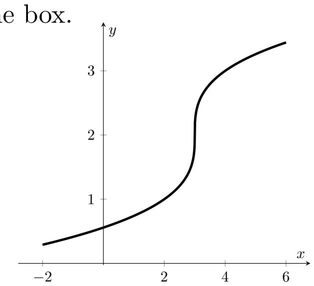
For example:



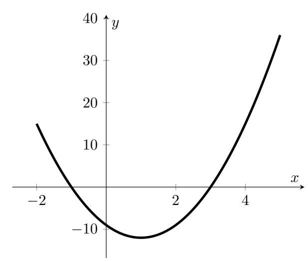
Discontinuity at $x = 0$

(b) There are two points of inflection, $x = -1$ and $x = 4$. We note that $x = -1$ is also a critical number and that by the first derivative test there is a local minimum at $x = -1$. If $f''(-1) = 0$, then $f'(-1)$ exists and $f'(-1) = 0$. This would imply that at this point the graph of f is *above* the tangent line at $x = -1$ which contradicts the fact that the curve *crosses* its tangent line at each inflection point. It follows that $f'(-1)$ does not exist and therefore $f''(-1)$ does not exist.

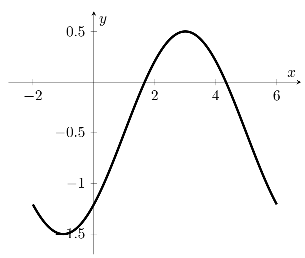
36. The graphs of r, s, f', and g' are shown below, as labelled (these functions are all unrelated). For each question, tick the box if the corresponding function r, s, f, or g has the stated property. Note that you can tick more than one box.



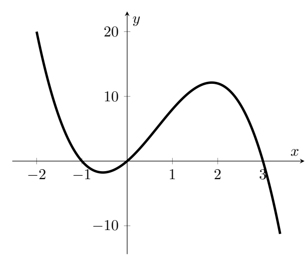
$r(x) = (x-1)^{1/3} + 2$



$f'(x) = 3x^2 - 6x - 9$



$s(x) = \sin \frac{\pi(x-1)}{4} - \frac{1}{2}$



$g'(x) = -2x^3 + 4x^2 + 6x$

(a) The function is increasing over the interval $(1, 3)$.

r ☐ s ☐ f ☐ g ☐ None of them ☐

(b) The function has a critical point when $x = 3$.

r ☐ s ☐ f ☐ g ☐ None of them ☐

(c) The function has an inflection point when $x = 1$.

r ☐ s ☐ f ☐ g ☐ None of them ☐

(d) The function is concave up over the interval $(0, 2)$.

r ☐ s ☐ f ☐ g ☐ None of them ☐

(e) The function has a point where the second derivative does not exist.

r ☐ s ☐ f ☐ g ☐ None of them ☐

Answer.

(a) $\surd$: r, s, g.

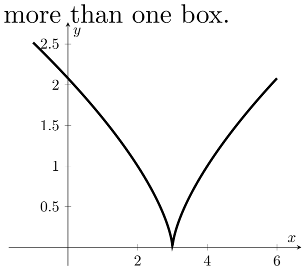
(b) $\surd$: r, s, f, g.

(c) $\surd$: r.

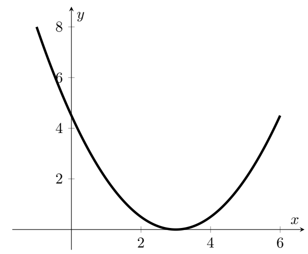
(d) $\surd$: g.

(e) $\surd$: r.

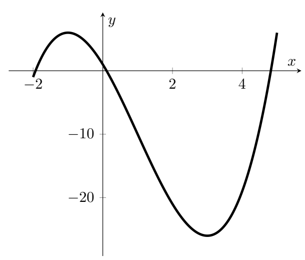
37. The graphs of four functions are shown below. For each question, tick the box if the corresponding function has the stated property. Note that you can tick more than one box.



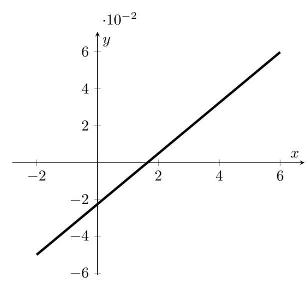
$A : y = (x-3)^{2/3}$



$B : y = 0.5(x-3)^2$



$C : y = x^3 - 3x^2 - 9x + 1$



$D : s(x) = \sin\frac{\pi(x-1)}{4} - \frac{1}{2}$

(a) The derivative of the function is zero at $x = -1$.

A ☐ B ☐ C ☐ D ☐ None of them ☐

(b) The function has a point where the second derivative does not exist.

A ☐ B ☐ C ☐ D ☐ None of them ☐

(c) The derivative of the function is negative on the whole interval $(-2, 0)$.

A ☐ B ☐ C ☐ D ☐ None of them ☐

(d) The function has a critical point when $x = 3$.

A ☐ B ☐ C ☐ D ☐ None of them ☐

(e) The second derivative is positive over the whole interval $(2, 5)$.

A ☐ B ☐ C ☐ D ☐ None of them ☐

Answer.

(a) $\surd$: C, D.

(b) $\surd$: A.

(c) $\surd$: A, D.

(d) $\surd$: A, B, C, D.

(e) $\surd$: B.

38. For what values of the constants a and b is $(1, 6)$ a point of inflection of the curve $y = x^3 + ax^2 + bc + 1$? Justify your answer.

Hint. Solve the system $y(1) = a + b + 2 = 6$ and $y"(1) = 6 + 2a = 0$.

Answer. $a = -3$, $b = 7$.

3.3 Optimization

Optimization problems in calculus are always asking to maximize or minimize some quantity. Typical phrases that indicate an optimization problem include: "Find the largest ..." or "Find the minimum ..."

To solve an optimization problem you need to do the following:

(a) Find a function of one variable that models the situation described in the given problem.

(b) Use the Calculus tools to find the critical numbers and determine whether they correspond to a local maximum or minimum.

Solve the following optimization problems:

1. Find the absolute maximum and minimum values of $f(x) = 3x^2 - 9x$ on the interval $[-1, 2]$.

Answer. The maximum value is $f(-1) = 12$ and the minimum value is $f\left(\frac{3}{2}\right) = -\frac{27}{4}$.

Solution. Note that the function f is continuous on the closed interval $[-1, 2]$. By the Intermediate Value Theorem the function f attains its maximum and minimum values on $[-1, 2]$. To find those global extrema we evaluate and compare the values of f at the endpoints and critical numbers that belong to $(-1, 2)$. From $f'(x) = 6x - 9 = 3(2x - 3)$ we conclude that the critical number is $x = \frac{3}{2}$. From $f(-1) = 12$, $f(2) = -6$, and $f\left(\frac{3}{2}\right) = -\frac{27}{4}$ we conclude that the maximum value is $f(-1) = 12$ and the minimum value is $f\left(\frac{3}{2}\right) = -\frac{27}{4}$.

2. Find the absolute maximum and minimum values of $f(x) = x^3 - 12x - 5$ on the interval $[-4, 6]$. Clearly explain your reasoning.

Answer. The global minimum value is $f(-4) = f(2) = -21$, and the global maximum value is $f(6) = 139$. Note that $f(2) = -21$ is also a local minimum and that $f(-2)$ is a local maximum. (*Reminder:* By our definition, for $x = c$ to be a local extremum of a function f it is necessary that c is an interior point of the domain of f. This means that there is an open interval I contained in the domain of f such that $c \in I$.)

3. If a and b are positive numbers, find the maximum value of $f(x) = x^a(1-x)^b$.

Answer. The maximum value of f is $f\left(\dfrac{a}{a+b}\right) = \left(\dfrac{a}{a+b}\right)^a \left(\dfrac{b}{a+b}\right)^b$.

Solution. From $f'(x) = ax^{a-1}(1-x)^b - bx^a(1-x)^{b-1} = x^{a-1}(1-x)^{b-1}(a-(a+b)x)$ and the fact that a and b are positive conclude that $x = \dfrac{a}{a+b} \in (0,1)$ is a critical number of the function f. Since $f(0) = f(1) = 0$ and $f(x) > 0$ for all $x \in (0,1)$ it follows that the maximum value of f is $f\left(\dfrac{a}{a+b}\right) = \left(\dfrac{a}{a+b}\right)^a \left(\dfrac{b}{a+b}\right)^b$.

4. Find all critical points of the function $f(x) = |3x-5|$ on the interval $[-3,2]$. Also find all maxima and minima of this function on $[-3,2]$, both local and global.

Answer. The global and local minimum is $f\left(\dfrac{5}{3}\right) = 0$ and that the global maximum is $f(-3) = 14$.

Solution. From $f(x) = \begin{cases} 3x-5 & \text{if } x \geq \frac{5}{3} \\ -3x+5 & \text{if } x < \frac{5}{3} \end{cases}$ we conclude that $f'(x) = \begin{cases} 3 & \text{if } x > \frac{5}{3} \\ -3 & \text{if } x < \frac{5}{3} \end{cases}$. Thus, for $x \neq \dfrac{5}{3}$, $f'(x) \neq 0$ and the derivative of f is not defined at $x = \dfrac{5}{3}$. We conclude that the only critical number of the function f on the interval $[-3,2]$ is $x = \dfrac{5}{3}$. Clearly, $f\left(\dfrac{5}{3}\right) = 0$. From $f(-3) = 14$ and $f(1) = 2$ it follows that the global and local minimum is $f\left(\dfrac{5}{3}\right) = 0$ and that the global maximum is $f(-3) = 14$.

5. The sum of two positive numbers is 12. What is the smallest possible value of the sum of their squares? Show your reasoning.

Answer. $f(6) = 72$.

Solution. The question is to find the minimum value of the function $f(x) = x^2 + (12-x)^2$, $x \in (0,12)$. From $f'(x) = 4(x-6)$ it follows that $x = 6$ is the only critical number. From $f''(6) = 4 > 0$, by the second derivative test, it follows that $f(6) = 72$ is the minimum value of the function f.

6. If a and b are positive numbers, find the x coordinate which gives the absolute maximum value of $f(x) = x^a(1-x)^b$ on the interval $[0,1]$.

Answer. $x = \dfrac{a}{a+b}$.

Solution. Note that $f(0) = f(1) = 0$ and that $f(x) > 0$ for $x \in (0,1)$. Thus by the Intermediate Value Theorem there is $c \in (0,1)$ such that $f(c)$ is the maximum value of f. Since f is differentiable on $(0,1)$, c must be a critical point. Note that $f'(x) = x^{a-1}(1-x)^{b-1}(a-(a+b)x)$. Since a and b are both positive we have that $x = \dfrac{a}{a+b} \in (0,1)$. Thus $x = \dfrac{a}{a+b}$ is the only critical point of the function f in the interval $(0,1)$ and $f\left(\dfrac{a}{a+b}\right) = \dfrac{a^a b^b}{(a+b)^{a+b}}$ is the maximum value.

7. Find the point on the curve $x+y^2=0$ that is closest to the point $(0,-3)$.

Answer. $(-1,-1)$.

Solution. The distance between a point (x,y) on the curve and the point $(0,-3)$ is $d=\sqrt{(x-0)^2+(y-(-3))^2}=\sqrt{y^4+(y+3)^2}$. The question is to minimize the function $f(y)=y^4+(y+3)^2$, $y\in\mathbb{R}$. From $f'(y)=2(2y^3+y+3)=2(y+1)(2y^2-2y+3)$we conclude that $y=-1$ is the only critical number of the function f. From $f"(-1)=10>0$, by the second derivative test we conclude that $f(-1)=5$ is the (local and global) minimum value of f. Thus the closest point is $(-1,-1)$.

8. A straight piece of wire 40 cm long is cut into two pieces. One piece is bent into a circle and the other is bent into a square. How should wire be cut so that the total area of both the circle and square is *minimized*?

Answer. The two pieces should be of the length $\dfrac{40\pi}{\pi+4}$ and $\dfrac{160}{\pi+4}$.

Solution. Let x be the radius of the circle. The question is to minimize the function $f(x)=\pi x^2+\left(\dfrac{40-2\pi x}{4}\right)^2$, $x\in\left(0,\dfrac{20}{\pi}\right)$. (We are given that there are TWO pieces.) The only critical number of the function f is $x=\dfrac{20}{\pi+4}$. To minimize the total area the two pieces should be of the length $\dfrac{40\pi}{\pi+4}$ and $\dfrac{160}{\pi+4}$.

9. A straight piece of wire 28 cm long is cut into two pieces. One piece is bent into a square (i.e. dimensions x times x.) The other piece is bent into a rectangle with aspect ratio three (i.e. dimensions y times $3y$.) What are the dimensions, in centimetres, of the square and the rectangle such that the sum of their areas is *minimized*?

Hint. The question is to minimize the function $f(x)=x^2+\dfrac{3(7-x)^2}{4}$, $x\in(0,7)$. (We are given that there are TWO pieces.)

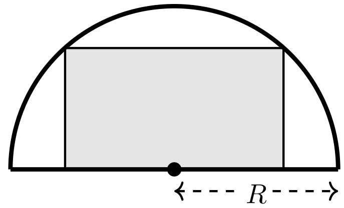
Answer. $x=3$ and $y=2$.

10. With a straight piece of wire 4m long, you are to create an equilateral triangle and a square, or either one only. Suppose a piece of wire of length x metres is bent into a triangle and the remainder is bent into a square. Find the value of x which maximizes the total area of both the triangle and the square.

Answer. The maximum total area is obtained when only the square is constructed.

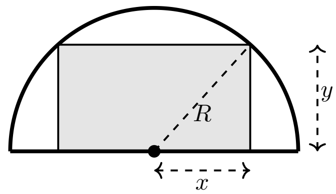
Solution. The question is to *maximize* the function $f(x)=\dfrac{x^2\sqrt{3}}{36}+\dfrac{(4-x)^2}{16}$, $x\in[0,4]$. Note that $f"(x)>0$ for $x\in(0,4)$ and conclude that the maximum value must occur at $x=0$ and/or $x=4$. Since $f(4)<f(0)$, the maximum total area is obtained when only the square is constructed.

11. Show that a perimeter of $2\sqrt{5}R$ is the largest possible perimeter of a rectangle inscribed in a semicircle of radius R, with one side of the rectangle lying along the diameter of the semicircle.



Hint. Maximize the function $P(x) = 2x + \sqrt{R^2 - x^2}$, where $2x$ represents the width of a rectangle inscribed in the semicircle.

Solution. Let x and y be as follows:



Then the perimeter is given by the function $P(x, y) = 4x + 2y$, for $0 \le x \le R$ and $0 \le y \le R$. From the constraint, we see that $R^2 = x^2 + y^2$, and so we can write $y = \sqrt{R^2 - x^2}$. Hence, we get

$$P(x) = 4x + 2\sqrt{R^2 - x^2}, \quad 0 \le x \le R.$$

We find one interior critical point at $x = \frac{2R}{\sqrt{5}}$. Since

$$P(0) = 2R, \quad P(R) = 4R, \quad P\left(\frac{2R}{\sqrt{5}}\right) = \frac{10R}{\sqrt{5}} = 2\sqrt{5}R.$$

Therefore, we see that the largest possible perimeter is $2\sqrt{5}R$, as desired.

12. A rectangle with sides parallel to the coordinate axes is to be inscribed in the region enclosed by the graphs of $y = x^2$ and $y = 4$ so that its perimeter has maximum length.

(a) Sketch the region under consideration.

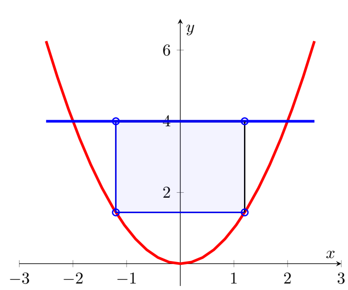
(b) Supposing that the x-coordinate of the bottom right vertex of the rectangle is a, find a formula which expresses P, the length of the perimeter, in terms of a.

(c) Find the value of a which gives the maximum value of P, and explain why you know that this value of a gives a maximum.

(d) What is the maximum value of P, the length of the perimeter of the rectangle?

Answer.

(a)

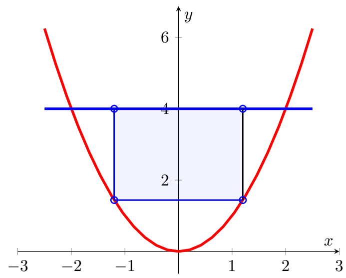


(b) $P = 4a + 2(4 - a^2) = 2(4 + 2a - a^2)$, $a \in (0, 2)$.

(c) $P(1)$.

(d) $P(1) = 10$.

Solution.

(a) 

(b) We let (a, a^2) be bottom right vertex of the rectangle for $a \in (0, 2)$. Then the remaining three vertices are given by $(-a, a^2)$, $(a, 4)$ and $(-a, 4)$. Therefore, the perimeter of the rectangle is given by

$$P(a) = 2(2a) + 2(4 - a^2) = 2(4 - 2a - a^2),$$

for $a \in (0, 2)$.

(c) From $\dfrac{dP}{da} = 4(1-a)$ it follows that $a = 1$ is the only critical number. The fact that $f''(a) = -8 < 0$ for all $a \in (0, 2)$ implies, by the second derivative test, that $P(1)$ is the maximum value.

(d) We have that $P(1) = 2(4 + 2 - 1) = 10$.

13. Find the dimensions of the rectangle of largest area that has its base on the x-axis and its other two vertices above the x-axis and lying on the parabola $y = 12 - x^2$.

Answer. The length of the rectangle with the largest area is 4 and its height is 8.

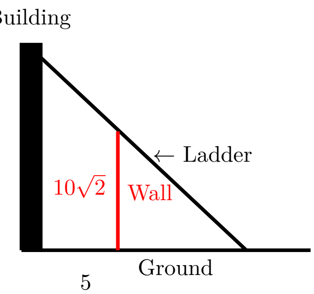
Solution. Let $(x, 0)$ be the bottom right vertex of the rectangle. The question is to maximize $f(x) = 2x(12 - x^2)$, $x \in (0, 2\sqrt{3})$. The only critical number is $x = 2$. The length of the rectangle with the largest area is 4 and its height is 8.

14. A farmer has 400 feet of fencing with which to build a rectangular pen. He will use part of an existing straight wall 100 feet long as part of one side of the perimeter of the pen. What is the maximum area that can be enclosed?

Answer. $f(150) = 15000$ ft^2.

Solution. Let x be the length of one side of the fence that is perpendicular to the wall. Note that the length of the side of the fence that is parallel to the wall equals $400 - 2x$ and that this number cannot be larger than 100.The question is to maximize the function $f(x) = x(400 - 2x)$, $x \in [150, 400)$. The only solution of the equation $f'(x) = 4(100 - x) = 0$ is $x = 100$ but this value is not in the domain of the function f. Clearly $f'(x) < 0$ for $x \in [150, 400)$ which implies that f is decreasing on its domain. Therefore the maximum area that can be enclosed is $f(150) = 15000$ ft^2.

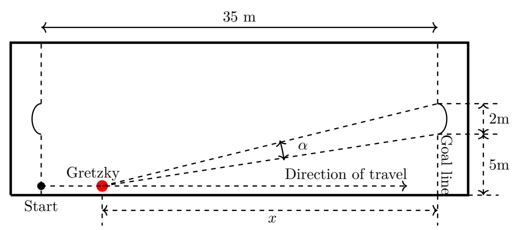
15. A $10\sqrt{2}$ ft wall stands 5 ft from a building. Find the length L of the shortest ladder, supported by the wall, that reaches from the ground to the building.



Hint. To minimize $L^2 = (x+5)^2 + y^2$, use the fact that $\dfrac{x}{10\sqrt{2}} = \dfrac{x+5}{y}$ and the first derivative.

Answer. $L = 15\sqrt{3}$.

16. An attacking player (Gretzky) is skating with the puck along the boards as shown. As Gretzky proceeds, the apparent angle α between the opponent's goal posts first increases, then decreases.



(a) Using dimensions given in the Figure above, find an expression for α in terms of the distance x from Gretzky to the goal line.

(b) Assume that Gretzky's chance of scoring is greatest when α is maximum (this may be the case if the opposing team has "pulled" their goalie). At which distance x from the goal line should Gretzky shoot the puck? It is clear that α is very small when $x = 35$ and $x = 0$, so there is no need to check the endpoints of the domain $[0, 35]$.

Answer.

(a) $\alpha = \tan^{-1}\left(\dfrac{7}{x}\right) - \tan^{-1}\left(\dfrac{5}{x}\right)$.

(b) $x = \sqrt{35}$.

17. In an elliptical sport field we want to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Find the length $2x$ and width $2y$ of the pitch (in terms of a and b) that maximize the area of the pitch. [Hint: express the area of the pitch as a function of x only.]

Answer. To maximize the area of the soccer field its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

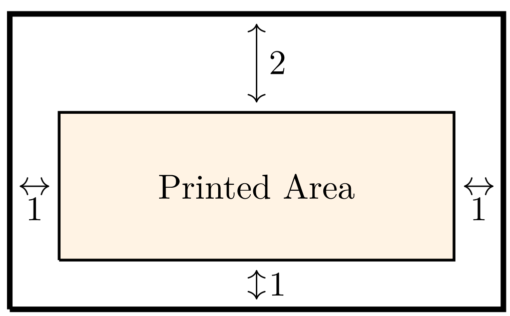
Solution. Let $(x, y) = \left(x, \dfrac{b}{a}\sqrt{a^2 - x^2}\right)$ be the upper right vertex of the rectangle. The question is to maximize the function $f(x) = \frac{4b}{a}x\sqrt{a^2 - x^2}$, $x \in (0, a)$. From $f'(x) = \frac{4b}{a}\frac{a^2 - 2x^2}{\sqrt{a^2 - x^2}}$ we conclude that the only critical number is $x = \frac{a}{\sqrt{2}}$. By the first derivative test, there is a local maximum at this critical number. Since $\lim_{x\to 0^+} f(x) = \lim_{x\to a^-} f(x) = 0$, it follows that $f\left(\frac{a}{\sqrt{2}}\right) = 2ab$ is the maximum value of the function f. Thus to maximize the area of the soccer field its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

18. The top and bottom margins of a poster are each 6 cm, and the side margins are each 4 cm. If the area of the printed material on the poster (that is, the area between the margins) is fixed at 384 cm^2, find the dimensions of the poster with the smallest total area.

Answer. The dimensions of the poster with the smallest area are $x = 24$ cm and $y = 36$ cm.

Solution. Let a be the length of the printed material on the poster. Then the width of this area equals $b = \frac{384}{a}$. It follows that the length of the poster is $x = a + 8$ and the width of the poster is $y = b + 12 = \frac{384}{a} + 12$. The question is to minimize the function $f(a) = xy = (a + 8)\left(\frac{384}{a} + 12\right) = 12\left(40 + a + \frac{256}{a}\right)$. It follows that the function has a local minimum at $a = 16$. The dimensions of the poster with the smallest area are $x = 24$ cm and $y = 36$ cm.

19. A poster is to have an area of 180 in^2 with 1-inch margins at the bottom and the sides and a 2 inch margin at the top. What dimensions will give the largest printed area?



Hint. Maximize the function $A(x) = (x - 2)\left(\frac{180}{x} - 3\right)$, where x represents the length of the poster.

Answer. $2\sqrt{30} \cdot 3\sqrt{30}$.

20. Each rectangular page of a book must contain 30 cm^2 of printed text, and each page must have 2 cm margins at top and bottom, and a 1 cm margin at each side. What is the minimum possible area of such a page?

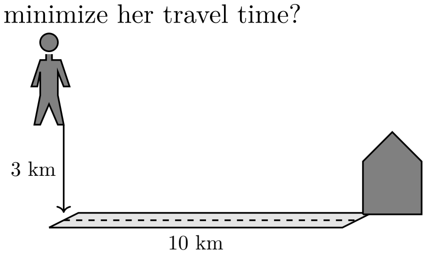
Answer. $(\sqrt{15} + 2) \times (2\sqrt{15} + 4)$.

21. Maya is 2 km offshore in a boat and wishes to reach a coastal village which is 6 km down a straight shoreline from the point on the shore nearest to the boat. She can row at 2 km/hr and run at 5 km/hr. Where should she land her boat to reach the village in the least amount of time?

Answer. Maya should land her boat $\dfrac{4}{3}$ km from the point initially nearest to the boat.

Solution. Let P be the point on the shore where Maya lands her boat and let x be the distance from P to the point on the shore that is closest to her initial position. Thus to reach the village she needs to row the distance $z = \sqrt{4 + x^2}$ and run the distance $y = 6 - x$. Time needed to row the distance z is given by $T_1 = \dfrac{z}{2}$ and time she needs to run is $T_2 = \dfrac{y}{5}$. Therefore the question is to minimize the function $T = T(x) = T_1 + T_2 = \dfrac{\sqrt{4+x^2}}{2} + \dfrac{6-x}{5}$, $x \in [0, 6]$. From $f'(x) = \dfrac{x}{2\sqrt{4+x^2}} - \dfrac{1}{5}$ it follows that the only critical number is $x = \dfrac{4}{3}$. From $T(0) = \dfrac{11}{5} = 2.2$, $T(6) = \sqrt{10}$, and $T\left(\dfrac{4}{3}\right) \approx 2.135183758$ it follows that the minimum value is $T\left(\dfrac{4}{3}\right)$. Maya should land her boat $\dfrac{4}{3}$ km from the point initially nearest to the boat.

22. A hiker is in the woods at a distance of 3 km from a straight road. She would like to reach a supply store which is located 10 km down the road from the nearest point on the road to her. Assuming that she hikes through the woods at a rate of 2 km/hr and walks along the road at a rate of 4 km/h, what point on the road should she hike through the woods to so as to minimize her travel time?



Hint. Minimize the function $T(x) = \frac{1}{2} \cdot \sqrt{9 + (10-x)^2} + \frac{x}{4}$, where x is the distance between a point on the road and the store.

Answer. $x = 10 - \sqrt{3}$.

23. A rectangular box has a square base with edge length x of at least 1 unit. The total surface area of its six sides is 150 square units.

(a) Express the volume V of this box as a function of x.

(b) Find the domain of $V(x)$.

(c) Find the dimensions of the box in part (a) with the greatest possible volume. What is this greatest possible volume?

Answer.

(a) Express the volume V of this box as a function of x.

(b) $[1, 5\sqrt{3})$.

(c) $V(5) = 125$ cube units.

Solution.

(a) Let y be the height of the box. Then the surface area is given by $S = 2x^2+4xy$. From $S = 150$ it follows that $y = \dfrac{1}{2}\left(\dfrac{75}{x} - x\right)$. Therefore the volume of the box is given by $V = V(x) = \dfrac{x}{2}\left(75 - x^2\right)$.

(b) From the fact that $y = \dfrac{1}{2}\left(\dfrac{75}{x} - x\right) > 0$ it follows that the domain of the function $V = V(x)$ is the interval $[1, 5\sqrt{3})$.

(c) Note that $\dfrac{dV}{dx} = \dfrac{3}{2}(25 - x^2)$ and that $\dfrac{d^2V}{dx^2} = -3x < 0$ for all $x \in (1, 5\sqrt{3})$. Thus the maximum value is $V(5) = 125$ cube units.

24. An open-top box is to have a square base and a volume of 10 m^3. The cost per square metre of material is \$5 for the bottom and \$2 for the four sides. Let x and y be lengths of the box's width and height respectively. Let C be the total cost of material required to make the box.

(a) Express C as a function of x and find its domain.

(b) Find the dimensions of the box so that the cost of materials is minimized. What is this minimum cost?

Answer.

(a) Note that $y = \frac{10}{x^2}$. The cost function is given by $C(x) = 5x^2 + 2 \cdot 4 \cdot x \cdot \dfrac{10}{x^2} = 5x^2 + \dfrac{80}{x}$, $x > 0$.

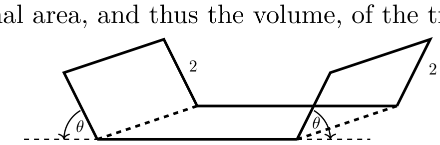
(b) $2 \times 2 \times \dfrac{5}{2}$. The minimum cost is $C(2) = \$60$.

25. An open-top box is to have a square base and a volume of 13500 cm^3. Find the dimensions of the box that minimize the amount of material used.

Answer. $x = 30$.

Solution. Let x be the length and the width of the box. Then its height is given by $y = \dfrac{13500}{x^2}$. It follows that the surface area is $S = x^2 + \dfrac{54000}{x}$ cm^2, $x > 0$. The question is to minimize S. From $\dfrac{dS}{dx} = 2x - \dfrac{54000}{x^2}$ and $\dfrac{d^2S}{dx^2} = 2 + \dfrac{3 \cdot 54000}{x^3} > 0$ for all $x > 0$ it follows that the function S has a local and global minimum at $x = 30$.

26. A water trough is to be made from a long strip of tin 6 ft wide by bending up at the angle θ a 2 ft strip at each side. What angle θ would maximize the cross sectional area, and thus the volume, of the trough?

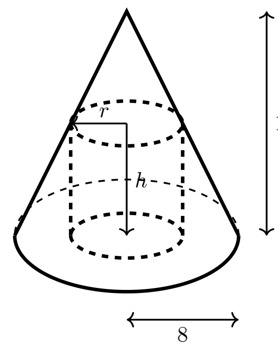


Answer. $\theta = \dfrac{\pi}{3}$.

Solution. We need to maximize the area of the trapezoid with parallel sides of lengths $a = 2$ and $c = 2 + 2 \cdot 2\cos\theta = 2 + 4\cos\theta$ and the height $h = 2\sin\theta$. Thus we maximize the function $A = A(\theta) = \dfrac{2 + (2 + 4\cos\theta)}{2} \cdot$

$2\sin\theta = 4(\sin\theta + \sin\theta\cos\theta)$, $\theta \in (0,\pi)$. From $\dfrac{dA}{d\theta} = 4(\cos\theta + \cos^2\theta - \sin^2\theta) = 4(2\cos^2\theta + \cos\theta - 1) = 4(2\cos\theta - 1)(\cos\theta + 1)$ we obtain the critical number $\theta = \dfrac{\pi}{3}$. The First Derivative Test confirms that $\theta = \dfrac{\pi}{3}$ maximizes the cross sectional area of the trough.

27. Find the dimensions of the right circular cylinder with greatest volume that can be inscribed in a right circular cone of radius 8 cm and height 12 cm.

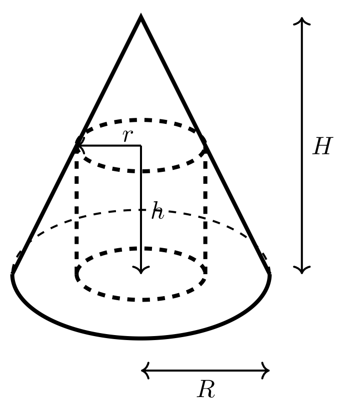


Answer. $r = \dfrac{16}{3}$ cm, $h = 4$ cm.

28. Find the dimension of the right circular cylinder of maximum volume that can be inscribed in a right circular cone of radius R and height H.

Answer. The dimensions are $r = \dfrac{2R}{3}$ and $h = \dfrac{H}{3}$.

Solution. Let r be the radius of the base of a cylinder inscribed in the cone and let h be its height. From $\dfrac{H}{R} = \dfrac{h}{R-r}$, we conclude that $h = \dfrac{H(R-r)}{R}$.



Thus the volume of the cylinder is $V = V(r) = \dfrac{\pi H}{R} r^2(R-r)$, $r \in (0,R)$. From $\dfrac{dV}{dr} = \dfrac{\pi H}{R} r(2R-3r)$ and $\dfrac{d^2V}{dr^2} = \dfrac{2\pi H}{R}(R-3r)$ it follows that the maximum value of the volume of the cylinder is $V\left(\dfrac{2R}{3}\right) = \dfrac{4\pi H R^2}{27}$. The dimensions are $r = \dfrac{2R}{3}$ and $h = \dfrac{H}{3}$.

29. A hollow plastic cylinder with a circular base and open top is to be made and 10 m^2 of plastic is available. Find the dimensions of the cylinder that give the maximum volume and find the value of the maximum volume.

Answer. $r = \sqrt{\dfrac{10}{3\pi}}$ m, $h = \left(\dfrac{5}{\pi}\sqrt{\dfrac{3\pi}{10}} - \dfrac{1}{2}\sqrt{\dfrac{10}{3\pi}}\right)$ m, $V = \dfrac{10}{3}\sqrt{\dfrac{10}{3\pi}}$ m^3.

30. An open-topped cylindrical pot is to have volume 250 cm^3.The material for the bottom of the pot costs 4 cents per cm^2; that for its curved side costs 2 cents per cm^2. What dimensions will minimize the total cost of this pot?

Answer. $r = \dfrac{5}{\sqrt[3]{\pi}}$.

Solution. Let r be the radius of the base of the pot. Then the height of the pot is $h = \dfrac{250}{\pi r^2}$. The cost function is $C(r) = 4\pi r^2 + \dfrac{1000}{r}$, $r > 0$. The cost function has its minimum at $r = \dfrac{5}{\sqrt[3]{\pi}}$.

31. A cylindrical can without a top is made to contain 1,000 cm^2 of liquid. Find the dimensions that will minimize the cost of the material to make the can.

Answer. $R = \dfrac{10}{\sqrt[3]{\pi}}$ cm, $h = \dfrac{10}{\sqrt[3]{\pi}}$ cm.

32. Cylindrical soup cans are to be manufactured to contain a given volume V. No waste is involved in cutting the material for the vertical side of each can, but each top and bottom which are circles of radius r, are cut from a square that measures $2r$ units on each side. Thus the material used to manufacture each soup can has an area of $A = 2\pi rh + 8r^2$ square units.

(a) How much material is wasted in making each soup can?

(b) Find the ratio of the height to diameter for the most economical can (i.e. requiring the least amount of material for manufacture.)

(c) Use either the first or second derivative test to verify that you have minimized the amount of material used for making each can.

Answer.

(a) The amount of material wasted is $2(4 - \pi)r^2$.

(b) $\dfrac{h}{r} = \dfrac{4}{\pi}$.

(c) Second Derivative Test.

Solution.

(a) The surface area of the can is $S = 2\pi rh + 2\pi r^2$. The amount of material wasted is $A - S = 2(4 - \pi)r^2$.

(b) From $V = \pi r^2 h$ it follows that the amount of material needed to make a can of the given volume V is $A = A(r) = \dfrac{2V}{r} + 8r^2$. This function has its minimum at $r = \dfrac{\sqrt[3]{V}}{2}$. The ratio of the height to diameter for the most economical can is $\dfrac{h}{r} = \dfrac{4}{\pi}$.

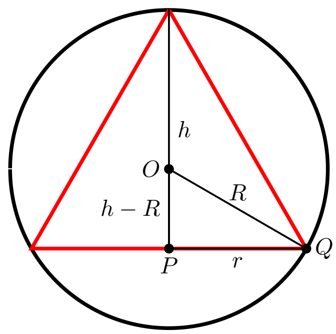
(c) $A"(r) = \dfrac{4V}{r^3} + 8 > 0$ for $r > 0$. Hence, by the Second Derivative Test, the amount of material is minimized.

33. A storage container is to be made in the form of a right circular cylinder and have a volume of 28π m^3. Material for the top of the container costs \$5 per square metre and material for the side and base costs \$2 per square metre. What dimensions will minimize the total cost of the container?

Answer. $r = \sqrt[3]{20}$, $h = \dfrac{14}{\sqrt[3]{50}}$. Minimize the cost function $C = C(r) = 7r^2\pi + \dfrac{280\pi}{r}$.

34. Show that the volume of the largest cone that can be inscribed inside a sphere of radius R is $\dfrac{32\pi R^3}{81}$.

Solution. From the Figure below, we conclude that $r^2 = R^2-(h-R)^2 = h(4R-h)$. Then the volume of the cone as a function of h is given by $V = \dfrac{\pi}{3}h^2(4R-h)$. Maximize.



35. The sound level measured in watts per square metre, varies in direct proportion to the power of the source and inversely as the square of the distance from the source, so that is given by $y = kPx^{-2}$, where y is the sound level, P is the source power, x is the distance form the source, and k is a positive constant. Two beach parties, 100 metres apart, are playing loud music on their portable stereos. The second party's stereo has 64 times as much power as the first. The music approximates the white noise, so the power from the two sources arriving at a point between them adds, without any concern about whether the sources are in or out of phase. To what point on the line segment between the two parties should I go, if I wish to enjoy as much quiet as possible? Demonstrate that you have found an absolute minimum, not just a relative minimum.

Solution. Let P be the source power of the first party's stereo and let x be the distance between the person and the first party. Since the power of the second party's stereo is $64P$, the sound level is $L(x) = kPx^{-2} + 64kP(100-x)^{-2}$, $x \in (0,100)$. From $ds\frac{dL}{dx} = 2kP\left(\frac{64}{(100-x)^3} - \frac{1}{x^3}\right)$ it follows that $x = 20$ is the only critical number for the function L. Since for $x \in (0,100)$

$$L'(x) > 0 \Leftrightarrow \frac{64}{(100-x)^3} - \frac{1}{x^3} > 0 \Leftrightarrow 64x^3 > (100-x)^3 \Leftrightarrow 4x > 100-x \Leftrightarrow x > 20$$

the function L is strictly increasing on the interval $(20,100)$ and strictly decreasing on the interval $(0,20)$. Therefore, $L(20)$ is the absolute minimum.

3.4 Mean Value Theorem

Use the Mean Value Theorem to solve the following problems.

1. Verify that the function

$$g(x) = \frac{3x}{x+7}$$

satisfies the hypothesis of the Mean Value Theorem on the interval $[-1, 2]$. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem. Leave your final answer(s) exact.

Solution. Since $x+7 \neq 0$ for all $x \in [-1, 2]$ it follows that the function g, as a rational function, is continuous on the closed interval $[-1, 2]$ and differentiable on the open interval $(-1, 2)$. Therefore the function g satisfies he hypothesis of the Mean Value Theorem on the interval $[-1, 2]$. By the Mean Value Theorem there is $c \in (-1, 2)$ such that $g'(c) = \dfrac{g(2) - g(-1)}{2 - (-1)}$. Thus the question is to solve $\dfrac{21}{(c+7)^2} = \dfrac{7}{18}$ for c. Hence $c = -7 \pm 3\sqrt{6}$. Clearly $-7 - 3\sqrt{6} < -1$ and this value is rejected. From

$$-7 + 3\sqrt{6} > -1 \Leftrightarrow 3\sqrt{6} > 6 \text{ and } -7 + 3\sqrt{6} < 2 \Leftrightarrow 3\sqrt{6} < 9$$

it follows that $c = -7+3\sqrt{6} \in (-1, 2)$ and it is the only value that satisfies the conclusion of the Mean Value Theorem.

2. Use the Mean Value Theorem to show that $|\sin b - \sin a| \leq |b - a|$, for all real numbers a and b.

Solution. The inequality is obviously satisfied if $a = b$. Let $a, b \in \mathbb{R}$, $a < b$, and let $f(x) = \sin x$, $x \in [a, b]$. Clearly the function f is continuous on the closed interval $[a, b]$ and differentiable on (a, b). Thus, by the Mean value Theorem, there is $c \in (a, b)$ such that $\cos c = \dfrac{\sin b - \sin a}{b - a}$. Since $|\cos c| \leq 1$ for all real numbers c it follows that $|\sin b - \sin a| \leq |b - a|$.

3. Two horses start a race at the same time and finish in a tie. Prove that at some time during the race they have the same speed.

Solution. Let $f(t)$ be the distance that the first horse covers from the start in time t and let $g(t)$ be the distance that the second horse covers from the start in time t. Let T be time in which the two horses finish the race. It is given that $f(0) = g(0)$ and $f(T) = g(T)$. Let $F(t) = f(t) - g(t)$, $t \in [0, T]$. As the difference of two position functions, the function F is continuous on the closed interval $[0, T]$ and differentiable on the open interval $(0, T)$. By the Mean value Theorem there is $c \in (0, T)$ such that $F'(c) = \dfrac{F(T) - F(0)}{T - 0} = 0$. It follows that $f'(c) = g'(c)$ which is the same as to say that at the instant c the two horse have the same speed. (Note: It is also possible to use Rolle's theorem.)

4. Complete the following statement of the Mean Value Theorem precisely. Let f be a function that is continuous on the interval . . . . and differentiable on the interval . . . . Then there is a number c in (a, b) such that $f(b) - f(a) = \dots\dots\dots\dots$

Answer. $[a, b]$; (a, b); $f'(c)(b - a)$.

5. Suppose that $-1 \leq f'(x) \leq 3$ for all x. Find similar lower and upper bounds for the expression $f(5) - f(3)$.

Answer. Note that all conditions of the Mean Value Theorem are satisfied. To get the bounds use the fact that, for some $c \in (1, 3)$, $f(5) - f(3) = 2f'(c)$.

6. Suppose $g(x)$ is a function that is differentiable for all x. Let $h(x)$ be a new function defined by $h(x) = g(x) + g(2 - x)$. Prove that $h'(x)$ has a root in the interval $(0, 2)$.

Answer. Note that $h(2) - h(0) = 0$ and apply the Mean Value theorem for the function h on the closed interval $[0, 2]$.

3.5 Linear Approximation and Newton's Method

Solve the following problems.

1. The circumference of a sphere is measured to be 24 cm, with a possible error of 0.25 cm. Use the differential dV to estimate the maximum error in the calculated volume V.

Hint. Observe that $dr = \dfrac{1}{8\pi}$.

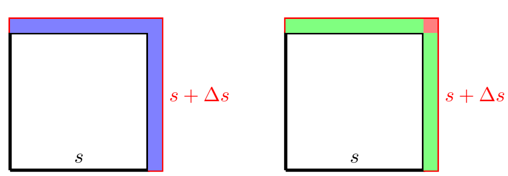
Answer. $dV = \dfrac{72}{\pi^2}$.

2. The area A of a square of side length s is $A = s^2$. Suppose s increases by an amount $\Delta s = ds$.

(a) Draw a square and then illustrate the quantity dA on your diagram.

(b) If dA is used to approximate ΔA, illustrate the error of approximation on the same diagram.

Answer. Observe that $\Delta A = A(s + \Delta s) - A(s) = 2s\Delta s + (\Delta s)^2$ and $dA = 2s\Delta s$.



3. Let $f(x) = \sqrt{(x + 4)^3}$.

(a) Find the linear approximation to the function $f(x) = \sqrt{(x + 4)^3}$ at $a = 0$.

(b) Use this approximation to estimate the number $\sqrt{(3.95)^3}$. Is your estimate an overestimate or an underestimate?

Hint. What is the concavity of the function $f(x)$?

Answer.

(a) $L(x) = 8 + 3x$.

(b) $\sqrt{(3.95)^3} \approx 7.85$. Underestimate.

Solution.

(a) Note that $f(0) = 8$. From $f'(x) = \frac{3}{2}\sqrt{x+4}$ it folows that $f'(0) = 3$. Thus the linearization of f at $a = 0$ is $L(x) = 8 + 3x$.

(b) For x "close" to 0 we have that $f(x) = \frac{3}{2}\sqrt{(x+4)^3} \approx L(x)$. Thus $\sqrt{(3.95)^3} = f(-0.05) \approx L(-0.05) = 8 - 0.15 = 7.85$. Since $f"(x) = \frac{3}{4\sqrt{x+4}} > 0$ we conclude that, in the neighbourhood of $x = 0$, the graph of the function f is above the tangent line at $x = 0$. Thus $L(-0.05)$ is an underestimate.

4.

(a) Use linear approximation to estimate $\sqrt[3]{65}$.

(b) Use concavity to state if your estimate in (a) is greater than or less than the exact value of $\sqrt[3]{65}$. Explain.

Answer.

(a) $\sqrt[3]{65} \approx L(65) = \frac{193}{48}$.

(b) Overestimate.

5. Use linear approximation to estimate the value of $\sqrt[3]{26^2}$. Express your answer a single fraction (for example, $\frac{16}{729}$).

Answer. $\sqrt[3]{26^2} \approx \frac{79}{9}$.

Solution. Let $f(x) = x^{\frac{2}{3}}$. Then $f(x) = \frac{2}{3}x^{-\frac{1}{3}}$, $f(27) = 9$, and $f'(27) = \frac{2}{9}$. Hence the linearization of the function f at $a = 27$ is $L(x) = 9 + \frac{2}{9}(x - 27)$. It follows that $\sqrt[3]{26^2} = f(26) \approx L(26) = 9 - \frac{2}{9} = \frac{79}{9}$. (Note: MAPLE gives $\frac{79}{9} \approx 8.777777778$ and $\sqrt[3]{26^2} \approx 8.776382955$.)

6. Use the linear approximation to approximate $(63)^{2/3}$. Then use differentials to estimate the error.

Answer. $(63)^{2/3} \approx \frac{95}{6}$.

Solution. Let $f(x) = x^{\frac{2}{3}}$. Then $f(x) = \frac{2}{3}x^{-\frac{1}{3}}$, $f(64) = 16$, and $f'(64) = \frac{1}{6}$. Hence the linearization of the function f at $a = 64$ is $L(x) = 16 + \frac{1}{6}(x - 64)$. It follows that $(63)^{2/3} = f(63) \approx L(63) =$

$16 - \frac{1}{6} = \frac{95}{6}$. The error is close to the absolute value of the differential $|dy| = |f'(64)\Delta x| = \frac{1}{6}$. (Note: MAPLE gives $\frac{95}{6} \approx 15.83333333$ and $\sqrt[3]{63^2} \approx 15.83289626$.)

7. Use linear approximation to estimate the value of $\sqrt{80}$.

Answer. $\sqrt{80} \approx \frac{161}{18}$.

Solution. Let $f(x) = \sqrt{x}$. Then $f(x) = \frac{1}{2\sqrt{x}}$, $f(81) = 9$, and $f'(81) = \frac{1}{18}$. Hence the linearization of the function f at $a = 81$ is $L(x) = 9 + \frac{1}{18}(x - 81)$. It follows that $\sqrt{80} = f(80) \approx L(80) = 9 - \frac{1}{18} = \frac{161}{18}$. (Note: MAPLE gives $\frac{161}{18} \approx 8.944444444$ and $\sqrt{80} \approx 8.944271910$.)

8. Assume that f is function such that $f(5) = 2$ and $f'(5) = 4$. Using a linear approximation to f near $x = 5$, find an approximation to $f(4.9)$.

Answer. The linearization of the function f at $a = 5$ is $L(x) = 2 + 4(x - 5)$. Thus $f(4.9) \approx L(4.9) = 2 - 0.4 = 1.6$.

9. Suppose that we don't have a formula for $g(x)$ but we know that $g(2) = -4$ and $g'(x) = \sqrt{x^2 + 5}$ for all x.

(a) Use linear approximation to estimate $g(2.05)$.

(b) Is your estimate in part (a) larger or smaller than the actual value? Explain.

Answer.

(a) -3.85.

(b) Larger.

Solution.

(a) The linearization of the function g at $a = 2$ is $L(x) = -4 + 3(x - 2)$. Thus $g(2.05) \approx L(2.05) = -3.85$.

(b) From $g"(2) = \frac{2}{3} > 0$ we conclude that the function g is concave downward at $a = 2$, i.e. the graph of the function lies below the tangent line. Thus, the estimate is larger than the actual value.

10. Let $f(x) = \sqrt{1 - x}$.

(a) Find a linear approximation for the function $f(x) = \sqrt{1 - x}$ valid for x close to 0.

(b) Use your answer to find an approximate value for $\sqrt{0.9}$.

(c) Find the tangent line to the graph of $f(x) = \sqrt{1 - x}$ at $x = 0$.

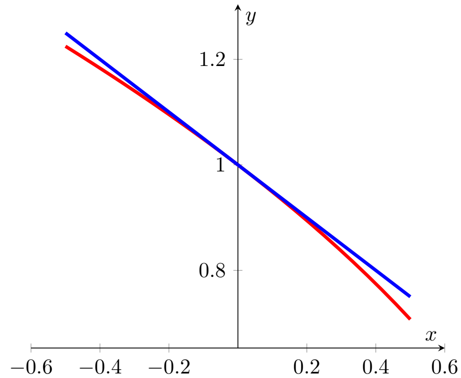
(d) Sketch a graph to illustrate the relationship between $f(x) = \sqrt{1 - x}$ and its linear approximation near $x = 0$.

Answer.

(a) $L(x) = 1 - \frac{x}{2}$.

(b) $\sqrt{0.9} \approx 1 - \dfrac{9}{20} = \dfrac{11}{20}$.

(c) $y = -\dfrac{x}{2} + 1$.

(d) 

11. Find the linear approximation of the function $f(x) = \sqrt{1+x}$ at $a = 3$, and use it to estimate the value of $\sqrt{5}$. Use a sketch to explain if this is an overestimate or underestimate of the actual value.

Answer. $\sqrt{5} \approx L(4) = 2.25$. Overestimate.

12. Let $f(x) = \sqrt{1+2x}$.

(a) Find the linear approximation of $f(x)$ at $x = 0$.

(b) Use your answer to estimate the value of $\sqrt{1.1}$.

(c) Is your estimate an over- or under-estimate?

Answer.

(a) $L(x) = 1 + x$.

(b) $\sqrt{1.1} = f(0.05) \approx L(0.05) = 1.05$.

(c) An over-estimate since f is concave-down. MAPLE gives $\sqrt{1.1} \approx 1.048808848$.

13. Let $f(x) = \sqrt[3]{x+8}$.

(a) Find a linear approximation to the function $f(x) = \sqrt[3]{x+8}$ at $a = 0$.

(b) Use this approximation to estimate the numbers $\sqrt[3]{7.95}$ and $\sqrt[3]{8.1}$.

Answer.

(a) $L(x) = 2 + \dfrac{x}{12}$.

(b) $\sqrt[3]{7.95} \approx L(-0.05) = 2 - \dfrac{1}{240} = \dfrac{479}{240}$ and $\sqrt[3]{8.1} \approx L(0.1) = 2 + \dfrac{1}{120} = \dfrac{243}{120}$. (Note: MAPLE gives $\dfrac{479}{240} \approx 1.995833333$ and $\sqrt[3]{7.95} \approx 1.995824623$. Also, $\dfrac{243}{120} \approx 2.025000000$ and $\sqrt[3]{8.1} \approx 2.008298850$.)

14. Let $f(x) = (1+x)^{100}$.

(a) Construct the linear approximation to $f(x) = (1+x)^{100}$.

(b) Use your approximation from (a) to estimate $(1.0003)^{100}$.

(c) Is your estimate from (b) higher or lower than the true value? Explain.

Answer.

(a) For $a \in \mathbb{R}$, $L(x) = (1+a)^{100} + 100(1+a)^{99}(x-a)$.

(b) $(1.0003)^{100} \approx L(0.0003) = 1.03$.

(c) Underestimate.

15. Let $f(x) = \sqrt[3]{27+3x}$.

(a) Find the equation of the tangent line to the graph of the function $f(x) = \sqrt[3]{27+3x}$ at $x = 0$.

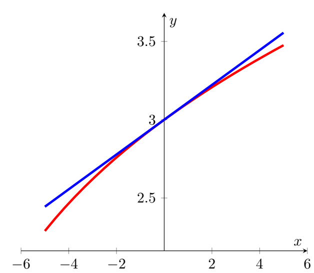
(b) Use your answer to estimate a value of $\sqrt[3]{30}$.

(c) Draw a sketch to show how the graph of f and its tangent line behave around the point where $x = 0$ and the value of x where the value in part (b) is obtained.

Answer.

(a) $y = \dfrac{x}{9} + 3$.

(b) $\sqrt[3]{30} \approx \dfrac{1}{9} + 3 = \dfrac{28}{9}$. (Note: MAPLE gives $\dfrac{28}{9} \approx 3.111111111$ and $\sqrt[3]{30} \approx 3.107232506$.)

(c) 

16. Use linear approximation to estimate the value of $\ln 0.9$.

Answer. $\ln 0.9 \approx -0.1$.

Solution. The linearization of the function $f(x) = \ln x$ at $a = 1$ is given by $L(x) = x - 1$. Thus $\ln 0.9 \approx L(0.9) = -0.1$. (Note: MAPLE gives $\ln 0.9 \approx -.1053605157$.)

17. Use a linear approximation to estimate the value of $e^{-0.015}$. Is your estimate too large or too small?

Hint. Take $f(x) = e^x$.

Answer. $e^{-0.015} \approx 0.985$. Underestimate.

18. Let $f(x) = \ln x$.

(a) Write the linear approximation for $f(x) = \ln x$ around 1.

(b) Compute the approximated value for $\exp(-0.1)$ using linear approximation.

Answer.

(a) $L(x) = x - 1$.

(b) Let $x = \exp(-0.1)$. Then $\ln x = -0.1 \approx L(x) = x-1$. Thus $x \approx 0.9$. (Note: MAPLE gives $\exp(-0.1) \approx 0.9048374180$.)

19. Using the function $f(x) = x^{1/3}$ and the technique of linear approximation, give an estimate for $1001^{1/3}$.

Solution. $L(x) = 10+\frac{1}{300}(x-1000)$ implies $1001^{1/3} \approx L(1001) = \frac{3001}{300}$. (Note: MAPLE gives $\frac{3001}{300} \approx 10.00333333$ and $\sqrt[3]{1001} \approx 10.00333222$.)

20. Let $f(x) = \sqrt{x} + \sqrt[5]{x}$.

(a) Use linear approximation to determine which of the following is nearest the value of $f(1.001)$:

2.0001	2.0002	2.0003	2.0005	2.0007
2.001	2.002	2.003	2.005	2.007

(b) At $x = 1$, is $f(x)$ concave up or concave down?

(c) Based on your answer above, is your estimate of $f(1.001)$ too high or too low?

Solution.

(a) The linearization of the function $f(x) = \sqrt{x} + \sqrt[5]{x}$ at $a = 1$ is given by $L(x) = 2+\frac{7}{10}(x-1)$. Thus $f(1.001) \approx L(1.001) = 2+0.7\cdot 0.001 = 2.0007$.

(b) Note that the domain of the function f is the interval $[0, \infty)$. From $f"(x) = -\frac{1}{4}x^{-\frac{3}{2}} - \frac{4}{25}x^{-\frac{9}{5}}$ it follows that f is concave downwards on the interval $(0, \infty)$.

(c) The graph of the function is below the tangent line at $a = 1$, so the estimate $f(1.001) \approx 2.0007$ is too high.

21.

(a) Find the linear approximation of $f(x) = \sin x$ about the point $x = \pi/6$.

(b) Explain why f satisfies the conditions of the Mean Value Theorem. Use the theorem to prove that $\sin x \leq \frac{1}{2} + (x - \frac{\pi}{6})$ on the interval $[\frac{\pi}{6}, x]$ where $x > \frac{\pi}{6}$

(c) Is the differential df larger or smaller than Δf from $x = \frac{\pi}{6}$ to $x = \frac{\pi}{2}$? Do not perform any calculations. Use only the results in part (a) and (b) to explain your answer.

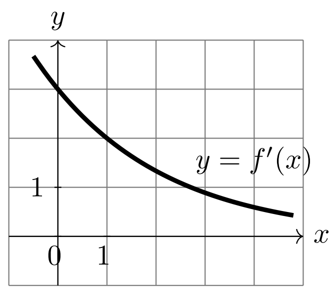
Answer.

(a) $L(x) = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$.

(b) By the Mean Value Theorem, for $x > \frac{\pi}{6}$ and some $c \in (\frac{\pi}{6}, x)$, $\frac{f(x) - f(\frac{\pi}{6})}{x - \frac{\pi}{6}} = \frac{\sin x - \frac{1}{2}}{x - \frac{\pi}{6}} = f'(c) = \cos c \leq 1$. Since $x - \frac{\pi}{6} > 0$, the inequality follows.

(c) From (a) and (b) it follows that, for $x > \frac{\pi}{6}$, $\sin x \leq \frac{1}{2} + (x - \frac{\pi}{6}) < \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) = L(x)$. Next, $\Delta f = f(x) - f(\frac{\pi}{6}) = \sin x - \frac{1}{2} < L(x) - \frac{1}{2} = \frac{\sqrt{3}}{2}(x - \frac{\pi}{6}) = f'(\frac{\pi}{6})\Delta x = df$.

22. Suppose that the only information we have about a function f is that $f(1) = 5$ and that the graph of its derivative is as shown in the Figure.



(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

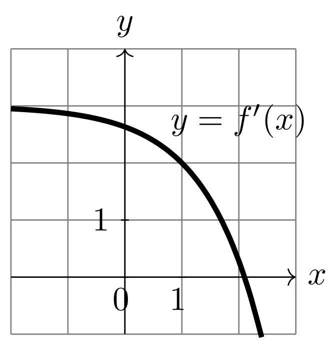
(b) Are your estimates in part (a) too large or too small?

Answer.

(a) $f(0.9) \approx L(0.9) = 5.2$, $f(1.1) \approx L(1.1) = 4.8$.

(b) Too large.

23. Suppose that the only information we have about a function f is that $f(1) = 3$ and that the graph of its derivative is as shown in the Figure.



(a) Use a linear approximation to estimate $f(0.9)$ and $f(1.1)$.

(b) Are your estimates in part (a) too large or too small?

Answer.

(a) $f(0.9) \approx L(0.9) = 2.8$, $f(1.1) \approx L(1.1) = 3.2$.

(b) Too large.

24.

(a) State Newton's iterative formula that produces a sequence of approximations $x_1, x_2, x_3, \ldots$ to a root of function $f(x)$.

(b) Find the positive root of the equation $\cos x = x^2$ using Newton's method, correct to 3 decimal points, with the first approximation

$x_1 = 1$.

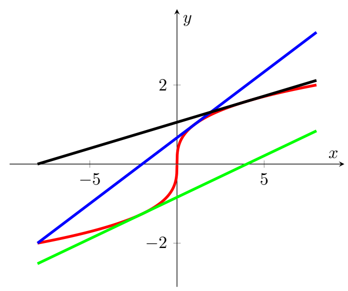
Solution. Let $f(x) = \cos x - x^2$. Then $f'(x) = -\sin x - 2x$. Thus $x_2 = 1 - \dfrac{\cos 1 - 1}{-\sin 1 - 2} \approx 0.8382184099$, $x_2 = 0.8382184099 - \dfrac{\cos 0.8382184099 - 0.8382184099^2}{-\sin 0.8382184099 - 2 \cdot 0.8382184099} \approx 0.8242418682$, and $x_3 = 0.8242418682 - \dfrac{\cos 0.8242418682 - 0.8242418682^2}{-\sin 0.8242418682 - 2 \cdot 0.8242418682} \approx 0.8241323190$. (Note: MAPLE gives $\cos 0.8241323190 - 0.8241323190^2 \approx -1.59 \cdot 10^{-8}$.)

25.

(a) State Newton's iterative formula that produces a sequence of approximations $x_0, x_1, x_2, \ldots$ to a solution of $f(x) = 0$, assuming that x_0 is given.

(b) Draw a labeled diagram showing an example of a function $f(x)$ for which Newton's iterative formula fails to find a solution of $f(x) = 0$. Mark on your diagram x_0, x_1, and x_2.

Solution. Take $f(x) = \sqrt[3]{x}$, $x_0 = 1$, $x_1 = -2$, $x_2 = 4$, and $x_3 = -8$.



26.

(a) Explain how you can use Newton's Method to approximate the value of $\sqrt{5}$.

(b) Explain which of the following choices is the best initial approximation when using Newton's Method as in (a):-1, 0, or 1?

(c) Find the fourth approximation x_4 to the value of $\sqrt{5}$ using Newton's Method with the initial approximation x_1 you chose in (b).

Answer.

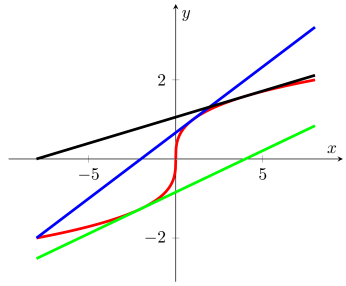
(a) We use Newton's Method to solve the equation $x^2 - 5 = 0$, $x > 0$. From $f(x) = x^2 - 5$ and $f'(x) = 2x$, Newton's Method gives $x_{n+1} = x_n - \dfrac{x_n^2 - 5}{2x_n} = \dfrac{1}{2}\left(x_n + \dfrac{5}{x_n}\right)$.

(b) A rough estimate of $\sqrt{5}$ gives a value that is a bit bigger than 2. Thus, take $x_1 = 1$.

(c) $x_2 = 3$, $x_3 = \dfrac{7}{3}$, $x_4 = \frac{47}{21} \approx 2.23809$. (Note: MAPLE gives $\sqrt{5} \approx 2.23606$.)

27. Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate x_1, x_2, x_3, x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.

Solution. Let $f(x) = x^{\frac{1}{3}}$. Then Newton's method gives $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)} = x_n - \dfrac{x_n^{\frac{1}{3}}}{\frac{1}{3}x_n^{-\frac{2}{3}}} = -2x_n$. So $|x_{n+1}| = 2|x_n|$. This implies that if $x_0 \neq 0$, $|x_n| = 2^n|x_0| \to \infty$ as $n \to \infty$; Newton's Method does not work in this case!



28.

(a) Find the Newton's method iteration formula to compute to estimate $\sqrt[3]{68}$.

(b) Provide an initial guess. Then explain, whether your initial guess will lead to an over- or underestimate after the first iteration.

Answer.

(a) $x_{n+1} = x_n - \dfrac{x_n^3 - 68}{3x_n^2}$.

(b) $x_0 = 4$. Underestimate.

29.

(a) Use linear approximation to estimate $\sqrt[3]{26}$.

(b) The value of $\sqrt[3]{26}$ is approximately $x_1 = 3$. Use Newton's method to find a better approximation, x_2, to $\sqrt[3]{26}$.

Answer.

(a) $\sqrt[3]{26} \approx L(26) \approx 2.962$.

(b) Take $f(x) = x^3 - 26$. $x_0 = 3$ implies $x_1 \approx 2.962$.

30. This question concerns finding zeros of the function

$$f(x) = \begin{cases} \sqrt{x} & \text{if } x \geq 0 \\ -\sqrt{-x} & \text{if } x < 0. \end{cases}$$

(a) If the initial approximation to the zero, for $f(x)$ given above, is x_1, what formula does Newton's method give for the next approximation?

(b) The root of the equation $f(x) = 0$ is $x = 0$. Explain why Newton's method fails to find the root no matter which initial approximation $x_1 \neq 0$ is used. Illustrate your explanation with a sketch.

Answer.

(a) Observe that $f(x) = \text{sign}(x)\cdot\sqrt{|x|}$ and $f'(x) = \dfrac{1}{2\sqrt{|x|}}$. $x_{n+1} = -x_n$.

(b) By induction, for all $n \in \mathbb{N}$, $|x_n| = |x_1|$.

31.

(a) Suppose k is a constant. Show that if we apply Newton's method to approximate the value of $\sqrt[5]{k}$, we get the following iterative formula:

$$x_{n+1} = \frac{x_n}{5}\left(4 + \frac{k}{x_n^5}\right).$$

(b) If $x_n = \sqrt[5]{k}$, what is the value of x_{n+1}?

(c) Take $x_1 = 2$ and use the formula in part (a) to find x_2, an estimate of the value of $\sqrt[5]{20}$ that is correct to one decimal place.

Answer.

(a) Take $f(x) = x^5 - k$. Then $f'(x) = 5x^4$ and $x_{n+1} = x_n - \dfrac{x_n^5 - k}{5x_n^4} = \dfrac{4x_n^5 + k}{5x_n^4} = \dfrac{x_n}{5}\left(4 + \dfrac{k}{x_n^5}\right)$.

(b) $x_{n+1} = \sqrt[5]{k}$.

(c) $x_2 = 1.85$. [MAPLE gives $\sqrt[5]{20} \approx 1.820564203$.]

32. Use Newton's method to find the second approximation x_2 of $\sqrt[5]{31}$ starting with the initial approximation $x_0 = 2$.

Solution. From $f(x) = x^5 - 31$ and $f'(x) = 5x^4$ it follows that $x_1 = \dfrac{159}{80}$ and $x_2 = \dfrac{159}{80} - \dfrac{\left(\frac{159}{80}\right)^5 - 31}{5 \cdot \left(\frac{159}{80}\right)^4} \approx 1.987340780$. (Note: MAPLE gives $\sqrt[5]{31} = 1.987340755$.)

33.

(a) Suppose x_0 is an initial estimate in Newton's method applied to the function $f(x)$. Derive Newton's formula for x_1, namely

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Support your derivation with a sketch showing a function $f(x)$, with x_0, x_1 and the line whose slope is $f'(x_0)$ clearly labeled.

(b) Using one iteration of Newton's method with $x_0 = \dfrac{\pi}{2}$ approximate the x-coordinate of the point where the function $g(x) = \sin x$ crosses the line $y = x$.

Solution. The question is approximate the solution of the equation $F(x) = \sin x - x = 0$ with $x_0 = \dfrac{\pi}{2}$. Thus $x_1 = \dfrac{\pi}{2} - \dfrac{\sin\frac{\pi}{2} - \frac{\pi}{2}}{\cos\frac{\pi}{2} - 1} = 1$. (Note: Clearly the solution of the given equation is $x = 0$. Newton's method with $x_0 = \frac{\pi}{2}$ gives $x_7 = 0.08518323251$.)

34. The equation

$$8x^3 - 12x^2 - 22x + 25 = 0$$

has a solution near $x_1 = 1$. Use Newton's Method to find a better approximation x_2 to this solution. Express your answer as a fraction.

Answer. $x_2 = 1 - \dfrac{-1}{-22} = \dfrac{21}{22}$. Note: MAPLE gives $\dfrac{21}{22} \approx 0.9545454545$

and approximates the solution of the equation as 0.9555894038)

35. The tangent line to the graph $y = f(x)$ at the point $A(2, -1)$ is given by $y = -1 + 4(x - 2)$. It is also known that $f''(2) = 3$.

(a) Assume that Newton's Method is used to solve the equation $f(x) = 0$ and $x_0 = 2$ is the initial guess. Find the next approximation, x_1, to the solution.

(b) Assume that Newton's Method is used to find a critical point for f and that $x_0 = 2$ is the initial guess. Find the next approximation, x_1, to the critical point.

Answer.

(a) $x_1 = 2 - \dfrac{-1}{4} = \dfrac{9}{4}$.

(b) The question is to approximate a solution of the equation $f'(x) = 0$ with the initial guess $x_0 = 2$, $f'(2) = 4$, and $f"(2) = 3$ given. Hence $x_1 = 2 - \frac{4}{3} = \frac{2}{3}$.

36.

(a) Apply Newton's method to the equation $\dfrac{1}{x} - a = 0$ to derive the following algorithm for finding reciprocals:

$$x_{n+1} = 2x_n - ax_n^2.$$

(b) Use the algorithm from part (a) to calculate $\dfrac{1}{1.128}$ correct to three decimal places, starting with the first approximation $x_1 = 1$.

Solution.

(a) From $f(x) = \dfrac{1}{x} - a$ and $f'(x) = -\dfrac{1}{x^2}$ it follows that $x_{n+1} = x_n - \dfrac{\frac{1}{x_n} - a}{-\frac{1}{x_n^2}} = 2x_n - ax_n^2$.

(b) Note that $\dfrac{1}{1.128}$ is the solution of the equation $\dfrac{1}{x} - 1.128 = 0$. Thus $x_2 = 2 - 1.128 = 0.872$, $x_3 = 2 \cdot 0.872 - 1.128 \cdot 0.872^2 = 0.886286848$, and $x_4 = 0.8865247589$. (Note: MAPLE gives $\dfrac{1}{1.128} \approx 0.8865248227$.)

37.

(a) Apply Newton's method to the equation $x^2 - a = 0$ to derive the following algorithm for the roots:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right).$$

(b) Approximate $\sqrt{2}$ by taking $x_1 = 2$ and calculating x_2.

Answer. $x_2 = \dfrac{3}{2}$.

38.

(a) State the formula for the linearization of f at a,

(b) Using linear approximation, approximate $\sqrt[4]{81.1}$.

(c) Approximate $\sqrt[4]{81.1}$ using one iteration of Newton's method.

Answer.

(a)

(b) Take $f(x) = \sqrt[4]{x}$ with $a = 81$. $\sqrt[4]{81.1} \approx L(81.1) \approx 3.000925$.

(c) Take $g(x) = x^4 - 81.1$. $x_2 = 2.9997685$.

39. You seek the approximate value of x which is near 1.8 for which $\sin x = \dfrac{x}{2}$. Your first guess is that $x \approx x_1 = \frac{\pi}{2}$. Use one iteration of Newton's method to find a better approximation to x. Simplify your answer as far as possible.

Solution. $x_2 = \dfrac{\pi}{2} - \dfrac{1 - \frac{\pi}{4}}{-\frac{1}{2}} = 2$. (Note: MAPLE estimates the positive solution of the equation $\sin x = \dfrac{x}{2}$ as 1.895494267. Newton's method with the initial guess $x_1 = \dfrac{\pi}{2}$ gives $x_3 \approx 1.900995594$.)

40.

(a) For the function $f(x) = x^3 - 3x + 5$ use the Intermediate Value Theorem, and any other tools you need to determine intervals of length 1 each of which contains a root of f.

(b) Pick one of the intervals found in part (a). Choose the left endpoint of this interval to be x_0. Now, use this as a starting value to find two new iterations to the root of f by using Newton's method. Determine from these whether Newton's method is working. Justify your answer carefully.

Solution.

(a) From $f'(x) = 3(x^2 - 1)$ it follows that the critical numbers are $x = \pm 1$. From $f(1) = 3$, $f(-1) = 7$, $\lim_{x\to-\infty} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) = \infty$ it follows that f has only one root and that root belongs to the interval $(-\infty, -1)$. From $f(-2) = 3 > 0$ and $f(-3) = -13 < 0$, by the Intermediate Value Theorem, we conclude that the root belongs to the interval $(-3, -2)$.

(b) Let $x_0 = -3$. Then $x_1 = -3 - \dfrac{-13}{504} = -\dfrac{1499}{504} \approx -2.974206349$ and $x_3 \approx -2.447947724$. It seems that Newton's method is working, the new iterations are inside the interval $(-3, -2)$ where we know that the root is. (Note: MAPLE estimates the solution of the equation $x^3 - 3x + 5 = 0$ as $x = -2.279018786$.)

41. Let $f(x) = x^3 + 3x + 1$.

(a) Show that f has at least one root in the interval $\left(-\dfrac{1}{2}, 0\right)$. Explain your reasoning.

(b) Use Newton's method to approximate the root that lies in the interval $\left(-\dfrac{1}{2}, 0\right)$. Stop when the next iteration agrees with the previous one to two decimal places.

Solution.

(a) The function f is continuous on the closed interval $\left[-\frac{1}{2}, 0\right]$ and $f\left(-\frac{1}{2}\right) = -\frac{5}{8} < 0$ and $f(0) = 1 > 0$. By the Intermediate Value Theorem, the function f has at least one root in the interval $\left(-\frac{1}{2}, 0\right)$.

(b) Take $x_1 = -\frac{1}{3}$. Then $x_2 = -\frac{1}{3} - \frac{-\frac{1}{27} - 3 \cdot \left(-\frac{1}{3}\right) + 1}{3\left(\frac{1}{9} + 1\right)} = -\frac{29}{90} \approx -.3222222222$ and $x_3 \approx -.3221853550$. (Note: MAPLE estimates the solution of the equation $x^3 + 3x + 1 = 0$ as $x = -.3221853546$.)

42. In this question we investigate the solution of the equation $\ln x = -x^2 + 3$ on the interval $[1, 3]$.

(a) Explain why you know the equation has at least one solution on $[1, 3]$.

(b) Show that the equation has exactly one solution on $[1, 3]$.

(c) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 1$ and finding x_2.

Answer.

(a) Take $f(x) = \ln x + x^2 - 3$, evaluate $f(1)$ and $f(3)$, and then use the Intermediate Value Theorem.

(b) Note that $f'(x) = \frac{1}{x} + 2x > 0$ for $x \in (1, 3)$.

(c) From $f(1) = -2$ and $f'(1) = 3$ it follows that $x_2 = \frac{5}{3} \approx 1.66$. [MAPLE gives 1.592142937 as the solution.]

43. In this question we investigate the positive solution of the equation $x^2 + x = 5 - \ln x$.

(a) Explain why you know the equation has at least one positive solution.

(b) Show that the equation has exactly one positive solution.

(c) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 1$ and finding x_2.

Answer.

(a) Use the Intermediate Value Theorem.

(b) Use Rolle's Theorem.

(c) $x_2 = 1.75$.

44. In this question we investigate the solution of the equation $2x = \cos x$.

(a) Explain why you know the equation has at least one solution.

(b) Show that the equation has exactly one solution.

(c) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 0$ and finding x_2.

Answer.

(a) Take $f(x) = 2x - \cos x$, evaluate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$, and then use the Intermediate Value Theorem.

(b) Note that $f'(x) = 2 + \sin x > 0$ for $x \in \mathbb{R}$.

(c) From $f(0) = -1$ and $f'(0) = 2$ it follows that $x_2 = \dfrac{1}{2}$. [MAPLE gives 0.4501836113 as the solution.]

45. In this question we investigate the solution of the equation $2x - 1 = \sin x$.

(a) Explain why you know the equation has at least one solution.

(b) Show that the equation has exactly one solution.

(c) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 0$ and finding x_2.

Answer.

(a) Take $f(x) = 2x - 1 - \sin x$, evaluate $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$, and then use the Intermediate Value Theorem.

(b) Note that $f'(x) = 2 - \cos x > 0$ for $x \in \mathbb{R}$.

(c) From $f(0) = -1$ and $f'(0) = 1$ it follows that $x_2 = 1$. [MAPLE gives 0.8878622116 as the solution.]

46. In this question we investigate the positive solution of the equation $e^x = 2\cos x$.

(a) Explain why you know the equation has at least one positive solution.

(b) Show that the equation has exactly one positive solution.

(c) Use Newton's Method to approximate the solution of the equation by starting with $x_1 = 0$ and finding x_2.

Answer.

(a) Take $f(x) = e^x - \cos 2x$ and then use the Intermediate Value Theorem.

(b) Observe that $f'(x) > 0$ for all $x > 0$.

(c) $x_2 = 1$.

47. Consider the equation

$$x^6 - x - 1 = 0.$$

(a) Apply the Intermediate Value Theorem to the function $f(x) = x^6 - x - 1$ to prove that the given equation has *a root greater than 1*. Make sure that you justify why the function f is continuous on its domain.

(b) Use the derivative of the function $f(x) = x^6 - x - 1$ to prove that the given equation has *only one root greater than 1*. Call that root a. Show all your work. Clearly explain your reasoning.

(c) State Newton's Method.

(d) Use Newton's Method with the initial approximation $x_1 = 1$ to find x_2 and x_3, the second and the third approximations to the root a of the equation $x^6 - x - 1 = 0$. You may use your calculator to find those values. Show all your work. Clearly explain your reasoning.

(e) WolframAlpha gives $a \approx 1.13472$. Use your calculator to evaluate the number $|x_3 - 1.13472|$. Are you satisfied with *your* approximation. Why yes or why not?

Answer.

(a) Take $f(x) = x^6 - x - 1$ and then use the Intermediate Value Theorem.

(b)

(c)

(d) $x_2 = 1.2$ and $x_3 \approx 1.143575843$.

(e) $|x_3 - 1.1347| \approx 0.008855843$.

48.

(a) State Rolle's theorem.

(b) Use Rolle's theorem to prove that $f(x)$ has a critical point in $[0, 1]$ where

$$f(x) = \sin\left(\frac{\pi x}{2}\right) - x^2.$$

(c) Set up the Newton's method iteration formula (x_{n+1} in terms of x_n) to approximate the critical point You do not need to simplify.

Answer. $x_{n+1} = x_n - \dfrac{\sin\left(\frac{\pi x_n}{2}\right) - x_n^2}{\frac{\pi}{2}\cos\left(\frac{\pi x_n}{2}\right) - 2x_n}$.

49.

(a) State the Mean Value Theorem.

(b) Using the Mean Value Theorem, prove that $f(x)$ has a critical point in $[0, 1]$ where

$$f(x) = \cos\left(\frac{\pi x}{2}\right) + x.$$

(c) Set up the Newton's method iteration formula to approximate the critical point You do not need to simplify.

Answer. $x_{n+1} = x_n - \dfrac{\cos\left(\frac{\pi x_n}{2}\right) + x_n}{\frac{-\pi}{2}\cos\left(\frac{\pi x_n}{2}\right) + 1}$.

50.

(a) State the Intermediate Value Theorem.

(b) State the Mean Value Theorem.

(c) Use the Intermediate Value Theorem and the Mean Value Theorem to show that the equation $1+2x+x^3+4x^5=0$ has exactly one real root.

Hint. Proof by contradiction.

51. A function $h(x)$ is said to have a fixed point at $x=c$ if $h(c)=c$. Suppose that the domain and range of a function $f(x)$ are both the interval $[0,1]$ and that f is continuous on this domain, with $f(0)\neq 0$ and $f(1)\neq 1$.

(a) Prove that f has at least one fixed point. That is, prove that $f(c)=c$ for some $c\in(0,1)$.

(b) Suppose that $f'(x)<1$ for all $x\in(0,1)$. Prove that f has exactly one fixed point in $[0,1]$.

(c) Use Newton's method to determine an iteration formula for the fixed point $x=c$.

Answer.

(a) Consider $g(x)=f(x)-x$.

(b)

(c) $x_{n+1}=x_n-\dfrac{f(x_n)-x_n}{f'(x_n)-1}$.

3.6 Antiderivatives and Differential Equations

Solve the following problems.

1. Suppose that h is a function such that $h'(x)=x^2+2e^x+3$ and $h(3)=0$. What is $h(1)$?

Answer. $-\dfrac{44}{3}+2e(1-6e^2)$.

2. Give an expression of the most general function g for which $g'(x)=\dfrac{5}{x}+\dfrac{1}{1+x^2}$.

Answer. $g(x)=\ln|x|+\arctan x+c$.

3. Find f if $f''(t)=2e^t+3\sin t$ and $f(0)=0,\ f'(0)=0$.

Answer. $f(t)=2e^t-3\sin t+t+2$.

4. Find f if $f'(x)=2\cos(8x)+e^{3x}-1$ and $f(0)=1$.

Answer. $f(x)=\dfrac{\sin 8x}{4}+\dfrac{1}{3}e^{3x}-x+\dfrac{2}{3}$.

5. If $f'(x)=kf(x)$ and $f(0)=A$, where k and A are constants, what is $f(x)$?

Answer. $f(x)=Ae^{kx}$.

6. Find f if $f'(x)=2\cos x+8x^3-e^x$ and $f(0)=7$.

Answer. $f(x)=2\sin x+2x^4-e^x+8$.

7. Find g if $g'(x)=\sin x+x^{-2}-e^x$ and $g(\pi)=1$.

Answer. $g(x)=-\cos x-x^{-1}-e^x+\pi^{-1}+e^{\pi}$.

8. Suppose f is a function such that $f'(x) = x^2 + 2x + 3$ and $f(5) = 1$. What is $f(1)$?

Answer. $f(x) = \frac{1}{3}x^3 + x^2 + 3x - \frac{242}{3}$ and $f(1) = -\frac{229}{3}$.

9. Suppose h is a function such that $h'(x) = x^2 + 2e^x + 3$ and $h(3) = 0$. What is $h(1)$?

Answer. $h(1) = 2e(1 - e^2) - \frac{44}{3}$.

10. Find the general form for the following antiderivative: $\int \frac{z}{z^2 + 9} dz$

Answer. $F(z) = \frac{1}{2} \ln(z^2 + 9)$.

11. Find a curve $y = f(x)$ with the following properties:

- $\frac{d^2y}{dx^2} = 6x$
- Its graph passes through the point $(0, 1)$ and has a horizontal tangent there.

Answer. It is given that $f(0) = 1$ and $f'(0) = 0$. Thus $f(x) = x^3 + 1$.

12. Find $\int \frac{dx}{x + x \ln x}$.

Answer. $\int \frac{dx}{x(1 + \ln x)} = \ln(1 + \ln x) + C$.

For each case compute the indefinite integral.

13. $\int (1 - x)^8 d$

Answer. $F(x) = -\frac{1}{9}(1-x)^9$

14. $\int \tan^2 x dx$

Answer. $\int \tan^2 x dx = \int (\sec^2 x - 1) dx = \tan x - x + C$

15. $\int \frac{dx}{2 + x^2}$

Answer. $F(x) = \frac{1}{\sqrt{2}} \arctan \frac{x}{\sqrt{2}} + C$

16. $\int e^{2x} \cosh x dx$

Answer. $F(x) = \frac{1}{6}e^{3x} + \frac{1}{2}e^x + C$

17. Find f if $f''(t) = 2e^t + 3 \sin t$ and $f(0) = 0$, $f'(0) = 0$.

Answer. $f(t) = 2e^t - 3 \sin t + t - 2$.

18. Given the acceleration of a particle is $a(t) = 2e^t + 3 \sin t$, and $s(0) = 0$, $s(\pi) = 0$, find the position function $s(t)$ of the particle.

Answer. $s(t) = 2e^t - 3 \sin t - (2e^\pi + 3)t - 2$.

19. A particle moves in a straight line and has an acceleration given by $a(t) = \sin t + 3 \cos t$. Its initial velocity $v(0) = 2$ cm/s and its initial displacement is $s(0) = 0$ cm. Find its position function $s(t)$.

Answer. $s(t) = -\sin t - 3 \cos t + 3t + 3$.

20. A stone is thrown downward off a building 30 metres high. Suppose that the stone has an initial velocity of 5 m/s, and that it falls with a constant acceleration due to gravity of 10 m/s^2.

(a) Find the velocity function $v(t)$ for the falling stone.

(b) Find the displacement function $s(t)$ for the falling stone.

(c) How many seconds does it take for the stone to hit the ground?

(d) How fast is the stone travelling when it hits the ground?

Answer.

(a) $v(t) = -10t + 5$.

(b) $s(t) = -5t^2 + 5t + 30$.

(c) $t = 6$ s.

(d) $|v(6)| = 5$ m/s.

21. A particle starts from rest (that is with initial velocity zero) at the point $x = 10$ and moves along the x-axis with acceleration function $a(t) = 12t$. Find the resulting position function $x(t)$.

Answer. $x(t) = 2t^3 + 10$.

Solution. It is given that $x(0) = 10$, $x'(0) = v(0) = 0$ and $x''(t) = 12t$. Hence $x(t) = 2t^3 + 10$.

22. At time $t = 0$ a car is moving at 6 m/s. and driver smoothly accelerates so that the acceleration after t seconds is $a(t) = 3t$ m/s^2.

(a) Write a formula for the speed $v(t)$ of the car after t seconds.

(b) How far did the car travel between during the time it took to accelerate from 6 m/s to 30 m/s?

Answer.

(a) $v(t) = \dfrac{3}{2}t^2 + 6$.

(b) 4 seconds. Solve $v(t) = 30$.

23. The skid marks made by a car indicate that its breaks were fully applied for a distance of 160 ft before it came to a stop. Suppose that the car in question had a constant deceleration of 20 ft/s^2 under the conditions of the skid. How fast was the car traveling when its breaks were applied?

Answer. $v(0) = 80$ ft/s.

24.

(a) You are standing at ground level, and you throw a ball upward into the air with an initial velocity of 64 feet per second. The acceleration due to gravity is 32 feet per second squared (towards the ground). How much time is the ball in the air for? Use antiderivatives.

(b) What is the velocity of the ball at the time that it hits the ground?

Answer.

(a) 4 seconds.

(b) -64 ft/sec^2.

Solution.

(a) Let $s(t)$ be the height of the ball after t seconds. It is given that $s(0) = 0$, $s'(0) = v(0) = 64$ ft/sec and $s''(0) = v'(0) = a(0) = -32$ ft/sec^2. Thus $s(t) = -16t^2 + 64t = 16t(4 - t)$. From $s(4) = 0$ it follows that the ball is in the air for 4 seconds.

(b) $v(4) = s'(4) = -64$ ft/sec^2.

25. In 1939, Joe Sprinz, a professional baseball player, attempted to catch a ball dropped from a blimp. This was done for the purpose of breaking the record for catching a ball dropped from the greatest height set the previous year by another player on another team.

(a) Ignoring air resistance, give an expression for the time it takes for a ball to drop H metres (from rest) under a constant gravitational acceleration g.

(b) Give an expression for the velocity of the ball after it has fallen H metres. What is this value if $H = 245$ m and $g = -10$ m/s^2? Would you try to catch this ball? *(Even with air resistance, the ball slammed Sprinz's glove into his face, fractured his upper jaw at 12 places, broke five teeth and knocked him unconscious. Worse still, he dropped the ball!)*

Answer.

(a) $\sqrt{\dfrac{2H}{g}}$s.

(b) -70 m/sec.

Solution.

(a) From the fact that the velocity of an falling object is approximated by $v(t) = -gt + v(0)$ and the fact that, in the given case, $v(t) = 0$, we conclude that the distance $y = y(t)$ between the ball and the surface of the Earth at time t is given by $\dfrac{dy}{dt} = -gt$. Hence, $y = -\dfrac{gt^2}{2} + H$, where H is the height of the blimp at the moment when the ball was dropped. At the moment when the ball hits surface we have that $0 = -\dfrac{gt^2}{2} + H$ which implies that it takes $t = \sqrt{\dfrac{2H}{g}}$ seconds for a ball to drop H metres.

(b) $v = -\sqrt{2Hg} = -10 \cdot 7 = -70$ m/sec.

Solve the initial value problem:

26. $\dfrac{dy}{dx} = 2\sin 3x + x^2 + e^{3x} + 1$, $y(0) = 0$.

Answer. $y = \dfrac{1}{3} \cdot (-2\cos 3x + x^3 + e^{3x} + 1) + x$.

27. $\dfrac{dy}{dx} = \sqrt{1 - y^2}$, $y(0) = 1$.

Answer. $y = \sin\left(x + \dfrac{\pi}{2}\right)$.

28. $\dfrac{dy}{dx} = y^2 + 1$, $y(\pi/4) = 0$.

Answer. $y = \tan\left(x - \dfrac{\pi}{4}\right)$.

29. $\dfrac{dy}{dx} = 1 + y$, $y(0) = 3$.

Answer. $y = 4e^x - 1$.

30. $\dfrac{dx}{dt} = \dfrac{36}{(4t-7)^4}$, $x(2) = 1$.

Answer. $x(t) = -3(4t-7)^{-3} + 4$.

31. $\dfrac{dy}{dx} = e^{-y}$, $y(0) = 2$.

Answer. $y = \ln(x + e^2)$.

32. $\dfrac{dy}{dt} = 2 - y$, $y(0) = 1$.

Answer. $y = 2 - e^{-t}$.

33.

(a) Find the solution to $\dfrac{dy}{dx} = 3\cos 2x + \exp(-4x)$ such that $y(0) = 1$.

(b) Use the change of variables $u = 2x + 1$ to find the general form for the antiderivative of $f(x) = \dfrac{1}{2x+1}$.

Answer.

(a) $y = \dfrac{3}{2}\sin 2x - \dfrac{1}{4}\exp(-4x) + \dfrac{5}{4}$.

(b) $F(x) = \dfrac{1}{2}\ln(2x+1) + C$.

34. A company is manufacturing and selling towels. The marginal cost for producing towels is 0.15 CAD per towel. A market survey has shown that for every 0.10 CAD increase in the price per towel, the company will sell 50 towels less per week. Currently the company sells 1000 towels per week against the price that maximizes their profit. What is the price of one towel? (Note: As usual, even though towels are only sold in integer units, assume we can use differentiable functions to describe the relevant quantities.)

Answer. The price that maximizes the profit is $p = -\dfrac{0.10 \cdot 1000}{50} + 4.15 = 2.15$ CAD.

Solution. Let x be the number of towels sold per week at the price $p = p(x)$. Let $C = C(x)$ be the cost of manufacturing x towels. It is given that $\dfrac{dC}{dx} = 0.15$ CAD/towel and $\dfrac{dp}{dx} = -\dfrac{0.10}{50}$ CAD/towel. Hence $C(x) = 0.15x + a$ and $p(x) = -\dfrac{0.10x}{50} + b$, for some constants a and b (in CAD). Then the profit is given by $P = P(x) = \text{Revenue} - \text{Cost} = x \cdot p(x) - C(x) = -\dfrac{0.10x^2}{50} + bx - 0.15x - a$. The quantity that maximizes revenue is $x = 1000$ towels and it must be a solution of the equation $\dfrac{dP}{dx} = -\dfrac{0.10x}{25} + b - 0.15 = 0$. Hence $-\dfrac{0.10 \cdot 1000}{25} + b - 0.15 = 0$ and $b = 4.15$ CAD. The price that maximizes the profit is $p = -\dfrac{0.10 \cdot 1000}{50} + 4.15 = 2.15$ CAD.

3.7 Exponential Growth and Decay

Recall that the solution of the initial-value problem $y' = ky$, $y(0) = A$, is given by $y = Ae^{kx}$.

Solve the following problems.

1.

(a) An amount of A_0 CAD is invested against yearly interest of $p\%$. Give the expression for $A(t)$, the value of the investment in CAD after t years if the interest is compounded continuously by writing down the differential equation that A satisfies and solving it.

(b) Jane invests 10,000 CAD against a yearly interest $p\%$, compounded continuously. After 4 years the value of her investment is 15,000 CAD. What is p?

Answer.

(a) $A = A_0 e^{pt}$.

(b) $p \approx 0.101$.

Solution.

(a) We notice that the value of the investment satisfies the differential equation,

$$\frac{dA}{dt} = pA,$$

with initial condition $A(0) = A_0$. Hence, we must have that

$$A(t) = A_0 e^{pt}.$$

(b) We solve $15,000 = 10,000 \cdot e^{4p}$ to get $p = \frac{1}{4}\log\left(\frac{3}{2}\right)$.

2. The rate at which a student learns new material is proportional to the difference between a maximum, M, and the amount she already knows at time t, $A(t)$. This is called a learning curve.

(a) Write a differential equation to model the learning curve described.

(b) Solve the differential equation you created in part (a).

(c) If took a student 100 hours to learn 50% of the material in Math 151 and she would like to know 75% in order to get a B, how much longer she should study? You may assume that the student began knowing none of the material and that the maximum she might achieve is 100%.

Answer.

(a) $\dfrac{dA}{dt} = k(M - A(t))$.

(b) $A(t) = M - (M - A_0)\, e^{-kt}$..

(c) 100 hours.

Solution.

(a) Let $A(t)$ be the amount of material a student knows. We are told that the rate $\frac{dA}{dt}$ is proportional to $M - A(t)$. That is, $\dfrac{dA}{dt} = k(M - A(t))$, for some constant k.

(b) We separate variables:

$$\int \frac{dA}{M - A} = k \int dt \implies -\ln|M - A| = kt + C,$$

for some constant C. Therefore, we get

$$A(t) = M - Ce^{-kt}.$$

Now let $A(0) = A_0$. Then we have $C = M - A_0$, and so

$$A(t) = M - (M - A_0)\, e^{-kt}.$$

(c) It is given that $M = 100$, $A(0) = 0$. Hence,

$$A(t) = 100(1 - e^{-kt}).$$

We first solve for k:

$$A(100) = 50 \implies 100(1 - e^{-k(100)}) = 50 \implies k = -\frac{\ln 2}{100}.$$

We now wish to solve

$$75 = 100(1 - e^{-\frac{t \ln 2}{100}})$$

for t. It follows that the student needs to study another 100 hours.

3. The concentration of alcohol (in %) in the blood, $C(t)$, obeys the decay differential equation:

$$\frac{dC}{dt} = -\frac{1}{k}C,$$

where $k = 2.5$ hours is called the elimination time. It is estimated that a male weighing 70 kg who drinks 3 pints of beer over a period of one hour has a concentration of 1% of alcohol in his blood. The allowed legal concentration for driving is a maximum of 0.5%.

(a) If a person has a blood alcohol concentration of 1%, how long should she/he wait before driving in order not to disobey the law. You may need the value $\ln 2 \approx 0.7$.

(b) What is the initial ($t = 0$) rate of change in the concentration?

Note: The permissible BAC limit in the Criminal Code of Canada is .08 (80 milligrams of alcohol in 100 millilitres of blood). Some advocate a lower criminal limit of .05 (50 milligrams of alcohol in 100 millilitres of blood).

Answer.

(a) $t = 2.5 \ln 2 \approx 1.75$ hours.

(b) $-\frac{2}{7}$ hours.

Solution.

(a) The model is $C(t) = C_0 e^{-\frac{t}{2.5}}$ where $C_0 = 1$ and the question is to solve $0.5 = e^{-\frac{t}{2.5}}$ for t. Hence $t = 2.5 \ln 2 \approx 1.75$ hours.

(b) $C'(0) = -\dfrac{1}{2.5} = -\dfrac{2}{7}$ hours.

4. The concentration of alcohol (in %) in the blood, obeys the decay equation $\dfrac{dC}{dt} = -0.4C$. If a person has a blood alcohol concentration of 2%, how long would it take for blood alcohol concentration to drop to 1%? Take that the elimination time is given in hours.

Answer. ≈ 1.73 hours.

Solution. The percentage of alcohol in the blood at time t can be modelled as $c(t) = c_0 e^{-0.4t}$. If $c(t) = 1$ and $c_0 = 2$, it follows that $t = \dfrac{5\ln 2}{2} \approx 1.73$ hours.

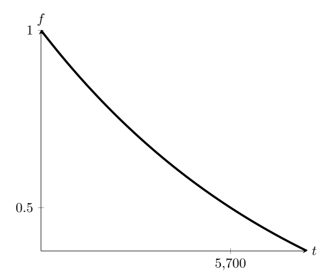
5. Carbon dating is used to estimate the age of an ancient human skull. Let $f(t)$ be the proportion of original ^{14}C atoms remaining in the scull after t years of radioactive decay. Since ^{14}C has a half life of 5700 years we have $f(0) = 1$ and $f(5700) = 0.5$.

(a) Sketch the graph of $f(t)$ versus t in the domain $0 \le t \le 20000$. Label at least two points of your plot and be sure to label the axes.

(b) Write an expression for $f(t)$ in terms of t and other numerical constants such as $\ln 2$, $\sin 5$, e^3, and $1/5700$. (Note: Not all of these constants need appear in your answer!)

(c) Suppose that only 15% of the original ^{14}C is found to remain in the skull. Derive from your previous answer, an expression for the estimated age of the skull.

Answer.

(a) 

(b) $f(t) = e^{-\frac{t\ln 2}{5700}}$.

(c) The question is to solve $0.15 = e^{-\frac{t\ln 2}{5700}}$ for t. Hence the age of the skull is $t = -\dfrac{5700\ln 0.15}{\ln 2} \approx 15600$ years.

6. The mass of a sample of a radioactive particle decays according to the rule $\dfrac{dm}{dt} = -5m$. Determine the half-life of this particle.

Answer. ≈ 0.138 units of time.

Solution. The proportion of radioactive sample remaining after time t can be modelled as $m(t) = m_0 e^{-5t}$. If $m(t) = 0.5$, $m_0 = 1$, it follows that the half-life of the particle is $t = \frac{\ln 2}{5} \approx 0.138$ units of time.

7. Plutonium-239 is part of the highly radioactive waste that nuclear power plans produce. The half-life of plutonium-239 is 24,110 years. Suppose that 10 kg of plutonium-239 has leaked into and contaminated a lake. Let $m(t)$ denote the mass of plutonium-239 that remains in the lake after t years.

(a) Find an expression for $m(t)$ based on the information given.

(b) How much mass remains in the lake after 1000 years.

(c) Suppose the lake is considered safe for use after only 1 kg of the plutonium-239 remains. How many years will this take?

Answer.

(a) $m(t) = 10e^{-\frac{t\ln 2}{24110}}$.

(b) $m(1000) = 10e^{-\frac{\ln 2}{24.11}} \approx 9.716$ kilograms.

(c) About 80091.68 years.

Solution.

(a) The model is $m(t) = 10e^{-kt}$ where t is in years, $m(t)$ is in kilograms, and k is a constant that should be determined from the fact that $m(24110) = 5$. Hence $k = -\dfrac{\ln 2}{24110}$ and $m(t) = 10e^{-\frac{t\ln 2}{24110}}$.

(b) $m(1000) = 10e^{-\frac{\ln 2}{24.11}} \approx 9.716$ kilograms.

(c) We solve $1 = 10e^{-\frac{t\ln 2}{24110}}$ to get $t = 24110\dfrac{\ln 10}{\ln 2} \approx 80091.68$ years.

8. On a certain day, a scientist had 1 kg of a radioactive substance X at 1:00 pm. After six hours, only 27 g of the substance remained. How much substance X was there at 3:00 pm that same day?

Answer. ≈ 0.1392476650 kg.

Solution. The model is $A = A(0)e^{-kt}$. It is given that $A(0) = 1$ kg and $A(6) = 0.027$ kg. Hence $A(t) = e^{\frac{t\ln 0.0027}{6}}$. It follows that at 3:00 there are $A(2) = e^{\frac{\ln 0.0027}{3}} \approx 0.1392476650$ kg of substance X.

9. In a certain culture of bacteria, the number of bacteria increased tenfold in 10 hours. Assuming natural growth, how long did it take for their number to double?

Answer. ≈ 3.01 hours.

Solution. The model is $P = P(t) = P_0e^{kt}$ where k is a constant, P_0 is the initial population and t is the time elapsed. It is given that $10P_0 = P_0e^{10k}$ which implies that $k = \dfrac{\ln 10}{10}$. The question is to solve $2 = e^{\frac{t\ln 10}{10}}$ for t. Hence $t = \dfrac{10\ln 2}{\ln 10} \approx 3.01$ hours.

10. A bacterial culture starts with 500 bacteria and after three hours there are 8000. Assume that the culture grows at a rate proportional to its size.

(a) Find an expression in t for the number of bacteria after t hours.

(b) Find the number of bacteria after six hours.

(c) Find an expression of the form $m\dfrac{\ln a}{\ln b}$ with m, a, and b positive integers for the number of hours it takes the number of bacteria to reach a million.

Answer.

(a) $P = 500e^{\frac{4t\ln 2}{3}}$.

(b) 128000 bacteria.

(c) ≈ 4.7414 hours.

Solution.

(a) The model is $P = 500e^{kt}$. From $8000 = 500e^{3k}$ it follows that $k = \dfrac{4\ln 2}{3}$. Thus the model is $P = 500e^{\frac{4t\ln 2}{3}}$.

(b) 128000 bacteria.

(c) Solve $10^6 = 500e^{\frac{4t\ln 2}{3}}$ for t. It follows that $t = \dfrac{3(4\ln 2 + 3\ln 5)}{4\ln 2} \approx$ 4.7414 hours.

11. A bacteria culture starts with 500 bacteria and grows at a rate proportional to its size. After three hours there are 8000 bacteria. Find the number of bacteria after four hours.

Answer. 20158.

12. The bacteria population $P(t)$ quadruples every 15 minutes. The initial bacteria population is $P(0) = 10$. You might need the following values: $\ln 6 \approx 1.6$, $\ln 8 \approx 2.08$, $(\ln 10)/(\ln 2) \approx 3.32$, $\ln 2 \approx 0.69$, $4^3 = 64$, $4^6 = 4096$, $4^9 = 262144$, $4^{12} = 16777216$.

(a) What is the population after three hours?

(b) How much time does it take for the population to grow to 1 billion?

Answer.

(a) $167,772,160$ bacteria.

(b) 3.32 hours.

Solution.

(a) The model is $P(t) = 10e^{kt}$ where t is time in hours. From $40 = 10e^{\frac{k}{4}}$ it follows that $k = 4\ln 4$. Hence $P(3) = 10e^{12\ln 4} = 167,772,160$ bacteria.

(b) 3.32 hours.

13. The population of a bacteria culture grows at a rate that is proportional to the size of the population.

(a) Let P denote the population of the culture at time t. Express dP/dt in terms of the proportional constant k and P.

(b) If the population is 240 at time $t = 1$ and is 360 at time $t = 2$, find a formula for the number of bacteria at time t. (t in hours.)

(c) How many bacteria were there at time $t = 0$. Your answer should be a positive integer.

(d) What is the value of dP/dt when $t = 0$.

Answer.

(a) $\dfrac{dP}{dt} = kP$.

(b) $P = 160e^{t\ln\frac{3}{2}}$.

(c) $P(0) = 160$ bacteria.

(d) $\dfrac{dP}{dt}(0) = 160\ln\dfrac{3}{2}$.

14. Assume that Math 151 in fall of 2000 had an enrolment of 500 students and in fall 2002 had an enrolment of 750 students. Assume also that if $P(t)$ is the enrolment at time t (let t be in years, with $t = 0$ corresponding to year 2000), then $P'(t) = kP(t)$ for some constant k. Calculate $P(500)$ (the enrolment in Math 151 in fall of 2500). Simplify your answer as much as possible. The answer will be quite large.

Answer. 5.27\cdot 10^{46}.

Solution. The model is $P(t) = 500e^{kt}$ where t is time in years. From $P(2) = 750$ it follows that $k = \dfrac{\ln 3 - \ln 2}{2}$. Thus $P(500) = 500e^{250(\ln 3 - \ln 2)} = 5.27 \cdot 10^{46}$.

15. A freshly brewed cup of coffee has temperature 95°C and is in a 20°C room. When its temperature is 70°C, it is cooling at the rate of 1°C per minute. When does this occur?

Hint. Use Newton's Law of Cooling.

Answer. $t \approx 20.2$ minutes.

16. A cup of coffee, cooling off in a room at temperature 20^0C, has cooling constant $k = 0.09\text{min}^{-1}$.

(a) How fast is the coffee cooling (in degrees per minute) when its temperature is $T = 80°$C?

(b) Use linear approximation to estimate the change in temperature over the next 6 seconds when $T = 80°$C.

(c) The coffee is served at a temperature of 90°. How long should you wait before drinking it if the optimal temperature is 65°C?

Answer.

(a) $\left.\dfrac{dT}{dt}\right|_{T=80^0} = -0.09 \cdot (80 - 20) = -5.4\ ^0\text{C/min}.$

(b) $T - 80 \approx -0.54^0$C.

(c) About 6.4 minutes.

Solution.

(a) $\left.\dfrac{dT}{dt}\right|_{T=80^0} = -0.09 \cdot (80 - 20) = -5.4\ ^0\text{C/min}.$

(b) Note that 6 seconds should be used as 0.1 minutes. From $T \approx 80 - 5.4\Delta T = 80 - 5.4 \cdot 0.1$ it follows that the change of temperature will be $T - 80 \approx -0.54^0$C.

(c) $t = -\dfrac{100}{9} \ln \dfrac{9}{16} \approx 6.4$ minutes. Find the function $T = T(t)$ that is the solution of the initial value problem $\dfrac{dT}{dt} = -0.09(T - 20)$, $T(0) = 90$, and then solve the equation $T(t) = 65$ for t.

17. A cold drink is taken from a refrigerator and placed outside where the temperature is 32°C. After 25 minutes outside its temperature is 14°C, and after 50 minutes outside its temperature is 20°C. assuming the temperature of drink obeys Newton's Law of Heating, what was the initial temperature of the drink?

Answer. $T_0 = 5^0$C.

Solution. The model is $\frac{dT}{dt} = k(T-32)$ where $T = T(t)$ is the temperature after t minutes and k is a constant. Hence $T = 32+(T_0-32)e^k t$ where T_0 is the initial temperature of the drink. From $14 = 32 + (T_0 - 32)e^{25k}$ and $20 = 32 + (T_0 - 32)e^{50k}$ it follows $\frac{3}{2} = e^{-25k}$ and $k = -\frac{1}{25}\ln\frac{3}{2}$. Therefore, the initial temperature was approximately $T_0 = 5^0$C.

18. On Hallowe'en night you go outside to sit on the porch to hand out candy. It is a cold night and the temperature is only 10°C so you have made a cup of hot chocolate to drink. If the hot chocolate is 90°C when you first go out, how long does it take until it is a drinkable 60°C given that $k = 0.03s^{-1}$?

Answer. $t \approx 4.5$ minutes.

19. In a murder investigation the temperature of the corpse was 32.5°C at 1:30PM and 30.3°C an hour later. Normal body temperature is 37°C and the temperature of the surroundings was 20°C. When did murder take place?

Answer. $t \approx 2.63$ hours.

3.8 Miscellaneous

Solve the following problems.

1. For what values of the constant c does $\ln x = cx^2$ have solutions. Assume that $c > 0$.

Answer. $c < \frac{1}{2e}$

Solution. Let $f(x) = \ln x - cx^2$. Note that the domain of f is the interval $(0, \infty)$ and that $\lim\limits_{x\to\pm\infty} f(x) = -\infty$. From $f'(x) = \frac{1}{x} - 2cx$ it follows that there is a local (and absolute) maximum at $x = \frac{1}{\sqrt{2c}}$. Since the function f is continuous on its domain, by the Intermediate Value Theorem, it will have a root if and only if $f\left(\frac{1}{\sqrt{2c}}\right) > 0$. Thus

$$\ln\frac{1}{\sqrt{2c}} - c\cdot\frac{1}{2c} > 0 \Leftrightarrow \ln\frac{1}{\sqrt{2c}} > \frac{1}{2} \Leftrightarrow c < \frac{1}{2e}.$$

2. Show that $y = 3x^3 + 2x + 12$ has a unique root.

Solution. Note that the function $y = 3x^3 + 2x + 12$, as a polynomial, is continuous on the set of real numbers. Also, $\lim\limits_{x\to-\infty}(3x^3 + 2x + 12) = -\infty$ and $\lim\limits_{x\to\infty}(3x^3 + 2x + 12) = \infty$. By the Intermediate Value Theorem, the function has at least one root. Next, note that $y' = 6x^2 + 2 > 0$ for all $x \in \mathbb{R}$. This means that the function is increasing on its domain and therefore has only one root. (If there is another root, by Rolle's theorem there will be a critical number.)

3. Show that the equation $x^3 + 9x + 5 = 0$ has exactly one real solution.

Hint. Take $y = x^3 + 9x + 5$.

Solution. Let $y = x^3 + 9x + 5$. This function is a polynomial, and so is continuous. Since

$$y(0) = 5 > 0, \quad y(-1) = -5,$$

by the Intermediate Value Theorem that y has at least one root in the interval $(-1,0)$. Furthermore, since

$$y' = 3x^2 + 9 = 3(x^2 + 3),$$

we see that y is increasing for all $x \in \mathbb{R}$. Therefore, y can have no additional roots. Hence, $x^3 + 9x + 5 = 0$ has exactly one real solution.

4. For which values of a and b is $(1,6)$ a point of inflection of the curve $y = x^3 + ax^2 + bx + 1$?

Hint. Solve $f(10) = 6$ and $f''(1) = 0$.

Answer. $a = -3$, $b = 1$.

5. Prove that $f(x) = \dfrac{1}{(x+1)^2} - 2x + \sin x$ has exactly one positive root.

Solution. Note that the function f is continuous on its domain $\mathbb{R}\backslash\{-1\}$. Since $\displaystyle\lim_{x\to\infty} \frac{1}{(x+1)^2} = 0$ and $|\sin x| \leq 1$, for all $x \in \mathbb{R}$, it follows that $\displaystyle\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(-2x) = -\infty$. Also, $f(0) = 1 > 0$. By the Intermediate Value Theorem, the function has at least one root in the interval $(0,\infty)$. Next, note that $f'(x) = -\dfrac{2}{(x+1)^3} - 2 + \cos x < 0$ for all $x \in (0,\infty)$. This means that the function is decreasing on $(0,\infty)$ and therefore has only one root.

6. A ball is thrown vertically upwards from a platform 12 feet above the ground so that after t seconds have elapsed the height s (in feet) of the ball above the ground is given by

$$s = 12 + 96t - t^2.$$

Compute the following quantities:

(a) The initial velocity.

(b) The time to highest point.

(c) The maximum height attained.

Answer.

(a) From $s'(t) = v(t) = 96 - 2t$ it follows that the initial velocity is $v(0) = 96$ ft/sec.

(b) The only critical number is $t = 48$. By the second derivative test there is a local (and absolute) maximum there.

(c) $s(48) = 2316$ feet.

7. The table below gives the values of certain functions at certain points. Calculate each of the following or write "insufficient information" if there is no enough information to compute the value.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	3	3	2	2
2	4	-1	4	0
3	6	1	0	4
4	-1	0	1	1
5	2	4	3	3

(a) What is $\lim_{h\to 0} \dfrac{f(3+h)-f(3)}{h}$?

(b) What is $\lim_{h\to 0} \dfrac{f(2+h)g(2+h)-f(2)g(2)}{h}$?

(c) Use differentials to find the approximate value of $f(0.98)$.

(d) What are the coordinates of any point on the graph of f at which there is a critical point?

Answer.

(a) $f'(3) = 1$.

(b) $(fg)'(2) = f'(2)g(2) + f(2)g'(2) = -4$.

(c) $f(0.98) \approx f(1) + f'(1)(0.98-1) = 3 + 3(-0.02) = 2.94$.

(d) There is a critical point at $(4, -1)$.

8.

(a) State domain and range of $f(x) = \arcsin x$.

(b) Derive the differentiation formula $\dfrac{d}{dx}[\arcsin x] = \dfrac{1}{\sqrt{1-x^2}}$.

(c) Let $g(x) = \arcsin(\sin x)$. Graph $g(x)$ and state its domain and range.

(d) For the function $g(x)$ as defined in part (c) find $g'(x)$ using any method you like. Simplify your answer completely.

(e) Explain carefully why the equation

$$4x - 2 + \cos\left(\frac{\pi x}{2}\right) = 0$$

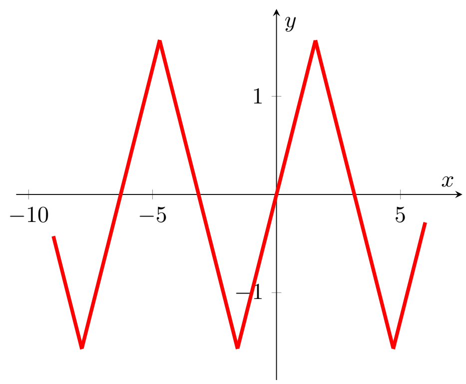
has exactly one real root.

Solution.

(a) The domain of $f(x) = \arcsin x$ is the interval $[-1, 1]$ and its range is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) For $x \in (-1, 1)$ let $y = \arcsin x$. Then $\sin y = x$ and $y' \cos y = 1$. Since $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ it follows that $\cos y > 0$. Thus $\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1-x^2}$.

(c) The domain of the function g is the set of all real numbers and its range is the set $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.



(d) $g'(x) = \begin{cases} 1 & \text{if } x \in (4n-1, 4n+1),\ n \in \mathbb{Z} \\ -1 & \text{if } x \in (4n+1, 4n+3),\ n \in \mathbb{Z} \end{cases}$. The derivative of g is not defined if $x = 2n+1$, $n \in \mathbb{Z}$.

(e) The function $F(x) = 4x - 2 + \cos\left(\frac{\pi x}{2}\right)$ is continuous on the set of real numbers. From $\lim\limits_{x \to \pm\infty} F(x) = \pm\infty$, by the Intermediate Value Theorem, the function F has at least one root. From $F'(x) = 1 - \dfrac{2}{\pi} \sin\left(\dfrac{\pi x}{2}\right) > 0$ we conclude that F is monotone.

9. Given that

$$\sinh x = \frac{e^x - e^{-x}}{2} \text{ and } \cosh x = \frac{e^x + e^{-x}}{2}$$

(a) Find $\lim\limits_{x \to \infty} \tanh x$.

(b) Find the equation of the tangent line to the curve $y = \cosh(x) + 3x + 4$ at the point $(0, 5)$.

Answer.

(a) $\lim\limits_{x \to \infty} \tanh x = 1$.

(b) $y_T = 3x + 5$.

Solution.

(a) We have

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}}.$$

Hence,

$$\lim_{x \to \infty} \tanh = \lim_{x \to \infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = 1.$$

(b) We differentiate:

$$\frac{dy}{dx}\Big|_{x=0} = \left[\frac{d}{dx} \cosh x\right]_{x=0} + 3 = 3.$$

Therefore, the equation of the tangent line to $y = \cosh(x) + 3x + 4$ at the point $(0, 5)$ is given by

$$y_T = 3x + 5.$$

10.

(a) Differentiate $f(x) = \dfrac{\ln x}{x}$, for $x > 0$.

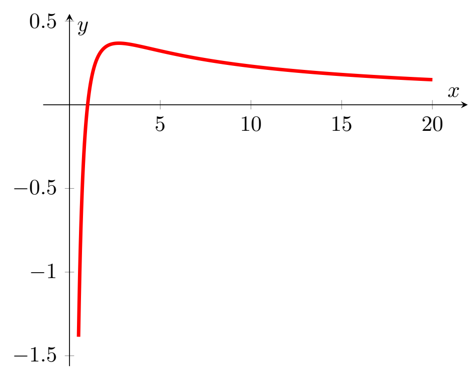
(b) Sketch the graph of $f(x) = \dfrac{\ln x}{x}$, showing all extrema.

(c) Use what you have done to decide which is larger, 99^{101} or 101^{99}.

Answer.

(a) $f'(x) = \dfrac{1 - \ln x}{x^2}$.

(b)



(c) Since $99 > e$ we have that $\dfrac{\ln 99}{99} < \dfrac{\ln 101}{101}$. This is the same as $101 \ln 99 < 99 \ln 101$. Hence $\ln 99^{101} < \ln 101^{99}$ and $99^{101} < 101^{99}$.

11. Answer the following questions. No justification necessary.

(a) What is the general antiderivative of $f(x) = 6x^2 + 2x + 5$?

(b) What is the derivative of $g(x) = \sinh(x)$ with respect to x?

(c) If $f'(x)$ changes from negative to positive at c the $f(x)$ has a (pick one)

i. local maximum at c.

ii. local minimum at c.

iii. global maximum at c.

iv. global minimum at c.

(d) If $x^5 + y^5 = 1$, what is y' in terms of x and y?

(e) If a point has polar coordinates $(r, \theta) = (3, 3\pi)$, what are its Cartesian coordinates?

Answer.

(a) $2x^3 + x^2 + 5x + c, c \in \mathbb{R}$.

(b) $\cosh(x)$.

(c) local minimum at c.

(d) $y' = -x^4 y^{-4}$.

(e) $(-3, 0)$.

12.

(a) State the definition of the derivative of a function g at a number x.

(b) State the Squeeze Theorem, clearly identifying any hypothesis and the conclusion.

(c) State Fermat's Theorem, clearly identifying any hypothesis and the conclusion.

(d) Give an example of a function with one critical point which is also an inflection point.

(e) Give an example of a function that satisfies $f(-1) = 0$, $f(10) = 0$, and $f'(x) > 0$ for all x in the domain of f'.

Answer.

(a) $g'(x) = \lim_{h\to 0} \dfrac{g(x+h) - g(x)}{h}$.

(b) If $f(x) \le g(x) \le h(x)$ when x is near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$ then $\lim_{x\to a} g(x) = L$.

(c) If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

(d) Take $f(x) = x^3$ and the point $(0, 0)$.

(e) For example, $f(x) = x - \dfrac{1}{x}$.

13.

(a) State the definition of a critical number of a function f.

(b) State the Mean Value Theorem, clearly identifying any hypothesis and the conclusion.

(c) State the Extreme Value Theorem, clearly identifying any hypothesis and the conclusion.

(d) State the definition of an inflection point of a function f.

(e) Give an example of a function with a local maximum at which the second derivative is 0.

(f) Give an example of a quadratic function of the form $f(x) = x^2 + bx + c$ whose tangent line is $y = 3x + 1$ at the point $(0, 1)$.

Answer.

(a) A critical number of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

(b) Let f be a function that satisfies the following hypotheses:

i. f is continuous on the closed interval $[a, b]$.
ii. f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$ or, equivalently, $f(b) - f(a) = f'(c)(b - a)$.

(c) If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.

(d) A point P on a curve $y = f(x)$ is called an inflection point if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

(e) Take $f(x) = -x^4$ and the point $(0, 0)$.

(f) $f(x) = x^2 + 3x + 1$.

14.

(a) Give an example of a function that is strictly decreasing on its domain.

(b) Give an example of a function f and an interval $[a, b]$ such that the conclusion of the Mean Value Theorem is not satisfied for f on this interval.

(c) Use a linear approximation to estimate $\sqrt{100.4}$. Is your estimate larger or smaller than the actual value?

Answer.

(a) Take $y = -x$ for all $x \in \mathbb{R}$.

(b) Take $y = \frac{1}{x^2}$ for $x \in [-1, 1]$.

(c) $\sqrt{102} \approx 10.1$. Overestimate.

Solution.

(a) Take $y = -x$ for all $x \in \mathbb{R}$. Then $y'(x) = -1 < 0$ for all $x \in \mathbb{R}$, and so the function is strictly decreasing on its entire domain.

(b) Take $y = \frac{1}{x^2}$ for $x \in [-1, 1]$. Then there is no number $c \in (-1, 1)$ such that $f'(c) = \frac{1-1}{-2} = 0$. The MVT fails because the function is not continuous on $[-1, 1]$.

(c) Let $f(x) = \sqrt{x}$. Then we know that $f(100) = 10$ and $f'(100) = 1/20$. Therefore, we compute:

$$\sqrt{102} = f(102) \approx f(100) + f'(100)(2) = 10 + \frac{1}{10} = 10.1.$$

Since $f(x)$ is concave down, this will be an overestimate. Indeed, using computer software, we find that $\sqrt{102} = 10.0995....$

Short definitions, theorems and examples. No part marks given.

15. Suppose for a function $y = f(x)$ we have that if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ for all x_1, x_2 in the domain of f. What does this tell you about f?

Solution. From the statement, it follows that if $f(x_1) = f(x_2)$, then $x_1 = x_2$ for all x_1, x_2 in the domain of f. Hence, we know that f is a one-to-one function, and that it is invertible.

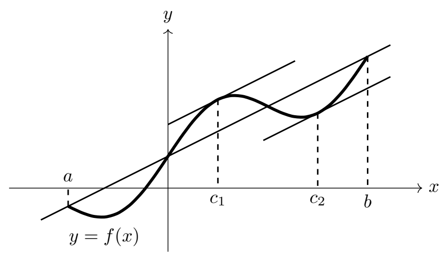
16. Suppose you need to show that $\lim_{x\to a} g(x) = L$ using the Squeeze Theorem with given functions $y = m(x)$ and $y = n(x)$. What condition(s) must the functions m and n satisfy?

Solution. We must have that $\lim_{x\to a} m(x) = \lim_{x\to a} n(x) = L$, and that for all x in some neighbourhood of a (but not necessarily at a), $g(x)$ is bounded above by one of $m(x), n(x)$, and bounded below by the other.

17. Identify the theorem that states the following: if any horizontal line $y = b$ is given between $y = f(a)$ and $y = f(c)$, then the graph of f cannot jump over the line; it must intersect $y = b$ somewhere provided that f is continuous.

Answer. The Intermediate Value Theorem.

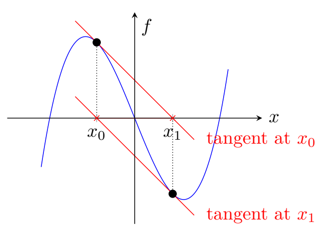
18. In the Figure below, which theorem is shown geometrically for a function f on $[a, b]$?



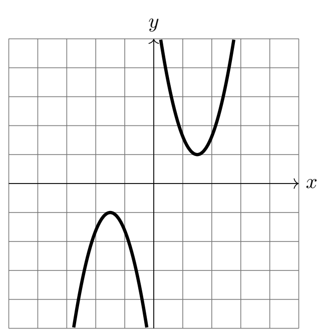
Answer. The Mean Value Theorem.

19. Given a function $y = f(x)$. Suppose that f has a root in $[a, b]$ and you want to approximate the root using Newton's method with initial value $x = x_1$. Show graphically, how Newton's method could fail.

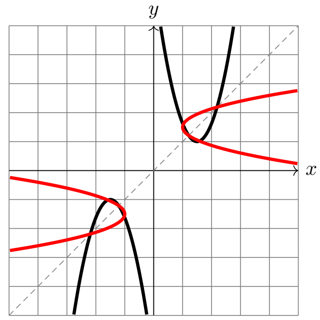
Answer.



20. The graph of the function f is given in the Figure below. Clearly graph a possible inverse function for f over its maximum domain directly on the given Cartesian coordinates.

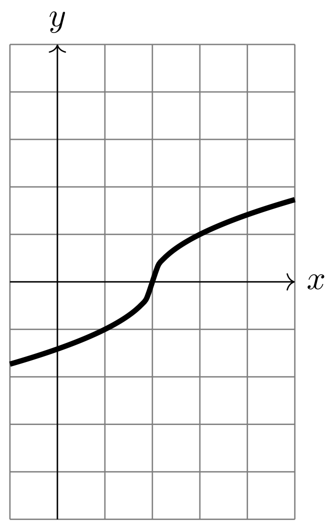


Answer.



21. Draw a graph of a function f that has an inflection point at $x = 2$ and $f'(2)$ does not exist.

Answer.



22. Draw a graph of a non-linear function f that satisfies the conclusion of the Mean value Theorem at $c = -2$ in $[-4, 2]$.

23. Let

$$f(x) = \begin{cases} x+4, & x < 1 \\ 5, & x = 1 \\ 2x^2+3, & x > 1. \end{cases}$$

What can you say about differentiability of f at $x = 1$?

Answer. Not differentiable.

Solution. Since

$$\lim_{h \to 0+} \frac{f(1+h) - f(1)}{h} = 6,$$

and

$$\lim_{h \to 0-} \frac{f(1+h) - f(1)}{h} = 1,$$

it follows that f is not differentiable at $x = 1$.

24. Show that $f(x) = \sinh x$ is an odd function.

Solution. For all $x \in \mathbb{R}$, we have that

$$f(-x) = \sinh(-x) = \frac{e^{-x} - e^{x}}{2} = -\frac{e^{x} - e^{-x}}{2} = -\sinh(x).$$

Hence, $\sinh(x)$ is an odd function.

Chapter 4

Parametric Equations and Polar Coordinates

4.1 Introduction

Use the following definitions and properties to solve the problems contained in this Chapter.

Parametric Curves - Vocabulary Let I be an interval and let f and g be continuous on I.

1. The set of points $C = \{(f(t), g(t)) : t \in I\}$ is called a *parametric curve.*
2. The variable t is called a *parameter.*
3. We say that the curve C is defined by *parametric equations* $x = f(t)$, $y = g(t)$.
4. We say that $x = f(t)$, $y = g(t)$ is a *parametrization* of C.
5. If $I = [a, b]$ then $(f(a), g(a))$ is called the *initial point* of C and $(f(b), g(b))$ is called the *terminal point* of C.

Derivative of Parametric Curves The derivative to the parametric curve $x = f(t)$, $y = g(t)$ is given by $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{g'(t)}{f'(t)}$.

Polar Coordinate System

1. Choose a point in the plane. Call it O, the *pole.*
2. Choose a ray starting at O. Call it the *polar axis.*
3. Take any point P, except O, in the plane. Measure the distance $d(O, P)$ and call this distance r.
4. Measure the angle between the polar axis and the ray starting at O and passing through P going from the polar axis in counterclockwise direction. Let θ be this measure in radians.
5. There is a bijection between the plane and the set

$$\mathbb{R}^+ \times [0, 2\pi) = \{(r, \theta) : r \in \mathbb{R}^+ \text{ and } \theta \in [0, 2\pi)\}.$$

This means that each point P, except O, in the plane is uniquely determined by a pair $(r, \theta) \in \mathbb{R}^+ \times [0, 2\pi)$.

6. r and θ are called *polar coordinates* of P.

Polar Curves The graph of a polar equation $r = f(\theta)$ consists of all points P whose polar coordinates satisfy the equation.

Derivative of Polar Curves Suppose that y is a differentiable function of x and that $r = f(\theta)$ is a differentiable function of θ. Then from the parametric equations $x = r\cos\theta$, $y = r\sin\theta$ it follows that

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}.$$

Parabola A parabola is a set of points in the plane that are equidistant from a fixed point F (called the focus) and a fixed line called the directrix). An equation of the parabola with focus $(0, p)$ and directrix $y = -p$ is $x^2 = 4py$.

Ellipse An ellipse is a set of point in plane the sum of whose distances from two fixed points F_1 and F_2 is constant. The fixed points are called foci. The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a \geq b > 0$, has foci $(\pm c, 0)$, where $c = \sqrt{a^2 - b^2}$, and vertices $(\pm a, 0)$.

Hyperbola A hyperbola is a set of points in plane the difference of whose distances from two fixed points F_1 and F_2 (the foci) is constant. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ has foci $(\pm c, 0)$, where $c = \sqrt{a^2 + b^2}$, vertices $(\pm a, 0)$, and asymptotes $y = \pm\frac{bx}{a}$.

Eccentricity Let F be a fixed point in the plane, let l be a fixed line in the plane, and let e be a fixed positive number (called the eccentricity). The set of all points P in the plane such that $\frac{|PF|}{|Pl|} = e$ is a conic section. The conic is an ellipse if $e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$.

4.2 Parametric Curves

Solve the following problems.

1. An object is moving counter-clockwise along a circle with the centre at the origin. At $t = 0$ the object is at point $A(0, 5)$ and at $t = 2\pi$ it is back to point A for the first time. Determine the parametric equations $x = f(t)$, $y = g(t)$ that describe the motion for the duration $0 \leq t \leq 2\pi$.

Answer. $x = -5\sin t,\ y = 5\cos t,\ t \in [0, 2\pi]$.

2. The trajectory of a particle in a plane as a function of the time t in seconds is given by the parametric equations

$$x = 3t^2 + 2t - 3, \quad y = 2t^3 + 2.$$

Prove that there is exactly one time when the particle crosses the line $y = x$.

Hint. From $3t^3 + 2t - 3 = 2t^3 + 2$ obtain $t^3 + 2t - 5 = 0$. What can you tell about the function $f(t) = t^3 + 2t - 5$?

3. Let $x = 2\sin t + 1$ and $y = 2t^3 - 3$ define a parametric curve. Find $\dfrac{d^2y}{dx^2}$ as a function of t, without simplifying your answer.

Answer. $\dfrac{d^2y}{dx^2} = \dfrac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \dfrac{3t(2\cos t + t\sin t)}{2\cos^3 t}$.

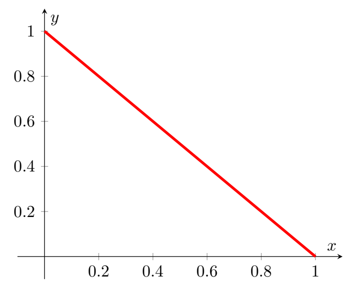
4. Sketch the curve

$$x = \sin^2 \pi t,\ y = \cos^2 \pi t,\ 0 \le t \le 2.$$

Clearly label the initial and terminal points and describe the motion of the point $(x(t), y(t))$ as t varies in the given interval.

Hint. Note that $x + y = 1$ and that $x, y \in [0, 1]$.

Answer.



5. Find an equation of the tangent line to the curve $x = t - t^{-1}$, $y = 1 + t^2$ at $t = 1$.

Answer. $y = x + 2$.

6. The parametric equations for a curve are given by

$$x = \theta - \sin\theta, \quad y = 1 - \cos\theta.$$

(a) Find $\dfrac{dx}{dy}$ as a function of θ.

(b) Find $\dfrac{d^2x}{dy^2}$ as a function of θ.

(c) Find the tangent line to the curve at the point of the curve obtained by setting $\theta = \dfrac{\pi}{3}$.

Answer.

(a) $\dfrac{dx}{dy} = \dfrac{1 - \cos\theta}{\sin\theta}$.

(b) $\dfrac{d^2x}{dy^2} = \dfrac{1 - \cos\theta}{\sin^3\theta}$.

(c) $y = \sqrt{3}x - \dfrac{\pi\sqrt{3}}{3} + 2$.

7. The graphs of the parametric equations $x = f(t)$, $y = g(t)$ are shown in Figure 4.1 Identify the corresponding parametric curves in Figure 4.2.

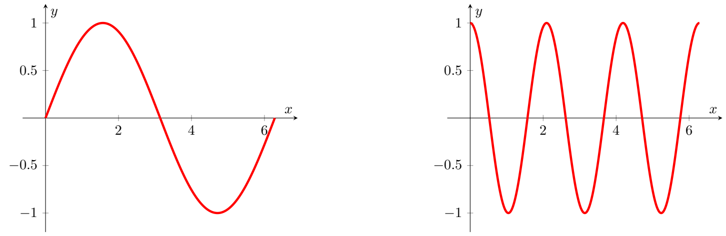


Figure 4.1 $x = f(t), y = g(t)$

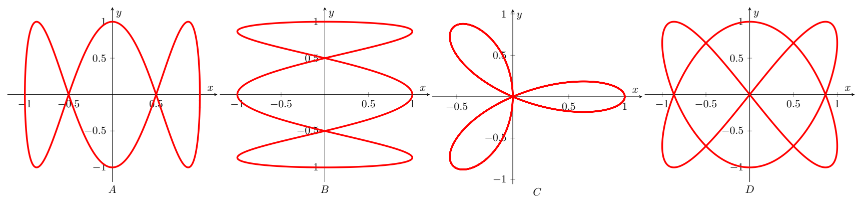


Figure 4.2 Only one is right

Answer. A. Follow the curve as t increases.

8. Consider the parametric curve $x(t) = -2 + 2\cos t$, $y(t) = 1 - 2\sin t$.

(a) State the Cartesian equation of the curve and the sketch the curve. Determine the direction of evolution of the curve for increasing t and indicate it on the graph.

(b) Find the points on the curve for which the tangent line has a slope of 1.

Answer.

(a) Express $\cos t$ and $\sin t$ in terms of x and y to get the circle $(x+2)^2 + (y-1)^2 = 4$. Check which points correspond to $t = 0$ and $t = \dfrac{\pi}{2}$ to get the orientation.

(b) Solve $\dfrac{dy}{dx} = \cot t = 1$ for $t \in (0, 2\pi)$.

9. This question concerns the curve $x = 2\cos 3t$, $y = 2\sin 2t$ for $0 \leq t \leq 2\pi$.

(a) Find dy/dx for this curve.

(b) Find the equation of the two tangent lines at the origin.

(c) Identify the graph on Figure 4.3 that corresponds to the parametric curve.

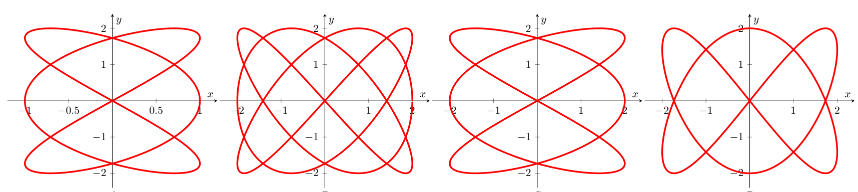


Figure 4.3 Which graph is right?

Answer.

(a) $\dfrac{dy}{dx} = -\dfrac{2\cos 2t}{3\sin 3t}$.

(b) Note that for both $t = \dfrac{\pi}{2}$ and $t = \dfrac{3\pi}{2}$ the curve passes through the origin. Thus, $y = \pm\dfrac{2x}{3}$.

(c) C.

10. For the following parametric curve $x = t - 1$, $y = t^2 - 2t + 2$:

(a) Find the derivative $\dfrac{dy}{dx}$ as a function of t.

(b) Eliminate the parametric dependance to determine an expression of the form $y = f(x)$ for the given curve.

(c) Find an expression for m and b (as functions of x_1) such that the equation of the line $y = mx + b$ is tangent to the curve $y = f(x)$ at $(x_1, f(x_1))$.

(d) Find an expression for all tangent lines to the curve $y = f(x)$ that pass through the point $(2, 0)$.

Answer.

(a) $\dfrac{dy}{dx} = 2t - 2$.

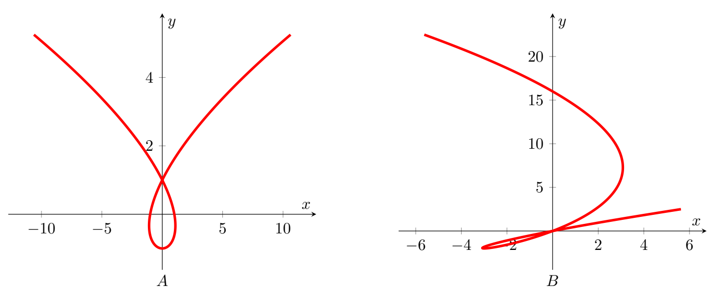
(b) $y = x^2 + 1$.

(c) $m = 2x_1$, $b = 1 - x_1^2$.

(d) Note that the point $(2, 0)$ does not belong to the curve. Since all tangent lines to the curve are given by $y = 2x_1 x + 1 - x_1^2$, $x_1 \in \mathbb{R}$, we need to solve $0 = 4x_1 + 1 - x_1^2$ for x_1. Hence $x_1 = 2 \pm \sqrt{5}$. The tangent lines are given by $y = 2(2 \pm \sqrt{5})x - 8 \mp 4\sqrt{5}$.

11. This question concerns the parametric curve $x = t^3 - 4t$, $y = 2t^2 - 4t$, $-\infty < t < \infty$.

(a) Which of the two graphs below corresponds to the given parametric curve?



(b) Find the y-coordinates of all points where the curve crosses the y-axis.

(c) This curve crosses itself at exactly one point. Find equations of both tangent lines at that point.

Answer.

(a) Right. Note that if $t = 0$ then $x = y = 0$.

(b) Solve $x = t^3 - 4t = t(t^2 - 4) = 0$. Next, $y(-2) = 16$, $y(0) = y(2) = 0$.

(c) From $\dfrac{dy}{dx} = \dfrac{4(t-1)}{3t^2-4}$ it follows that $\left.\dfrac{dy}{dx}\right|_{t=0} = 1$ and $\left.\dfrac{dy}{dx}\right|_{t=2} = \dfrac{1}{2}$. The tangent lines are $y = x$ and $y = \dfrac{x}{2}$.

12. Consider the parametric curve

$$x = \frac{3t}{1+t^3},\ y = \frac{3t^2}{1+t^3},\ t \in \mathbb{R}\backslash\{-1\}.$$

(a) Find an ordinary equation on x and y for this curve by eliminating the parameter t.

(b) Find the slope of the tangent line to the curve at the point where $t = 1$.

(c) Find $\dfrac{d^2y}{dx^2}$ at $t = 1$.

Answer.

(a) $x^3 + y^3 = 3xy$.

(b) -1.

(c) $-\frac{32}{3}$.

Solution.

(a) $x^3 + y^3 = 3xy$.

(b) From $\left.\dfrac{dx}{dt}\right|_{t=1} = -\dfrac{3}{4}$ and $\left.\dfrac{dy}{dt}\right|_{t=1} = \dfrac{3}{4}$ it follows that the slope equals to $\left.\dfrac{dy}{dx}\right|_{t=1} = -1$. Alternatively, differentiate the expression in (a) with respect to x.

(c) Use the fact that $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{1}{\frac{dx}{dt}} \cdot \dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)$ to obtain $\left.\dfrac{d^2y}{dx^2}\right|_{t=1} = -\dfrac{32}{3}$. Alternatively, differentiate the expression in (a) with respect to x twice.

13. Consider the parametric curve C described by

$$x = e^{-t},\ y = 2^{2t},\quad \text{where } -\infty < t < \infty.$$

(a) Calculate $\dfrac{dy}{dx}$ (as a function of t) directly from these equations.

(b) Calculate $\dfrac{d^2y}{dx^2}$ (as a function of t) directly from these equations.

(c) Does the graph of C have any points of inflection?

(d) Eliminate t from the equations of C thus obtaining a description of the graph C of the form $y = f(x)$.

(e) Sketch C. Label the axis. Plot at least three points of the curve C, labeling the points with the corresponding values of t. Draw an arrow on your curve indicating the direction of motion as t increases toward $+\infty$.

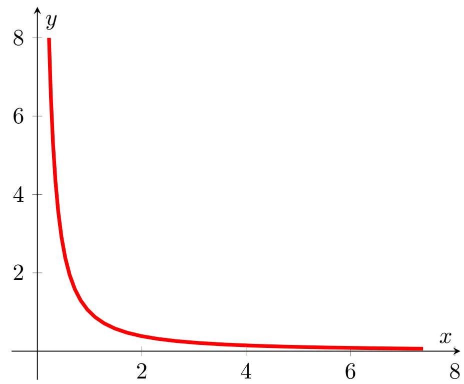
Answer.

(a) $\dfrac{dy}{dx} = -e^t \cdot 2^{2t+1} \ln 2$.

(b) $\dfrac{d^2y}{dx^2} = e^{2t} \cdot 2^{2t+1} \cdot (1 + 2\ln 2) \ln 2$.

(c) No, the second derivative is never zero.

(d) $y = 2^{-2\ln x}$.

(e) 

14. Consider the parametric curve described by

$$x(t) = e^t,\ y(t) = te^t, \quad \text{where } t \in \mathbb{R}.$$

(a) Find $\dfrac{dy}{dx}$ in terms of t.

(b) Find the tangent line to the curve at the point where $t = 0$.

(c) Find the second derivative $\dfrac{d^2y}{dx^2}$ in terms of t.

(d) For which values of t is the curve concave upward?

Answer.

(a) $\dfrac{dy}{dx} = 1 + t$.

(b) $y = x - 1$.

(c) $\dfrac{d^2y}{dx^2} = e^{-t}$.

(d) Observe that $\dfrac{d^2y}{dx^2} > 0$ for all $t \in \mathbb{R}$.

15. Consider the parametric curve C described by

$$x(t) = -e^{4t},\ y(t) = e^{1-t}, \quad \text{where } t \in \mathbb{R}.$$

(a) Find $\dfrac{dy}{dx}$ as a function of t directly from the above equations.

(b) Find $\dfrac{d^2y}{dx^2}$ as a function of t. Simplify your answer.

(c) Determine if C is concave up or concave down at $t = 0$.

Answer.

(a) $\dfrac{dy}{dx} = \dfrac{1}{4}e^{1-5t}$.

(b) $\dfrac{d^2y}{dx^2} = \dfrac{5}{16}e^{1-9t}$.

(c) Note that $\left.\dfrac{d^2y}{dx^2}\right|_{t=0} = \dfrac{5}{16}e > 0$.

16. Consider the parametric curve C described by

$$x(t) = 1 + 3t, \ y(t) = 2 - t^3, \quad \text{where } t \in \mathbb{R}.$$

(a) Find $\dfrac{dy}{dx}$ as a function of t directly from the above equations.

(b) Eliminate t from the equations of C thus obtaining a description of the graph of C having the form $y = f(x)$.

Answer.

(a) $\dfrac{dy}{dx} = -t^2$.

(b) $y = 2 - \dfrac{(x-1)^2}{3}$.

17. For the following parametric curve:

$$\begin{cases} x = t\sin t \\ y = t\cos t \end{cases}$$

(a) Find the derivative $\dfrac{dy}{dx}$ as a function of t.

(b) Find the equation of the tangent line at $t = \dfrac{\pi}{2}$.

(c) Determine if the curve is concave upwards or downwards at $t = \dfrac{\pi}{2}$.

Answer.

(a) $\dfrac{dy}{dx} = \dfrac{\cos t - t\sin t}{\sin t + t\cos t}$.

(b) $y = -\dfrac{\pi}{2}\left(x - \dfrac{\pi}{2}\right)$.

(c) $\left.\dfrac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = -2 - \dfrac{\pi^2}{4} < 0$.

18. A small ball is fastened to a long rubber band and twirled around in such a way that the ball moves in a circular path given by the equation $\overrightarrow{r} = b\langle\cos\omega t, \sin\omega t\rangle$, where b and ω are constants.

(a) Find the velocity and speed.

(b) Find the acceleration.

Answer.

(a) $\overrightarrow{v} = b\omega\langle-\sin\omega t, \cos\omega t\rangle$; speed $= |\overrightarrow{v}| = |b\omega|$.

(b) $\overrightarrow{a} = -b\omega^2\langle\cos\omega t, \sin\omega t\rangle$.

19. A curve is defined by the parametric equations

$$x = 3(t^2 - 3),\ \ y = t^3 - 3t.$$

(a) Calculate $\dfrac{dy}{dx}$ in terms of t.

(b) Find the equation of the tangent line to the curve at the point corresponding to $t = 2$.

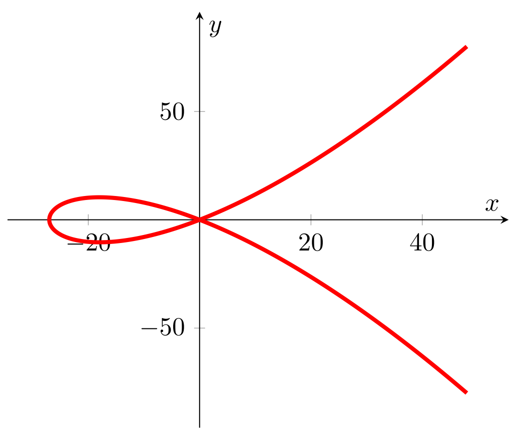
Answer.

(a) $\dfrac{dy}{dx} = \dfrac{t^2 - 1}{2t}$.

(b) $y - 2 = \dfrac{3}{4}(x - 3)$.

20. Consider the parametric curve

$$x = 3(t^2 - 9),\ \ y = t(t^2 - 9),\ \ -\infty < t < \infty.$$



(a) Find the x-coordinate of all points where the curve crosses the x-axis.

(b) Find the coordinate of the point on the curve corresponding to $t = -1$.

(c) Find the tangent line to the curve at the point where $t = -1$.

(d) Find the second derivative at the point where $t = -1$.

Answer.

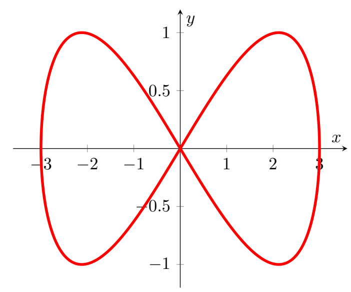
(a) -27, 0.

(b) $(-24, 8)$.

(c) $y = x + 32$.

(d) $\dfrac{d^2y}{dx^2} = \dfrac{t^2 + 3}{12t^3}$.

21. Consider the parametric curve:



$$x = 3\cos(2t),\ \ y = \sin(4t),\ \ 0 \le t \le \pi.$$

(a) Find the coordinates of the point on the curve corresponding to $t = 0$, and draw this point on the curve.

(b) Find the coordinates of the point on the curve corresponding to $t = \dfrac{\pi}{4}$, and draw this point on the curve.

(c) Draw arrows indicating the direction the curve is sketched as the t values increase from 0 to π.

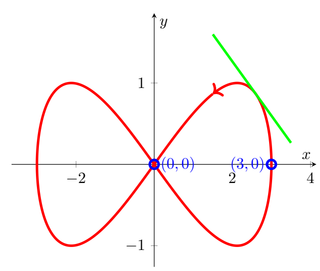
(d) Find the slope of the tangent line to the curve at the point $t = \dfrac{\pi}{12}$.

(e) State the intervals of t for which the curve is concave up, and the intervals of t for which the curve is concave down.

Answer.

(a) $(3, 0)$.

(b) $(0, 0)$.

(c) 

(d) $4x + 6y = 9\sqrt{3}$.

(e) Conclude from the graph and from what you have observed in (a) — (c) that the curve is concave up if $t \in \left(\dfrac{\pi}{4}, \dfrac{\pi}{2}\right) \cup \left(\dfrac{3\pi}{4}, \pi\right)$. Alternatively you may discuss the sign of the second derivative.

22. Consider the perimetrically defined curve $x(t) = t - \sin 2t$, $y = 4 - 3\cos t$, $0 \le t \le 10$. Find the values for the parameter t where the tangent line of the curve is vertical.

Answer. $\theta \in \{\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{11\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}\}$.

Solution. From $\dfrac{dy}{dx} = \dfrac{3\sin t}{1 - 2\cos 2t}$ we conclude that $\dfrac{dy}{dx}$ is not defined if $1 - 2\cos 2t = 0$, $0 \le t \le 10$. Thus, $\theta \in \{\frac{\pi}{6}, \frac{5\pi}{6}, \frac{7\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}\}$.

23. Consider the parametric curve C described by

$$x = e^{-t},\ y = e^{2t},\ \text{where } -\infty < t < \infty.$$

(a) Calculate $\frac{dy}{dx}$ (as a function of t) directly from the above equations.

(b) Calculate $\frac{d^2y}{dx^2}$ (as a function of t).

(c) Find all inflection points of C or otherwise show that C has no inflection points.

(d) Eliminate t from the equations C to obtain a function of the form $y = f(x)$ that describes C.

Answer.

(a) $\frac{dy}{dx} = -2e^{3t}$.

(b) $\frac{d^2y}{dx^2} = 6e^{4t}$.

(c) $\frac{d^2y}{dx^2} > 0$.

(d) $y = x^{-2}$.

24. Given the parametric curve

$$x = \cos^3 t,\ y = \sin^3 t.$$

(a) Without eliminating the parameter t, show that $\frac{dy}{dx} = -\tan t$.

(b) Determine the concavity of this curve when $t = 1$.

Answer.

(a) $\frac{dy}{dx} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\tan t$.

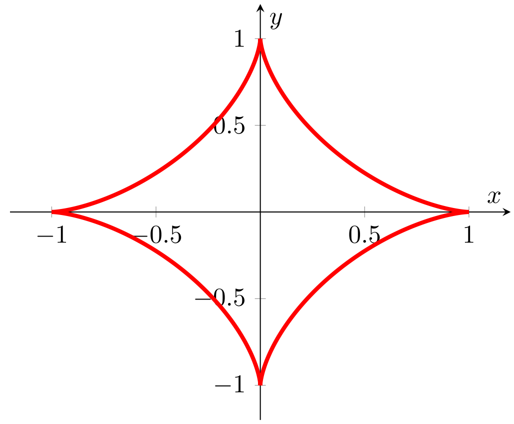
(b) concave up.

Solution.

(a) $\frac{dy}{dx} = \frac{3\sin^2 t \cos t}{-3\cos^2 t \sin t} = -\tan t$.

(b) From $\frac{d^2y}{dx^2} = -\frac{\sec^2 t}{-3\cos^2 t \sin t} = \frac{1}{3\cos^4 t \sin t}$ we get $\left.\frac{d^2y}{dx^2}\right|_{t=1} = \frac{1}{3\cos^4 1 \sin 1} > 0$.

25. Consider the parametric curve:



$$x = \cos^3 t,\ \ y = \sin^3 t,\ \ 0 \leq t \leq 2\pi.$$

(a) Find the x and y coordinates of the point on the curve corresponding to $t = 0$, and draw and clearly label this point on the graph.

(b) Find the x and y coordinates of the point on the curve corresponding to $t = 3\pi/4$, and draw and clearly label this point on the graph.

(c) On the graph draw arrows indicating the direction the curve is sketched as the t value increases from 0 to 2π.

(d) Find the tangent line to the curve at the point where $t = 3\pi/4$.

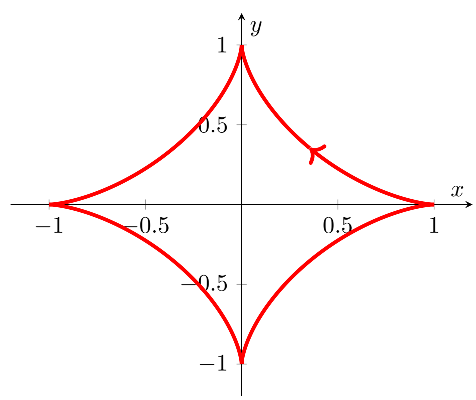
(e) State the intervals of t for which the curve is concave up, and the intervals of t for which the curve is concave down.

Answer.

(a) $(1, 0)$.

(b) $\left(-\dfrac{\sqrt{2}}{4}, \dfrac{\sqrt{2}}{4}\right)$.

(c)



(d) Observe that $\dfrac{dy}{dx} = -\tan t$, $t \notin \{0, \pi/2, \pi, 3\pi/2\}$. $2x - 2y + \sqrt{2} = 0$.

(e) Concave down.

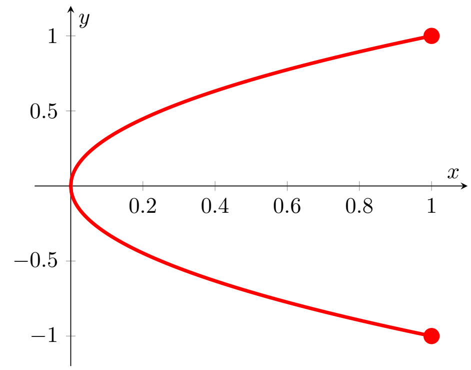
26. Consider the parametric curve $x = \cos^2 t$, $y = \cos t$, for $0 \le t \le \pi$.

(a) Sketch the curve.

(b) Write the equation of the tangent line where the slope is $\frac{1}{\sqrt{3}}$.

Answer.

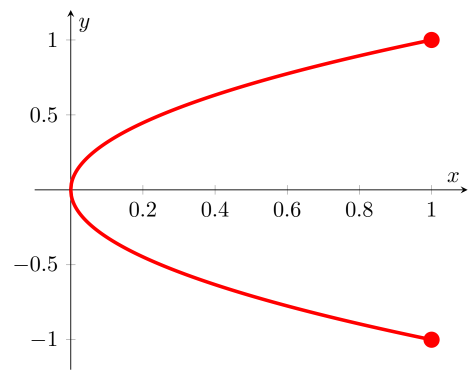
(a)



(b) $4x - 4\sqrt{3}y + 3 = 0$.

Solution.

(a) Observe that by eliminating the parameter t, the equation becomes $y^2 = x$, $(x, y) \in [0, 1] \times [-1, 1]$.



(b) Observe that solving $\frac{dy}{dx} = \frac{1}{2\cos t} = \frac{1}{\sqrt{3}}$ gives $t = \frac{\pi}{6}$. $4x - 4\sqrt{3}y + 3 = 0$.

27. Given the parametric curve

$$x = e^t, \ y = e^{-t}.$$

(a) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

(b) Find the equation of the line tangent to the curve that is parallel to the line $y + x = 1$.

Answer.

(a) $\frac{dy}{dx} = -e^{-2t}$, $\frac{d^2y}{dx^2} = e^{-3t}$.

(b) From $\dfrac{dy}{dx} = -e^{-2t} = -1$ we get that $t = 0$. Thus the tangent line is $y = -x$.

28. Given the parametric curve

$$x = t(t^2 - 3), \;\; y = 3(t^2 - 3).$$

(a) Find the y-intercepts of the curve.

(b) Find the points on the curve where the tangent line is horizontal or vertical.

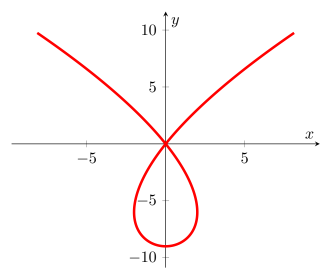
(c) Sketch the curve.

Answer.

(a) $(0, 0)$, $(0, -9)$.

(b) From $\dfrac{dy}{dx} = \dfrac{2t}{t^2 - 1}$ we get that the tangent line is horizontal at the point $(0, -9)$ and vertical at the points $(-2, -6)$ and $(2, -6)$.

(c)



29. A parametric curve C is described by $x(t) = -e^{3t+2}$ and $y(t) = 2e^{1-t}$. where t is a real number.

(a) Find $\dfrac{dy}{dx}$ as a function of t directly from the equation above. Simplify your answer.

(b) Find $\dfrac{d^2y}{dx^2}$ as a function of t. Simplify your answer.

(c) Determine if C is concave up or concave down at $t = 0$.

Answer.

(a) $\dfrac{dy}{dx} = \dfrac{2}{3}e^{-4t-1}$.

(b) $\dfrac{d^2y}{dx^2} = \dfrac{8}{9}e^{-7t-3}$.

(c) concave up.

4.3 Polar Coordinates

Use polar coordinates to solve the following problems.

1. Express the polar equation $r = \cos 2\theta$ in rectangular coordinates.

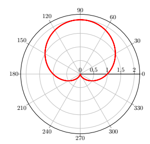
Hint. Multiply by r^2 and use the fact that $\cos 2\theta = cos^2\theta - \sin^2\theta$.

Answer. $(x^2 + y^2)^3 = (y^2 - x^2)$.

Sketch polar graphs of:

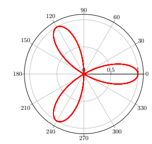
2. $r = 1 + \sin\theta$.

Answer.



3. $r = \cos 3\theta$.

Answer.



For the each of the following circles find a polar equation, i.e. an equation in r and θ:

4. $x^2 + y^2 = 4$

Answer. $r = 2$.

5. $(x-1)^2 + y^2 = 1$

Answer. $r = 2\cos\theta$.

6. $x^2 + (y-0.5)^2 = 0.25$

Answer. $r = \sin\theta$.

7. Find the maximum height above the x-axis of the cardioid $r = 1 + \cos\theta$.

Answer. $y = \dfrac{3\sqrt{3}}{4}$.

Solution. On the given cardioid, $x = (1+\cos\theta)\cos\theta$ and $y = (1+\cos\theta)\sin\theta$. The question is to find the maximum value of y. Note that $y > 0$ is equivalent to $\sin\theta > 0$. From $\dfrac{dy}{d\theta} = 2\cos^2\theta + \cos\theta - 1$ we get that the critical numbers of the function $y = y(\theta)$ are the values of θ for which $\cos\theta = \dfrac{-1 \pm 3}{4}$. It follows that the critical numbers are the values of θ for which $\cos\theta = -1$ or $\cos\theta = \dfrac{1}{2}$. Since $y_{\text{max}} > 0$ it follows that $\sin\theta = \sqrt{1 - \left(\dfrac{1}{2}\right)^2} = \dfrac{\sqrt{3}}{2}$ and the maximum height equals $y = \dfrac{3\sqrt{3}}{4}$.

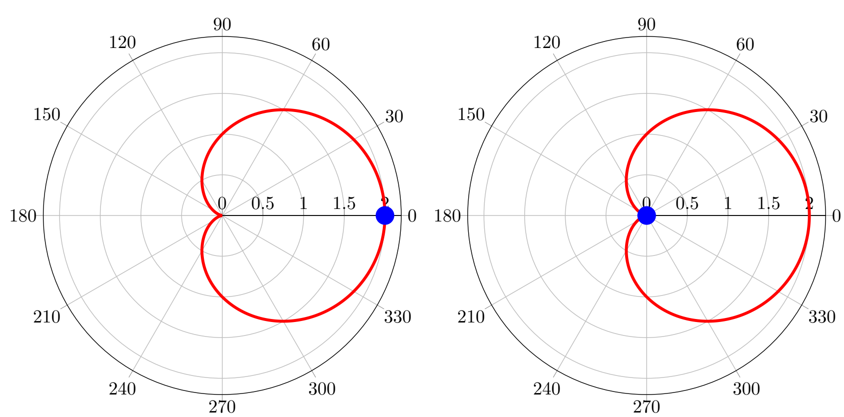
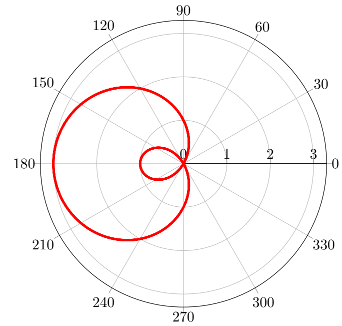


Figure 4.4 Curves $r = 1 + \cos\theta$ and $r = -1 + \cos\theta$ and the points that correspond to $\theta = 0$

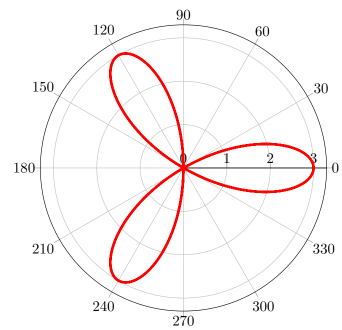
8. Sketch the graph of the curve whose equation in polar coordinates is $r = 1 - 2\cos\theta$, $0 \le \theta < 2\pi$.

Answer.



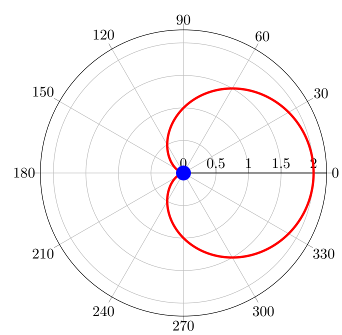
9. Sketch the graph of the curve whose equation in polar coordinates is $r = 3\cos 3\theta$.

Answer.



10. Sketch the curve whose polar equation is $r = -1 + \cos\theta$, indicating any symmetries. Mark on your sketch the polar coordinates of all points where the curve intersects the polar axis.

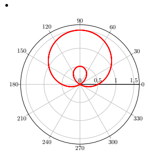
Answer.



Sketch a polar coordinate plot of:

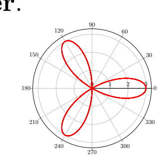
11. $r = \dfrac{1}{2} + \sin\theta$

Answer.



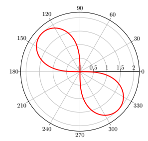
12. $r = 2\cos 3\theta$

Answer.



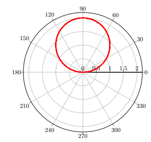
13. $r^2 = -4 \sin 2\theta$

Answer.



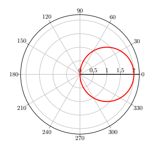
14. $r = 2 \sin \theta$

Answer.



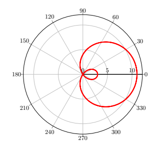
15. $r = 2 \cos \theta$

Answer.



16. $r = 4 + 7 \cos \theta$

Answer.



17. Consider the curve given by the polar equation $r = 1 - \cos \theta$, for $0 \leq \theta < 2\pi$.

(a) Given a point P on this curve with polar coordinates (r, θ), represent its Cartesian coordinates (x, y) in terms of θ.

(b) Find the slope of the tangent line to the curve where $\theta = \dfrac{\pi}{2}$.

(c) Find the points on this curve where the tangent line is horizontal or vertical.

Answer.

(a) $(x, y) = ((1 - \cos \theta) \cdot \cos \theta, (1 - \cos \theta) \cdot \sin \theta)$.

(b) $\left.\dfrac{dy}{dx}\right|_{\theta = \frac{\pi}{2}} = -1$.

(c) Horizontal tangent lines at $\left(-\dfrac{3}{4}, \dfrac{3\sqrt{3}}{4}\right)$, $\left(-\dfrac{3}{4}, -\dfrac{3\sqrt{3}}{4}\right)$; Vertical tangent line at $(0,0)$, $(-2,0)$, $\left(\dfrac{1}{4}, \dfrac{\sqrt{3}}{4}\right)$, $\left(\dfrac{1}{4}, -\dfrac{\sqrt{3}}{4}\right)$.

18. Consider the curve given by the polar equation $r = \cos(2\theta)$, for $0 \leq \theta < 2\pi$.

(a) Find $\dfrac{dy}{dx}$ in terms of θ.

(b) Find the Cartesian coordinates for the point on the curve corresponding to $\theta = \dfrac{\pi}{8}$.

(c) Find the tangent line to the curve at the point corresponding to $\theta = \dfrac{\pi}{8}$.

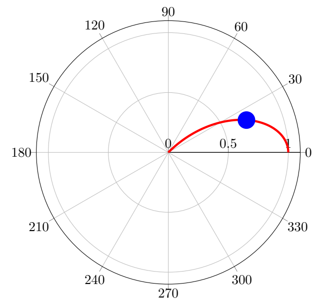
(d) Sketch this curve for $0 \leq \theta \leq \dfrac{\pi}{4}$ and label the point from part (b) on your curve.

Answer.

(a) $\dfrac{dy}{dx} = \dfrac{-2 \sin 2\theta \sin \theta + \cos \theta \cos 2\theta}{-2 \sin 2\theta \cos \theta - \sin \theta \cos 2\theta}$.

(b) $\left(\frac{1}{2}\sqrt{1+\frac{1}{\sqrt{2}}}, \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{2}}}\right)$.

(c) $y - \frac{1}{2}\sqrt{1-\frac{1}{\sqrt{2}}} = \frac{2\sqrt{2-\sqrt{2}}-\sqrt{2+\sqrt{2}}}{2\sqrt{2+\sqrt{2}}+\sqrt{2-\sqrt{2}}} \cdot \left(x - \frac{1}{2}\sqrt{1+\frac{1}{\sqrt{2}}}\right)$.

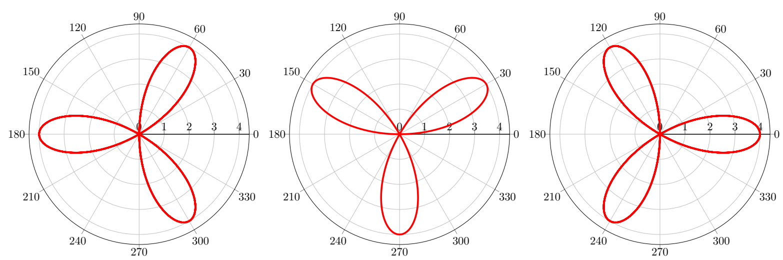


(d)

19. Consider the curve given by the polar equation $r = 4\cos(3\theta)$, for $0 \leq \theta < 2\pi$.

(a) Find the Cartesian coordinates for the point on the curve corresponding to $\theta = \frac{\pi}{3}$.

(b) One of graphs in the Figure below is the graph of $r = 4\cos(3\theta)$. Indicate which one by circling it.



(c) Find the slope of the tangent line to the curve where $\theta = \frac{\pi}{3}$.

Answer.

(a) $(-2, -2\sqrt{3})$.

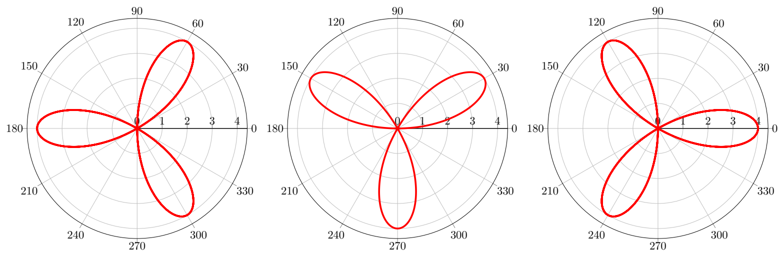
(b) Right.

(c)

20. Consider the curve given by the polar equation $r = 4\sin(3\theta)$, for $0 \leq \theta < 2\pi$.

(a) Find the Cartesian coordinates for the point on the curve corresponding to $\theta = \frac{\pi}{6}$.

(b) One of graphs in the Figure below is the graph of $r = 4\sin(3\theta)$. Indicate which one by circling it.



(c) Find the slope of the tangent line to the curve where $\theta = \dfrac{\pi}{3}$.

Answer.

(a) $(2\sqrt{3}, 2)$.

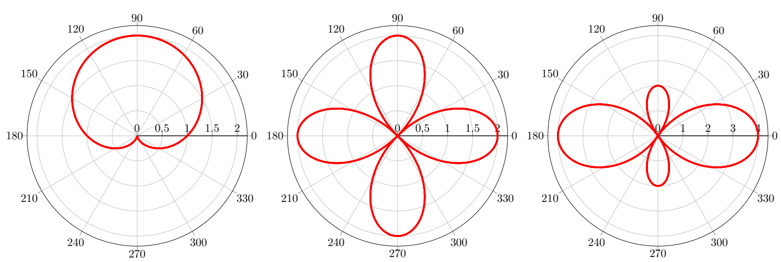
(b) Middle.

(c) $\sqrt{3}$.

21. Consider the curve given by the polar equation $r = 1 + 3\cos(2\theta)$, for $0 \leq \theta < 2\pi$.

(a) Find the Cartesian coordinates for the point on the curve corresponding to $\theta = \dfrac{\pi}{6}$.

(b) One of graphs in the Figure below is the graph of $r = 1 + 3\cos(2\theta)$. Indicate which one by putting a checkmark in the box below the graph you chose.



(c) Find the slope of the tangent line to the curve where $\theta = \dfrac{\pi}{6}$.

Answer.

(a) $\left(\frac{5\sqrt{3}}{4}, \frac{5}{4}\right)$.

(b) Right. Observe that $r(0) = r(\pi) = 4$ and $r\left(\dfrac{\pi}{2}\right) = r\left(\dfrac{3\pi}{2}\right) = 1$.

(c) $\dfrac{\sqrt{3}}{23}$.

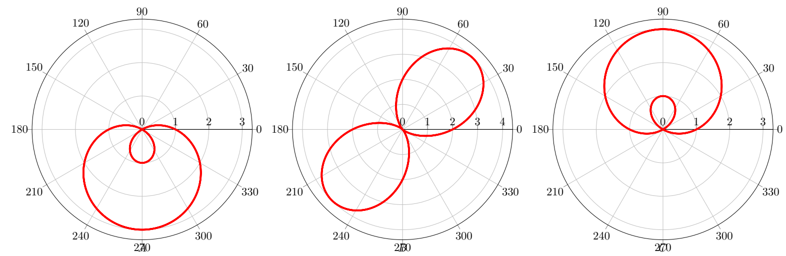
22. Consider the curve given by the polar equation $r = 1 - 2\sin\theta$, for $0 \leq \theta < 2\pi$.

(a) Find the Cartesian coordinates for the point on the curve corresponding to $\theta = \dfrac{3\pi}{2}$.

(b) The curve intersects the x-axis at two points other than the pole. Find polar coordinates for these other points.

(c) On the Figure below, identify the graphs that correspond to the following two polar curves.

$$\square\ r = 1 - 2\sin\theta \quad \square\ r = 1 + 2\sin\theta$$



Answer.

(a) $(0, -3)$.

(b) $(\pm 1, 0)$. Solve $y = (1 - 2\sin\theta)\sin\theta = 0$.

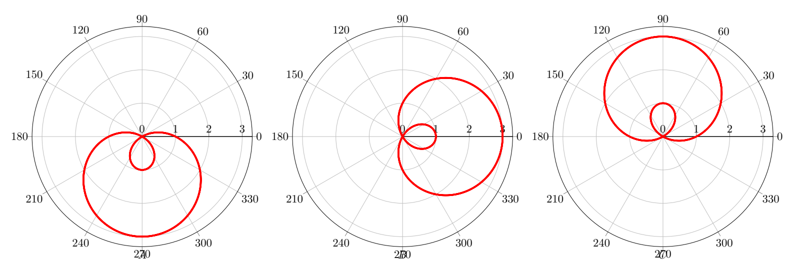
(c) The middle graph corresponds to $r = 1 + \sin 2\theta$ and the right graph corresponds to $r = 1 - 2\sin\theta$.

23. Consider the curve C given by the polar equation $r = 1 + 2\cos\theta$, for $0 \le \theta < 2\pi$.

(a) Find the Cartesian coordinates for the point on the curve corresponding to $\theta = \dfrac{\pi}{3}$.

(b) Find the slope of the tangent line where $\theta = \dfrac{\pi}{3}$.

(c) On the Figure below, identify the graph of C.



Answer.

(a) $(1, \sqrt{3})$.

(b) $\dfrac{1}{3\sqrt{3}}$.

(c) B.

24.

(a) Sketch a polar coordinate plot of

$$r = 1 + 2\sin 3\theta,\ 0 \le \theta \le 2\pi.$$

(b) How many points lie in the intersection of the two polar graphs

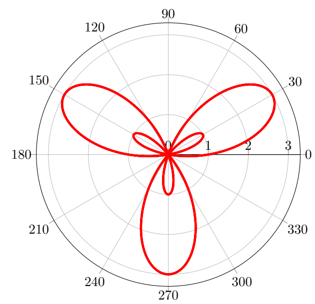
$$r = 1 + 2\sin 3\theta,\ 0 \le \theta \le 2\pi$$

and
$$r = 1?$$

(c) *Algebraically* find all values of θ that
$$1 = 1 + 2\sin 3\theta,\ 0 \le \theta \le 2\pi.$$

(d) Explain in a sentence or two why the answer to part (b) differs from (or is the same as) the number of solutions you found in part (c).

Answer.

(a) 

(b) 9.

(c) $\theta = 0, \dfrac{\pi}{3}, \dfrac{2\pi}{3}, \pi, \dfrac{4\pi}{3}, \dfrac{5\pi}{3}, 2\pi.$

(d) The remaining points of intersection are obtained by solving $-1 = 1 + 2\sin 3\theta$.

25. Consider the following curve C given in polar coordinates as
$$r(\theta) = 1 + \sin\theta + e^{\sin\theta},\ 0 \le \theta \le 2\pi.$$

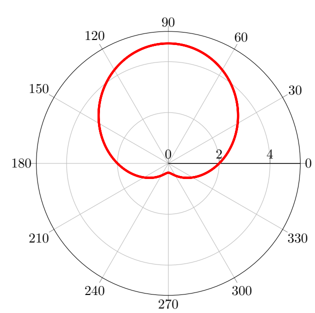
(a) Calculate the value of $r(\theta)$ for $\theta = 0, \dfrac{\pi}{2}, \dfrac{3\pi}{2}$.

(b) Sketch a graph of C.

(c) What is the minimum distance from a point on the curve C to the origin? (i.e. determine the minimum of $|r(\theta)| = r(\theta) = 1 + \sin\theta + e^{\sin\theta}$ for $\theta \in [0, 2\pi]$).

Answer.

(a) $r(0) = 2$, $r\left(\dfrac{\pi}{2}\right) = 2 + e$, $r\left(\dfrac{3\pi}{2}\right) = e^{-1}$.

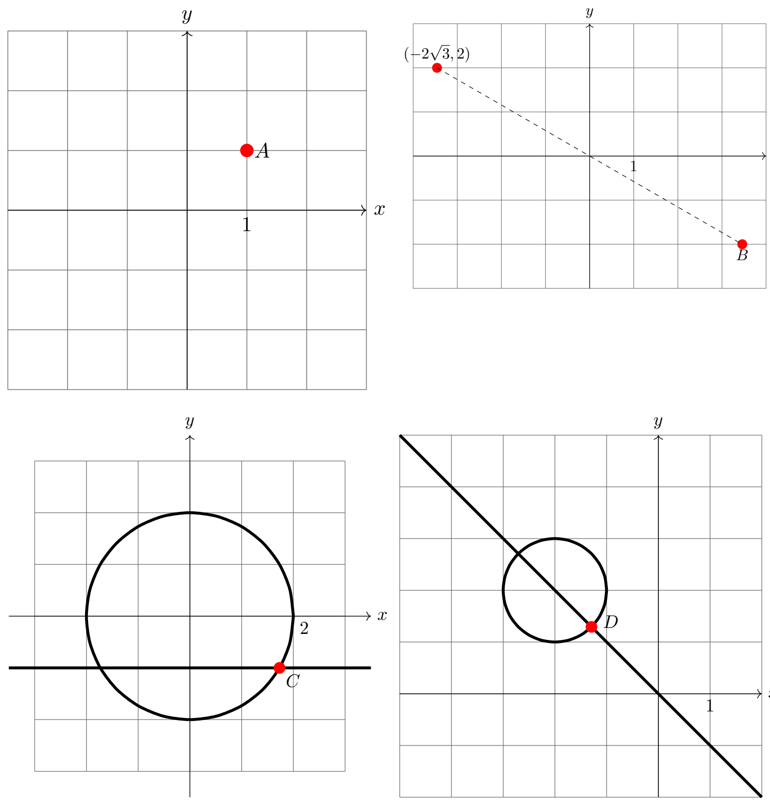
(b) 

(c) e^{-1}.

Solution. (c) From $\dfrac{dr}{d\theta} = \cos\theta(1 + e^{\sin\theta}) = 0$ we conclude that the critical numbers are $\dfrac{\pi}{2}$ and $\dfrac{3\pi}{2}$. By the Extreme Value Theorem, the minimum distance equals e^{-1}.

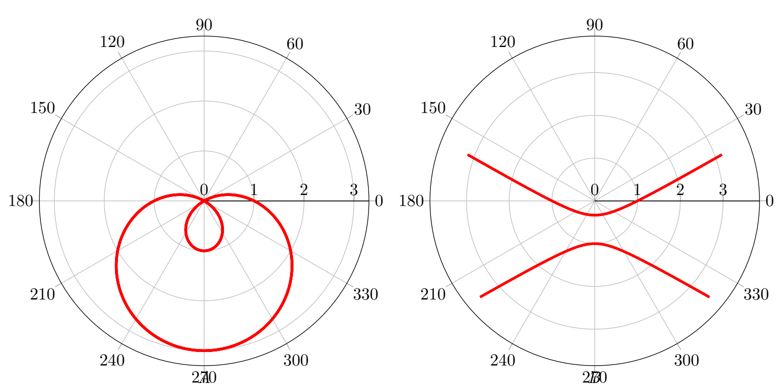
26.

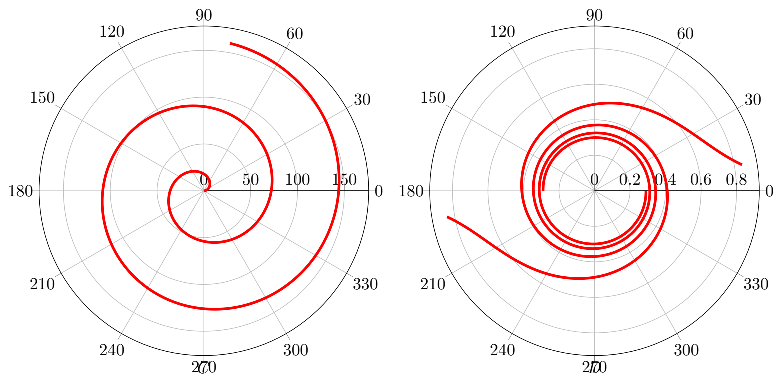
(a) Give polar coordinates for each of the points A, B, C and D on the Figure below.



(b) On the Figure below identify the graphs that correspond to the following three polar curves.

$$\square\ r = 1 - 2\sin\theta \quad \square\ r^2\theta = 1 \quad \square\ r = \frac{1}{1 - 2\sin\theta}$$





Answer.

(a) $A = \left(r = \sqrt{2}, \theta = \dfrac{\pi}{4}\right)$, $B = \left(4, \dfrac{5\pi}{3}\right)$, $C = \left(2, \dfrac{7\pi}{6}\right)$, $D = \left(2\sqrt{2} - 1, \dfrac{3\pi}{4}\right)$.

(b) A, B, D.

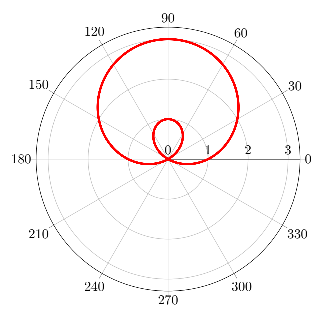
27.

(a) Sketch the curve defined by $r = 1 + 2\sin\theta$.

(b) For what values of θ, $\theta \in [-\pi, \pi)$, is the radius r positive?

(c) For what values of θ, $\theta \in [-\pi, \pi)$, is the radius r maximum and for what values is it minimum?

Answer.

(a) 

(b) $\theta \in \left[-\pi, -\dfrac{5\pi}{6}\right) \cup \left(-\dfrac{\pi}{6}, \pi\right)$. Solve $\sin\theta > -\dfrac{1}{2}$.

(c) To find critical numbers solve $\dfrac{dr}{d\theta} = 2\cos\theta = 0$ in $[-\pi, \pi)$. It follows that $\theta = -\dfrac{\pi}{2}$ and $\theta = \dfrac{\pi}{2}$ are critical numbers. Compare $r(-\pi) = r(\pi) = 1$, $r\left(-\dfrac{\pi}{2}\right) = -1$, and $r\left(\dfrac{\pi}{2}\right) = 3$ to answer the question.

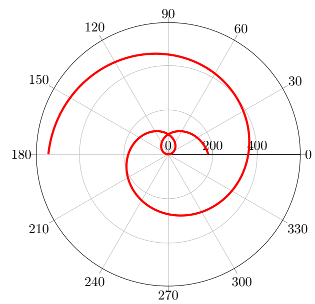
28.

(a) Sketch the graph described in polar coordinates by the equation $r = \theta$ where $-\pi \leq \theta \leq 3\pi$.

(b) Find the slope of this curve when $\theta = \dfrac{5\pi}{2}$. Simplify your answer for full credit.

(c) Express the polar equation $r = \theta$ in cartesian coordinates, as an equation in x and y.

Answer.

(a) 

(b) $-\frac{2}{5\pi}$.

(c) $\sqrt{x^2+y^2} = \arctan\dfrac{y}{x}$.

Solution. (b) The slope is given by $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{5\pi}{2}}$. From $x = r\cos\theta = \theta\cos\theta$ and $y = \theta\sin\theta$ it follows that $\dfrac{dy}{dx} = \dfrac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \dfrac{\sin\theta + \theta\cos\theta}{\cos\theta - \theta\sin\theta}$. Thus $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{5\pi}{2}} = -\dfrac{2}{5\pi}$.

Answer the following.

29. Let C denote the graph of the polar equation $r = 5\sin\theta$. Find the rectangular coordinates of the point on C corresponding to $\theta = \dfrac{3\pi}{2}$.

Answer. $(0,5)$.

30. Write a rectangular equation (i.e. using the variables x and y) for C. (in other words, convert the equation for C into rectangular coordinates.)

Answer. $x^2 + y^2 = 5y$.

31. Rewrite the equation of C in parametric form, i.e. express both x and y as functions of θ.

Answer. $x = 5\sin\theta\cos\theta$, $y = 5\sin^2\theta$.

32. Find an expression for $\dfrac{dy}{dx}$ in terms of θ.

Answer. $\dfrac{dy}{dx} = \dfrac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta} = \tan 2\theta$.

33. Find the equation of the tangent line to C at the point corresponding to $\theta = \dfrac{\pi}{6}$.

Answer. $y - \dfrac{5}{4} = \sqrt{3}\left(x - \dfrac{5\sqrt{3}}{4}\right)$.

34. Find the slope of the tangent line to the polar curve $r = 2$ at the points where it intersects the polar curve $r = 4\cos\theta$. (Hint: After you find the intersection points, convert one of the curves to a pair of parametric equations with θ as the perimeter.)

Answer. $-\frac{\sqrt{3}}{3}$.

Solution. Solve $2 = 4\cos\theta$ to get that curve intersect at $(1, \sqrt{3})$. To find the slope we note that the circle $r = 2$ is given by parametric equations $x = 2\cos\theta$ and $y = 2\sin\theta$. It follows that $\dfrac{dy}{dx} = \dfrac{2\cos\theta}{-2\sin\theta} = -\cot\theta$. The slope of the tangent line at the intersection point equals $\left.\dfrac{dy}{dx}\right|_{\theta=\frac{\pi}{3}} = -\dfrac{\sqrt{3}}{3}$.

35. A bee goes out from its hive in a spiral path given in polar coordinates by $r = be^{kt}$ and $\theta = ct$, where b, k, and c are positive constants. Show that the angle between the bee's velocity and acceleration remains constant as the bee moves outward.

Solution. See the Figure below for the graph of the case $b = 1$, $k = 0.01$, and $c = 2$. The position in $(x, y)-$plane of the bee at time t is given by a vector function $\vec{s}(t) = \langle be^{kt}\cos ct, be^{kt}\sin ct\rangle$. Recall that the angle α between the velocity and acceleration is given by $\cos\alpha = \dfrac{\vec{v}\cdot\vec{a}}{|\vec{v}||\vec{a}|}$, where $\vec{v}(t) = \vec{s}'(t)$ and $\vec{a}(t) = \vec{s}''(t)$. One way to solve this problem is to consider that the bee moves in the complex plane. In that case its position is given by

$$F(t) = be^{kt}\cos ct + i\cdot be^{kt}\sin ct = be^{kt}(\cos ct + i\sin ct) = be^{(k+ic)t},$$

where i is the imaginary unit. Observe that $\vec{v}(t) = \langle \mathrm{Re}(F'(t)), \mathrm{Im}(F'(t))\rangle$ and $\vec{a}(t) = \langle \mathrm{Re}(F''(t)), \mathrm{Im}(F''(t))\rangle$. Next, observe that $F'(t) = (k + ic)F(t)$ and $F''(t) = (k+ic)^2F(t)$. From $F''(t) = (k+ic)F'(t)$ it follows that $\mathrm{Re}(F''(t)) = k\cdot\mathrm{Re}(F'(t)) - c\cdot\mathrm{Im}(F'(t))$ and $\mathrm{Im}(F''(t)) = k\cdot\mathrm{Im}(F'(t)) + c\cdot\mathrm{Re}(F'(t))$. Finally, $\vec{v}\cdot\vec{a} = \mathrm{Re}(F'(t))\cdot\mathrm{Re}(F''(t)) + \mathrm{Im}(F'(t))\cdot\mathrm{Im}(F''(t)) = k((\mathrm{Re}(F'(t)))^2 + (\mathrm{Im}(F'(t)))^2) = k|F'(t)|^2$ which immediately implies the required result.

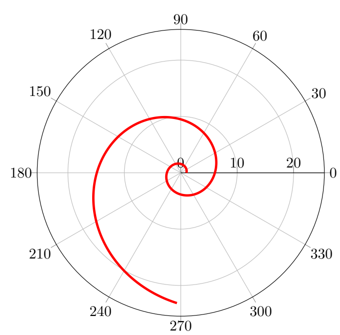


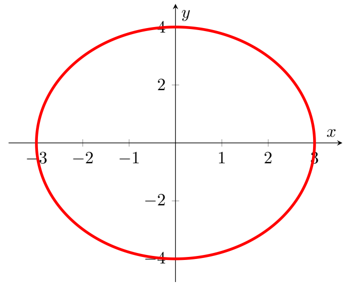
Figure 4.5 $r(t) = e^{0.01t}$, $\theta = 2t$, $0 \le t \le 10\pi$

4.4 Conic Sections

Solve the following problems.

1. Sketch the graph of the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{16} = 1$ and determine its foci.

Answer. Focci: $(0, -\sqrt{7})$, $(0, \sqrt{7})$.



2. Let C be the conic which consists of all points $P = (x, y)$ such that the distance from P to $F = (0, 1)$ is $\frac{1}{2}$ the perpendicular distance from P to the line $y = 4$.

(a) What is the eccentricity of the conic C?

(b) Show that the equation of the conic C is

$$\frac{x^2}{3} + \frac{y^2}{4} = 1.$$

(c) What is the equation of the tangent line to the point $A = (\sqrt{3}/2, \sqrt{3})$ on the conic C?

Answer.

(a) $e = \frac{1}{2}$.

(b) Use the fact that, for $P = (x, y)$, $|PF|^2 = x^2 + (y - 1)^2$ and $|Pl| = \frac{1}{2}|y - 4|$.

(c) From $\frac{dy}{dx} = -\frac{4x}{3y}$ it follows that the slope of the tangent line is $\left.\frac{dy}{dx}\right|_{x=\frac{\sqrt{3}}{2}} = -\frac{2}{3}$.

3.

(a) Find the equation of the tangent line to the polar curve $r = 1 + \cos\theta$ at $\theta = \frac{\pi}{2}$.

(b) Find an equation of the ellipse with horizontal and vertical axes that passes through the points $(2, 0)$, $(-2, 0)$, $(0, -1)$, and $(0, 3)$.

Answer.

(a) $y = x + 1$.

(b) $\frac{x^2}{\frac{16}{3}} + \frac{(y - 1)^2}{4} = 1$.

4. Consider the polar equation $r(3 - k\cos\theta) = 4$.

(a) For what values of the constant k is this conic section an ellipse?

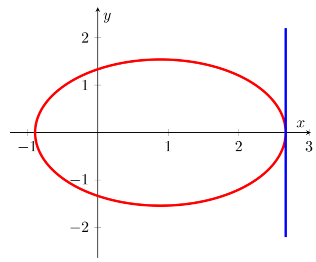
(b) Now assume that k has the value in the middle of the interval found in part (a). Determine the location of the focus and directrix of this conic section.

(c) Sketch the graph of this conic section which shows the focus, directrix and vertices.

Answer.

(a) $0 < k < 3$.

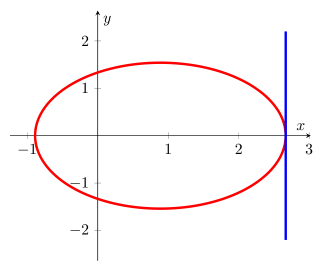
(b) The directrix is $x = \dfrac{8}{3}$. The focus is $\frac{8}{9}$.

(c) 

Solution.

(a) From $r = \dfrac{\frac{4}{3}}{1 - \frac{k}{3}\cos\theta}$ it follows that this conic section is an ellipse if $0 < k < 3$.

(b) If $k = \dfrac{3}{2}$ then the eccentricity is given by $e = \dfrac{k}{3} = \dfrac{1}{2}$. The directrix is $x = d$ where $ed = \dfrac{4}{3}$. Thus the directrix is $x = \dfrac{8}{3}$. Let $c > 0$ be such that $(c, 0)$ is a focus of the ellipse. Then $\dfrac{c}{e}(1 - e^2) = ed$ and $c = \dfrac{8}{9}$.

(c) 

Answer the following.

5. Assume that the earth is a sphere with radius s. (s is about 6400 kilometres, but this has nothing to do with the solution of this problem. Express everything, including your answer, in terms of s.) A satellite has an elliptical orbit with the centre of the earth at one focus. The lowest point of the orbit is 5s above the surface of the earth, when the satellite is *directly above the North Pole*. The highest point of the orbit is 11s above the surface of the earth, when the satellite is *directly above the South Pole*. What is the height of the satellite above the surface of the earth, when the satellite is *directly above the equator*? [Suggestion: Choose a coordinate system with the centre of the earth on the y-axis and the centre of the satellite's orbit at the origin.]

Answer. $8s$.

Solution. Let the centre of the earth (and a focus of the ellipse) be at the the point $(0, c)$, $c > 0$. Let the equation of the ellpse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. It is given that the vertices of the ellipse on the y-axis are $(0, (c+s)+5s)$ and $(0, (c-s)-11s)$. It follows that the length of the major axis on the y-axis is $2b = 6s+12s = 18s$. Thus, $b = 9s$ and $c = b - 6s = 3s$. From $c^2 = b^2 - a^2$ we get that $a^2 = 72s^2$. Thus the equation of the ellipse is $\frac{x^2}{72s^2} + \frac{y^2}{81s^2} = 1$. The question is to evaluate the value of $|x|$ when $y = c = 3s$. From $\frac{x^2}{72s^2} + \frac{9s^2}{81s^2} = 1$ it follows that $|x| = 8s$.

6. Let the focus F of a conic section with eccentricity e be the origin and let the directrix L be the vertical line $x = -p$, $p > 0$. Thus the conic consists of all those points $P = (x, y)$ such that $|PF| = e|PL|$. Find the polar equation of the conic section. (With the origin as the pole, and the positive x-axis as the polar axis.)

Answer. $r = \frac{ep}{1 - e\cos\theta}$.

7.

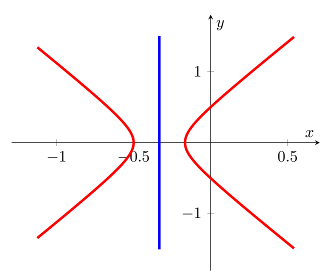
(a) Identify and sketch the graph of the conic section defined by

$$r = \frac{1}{2 - 4\cos\theta}$$

in the x, y, plane without converting it to cartesian coordinates. Clearly label all foci and vertices. (Hint: You may use the equation $r = \frac{ep}{1 - e\cos\theta}$ without deriving it.)

(b) Convert the polar coordinates equation given above to a cartesian coordinates equation of the form $Ax^2 + Bx + Cy^2 + Dy = E$.

Answer.

(a) 

(b) $12x^2 - 4y^2 + 8x + 1 = 0$.

Solution.

(a) From $r = \frac{\frac{1}{2}}{1 - 2\cos\theta}$ we see that the eccentricity is $e = 2$ and the equation therefore represents a hyperbola. From $ed = \frac{1}{2}$ we conclude that the directrix is $x = -\frac{1}{4}$. The vertices occur when $\theta = 0$ and $\theta = \pi$. Thus the vertices are $\left(-\frac{1}{2}, 0\right)$ and $\left(-\frac{1}{6}, 0\right)$.The y-intercepts occur when $\theta = \frac{\pi}{2}$ and $\theta = \frac{3\pi}{2}$. Thus $\left(0, \frac{1}{2}\right)$ and

$\left(0, -\frac{1}{2}\right)$. We note that $r \to \infty$ when $\cos\theta \to \frac{1}{2}$. Therefore the asymptotes are parallel to the rays $\theta = \frac{\pi}{3}$ and $\theta = \frac{5\pi}{3}$.

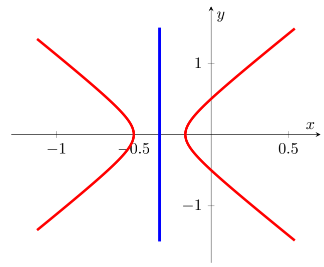


Figure 4.6 $r = \dfrac{1}{2 - 4\cos\theta}$ and $x = -\dfrac{1}{3}$

(b) From $\sqrt{x^2+y^2} = \dfrac{1}{2 - \frac{4x}{\sqrt{x^2+y^2}}}$ it follows that the conic section is given by $12x^2 - 4y^2 + 8x + 1 = 0$.

8.

(a) Find an equation of the set of all points (x, y) that satisfy this condition: The distance from (x, y) to $(5, 0)$ is exactly half the distance from (x, y) to the line $x = -5$.

(b) Simplify your answer from part (a) enough to be able to tell what type of conic section it is.

Answer.

(a) $(x-5)^2 + y^2 = \dfrac{(x+5)^2}{4}$.

(b) $3\left(x - \dfrac{25}{3}\right)^2 + 4y^2 = \dfrac{400}{3}$. This is an ellipse.

9. Derive the equation of the set of all points $P(x, y)$ that are equidistant from the point $A(1, 0)$ and the line $x = -5$. Provide a diagram with your work. Simplify the equation.

Answer. From $(x-1)^2 + y^2 = (x+5)^2$ we get that $y^2 = 24 + 12x$.

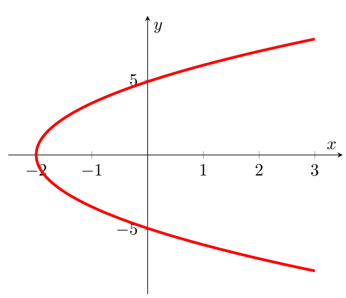


Figure 4.7 $y^2 = 24 + 12x$

10. Let C be the conic section described by the equation

$$x^2 - 2y^2 + 4x = 0.$$

(a) Using the method of completing the square, identify the conic section

C.

(b) Sketch a graph of C. Find, foci and asymptotes (if any).

Answer.

(a) $\dfrac{(x+2)^2}{4} - \dfrac{y^2}{2} = 1$. This is a hyperbola.

(b) Foci are $(-2-\sqrt{6}, 0)$ and $(-2+\sqrt{6}, 0)$. The asymptotes are $y = \pm\dfrac{\sqrt{2}}{2}(x+2)$.

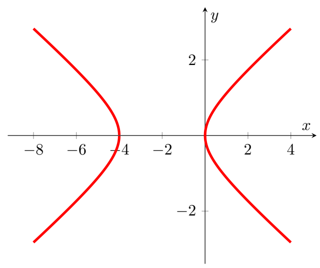


Figure 4.8 $x^2 - 2y^2 + 4x = 0$

11. Given is the polar equation $r = \dfrac{2}{1 - 2\cos\theta}$.

(a) Which type of conic section does this polar equation represent: Parabola, ellipse, or hyperbola?

(b) Show that the polar equation implies the following equation in Cartesian coordinates:

$$9\left(x + \frac{4}{3}\right)^2 - 3y^2 = 4.$$

(c) Give the foci, vertices, and asymptotes of the conic if it has any.

(d) Sketch the conic section based on the information found above. Indicate the features you found in (c) in your sketch.

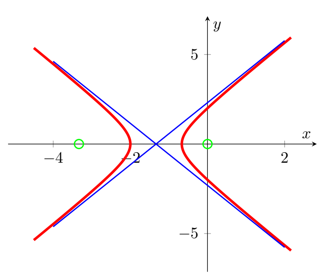
Answer.

(a) Hyperbola.

(b)

(c) The foci are given by $\left(-\dfrac{4}{3} \pm \dfrac{4}{3}, 0\right)$, the vertices are given by $\left(-\dfrac{4}{3} \pm \dfrac{2}{3}, 0\right)$, and the asymptotes are given by $y = \pm\sqrt{3}\left(x + \dfrac{4}{3}\right)$.

(d)



Solution. (a) A hyperbola, since the eccentricity is $e = 2 > 1$. (b) From $r(1 - 2\cos\theta) = 2$ conclude that $\sqrt{x^2 + y^2} = 2(x + 1)$. Square both sides, rearrange the expression, and complete the square.

Chapter 5

True or False and Multiple Choice Problems

5.1 True Or False

5.1.1 Exercises

Answer the following questions.

For each of the following ten statements answer TRUE or FALSE as appropriate:

1. If f is differentiable on $[-1, 1]$ then f is continuous at $x = 0$.
Answer. True.

2. If $f'(x) < 0$ and $f"(x) > 0$ for all x then f is concave down.
Answer. False.

3. The general antiderivative of $f(x) = 3x^2$ is $F(x) = x^3$.
Answer. False.

4. $\ln x$ exists for any $x > 1$.
Answer. True.

5. $\ln x = \pi$ has a unique solution.
Answer. True.

6. e^{-x} is negative for some values of x.
Answer. False.

7. $\ln e^{x^2} = x^2$ for all x.
Answer. True.

8. $f(x) = |x|$ is differentiable for all x.
Answer. False.

9. $\tan x$ is defined for all x.
Answer. False.

10. All critical points of $f(x)$ satisfy $f'(x) = 0$.
Hint. Take $f(x) = |x|$.
Answer. False.

Answer each of the following either TRUE or FALSE.

11. The function $f(x) = \begin{cases} 3 + \frac{\sin(x-2)}{x-2} & \text{if} \quad x \neq 2 \\ 3 & \text{if} \quad x = 2 \end{cases}$ is continuous at all real numbers x.

Hint. Find $\lim_{x\to 2} f(x)$.

Answer. False.

12. If $f'(x) = g'(x)$ for $0 < x < 1$, then $f(x) = g(x)$ for $0 < x < 1$.

Hint. Take $f(x) = 1$ and $g(x) = 2$.

Answer. False.

13. If f is increasing and $f(x) > 0$ on I, then $g(x) = \dfrac{1}{f(x)}$ is decreasing on I.

Answer. True.

14. There exists a function f such that $f(1) = -2$, $f(3) = 0$, and $f'(x) > 1$ for all x.

Hint. Apply the Mean Value Theorem.

Answer. False.

15. If f is differentiable, then $\dfrac{d}{dx} f(\sqrt{x}) = \dfrac{f'(x)}{2\sqrt{x}}$.

Hint. Apply the chain rule.

Answer. False.

16. $\dfrac{d}{dx} 10^x = x10^{x-1}$

Answer. False.

17. Let $e = \exp(1)$ as usual. If $y = e^2$ then $y' = 2e$.

Answer. False.

18. If $f(x)$ and $g(x)$ are differentiable for all x, then $\dfrac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.

Answer. True.

19. If $g(x) = x^5$, then $\lim_{x\to 2} \dfrac{g(x) - g(2)}{x - 2} = 80$.

Hint. The limit equals $g'(2)$.

Answer. True.

20. An equation of the tangent line to the parabola $y = x^2$ at $(-2, 4)$ is $y - 4 = 2x(x + 2)$.

Answer. False.

21. $\dfrac{d}{dx} \tan^2 x = \dfrac{d}{dx} \sec^2 x$

Hint. $\tan^2 x - \sec^2 x = -1$.

Answer. True.

22. For all real values of x we have that $\dfrac{d}{dx}|x^2 + x| = |2x + 1|$.

Answer. False. $y = |x^2 + x|$ is not differentiable for all real numbers.

23. If f is one-to-one then $f^{-1}(x) = \dfrac{1}{f(x)}$.

Answer. False.

24. If $x > 0$, then $(\ln x)^6 = 6 \ln x$.

Answer. False.

25. If $\lim_{x\to 5} f(x) = 0$ and $\lim_{x\to 5} g(x) = 0$, then $\lim_{x\to 5} \dfrac{f(x)}{g(x)}$ does not exist.

Hint. Take $\lim_{x\to 5} \dfrac{x-5}{x-5}$.

Answer. False.

26. If the line $x = 1$ is a vertical asymptote of $y = f(x)$, then f is not defined at 1.

Hint. Take $f(x) = \dfrac{1}{x-1}$ if $x > 1$ and $f(x) = 0$ if $x \le 1$.

Answer. False.

27. If $f'(c)$ does not exist and $f'(x)$ changes from positive to negative as x increases through c, then $f(x)$ has a local minimum at $x = c$.

Answer. False.

28. $\sqrt{a^2} = a$ for all $a > 0$.

Answer. True.

29. If $f(c)$ exists but $f'(c)$ does not exist, then $x = c$ is a critical point of $f(x)$.

Answer. False. c might be an isolated point.

30. If f"(c) exists and $f'''(c) > 0$, then $f(x)$ has a local minimum at $x = c$.

Hint. Take $f(x) = x^3$.

Answer. False.

Are the following statements TRUE or FALSE.

31. $\lim_{x\to 3} \sqrt{x-3} = \sqrt{\lim_{x\to 3}(x-3)}$.

Answer. True.

32. $\dfrac{d}{dx}\left(\dfrac{\ln 2^{\sqrt{x}}}{\sqrt{x}}\right) = 0$

Answer. True. $\frac{1}{\sqrt{x}} \cdot \ln 2^{\sqrt{x}} = \ln 2$, $x > 0$

33. If $f(x) = (1+x)(1+x^2)(1+x^3)(1+x^4)$, then $f'(0) = 1$.

Answer. True.

34. If $y = f(x) = 2^{|x|}$, then the range of f is the set of all non-negative real numbers.

Answer. False. $f(x) \ge 1$.

35. $\dfrac{d}{dx}\left(\dfrac{\log x^2}{\log x}\right) = 0$.

Answer. True.

36. If $f'(x) = -x^3$ and $f(4) = 3$, then $f(3) = 2$.

Answer. False $f(x) = -\dfrac{x^4 - 256}{4} + 3$.

37. If f"(c) exists and if f"$(c) > 0$, then $f(x)$ has a local minimum at $x = c$.

Answer. False. Take $f(x) = x^2$ and $c = 1$.

38. $\dfrac{d}{du}\left(\dfrac{1}{\csc u}\right) = \dfrac{1}{\sec u}$.

Answer. True. $\dfrac{1}{\csc u} = \sin u$ with $\sin u \ne 0$.

39. $\dfrac{d}{dx}(\sin^{-1}(\cos x) = -1$ for $0 < x < \pi$.

Hint. Use the chain rule.

Answer. True.

40. $\sinh^2 x - \cosh^2 x = 1$.

Answer. False. $\sinh^2 x - \cosh^2 x = -1$.

41. $\displaystyle\int \frac{dx}{x^2+1} = \ln(x^2+1) + C$.

Answer. False. $\displaystyle\int \frac{dx}{x^2+1} = \arctan x + C$.

42. $\displaystyle\int \frac{dx}{3-2x} = \frac{1}{2}\ln|3-2x| + C$.

Answer. False. $\displaystyle\int \frac{dx}{3-2x} = -\frac{\ln|3-2x|}{2} + C$.

Answer each of the following either TRUE or FALSE.

43. For *all* functions f, if f is continuous at a certain point x_0, then f is differentiable at x_0.

Answer. False.

44. For *all* functions f, if $\lim\limits_{x\to a^-} f(x)$ exist, and $\lim\limits_{x\to a^+} f(x)$ exist, then f is continuous at a.

Hint. Take $f(x) = \dfrac{x^2}{x}$ and $a = 0$.

Answer. False.

45. For *all* functions f, if $a < b$, $f(a) < 0$, $f(b) > 0$, then there must be a number c, with $a < c < b$ and $f(c) = 0$.

Hint. Take $f(x) = \dfrac{x^2}{x}$, $a = -1$, and $b = 1$.

Answer. False.

46. For *all* functions f, if $f'(x)$ exists for all x, then $f''(x)$ exists for all x.

Hint. Take $f(x) = x|x|$.

Answer. False.

47. It is impossible for a function to be discontinuous at *every* number x.

Hint. Take $f(x) = 1$ if x is rational and $f(x) = 0$ if x is irrational.

Answer. False.

48. If f, g, are *any* two functions which are continuous for all x, then $\dfrac{f}{g}$ is continuous for all x.

Hint. Take $g(x) = 0$.

Answer. False.

49. It is possible that functions f and g are *not* continuous at a point x_0, but $f + g$ *is* continuous at x_0.

Hint. Take $f(x) = \frac{1}{x}$ if $x \neq 0$, $f(0) = 0$, and $g(x) = -f(x)$.

Answer. True.

50. If $\lim\limits_{x\to\infty}(f(x)+g(x))$ exists, then $\lim\limits_{x\to\infty} f(x)$ exists and $\lim\limits_{x\to\infty} g(x)$ exists.

Hint. Take $f(x) = \sin x$ and $g(x) = -\sin x$.

Answer. False.

51. $\lim_{x\to\infty} \dfrac{(1.00001)^x}{x^{100000}} = 0$

Answer. False. The numerator is an exponential function with a base greater than 1 and the denominator is a polynomial.

52. Every continuous function on the interval $(0,1)$ has a maximum value and a minimum value on $(0,1)$.

Hint. Take $f(x) = \tan \dfrac{\pi x}{2}$.

Answer. False.

Answer each of the following either TRUE or FALSE.

53. Let f and g be *any* two functions which are continuous on $[0,1]$, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 10$. Then there must exist $c, d \in [0,1]$ such that $f'(c) = g'(d)$.

Hint. Take $f(x) = 10x$ and $g(x) = 20x$ if $x \in [0, 0.5]$ and $g(x) = 10x$ if $x \in (0.5, 1]$.

Answer. False.

54. Let f and g be *any* two functions which are continuous on $[0,1]$ and differentiable on $(0,1)$, with $f(0) = g(0) = 0$ and $f(1) = g(1) = 10$. Then there must exist $c \in [0,1]$ such that $f'(c) = g'(c)$.

Hint. Take $F(x) = f(x) - g(x)$ and apply Rolle's Theorem.

Answer. True.

55. For all x in the domain of $\sec^{-1} x$,

$$\sec(\sec^{-1}(x)) = x.$$

Answer. True.

Answer each of the following either TRUE or FALSE.

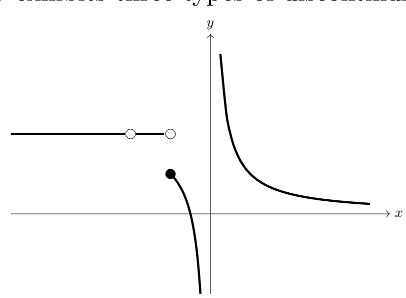
56. The slope of the tangent line of $f(x)$ at the point $(a, f(a))$ is given by $\dfrac{f(a+h) - f(a)}{h}$.

Answer. False. The limit is missing.

57. Using the Intermediate Value Theorem it can be shown that $\lim_{x\to 0} x \sin \dfrac{1}{x} = 0$.

Answer. False. The Squeeze Theorem.

58. The graph below exhibits three types of discontinuities.



Answer. True.

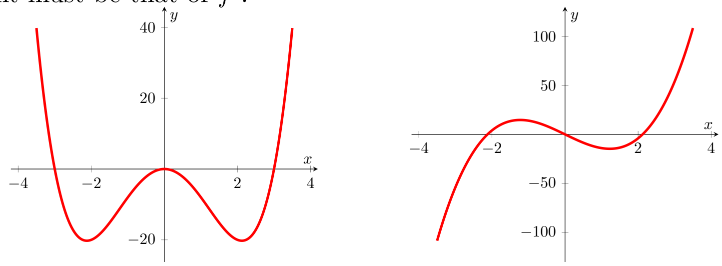
59. If $w = f(x)$, $x = g(y)$, $y = h(z)$, then $\dfrac{dw}{dz} = \dfrac{dw}{dx} \cdot \dfrac{dx}{dy} \cdot \dfrac{dy}{dz}$.

Answer. True.

60. Suppose that on the open interval I, f is a differentiable function that has an inverse function f^{-1} and $f'(x) \neq 0$. Then f^{-1} is differentiable and $\left(f^{-1}(x)\right)' = \dfrac{1}{f'(f^{-1}(x))}$ for all x in the domain of f^{-1}.

Answer. True.

61. If the graph of f is on the Figure below, to the left, the graph to the right must be that of f'.



Answer. False. For $x < 3$ the function is decreasing.

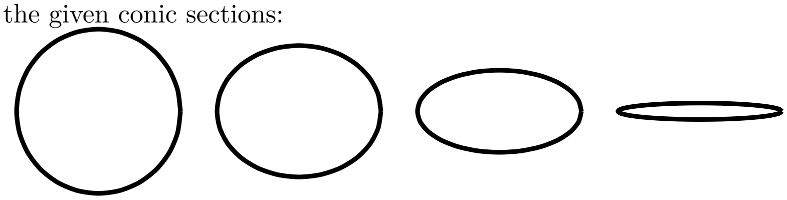
62. The conclusion of the Mean Value Theorem says that the graph of f has at least one tangent line in (a, b), whose slope is equal to the average slope on $[a, b]$.

Answer. True .

63. The linear approximation $L(x)$ of a function $f(x)$ near the point $x = a$ is given by $L(x) = f'(a) + f(a)(x - a)$.

Answer. False. It should be $L(x) = f(a) + f'(a)(x - a)$.

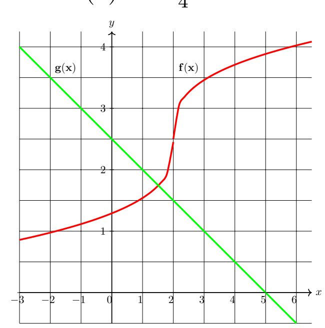
64. The graphs below are labeled correctly with possible eccentricities for the given conic sections:



$e = 1$ $\quad$ $e = 0.8$ $\quad$ $e = 0.5$ $\quad$ $e = 0.1$

Answer. False. The eccentricity of a circle is $e = 0$.

65. Given $h(x) = g(f(x))$ and the graphs of f and g on the Figure below, then a good estimate for $h'(3)$ is $-\frac{1}{4}$.



Hint. Note that $g'(x) = -0.5$ and $f'(3) \approx 0.5$.

Answer. True.

Answer TRUE or FALSE to the following questions.

66. If $f(x) = 7x + 8$ then $f'(2) = f'(17.38)$.

Answer. True.

67. If $f(x)$ is any function such that $\lim_{x\to 2} f(x) = 6$ the $\lim_{x\to 2^+} f(x) = 6$.

Answer. True.

68. If $f(x) = x^2$ and $g(x) = x + 1$ then $f(g(x)) = x^2 + 1$.

Answer. False. $f(g(x)) = (x + 1)^2$.

69. The average rate of change of $f(x)$ from $x = 3$ to $x = 3.5$ is $2(f(3.5) - f(3))$.

Answer. True.

70. An equivalent precise definition of $\lim_{x\to a} f(x) = L$ is: For any $0 < \epsilon < 0.13$ there is $\delta > 0$ such that

$$\text{if } |x - a| < \delta \text{ then } |f(x) - L| < \epsilon.$$

The last four True/False questions ALL pertain to the following function. Let

$$f(x) \begin{cases} x - 4 & \text{if} \quad x < 2 \\ 23 & \text{if} \quad x = 2 \\ x^2 + 7 & \text{if} \quad x > 2 \end{cases}$$

Answer. True.

71. $f(3) = -1$

Answer. False. $f(3) = 16$.

72. $f(2) = 11$

Answer. False.

73. f is continuous at $x = 3$.

Answer. True.

74. f is continuous at $x = 2$.

Answer. False.

Answer TRUE or FALSE to the following questions.

75. If a particle has a constant acceleration, then its position function is a cubic polynomial.

Answer. False. It is a quadratic polynomial.

76. If $f(x)$ is differentiable on the open interval (a, b) then by the Mean Value Theorem there is a number c in (a, b) such that $(b - a)f'(c) = f(b) - f(a)$.

Answer. False. The function should be also continuous on $[a, b]$.

77. If $\lim_{x\to\infty} \left(\dfrac{k}{f(x)}\right) = 0$ for every number k, then $\lim_{x\to\infty} f(x) = \infty$.

Hint. Take $f(x) = -x$.

Answer. False.

78. If $f(x)$ has an absolute minimum at $x = c$, then $f'(c) = 0$.

Hint. Take $f(x) = -|x|$.

Answer. False.

True or False. Give a brief justification for each answer.

79. There is a differentiable function $f(x)$ with the property that $f(1) = -2$ and $f(5) = 14$ and $f'(x) < 3$ for every real number x.

Hint. Use the Mean Value Theorem.

Answer. False.

80. If $f"(5) = 0$ then $(5, f(5))$ is an inflection point of the curve $y = f(x)$.

Hint. Take $y = (x - 5)^4$.

Answer. False.

81. If $f'(c) = 0$ then $f(x)$ has a local maximum or a local minimum at $x = c$.

Hint. Take $f(x) = x^3$, $c = 0$.

Answer. False.

82. If $f(x)$ is a differentiable function and the equation $f'(x) = 0$ has 2 solutions, then the equation $f(x) = 0$ has no more than 3 solutions.

Answer. True. Since f is differentiable, by Rolle's Theorem there is a local extremum between any two isolated solutions of $f(x) = 0$.

83. If $f(x)$ is increasing on $[0, 1]$ then $[f(x)]^2$ is increasing on $[0, 1]$.

Hint. Take $f(x) = x - 1$.

Answer. False.

Answer the following questions TRUE or False.

84. If f has a vertical asymptote at $x = 1$ then $\lim\limits_{x\to 1} f(x) = L$, where L is a finite value.

Answer. False.

85. If has domain $[0, \infty)$ and has no horizontal asymptotes, then $\lim_{x\to\infty} f(x) = \pm\infty$.

Hint. Take $f(x) = \sin x$.

Answer. False.

86. If $g(x) = x^2$ then $\lim\limits_{x\to 2} \dfrac{g(x) - g(2)}{x - 2} = 0$.

Answer. False. $g'(2) = 4$.

87. If $f"(2) = 0$ then $(2, f(2))$ is an inflection point of $f(x)$.

Answer. False.

88. If $f'(c) = 0$ then f has a local extremum at c.

Answer. False.

89. If f has an absolute minimum at c then $f'(c) = 0$.

Hint. Take $f(x) = |x|$.

Answer. False.

90. If $f'(c)$ exists, then $\lim\limits_{x\to c} f(x) = f(c)$.

Answer. True. If if is differentiable at c then f is continuous at c.

91. If $f(1) < 0$ and $f(3) > 0$, then there exists a number $c \in (1, 3)$ such that $f(c) = 0$.

Answer. False. It is not given that f is continuous.

92. If $f'(g) = \dfrac{1}{(3 - g)^2}$, then $f(g)$ is differentiable on $(-\infty, 3) \cup (3, \infty)$.

Answer. True.

93. If $f'(g) = \dfrac{1}{(3 - g)^2}$, the equation of the tangent line to $f(g)$ at $(0, 1/3)$ is $y = \frac{1}{9}g + \frac{1}{3}$.

Answer. True.

Are the following statements true or false?

94. The points described by the polar coordinates $(2, \pi/4)$ and $(-2, 5\pi/4)$ are the same.

Answer. True.

95. If the limit $\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$ does not exist, then the limit $\lim_{x\to\infty} \frac{f(x)}{g(x)}$ does not exist.

Hint. Take functions $f(x) = xe^{-1/x^2}\sin(x^{-4})$ and $g(x) = e^{-1/x^2}$.

Answer. False.

96. If f is a function for which $f"(x) = 0$, then f has an inflection point at x.

Hint. Take $f(x) = x^4$.

Answer. False.

97. If f is continuous at the number x, then it is differentiable at x.

Hint. Take $f(x) = |x|$ and $x = 0$.

Answer. False.

98. Let f be a function and c a number in its domain. The graph of the linear approximation of f at c is the tangent line to the curve $y = f(x)$ at the point $(c, f(c))$.

Answer. True.

99. Every function is either an odd function or an even function.

Hint. Take $f(x) = e^x$.

Answer. False.

100. A function that is continuous on a closed interval attains an absolute maximum value and an absolute minimum value at numbers in that interval.

Answer. True.

101. An ellipse is the set of all points in the plane the sum of whose distances from two fixed points is a constant.

Answer. True.

For each statement indicate whether is True or False.

102. There exists a function g such that $g(1) = -2$, $g(3) = 6$ and $g'(x) > 4$ for all x.

Hint. Use the Mean Value Theorem.

Answer. False.

103. If $f(x)$ is continuous and $f'(2) = 0$ then f has either a local maximum to minimum at $x = 2$.

Hint. Take $f(x) = (x-2)^2$.

Answer. False.

104. If $f(x)$ does not have an absolute maximum on the interval $[a, b]$ then f is not continuous on $[a, b]$.

Answer. True.

105. If a function $f(x)$ has a zero at $x = r$, then Newton's method will find r given an initial guess $x_0 \neq r$ when x_0 is close enough to r.

Hint. Take $f(x) = \sqrt[3]{x}$.

Answer. False.

106. If $f(3) = g(3)$ and $f'(x) = g'(x)$ for all x, then $f(x) = g(x)$.

Answer. True.

107. The function $g(x) = \dfrac{7x^4 - x^3 + 5x^2 + 3}{x^2 + 1}$ has a slant asymptote.

Answer. False.

For each statement indicate whether is True or False.

108. If $\lim_{x \to a} f(x)$ exists then $\lim_{x \to a} \sqrt{f(x)}$ exists.

Hint. Take $f(x) = x^2 - 2$ and $a = 0$.

Answer. False.

109. If $\lim_{x \to 1} f(x) = 0$ and $\lim_{x \to 1} g(x) = 0$ then $\lim_{x \to 1} \dfrac{f(x)}{g(x)}$ does not exist.

Hint. Take $f(x) = g(x) = x$.

Answer. False.

110. $\sin^{-1}\left(\sin\left(\dfrac{7\pi}{3}\right)\right) = \dfrac{7\pi}{3}$.

Answer. False. Recall, $\sin^{-1} x \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$ for all $x \in [-1, 1]$.

111. If $h(3) = 2$ then $\lim_{x \to 3} h(x) = 2$.

Hint. Take $f(x) = \dfrac{1}{x-3}$ if $x \neq 3$ and $f(3) = 2$.

Answer. False.

112. The equation $e^{-x^2} = x$ has a solution on the interval $(0, 1)$.

Answer. True.

113. If $(4, 1)$ is a point on the graph of h then $(4, 0)$ is a point on the graph $f \circ h$ where $f(x) = 3^x + x - 4$.

Answer. True.

114. If $-x^3 + 3x^2 + 1 \leq g(x) \leq (x-2)^2 + 5$ for $x \geq 0$ then $\lim_{x \to 2} g(x) = 5$.

Answer. True.

115. If $g(x) = \ln x$, then $g(g^{-1}(0)) = 0$.

Answer. True.

5.2 Multiple Choice

5.2.1 Exercises

For each of the following, circle only one answer.

1. If $h(x) = \ln(1 - x^2)$ where $-1 < x < 1$, then $h'(x) =$

A. $\dfrac{1}{1-x^2}$, B. $\dfrac{1}{1+x} - \dfrac{1}{1-x}$, C. $\dfrac{2}{1-x^2}$, D. None of these.

Answer. B.

2. The derivative of $f(x) = x^2 \tan x$ is

A. $2x \sec^2 x$, B. $2x \tan x + x^2 \cot x$, C. $2x \tan x + (x \sec x)^2$, D. None of these.

Answer. C.

3. If $\cosh y = x + x^3 y$, then at the point $(1, 0)$ we have y'

A. 0, B. 3, C. -1, D. Does not exist.

Answer. C.

4. The derivative of $g(x) = e^{\sqrt{x}}$ is

A. $e^{\sqrt{x}}$, B. $\sqrt{x}e^{\sqrt{x}-1}$, C. $\frac{0.5e^{\sqrt{x}}}{\sqrt{x}}$, D. None of these.

Answer. C.

For each of the following, circle only one answer.

5. Suppose $y'' + y = 0$. Which of the following is a possibility for $y = f(x)$.

A. $y = \tan x$, B. $y = \sin x$, C. $y = \sec x$, D. $y = 1/x$, E. $y = e^x$

Answer. B.

6. Which of the following is $\arcsin\left(\sin\left(\frac{3\pi}{4}\right)\right)$?

A. 0, B. $\frac{\pi}{4}$, C. $-\frac{\pi}{4}$, D. $\frac{5\pi}{4}$, E. $\frac{3\pi}{4}$

Answer. C. The range of $y = \arcsin x$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

7. Let $f(x)$ be a continuous function on $[a, b]$ and differentiable on (a, b) such that $f(b) = 10$, $f(a) = 2$. On which of the following intervals $[a, b]$ would the Mean Value Theorem guarantee a $c \in (a, b)$ such that $f'(c) = 4$.

A. $[0, 4]$, B. $[0, 3]$, C. $[2, 4]$, D. $[1, 10]$, E. $(0, \infty)$

Answer. C. $\frac{10-2}{4-2} = 4$.

8. Let $P(t)$ be the function which gives the population as a function of time. Assuming that $P(t)$ satisfies the natural growth equation, and that at some point in time t_0, $P(t_0) = 500$, $P'(t_0) = 1000$, find the growth rate constant k.

A. $-\frac{1}{2}$, B. $\ln\left(\frac{1}{2}\right)$, C. $\frac{1}{2}$, D. 2, E. $\ln 2$

Answer. C. Use $\frac{dP}{dt} = kP$.

9. Suppose that f is continuous on $[a, b]$ and differentiable on (a, b). If $f'(x) > 0$ on (a, b). Which of the following is *necessarily* true?

A. f is decreasing on $[a, b]$,
B. f has no local extrema on (a, b),
C. f is a constant function on (a, b),
D. f is concave up on (a, b),
E. f has no zero on (a, b)

Answer. B. f is increasing.

For each of the following, circle only one answer.

10. The equation $x^5 + 10x + 3 = 0$ has

A. no real roots, B. exactly one real root, C. exactly two real roots, D. exactly three real roots, E. exactly five real roots

Answer. B. Consider $f(x) = x^5 + 10x + 3$ and its first derivative.

11. The value of $\cosh(\ln 3)$ is

A. $\frac{1}{3}$, B. $\frac{1}{2}$, C. $\frac{2}{3}$, D. $\frac{4}{3}$, E. $\frac{5}{3}$

Answer. E. $\cosh(\ln 3) = \frac{3 + \frac{1}{3}}{2}$.

12. The function f has the property that $f(3) = 2$ and $f'(3) = 4$. Using a linear approximation to f near $x = 3$, an approximation to $f(2.9)$ is

A. 1.4, B. 1.6, C. 1.8, D. 1.9, E. 2.4

Answer. B. $f(2.9) \approx 2 + 4(2.9 - 3)$.

13. Suppose F is an antiderivative of $f(x) = \sqrt[3]{x}$. If $F(0) = \dfrac{1}{4}$, then $F(1)$ is

A. -1, B. $-\dfrac{3}{4}$, C. 0, D. $\dfrac{3}{4}$, E. 1

Answer. E. $F(x) = \frac{3}{4}x^{\frac{4}{3}} + \frac{1}{4}$.

14. Suppose f is a function such that $f'(x) = 4x^3$ and $f"(x) = 12x^2$. Which of the following is true?

A. f has a local maximum at $x = 0$ by the first derivative test
B. f has a local minimum at $x = 0$ by the first derivative test.
C. f has a local maximum at $x = 0$ by the second derivative test.
D. f has a local minimum at $x = 0$ by the second derivative test.
E. f has an inflection point at $x = 0$

Answer. B.

Circle clearly your answer to the following 10 multiple choice question.

15. Evaluate $\dfrac{d}{dx}\sin(x^2)$

A. $2x\cos(x^2)$, B. $2x\sin(x^2)$, C. $2x\cos(x)$, D. $2x\cos(2x)$, E. $2x\cos(2x)$

Answer. A.

16. Evaluate $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x}$

A. 0, B. ∞, C. 1, D. -1, E. none of the above

Answer. E. $\lim\limits_{x\to 0^+} \dfrac{\ln x}{x} = -\infty$.

17. Evaluate $\lim\limits_{x\to 0^+} \dfrac{1 - e^x}{\sin x}$

A. 1, B. -1, C. 0, D. ∞, E. $\sin e$

Answer. B. Use L'Hospital's rule.

18. The circle described by the equation $x^2 + y^2 - 2x - 4 = 0$ has center (h, k) and radius r. The values of h, k, and r are

A. 0, 1, and $\sqrt{5}$, B. 1, 0, and 5, C. 1, 0, and $\sqrt{5}$, D. -1, 0, and 5, E. -1, 0, and $\sqrt{5}$

Answer. C. $(x - 1)^2 + y^2 = 5$.

19. The edge of the cube is increasing at a rate of 2 cm/hr. How fast is the cube's volume changing when its edge is $\sqrt{x}$ cm in length?

A. 6 cm^3/hr, B. 12 cm^3/hr, C. $3\sqrt{2}$ cm^3/hr, D. $6\sqrt{2}$ cm^3/hr, E. none of the above

Answer. B. $\dfrac{dV}{dt} = 3x^2\dfrac{dx}{dt}$.

20. Given the polar equation $r = 1$, find $\dfrac{dy}{dx}$

A. $\cot\theta$, B. $-\tan\theta$, C. 0, D. 1, E. $-\cot\theta$

Answer. E. $\dfrac{dy}{dt} = \dfrac{\frac{dy}{d\theta}}{\frac{dy}{d\theta}}$.

21. Let $A(t)$ denote the amount of a certain radioactive material left after time t. Assume that $A(0) = 16$ and $A(1) = 12$. How much time is left after time $t = 3$?

A. $\frac{16}{9}$, B. 8, C. $\frac{9}{4}$, D. $\frac{27}{4}$, E. 4

Answer. D. $A(t) = 16\left(\frac{3}{4}\right)^t$.

22. Which of the following statements is always true for a function $f(x)$?

(i) If $f(x)$ is concave up on the interval (a, b), then $f(x)$ has a local minimum (a, b).

(ii) It is possible for $y = f(x)$ to have an inflection point at $(a, f(a))$ even if $f'(x)$ does not exist at$x = a$.

(iii) It is possible for $(a, f(a))$ to be both a critical point and an inflection point of $f(x)$.

A. i. and ii.
B. only iii.
C. i., ii., and iii.
D. ii. and iii.
E. only i.

Answer. D. For (1) take $f(x) = x^3$ on $(0, 1)$. For (2) take $f(x) = \sqrt[3]{x}$. For (3) take $f(x) = x^4$.

23. Which of the following statements is always true for a function $f(x)$?

(i) If $f(x)$ and $g(x)$ are continuous at $x = a$, then $\frac{f(x)}{g(x)}$ is continuous at $x = a$.

(ii) If $f(x) + g(x)$ is continuous at $x = a$ and $f'(a) = 0$, then $g(x)$ is continuous ta $x = a$.

(iii) If $f(x) + g(x)$ is differentiable at $x = a$, then $f(x)$ and $g(x)$ are both differentiable at $x = a$.

A. only i.
B. only ii.
C. only iii.
D. i. and ii.
E. ii. and iii.

Answer. B. For (1) take $g(x) = 0$. For (3) take $f(x) = |x|$, $g(x) = -|x|$, and $a = 0$.

24. The slant asymptote of the function $f(x) = \frac{x^2 + 3x - 1}{x - 1}$ is
A. $y = x + 4$, B. $y = x + 2$, C. $y = x - 2$, D. $y = x - 4$, E. none of the above

Answer. A.

This is a multiple choice question. No explanation is required.

25. The derivative of $g(x) = e^{\sqrt{x}}$ is
A. $\sqrt{x}e^{\sqrt{x}-1}$,
B. $2e^{\sqrt{x}}x^{-0.5}$,
C. $\frac{0.5e^{\sqrt{x}}}{\sqrt{x}}$,
D. $e^{\sqrt{x}}$,
E. None of these

Answer. C.

26. If $\cosh y = x + x^3 y$, then at the point $(1, 0)$ $y' =$

A. 0,
B. -1,
C. 1,
D. 3,
E. Does not exist

Answer. B. Note $y' \sinh y = 1 + 3x^2 y + x^3 y'$.

27. An antiderivative of $f(x) = x - \sin x + e^x$ is

A. $1 - \cos x + e^x$,
B. $x^2 + \ln x - \cos x$,
C. $0.5x^2 + e^x - \cos x$,
D. $\cos x + e^x + 0.5x^2$,
E. None of these

Answer. D.

28. If $h(x) = \ln(1 - x^2)$ where $-1 < x < 1$, then $h'(x) =$

A. $\dfrac{1}{1 - x^2}$,
B. $\dfrac{1}{1 + x} + \dfrac{1}{1 - x}$,
C. $\dfrac{2}{1 - x^2}$,
D. $\dfrac{1}{1 + x} - \dfrac{1}{1 - x}$,
E. None of these

Answer. D.

29. The linear approximation to $f(x) = \sqrt[3]{x}$ at $x = 8$ is given by

A. 2,
B. $\dfrac{x + 16}{12}$,
C. $\dfrac{1}{3x^{2/3}}$,
D. $\dfrac{x - 2}{3}$,
E. $\sqrt[3]{x} - 2$

Answer. B.

This is a multiple choice question. No explanation is required.

30. If a function f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), then there exists $c \in (a, b)$ such that $f(b) - f(a) = f'(c)(b - a)$ is:

A. The Extreme Value Theorem,
B. The Intermediate Value Theorem,
C. The Mean Value Theorem,
D. Rolle's Theorem,
E. None of these

Answer. C.

31. If f is continuous function on the closed interval $[a, b]$, and N is a number between $f(a)$ and $f(b)$, then there is $c \in [a, b]$ such that $f(c) = N$ is:

A. Fermat's Theorem
B. The Intermediate Value Theorem
C. The Mean Value Theorem
D. Rolle's Theorem

E. The Extreme Value Theorem

Answer. B.

32. If f is continuous function on the open interval (a, b) then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in (a, b)$ is:

A. The Extreme Value Theorem,
B. The Intermediate Value Theorem,
C. The Mean Value Theorem,
D. Rolle's Theorem,
E. None of these

Answer. E.

33. A function f is continuous at a number a ...

A. ... if f is defined at a,
B. ... if $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists,
C. ... if $\lim\limits_{x\to a} f(x)$ exists,
D. ... if f is anti-differentiable at a,
E. ... if $\lim\limits_{x\to a} f(x) = f(a)$

Answer. E.

34. A function f is differentiable at a number a ...

A. ... if $\lim\limits_{x\to a} f(x) = f(a)$,
B. ... if $\lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a}$ exists,
C. ... if f is defined at a,
D. ... if f is continuous at a,
E. ... if we can apply the Intermediate Value Theorem

Answer. B.

35. An antiderivative of a function f ...

A. ... is a function F such that $F'(x) = f(x)$,
B. ... is a function F such that $F(x) = f'(x)$,
C.... is a function F such that $F'(x) = f'(x)$,
D. ... is a function F such that $F(x) = f(x)$,
E. ... is a function F such that $F''(x) = f(x)$

Answer. A.

36. A critical number of a function f is a number c in the domain of f such that ...

A. ... $f'(c) = 0$,
B. ... $f(c)$ is a local extremum,
C. ... either $f'(c) = 0$ or $f'(x)$ is not defined,
D.... $(c, f(c))$ is an inflection point,
E. ... we can apply the Extreme Value Theorem in the neighbourhood of the point $(c, f(c))$

Answer. C.

Answer the following questions. You need not show work for this section.

37. What is the period of $f(x) = \tan x$?

Answer. π.

38. What is the derivative of $f(x) = x \ln |x| - x$?

Answer. $f'(x) = \ln |x|$.

39. If $y = \sin^2 x$ and $\dfrac{dx}{dt} = 4$, find $\dfrac{dy}{dt}$ when $x = \pi$.

Answer. 0. $\frac{dy}{dt} = 2\sin x \cdot \cos x \cdot \frac{dx}{dt}$.

40. What is the most general antiderivative of $f(x) = 2xe^{x^2}$?

Answer. $F(x) = e^{x^2} + C$.

41. Evaluate $\lim_{t\to\infty}(\ln(t+1) - \ln t)$?

Answer. 0. $\lim_{t\to\infty} \ln \frac{t+1}{t}$.

42. Does differentiability imply continuity?

Answer. Yes.

43. Convert the Cartesian equation $x^2 + y^2 = 25$ into a polar equation.

Answer. $r = 5$.

44. Simplify $\cosh^2 x - \sinh^2 x$.

Answer. 1.

Give an example for the each of the following:

45. Function $F = f \cdot g$ so that the limits of F and f at a exist and the limit of g at a does not exist.

Answer. $F = x \cdot \sin \frac{1}{x}$ and $a = 0$.

46. Function that is continuous but not differentiable at a point.

Answer. $f(x) = |x|$.

47. Function with a critical number but no local maximum or minimum.

Answer. $f(x) = x^3$.

48. Function with a local minimum at which its second derivative equals 0.

Answer. $f(x) = x^4$.

Answer the following.

49. State the definition of the derivative of function f at a number a.

Answer. The derivative of function f at a number a, denoted by $f'(a)$, is $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ if this limit exists.

50. State the definition of a critical number of a function.

Answer. A critical number of a function f is a number c in the domain of f such that $f'(c) = 0$ or $f'(c)$ does not exist.

51. State the Extreme Value Theorem.

Answer. If f is continuous on a closed interval $[a, b]$, the f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers c and d in $[a, b]$.

Match the start of each definition/theorem with its conclusion.

(a) ... if f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

(b) ... if f is a function that satisfies the following hypotheses:

(i) f is continuous on the closed interval $[a, b]$.

(ii) f is differentiable on the open interval (a, b).

Then there is a number c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

(c) ... $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$ if this limit exists.

(d) ... If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.

(e) ... is in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

(f) ... on a continuous curve where the curve changes from concave upward to concave downward or from concave downward to concave upward.

(g) ... the base of the exponential function which has a tangent line of slope 1 at $(0, 1)$.

(h) ... If f and g are both differentiable then $\dfrac{d}{dx}[f(g(x))] = f'(g(x)) \cdot g'(x)$.

(i) ... If $f(x) \leq g(x) \leq h(x)$ and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$ then $\lim_{x \to a} g(x) = L$.

52. The *Mean Value Theorem states* that ...

Answer. ii.

53. The *chain rule* states that ...

Answer. viii.

54. A *critical number* is a number that ...

Answer. v.

55. The *Extreme Value Theorem* states that ...

Answer. iv.

56. *Fermat's Theorem* states that

Answer. no match.

57. An *antiderivative* of a function f is ...

Answer. no match.

58. The *natural number* e is ...

Answer. vii.

59. An *inflection point* is a point ...

Answer. vi.

60. The *derivative* of a function f at a number a is ...

Answer. iii.

61. *L'Hospital's Rule* states that ...

Answer. no match.

62. The *Intermediate Value Theorem* states that ...

Answer. i.

63. A function f is continuous at a number a ...

Answer. no match.

64. The Squeeze Theorem states that ...

Answer. ix.

Chapter 6

Recommendations for Success in Mathematics

The following is a list of various categories gathered by the Department of Mathematics. This list is a recommendation to all students who are thinking about their well-being, learning, and goals, and who want to be successful academically.

6.1 Tips for Reading these Recommendations

- Do not be overwhelmed with the size of this list. You may not want to read the whole document at once, but choose some categories that appeal to you.
- You may want to make changes in your habits and study approaches after reading the recommendations. Our advice is to take small steps. Small changes are easier to make, and chances are those changes will stick with you and become part of your habits.
- Take time to reflect on the recommendations. Look at the people in your life you respect and admire for their accomplishments. Do you believe the recommendations are reflected in their accomplishments?

6.2 Habits of a Successful Student

- *Acts responsibly.*

  This student

  - reads the documents (such as course outline) that are passed on by the instructor and acts on them.
  - takes an active role in their education.
  - does not cheat and encourages academic integrity in others.
- *Sets goals.*

  This student

  - sets attainable goals based on specific information such as the academic calendar, academic advisor, etc..

  - is motivated to reach the goals.
  - is committed to becoming successful.
  - understands that their physical, mental, and emotional well-being influences how well they can perform academically.

- *Is reflective.*

  This student

  - understands that deep learning comes out of reflective activities.
  - reflects on their learning by revisiting assignments, midterm exams, and quizzes and comparing them against posted solutions.
  - reflects why certain concepts and knowledge are more readily or less readily acquired.
  - knows what they need to do by having analyzed their successes and their failures.

- *Is inquisitive.*

  This student

  - is active in a course and asks questions that aid their learning and build their knowledge base.
  - seeks out their instructor after a lecture and during office hours to clarify concepts and content and to find out more about the subject area.
  - shows an interest in their program of studies that drives them to do well.

- *Can communicate.*

  This student

  - articulates questions.
  - can speak about the subject matter of their courses, for example by explaining concepts to their friends.
  - takes good notes that pay attention to detail but still give a holistic picture.
  - pays attention to how mathematics is written and attempts to use a similar style in their written work.
  - pays attention to new terminology and uses it in their written and oral work.

- *Enjoys learning.*

  This student

  - is passionate about their program of study.
  - is able to cope with a course they don't like because they see the bigger picture.
  - is a student because they made a positive choice to be one.
  - reviews study notes, textbooks, etc..
  - works through assignments individually at first and way before the due date.
  - does extra problems.

  - reads course related material.

- *Is resourceful.*

  This student

  - uses the resources made available by the course and instructor such as the Math Workshop, the course container on WebCT, course websites, etc..
  - researches how to get help in certain areas by visiting the instructor, or academic advisor, or other support structures offered through the university.
  - uses the library and internet thoughtfully and purposefully to find additional resources for a certain area of study.

- *Is organized.*

  This student

  - adopts a particular method for organizing class notes and extra material that aids their way of thinking and learning.

- *Manages his/her time effectively.*

  This student

  - is in control of their time.
  - makes and follows a schedule that is more than a timetable of course. It includes study time, research time, social time, sports time, etc..

- *Is involved.*

  This student

  - is informed about their program of study and their courses and takes an active role in them.
  - researches how to get help in certain areas by visiting the instructor, or academic advisor, or other support structures offered through the university.
  - joins a study group or uses the support that is being offered such as a Math Workshop (that accompanies many first and second year math courses in the Department of Mathematics) or the general SFU Student Learning Commons Workshops.
  - sees the bigger picture and finds ways to be involved in more than just studies. This student looks for volunteer opportunities, for example as a Teaching Assistant in one of the Mathematics Workshops or with the MSU (Math Student Union).

6.3 How to Prepare for Exams

- Start preparing for an exam on the FIRST DAY OF LECTURES!
- Come to all lectures and listen for where the instructor stresses material or points to classical mistakes. Make a note about these pointers.
- Treat each chapter with equal importance, but distinguish among items within a chapter.

- Study your lecture notes in conjunction with the textbook because it was chosen for a reason.
- Pay particular attention to technical terms from each lecture. Understand them and use them appropriately yourself. The more you use them, the more fluent you will become.
- Pay particular attention to definitions from each lecture. Know the major ones by heart.
- Pay particular attention to theorems from each lecture. Know the major ones by heart.
- Pay particular attention to formulas from each lecture. Know the major ones by heart.
- Create a "cheat sheet" that summarizes terminology, definitions, theorems, and formulas. You should think of a cheat sheet as a very condensed form of lecture notes that organizes the material to aid your understanding. (However, you may not take this sheet into an exam unless the instructor specifically says so.)
- Check your assignments against the posted solutions. Be critical and compare how you wrote up a solution versus the instructor/textbook.
- Read through or even work through the paper assignments, online assignments, and quizzes (if any) a second time.
- Study the examples in your lecture notes in detail. Ask yourself, why they were offered by the instructor.
- Work through some of the examples in your textbook, and compare your solution to the detailed solution offered by the textbook.
- Does your textbook come with a review section for each chapter or grouping of chapters? Make use of it. This may be a good starting point for a cheat sheet. There may also be additional practice questions.
- Practice writing exams by doing old midterm and final exams under the same constraints as a real midterm or final exam: strict time limit, no interruptions, no notes and other aides unless specifically allowed.
- Study how old exams are set up! How many questions are there on average? What would be a topic header for each question? Rate the level of difficulty of each question. Now come up with an exam of your own making and have a study partner do the same. Exchange your created exams, write them, and then discuss the solutions.

6.4 Getting and Staying Connected

- Stay in touch with family and friends:
    - ◦ A network of family and friends can provide security, stability, support, encouragement, and wisdom.
    - ◦ This network may consist of people that live nearby or far away. Technology — in the form of cell phones, email, facebook, etc. — is allowing us to stay connected no matter where we are. However, it is up to us at times to reach out and stay connected.

  - Do not be afraid to talk about your accomplishments and difficulties with people that are close to you and you feel safe with, to get different perspectives.

- Create a study group or join one:
  - Both the person being explained to and the person doing the explaining benefit from this learning exchange.
  - Study partners are great resources! They can provide you with notes and important information if you miss a class. They may have found a great book, website, or other resource for your studies.

- Go to your faculty or department and find out what student groups there are:
  - The Math Student Union (MSU) seeks and promotes student interests within the Department of Mathematics at Simon Fraser University and the Simon Fraser Student Society. In addition to open meetings, MSU holds several social events throughout the term. This is a great place to find like-minded people and to get connected within mathematics.
  - Student groups or unions may also provide you with connections after you complete your program and are seeking either employment or further areas of study.

- Go to your faculty or department and find out what undergraduate outreach programs there are:
  - There is an organized group in the Department of Mathematics led by Dr. Jonathan Jedwab that prepares for the William Lowell Putnam Mathematical Competition held annually the first Saturday in December: http://www.math.sfu.ca/ ugrad/putnam.shtml
  - You can apply to become an undergraduate research assistant in the Department of Mathematics, and (subject to eligibility) apply for an NSERC USRA (Undergraduate Student Research Award): http://www.math.sfu.ca/ugrad/ awards/nsercsu.shtml
  - You can attend the Math: Outside the Box series which is an undergraduate seminar that presents on all sorts of topics concerning mathematics.

6.5 Staying Healthy

- A healthy mind, body, and soul promote success. Create a healthy lifestyle by taking an active role in this lifelong process.

- Mentally:
  - Feed your intellectual hunger! Choose a program of study that suits your talents and interests. You may want to get help by visiting with an academic advisor: math_advice@sfu.ca.
  - Take breaks from studying! This clears your mind and energizes you.

- Physically:

  - Eat well! Have regular meals and make them nutritious.
  - Exercise! You may want to get involved in a recreational sport.
  - Get out rain or shine! Your body needs sunshine to produce vitamin D, which is important for healthy bones.
  - Sleep well! Have a bed time routine that will relax you so that you get good sleep. Get enough sleep so that you are energized.

- Socially:

  - Make friends! Friends are good for listening, help you to study, and make you feel connected.
  - Get involved! Join a university club or student union.

6.6 Resources

- Old exams for courses serviced through a workshop that are maintained by the Department of Mathematics: Old Exams
- WolframAlpha Computational Knowledge Engine: Wolfram Alpha
- Survival Guide to 1st Year Mathematics at SFU: First year mathematics guides
- Problem Solving and Quantitative Courses: Resources to help you with problem-solving and quantitative (Q) courses
- SFU Student Learning Commons: Student Learning Commons: Home
- SFU Student Success Programs: Four efficient and effective methods for studying
- SFU Writing for University: Services and resources that support academic writing
- SFU Health & Counselling Services: Services and resources that support academic writing

Bibliography

[**1**] J. Stewart. \textbf{Calculus, Early Transcendentals}, 6th Edition, Thomson, 2008. \end{thebibliography} IIIII \addcontentsline{toc}{chapter}{Index}