Chapter 6

Vectors

6.1 Introduction

Definition 6.1. *A* vector *is a quantity with both a* <u>*magnitude*</u> *(size) and* <u>*direction*</u>*.*

Many quantities in engineering applications can be described by vectors, e.g. force, velocity, magnetic field.

They can be represented by arrows, for example...

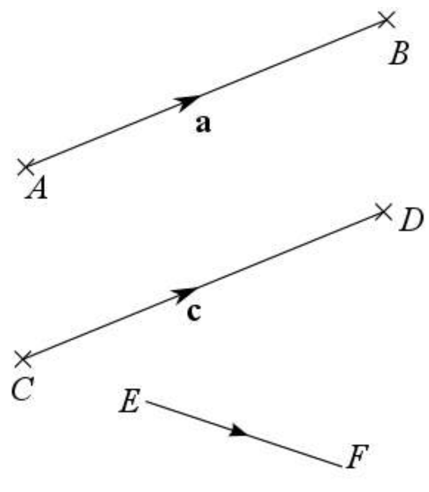


Figure 6.1: Some vectors.

Magnitude=Length of AB

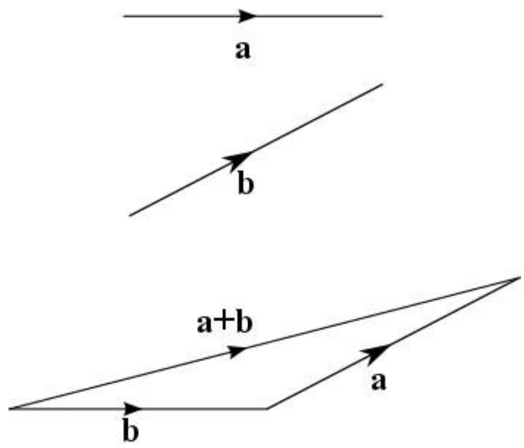
Direction is shown in the Figure 6.1.

We will write $\overrightarrow{AB}$ or $\mathbf{a}$ to represent the top vector in the figure.

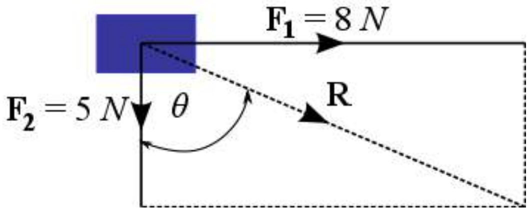
Two vectors are equal when they have both the same magnitude and direction. So $\overrightarrow{AB} = \overrightarrow{CD}$.

But $\overrightarrow{AB} \neq \overrightarrow{EF}$, since both the magnitude and direction are different.

The sum of two vectors **a** and **b** is found by adding the vectors "head to tail":



Example 6.1 (Forces on an object)**.** Consider the following forces acting on an object:



Forces add to give a net effect or resultant force.

$$\mathbf{R} = \mathbf{F_1} + \mathbf{F_2}$$

$$\text{Magnitude: } |\mathbf{R}| = \sqrt{8^2 + 5^2} \approx 9.4\text{N}.$$

$$\text{Direction: Use } \tan\theta = \frac{|\mathbf{F_1}|}{|\mathbf{F_2}|} = \frac{8}{5} = 1.6$$
$$\Rightarrow \theta = 58°.$$

You can multiply a vector **a** by a scalar (number) k. Then, as shown in Figure 6.2, if $k > 0$, $k\mathbf{a}$ is a vector in the same direction as **a**, and the magnitude is $k|\mathbf{a}|$... BUT if $k < 0$, $k\mathbf{a}$ is in the opposite direction!

Example 6.2. Two points A and B have position vectors (i.e. relative to a fixed origin O) **a** and **b** respectively. What is the position vector of a point on the line joining A and B, equidistant from A and B?

Well, the first thing we need is a sketch of the problem, like in Figure 6.3.

Next, note that $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$.

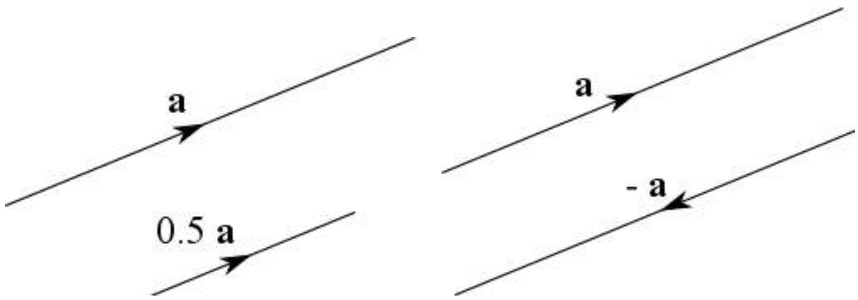


Figure 6.2: Two examples of scalar multiplication of the vector **a**.

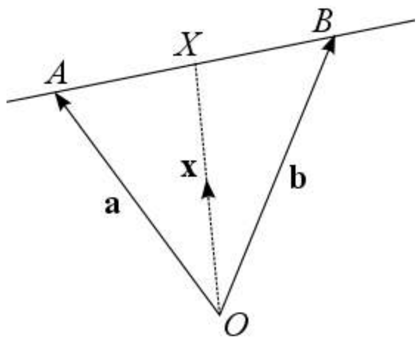


Figure 6.3: In this sketch, X is the midpoint of the line joining A and B

$$\begin{aligned}\mathbf{x} &= \mathbf{a} + \overrightarrow{AX} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB} \\ &= \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) \\ &= \frac{1}{2}(\mathbf{a} + \mathbf{b}).\end{aligned}$$

Definition 6.2. *A* unit vector *is a vector with magnitude 1.*

Often represented using a hat symbol:

For any vector **a**,

$$\hat{\mathbf{a}} = \frac{\mathbf{a}}{|\mathbf{a}|} \quad \text{is a unit vector since}$$

$$|\hat{\mathbf{a}}| = \left|\frac{\mathbf{a}}{|\mathbf{a}|}\right| = \frac{|\mathbf{a}|}{|\mathbf{a}|} = 1.$$

Unit vectors in the x, y, z idrections are denoted **i**, **j**, **k** respectively.

Then the position of a point P from the origin, with coordinates (x, y, z), is

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}.$$

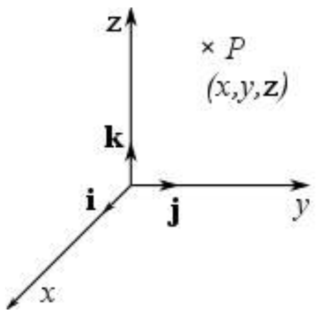


Figure 6.4: ijk

Example 6.3.

$$\mathbf{a} = 6\mathbf{i} - 3\mathbf{j} + \mathbf{k},$$
$$\mathbf{b} = 4\mathbf{i} + 2\mathbf{j}.$$

Then

$$\mathbf{a} + \mathbf{b} = 10\mathbf{i} - \mathbf{j} - \mathbf{k}$$
$$\mathbf{b} - \mathbf{a} = -2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$$
$$3\mathbf{a} = 18\mathbf{i} - 9\mathbf{j} + 3\mathbf{k}.$$

For a position vector $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, the magnitude is

$$|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2}.$$

Then for the previous example,

$$|\mathbf{a}| = \sqrt{6^2 + (-3)^2 + 1^2} = \sqrt{46},$$
$$|\mathbf{b}| = \sqrt{4^2 + 2^2 + 0^2} = 2\sqrt{5}.$$

So far we've seen how to add two vectors. Now we have a question. . .

Q: How can we multiply two vectors together?

I'm going to show you that there are in fact two ways to multiply vectors. . .

6.2 The Dot Product

Let us consider the origin of the dot product:

We take two vectors **a** and **b**:

We might be interested in the length of the component of **a** which is in the same direction as **b**.

Here $0 \leq \theta < \pi$ is the angle between **a** and **b**.

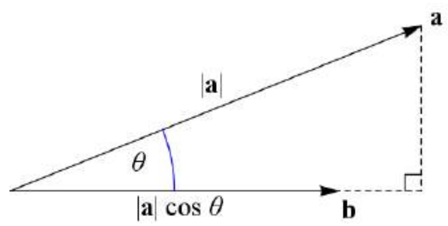


Figure 6.5: The two vectors **a** *and* **b**. *We see that the length of the component of* **a** *which is in the same direction as* **b** *is* $|\mathbf{a}|\cos\theta$.

Compare with the *dot product* formula:

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

Looks almost like the length of the component of **a**, but is rescaled such that we have the symmetry:

$$\mathbf{a}\cdot\mathbf{b} = \mathbf{b}\cdot\mathbf{a}$$

So the dot product also gives us a rescaling of the length of the component of **b** in the same direction as **a**. But we expected that in the first place, because of the above symmetry rule!

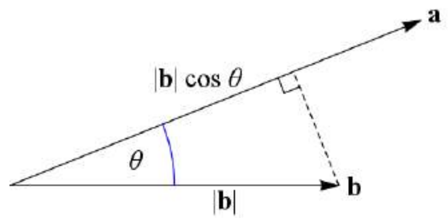


Figure 6.6: This time, we would like the length of the component of **b** *which is in the same direction as* **a**. *That length is* $|\mathbf{b}|\cos\theta$.

Note that

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta \quad\Rightarrow\quad \cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}||\mathbf{b}|};$$

which is a useful method for calculating θ if you know **a** and **b**.

Two non-zero vectors are perpendicular (orthogonal) if and only if their dot product is zero, i.e.

$$\begin{aligned}\mathbf{a}.\mathbf{b} = 0 \quad &\Rightarrow \quad |\mathbf{a}||\mathbf{b}|\cos\theta = 0\\ &\Rightarrow \quad \cos\theta = 0\\ &\Rightarrow \quad \theta = \frac{\pi}{2} \quad (90^\circ)\end{aligned}$$

Now consider **i**, **j**, **k**. These are unit vectors, and are mutually perpendicular. These two facts combined show that, e.g.

$$\mathbf{i}\cdot\mathbf{i} = 1, \quad \mathbf{i}\cdot\mathbf{j} = 0, \quad \text{etc.},$$

so if you then let

$$\begin{aligned} \mathbf{a} &= (a_1, a_2, a_3) && (= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}) \\ \mathbf{b} &= (b_1, b_2, b_3) && (= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}), \end{aligned}$$

and multiply out $\mathbf{a} \cdot \mathbf{b}$, you obtain

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3.$$

Note:

$$\begin{aligned} \mathbf{a} \cdot \mathbf{a} =& |\mathbf{a}||\mathbf{a}| \cos 0 = |\mathbf{a}|^2 \\ \text{i.e.} \quad & |\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}. \end{aligned}$$

Let's try this with $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + x\mathbf{k}$. Then:

$$|\mathbf{r}| = \sqrt{\mathbf{r} \cdot \mathbf{r}} = \sqrt{x^2 + y^2 + z^2},$$

which is consistent with the earlier formula for the magnitude of $\mathbf{r}$.

Example 6.4. For

$$\begin{aligned} \mathbf{a} &= 6\mathbf{i} - 3\mathbf{j} + \mathbf{k} \\ \mathbf{b} &= 4\mathbf{i} + 2\mathbf{j}, \end{aligned}$$

calculate $\mathbf{a} \cdot \mathbf{b}$ and find the angle between the two vectors.

$$\mathbf{a}.\mathbf{b} = 6 \times 4 + (-3) \times 2 + 1 \times (0) = 18.$$

But recall

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta,$$

and that

$$|\mathbf{a}| = \sqrt{46}, \quad |\mathbf{b}| = 2\sqrt{5},$$

therefore

$$\begin{aligned} \cos \theta &= \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{18}{2\sqrt{5}\sqrt{46}} = 0.593. \\ \therefore \quad \theta &= \cos^{-1}(0.593) = 53.6°. \end{aligned}$$

Example 6.5. Points A, B and C have coordinates $(3, 2), (4, -3), (7, -5)$ respectively.

i Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

ii Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$.

iii Deduce the angle between $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

i

$$\begin{aligned} \overrightarrow{AB} &= (4\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = \mathbf{i} - 5\mathbf{j}, \\ \overrightarrow{AC} &= (7\mathbf{i} - 5\mathbf{j}) - (3\mathbf{i} + 2\mathbf{j}) = 4\mathbf{i} - 7\mathbf{j}. \end{aligned}$$

ii Now for the dot product:

$$\overrightarrow{AB}\cdot\overrightarrow{AC} = 4 \times 1 + (-5) \times (-7) = 4 + 35 = 39.$$

iii To calculate the angle, note that

$$|\overrightarrow{AB}| = \sqrt{1^2 + (-5)^2} = \sqrt{26},$$
$$|\overrightarrow{AC}| = \sqrt{4^2 + (-7)^2} = \sqrt{65}.$$

Then

$$\cos\theta = \frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{|\overrightarrow{AB}||\overrightarrow{AC}|} = \frac{39}{\sqrt{26}\sqrt{65}} = 0.949 \quad (3\,\text{d.p.}),$$

which gives $\theta = 18°$.

So far, we have seen one way to multiply two vectors together. However, that first way, the dot product, spits out a number. It would be nice if there was a way to multiply two vectors together such that the result is another vector (Guess what? There is one!)

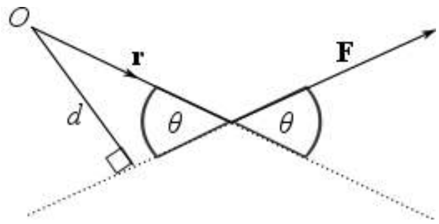
6.3 The Cross Product

Take any two vectors $\mathbf{a}$ and $\mathbf{b}$. Then the *cross product* is denoted as

$$\mathbf{a} \times \mathbf{b}.$$

Before giving the definition, let's consider the motivation behind it using a physics context...

Example 6.6 (Moments)**.** Consider a seesaw. If I apply a force on it at some point away from the pivot, it will turn. Also, if the force is applied farther away from the pivot, the seesaw will turn more easily.



$$\mathbf{r} = \text{Position where the force is exerted}$$
$$\mathbf{F} = \text{The force applied,}$$

then the moment of $\mathbf{F}$ about a point O is

$$m = |\mathbf{F}|d,$$

where

$$d = |\mathbf{r}|\sin\theta$$

is the perpendicular distance between O and the line of action of $\mathbf{F}$.

$$\therefore \quad m = |\mathbf{r}||\mathbf{F}|\sin\theta.$$

In fact, the moment <u>vector</u> of $\mathbf{F}$ about O, i.e. $\mathbf{m}$, is

$$\mathbf{m} = \mathbf{r} \times \mathbf{F},$$

which is perpendicular to both $\mathbf{r}$ and $\mathbf{F}$. Moreover, $\mathbf{m}$ points in the same direction as the axis of rotation for the seesaw (here, $\mathbf{m}$ points out of the page).

Now, $m = |\mathbf{m}|$, hence the magnitude of $\mathbf{m}$ is:

$$|\mathbf{m}| = |\mathbf{r}||\mathbf{F}|\sin\theta.$$

Okay, now I can define the vector product:

Definition 6.3. *The* cross product *of two vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is*

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}|\sin\theta\,\hat{\mathbf{n}},$$

which is a VECTOR, not a NUMBER. So try not to confuse this with the dot product.

$$\text{Length of } \mathbf{a} \times \mathbf{b}: \quad |\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin\theta.$$
$$\text{Direction of } \mathbf{a} \times \mathbf{b}: \quad \hat{\mathbf{n}}, \text{ found using the right hand rule.}$$

$\hat{\mathbf{n}}$ *is a unit vector perpendicular to* $\mathbf{a}$ *and* $\mathbf{b}$*.*

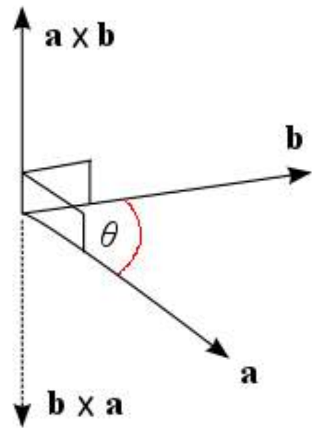


Figure 6.7: The vectors $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{a} \times \mathbf{b}$*. If you put your thumb on* $\mathbf{a}$ *and your index finger on* $\mathbf{b}$*, then your middle finger will tell you the direction of* $\mathbf{a} \times \mathbf{b}$*.*

This definition only works for 3D vectors!

Q: Now, does $\mathbf{a} \times \mathbf{b} = \mathbf{b} \times \mathbf{a}$?

A: NO!

To see this, let $\mathbf{v} = \mathbf{a} \times \mathbf{b}$ and $\mathbf{w} = \mathbf{b} \times \mathbf{a}$. By definition, we will have that $|\mathbf{v}| = |v|$, but what about their directions? Well, the right hand rule shows us that $\mathbf{v} = -\mathbf{w}$. Hence

$$\mathbf{b} \times \mathbf{a} \neq \mathbf{a} \times \mathbf{b}!$$

Suppose we have any two vectors $\mathbf{a}$ and $\mathbf{b}$. If:

$$\begin{aligned} \mathbf{a} &= a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k} && = (a_1, a_2, a_3) \\ \mathbf{b} &= b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k} && = (b_1, b_2, b_3), \end{aligned}$$

then the three components of $\mathbf{a} \times \mathbf{b}$ are:

$$\mathbf{a} \times \mathbf{b} = (a_2b_3 - a_3b_2)\mathbf{i} + (a_3b_1 - a_1b_3)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}.$$

This can be conveniently represented using a 3×3 matrix determinant:

$$\mathbf{a} \times \mathbf{b} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ a_1 & a_2 & a_3 & a_1 & a_2 \\ b_1 & b_2 & b_3 & b_1 & b_2. \end{array} \right.$$

A trick to calculate the determinant is to multiply along each of the six diagonal lines. Next, add all the products corresponding to the green diagonals, and then subtract all the products for the red diagonals. In other words,

$$\text{Determinant} = \text{Sum of the green products} - \text{Sum of red products}.$$

Example 6.7. Compute $\mathbf{a} \times \mathbf{b}$, where

$$\begin{aligned} \mathbf{a} &= 4\mathbf{i} - \mathbf{k} \\ \mathbf{b} &= -2\mathbf{i} + \mathbf{j} + 3\mathbf{k} \end{aligned}$$

$$\mathbf{a} \times \mathbf{b} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 4 & 0 & -1 & 4 & 0 \\ -2 & 1 & 3 & -2 & 1 \end{array} \right.$$

$$\begin{aligned} &= 0\mathbf{i} + 2\mathbf{j} + 4\mathbf{k} - 0\mathbf{k} - (-\mathbf{i}) - 12\mathbf{j} \\ &= \mathbf{i} - 10\mathbf{j} + 4\mathbf{k}. \end{aligned}$$

Example 6.8. Show that $\mathbf{i} \times \mathbf{j} = \mathbf{k}$.

$$\mathbf{i} \times \mathbf{j} = \left| \begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{array} \right.$$

$$\begin{aligned} &= 0\mathbf{i} + 0\mathbf{j} + 1\mathbf{k} - 0\mathbf{i} - 0\mathbf{j} - 0\mathbf{k} \\ &= \mathbf{k}. \end{aligned}$$

Remark 6.1. *A nice interpretation of the length* $|\mathbf{a} \times \mathbf{b}|$ *is that if* θ *is the angle between* $\mathbf{a}$ *and* $\mathbf{b}$*, then this is the area of the parallelogram with sides* $\mathbf{a}$ *and* $\mathbf{b}$*, i.e.*

$$A = \underbrace{|\mathbf{a}|}_{\textit{Base length}} \underbrace{|\mathbf{b}|\sin\theta}_{\textit{Height}}$$

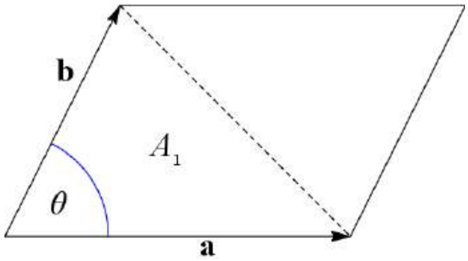


Figure 6.8: A paralellogram, whose sides correspond to vectors **a** *and* **b**. *It can be split into two triangles.*

Proof:

$$A = 2A_1,$$

but

$$\begin{aligned} A_1 &= \frac{1}{2}|\mathbf{a}||\mathbf{b}|\sin\theta, \quad \text{[Anyone recognise this trigonometric formula?]} \\ &= \frac{1}{2}|\mathbf{a}\times\mathbf{b}|, \end{aligned}$$

hence

$$A = |\mathbf{a}\times\mathbf{b}|.$$

□

Example 6.9 (Recycled exam question!)**.** Find the area of a triangle with adjacent sides given by

$$\begin{aligned} \mathbf{a} &= \mathbf{i} + 2\mathbf{j} - \mathbf{k} \\ \mathbf{b} &= \mathbf{j} + \mathbf{k}. \end{aligned}$$

Note that

$$\mathbf{i}\times\mathbf{j} = \left|\begin{array}{ccc|cc} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{i} & \mathbf{j} \\ 1 & 2 & -1 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \end{array}\right.$$

$$\begin{aligned} &= 2\mathbf{i} + 0\mathbf{j} + \mathbf{k} - (-\mathbf{i}) - \mathbf{j} - 0\mathbf{k} \\ &= 3\mathbf{i} - \mathbf{j} + \mathbf{k}. \end{aligned}$$

We want the area of the shaded region A, but

$$\begin{aligned} |\mathbf{a}\times\mathbf{b}| &= 2A \\ \Rightarrow A &= \frac{1}{2}|\mathbf{a}\times\mathbf{b}| \\ &= \frac{1}{2}\sqrt{3^2 + (-1)^2 + 1^2} \\ &= \frac{1}{2}\sqrt{11}. \end{aligned}$$