Derivation of Kinematic Equations

View this after

Motion on an Incline Lab

Constant velocity

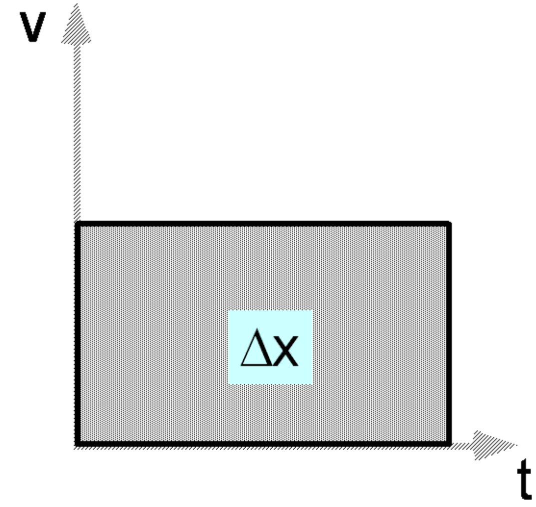
Average velocity equals the **slope** of a position vs time graph when an object travels at constant velocity.

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Displacement when object moves with constant velocity

The displacement is the area under a velocity vs time graph

$$\Delta x = \bar{v}\Delta t$$



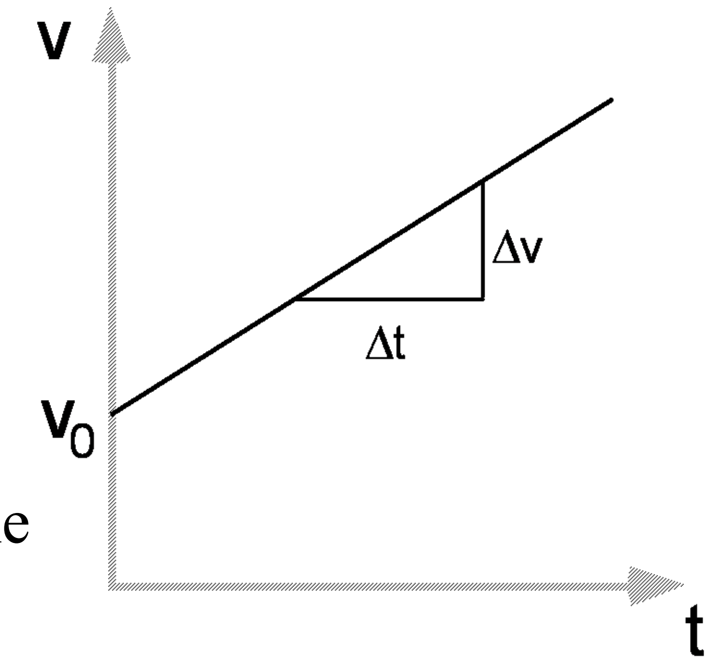
Uniform acceleration

This is the equation of the line of the **velocity** vs **time** graph when an object is undergoing uniform acceleration.

$$v_f = at + v_0$$

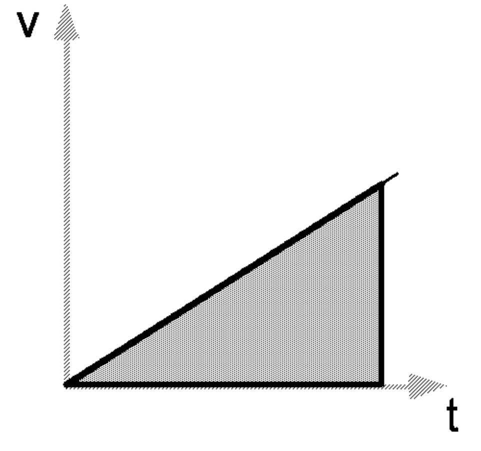
The slope is the acceleration

The intercept is the initial velocity



Displacement when object accelerates from rest

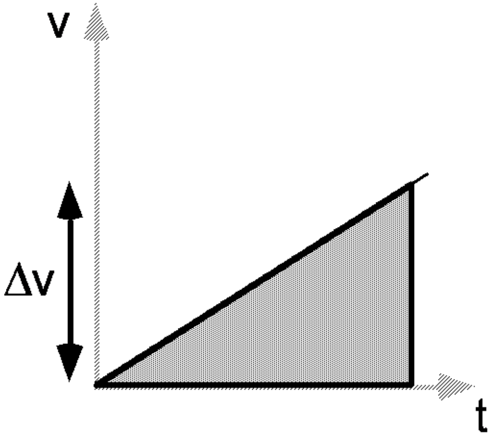
Displacement is still the area under the velocity vs time graph. However, velocity is constantly changing.



Displacement when object accelerates from rest

Displacement is still the area under the velocity vs time graph. Use the formula for the area of a triangle.

$$\Delta x = \frac{1}{2}\Delta v \Delta t$$



Displacement when object accelerates from rest

From slope of v-t graph

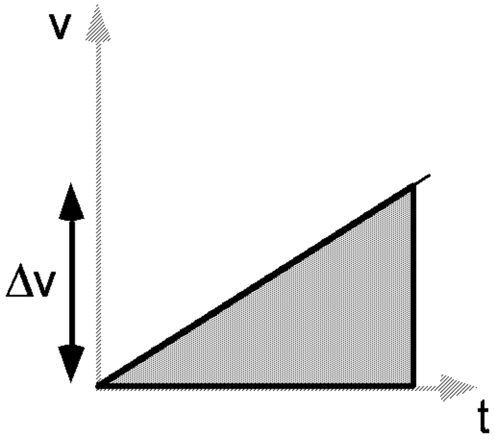
$$\bar{a} = \frac{\Delta v}{\Delta t}$$

Rearrange to get

$$\Delta v = \bar{a}\Delta t$$

Now, substitute for Δv

$$\Delta x = \frac{1}{2}\bar{a}\Delta t \cdot \Delta t$$



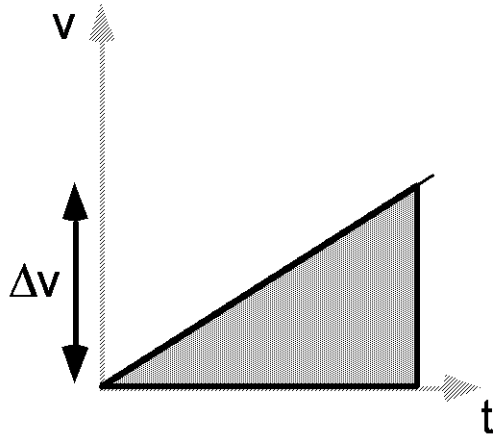
Displacement when object accelerates from rest

Simplify

$$\Delta x = \tfrac{1}{2}\bar{a}(\Delta t)^2$$

Assuming uniform acceleration and a starting time = 0, the equation can be written:

$$\Delta x = \tfrac{1}{2}at^2$$

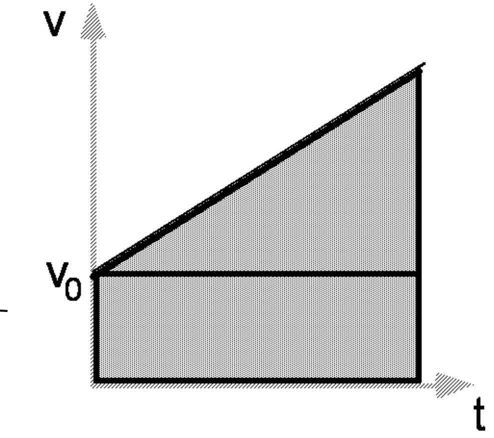


Displacement when object accelerates with initial velocity

Break the area up into two parts:

the rectangle representing displacement due to initial velocity

$$\Delta x = v_0 \Delta t$$

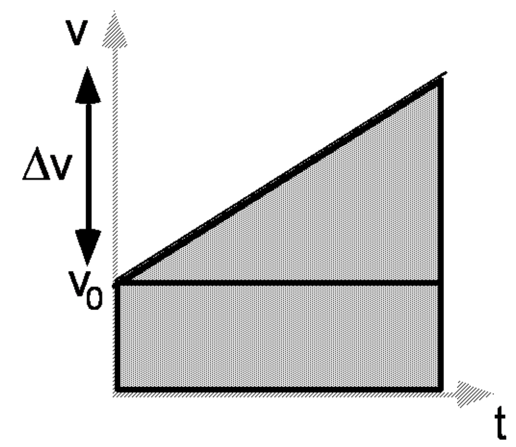


Displacement when object accelerates with initial velocity

Break the area up into two parts:

and the triangle representing displacement due to acceleration

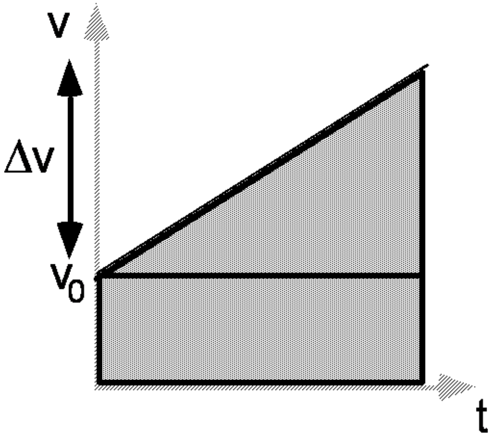
$$\Delta x = \tfrac{1}{2} a (\Delta t)^2$$



Displacement when object accelerates with initial velocity

Sum the two areas:

$$\Delta x = v_0 \Delta t + \tfrac{1}{2} a (\Delta t)^2$$



Or, if starting time = 0, the equation can be written:

$$\Delta x = v_0 t + \tfrac{1}{2} a t^2$$

Time-independent relationship between Δx, v and a

Sometimes you are asked to find the final velocity or displacement when the length of time is not given.

To derive this equation, we must start with the definition of average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Time-independent relationship between Δx, v and a

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

Another way to express average velocity is:

$$\bar{v} = \frac{v_f + v_0}{2}$$

Time-independent relationship between Δx, v and a

We have defined acceleration as:

$$\bar{a} = \frac{\Delta v}{\Delta t}$$

This can be rearranged to:

$$\Delta t = \frac{\Delta v}{\bar{a}}$$

and then expanded to yield : $$\Delta t = \frac{v_f - v_0}{\bar{a}}$$

Time-independent relationship between Δx, v and a

Now, take the equation for displacement

$$\Delta x = \bar{v}\Delta t$$

and make substitutions for average velocity and Δt

Time-independent relationship between Δx, v and a

$$\Delta x = \bar{v}\Delta t$$

$$\bar{v} = \frac{v_f + v_0}{2} \qquad \Delta t = \frac{v_f - v_0}{\bar{a}}$$

Time-independent relationship between Δx, v and a

$$\Delta x = \bar{v}\Delta t$$

$$\Delta x = \frac{v_f + v_0}{2} \cdot \frac{v_f - v_0}{\bar{a}}$$

Time-independent relationship between Δx, v and a

$$\Delta x = \frac{v_f + v_0}{2} \cdot \frac{v_f - v_0}{\bar{a}}$$

Simplify

$$\Delta x = \frac{v_f^2 - v_0^2}{2\bar{a}}$$

Time-independent relationship between Δx, v and a

$$\Delta x = \frac{v_f^2 - v_0^2}{2\bar{a}}$$

Rearrange

$$2\bar{a}\Delta x = v_f^2 - v_0^2$$

Time-independent relationship between Δx, v and a

$$2\bar{a}\Delta x = v_f^2 - v_0^2$$

Rearrange again to obtain the more common form:

$$v_f^2 = v_0^2 + 2\bar{a}\Delta x$$

Which equation do I use?

- First, decide what model is appropriate
  - Is the object moving at constant velocity?
  - Or, is it accelerating uniformly?
- Next, decide whether it’s easier to use an algebraic or a graphical representation.

Constant velocity

If you are looking for the velocity,

- use the algebraic form

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

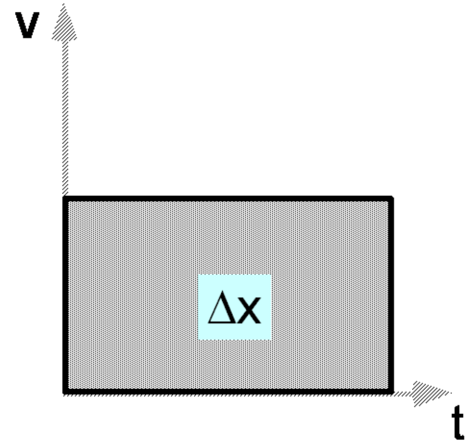
- or find the slope of the graph (actually the same thing)

Constant velocity

- If you are looking for the displacement,
  - use the algebraic form

$$\Delta x = \bar{v}\Delta t$$

  - or find the area under the curve



Uniform acceleration

- If you want to find the final velocity,
  - use the algebraic form

$$v_f = at + v_0$$

- If you are looking for the acceleration
  - rearrange the equation above

$$a = \frac{\Delta v}{\Delta t}$$

  - which is the same as finding the slope of a velocity-time graph

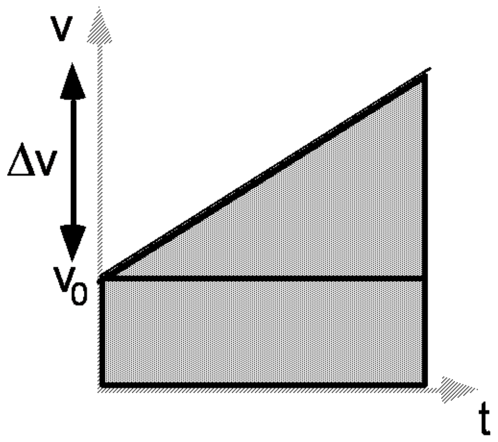
Uniform acceleration

If you want to find the displacement,

- use the algebraic form
- eliminate initial velocity if the object starts from rest

$$\Delta x = v_0 t + \tfrac{1}{2} a t^2$$

- Or, find the area under the curve



If you don't know the time…

You can solve for Δt using one of the earlier equations, and then solve for the desired quantity, or

You can use the equation

$$v_f^2 = v_0^2 + 2\bar{a}\Delta x$$

– rearranging it to suit your needs

All the equations in one place

constant velocity

$$\bar{v} = \frac{\Delta x}{\Delta t}$$

$$\Delta x = \bar{v}\Delta t$$

uniform acceleration

$$a = \frac{\Delta v}{\Delta t}$$

$$v_f = at + v_0$$

$$\Delta x = v_0 t + \tfrac{1}{2} a t^2$$

$$v_f^2 = v_0^2 + 2a\Delta x$$