MATH6103 Differential & Integral Calculus
Notes in Brief

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Contents

Using These Notes

These notes are intended as a revision aid. I started with the full lecture notes of the course and taken out all unnecessary detail, examples, etc. to leave a minimal outline of the course. If you need more detail on a topic, look at the relevant section in the full notes!

Throughout these notes in brief, you will find boxes that look like this:

A-level C1 Differentiation

These boxes contain references to the parts of GCSE and A-level maths that are relevant to the section. These may be useful as there is a great range of GCSE and A-level revision material online. These are all based on the syllabus of the EdExcel exam board, as this is the one I am familiar with. Other exam boards are mostly the same.

Chapter 1

Functions

A-level C1 functions

$\mathbb{Z} = \{\text{all whole numbers}\} = \{\dots, -2, -1, 0, 1, 2, \dots\}$
$\mathbb{N} = \{\text{all positive whole numbers}\} = \{0, 1, 2, 3, \dots\}$
$\mathbb{R} = \{\text{all real numbers}\}$

$f : A \to B$
A is the **domain** of f. B[1] is the **range** of f.

If $x \neq y$ implies $f(x) \neq f(y)$, the function is **one-to-one**.
Otherwise, the function is **many-to-one**.

If $f(-x) = f(x)$, f is **even**.
If $f(-x) = -f(x)$, f is **odd**.

If $f(x + T) = f(x)$ for all x, then $f(x)$ is a **periodic function** with period T.

Vertical/horizontal asymptotes are vertical/horizontal lines that the function approached but never reaches.

1.1 Polynomials

A polynomial is a function P with a general form

$$P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$$

[1] or a subset of B

1.1.1 Some polynomial degrees

Degree 0

This polynomial is simply a constant.

Degree 1

$P_1(x) = ax + b$ $(a \neq 0)$. a is the gradient. b is the y-intercept

Degree 2

$P_2(x) = ax^2 + bx + c$, $a \neq 0$ are called quadratics.

GCSE quadratics
A-level C1 quadratics

1. Factorising

Example

$$P(x) = x^2 - 3x + 2 \qquad = (x-2)(x-1)$$

The solutions of $P(x) = 0$ are $x = 2$ and $x = 1$.

2. Completing the square

Example

$$\begin{aligned} P(x) &= x^2 - 3x + 2 \\ &= \left(x - \tfrac{3}{2}\right)^2 - \left(\tfrac{-3}{2}\right)^2 + 2 \end{aligned}$$

$$\begin{aligned} \left(x - \tfrac{3}{2}\right)^2 - \left(\tfrac{-3}{2}\right)^2 + 2 &= 0 \\ \left(x - \tfrac{3}{2}\right)^2 &= \tfrac{1}{4} \\ x - \tfrac{3}{2} &= \pm\tfrac{1}{2} \\ x &= 1 \text{ or } 2 \end{aligned}$$

3. The quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Degree ≥ 3

A-level C2 Remainder theorem

Theorem: Factor theorem

$$P(a) = 0 \quad \text{if and only if} \quad P(x) = (x-a)Q(x)$$

1.2 Exponentials

GCSE Indices and powers

$$\begin{aligned} a^{x+y} &= a^x \cdot a^y \\ (a^x)^y &= a^{xy} \\ a^x \cdot b^x &= (ab)^x \\ a^0 &= 1 \\ a^{-x} &= \frac{1}{a^x} \\ a^{\frac{1}{n}} &= \sqrt[n]{a} \\ a^{\frac{m}{n}} &= \left(\sqrt[n]{a}\right)^m \end{aligned}$$

1.3 Trigonometric functions

1.3.1 Measuring angles

GCSE Trigonometry
A-level C3 Radians

$$\begin{aligned} 1 \text{ turn} &= 360^\circ = 2\pi \,\text{rad} \\ \tfrac{1}{2} \text{ turn} &= 180^\circ = \pi \,\text{rad} \end{aligned}$$

"SOH CAH TOA"

1.3.2 Properties of sin, cos and tan

A-level C2 Trigonometry
A-level C3 Trigonometry

$\cos^2\theta + \sin^2\theta = 1$

cos and sin are periodic functions with period 2π (i.e. for any x, $\cos(x + 2\pi) = \cos x$, $\sin(x + 2\pi) = \sin x$).

$\cos : \mathbb{R} \to [-1, 1]$ and $\sin : \mathbb{R} \to [-1, 1]$.

cos is an even function. sin is an odd function.

cos and sin are the same shape but shifted by $\pi/2$, which means

$$\cos\left(\theta - \frac{\pi}{2}\right) = \sin\theta$$
$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi$$
$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

$$\sin 2\theta = 2\sin\theta\cos\theta$$
$$\cos 2\theta = 1 - 2\sin^2\theta$$
$$\cos 2\theta = 2\cos^2\theta - 1$$

$$\cos^2\left(\frac{\alpha}{2}\right) = \frac{1 + \cos\alpha}{2}$$
$$\sin^2\left(\frac{\alpha}{2}\right) = \frac{1 - \cos\alpha}{2}$$

tan has vertical asymptotes at $\theta = \frac{\pi}{2}(2N - 1)$ for $N \in \mathbb{Z}$.

$\tan : \mathbb{R} \setminus \{\frac{\pi}{2}(2N - 1) : N \in \mathbb{Z}\} \to \mathbb{R}$

tan is periodic with period π.

$$\tan(\theta + \phi) = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi}$$

The secant, cosecant and cotangent functions are defined as

$$\sec x = \frac{1}{\cos x}, \quad \mathrm{cosec} x = \frac{1}{\sin x}, \quad \mathrm{cot} x = \frac{1}{\tan x}.$$

$$1 + \tan^2 x = \sec^2 x.$$

Angle (°)	**Angle (c)**	**sin**	**cos**	**tan**
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{3}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	∞

1.4 Polar co-ordinates

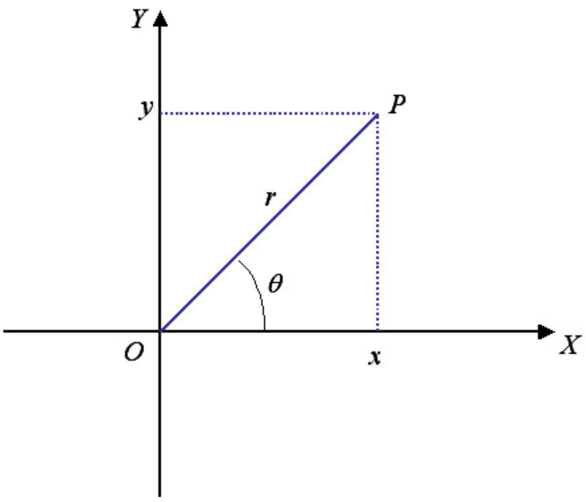


Figure 1.1: Polar co-ordinates are given by r and θ.

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Chapter 2

Differentiation

$$\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$$

2.1 Finding the gradient

A-level C1 Differentiation
A-level C2 Differentiation
A-level C3 Differentiation

Definition
Differentiation by first principles:

$$f'(c) = \lim_{h \to 0} \frac{f(c+h) - f(c)}{h}.$$

2.2 Some common derivatives

IMPORTANT: Always use radians!

$f(x)$	$f'(x)$
x^n	nx^{n-1}
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$

2.3 Rules for differentiation

A-level C4 Differentiation

The sum rule

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}(f(x)) + \frac{d}{dx}(g(x))$$

The product rule

$$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x))\,g(x) + f(x)\frac{d}{dx}(g(x))$$

The chain rule

$$\frac{d}{dx}(f(g(x))) = f'(g(x))g'(x)$$

The quotient rule

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$

2.4 Polar co-ordinates

The chain rule can be rearranged to give:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Example
$x = t^2 + 4$; $y = e^t$

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} \\ &= \frac{e^t}{2t} \end{aligned}$$

2.5 Uses of differentiation

2.5.1 Finding the gradient at a point

The gradient of f at $x = k$ is $f(k)$.

2.5.2 Finding the maximum and minimum points

- Local maximum
$$f'(x) = 0$$
$$f''(x) < 0$$
- Local minimum
$$f'(x) = 0$$
$$f''(x) > 0$$
- Need more information
$$f'(x) = 0$$
$$f''(x) = 0$$

2.6 Differentiating inverse functions

Definition
f^{-1} is the inverse of f:
$$f^{-1}(f(x)) = x$$

note: Sometimes, arcsin, arccos and arctan are used to represent $\sin^{-1}$, $\cos^{-1}$ and $\tan^{-1}$.

Finding the derivative of an inverse
$$\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}$$

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

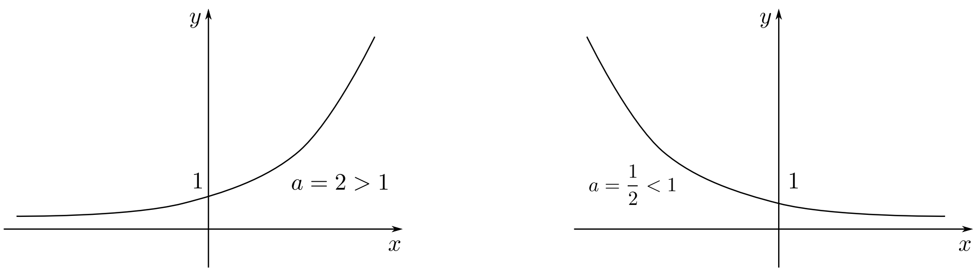
Chapter 3

Exponentials and Logarithms

3.1 Exponentials

A-level C3 Exponentials and Logarithms

$$f(x) = a^x,$$



(a) $y = 2^x$ (b) $y = \left(\frac{1}{2}\right)^x$

$e \approx 2.718281828459\ldots$

Definition
$f(x) = e^x = \exp(x)$ is the **exponential function**.

Property

$$\frac{d}{dx}(e^x) = e^x.$$

3.2 Logarithms

A-level C2 Exponentials and Logarithms
A-level C3 Exponentials and Logarithms

Definition
The inverse of a^x is $\log_a x$.

Laws of Logs

1. $\log_a(MN) = \log_a M + \log_a N$.
2. $\log_a(M^p) = p \log_a M$.

Example
Find x, given $3^x = 7$.

$$\begin{aligned} \ln(3^x) &= \ln 7 \\ x \ln 3 &= \ln 7. \\ x &= \frac{\ln 3}{\ln 7} \\ &\approx 1.77 \end{aligned}$$

3.2.1 The natural logarithm

A-level C3 Exponentials and Logarithms

Definition
The inverse of $f(x) = e^x$ is the **natural logarithm**, $\ln x$.

Property

$$\frac{d}{dx}(\ln x) = \frac{1}{x}.$$

3.2.2 Differentiation of other logarithms

A-level C3 Exponentials and Logarithms

Property: Change of base

$$\log_a x = \frac{\log_b x}{\log_b a}$$

Property

$$\frac{d}{dx}(\log_a x) = \frac{1}{x \ln a}.$$

3.3 Differentiation of other exponentials

A-level C3 Exponentials and Logarithms

In general, for any positive constant a

$$\frac{d}{dx}(a^x) = a^x \ln a.$$

Chapter 4

Integration

Integration = Finding the area under the curve.

Theorem: Fundamental Theorem of Calculus

$$\int_a^b g'(x)\ dx = g(a) - g(b)$$

4.1 Finding integrals

A-level C1 integration
A-level C2 integration
A-level C4 integration

$f(x)$	$\int f(x)\ dx$
ax^b	$\frac{ax^{b+1}}{b+1} + c$
$\frac{1}{x}$	$\ln\lvert x\rvert + c$
e^x	$e^x + c$
a^x	$\frac{a^x}{\ln a} + c$
$\cos x$	$\sin x + c$
$\sin x$	$-\cos x + c$

4.2 Rules for integration

Sum Rule

$$\int (f(x) + g(x))\ dx = \int f(x)\ dx + \int g(x)\ dx \tag{4.1}$$

Multiplication by a constant

$$\int Kf(x)\ dx = K\int f(x)\ dx \tag{4.2}$$

A special case

$$\int \frac{f'(x)}{f(x)}\ dx = \ln(f(x)) + c$$

Example: Integration by Substitution

A-level C4 Integration

$$\int (2x+3)^{100}\ dx$$

$$u = 2x + 3.$$

$$\frac{du}{dx} = 2$$

$$dx = \frac{1}{2}\ du.$$

$$\begin{aligned} \int (2x+3)^{100}\ dx &= \int u^{100}\cdot\frac{1}{2}\ du \\ &= \frac{1}{2}\int u^{100}\ du \\ &= \frac{1}{2}\cdot\frac{1}{101}u^{101} \\ &= \frac{1}{202}(2x+3)^{101} + c. \end{aligned}$$

Integration by Parts

A-level C4 Integration

$$\int u\frac{dv}{dx}\ dx = uv - \int v\frac{du}{dx}\ dx.$$

4.2.1 Partial fractions

A-level C4 Integration

Example

$$\frac{1}{x^2-1}$$
$$x^2-1=(x+1)(x-1)$$
$$\frac{1}{x^2-1}=\frac{A}{x-1}+\frac{B}{x+1},$$
$$1=A(x+1)+B(x-1)$$

Substituting in $x=1$ gives $1=2A$.
Substituting in $x=-1$ gives $1=-2B$.
$A=\frac{1}{2}$; $B=-\frac{1}{2}$.

$$\frac{1}{x^2-1}=\frac{1}{2(x-1)}-\frac{1}{2(x+1)}$$

4.2.2 Trapezium method

A-level C2 Integration
A-level C4 Integration

This is a method for approximating an integral.

$$\int_a^b f(x)\ dx \approx \frac{h}{2}\left(f(a)+2f(a+h)+...+2f(a+(n-1)h)+f(b)\right)$$

Chapter 5

Differential Equations

5.1 First order differential equations

A-level C4 Integration

Here we will consider different techniques to solve first order ODEs.

Definition
A function $f(x, y)$ is **separable** if it can be written as

$$f(x, y) = g(x)h(y).$$

Example: Separating the variables

$$\frac{dy}{dx} = xy,$$

$$\frac{1}{y}\, dy = x\, dx$$

$$\int \frac{1}{y}\, dy = \int x\, dx$$

$$\ln y = \frac{1}{2}x^2 + C$$

$$y = e^{\frac{1}{2}x^2 + C} = Ae^{\frac{1}{2}x^2}, \quad A = e^C.$$

If boundary conditions are given, substitute them in to find the constant(s).

5.1.1 Integrating factors

A-level FP2 First Order Differential Equations

Definition
The **integrating factor** of the ODE

$$\frac{dy}{dx} + g(x)y = f(x).$$

is

$$\exp\left(\int g(x)\ dx\right).$$

Example

$$\frac{dy}{dx} + \frac{y}{x} = x.$$

The integrating factor is:

$$\begin{aligned}\exp\left(\int \frac{1}{x}\ dx\right) &= \exp\left(\ln x\right)\\ &= x\end{aligned}$$

Multiplying through by the integrating factor gives:

$$x\frac{dy}{dx} + y = x^2$$

Notice that:

$$\frac{d}{dx}\left(xy\right) = x\frac{dy}{dx} + y$$

Therefore:

$$\begin{aligned}\frac{d}{dx}\left(xy\right) &= x^2\\ xy &= \int x^2\ dx\\ &= \frac{x^3}{3} + c\\ y &= \frac{x^2}{3} + \frac{c}{x}\end{aligned}$$

5.2 Complementary functions and particular integrals

A-level FP2 Second Order Differential Equations

Definition
When $y = f(x) + cg(x)$ is the solution of an ODE, f is called the **particular integral** (P.I.) and g is called the **complementary function** (C.F.).

1. The complementary function (g) is the solution of the homogenous ODE.
2. The particular integral (f) is any solution of the non-homogenous ODE.

5.2.1 Finding complementary functions

Aim: find two independent solutions to

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$$

Definition

$$\lambda^2 + A\lambda + B = 0$$

is the **characteristic equation** or **auxiliary equation** of

$$\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0.$$

Case 1: Two distinct real roots

$$\lambda_1 = \frac{-r + \sqrt{A^2 - 4B}}{2} \quad \text{and} \quad \lambda_2 = \frac{-r - \sqrt{A^2 - 4B}}{2}$$

$$g(x) = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}.$$

Case 2: Repeated root

$$\lambda_1 = \frac{A}{B}.$$

$$g(x) = (c_1 + c_2 x)e^{\lambda_1 x}.$$

Case 3: No real roots

$$g(x) = e^{\alpha x}\left(c_1 \cos \beta x + c_2 \sin \beta x\right),$$

where $\alpha = -\frac{A}{2}$ and $\beta = \frac{\sqrt{4B-A^2}}{2}$.

5.2.2 Finding a particular integral

The particular integral is found by guessing its form, then finding the constants. It depends on the right hand side, $p(x)$.

$p(x)$	guess
1	c
x	$ax + b$
x^2	$ax^2 + bx + c$
sin or cos	$a \sin x + b \sin x$
e^{ax} and a is not a solution of the characteristic equation	$Ae^a x$
e^{ax} and a is a solution of the characteristic equation	$Axe^a x$
e^{ax} and a is a repeated solution of the characteristic equation	$Ax^2 e^a x$

5.2.3 Euler's method

Not in GCSE or A-level

This is a method for approximately solving an ODE. Given:

$$\frac{dy}{dx} = f(x, y), \quad y(a) = y_0.$$

We want to find $y(b)$.

Let $x_k = a + kh$. We use:

$$y_{k+1} = y_k + hf(x_k, y_k)$$