Electrical circuits as manifolds

Introduction

Once we have shown how to make a link between electrical networks and parametrized surfaces, it becomes interesting to look at circuits like manifolds. Kron's technique becomes under this idea, the method to construct some systems made of "primitive manifolds". The purpose of this presentation is to introduce the approach, starting from simple and usual components like resistances, inductances, etc.

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Electrical circuits as manifolds: resistances

A resistance a gives a simple law under first order hypothesis between a current x and a potential difference y: $y = ax$. In order to draw the figures described by this kind of law, we have to choose a waveform for x. In major applications, waveforms can be set as a sum of Gaussian pulses. Considering one, we can compute $x(t)$ for t given by:

$$x(t) = e^{-\left(\frac{t-\tau}{\sigma}\right)^2} \tag{1}$$

Knowing $x(t)$ we can draw the three dimensions surface (t, x, y) for various values of R. It gives the kind of image shown figure 1.

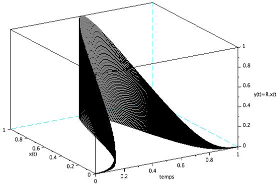

Figure 1

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Resistances

Taking a look to figure 2, we understand that, whatever x and t values, a common axe belongs to all the graphic giving the increasing of y depending on R.

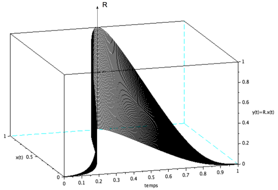

Figure 2

If $x(\theta) = x_0 sin(\theta)$, the surface drawn is a cylinder Cy where R is the straight line L coefficient; line multiplied by the circle C_0 made by (x, t): $Cy = L \times C_0$.

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If we draw the vectors associated with the time running and the common axe R, we can find a normal vector to the surface n as shown figure 3. The surface can be faced that all along, **n** points out in the same side of the surface.

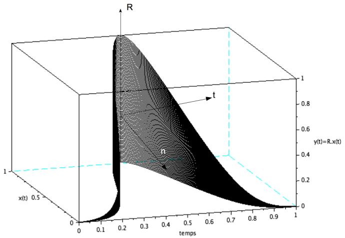

Figure 3

It must be noted that all the points that doesn't belongs to the surface give all the cases that are not solutions of the simple circuit attached with the Ohm's law, for the chosen $x(t)$ waveform.

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Capacitor

Same exercize can be done with a capacitor. The surface obtained for various values of $1/C$ can be faced and has a common axe of increasing linked with $1/C$. Figure 4 shows the computation.

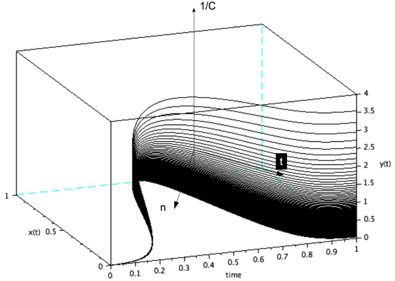

Figure 4

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Inductance

For inductance, a problem appears. It stills possible to find a coefficient followed for the amplitude: L. But the total surface cannot be faced. As shown figure 5, the direction of the normal vector $\mathbf{n}$ changes when going through the surface from the positive time derivative part to the negative one. By the fact, one common direction cannot be find.

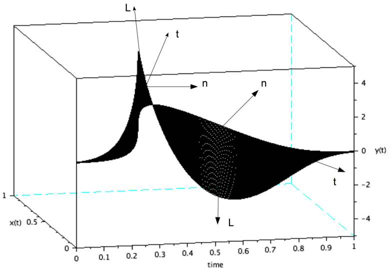

Figure 5

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Inductance

This appears clearly on figure 6 where the curve has been turned in order to present its front side. Note that the variable derivative of the inductance operator leads to 0: $\partial_x L \partial_t x = 0$.

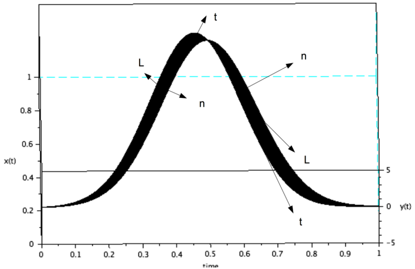

Figure 6

Electrical circuits as manifolds

In all the cases where a direction can be found, it is clear that this direction says how change y when x changes. So, these directions can be seen as directions of a space: the basic vector of a parametrized surface. When we look to the resistance, it is written $y = R.x$. If we compute $\partial_x y = R$. A basic vector component for the parameter x becomes R:

$$\mathbf{b} = (R) \tag{2}$$

It means that $y = R.x$ becomes the coordinate x of a vector $\mathbf{y}$ on a line of base R. For a capacitor, it's a little more complicated:

$$\partial_x \frac{1}{C}\int_t^T dt x(t) = \frac{1}{C}\int_t^T dt \partial_x x(t) = \frac{T}{C} = b_i \tag{3}$$

So that:

$$y = \frac{1}{C}\int_t^T dt x(t) = \frac{b_i}{T}\int_t^T dt x(t) = \mathcal{F}_0 b_i \tag{4}$$

$\mathcal{F}_0$ being the first Fourier's coefficient for the periodic signal $x(t)$.

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Another way to see the direction is to cut the time axis in a (x, y, t) representation. For a given time, the surface becomes a straight line that we can project in the (x, y) plan. For two of these projections, and for a given value of R, R appears clearly as the director coefficient of the tangential function to the curve (figure 6 bis. This time R is one chosen fixed value, not a direction).

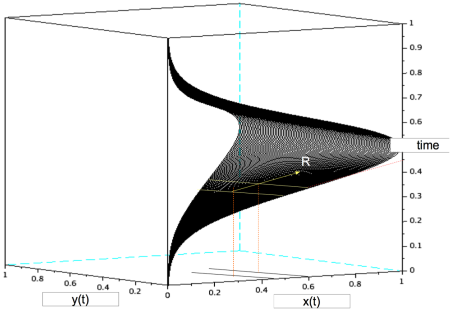

Figure 6bis

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Figure 7 shows the global strategy to define understandable base and metric.

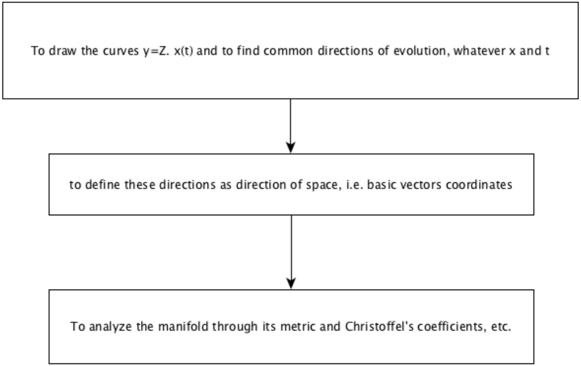

Figure 7

Electrical circuits as manifolds: chords as fundamental metric

Kron's considered that inductances give the metric of the mesh space, by analogy with the ds^2 expression in relativity, and due to the fact that the magnetic field can present similar expressions to general relativity in electrical machines. But this hypothesis wasn't directly confirmed by equations. Starting from parametrized surface theory in differential geometry, we can find a rigorous definition for a metric in Kron's method. Seeing that impedances can be seen as space direction, we consider the equations obtained under the Kron's method: $e_\mu = z_{\mu\nu} i_\nu$ (using mute index notation as Feynman) as a parametrized surface. Taking $e_\mu = 0$ we obtain N equations $z_{\mu\nu} i_\nu$. Various equations can come from various physics involved in the system studied. With a new space configuration, we associate $i_\nu \to x^\nu$. All equations can be seen as a vector of functions ψ_k. This allows to generate basic vectors $\mathbf{b}_i = \partial \psi_k \left(\partial x^i \right)^{-1}$. From the $\mathbf{b}_i$ can be defined a jacobian matrix $J_{ki} = [\mathbf{b}_i]$.

Electrical circuits as manifolds: chords as fundamental metric

The exercise now is to find the original equation established in the mesh space $e_\mu = z_{\mu\nu} i_\nu$ with J. If the system can be written (excluding the inductance, as we have seen they generate 0 for basic vector coordinate):

$$e_\mu - L_{\mu\nu}\dot{x}^\nu = J_{\mu\nu}x^\nu \Rightarrow \Gamma_{\beta\mu}\left(e_\mu - L_{\mu\nu}\dot{x}^\nu\right) = \Gamma_{\beta\mu}J_{\mu\nu}x^\nu \qquad (5)$$

with $\Gamma_{\beta\nu} = \left(J_{\mu\nu}\right)^T$. If $\mathcal{L}_{\beta\nu} = \Gamma_{\beta\mu}L_{\mu\nu}$ and $\Gamma_{\beta\mu}J_{\mu\nu} = G_{\beta\nu}$. Setting $\Gamma_{\beta\mu}e_\mu = T_\beta$ we obtain:

$$T_\beta - \mathcal{L}_{\beta\nu}\dot{x}^\nu = G_{\beta\nu}x^\nu \qquad (6)$$

Which is a fundamental result, showing that inductances can be linked with a rigorous and classical metric definition but not directly equal to it, using the jacobian transposed matrix.

Simple RL example

If we consider two resistance - inductance circuits (R, L_1) and (σ, L_2), cross talked by a mutual inductance M. It leads to the system in the Kron's mesh space:

$$\left\{ \begin{array}{l} e_1 = Ri_1 + L_1 \frac{di_1}{dt} - M \frac{di_2}{dt} \\ e_2 = -M \frac{di_1}{dt} + \sigma i_2 + L_2 \frac{di_2}{dt} \end{array} \right. \tag{7}$$

We look at this system as if it was a surface ψ_k made of k equations of parameters u^k giving point coordinates x_k through the system:

$$\left\{ \begin{array}{l} x_1 = Ru^1 + L_1 \frac{du^1}{dt} - M \frac{du^2}{dt} \\ x_2 = -M \frac{du^1}{dt} + \sigma u^2 + L_2 \frac{du^2}{dt} \end{array} \right. \tag{8}$$

Making the analogy $i_k \to u^k$ and $e_k \to x_k$. This parametrized surface allows to define the basic vectors **b** with:

$$b_\alpha|_k = \frac{\partial \psi_k}{\partial u^\alpha} \tag{9}$$

Simple RL example

We find:

$$\mathbf{b}_1 = \begin{pmatrix} R \\ 0 \end{pmatrix} \quad \mathbf{b}_2 = \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \tag{10}$$

and so:

$$[J] = [\mathbf{b}_1, \mathbf{b}_2] = \begin{pmatrix} R & 0 \\ 0 & \sigma \end{pmatrix}$$

With $[M] = \begin{pmatrix} L_1 & -M \\ -M & L_2 \end{pmatrix}$, we obtain:

$$T_\beta = [J]^T \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} Rx_1 \\ \sigma x_2 \end{pmatrix} \quad [\mathcal{L}] = [J]^T[M] = \begin{pmatrix} RL_1 & -RM \\ -\sigma M & \sigma L_2 \end{pmatrix}$$

$$[G] = [J]^T[J] = \begin{pmatrix} R^2 & 0 \\ 0 & \sigma^2 \end{pmatrix}$$

You can verify that:

$$T_\beta = G_{\beta\nu} u^\nu + \mathcal{L}_{\beta\nu} u^\nu \Leftrightarrow e_\mu = z_{\mu\nu} i_\nu \tag{11}$$

Simple RL example

One could say: what's the advantage of this new formulation?
Firstly it gives us a theoretical access to the manifold properties. Here, the space is flat. It means that in the two axes reference R, σ, the current amplitudes are the coordinates of points that belong to the manifold. If $\mathcal{D}_1$ and $\mathcal{D}_2$ are the domains giving all the values possible for u^1 and u^2, the geometrical form obtain from (11) computing x_k for all possible values of u^k is the space where all functional states of the system are. If an external excitation x'_k is applied and do not belongs to this space, it will make the system in an abnormal state of unknown behaviour.

One way to verify this risk, we can compare the distance $\sqrt{G_{\beta\nu} u^\beta u^\nu}$ with $\sqrt{T'_\beta u^\beta - \mathcal{L}_{\beta\nu} \frac{\partial u^\nu}{\partial t} u^\beta}$ where T'_β is a new constraint hypothesis. If this second distance is greater than the first one (in average, instantaneously, et., depending on the criterion chosen), the risk is detected for the system.

Simple RL example

We apply the previous technique to the RL circuit. In a first phase we compute the currents for a "normal" circuit activity. These gives the waited $\sqrt{G_{\beta\nu}u^\beta u^\nu}$ limit. Then we compute $\sqrt{T'_\beta u^\beta - \mathcal{L}_{\beta\nu}\frac{\partial u^\nu}{\partial t}u^\beta}$ for a new external excitation of different time waveform. We obtain the difference between the two curves shown figure 9 (in blue the original curve).

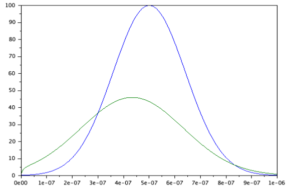

Figure 9

Computing $qt_{max}^{-1}\int_t dt\sqrt{G_{\beta\nu}u^\beta u^\nu}$ and $qt_{max}^{-1}\int_t dt\sqrt{T'_\beta u^\beta - \mathcal{L}_{\beta\nu}\frac{\partial u^\nu}{\partial t}u^\beta}$ we obtain respectively here 35,4[J] and 22,6[J]. No risk here!

RLC example

Is the average energy the best criterion? Let's take another example with two RLC circuits cross talked through a mutual inductance. The fundamental tensor in the mesh space is given by (s is the Laplace's operator):

$$z = \begin{pmatrix} R + \frac{1}{sC} + L_1 s & -Ms \\ -Ms & \sigma + \frac{1}{sD} + L_2 s \end{pmatrix} \tag{12}$$

This leads to next objects:

$$J = \begin{pmatrix} R + \frac{1}{sC} & 0 \\ 0 & \sigma + \frac{1}{sD} \end{pmatrix} \quad M = \begin{pmatrix} L_1 & -M \\ -M & L_2 \end{pmatrix} \quad G = J^2 \tag{13}$$

so that:

$$T = \begin{pmatrix} \left(R + \frac{1}{sC}\right) y_1 \\ \left(\sigma + \frac{1}{sD}\right) y_2 \end{pmatrix} \quad \mathcal{L} = \begin{pmatrix} \left(R + \frac{1}{sC}\right) L_1 & -M\left(R + \frac{1}{sC}\right) \\ -M\left(\sigma + \frac{1}{sD}\right) & \left(\sigma + \frac{1}{sD}\right) L_2 \end{pmatrix} \tag{14}$$

RLC example

As previously we obtain the same equations as in (11) and this time the distance is given by:

$$ds = \sqrt{R^2(u^1)^2 + \sigma^2(u^2)^2 + \frac{(q^1)^2}{C^2} + \frac{(q^2)^2}{D^2}}$$

$(sq^k = u^k)$ which includes not only the joule effects but also the electric energy stored in capacitors and that can create spar gaps.
We can take a look to some extrema:

$$\int_t dt \sqrt{G_{\beta\nu} u^\beta u^\nu} \tag{15}$$

which is an image of the cumulated energy normalized on q. So, it's similar to the classical ds^2.

RLC example

But another criterion that can be interesting is the action written using past dissipated and stored energy and new one coming from a new constraint. This is given by:

$$\delta S_m = \frac{\partial}{\partial u^m} \int_t dt \left\{ \sqrt{T'_m u^m - \mathcal{L}_{km} \dot{u}^k u^m} - \sqrt{G_{km} u^k u^m} \right\} \tag{16}$$

This equation gives the variations of constraint on each current, i.e. on each mesh.

What can be noticed is that under this relation, the sources T cannot be inferior to $\mathcal{L}_{km} \dot{u}^k u^m$. But in fact, without sources, there isn't any currents...

Previous expression leads in the first case to:

$$\delta S_m = \int_t dt \left[\frac{1}{2\sqrt{\cdots}_{T,\mathcal{L}}} \left\{ \begin{pmatrix} Rx^1 \\ 0 \end{pmatrix} - \begin{pmatrix} RL_1\dot{u}^1 - RM\dot{u}^2 \\ -\sigma M\dot{u}^1 + \sigma L_2\dot{u}^2 \end{pmatrix} \right\} \right. \\ \left. - \frac{1}{2\sqrt{\cdots}_G} \left\{ \begin{pmatrix} R^2u^1 \\ \sigma^2u^2 \end{pmatrix} \right\} \right] \quad (17)$$

with $\sqrt{\cdots}_{T,\mathcal{L}} = \sqrt{T'_m u^m - \mathcal{L}_{km}\dot{u}^k u^m}$ and $\sqrt{\cdots}_G = \sqrt{G_{km}u^k u^m}$.

RLC example

Figure 10 shows the curve obtained for the first case, to compare with figure 9. It clearly shows where the new constraint is higher than the original one supported and give the deviation in ohm. It means that at the beginning for example, all appears as if the *R* axis of the base was multiplied in dimension by 3700 !

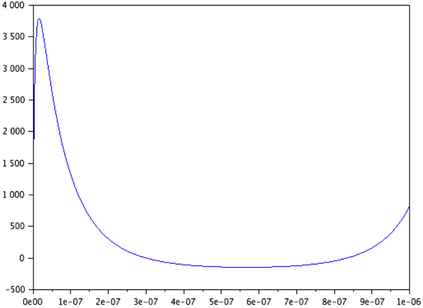

Figure 10

Conclusion

This last result is very incentive. It gives a clear criterion to compute the real time risk applied on a system in front of a new environment. Depending on the expression if the metric G and inertia $\mathcal{L}$, it gives a theory to detect where are the weakness of the system. More particularly, often only some ports are directly concerned by the environment. Studying them will give all the information to detect the risk without looking at all the currents of the system.

Another important point is that expression (16) will make appear the curvature of space through the derivatives of G and L versus u^k. This means that changing the environment can change this curvature and increase the distance to the reference.