Higher Order Linear Differential Equations

Math 240 — Calculus III

Summer 2015, Session II

Tuesday, July 28, 2015

Agenda

1. Linear differential equations of order n
   - Linear differential operators
   - Familiar stuff
   - An example

2. Homogeneous constant-coefficient linear differential equations

Introduction

We now turn our attention to solving **linear differential equations of order** n. The general form of such an equation is

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0, a_1, \ldots, a_n,$ and F are functions defined on an interval I.

The general strategy is to reformulate the above equation as

$$Ly = F,$$

where L is an appropriate linear transformation. In fact, L will be a *linear differential operator*.

Linear differential operators

Recall that the mapping $D : C^k(I) \to C^{k-1}(I)$ defined by $D(f) = f'$ is a linear transformation. This D is called the **derivative operator.** Higher order derivative operators $D^k : C^k(I) \to C^0(I)$ are defined by composition:

$$D^k = D \circ D^{k-1},$$

so that

$$D^k(f) = \frac{d^k f}{dx^k}.$$

A **linear differential operator of order** n is a linear combination of derivative operators of order up to n,

$$L = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n,$$

defined by

$$Ly = y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y,$$

where the a_i are continous functions of x. L is then a linear transformation $L : C^n(I) \to C^0(I)$. (Why?)

Examples

Example

If $L = D^2 + 4xD - 3x$, then

$$Ly = y'' + 4xy' - 3xy.$$

We have

$$\begin{aligned} L(\sin x) &= -\sin x + 4x\cos x - 3x\sin x, \\ L\left(x^2\right) &= 2 + 8x^2 - 3x^3. \end{aligned}$$

Example

If $L = D^2 - e^{3x}D$, determine

1. $L\left(2x - 3e^{2x}\right) = -12e^{2x} - 2e^{3x} + 6e^{5x}$
2. $L\left(3\sin^2 x\right) = -3e^{3x}\sin 2x - 6\cos 2x$

Homogeneous and nonhomogeneous equations

Consider the general n-th order linear differential equation

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

where $a_0 \neq 0$ and $a_0, a_1, \ldots, a_n,$ and F are functions on an interval I.

If $a_0(x)$ is nonzero on I, then we may divide by it and relabel, obtaining

$$y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_{n-1}(x)y' + a_n(x)y = F(x),$$

which we rewrite as

$$Ly = F(x),$$

where $L = D^n + a_1 D^{n-1} + \cdots + a_{n-1}D + a_n$.

If $F(x)$ is identically zero on I, then the equation is **homogeneous**, otherwise it is **nonhomogeneous**.

The general solution

If we have a homogeneous linear differential equation

$$Ly = 0,$$

its solution set will coincide with $\operatorname{Ker}(L)$. In particular, the kernel of a linear transformation is a subspace of its domain.

Theorem

The set of solutions to a linear differential equation of order n is a subspace of $C^n(I)$. It is called the **solution space**. *The dimension of the solutions space is n.*

Being a vector space, the solution space has a basis $\{y_1(x), y_2(x), \ldots, y_n(x)\}$ consisting of n solutions. Any element of the vector space can be written as a linear combination of basis vectors

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x).$$

This expression is called the **general solution**.

The Wronskian

We can use the Wronskian

$$W[y_1, y_2, \ldots, y_n](x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}$$

to determine whether a set of solutions is linearly independent.

Theorem

Let $y_1, y_2, \ldots, y_n$ be solutions to the n-th order differential equation $Ly = 0$ whose coefficients are continuous on I. If $W[y_1, y_2, \ldots, y_n](x) = 0$ at any single point $x \in I$, then $\{y_1, y_2, \ldots, y_n\}$ is linearly dependent.

To summarize, the vanishing or nonvanishing of the Wronskian on an interval *completely characterizes* the linear dependence or independence of a set of solutions to $Ly = 0$.

The Wronskian

Example

Verify that $y_1(x) = \cos 2x$ and $y_2(x) = 3 - 6\sin^2 x$ are solutions to the differential equation $y'' + 4y = 0$ on $(-\infty, \infty)$.

Determine whether they are linearly independent on this interval.

$$W[y_1, y_2](x) = \begin{vmatrix} \cos 2x & 3 - 6\sin^2 x \\ -2\sin 2x & -12\sin x\cos x \end{vmatrix}$$
$$= -6\sin 2x\cos 2x + 6\sin 2x\cos 2x = 0$$

They are linearly dependent. In fact, $3y_1 - y_2 = 0$.

Nonhomogeneous equations

Consider the nonhomogeneous linear differential equation $Ly = F$. The **associated homogeneous equation** is $Ly = 0$.

Theorem

Suppose $\{y_1, y_2, \ldots, y_n\}$ *are* n *linearly independent solutions to the* n*-th order equation* $Ly = 0$ *on an interval* I, *and* $y = y_p$ *is any particular solution to* $Ly = F$ *on* I. *Then every solution to* $Ly = F$ *on* I *is of the form*

$$\begin{aligned} y &= \underbrace{c_1y_1 + c_2y_2 + \cdots + c_ny_n}_{y_c} + y_p, \\ &= \qquad\qquad y_c \qquad\qquad + y_p \end{aligned}$$

for appropriate constants $c_1, c_2, \ldots, c_n$.

This expression is the **general solution** to $Ly = F$. The components of the general solution are

- the **complementary function**, y_c, which is the general solution to the associated homogeneous equation,
- the **particular solution**, y_p.

Something slightly new

Theorem

If $y = u_p$ and $y = v_p$ are particular solutions to $Ly = f(x)$ and $Ly = g(x)$, respectively, then $y = u_p + v_p$ is a solution to $Ly = f(x) + g(x)$.

Proof.

We have $L(u_p + v_p) = L(u_p) + L(v_p) = f(x) + g(x)$. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

An example

Example

Determine all solutions to the differential equation $y'' + y' - 6y = 0$ of the form $y(x) = e^{rx}$, where r is a constant.

Substituting $y(x) = e^{rx}$ into the equation yields

$$e^{rx}(r^2 + r - 6) = r^2 e^{rx} + r e^{rx} - 6e^{rx} = 0.$$

Since $e^{rx} \neq 0$, we just need $(r+3)(r-2) = 0$. Hence, the two solutions of this form are

$$y_1(x) = e^{2x} \quad \text{and} \quad y_2(x) = e^{-3x}.$$

Could this be a basis for the solution space? Check linear independence. Yes! The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x}.$$

An example

Example

Determine the general solution to the differential equation

$$y'' + y' - 6y = 8e^{5x}.$$

We know the complementary function,

$$y_c(x) = c_1 e^{2x} + c_2 e^{-3x}.$$

For the particular solution, we might guess something of the form $y_p(x) = ce^{5x}$. What should c be? We want

$$8e^{5x} = y_p'' + y_p' - 6y_p = (25c + 5c - 6c)e^{5x}.$$

Cancel e^{5x} and then solve $8 = 24c$ to find $c = \frac{1}{3}$.

The general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-3x} + \frac{1}{3} e^{5x}.$$

Introduction

We just found solutions to the linear differential equation

$$y'' + y' - 6y = 0$$

of the form $y(x) = e^{rx}$. In fact, we found all solutions.
This technique will often work. If $y(x) = e^{rx}$ then

$$y'(x) = re^{rx}, \quad y''(x) = r^2e^{rx}, \quad \ldots, \quad y^{(n)}(x) = r^ne^{rx}.$$

So if $r^n + a_1r^{n-1} + \cdots + a_{n-1}r + a_n = 0$ then $y(x) = e^{rx}$ is a solution to the linear differential equation

$$y^{(n)} + a_1y^{(n-1)} + \cdots + a_{n-1}y' + a_ny = 0.$$

Let's develop this approach more rigorously.

The auxiliary polynomial

Consider the homogeneous linear differential equation

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0$$

with *constant coefficients* a_i. Expressed as a linear differential operator, the equation is $P(D)y = 0$, where

$$P(D) = D^n + a_1 D^{n-1} + \cdots + a_{n-1} D + a_n.$$

Definition

A linear differential operator with constant coefficients, such as $P(D)$, is called a **polynomial differential operator**. The polynomial

$$P(r) = r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n$$

is called the **auxiliary polynomial**, and the equation $P(r) = 0$ the **auxiliary equation**.

The auxiliary polynomial

Example

The equation $y'' + y' - 6y = 0$ has auxiliary polynomial

$$P(r) = r^2 + r - 6.$$

Examples

Give the auxiliary polynomials for the following equations.

1. $y'' + 2y' - 3y = 0$ $\qquad r^2 + 2r - 3$
2. $(D^2 - 7D + 24)y = 0$ $\qquad r^2 - 7r + 24$
3. $y''' - 2y'' - 4y' + 8y = 0$ $\qquad r^3 - 2r^2 - 4r + 8$

The roots of the auxiliary polynomial will determine the solutions to the differential equation.

Polynomial differential operators commute

The key fact that will allow us to solve constant-coefficient linear differential equations is that polynomial differential operators commute.

Theorem

If $P(D)$ and $Q(D)$ are polynomial differential operators, then

$$P(D)Q(D) = Q(D)P(D).$$

Proof.

For our purposes, it will suffice to consider the case where P and Q are linear. $\mathcal{Q.E.D.}$

Commuting polynomial differential operators will allow us to turn a root of the auxiliary polynomial into a solution to the corresponding differential equation.

Linear polynomial differential operators

In our example,

$$y'' + y' - 6y = 0,$$

with auxiliary polynomial

$$P(r) = r^2 + r - 6,$$

the roots of $P(r)$ are $r = 2$ and $r = -3$. An equivalent statement is that $r - 2$ and $r + 3$ are linear factors of $P(r)$.

The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions to

$$y_1' - 2y_1 = 0 \quad \text{and} \quad y_2' + 3y_2 = 0,$$

respectively.

Theorem

The general solution to the linear differential equation

$$y' - ay = 0$$

is $y(x) = ce^{ax}$.

Theorem

Suppose $P(D)$ and $Q(D)$ are polynomial differential operators

$$P(D)y_1 = 0 = Q(D)y_2.$$

If $L = P(D)Q(D)$, then

$$Ly_1 = 0 = Ly_2.$$

Proof.

$$P(D)Q(D)y_2 = P(D)\Big(Q(D)y_2\Big) = P(D)0 = 0$$

$$\begin{aligned} P(D)Q(D)y_1 &= Q(D)P(D)y_1 \\ &= Q(D)\Big(P(D)y_1\Big) = Q(D)0 = 0 \end{aligned} \qquad \mathcal{Q.E.D.}$$

Example

The theorem implies that, since

$$(D-2)y_1 = 0 \quad \text{and} \quad (D+3)y_2 = 0,$$

the functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$ are solutions to

$$y'' + y' - 6y = (D^2 + D - 6)y = (D-2)(D+3)y = 0.$$

Linear polynomial differential operators

Furthermore, solutions produced from different roots of the auxiliary polynomial are independent.

Example

If $y_1(x) = e^{2x}$ and $y_2(x) = e^{-3x}$, then

$$W[y_1, y_2](x) = \begin{vmatrix} e^{2x} & e^{-3x} \\ 2e^{2x} & -3e^{-3x} \end{vmatrix}$$

$$= e^{-x} \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -5e^{-x} \neq 0.$$

Distinct linear factors

If we can factor the auxiliary polynomial into distinct linear factors, then the solutions from each linear factor will combine to form a fundamental set of solutions.

Example

Determine the general solution to $y'' - y' - 2y = 0$.

The auxiliary polynomial is

$$P(r) = r^2 - r - 2 = (r-2)(r+1).$$

Its roots are $r_1 = 2$ and $r_2 = -1$. The functions $y_1(x) = e^{2x}$ and $y_2(x) = e^{-x}$ satisfy

$$(D-2)y_1 = 0 = (D+1)y_2.$$

Therefore, y_1 and y_2 are solutions to the original equation. Since we have 2 solutions to a 2^{nd} degree equation, they constitute a fundamental set of solutions; the general solution is

$$y(x) = c_1 e^{2x} + c_2 e^{-x}.$$

Multiple roots

What can go wrong with this process? The auxiliary polynomial could have a multiple root. In this case, we would get one solution from that root, but not enough to form the general solution. Fortunately, there are more.

Theorem

The differential equation $(D-r)^m y = 0$ has the following m linearly independent solutions:

$$e^{rx},\ xe^{rx},\ x^2e^{rx},\ \ldots,\ x^{m-1}e^{rx}.$$

Proof.

Check it. $\mathcal{Q}.\mathcal{E}.\mathcal{D}.$

Multiple roots

Example

Determine the general solution to $y'' + 4y' + 4y = 0$.

1. The auxiliary polynomial is $r^2 + 4r + 4$.
2. It has the multiple root $r = -2$.
3. Therefore, two linearly independent solutions are
$$y_1(x) = e^{-2x} \quad \text{and} \quad y_2(x) = xe^{-2x}.$$
4. The general solution is
$$y(x) = e^{-2x}(c_1 + c_2 x).$$

Complex roots

What happens if the auxiliary polynomial has complex roots? Can we recover real solutions? Yes!

Theorem

If $P(D)y = 0$ is a linear differential equation with real constant coefficients and $(D-r)^m$ is a factor of $P(D)$ with $r = a+bi$ and $b \neq 0$, then

1. *$P(D)$ must also have the factor $(D-\overline{r})^m$,*
2. *this factor contributes the complex solutions*
$$e^{(a\pm bi)x}, \; xe^{(a\pm bi)x}, \; \ldots, \; x^{m-1}e^{(a\pm bi)x},$$
3. *the real and imaginary parts of the complex solutions are linearly independent real solutions*
$$x^k e^{ax}\cos bx \quad \text{and} \quad x^k e^{ax}\sin bx$$
for $k = 0, 1, \ldots, m-1$.

Complex roots

Example

Determine the general solution to $y'' + 6y' + 25y = 0$.

1. The auxiliary polynomial is $r^2 + 6r + 25$.
2. Its has roots $r = -3 \pm 4i$.
3. Two independent real-valued solutions are

$$y_1(x) = e^{-3x} \cos 4x \quad \text{and} \quad y_2(x) = e^{-3x} \sin 4x.$$

4. The general solution is

$$y(x) = e^{-3x}(c_1 \cos 4x + c_2 \sin 4x).$$