Problems and Solutions for Partial Differential Equations

by

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Chapter 1

Linear Partial Differential Equations

Problem 1. Show that the fundamental solution of the *drift diffusion equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}$$

is given by

$$u(x,t) = \exp\left(-\frac{1}{\sqrt{4\pi t}}\frac{(x-x_0+2t)^2}{4t}\right).$$

Solution 1.

Problem 2. (i) Show that

$$D_x^m(f\cdot 1) = \frac{\partial^m f}{\partial x^m}. \tag{1}$$

(ii) Show that

$$D_x^m(f\cdot g) = (-1)^m D_x^m(g\cdot f). \tag{2}$$

(iii) Show that

$$D_x^m(f\cdot f) = 0, \quad \text{for} \quad m \quad \text{odd} \tag{3}$$

Solution 2.

Problem 3. (i) Show that

$$D_x^m D_t^n(\exp(k_1x - \omega_1 t)\cdot\exp(k_2x-\omega_2 t)) =$$
$$(k_1-k_2)^m(-\omega_1+\omega_2)^n\exp((k_1+k_2)x-(\omega_1+\omega_2)t) \qquad (1)$$

This property is very useful in the calculation of soliton solutions.

(ii) Let $P(D_t, D_x)$ be a polynomial in D_t and D_x. Show that

$$P(D_x, D_t)(\exp(k_1x-\omega_1 t)\cdot\exp(k_2x-\omega_2 t)) =$$
$$\frac{P(k_1-k_2,-\omega_1+\omega_2)}{P(k_1+k_2,-\omega_1-\omega_2)}P(D_x,D_t)(\exp((k_1+k_2)x-(\omega_1+\omega_2)t)\cdot 1) \qquad (2)$$

Solution 3.

Problem 4. Consider a free particle in two dimensions confined by the boundary

$$G := \{\,(x,y)\,:\,|xy| = 1\,\}.$$

Solve the eigenvalue problem

$$\Delta\psi + k^2\psi = 0$$

where

$$k^2 = \frac{2mE}{\hbar^2}$$

with

$$\psi_G = 0.$$

Solution 4.

Problem 5. Consider an electron of mass m confined to the $x-y$ plane and a constant magnetic flux density $\mathbf{B}$ parallel to the z-axis, i.e.

$$\mathbf{B} = \begin{pmatrix} 0 \\ 0 \\ B \end{pmatrix}.$$

The Hamilton operator for this two-dimensional electron is given by

$$\hat{H} = \frac{(\hat{\mathbf{p}} + e\mathbf{A})^2}{2m} = \frac{1}{2m}((\hat{p}_x + eA_x)^2 + (\hat{p}_y + eA_y)^2)$$

where $\mathbf{A}$ is the vector potential with

$$\mathbf{B} = \nabla \times \mathbf{A}$$

and

$$\hat{p}_x = -i\hbar\frac{\partial}{\partial x}, \qquad \hat{p}_y = -i\hbar\frac{\partial}{\partial y}.$$

(i) Show that $\mathbf{B}$ can be obtained from

$$\mathbf{A} = \begin{pmatrix} 0 \\ xB \\ 0 \end{pmatrix}$$

or

$$\mathbf{A} = \begin{pmatrix} -yB \\ 0 \\ 0 \end{pmatrix}.$$

(ii) Use the second choice for $\mathbf{A}$ to find the Hamilton operator $\hat{H}$.
(iii) Show that

$$[\hat{H}, \hat{p}_x] = 0.$$

(iv) Let $k = p_x/\hbar$. Make the ansatz for the wave function

$$\psi(x,y) = e^{ikx}\phi(y)$$

and show that the eigenvalue equation $\hat{H}\psi = E\psi$ reduces to

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega_c^2}{2}(y-y_0)^2\right)\phi(y) = E\phi(y)$$

where

$$\omega_c := \frac{eB}{m}, \qquad y_0 := \frac{\hbar k}{eB}.$$

(v) Show that the eigenvalues are given by

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \qquad n = 0, 1, 2, \ldots.$$

Solution 5. (i) Since

$$\nabla \times \mathbf{A} := \begin{pmatrix} \partial A_z/\partial y - \partial A_y/\partial z \\ \partial A_x/\partial z - \partial A_z/\partial x \\ \partial A_y/\partial x - \partial A_x/\partial y \end{pmatrix}$$

we obtain the desired result.
(ii) Inserting $A_x = -yB$, $A_y = 0$, $A_z = 0$ into the Hamilton operator we find

$$\hat{H} = \frac{1}{2m}\hat{p}_y^2 + \frac{1}{2m}(\hat{p}_x - eyB)^2.$$

(iii) Since coordinate x does not appear in the Hamilton operator $\hat{H}$ and

$$[\hat{y}, \hat{p}_x] = [\hat{p}_y, \hat{p}_x] = 0$$

it follows that $[\hat{H}, \hat{p}_x] = 0$.
(iv) From (iii) we have

$$\hat{p}_x \psi(x,y) \equiv -i\hbar \frac{\partial}{\partial x}\psi(x,y) = \hbar k \psi(x,y), \qquad \hat{H}\Psi(x,y) = E\psi(x,y).$$

Inserting the ansatz $\psi(x,y) = e^{ikx}\psi(y)$ into the first equation we find

$$\psi(x,y) = e^{ikx}\phi(y).$$

(v) We have

$$\begin{aligned} \hat{\psi} &= \frac{1}{2m}((\hat{p}_x - eyB)^2 + \hat{p}_y)e^{ikx}\phi(y) \\ &= \frac{1}{2m}e^{ikx}((\hbar k - eyB)^2 + \hat{p}_y^2)\phi(y) \\ &= \frac{1}{2m}e^{ikx}(e^2B^2(y - \hbar k/eB)^2 + \hat{p}_y^2)\phi(y) \\ &= e^{ikx}\left(\frac{m\omega_c^2}{2}(y-y_0)^2 + \frac{\hat{p}_y^2}{2m}\right)\phi(y). \end{aligned}$$

Now the right-hand side must be equal to $E\psi(x,y) = Ee^{ikx}\phi(y)$. Thus since

$$\hat{p}_y^2 = -\hbar^2 \frac{\partial^2}{\partial y^2}$$

the second order ordinary differential equation follows

$$\left(-\frac{\hbar^2}{2m}\frac{d^2}{dy^2} + \frac{m\omega_c^2}{2}(y-y_0)^2\right)\phi(y) = E\phi(y).$$

(vi) The eigenvalue problem of (v) is essentially the harmonic oscillator except the term is $(y-y_0)^2$ instead of y^2. This means that the centre of oscillation is at $y = y_0$ instead of 0. This has no influence on the eigenvalues which are the same as for the harmonic oscillator,namely

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega_c, \quad n = 0, 1, 2, \ldots.$$

Problem 6. Consider the Schrödinger equation

$$i\hbar\frac{\partial \psi}{\partial t} = \left(-\frac{1}{2m}\Delta + V(x)\right)\psi.$$

Find the coupled system of partial differential equations for

$$\rho := \psi^* \psi, \qquad v := \Im \left(\frac{\nabla \psi}{\psi} \right).$$

Solution 6. First we calculate

$$\frac{\partial \rho}{\partial t} = \frac{\partial}{\partial t}(\psi\psi*) = \frac{\partial \psi}{\partial t}\psi^* + \psi\frac{\partial \psi^*}{\partial t}.$$

Inserting the Schrödinger equation and

$$-i\hbar\frac{\partial \psi^*}{\partial t} = \left(-\frac{1}{2m}\Delta + V(x)\right)\psi^*$$

yields

$$\frac{\partial \rho}{\partial t} = -\frac{1}{2im\hbar}\Delta(\psi\psi^*) + \frac{1}{2im\hbar}\psi\Delta\psi^*.$$

We set $\psi = e^{R(\mathbf{x})+iS(\mathbf{x}}$, where R and S are real-valued. Next we separate real and imaginary part after inserting this ansatz into the Schrödinger equation. Then we set $v = \nabla S$.

Problem 7. Consider the conservation law

$$\frac{\partial c(x,t)}{\partial t} + \frac{\partial j(x,t)}{\partial x} = 0$$

where

$$j(x,t) = -D(x)\frac{\partial c(x,t)}{\partial x}.$$

The diffusion coefficient $D(x)$ depends as follows on x

$$D(x) = D_0(1+gx) \qquad g \geq 0.$$

Thus $dD(x)/dx = D_0 g$. Inserting the current j into the convervation law we obtain a *drift diffusion equation*

$$\frac{\partial c(x,t)}{\partial t} = D_0 g\frac{\partial c(x,t)}{\partial x} + D_0(1+gx)\frac{\partial^2 c(x,t)}{\partial x^2}.$$

The initial condition for this partial differential equation is

$$c(x,t=0) = M_0\delta(x)$$

where δ denotes the delta function. The boundary conditions are

$$j(x=x_0,t) = j(x=+\infty,t) = 0.$$

Solve the one-dimensional drift-diffusion partial differential equation for these initial and boundary conditions using a *product ansatz* $c(x,t) = T(t)X(x)$.

Solution 7. (Martin) Inserting the product ansatz into the one-dimensional drift diffusion equation yields

$$\frac{1}{T(t)}\frac{dT(t)}{dt} = D_0 g \frac{1}{X(x)}\frac{dX(x)}{dx} + D_0(1+gx)\frac{1}{X(x)}\frac{d^2X(x)}{dx^2}.$$

Setting the left-hand and right-hand side to the constant $-D_0 g^2 \chi/4$, where χ is the the speration constant ($\chi > 0$) we obtain for $T(t)$ the solution

$$T(t) = C(\chi)\exp\left(-\frac{\chi D_0 g^2}{4}t\right)$$

where the $C(\chi)$ has to be calculated by the normalization. For $X(x)$ we find a second order linear differential equation

$$(1+gx)\frac{d^2X}{dx^2} + g\frac{dX}{dx} + \frac{\chi g^2}{4}X = 0.$$

Introducing the new variable $u = 1+gx$ and $Y(u) = u^{-1}X(u)$ we obtain

$$u^2\frac{d^2Y}{du^2} + 3u\frac{dY}{du} + \left(1+\frac{\chi}{4}u\right)Y = 0.$$

The solution of this second order ordinary differential equation is

$$Y(u) = \frac{1}{u}J_0(\sqrt{\chi u}), \qquad X(u) = uY(u).$$

Here $J_0(\sqrt{\chi u})$ is the Bessel function of the first kind of order $n=0$. We have

$$J_n(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!(n-k)!}\left(\frac{x}{2}\right)^{n+2k}.$$

Thus

$$J_0(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!k!}\left(\frac{x}{2}\right)^{2k}.$$

It follows that

$$X(u)T(t) = C(\chi)J_0(\sqrt{\chi u})\exp\left(-\frac{\chi D_0 g^2}{4}t\right).$$

Now we integrate over all values of the separation constant $\chi \geq 0$

$$c(x,t) = \int_0^{\infty} d\chi C(\chi)J_0(\sqrt{\chi(1+gx)})\exp\left(-\frac{\chi D_0 g^2}{4}t\right).$$

The normalization constant $C(\chi)$ is determined as follows. Inserting $t = 0$ (initial condition) we obtain

$$c(x, t = 0) = \int_0^\infty d\chi C(\chi) J_0(\sqrt{\chi(1 + gx)}).$$

Multiplying with $\int dx J_0(\sqrt{\chi'(1 + gx)})$ and using the orthogonality of the Bessel functions we obtain

$$C(\chi) = \frac{1}{4} M_0 g J_0(\sqrt{\chi}).$$

It follows that

$$c(x, t) = \frac{M_0 g}{4} \int_0^\infty d\chi J_0(\sqrt{\chi}) J_0(\sqrt{\chi(1 + gx)}) \exp(-\chi D_0 g^2 t/4).$$

The χ integration can be performed and we obtain

$$c(x, t) = \frac{M_0}{D_0 g t} I_0 \left(\frac{2}{D_0 g^2 t} \sqrt{1 + gx} \right) \exp \left(- \frac{2 + gx}{D_0 g^2 t} \right)$$

where I_0 is a modified Bessel function of order 0. I_ν is defined by

$$I_\nu(x) := i^{-\nu} J_\nu(ix) = \sum_{k=0}^\infty \frac{1}{k!\Gamma(\nu + k + 1)} \left(\frac{x}{2} \right)^{\nu + 2k}.$$

For $g \to 0$ we obtain

$$c(x, t) = \frac{M_0}{\sqrt{4\pi D_0 t}} \exp \left(- \frac{x^2}{4 D_0 t} \right).$$

Problem 8. Consider the time-dependent Schrödinger equation

$$i \frac{\partial}{\partial t} \psi = \left(-\frac{1}{2} \Delta + V(\mathbf{x}) \right) \psi \tag{1}$$

where

$$\Delta = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}.$$

Let

$$\rho := |\psi|^2 = \psi \psi^*, \qquad \mathbf{v} := \Im \left(\frac{\nabla \psi}{\psi} \right) = \Im \begin{pmatrix} \frac{1}{\psi} \frac{\partial \psi}{\partial x_1} \\ \frac{1}{\psi} \frac{\partial \psi}{\partial x_2} \\ \frac{1}{\psi} \frac{\partial \psi}{\partial x_3} \end{pmatrix}.$$

Show that the time-dependent Schrödinger equation can be written as the system of partial differential equations (Madelung equations)

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\mathbf{v}\rho) = -\left(\frac{\partial (v_1 \rho)}{\partial x_1} + \frac{\partial (v_2 \rho)}{\partial x_2} + \frac{\partial (v_3 \rho)}{\partial x_3} \right) \tag{2}$$

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\nabla \left(V(\mathbf{x}) - \frac{\Delta(\rho^{1/2})}{2\rho^{1/2}} \right). \tag{3}$$

Solution 8. To find (2) we start from (1) and

$$-i\hbar \frac{\partial \psi^*}{\partial t} = \left(-\frac{1}{2m}\Delta + V(\mathbf{x}) \right) \psi^*. \tag{4}$$

Now from $\rho = \psi\psi^*$ we obtain

$$\frac{\partial \rho}{\partial t} = \frac{\partial \psi}{\partial t}\psi^* + \psi \frac{\partial \psi^*}{\partial t}.$$

Inserting the time-dependent Schrödinger equations (1) and (4) we obtain

$$\begin{aligned} \frac{\partial \rho}{\partial t} &= \frac{1}{i\hbar}\left(-\frac{1}{2m}\Delta + V(\mathbf{x}) \right)\psi\psi^* + \psi\frac{1}{-i\hbar}\left(-\frac{1}{2m}\Delta + V(\mathbf{x}) \right)\psi^* \\ &= \frac{1}{i\hbar}\left(-\frac{1}{2m}\Delta(\psi\psi^*) \right) - \frac{1}{i\hbar}\left(-\frac{1}{2m}\psi\Delta\psi^* \right) \\ &= \frac{1}{i\hbar}\left(-\frac{1}{2m}\left((\Delta\psi)\psi^* + 2\sum_{j=1}^{3} \frac{\partial \psi}{\partial x_j}\frac{\partial \psi^*}{\partial x_j} \right) \right). \end{aligned}$$

Equation (3) can be derived by writing ψ in to so-called Madelung form

$$\psi(\mathbf{x}, t) = \exp(R(\mathbf{x}, t) + iS(\mathbf{x}, t))$$

where R and S are real-valued functions. Substituting this expression into the Schrödinger equation, dividing by ψ, separating real and imaginary part, and taking the gradient of the equation in S, where $\mathbf{v} = \nabla S$. The Madelung equations are not defined on the nodal set since we divide by ψ.

Problem 9. Consider the time-dependent Schrödinger equation

$$i\hbar \frac{\partial \psi(\mathbf{x}, t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(\mathbf{x}, t) + V(\mathbf{x}, t)\psi(\mathbf{x}, t).$$

Consider the ansatz

$$\psi(\mathbf{x}, t) = \phi(\mathbf{x}, t)\exp(imS(\mathbf{x}, t)/\hbar)$$

where the functions ϕ and S are real. Find the partial differential equations are ϕ and S.

Solution 9. Since

$$\frac{\partial \psi}{\partial t} =$$

and

$$\frac{\partial^2 \psi}{\partial x_j^2} =$$

we obtain the coupled system of partial differential equations

$$\frac{\partial}{\partial t}\phi^2 + \nabla \cdot (\phi^2 \nabla S) = 0$$
$$\frac{\partial}{\partial t}\nabla S + (\nabla S \cdot \nabla)\nabla S = -\frac{1}{m}\left(\nabla \frac{-(\hbar^2/2m)\nabla^2 \phi}{\phi} + \nabla V\right).$$

This is the Madelung representation of the Schrödinger equation. The term

$$-\frac{(\hbar^2/2m)\nabla^2 \phi}{\phi}$$

of the right-hand side of the last equation is known as the Bohm potential in the theory of hidden variables.

Problem 10. Consider the Schrödinger equation $\hat{H}\Psi = E\Psi$ of a particle on the torus. A *torus* surface can be parametrized by the azimuthal angle ϕ and its polar angle θ

$$x(\phi, \theta) = (R + a\cos(\theta))\cos(\phi)$$
$$y(\phi, \theta) = (R + a\cos(\theta))\sin(\phi)$$
$$z(\phi, \theta) = a\sin(\theta)$$

where R and a are the outer and inner radius of the torus, respectively such that the ratio a/R lies between zero and one.
(i) Find $\hat{H}$.
(ii) Apply the separation ansatz

$$\Psi(\theta, \phi) = \exp(im\phi)\psi(\theta)$$

where m is an integer.

Solution 10. (i) We find

$$\hat{H} = -\frac{1}{2}\left(\frac{1}{a^2}\frac{\partial^2}{\partial \theta^2} - \frac{\sin(\theta)}{a(R + a\cos(\theta))}\frac{\partial}{\partial \theta} + \frac{1}{(R + a\cos(\theta))^2}\frac{\partial^2}{\partial \phi^2}\right).$$

(ii) Let $\alpha = a/R$. We obtain the ordinary differential equation

$$\frac{d^2\psi}{d\theta^2} - \frac{a\sin(\theta)}{1+\alpha\cos(\theta)}\frac{d\psi}{d\theta} - \frac{m^2\alpha^2}{(1+\alpha\cos(\theta))^2}\psi(\theta) + \beta\psi = 0$$

where $\theta \in [0, 2\pi]$, $\alpha \in (0,1)$, and β is a real number with $\beta = 2Ea^2$.

Problem 11. The linear one-dimensional *diffusion equation* is given by

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}, \qquad t \geq 0, \quad -\infty < x < \infty$$

where $u(x,t)$ denotes the concentration at time t and position $x \in \mathbb{R}$. D is the diffusion constant which is assumed to be independent of x and t. Given the initial condition $c(x,0) = f(x)$, $x \in \mathbb{R}$ the solution of the one-dimensional diffusion equation is given by

$$u(x,t) = \int_{-\infty}^{\infty} G(x,t|x',0)f(x')dx'$$

where

$$G(x,t|x',t') = \frac{1}{\sqrt{4\pi D(t-t')}}\exp\left(-\frac{(x-x')^2}{4D(t-t')}\right).$$

Here $G(x,t|x',t')$ is called the fundamental solution of the diffusion equation obtained for the initial data $\delta(x-x')$ at $t = t'$, where δ denotes the Dirac delta function.
(i) Let $u(x,0) = f(x) = \exp(-x^2/(2\sigma))$. Find $u(x,t)$.
(ii) Let $u(x,0) = f(x) = \exp(-|x|/\sigma)$. Find $u(x,t)$.

Solution 11.

Problem 12. Let $\mathbf{f} : \mathbb{R}^3 \to \mathbb{R}^3$ be a differentiable function. Consider the first order vector partial differential equation

$$\nabla \times \mathbf{f} = k\mathbf{f}$$

where k is a positive constant.
(i) Find $\nabla \times (\nabla \times \mathbf{f})$.
(ii) Show that $\nabla \mathbf{f} = 0$.

Solution 12.

Problem 13. Consider *Maxwell's equation*

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}, \quad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \quad \mathrm{div}\mathbf{E} = 0, \quad \mathrm{div}\mathbf{B} = 0$$

with $\mathbf{B} = \mu_0 \mathbf{H}$.
(i) Assume that $E_2 = E_3 = 0$ and $B_1 = B_3 = 0$. Simplify Maxwell's equations.
(ii) Now assume that E_1 and B_2 only depends on x_3 and t with

$$E_1(x_3, t) = f(t)\sin(k_3 x_3), \qquad B_2(x_3, t) = g(t)\cos(k_3 x_3)$$

where k_3 is the third component of the wave vector $\mathbf{k}$. Find the system of ordinary differential equations for $f(t)$ and $g(t)$ and solve it and thus find the *dispersion relation*. Note that

$$\nabla \times \mathbf{E} = \begin{pmatrix} \partial E_3/\partial x_2 - \partial E_2/\partial x_3 \\ \partial E_1/\partial x_3 - \partial E_3/\partial x_1 \\ \partial E_2/\partial x_1 - \partial E_1/\partial x_2 \end{pmatrix}.$$

Solution 13. (i) Maxwell's equation written down in components take the form

$$\frac{\partial B_1}{\partial t} - \frac{\partial E_2}{\partial x_3} + \frac{\partial E_3}{\partial x_2} = 0$$
$$\frac{\partial B_2}{\partial t} - \frac{\partial E_3}{\partial x_1} + \frac{\partial E_1}{\partial x_3} = 0$$
$$\frac{\partial B_3}{\partial t} - \frac{\partial E_1}{\partial x_2} + \frac{\partial E_2}{\partial x_1} = 0$$
$$\frac{1}{c^2}\frac{\partial E_1}{\partial t} - \frac{\partial B_3}{\partial x_2} + \frac{\partial B_2}{\partial x_3} = 0$$
$$\frac{1}{c^2}\frac{\partial E_2}{\partial t} - \frac{\partial B_1}{\partial x_3} + \frac{\partial B_3}{\partial x_1} = 0$$
$$\frac{1}{c^2}\frac{\partial E_3}{\partial t} - \frac{\partial B_2}{\partial x_1} + \frac{\partial B_1}{\partial x_2} = 0$$

and

$$\frac{\partial E_1}{\partial x_1} + \frac{\partial E_2}{\partial x_2} + \frac{\partial E_3}{\partial x_3} = 0, \qquad \frac{\partial B_1}{\partial x_1} + \frac{\partial B_2}{\partial x_2} + \frac{\partial B_3}{\partial x_3} = 0.$$

With the simplification $E_2 = E_3 = 0$ and $B_1 = B_3 = 0$ we arrive at

$$\frac{\partial E_1}{\partial x_2} = 0, \qquad \frac{\partial B_2}{\partial t} + \frac{\partial E_1}{\partial x_3} = 0, \qquad \frac{\partial B_2}{\partial x_1} = 0, \qquad \frac{\partial E_1}{\partial t} + c^2\frac{\partial B_2}{\partial x_3} = 0$$

and

$$\frac{\partial E_1}{\partial x_1} = 0, \qquad \frac{\partial B_2}{\partial x_2} = 0.$$

(ii) With the assumption that E_1 and B_2 only depend on x_3 and t the equations reduce to two equations

$$\frac{\partial B_2}{\partial t} + \frac{\partial E_1}{\partial x_3} = 0, \qquad \frac{\partial E_1}{\partial t} + c^2\frac{\partial B_2}{\partial x_3} = 0.$$

Inserting the ansatz for E_1 and B_2 yields the system of differential equations for f and g

$$\frac{df}{dt} - c^2 k_3 g = 0, \qquad \frac{dg}{dt} + k_3 f = 0.$$

These are the equations for the harmonic oscillator. Starting from the ansatz for the solution

$$f(t) = A\sin(\omega t) + B\cos(\omega t)$$

where ω is the frequency, yields

$$g(t) = \frac{A\omega}{c^2 k_3}\cos(\omega t) - \frac{B\omega}{c^2 k_3}\sin(\omega t)$$

and the dispersion relation is

$$\omega^2 = c^2 k_3^2.$$

Problem 14. Let k be a constant. Show that the vector partial differential equation

$$\nabla \times \mathbf{u} = k\mathbf{u}$$

has the general solution

$$\mathbf{u}(x_1, x_2, x_3) = \nabla \times (\mathbf{c}v) + \frac{1}{k}\nabla \times \nabla(\mathbf{c}v)$$

where $\mathbf{c}$ is a constant vector and v satisfies the partial differential equation

$$\nabla^2 v + k^2 v = 0.$$

Solution 14.

Problem 15. Consider the partial differential equation of first order

$$\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3}\right) f = 0.$$

Show that f is of the form

$$f(x_1 - x_2, x_2 - x_3, x_3 - x_1).$$

Solution 15. We set $y_{12} = x_1 - x_2$, $y_{23} = x_2 - x_3$, $y_{31} = x_3 - x_1$. Then applying the chain rule we have

$$\frac{\partial f}{\partial x_1} = \frac{\partial f}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_1} + \frac{\partial f}{\partial y_{31}}\frac{\partial y_{31}}{\partial x_1} = \frac{\partial f}{\partial y_{12}} - \frac{\partial f}{\partial y_{31}}.$$

Analogously we have

$$\frac{\partial f}{\partial x_2} = \frac{\partial}{\partial} - \frac{\partial}{\partial}, \qquad \frac{\partial f}{\partial x_3} = \frac{\partial}{\partial} - \frac{\partial}{\partial}.$$

Thus $f(x-x, x-x, x-x)$ satisfies the partial differential equation.

Problem 16. Consider the operators

$$D_x + \frac{\partial}{\partial x}, \qquad D_t = \frac{\partial}{\partial t} + u(x,t)\frac{\partial}{\partial x}.$$

Find the commutator

$$[D_x, D_t]f(x,t)$$

Solution 16. We obtain

$$[D_x, D_t]f = \frac{\partial u}{\partial x} D_x f.$$

Thus

$$[D_x, D_t] = u_x D_x.$$

Problem 17. Let $\mathbf{C}$ be a constant column vector in $\mathbb{R}^n$ and $\mathbf{x}$ be a column vector in $\mathbb{R}^n$. Show that

$$\mathbf{C}e^{\mathbf{x}^T\mathbf{C}} \equiv \nabla e^{\mathbf{x}^T\mathbf{C}}$$

where ∇ denotes the gradient.

Solution 17. Since

$$\mathbf{x}^T\mathbf{C} = \sum_{j=1}^{n} x_j C_j$$

and

$$\frac{\partial}{\partial x_j}\mathbf{x}^T\mathbf{C} = C_j$$

we find the identity.

Problem 18. Consider the partial differential equation (*Laplace equation*)

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{on} \quad [0,1]\times[0,1]$$

with the boundary conditions

$$u(x,0) = 1, \quad u(x,1) = 2, \quad u(0,y) = 1, \quad u(1,y) = 2.$$

Apply the *central difference scheme*

$$\left(\frac{\partial^2 u}{\partial x^2}\right)_{j,k} \approx \frac{u_{j-1,k} - 2u_{j,k} + u_{j+1,k}}{(\Delta x)^2}, \quad \left(\frac{\partial^2 u}{\partial y^2}\right)_{j,k} \approx \frac{u_{j,k-1} - 2u_{j,k} + u_{j,k+1}}{(\Delta y)^2}$$

and then solve the linear equation. Consider the cases $\Delta x = \Delta y = 1/3$ and $\Delta x = \Delta y = 1/4$.

Solution 18. Case $\Delta x = \Delta y = 1/3$: We have the 16 grid points (j,k) with $j,k = 0,1,2,3$ and four interior grid points $(1,1)$, $(2,1)$, $(1,2)$, $(2,2)$. For $j = k = 1$ we have

$$u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} = 0.$$

For $j = 1$, $k = 2$ we have

$$u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} = 0.$$

For $j = 2$, $k = 1$ we have

$$u_{1,1} + u_{3,1} + u_{2,0} + u_{2,3} - 4u_{2,1} = 0.$$

For $j = 2$, $k = 2$ we have

$$u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} = 0.$$

From the boundary conditions we find that

$$u_{0,1} = 1, \quad u_{1,0} = 1, \quad u_{0,2} = 1, \quad u_{1,3} = 2$$

$$u_{3,1} = 2, \quad u_{2,0} = 1, \quad u_{3,2} = 2, \quad u_{2,3} = 2.$$

Thus we find the linear equation in matrix form $A\mathbf{u} = \mathbf{b}$

$$\begin{pmatrix} -4 & 1 & 1 & 0 \\ 1 & -4 & 0 & 1 \\ 1 & 0 & -4 & 1 \\ 0 & 1 & 1 & -4 \end{pmatrix} \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ -3 \\ -4 \end{pmatrix}.$$

The matrix A is invertible with

$$A^{-1} = -\frac{1}{12} \begin{pmatrix} 7/2 & 1 & 1 & 1/2 \\ 1 & 7/2 & 1/2 & 1 \\ 1 & 1/2 & 7/2 & 1 \\ 1/2 & 1 & 1 & 7/2 \end{pmatrix}$$

and thus we obtain the solution

$$\mathbf{u} = \begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{2,1} \\ u_{2,2} \end{pmatrix} = \begin{pmatrix} 5/4 \\ 3/2 \\ 3/2 \\ 7/4 \end{pmatrix}.$$

Case $\Delta x = \Delta y = 1/4$. We have 25 grid points and 9 interior grid points. The equations for these nine grid points are as follows

$$\begin{aligned}
u_{0,1} + u_{2,1} + u_{1,0} + u_{1,2} - 4u_{1,1} &= 0 \\
u_{0,2} + u_{2,2} + u_{1,1} + u_{1,3} - 4u_{1,2} &= 0 \\
u_{0,3} + u_{2,3} + u_{1,2} + u_{1,4} - 4u_{1,3} &= 0 \\
u_{1,1} + u_{3,1} + u_{2,0} + u_{2,2} - 4u_{2,1} &= 0 \\
u_{1,2} + u_{3,2} + u_{2,1} + u_{2,3} - 4u_{2,2} &= 0 \\
u_{1,3} + u_{3,3} + u_{2,2} + u_{2,4} - 4u_{2,3} &= 0 \\
u_{2,1} + u_{4,1} + u_{3,0} + u_{3,2} - 4u_{3,1} &= 0 \\
u_{2,2} + u_{4,2} + u_{3,1} + u_{3,3} - 4u_{3,2} &= 0 \\
u_{2,3} + u_{4,3} + u_{3,2} + u_{3,4} - 4u_{3,3} &= 0.
\end{aligned}$$

From the boundary conditions we find the 12 values

$$u_{0,1} = 1,\ \ u_{1,0} = 1,\ \ u_{0,2} = 1,\ \ u_{0,3} = 1,\ \ u_{1,4} = 2,\ \ u_{2,0} = 1,$$

$$u_{2,4} = 2,\ \ u_{4,1} = 2,\ \ u_{3,0} = 1,\ \ u_{4,2} = 2,\ \ u_{4,3} = 2,\ \ u_{3,4} = 2.$$

Inserting the boundary values the linear equation in matrix form $A\mathbf{u} = \mathbf{b}$ is given by

$$\begin{pmatrix}
-4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -4 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & -4 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & -4 & 1 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & -4 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & -4 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & -4
\end{pmatrix}
\begin{pmatrix} u_{1,1} \\ u_{1,2} \\ u_{1,3} \\ u_{2,1} \\ u_{2,2} \\ u_{2,3} \\ u_{3,1} \\ u_{3,2} \\ u_{3,3} \end{pmatrix}
=
\begin{pmatrix} -2 \\ -1 \\ -3 \\ -1 \\ 0 \\ -2 \\ -3 \\ -2 \\ -4 \end{pmatrix}.$$

The matrix is invertible and thus the solution is $\mathbf{u} = A^{-1}\mathbf{b}$.

Problem 19. Consider the partial differential equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi + k^2 \psi = 0.$$

(i) Apply the transformation

$$u(x,y) = x^2 - y^2, \qquad v(x,y) = 2xy, \qquad \psi(x,y) = \widetilde{\psi}(u(x,y), v(x,y)).$$

(ii) Then introduce polar coordinates $u = r\cos\phi$, $v = r\sin\phi$.

Solution 19. (i) Note that

$$u^2+v^2=(x^2+y^2)^2, \qquad x^2+y^2=\sqrt{u^2+v^2}.$$

Applying the chain rule we have

$$\frac{\partial\psi}{\partial x}=\frac{\partial\widetilde{\psi}}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial\widetilde{\psi}}{\partial v}\frac{\partial v}{\partial x}=2x\frac{\partial\widetilde{\psi}}{\partial u}+2y\frac{\partial\widetilde{\psi}}{\partial v}$$

and

$$\frac{\partial^2\psi}{\partial x^2}=2\frac{\partial\widetilde{\psi}}{\partial u}+4x^2\frac{\partial^2\psi}{\partial u^2}+8xy\frac{\partial^2\widetilde{\psi}}{\partial u\partial v}+4y^2\frac{\partial^2\widetilde{\psi}}{\partial v^2}.$$

Analogously we have

$$\frac{\partial\psi}{\partial y}=\frac{\partial\widetilde{\psi}}{\partial u}\frac{\partial u}{\partial y}+\frac{\partial\widetilde{\psi}}{\partial v}\frac{\partial v}{\partial y}=-2y\frac{\partial\widetilde{\psi}}{\partial u}+2x\frac{\partial\widetilde{\psi}}{\partial v}$$

and

$$\frac{\partial^2\psi}{\partial y^2}=-2\frac{\partial\widetilde{\psi}}{\partial u}+4y^2\frac{\partial^2\widetilde{\psi}}{\partial u^2}-8xy\frac{\partial^2\widetilde{\psi}}{\partial u\partial v}+4x^2\frac{\partial^2\widetilde{\psi}}{\partial v^2}.$$

Consequently we find the partial differential equation

$$4\sqrt{u^2+v^2}\left(\frac{\partial^2\widetilde{\psi}}{\partial u^2}+\frac{\partial^2\widetilde{\psi}}{\partial v^2}\right)+k^2\widetilde{\psi}=0.$$

Introducing polar coordinates we have

$$\frac{\partial}{\partial r}\left(r\frac{\partial\widetilde{\psi}}{\partial r}\right)+\frac{1}{r}\frac{\partial^2\widetilde{\psi}}{\partial\phi^2}+\frac{k^2}{4}=0.$$

One can now try a separation ansatz

$$\widetilde{\psi}(r,\phi)=f(r)g(\phi)$$

which yields

$$\frac{r}{f(r)}\frac{d}{dr}\left(r\frac{df}{dr}\right)+\frac{1}{g(\phi)}\frac{d^2g}{d\phi^2}+r\frac{k^2}{4}=0.$$

Thus we find the two ordinary differential equations

$$\frac{r}{f}\left(\frac{df}{dr}+r\frac{d^2f}{dr^2}\right)+r\frac{k^2}{4}=C$$

$$-\frac{1}{g}\frac{d^2g}{d\phi^2}=C$$

where C is a constant.

Problem 20. Starting from Maxwell's equations in vacuum show that

$$\mathbf{k} \times \mathbf{E} = \omega \mathbf{B}.$$

Solution 20.

Problem 21. Show that the linear partial differential equation

$$\frac{\partial^2 u}{\partial \theta^2} - \frac{\partial^2 u}{\partial t^2} = 0$$

admits the solution $u(t,\theta) = \sin(\theta)\sin(t)$. Does this solution satisfies the boundary condition $u(t, \theta = \pi) = 0$ and the initial condition $u(t = 0, \theta) = 0$?

Solution 21.

Problem 22. Consider the Hamilton operator

$$\hat{H} = -\frac{\hbar^2}{2} \sum_{j=1}^{3} \frac{1}{m_j} \frac{\partial^2}{\partial q_j^2} + \frac{k}{2}((q_2 - q_1 - d)^2 + (q_3 - q_2 - d)^2)$$

where d is the distance between two adjacent atoms. Apply the linear transformation

$$\xi(q_1, q_2, q_3) = (q_2 - q_1) - d$$
$$\eta(q_1, q_2, q_3) = (q_3 - q_2) - d$$
$$X(q_1, q_2, q_3) = \frac{1}{M}(m_1 q_1 + m_2 q_2 + m_3 q_3)$$

where $M := m_1 + m_2 + m_3$ and show that the Hamilton operator decouples.

Solution 22. The Hamilton operator $\hat{H}$ takes the form $\hat{H} = \hat{H}_{CM} + \hat{H}_h$, where

$$\hat{H}_{CM} = \frac{1}{2M}\hat{p}_M^2, \qquad p_M := -i\hbar \frac{\partial}{\partial X}$$
$$\hat{H}_h = -\frac{\hbar^2}{2\mu_1}\frac{\partial^2}{\partial \xi^2} - \frac{\hbar^2}{2\mu_2}\frac{\partial^2}{\partial \eta^2} + \frac{\hbar^2}{m_2}\frac{\partial^2}{\partial \xi \partial \eta} + \frac{k}{2}(\xi^2 + \eta^2)$$

where $\mu_1 = (m_1 m_2)/(m_1 + m_2)$, $\mu_2 = (m_2 m_3)/(m_2 + m_3)$. Thus the centre of mass motion is decoupled.

Problem 23. Consider the Hamilton operator for three particles

$$\hat{H} = -\frac{1}{2m}\left(\frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\right) - 6g\delta(x_1 - x_2)\delta(x_2 - x_3)$$

and the eigenvalue problem $\hat{H}u(x_1, x_2, x_3) = Eu(x_1, x_2, x_3)$. Apply the transformation

$$\begin{aligned} y_1(x_1, x_2, x_3) &= \sqrt{\frac{2}{3}}\left(\frac{1}{2}(x_1 + x_2) - x_3\right) \\ y_2(x_1, x_2, x_3) &= \frac{1}{\sqrt{2}}(x_1 - x_2) \\ y_3(x_1, x_2, x_3) &= \frac{1}{3}(x_1 + x_2 + x_3) \\ \widetilde{u}(y_1(x_1, x_2, x_3), y_2(x_1, x_2, x_3), y_3(x_1, x_2, x_3)) &= u(x_1, x_2, x_3) \end{aligned}$$

where y_3 is the centre-of mass position of the three particles and y_1, y_2 give their relative positions up to constant factors. Find the Hamilton operator for the new coordinates.

Solution 23. First we note that

$$\frac{\partial(x_1, x_2, x_3)}{\partial(y_3, y_1, y_2)} = -\sqrt{3}.$$

The Hamilton operator takes the form

$$\widetilde{H} = -\frac{1}{6m}\frac{\partial^2}{\partial y_3^2} - \frac{1}{2m}\left(\frac{\partial^2}{\partial y_1^2} + \frac{\partial^2}{\partial y_2^2}\right) - 2\sqrt{3}g\delta(y_1)\delta(y_2).$$

Consequently the variable can be y_3 can be separated out.

Problem 24. Consider the four dimensional Laplace equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} + \frac{\partial^2 u}{\partial x_3^2} + \frac{\partial^2 u}{\partial x_4^2} = 0.$$

Transform the equation into polar coordinates (r, θ, ϕ, χ)

$$\begin{aligned} x_1(r, \theta, \phi, \chi) &= r\sin(\theta/2)\sin((\phi - \chi)/2) \\ x_2(r, \theta, \phi, \chi) &= r\sin(\theta/2)\cos((\phi - \chi)/2) \\ x_3(r, \theta, \phi, \chi) &= r\cos(\theta/2)\sin((\phi + \chi)/2) \\ x_4(r, \theta, \phi, \chi) &= r\cos(\theta/2)\cos((\phi + \chi)/2) \\ u(\mathbf{x}) &= \widetilde{u}(r(\mathbf{x}), \theta(\mathbf{x}), \phi(\mathbf{x}), \chi(\mathbf{x})) \end{aligned}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4)$.

Solution 24. In polar coordinates we find

$$\nabla_r^2\widetilde{u} + \frac{4}{r^2}\nabla_\Omega^2\widetilde{u} = 0$$

where

$$\nabla_r^2 := \frac{1}{r^3}\frac{\partial}{\partial r}r^3\frac{\partial}{\partial r}$$
$$\nabla_\Omega^2 := \sin\theta\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\left(\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial\chi^2} - 2\cos\theta\frac{\partial^2}{\partial\phi\partial\chi}\right).$$

Problem 25. A nonrelativistic particle is described by the Schrödinger equation

$$\left(\frac{\hat{\mathbf{p}}^2}{2m} + V(\mathbf{x},t)\right)\psi(\mathbf{x},t) = i\hbar\frac{\partial\psi}{\partial t}$$

where $\hat{\mathbf{p}} := -i\hbar\nabla$. Write the wave function ψ in *polar form*

$$\psi(\mathbf{x},t) = R(\mathbf{x},t)\exp(iS(\mathbf{x},t)/\hbar)$$

where R, S are real functions and $R(\mathbf{x},t) \geq 0$. Give an interpretation of $\rho = R^2$.

Solution 25. The wave equation is equivalent to a set of two real differential equations

$$\frac{(\nabla S)^2}{2m} + V + Q = -\frac{\partial S}{\partial t}$$
$$\frac{\partial\rho}{\partial t} + \nabla\left(\rho\frac{\nabla S}{m}\right) = 0$$

where

$$Q := -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R}.$$

The equation for the time evolution of ρ is a conservation equation that provides the consistency of the interpretation of ρ as the probabilty density.

Problem 26. The time-dependent Schrödinger equation for the one-dimensional free particle case can be written in either position or momentum space as

$$-\frac{\hbar^2}{2m}\frac{\partial^2\psi(x,t)}{\partial x^2} = i\hbar\frac{\partial\psi(x,t)}{\partial t}$$
$$\frac{p^2}{2m}\phi(p,t) = i\hbar\frac{\partial\phi(x,t)}{\partial t}.$$

Consider the momentum space approach with the solution

$$\phi(p,t) = \phi_0(p)\exp(-ip^2t/2m\hbar)$$

where $\phi(p,0) = \phi_0(p)$ is the initial momentum distribution. We define $\hat{x} := i\hbar(\partial/\partial p)$. Find $\langle \hat{x} \rangle(t)$.

Solution 26. We have

$$\begin{aligned}\langle \hat{x} \rangle(t) &= \int_{-\infty}^{\infty} \phi^*(p,t) \left(i\hbar \frac{\partial}{\partial p} \phi(p,t) \right) dp \\ &= \int_{-\infty}^{\infty} \phi_0^*(p) \hat{x} \phi_0(p) dp + \frac{t}{m} \int_{-\infty}^{\infty} p |\phi_0(p,t)|^2 dp \\ &= \langle \hat{x} \rangle_0 + \frac{\langle p \rangle_0 t}{m}.\end{aligned}$$

Problem 27. Consider the partial differential equation

$$\frac{\partial^2}{\partial \mathbf{x} \partial t} (\ln(\det(I_n + tD\mathbf{f}(\mathbf{x}))) = \mathbf{0}$$

where $D\mathbf{f}(\mathbf{x})$ ($\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$) is the Jacobian matrix. Find the solution of the initial value problem.

Solution 27. The general solution is

$$\det(I_n + tD\mathbf{f}(\mathbf{x})) = g(\mathbf{x})h(t)$$

where $g : \mathbb{R}^n \to \mathbb{R}$, $h : \mathbb{R} \to \mathbb{R}$. At $t = 0$ since $\det(I_n) = 1$ we have $g(\mathbf{x}) = h^{-1}(0)$. Thus

$$\det(I_n + tD\mathbf{f}(\mathbf{x})) = \frac{h(t)}{h(0)}.$$

Thus

$$\frac{\partial}{\partial \mathbf{x}} \det(I_n + tD\mathbf{f}(\mathbf{x})) = \mathbf{0}.$$

Problem 28. The partial differential equation

$$\begin{aligned}&\left(\frac{\partial u}{\partial x_1}\right)^2 \left(\frac{\partial^2 u}{\partial x_2 \partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_3}\right)^2 + \left(\frac{\partial u}{\partial x_3}\right)^2 \left(\frac{\partial^2 u}{\partial x_1 \partial x_2}\right)^2 \\ &-2 \left(\frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_3} \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_3} \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_2 \partial x_3} + \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_3} \frac{\partial^2 u}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_3} \right) \\ &+4 \frac{\partial u}{\partial x_1} \frac{\partial u}{\partial x_2} \frac{\partial u}{\partial x_3} \frac{\partial^3 u}{\partial x_1 \partial x_2 \partial x_3} = 0\end{aligned}$$

has some link to the Bateman equation. Find the Lie symmetries. There are Lie-Bäcklund symmetries? Is there a Legendre transformation to linearize this partial differential equation?

Solution 28.

Problem 29. Consider the two-dimensional heat equation

$$\frac{\partial T}{\partial t} = \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2}$$

Show that

$$T(x,y,t) = \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{x^2+(y-1)^2}{4t}\right) - \frac{1}{\sqrt{\pi t}} \exp\left(-\frac{x^2+(y+1)^2}{4t}\right)$$

satisfies this partial differential equation.

Solution 29.

$$=$$

Problem 30. Let $\epsilon > 0$. Consider the *Fokker-Planck equation*

$$\frac{\partial u}{\partial t} = \epsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(xu)$$

with $u \geq 0$ for all x.
(i) Find steady state solutions, i.e. find solutions of the ordinary differential equation

$$\epsilon \frac{\partial^2 u}{\partial x^2} + \frac{\partial}{\partial x}(xu) = 0.$$

(ii) Find time dependent solution of the inital value problem with

$$u(t=0,x) = N_{\sigma_0}(x-\mu_0) = \frac{1}{\sqrt{2\pi\sigma_0}} e^{-(x-\mu_0)^2/(2\sigma_0)}$$

with $\sigma_0 \geq 0$ the variance and μ_0 the mean. With $\sigma = 0$ we have the delta function at 0.

Solution 30. (i) We have

$$\frac{d}{dx}\left(\frac{du}{dx} + \frac{xu}{\epsilon}\right) = 0.$$

Integration yields

$$\frac{du}{dx} + \frac{xu}{\epsilon} = C_1$$

where C_1 is a constant of integration. Consequently

$$\frac{d}{dx}(ue^{x^2/(2\epsilon)}) = C_1 e^{x^2/(2\epsilon)}.$$

Integrating once more yields

$$u(x) = \left(C_1 \int_0^x e^{s^2/(2\epsilon)} ds + C_2 \right) e^{-x^2/(2\epsilon)}$$

where C_2 is another constant of integration. The condition $u \geq 0$ implies $C_1 = 0$ and the condition

$$\int_{-\infty}^{\infty} u(x) dx = 1$$

implies

$$u(x) = \frac{1}{\sqrt{2\pi\epsilon}} e^{-x^2/(2\epsilon)}.$$

Thus this time-independent solution is the normal distribution with mean 0 and variance ϵ.

(ii) We obtain

$$u(t,x) = N_{\sigma(t)}(x - \mu(t))$$

with

$$\mu(t) = \mu_0 e^{-t}, \qquad \sigma(t) = \epsilon + (\sigma_0 - \epsilon)e^{-2t}.$$

Problem 31. Consider the Schrödinger equation

$$\left(-\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2c \sum_{j<k} \delta(x_j - x_k) \right) u(\mathbf{x}) = E u(\mathbf{x})$$

describing a one-dimensional Bose gas with the δ-fnction repulsive interaction. Show that for $N = 2$ the eigenvalue problem can be solved with a exponential ansatz.

Solution 31. For $N = 2$ we have the unnormalized wave function

$$u(x_1, x_2) = e^{ik_1x_1 + ik_2x_2} + e^{i\theta_{12}} e^{ik_2x_1 + ik_1x_2}$$

for $x_1 < x_2$ where the phase shift θ_{12} is given by

$$e^{i\theta_{12}} = -\frac{c - i(k_1 - k_2)}{c + i(k_1 - k_2)}.$$

The wave function for $x_1 > x_2$ is obtained by using the symmetry relation $u(x_1, x_2) = u(x_2, x_1)$. For $k_1 = k_2$ we have $u(x_1, x_2) = 0$.

Problem 32. (i) Show that the power series

$$P(t,\lambda) = \sum_{n=0}^{\infty} \rho_n \lambda^n \cos(\sqrt{n}t)$$

satisfies the linear partial differential equation (diffusion type equation)

$$\frac{\partial^2 P(t,\lambda)}{\partial t^2} = -\lambda \frac{\partial}{\partial \lambda} P(t,\lambda)$$

with boundary conditions

$$P(0,\lambda) = \sum_{n=0}^{\infty} \rho_n \lambda^n = \rho(\lambda), \quad \frac{\partial P(0,\lambda)}{\partial t} = 0, \quad P(t,0) = \rho(0).$$

(ii) Perform a *Laplace transformation*

$$\widetilde{P}(z,\lambda) := \int_0^{\infty} e^{-zt} P(t,\lambda) dt$$

and show that $\widetilde{P}(z,\lambda)$ obeys the differential equation

$$z^2 \widetilde{P}(z,\lambda) - z\rho(\lambda) = -\lambda \frac{\partial}{\partial \lambda} \widetilde{P}(z,\lambda)$$

with boundary condition

$$\widetilde{P}(z,0) = \frac{\rho(0)}{z}$$

and

$$\widetilde{P}(z,\lambda) = \int_0^{\lambda} \exp(-z^2 \ln(\lambda/x)) \frac{\rho(x)}{x} dx.$$

Solution 32.

Problem 33. Consider the Schrödinger equation (eigenvalue equation) for an identical three-body harmonic oscillator

$$-\frac{\hbar^2}{2M} \sum_{j=1}^{3} \Delta_{\mathbf{r}_j} \Psi + \frac{c}{2} \sum_{k>j=1}^{3} (\mathbf{r}_j - \mathbf{r}_k)^2 \Psi = E\Psi$$

where M denotes the mass and $\Delta_{\mathbf{r}_j}$ is the Laplace operator with respect to the position vector $\mathbf{r}_j$. In the center-of-mass frame the position vector

$\mathbf{r}_j$ is replaced by the Jacobi coordinate vectors $\mathbf{R}$, $\mathbf{x}$, $\mathbf{y}$

$$\mathbf{R} = \frac{1}{\sqrt{3}}(\mathbf{r}_1 + \mathbf{r}_2 + \mathbf{r}_3) = \mathbf{0}$$

$$\mathbf{x} = \frac{1}{\sqrt{6}}(2\mathbf{r}_1 - \mathbf{r}_2 - \mathbf{r}_3) = \frac{\sqrt{3}}{\sqrt{2}}\mathbf{r}_1$$

$$\mathbf{y} = \frac{1}{\sqrt{2}}(\mathbf{r}_2 - \mathbf{r}_3)$$

$$\Psi(\mathbf{R}(\mathbf{r}_j), \mathbf{x}(\mathbf{r}_j), \mathbf{y}(\mathbf{r}_j)) = \Psi(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3).$$

Show that the eigenvalue equation takes the form

$$-\frac{\hbar^2}{2M}(\Delta_{\mathbf{x}} + \Delta_{\mathbf{y}})\Psi + \frac{3c}{2}(\mathbf{x}^2 + \mathbf{y}^2)\Psi = E\Psi$$

under this transformation.

Solution 33.

Problem 34. Show that

$$u(x,t) = \frac{1}{\sqrt{4\pi D t}} \exp\left(-\frac{(x-x_0)^2}{4Dt}\right)$$

satisfies the one-dimensional diffusion equation

$$\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$$

with the initial condition

$$u(x,0) = \delta(x - x_0).$$

Solution 34.

$$=$$

Problem 35. Consider the integral equation

$$\frac{\partial c(x,t)}{\partial t} = -xc(x,t) + 2\int_x^\infty c(y,t)dy.$$

Let $s > 0$. Show that

$$c(x,t) = \left(1 + \frac{t}{s}\right)^2 e^{-x(s+t)}$$

with $c(x,0) = e^{-sx}$ satisfies the integral equation. Show that taking the derivative with respect to x we obtain the linear partial differential equation

$$\frac{\partial^2 c}{\partial t \partial x} = -x\frac{\partial c}{\partial x} - 3c.$$

Solution 35.

Problem 36. Consider the one-dimensional diffusion equation

$$\frac{\partial u'}{\partial t} = D\frac{\partial^2 u'}{\partial x^2}, \quad u'(x, t = 0) = u(x)$$

Show that

$$u'(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{x} u(x + 2\sqrt{Dt}s)e^{-s^2}ds, \quad x > 0$$

is a solution.

Solution 36.

Chapter 2

Nonlinear Partial Differential Equations

Problem 1. Consider the system of quasi-linear partial differential equations

$$\frac{\partial u}{\partial t} + \frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} - v\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0.$$

Show that this system arise as compatibility conditions $[L, M] = 0$ of an overdetermined system of linear equations $L\Psi = 0$, $M\Psi = 0$, where $\Psi(x, y, t, \lambda$ is a function, λ is a spectral parameter, and the *Lax pair* is given by

$$L = \frac{\partial}{\partial t} - v\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial y}, \qquad M = \frac{\partial}{\partial y} + u\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial x}.$$

Solution 1. Using the product rule we have

$$\begin{aligned}[L, M]\Psi &= L(M\Psi) - M(L\Psi)\\
&= \left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial y}\right)\left(\frac{\partial}{\partial y} + u\frac{\partial}{\partial x} - \lambda\frac{\partial}{\partial x}\right)\Psi\\
&= \frac{\partial u}{\partial t}\frac{\partial \Psi}{\partial x} - v\frac{\partial u}{\partial x}\frac{\partial \Psi}{\partial x} - \lambda\frac{\partial u}{\partial y}\frac{\partial \Psi}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial \Psi}{\partial x} + u\frac{\partial v}{\partial x}\frac{\partial \Psi}{\partial x} - \lambda\frac{\partial v}{\partial x}\frac{\partial \Psi}{\partial x}\end{aligned}$$

$$= \left(\frac{\partial u}{\partial t} - v\frac{\partial u}{\partial x} - \lambda\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} - \lambda\frac{\partial v}{\partial x} \right) \frac{\partial \Psi}{\partial x}$$
$$= \left(\frac{\partial u}{\partial t} - v\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + u\frac{\partial v}{\partial x} - \lambda\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \right) \right) \frac{\partial \Psi}{\partial x}.$$

Thus from $[L, M]\Psi = 0$ the system of partial differential equations follows.

Problem 2. Find the *traveling wave solution*

$$u(x,t) = f(x - ct) \qquad c = \text{constant} \tag{1}$$

of the one-dimensional *sine-Gordon equation*

$$\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial t^2} = \sin(u). \tag{2}$$

Solution 2. Let

$$s(x,t) := x - ct. \tag{3}$$

Then we find the ordinary differential equation for the function f

$$(1 - c^2)\frac{d^2 f}{ds^2} = \sin(f). \tag{4}$$

Solutions to (4) can be written as elliptic integrals

$$\int^f \frac{dz}{\sqrt{2(E - \cos z)}} = \frac{s}{\sqrt{1 - c^2}} \tag{5}$$

where E is an arbitrary constant of integration. From (5) it follows that

$$f(s) = \cos^{-1}\left(2\text{cd}^2\left(\frac{s}{\gamma\sqrt{1 - c^2}} \right) - 1 \right) \tag{6}$$

where cd is an elliptic function of modulus $\gamma = 2/(E + 1)$.

Problem 3. Show that the nonlinear nondispersive part of the *Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} + (\alpha + \beta u)\frac{\partial u}{\partial x} = 0 \tag{1}$$

possesses *shock wave solutions* that are intrinsically implicit

$$u(x,t) = f(x - (\alpha + \beta u(x,t))t) \tag{2}$$

with the initial value problem $u(x, t = 0) = f(x)$, where α, $\beta \in \mathbb{R}$.

Solution 3. Since

$$\frac{\partial u}{\partial t} = \left(-\alpha - \beta u - \beta t \frac{\partial u}{\partial t}\right) f'$$
$$\frac{\partial u}{\partial x} = \left(1 - \beta t \frac{\partial u}{\partial x}\right) f'$$

where f' is the derivative of f with respect to the argument we obtain

$$\left(1 - \beta t \frac{\partial u}{\partial x}\right) \frac{\partial u}{\partial t} = \left(-\alpha - \beta u - \beta t \frac{\partial u}{\partial t}\right) \frac{\partial u}{\partial x}. \tag{4}$$

From (4) it follows that

$$\frac{\partial u}{\partial t} = (-\alpha - \beta u) \frac{\partial u}{\partial x} \tag{6}$$

which is (1).

Problem 4. Consider the *Korteweg-de Vries-Burgers equation*

$$\frac{\partial u}{\partial t} + a_1 u \frac{\partial u}{\partial x} + a_2 \frac{\partial^2 u}{\partial x^2} + a_3 \frac{\partial^3 u}{\partial x^3} = 0 \tag{1}$$

where a_1, a_2 and a_3 are non-zero constants. It contains dispersive, dissipative and nonlinear terms.
(i) Find a solution of the form

$$u(x,t) = \frac{b_1}{(1 + \exp(b_2(x + b_3 t + b_4)))^2} \tag{2}$$

where b_1, b_2, b_3 and b_4 are constants determined by a_1, a_2, a_3 and a_4.
(ii) Study the case $t \to \infty$ and $t \to -\infty$.

Solution 4. (i) This is a typical problem for computer algebra. Inserting the ansatz (2) into (1) yields the conditions

$$b_3 = -(a_1 b_1 + a_2 b_2 + a_3 b_2^2) \tag{3}$$

and

$$(2a_1b_1+3a_2b_2+9a_3b_2^2)\exp((a_1b_1b_2+a_2b_2^2+a_3b_2^3)t)+(a_1b_1+3a_2b_2-3a_3b_2^2) = 0. \tag{4}$$

The quantity b_4 is arbitrary. Thus

$$2a_1b_1 = -3a_2b_2 - 9a_3b_2^2, \qquad a_1b_1 = -3a_2b_2 + 3a_3b_2^2. \tag{5}$$

Solving (5) with respect to b_1 and b_2 yields

$$b_1 = -\frac{12a_2^2}{25a_1a_3}, \qquad b_2 = \frac{a_2}{5a_3}. \tag{6}$$

Inserting (6) into (3) gives

$$b_3 = \frac{6}{25}\frac{a_2^2}{a_3}. \tag{7}$$

Thus the solution is

$$u(x,t) = -\frac{12a_2^2}{25a_1a_3}\left(1 + \exp\left(\frac{a_2}{5a_3}\left(x + \frac{6a_2^2}{25a_3}t + b_4\right)\right)\right)^{-2}. \tag{8}$$

(ii) Two asymptotic values exist which are for

$$t \to -\infty, \qquad u_I = \frac{2v_c}{a} \tag{9a}$$

and for

$$t \to \infty, \qquad u_{II} = 0. \tag{9b}$$

Problem 5. The *Fisher equation* is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + au(1-u) \tag{1}$$

where the positive constant a is a measure of intensity of selection. (i) Consider the substitution

$$v(x,\tau(t)) = \frac{1}{6}u(x,t), \qquad \tau(t) = 5t. \tag{2}$$

(i) Show that (1) takes the form

$$5\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + av(1-6v). \tag{3}$$

(ii) Consider the following ansatz

$$v(x,t) := \sum_{j=0}^{\infty} v_j(x,t)\phi^{j-p}(x,t) \tag{4}$$

is single-valued about the solution movable singular manifold $\phi = 0$. This means p is a positive integer, recursion relationships for v_j are self-consistent, and the ansatz (4) has enough free functions in the sense of the Cauchy-Kowalevskia theorem.
(iii) Try to truncate the expansion (4) with $v_j = 0$ for $j \geq 2$.

Solution 5. (i) Using the chain rule we find that substituting (2) into (1) provides (3).
(ii) By substituting (4) into (3), we have $p = 2$ and

$$5\left(\frac{\partial v_{j-2}}{\partial \tau} + (j-3)v_{j-1}\frac{\partial \phi}{\partial \tau}\right) = \frac{\partial^2 v_{j-2}}{\partial x^2} + 2(j-3)\frac{\partial v_{j-1}}{\partial x}\frac{\partial \phi}{\partial x} + (j-3)v_{j-1}\frac{\partial^2 \phi}{\partial x^2}$$

$$+(j-3)(j-2)v_j\left(\frac{\partial \phi}{\partial x}\right)^2 + av_{j-2} - 6a\sum_{k=0}^{j} v_{j-k}v_k \qquad (6)$$

for all $j \geq 0$, where $v_j \equiv 0$ for $j < 0$. For $j = 0$, we have

$$v_0 = \frac{1}{a}\left(\frac{\partial \phi}{\partial x}\right)^2. \qquad (7)$$

Thus using (7), (6) turns into

$$(j-6)(j+1)v_j\left(\frac{\partial \phi}{\partial x}\right)^2 = 5\left(\frac{\partial v_{j-2}}{\partial \tau} + (j-3)v_{j-1}\frac{\partial \phi}{\partial \tau}\right) - \frac{\partial^2 v_{j-2}}{\partial x^2}$$

$$-2(j-3)\frac{\partial v_{j-1}}{\partial x}\frac{\partial \phi}{\partial x} - (j-3)v_{j-1}\frac{\partial^2 \phi}{\partial x^2} + av_{j-2} + 6a\sum_{k=1}^{j-1} v_{j-k}v_k \qquad (8)$$

for $j \geq 2$. The resonance points are $j = -1$ and $j = 6$. The point -1 corresponds to the arbitrary singular manifold function ϕ, while the point 6 corresponds to the free function v_6. For $j = 1$, we obtain from (6) that

$$v_1 = \frac{1}{a}\left(\frac{\partial \phi}{\partial \tau} - \frac{\partial^2 \phi}{\partial x^2}\right). \qquad (9)$$

We find all functions of v_j using (8).
(iii) Using (6), we let $v_j = 0$, for $j \geq 3$. Then

$$v = \frac{v_0}{\phi^2} + \frac{v_1}{\phi} + v_2. \qquad (10)$$

Hence (8) gives

$$v_2 = \frac{1}{12} + \frac{1}{a}\left(-\frac{1}{4}\left(\frac{\phi_{xx}}{\phi_x}\right)^2 + \frac{1}{3}\left(\frac{\phi_{xxx}}{\phi_x}\right) - \frac{\phi_{x\tau}}{\phi_x} + \frac{1}{2}\frac{\phi_{xx}\phi_\tau}{\phi_x^2} - \frac{1}{12}\left(\frac{\phi_\tau}{\phi_x}\right)^2\right) \qquad (11)$$

$$\frac{\phi_{xx} - \phi_\tau}{\phi_x}v_2 = \frac{1}{12}\left(\frac{\phi_{xx} - \phi_\tau}{\phi_x}\right) + \frac{1}{a}\left(-\frac{1}{2}\frac{\phi_{xx\tau}}{\phi_x} + \frac{5}{12}\frac{\phi_{\tau\tau}}{\phi_x} + \frac{1}{12}\frac{\phi_{xxxx}}{\phi_x}\right) \qquad (12)$$

and

$$5\frac{\partial v_2}{\partial \tau} = \frac{\partial^2 v_2}{\partial x^2} + av_2(1 - 6v_2) \qquad (13)$$

where we set $\phi_x \equiv \partial\phi/\partial x$. Equation (13) is the same as (3). Thus (10) is an auto-Bäcklund transformation for the Fisher equation. We have from (7), (9) and (10) that

$$v = \frac{1}{a}\left(\frac{\partial}{\partial \tau} + \frac{\partial^2}{\partial x^2}\right)\ln\phi + v_2 \tag{14}$$

where ϕ satisfies (12), (13) with (11).

Remark. The technique described above is called the *Painlevé test* in literature.

Problem 6. Consider Fisher's equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u).$$

Show that

$$u(x,y,t) = \left(1 + \exp\left(\frac{(x - y/\sqrt{2}) - (5/\sqrt{6})t}{\sqrt{6}}\right)\right)^{-2}$$

is a traveling wave solution of this equation. Is the the solution an element of $L_2(\mathbb{R}^2)$ for a fixed t?

Solution 6.

Problem 7. Consider the nonlinear partial differential equation

$$\left(\frac{\partial\phi}{\partial y}\right)^2 \frac{\partial^2\phi}{\partial x^2} + \left(\frac{\partial\phi}{\partial x}\right)^2 \frac{\partial^2\phi}{\partial y^2} - 2\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial x\partial y} = 0. \tag{1}$$

(i) Show that this equation has an implicit solution for ϕ of the form

$$x = X(\phi, y, t) = f(\phi, t)y + h(\phi, t) \tag{2}$$

where f and h are arbitrary differentiable functions of ϕ and t.
(ii) Show that (1) may also be solved by means of a Legendre transformation.

Solution 7.

Problem 8. The *sine-Gordon equation* is the equation of motion for a theory of a single, dimensionless scalar field u, in one space and one time dimension, whose dynamics is determined by the Lagrangian density

$$\mathcal{L} = \frac{1}{2}(u_t^2 - c^2u_x^2) + \frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}u\right) - \mu. \tag{1}$$

Here c is a limiting velocity while m, λ, and μ are real parameters. u_t and u_x are the partial derivatives of ϕ with respect to t and x, respectively. In the terminology of quantum field theory, m is the mass associated with the normal modes of the linearized theory, while λ/m^2 is a dimensionless, coupling constant that measures the strength of the interaction between these normal modes. In classical theory m is proportional to the characteristic frequency of these normal modes.

(i) Let

$$x \to \frac{x}{m}, \qquad t \to \frac{t}{m}, \qquad u \to mu\sqrt{\lambda} \tag{2}$$

and set $c = 1$. Show that then the Lagrangian density becomes

$$\mathcal{L} = \frac{m^4}{2\lambda}((u_t^2 - u_x^2) + 2\cos u) - \mu \tag{3}$$

with the corresponding Hamiltonian density being given by

$$\mathcal{H} = \frac{m^2}{2\lambda}(u_t^2 + u_x^2 - 2\cos u) + \mu. \tag{4}$$

(ii) By choosing

$$\mu = \frac{m^4}{\lambda} \tag{5}$$

show that the minimum energy of the theory is made zero and (4) can be written as

$$\mathcal{H} = \frac{m^4}{2\lambda}(u_t^2 + u_x^2 + 2(1 - \cos u)). \tag{6}$$

Solution 8.

Problem 9. Find the solution to the nonlinear partial differential equation

$$\left(1 - \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x \partial t} - \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial t^2} = 0 \tag{1}$$

which satisfies the initial conditions (Cauchy problem)

$$u(t=0) = a(x), \qquad \frac{\partial u}{\partial t}(t=0) = b(x). \tag{2}$$

Equation (1) can describe processes which develop in time, since it is of the hyperbolic type if $1 + (\partial u/\partial x)^2 - (\partial u/\partial t)^2 > 0$. Show that the hyperbolic condition for (1) implies for the initial conditions that

$$1 + a'^2(x) - b^2(x) > 0. \tag{3}$$

Solution 9. We simplify (1) by introducing the new variables α, β

$$x = x(\alpha,\beta), \qquad t = t(\alpha,\beta), \qquad z = u(x(\alpha,\beta), t(\alpha,\beta)) = z(\alpha,\beta). \tag{4}$$

Thus we seek a solution of (1) in parametric form

$$\mathbf{r} = \mathbf{r}(\alpha,\beta) \tag{5}$$

where $\mathbf{r}(\alpha,\beta) = (t(\alpha,\beta), x(\alpha,\beta), z(\alpha,\beta))$ is a vector with components t, x, and z. If we denote the scalar product of the vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ by $\mathbf{r}_1\mathbf{r}_2$,

$$\mathbf{r}_1\mathbf{r}_2 := t_1t_2 - x_1x_2 - z_1z_2 \tag{6}$$

then (1) can be written in the following form

$$\mathbf{r}_\alpha^2 D_{\beta\beta} - 2\mathbf{r}_\alpha\mathbf{r}_\beta D_{\alpha\beta} + \mathbf{r}_\beta^2 D_{\alpha\alpha} = 0 \tag{7}$$

where

$$\mathbf{r}_\alpha := \frac{\partial \mathbf{r}}{\partial\alpha}, \qquad \mathbf{r}_\beta := \frac{\partial \mathbf{r}}{\partial\beta}, \qquad \mathbf{r}_{\alpha,\beta} := \frac{\partial^2 \mathbf{r}}{\partial\alpha\partial\beta} \tag{8}$$

and

$$D_{ik} := \begin{vmatrix} t_{ik} & x_{ik} & z_{ik} \\ t_\alpha & x_\alpha & z_\alpha \\ t_\beta & x_\beta & z_\beta \end{vmatrix}. \tag{9}$$

Next we construct the hyperbolic solution of (7) which satisfies the initial conditions. The hyperbolic nature of (7) implies that

$$(\mathbf{r}_\alpha\mathbf{r}_\beta)^2 - \mathbf{r}_\alpha^2\mathbf{r}_\beta^2 > 0. \tag{10}$$

Equation (7) has the following equations for the characteristics

$$(\mathbf{r}_\alpha)^2 = 0, \qquad (\mathbf{r}_\beta)^2 = 0. \tag{11}$$

These characteristic equations together with (7) are a system of three equations for the three functions $t(\alpha,\beta)$, $x(\alpha,\beta)$, and $z(\alpha,\beta)$. It follows from (7), (8) and (9) that

$$D_{\alpha,\beta} = 0. \tag{12}$$

Equation (12) describes a linear dependence between rows of the determinant $D_{\alpha,\beta}$ i.e.,

$$\mathbf{r}_{\alpha,\beta} = A(\alpha,\beta)\mathbf{r}_\alpha + B(\alpha,\beta)\mathbf{r}_\beta. \tag{13}$$

Taking the scalar product $\mathbf{r}_{\alpha,\beta}\mathbf{r}_\alpha$ and take into account that (9) and (11) are valid for all α, β we have

$$\mathbf{r}_{\alpha,\beta}\mathbf{r}_\alpha = \mathbf{r}_\alpha\mathbf{r}_\beta B = \frac{1}{2}\frac{\partial}{\partial\beta}\mathbf{r}_\alpha^2 = 0. \tag{14}$$

With $\mathbf{r}_\alpha \mathbf{r}_\beta \neq 0$ we conclude from (14) that $A = B = 0$. Thus we obtain

$$\mathbf{r}_\alpha^2 = 0, \qquad \mathbf{r}_\beta^2 = 0, \qquad \mathbf{r}_{\alpha,\beta} = 0 \tag{15}$$

or written in components

$$\left(\frac{\partial t}{\partial \alpha}\right)^2 - \left(\frac{\partial x}{\partial \alpha}\right)^2 - \left(\frac{\partial z}{\partial \alpha}\right)^2 = 0$$

$$\left(\frac{\partial t}{\partial \beta}\right)^2 - \left(\frac{\partial x}{\partial \beta}\right)^2 - \left(\frac{\partial z}{\partial \beta}\right)^2 = 0$$

$$\frac{\partial^2 t}{\partial \alpha \partial \beta} = 0, \quad \frac{\partial^2 x}{\partial \alpha \partial \beta} = 0, \quad \frac{\partial^2 z}{\partial \alpha \partial \beta} = 0. \tag{16}$$

Thus the general solution of (15) (or (16)) is

$$\mathbf{r}(\alpha, \beta) = \mathbf{r}_1(\alpha) + \mathbf{r}_2(\beta).$$

Problem 10. Consider the *Korteweg-de Vries equation*

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

From (1) we can derive the iteration scheme

$$u^{(j+1)} = \frac{1}{6}\frac{\left(u_t^{(j)} + u_{xxx}^{(j)}\right)}{u_x^{(j)}}, \qquad j = 0, 1, 2, \ldots \tag{2}$$

where

$$u_x^{(j)} := \frac{\partial u^{(j)}}{\partial x}. \tag{3}$$

(i) Let

$$u^{(0)}(x,t) = \ln(x - ct). \tag{4}$$

Show that (4) converges within two steps to an exact solution

$$u(x,t) = -\frac{c}{6} + 2(x - ct)^{-2} \tag{5}$$

of the Korteweg-de Vries equation (1).
(ii) Show that

$$u^{(0)}(x,t) = (x - ct)^3 \tag{6}$$

also converges to the solution (5).
(iii) Let

$$u^{(0)}(x,t) = \cos(a(x - ct))^{-k}. \tag{7}$$

Show that within two steps of the iteration we arrive at

$$u(x,t) = \frac{4}{3}a^2 - \frac{c}{6} + 2a^2 \tan^2(a(x-ct)). \tag{8}$$

(iv) Show that by demanding that $u \to 0$ for $|x| \to \infty$, provides $a = i\sqrt{c}/2$ and (5) becomes the well known soliton solution

$$u(x,t) = -\frac{c}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right). \tag{9}$$

Solution 10. Blender J. Phys. A : Math. Gen. 24 1991 L509

Problem 11. Solve the initial value problem of the partial differential equation

$$\frac{\partial u}{\partial x} = x^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

with $u(0,y) = y^2$ applying a Taylor series expansion.

Solution 11.

Problem 12. The *Zakharov-Kuznetsov equation* for ion acoustic waves and solitons propagating along a very strong external and uniform magnetic field is, for a two-component plasma

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{\partial}{\partial x}(\Delta u) = 0 \tag{1}$$

$$\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{2}$$

Here u is the normalized deviation of the ion density from the average. Exact solitonlike solutions exist in one, two and three space dimensions. They depend on the independent variables through the combination

$$x - ct, \qquad \rho := ((x-ct)^2 + y^2)^{1/2}, \quad r := ((x-ct)^2 + y^2 + z^2)^{1/2}, \tag{3}$$

respectively, where

$$\Delta u - \left(c - \frac{u}{2}\right) u = 0 \tag{4}$$

and

$$\Delta := \frac{\partial^2}{\partial x^2}, \qquad \Delta := \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial}{\partial \rho}\right), \qquad \Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) \tag{5}$$

for the three cases. (i) Show that for the one-dimensional case we obtain the soliton solution

$$u(x,t) = 3c\,\mathrm{sech}^2(c^{1/2}(x - ct - x_0)/2). \qquad (6)$$

(ii) Show that the flat soliton (6) is unstable with respect to nonaligned perturbations.

Solution 12.

Problem 13. Show that the *nonlinear Schrödinger equation*,

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + 2\sigma |w|^2 w = 0, \qquad \sigma = \pm 1 \qquad (1)$$

has one-zone solutions

$$w(x,t) = \sqrt{f(\theta(x,t))}\exp(i\varphi(x,t)), \qquad \varphi = \psi + h(\theta), \qquad (2)$$

$$\theta(x,t) := kx - \omega t, \qquad \psi := \kappa x - \Omega t \qquad (3)$$

where $f(\theta)$ and $h(\theta)$ are elliptic functions.

Solution 13.

Problem 14. Consider the partial differential equation

$$\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 = u.$$

Show that

$$u(x,y) = \frac{1}{4}(x+a)^2 + \frac{1}{4}(y+b)^2$$

is a solution.

Solution 14.

Problem 15. We consider the diffusion equation with nonlinear quadratic recombination

$$\frac{\partial u}{\partial t} = -\alpha u^2 + D_x \frac{\partial^2 u}{\partial x^2} + D_y \frac{\partial^2 u}{\partial y^2} + D_z \frac{\partial^2 u}{\partial z^2} \qquad (1)$$

where D_x, D_y and D_z are constants. This equation is relevant for the evolution of plasmas or of charge carries in solids, of generating functions. Find the condition on c_1, c_2 such that

$$u(x,y,z,t) = \frac{6}{5\alpha}\frac{c_1}{x^2/D_x + y^2/D_y + z^2/D_z + c_2(t+t_0)} +$$

$$\frac{24}{\alpha}\frac{x^2/D_x + y^2/D_y + z^2/D_z}{(x^2/D_x + y^2/D_y + z^2/D_z + c_2(t-t_0))^2} \tag{2}$$

satisfies (1).

Solution 15.

Problem 16. The Burgers equation determines the motion of a pressureless fluid subjected to dissipation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

(i) Show that any solution v of the diffusion equation

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} \tag{2}$$

yields a solution of the Burgers equation via the *Hopf-Cole transformation*

$$u = -\frac{2}{v}\frac{\partial v}{\partial x}. \tag{3}$$

(ii) Solving (1) with respect to u we can introduce the iteration formula

$$u^{(j+1)} = \frac{1}{u_x^{(j)}}(u_{xx}^{(j)} - u_t^{(j)}), \qquad j = 0, 1, 2, \ldots \tag{4}$$

where $u_t \equiv \partial u/\partial t$ etc.. Show that for $u^{(0)} = v^n$ we find the sequence

$$u^{(1)} = (n-1)\frac{1}{v}\frac{\partial v}{\partial x} \tag{5}$$

$$u^{(2)} = -\frac{2}{v}\frac{\partial v}{\partial x} \qquad \text{fixed point} \tag{6}$$

Remark. Thus the Hopf-Cole transformation is an attractor (fixed point) of the iteration (4).

Problem 17. Consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(u\frac{\partial u}{\partial x}\right). \tag{1}$$

(i) Show that this equation admits a solution of the form

$$u(x,t) = \begin{cases} \dfrac{1}{x_1}\left(1 - \left(\dfrac{x}{x_1}\right)^2\right) & \text{for } t > 0, \left|\dfrac{x}{x_1}\right| < 1 \\ 0 & \text{for } t > 0, \left|\dfrac{x}{x_1}\right| > 1 \end{cases} \tag{2}$$

where

$$x_1 := (6t)^{1/3} \tag{3}$$

Remark. The solution exhibits a wave-like behaviour although it is not a wave of constant shape. The leading edge of this wave, that is where $u = 0$, is at $x = x_1$ and the speed of propagation is proportional to $t^{-2/3}$.

Solution 17.

Problem 18. Show that the nonlinear partial differential equation (Fisher's equation)

$$\left(\frac{\partial^2}{\partial x^2} - \frac{\partial}{\partial t}\right) u + u = u^2 \tag{1}$$

admits the travelling wave solution

$$u(x,t) = \frac{1}{4}\left(1 - \tanh\left(\frac{1}{2\sqrt{6}}\left(x - \frac{5}{\sqrt{6}}vt\right)\right)\right)^2. \tag{2}$$

Solution 18.

Problem 19. Show that the coupled system of partial differential equations

$$i\frac{\partial E}{\partial t} + \frac{\partial^2 E}{\partial x^2} = nE, \qquad \frac{\partial^2 n}{\partial t^2} - \frac{\partial^2 n}{\partial x^2} = \frac{\partial^2}{\partial x^2}(|E|^2) \tag{1}$$

admits the solution

$$E(x,t) = E_0(x - st)\exp\left(\frac{is}{2}x + i\left(\lambda^2 - \frac{s^2}{4}\right)t\right) \tag{2a}$$

$$n(x,t) = -\frac{2\lambda^2}{\cosh^2\lambda(x - st - x_0)} \tag{2b}$$

where

$$E_0(x - st) = \frac{\lambda\sqrt{2(1-s)^2}}{\cosh(\lambda(x - st - x_0))} \tag{2c}$$

The soliton solution represents a moving one-dimensional plasma density well which *Langmuir oscillations* are locked.

Solution 19.

Problem 20. The *Navier-Stokes equation* is given by

$$\frac{\partial \mathbf{u}}{\partial t} = -\mathbf{u}\cdot\nabla\mathbf{u} + \nu\nabla^2\mathbf{u} - \frac{1}{\rho}\nabla p$$

where $\mathbf{u}$ denotes the solenoidal

$$\nabla \cdot \mathbf{u} \equiv \mathrm{div}\mathbf{u} = \frac{\partial u_1}{\partial x} + \frac{\partial u_2}{\partial y} + \frac{\partial u_3}{\partial z} = 0$$

flow velocity field, ν and ρ are the constant kinematic viscosity and fluid density. Here p is the fluid pressure. We have

$$\mathbf{u} \cdot \nabla \mathbf{u} := \begin{pmatrix} u_1 \frac{\partial u_1}{\partial x} + u_2 \frac{\partial u_1}{\partial y} + u_3 \frac{\partial u_1}{\partial z} \\ u_1 \frac{\partial u_2}{\partial x} + u_2 \frac{\partial u_2}{\partial y} + u_3 \frac{\partial u_2}{\partial z} \\ u_1 \frac{\partial u_3}{\partial x} + u_2 \frac{\partial u_3}{\partial y} + u_3 \frac{\partial u_3}{\partial z} \end{pmatrix}$$

Find the time evolution of

$$\mathbf{v} := \mathrm{curl}\mathbf{u} \equiv \nabla \times \mathbf{u} = \begin{pmatrix} \frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \\ \frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \\ \frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \end{pmatrix}.$$

Solution 20. Using that $\mathrm{div}\mathbf{u} = 0$ we obtain

$$\frac{\partial \mathbf{v}}{\partial t} = \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v} + \nu \nabla^2 \mathbf{v}$$

where

$$\mathbf{v} \cdot \nabla \mathbf{u} := \begin{pmatrix} v_1 \frac{\partial u_1}{\partial x} + v_2 \frac{\partial u_1}{\partial y} + v_3 \frac{\partial u_1}{\partial z} \\ v_1 \frac{\partial u_2}{\partial x} + v_2 \frac{\partial u_2}{\partial y} + v_3 \frac{\partial u_2}{\partial z} \\ v_1 \frac{\partial u_3}{\partial x} + v_2 \frac{\partial u_3}{\partial y} + v_3 \frac{\partial u_3}{\partial z} \end{pmatrix}$$

and analogously for $\mathbf{u} \cdot \nabla \mathbf{v}$. Note that

$$\{\, \mathbf{v}, \mathbf{u} \,\} := \mathbf{v} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{v}$$

maps an ordered pair of solenoidal vector fields into a solenoidal vector field.

Problem 21. Show that

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = 0 \tag{1}$$

can be linearized into

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \bar{x} \tag{2}$$

with

$$\bar{t}(x,t) = t, \qquad \bar{x}(x,t) = u(x,t), \qquad \bar{u}(\bar{x}(x,t), \bar{t}(x,t)) = x. \tag{3}$$

Solution 21. Applying the *chain rule* yields

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial t} + \frac{\partial \bar{u}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial u}{\partial t} + \frac{\partial \bar{u}}{\partial \bar{t}} = 0 \tag{4}$$

and

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial \bar{u}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial x} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial u}{\partial x} = 1. \tag{5}$$

Solving (4) and (5) with respect to $\partial u/\partial t$ and $\partial u/\partial x$, respectively, and inserting into (1) we obtain (2), where we used (3).

Problem 22. Consider the partial differential equation

$$\frac{\partial u}{\partial t} = u^2 \frac{\partial^2 u}{\partial x^2}. \tag{1}$$

Show that (1) linearizes under the transformation

$$\bar{x}(x,t) = \int_{-\infty}^{x} \frac{1}{u(s,t)} ds, \qquad \bar{t}(x,t) = t, \qquad \bar{u}(\bar{x}(x,t), \bar{t}(x,t)) = u(x,t). \tag{2}$$

It is assumed that u and all its spatial derivatives vanish at $-\infty$.

Solution 22. From (2) it follows that

$$\frac{\partial \bar{t}}{\partial t} = 1, \qquad \frac{\partial \bar{x}}{\partial x} = \frac{1}{u}, \qquad \frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial t} + \frac{\partial \bar{u}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t} = \frac{\partial u}{\partial t} \tag{3}$$

and

$$\frac{\partial \bar{x}}{\partial t} = -\int_{-\infty}^{x} \frac{1}{u^2(s,t)}\frac{\partial u(s,t)}{\partial t} ds = -\int_{-\infty}^{x} \frac{\partial^2 u(s,t)}{\partial s^2} ds = -\frac{\partial u}{\partial x} \tag{4}$$

and

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial \bar{u}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial x} = \frac{\partial u}{\partial x}. \tag{5}$$

Therefore

$$\frac{\partial \bar{u}}{\partial \bar{x}} = u\frac{\partial u}{\partial x} \tag{6}$$

Analogously

$$\frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial t} + \frac{\partial \bar{u}}{\partial \bar{t}}\frac{\partial \bar{t}}{\partial t} = \frac{\partial u}{\partial t} \tag{7}$$

and hence

$$-\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial u}{\partial x} + \frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial u}{\partial t}. \tag{8}$$

From (5) we obtain

$$\frac{\partial}{\partial x}\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}. \tag{9}$$

It follows that

$$\left(\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\frac{\partial \bar{x}}{\partial x} + \frac{\partial^2 \bar{u}}{\partial \bar{x}\bar{t}}\frac{\partial \bar{t}}{\partial \bar{x}}\right) = \left(\frac{\partial u}{\partial x}\right)^2 + u\frac{\partial^2 u}{\partial x^2}. \tag{10}$$

Since $\partial \bar{t}/\partial x = 0$ and $\partial \bar{x}/\partial x = 1/u$ we obtain

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2} = u\left(\frac{\partial u}{\partial x}\right)^2 + u^2\frac{\partial^2 u}{\partial x^2}. \tag{11}$$

Inserting (1c), (5), (7) and (12) into (2) gives the linear diffusion equation

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2}. \tag{12}$$

Problem 23. Show that the equation

$$\frac{\partial u}{\partial t} = u^2\frac{\partial^2 u}{\partial x^2} \tag{1}$$

is transformed under the *Cole-Hopf transformation*

$$\bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = \frac{1}{u}\frac{\partial u}{\partial x}, \qquad \bar{t}(x,t) = t, \qquad \bar{x}(x,t) = x \tag{2}$$

into *Burgers equation*

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial}{\partial \bar{x}}\left(\bar{u}^2 + \frac{\partial \bar{u}}{\partial \bar{x}}\right). \tag{3}$$

Solution 23.

Problem 24. Show that the the equation

$$\frac{\partial u}{\partial t} = u^2\frac{\partial^2 u}{\partial x^2} + cu^2\frac{\partial u}{\partial x} \tag{1}$$

is linearized by the transformation

$$\bar{x}(x,t) = \int_{-\infty}^{x} u(s,t)^{-1}ds, \qquad \bar{t}(x,t) = t, \qquad \bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = u(x,t). \tag{2}$$

Solution 24.

Problem 25. Show that the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{3}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{3}{2}u\frac{\partial^2 u}{\partial x^2} + \frac{3}{4}u^2\frac{\partial u}{\partial x} \tag{1}$$

can be derived from the linear equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial t^3} \tag{2}$$

and the transformation

$$\phi(x,t) = \exp\left(\frac{1}{2}\int^x u(s,t)ds\right). \tag{3}$$

Solution 25.

Problem 26. Using the transformation

$$\bar{x}(x,t) = \int_{-\infty}^x \frac{ds}{u(s,t)}, \qquad \bar{t}(x,t) = t, \qquad \bar{u}(\bar{x}(x,t),\bar{t}(x,t)) = u(x,t) \tag{1}$$

the equation

$$\frac{\partial u}{\partial t} = u^3\frac{\partial^3 u}{\partial x^3} \tag{2}$$

can be recast into the differential equation

$$\frac{\partial \bar{u}}{\partial \bar{t}} - \frac{\partial^3 \bar{u}}{\partial \bar{x}^3} - 3\frac{1}{\bar{u}^2}\frac{\partial \bar{u}}{\partial \bar{x}}\left(\frac{1}{2}\left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 - \bar{u}\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\right) = 0. \tag{3}$$

(ii) Show that using the Cole-Hopf transformation

$$\bar{u}(\bar{x},\bar{t}) = \frac{1}{v(\bar{x},\bar{t})}\frac{\partial v(\bar{x},\bar{t})}{\partial \bar{x}} \tag{4}$$

(3) can further be reduced to the modified Korteweg-de Vries equation in v. Hint. We have

$$\frac{\partial \bar{x}}{\partial t} = -\int_{-\infty}^x \frac{1}{u^2(s,t)}\frac{\partial u(s,t)}{\partial t}ds = -\int_{-\infty}^x u(s,t)\frac{\partial^3 u(s,t)}{\partial s^3}ds = u\frac{\partial^2 u}{\partial x^2} - \frac{1}{2}\left(\frac{\partial u}{\partial x}\right). \tag{5}$$

Solution 26.

Problem 27. Show that the transformation

$$\bar{u}(\bar{x},\bar{t}) = \frac{\partial u}{\partial x}, \qquad \bar{x}(x,t) = u(x,t), \qquad \bar{t}(x,t) = t \tag{1}$$

transforms the nonlinear heat equation

$$\frac{\partial u}{\partial t} = A(u)\frac{\partial^2 u}{\partial x^2} \tag{2}$$

into

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \bar{u}^2 \frac{\partial}{\partial \bar{x}}\left(A(\bar{x})\frac{\partial \bar{u}}{\partial \bar{x}}\right). \tag{3}$$

Solution 27.

Problem 28. Consider the nonlinear partial differential equation

$$\Delta u - f(u)(\nabla u)^2 + \mathbf{a}(\mathbf{x},t)\nabla u + b(\mathbf{x},t)\frac{\partial u}{\partial t} = 0 \tag{1}$$

where ∇ is the gradient operator in the variables $x_1,\ldots,x_n$, $\Delta := \nabla\nabla$, $f(u)$ and $b(\mathbf{x},t)$ are given functions, and $\mathbf{a}(\mathbf{x},t)$ is a given n-dimensional vector. Show that the transformation

$$\int_{u_0}^{u(\mathbf{x},t)} \left(\exp\left(-\int_{u_0}^{s} f(z)dz\right)\right) ds - v(\mathbf{x},t) = 0. \tag{2}$$

reduces (1) to the linear partial differential equation

$$\Delta v + \mathbf{a}(\mathbf{x},t)\nabla v + b(\mathbf{x},t)\frac{\partial v}{\partial t} = 0. \tag{3}$$

Hint. From (2) we find

$$\frac{\partial v}{\partial t} = \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz\right)\frac{\partial u}{\partial t} \tag{4a}$$

$$\frac{\partial v}{\partial x_1} = \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz\right)\frac{\partial u}{\partial x_1} \tag{4b}$$

$$\frac{\partial^2 v}{\partial x_1^2} = \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz\right)\frac{\partial^2 u}{\partial x_1^2} - \exp\left(-\int_{u_0}^{u(\mathbf{x},t)} f(z)dz\right) f(u)\left(\frac{\partial u}{\partial x_1}\right)^2. \tag{4c}$$

Solution 28.

Problem 29. Consider the two-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 u}{\partial t^2} = \sin(u)$$

It may be regarded as describing solitary waves in a two-dimensional Josephson junction. The *Lamb substitution* is given by

$$u(x,y,t) = 4\tan^{-1}(M(x,y,t)). \tag{1}$$

Find the equation for M.

Solution 29. Since

$$\sin(4\arctan w) \equiv \frac{4w(1-w^2)}{w^4+2w^2+1} \tag{2}$$

we obtain

$$(1+M^2)\left(\frac{\partial^2 M}{\partial x^2} + \frac{\partial^2 M}{\partial y^2} - \frac{\partial^2 M}{\partial t^2}\right) - 2M\left(\left(\frac{\partial M}{\partial x}\right)^2 + \left(\frac{\partial M}{\partial y}\right)^2 - \left(\frac{\partial M}{\partial t}\right)^2\right) = M(1-M^2). \tag{3}$$

Problem 30. We consider the nonlinear *d'Alembert equation*

$$\Box u = F(u) \tag{1}$$

where $u = u(\mathbf{x})$, $\mathbf{x} = (x_0, x_1, \ldots, x_n)$,

$$\Box := \frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \cdots - \frac{\partial^2}{\partial x_n^2} \tag{2}$$

and $F(u)$ is an arbitrary differentiable function.
(i) Consider the transformation

$$u(\mathbf{x}) = \Phi(w(\mathbf{x})) \tag{3}$$

where $w(\mathbf{x})$ and $\Phi(w)$ are new unknown functions. Show that (1) takes the form

$$\frac{d\Phi}{dw}\Box w + \frac{d^2\Phi}{dw^2} w_\mu w^\mu = F(\Phi) \tag{4}$$

where

$$w_\mu w^\mu := \left(\frac{\partial w}{\partial x_0}\right)^2 - \left(\frac{\partial w}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial w}{\partial x_n}\right)^2. \tag{5}$$

(ii) Show that (4) is equivalent to the following equation

$$\frac{d\Phi}{dw}\left(\Box w - \lambda \frac{\dot{P}_n}{P_n}\right) + \frac{d^2\Phi}{dw^2}(w_\mu w^\mu - \lambda) + \lambda\left(\frac{d^2\Phi}{dw^2} + \frac{d\Phi}{dw}\frac{\dot{P}_n}{P_n}\right) - F(\Phi) = 0 \tag{6}$$

where $P_n(w)$ is an arbitrary polynomial of degree n in w, and $\lambda = -1, 0, 1$. Moreover $\dot{P}_n \equiv dP_n/dw$.
(iii) Assume that Φ satisfies

$$\lambda \left(\frac{d^2\Phi}{dw^2} + \frac{d\Phi}{dw}\frac{\dot{P}_n}{P_n} \right) = F(\Phi). \tag{7}$$

Show that (6) takes the form

$$\frac{d\Phi}{dw}\left(\Box w - \lambda \frac{\dot{P}_n}{P_n} \right) + \frac{d^2\Phi}{dw^2}(w_\mu w^\mu - \lambda) = 0. \tag{8}$$

(iv) Show that a solution of the system

$$\Box w = \lambda \frac{\dot{P}_n}{P_n}, \qquad w_\mu w^\mu = \lambda \tag{9}$$

is also a solution of (8), and in this way we obtain a solution of (1) provided Φ satisfies (7).

Solution 30.

Problem 31. Consider the nonlinear wave equation in one space dimension

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u + u^3 = 0. \tag{1}$$

(i) Show that (1) can be derived from the Lagrangian density

$$\mathcal{L}(u_t, u_x, u) = \frac{1}{2}(u_t^2 - u_x^2) - g(u) \tag{2}$$

with $g(u)$ the potential function

$$g(u) = \frac{1}{4}(u^2 - 1)^2 \equiv \frac{1}{4}(u^4 - 2u^2 + 1). \tag{3}$$

(ii) **Definition.** A *conservation law* associated to (1) is an expression of the form

$$\frac{\partial T(u(x,t))}{\partial t} + \frac{\partial X(u(x,t))}{\partial x} = 0 \tag{4}$$

where T is the *conserved density* and X the *conserved flux*. T and X are functionals of u and its derivatives.
(ii) Show that the quantities

$$T_1 := \frac{1}{2}\left(\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial u}{\partial x} \right)^2 \right) + \frac{1}{4}(u^2 - 1)^2 \geq 0, \qquad T_2 := -\frac{\partial u}{\partial x}\frac{\partial u}{\partial t} \tag{5}$$

are conserved densities of (1), provided that u^2-1, $\partial u/\partial t$, and $\partial u/\partial x$ tend to zero sufficiently fast as $|x| \to +\infty$.

Hint. The corresponding fluxes are

$$X_1 = -\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}, \qquad X_2 = \frac{1}{2}\left(\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial u}{\partial x}\right)^2\right) - \frac{1}{4}(u^2-1)^2. \qquad (6)$$

T_1 and T_2 correspond to the energy and momentum densities, respectively.

The *Euler-Lagrange equation* is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial u_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial u_x}\right) - \frac{\partial \mathcal{L}}{\partial u} = 0 \qquad (7)$$

where $\mathcal{L}$ is the Lagrange density (2).

Solution 31. (i) Inserting (2) into (7) we find (1).
(ii) Inserting T_1 and X_1 into (4) yields

$$\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} + (u^2-1)u\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2}\frac{\partial u}{\partial t} - \frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial t} =$$
$$\frac{\partial u}{\partial t}\left(\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + u^3 - u\right) = 0$$

where we used (1).

Problem 32. Consider the nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u + u^3 = 0. \qquad (1)$$

(i) Show that (1) admits the solution

$$u_K(x-vt) = \tanh((\gamma/\sqrt{2})(x-vt-x_0)) = -u_{\bar{K}}(x-vt) \qquad (2)$$

where

$$\gamma^2 \equiv (1-v^2)^{-1} \qquad (3)$$

and x_0 is a constant. These solutions are of *kink* (K) and *antikink* ($\bar{K}$) type traveling at constant velocity v. These solutions do not tend to zero at infinity, but they do connect two minimum states of the potential $\frac{1}{4}(u^2-1)^2$.
(ii) The energy and momentum densities T_1 and T_2 for the kink and antikink are obtained by substituting (2) into (5) of problem 1. Show that

$$T_1(x,t) = \frac{\gamma^2}{2}\cosh^{-4}((\gamma/\sqrt{2})(x-vt-x_0)) \qquad (4)$$

$$T_2(x,t) = \frac{\gamma^2 v}{2} \cosh^{-4}((\gamma/\sqrt{2})(x - vt - x_0)). \tag{5}$$

(iii) Show that by integrating over x we find the energy and the momentum

$$E := \int_{-\infty}^{+\infty} T_1 dx = \frac{4}{3}\frac{\gamma}{\sqrt{2}}, \qquad P := \int_{-\infty}^{+\infty} T_2 dx = \frac{4}{3}\frac{\gamma v}{\sqrt{2}}. \tag{6}$$

(iv) Show that the densities T_1 and T_2 are localized in space, in contrast with u_K and $u_{\bar{K}}$.
(v) We associate the mass

$$M^2 := E^2 - P^2 \tag{7}$$

to (1) and the *energy center*

$$X_c := \frac{\displaystyle\int_{-\infty}^{+\infty} x T_1 dx}{\displaystyle\int_{-\infty}^{+\infty} T_1 dx}. \tag{8}$$

Show that the kink and antikink solutions of (6) take the values

$$M^2 = \frac{8}{9}, \qquad X_c = vt + x_0. \tag{9}$$

Solution 32.

Problem 33. We say that a partial differential equation described by the field $u(x,y)$ is *hodograph invariant* if it does not change its form by the *hodograph transformations*

$$\bar{x}(x,y) = u(x,y), \qquad \bar{y}(x,y) = y, \qquad \bar{u}(\bar{x}(x,y), \bar{y}(x,y)) = x. \tag{1}$$

Show that the *Monge-Ampere equation* for the surface $u(x,y)$ with a constant total curvature K

$$\frac{\partial^2 u}{\partial x^2}\frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial^2 u}{\partial x \partial y}\right)^2 = K\left(1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2\right) \tag{2}$$

is hodograph invariant.

Solution 33. We have to show that

$$\frac{\partial^2 \bar{u}}{\partial \bar{x}^2}\frac{\partial^2 \bar{u}}{\partial \bar{y}^2} - \left(\frac{\partial^2 \bar{u}}{\partial \bar{x} \partial \bar{y}}\right)^2 = K\left(1 + \left(\frac{\partial \bar{u}}{\partial \bar{x}}\right)^2 + \left(\frac{\partial \bar{u}}{\partial \bar{y}}\right)^2\right). \tag{3}$$

Applying the chain rule we have

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial x} + \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial x} = 1. \tag{4a}$$

Thus

$$\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial u}{\partial x} = 1. \tag{4b}$$

Here we used

$$\frac{\partial \bar{x}}{\partial x} = \frac{\partial u}{\partial x}, \qquad \frac{\partial \bar{x}}{\partial y} = \frac{\partial u}{\partial y}, \qquad \frac{\partial \bar{y}}{\partial x} = 0, \qquad \frac{\partial \bar{y}}{\partial y} = 1. \tag{5}$$

Analogously

$$\frac{\partial \bar{u}}{\partial y} = \frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial \bar{x}}{\partial y} + \frac{\partial \bar{u}}{\partial \bar{y}}\frac{\partial \bar{y}}{\partial y} = 0. \tag{6a}$$

Thus

$$\frac{\partial \bar{u}}{\partial \bar{x}}\frac{\partial u}{\partial x} = 1. \tag{7b}$$

For the second order deriatives we find

$$\frac{\partial^2 \bar{u}}{\partial x^2} = \frac{\partial \bar{u}}{\partial x} = 0 \tag{8a}$$

and

$$\frac{\partial^2 \bar{u}}{\partial y^2} = \tag{8b}$$

It follows that

$$\frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x} = \frac{\partial \bar{u}}{\partial x} =$$

Problem 34. Show that the nonlinear equation of the *Born-Infeld type* for a scalar field u is obtained by varying the Lagrangian

$$\mathcal{L}(u_x, u_y, u_z, u_t) = 1 - (1 + u_x^2 + u_y^2 + u_z^2 - u_t^2)^{1/2} \tag{1}$$

and has the following form

$$\left(1 + \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2 + \left(\frac{\partial u}{\partial z}\right)^2 - \left(\frac{\partial u}{\partial t}\right)^2\right)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} - \frac{\partial^2 u}{\partial t^2}\right)$$
$$- \left(\frac{\partial u}{\partial x}\right)^2 \frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial u}{\partial y}\right)^2 \frac{\partial^2 u}{\partial y^2} - \left(\frac{\partial u}{\partial z}\right)^2 \frac{\partial^2 u}{\partial z^2} - \left(\frac{\partial u}{\partial t}\right)^2 \frac{\partial^2 u}{\partial t^2} - 2\frac{\partial^2 u}{\partial x \partial y}\frac{\partial^2 u}{\partial x \partial y}$$
$$-2\frac{\partial u}{\partial x}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x \partial z} - 2\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial y \partial z} + 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial t \partial x} + 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial y}\frac{\partial^2 u}{\partial t \partial y} + 2\frac{\partial u}{\partial t}\frac{\partial u}{\partial z}\frac{\partial^2 u}{\partial t \partial z} = 0. \tag{2}$$

Solution 34. The *Euler-Lagrange equation* is given by

$$\frac{\partial}{\partial x}\frac{\partial \mathcal{L}}{\partial u_x}+\frac{\partial}{\partial y}\frac{\partial \mathcal{L}}{\partial u_y}+\frac{\partial}{\partial z}\frac{\partial \mathcal{L}}{\partial u_z}+\frac{\partial}{\partial t}\frac{\partial \mathcal{L}}{\partial u_x}-\frac{\partial \mathcal{L}}{\partial u}=0. \qquad (3)$$

Problem 35. The *sine-Gordon equation* is the equation of motion for a theory of a single, dimensionless scalar field u, in one space and one time dimension, whose dynamics is determined by the Lagrangian density

$$\mathcal{L}(u_t,u_x,u)=\frac{1}{2}(u_t^2-c^2u_x^2)+\frac{m^4}{\lambda}\cos\left(\frac{\sqrt{\lambda}}{m}u\right)-\mu. \qquad (1)$$

Here c is a limiting velocity while m, λ, and μ are real parameters. u_t and u_x are the partial derivatives of u with respect to t and x, respectively. In the terminology of quantum field theory, m is the mass associated with the normal modes of the linearized theory, while λ/m^2 is a dimensionless, coupling constant that measures the strength of the interaction between these normal modes. In classical theory m is proportional to the characteristic frequency of these normal modes. Let

$$x\to\frac{x}{m},\qquad t\to\frac{t}{m},\qquad u\to\frac{mu}{\sqrt{\lambda}} \qquad (2)$$

and set $c=1$. (i) Show that then the Lagrangian density becomes

$$\mathcal{L}(u_t,u_x,u)=\frac{m^4}{2\lambda}((u_t^2-u_x^2)+2\cos u)-\mu \qquad (3)$$

with the corresponding Hamiltonian density being given by

$$\mathcal{H}=\frac{m^4}{2\lambda}(u_t^2+u_x^2-2\cos u)+\mu. \qquad (4)$$

(ii) Show that by choosing

$$\mu=m^4/\lambda \qquad (5)$$

the minimum energy of the theory is made zero and (4) can be written as

$$\mathcal{H}=\frac{m^4}{2\lambda}(u_t^2+u_x^2+2(1-\cos u)). \qquad (6)$$

Solution 35.

Problem 36. Show that the following theorem holds. The *conservation law*

$$\frac{\partial}{\partial t}(T(\partial u/\partial x,\partial u/\partial t,u))+\frac{\partial}{\partial x}(F(\partial u/\partial x,\partial u/\partial t,u))=0 \qquad (1)$$

is transformed to the reciprocally associated conservation law

$$\frac{\partial}{\partial t'}(T'(\partial u/\partial x', \partial u/\partial t', u)\} + \frac{\partial}{\partial x'}(F'(\partial u/\partial x', \partial u/\partial t', u)\} = 0 \qquad (2)$$

by the reciprocal transformation

$$dx'(x,t) = Tdx - Fdt, \qquad t'(x,t) = t, \qquad (3)$$

$$T'(\partial u/\partial x', \partial u/\partial t', u) = \frac{1}{T(\partial u/\partial x, \partial u/\partial t, u)} \qquad (4a)$$

$$F'(\partial u/\partial x', \partial/\partial t', u) = \frac{-F(\partial u/\partial x, \partial u/\partial t, u)}{T(\partial/\partial x; \partial/\partial t; u)}. \qquad (4b)$$

(ii) Show that

$$\frac{\partial}{\partial x} = \frac{1}{T'}\frac{\partial}{\partial x'}, \qquad \frac{\partial}{\partial t} = \frac{F'}{T'}\frac{\partial}{\partial x'} + \frac{\partial}{\partial t'}. \qquad (5)$$

Solution 36. (i) From (3) and (4) we find

$$dx'' = T'dx' - F'dt' = T'Tdx - (T'F + F')dt = dx \qquad (5)$$

if and only if

$$T'(\partial u/\partial x', \partial u/\partial t', u)T(\partial u/\partial x, \partial u/\partial t, u) = 1 \qquad (6)$$

and

$$T'(\partial u/\partial x', \partial u/\partial t', u)F(\partial u/\partial u, \partial u/\partial t, u) + F'(\partial u/\partial x', \partial u/\partial t', u) = 0. \quad (7)$$

The result follows.

Problem 37. Consider the coupled *two-dimensional nonlinear Schrödinger equation*

$$i\frac{\partial u}{\partial t} - \beta\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^2 u}{\partial y^2} - \delta u^* uu - 2wu = 0 \qquad (1a)$$

$$\beta\frac{\partial^2 w}{\partial x^2} + \gamma\frac{\partial^2 w}{\partial y^2} + \beta\delta\frac{\partial^2}{\partial x^2}(u^* u) = 0 \qquad (1b)$$

where β, γ, δ are arbitrary constants. Consider the following transformation

$$\bar{x}(x,y,t) = \frac{x}{t}, \qquad \bar{y}(x,y,t) = \frac{y}{t}, \qquad \bar{t}(x,y,t) = -\frac{1}{t}, \qquad (2a)$$

$$u(x,y,t) = \frac{1}{t}\exp\left(\frac{-ix^2}{4\beta t} + \frac{iy^2}{4\gamma t}\right)\bar{u}(\bar{x}(x,y,t), \bar{y}(x,y,t), \bar{t}(x,y,t)) \qquad (2b)$$

$$w(x,y,t) = \frac{1}{t^2} w(\bar{x}(x,y,t), \bar{y}(x,y,t), \bar{t}(x,y,t)). \tag{2c}$$

Show that $\bar{u}$ and $\bar{w}$ satisfy the same equation with the subscripts x, y, t replaced by $\bar{x}$, $\bar{y}$, $\bar{t}$.

Solution 37.

Problem 38. Find a solution of the partial differential equation

$$\frac{\partial u}{\partial x_1}\left(1 - \left(\frac{\partial u}{\partial x_2}\right)^2\right) = \frac{\partial u}{\partial x_2}(1-u)$$

with the ansatz

$$u(x_1, x_2) = f(s) = f(x_1 + cx_2), \quad s = x_1 + cx_2$$

where c is a nonzero constant.

Solution 38. We have

$$\frac{\partial u}{\partial x_1} = \frac{df}{ds}, \quad \frac{\partial u}{\partial x_2} = c\frac{df}{ds}.$$

Thus

$$\frac{df}{ds}\left(1 - c + cf - c^2\left(\frac{df}{ds}\right)^2\right) = 0.$$

Integrating

$$\frac{cdf}{\sqrt{1-c+cf}} = ds$$

provides

$$1 - c + cf = \frac{1}{4}(x_1 + cx_2 + b)^2$$

where b is a constant of integration. Hence

$$u(x_1, x_2) = f(x + 1 + cx_2) = \frac{1}{4c}(x_1 + cx_2 + b)^2 + 1 - \frac{1}{c}.$$

Problem 39. An simplified analog of the *Boltzmann equation* is constructed as follows. It is one-dimensional and the velocities of the molecules are allowed to take two discrete values, $\pm c$, only. Thus the distribution function in the Boltzmann equation, $f(x,v,t)$, is replaced by two functions $u_+(x,t)$ and $u_-(x,t)$ denoting the density of particles with velocity $+c$ or $-c$, repsectively, at point x and time t. The gas is not confined, but x varies over all points of the real line $(-\infty, \infty)$. It is further assumed that there are only two types of interaction, viz., two + particles go over into two − particles, and vice versa, the probalitity for both processes to occur within one unit of time being the same number σ. Then the Boltzmann equation, which in absence of external forces reads

$$\frac{\partial f(\mathbf{x},\mathbf{v},t)}{\partial t} + \mathbf{v}\nabla f(\mathbf{x},\mathbf{v},t) = \int d^3\mathbf{v}_1 d\Omega |\mathbf{v}-\mathbf{v}_1| \sigma(|\mathbf{v}-\mathbf{v}_1|,\theta)[f' f_1' - f f_1] \quad (1)$$

where f' is the final distribution, i.e., the distribution after a collision. Under the assumption described above (1) translates into a system of two equations

$$\frac{\partial u_+}{\partial t} + c\frac{\partial u_+}{\partial x} = \sigma(u_-^2 - u_+^2), \qquad \frac{\partial u_-}{\partial t} - c\frac{\partial u_-}{\partial x} = \sigma(u_+^2 - u_-^2) \quad (2)$$

where c and σ are positive constants. This model is called the *Carleman model*. The Carleman model is rather unphysical. However with its aid one can prove almost all those results which one would like to obtain for the Boltzmann equation itself – as, for instance, the existence of solutions for a wide class of initial conditions or a rigorous treatment of the hydrodynamic limit. (i) Show that as for the Boltzmann equation, the *H theorem* holds for the Caleman model: The quantity

$$-\int (u_+(x,t)\ln u_+(x,t) + u_-(x,t)\ln u_-(x,t))dx \quad (3)$$

never decreases in time. (ii) Show that there exists the following generalizations of the H theorem. Let f be concave function, defined on the half-line $(0,\infty)$ which is once continuously differentiable. Let

$$S_f := \int (f(u_+) + f(u_-))dx \quad (4)$$

Show that

$$\frac{d}{dt} S_f(u) \geq 0. \quad (5)$$

Thus, not only does entropy never decrease, but the same is true for all quasientropies.
(iii) Show that as a consequence, all *Renyi entropies*

$$S_\alpha := (1-\alpha)^{-1} \ln \int (u_+^\alpha(x,t) + u_-^\alpha(x,t))dx \quad (6)$$

never decrease. In information-theretical language, this means that all sensible measures of the lack of information are nondecreasing. In other words, information is lost, or chaos is approached, in the strongest possible way.

Solution 39.

Problem 40. For the functions $u, v : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ we consider the Cauchy problem

$$\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = v^2 - u^2, \qquad \frac{\partial v}{\partial t} - \frac{\partial v}{\partial x} = u^2 - v^2 \tag{1a}$$

$$u(0, x) = u_0(x), \qquad v(0, x) = v_0(x). \tag{1b}$$

This is the *Carleman model* introduced above.
(i) Define

$$S := u + v, \qquad D := u - v \tag{2}$$

Show that S and D satisfy the system of partial differential equations

$$\frac{\partial S}{\partial t} + \frac{\partial D}{\partial x} = 0, \qquad \frac{\partial D}{\partial t} + \frac{\partial S}{\partial x} = -2DS \tag{3}$$

and the conditions $u \geq 0$, $v \geq 0$ take the form

$$S \geq 0 \quad \text{and} \quad S^2 - D^2 \geq 0. \tag{4}$$

(ii) Find explict solutions assuming that S and D are conjugate harmonic functions.

Solution 40. (ii) Assume that S und D conjugate harmonic functions, i.e.,

$$\frac{\partial S}{\partial t} + \frac{\partial D}{\partial x} = 0, \qquad \frac{\partial S}{\partial x} - \frac{\partial D}{\partial t} = 0. \tag{5}$$

Let $z := x + it$ and $f(z) := S + iD$. Then

$$\frac{df}{dz} = -\frac{i}{2}(f^2(z) + c), \qquad c \in \mathbb{R} \tag{6}$$

since

$$\frac{df}{dz} = \frac{\partial S}{\partial x} + i\frac{\partial D}{\partial x} = -\frac{i}{2}(S^2 - D^2 + 2iDS + c), \qquad \frac{\partial S}{\partial x} = DS. \tag{7}$$

Owing to $\partial S/\partial x = \partial D/\partial t$ the second equation is also satsified. Let

$$g(z) := \alpha f(\alpha z), \qquad \alpha \in \mathbb{R} \tag{8}$$

then

$$\frac{dg(z)}{dz} = \alpha^2 f'(\alpha z) = -\frac{i\alpha^2}{2}(f^2(\alpha z) + c) = -\frac{i}{2}(g^2(z) + \alpha^2 c). \tag{9}$$

For the solution

$$f(z) = \begin{cases} \dfrac{2f(z_0)}{2 + if(z_0)z_0 - if(z_0)z} & c = 0 \\ id\dfrac{(f(z_0) + id)e^{d(z-z_0)} + f(z_0) - id}{(f(z_0) + id)e^{d(z-z_0} - f(z_0) + id} & d = \sqrt{c} \neq 0 \end{cases} \tag{10}$$

of (6) we only have to study the three cases $c = -1, 0, 1$. Since the problem is analytic we can consider

$$c = -1, \quad z_0 = 0, \quad f(z_0) = \frac{1+\theta}{1-\theta}, \qquad \theta \in \mathbb{R}, \quad \theta \neq 1 \tag{11}$$

$$f(z) = -\frac{\theta e^{iz} + 1}{\theta e^{iz} - 1} = \frac{1 - \theta^2 e^{-2t} + i2\theta e^{-t} \sin x}{1 - 2\theta e^{-t} \cos x + \theta^2 e^{-2t}}. \tag{12}$$

We can check that for $|\theta| \leq \sqrt{2} - 1$ the conditions (4) are satisfied. Thus a particular solution of (1) is given by

$$u(t, x) = \frac{1}{2} \frac{\text{sgn}\theta \sinh(t - \log|\theta|) + \sin x}{\text{sgn}\theta \cosh(t - \log|\theta| - \cos x} \tag{12a}$$

$$v(t, x) = \frac{1}{2} \frac{\text{sgn}\theta \sinh(t - \log|\theta|) - \sin x}{\text{sgn}\theta \cosh(t - \log|\theta|) - \cos x}. \tag{12b}$$

Problem 41. The Carleman model as an approximation of the Boltzmann equation is given by the non-linear equations

$$\frac{\partial u_+}{\partial t} + c\frac{\partial u_+}{\partial x} = \sigma(u_-^2 - u_+^2), \tag{1a}$$

$$\frac{\partial u_-}{\partial t} - c\frac{\partial u_-}{\partial x} = \sigma(u_+^2 - u_-^2). \tag{1b}$$

(i) Show that if $u_\pm(x, 0) \geq 0$, then $u_\pm(x, t) \geq 0$ for all $t > 0$.
(ii) Show that

$$\frac{\partial}{\partial t} \int_{\mathbb{R}} (u_+ + u_-) dx = 0. \tag{2}$$

The two properties have to hold, of course, in any model that is considered to be of some physical relevance.
(iii) Let

$$H \equiv \int_{\mathbb{R}} (u_+ \ln u_+ + u_- \ln u_-) dx \tag{3}$$

Show that the analogue of Boltzmann's H-function is given by

$$\frac{dH}{dt} \leq 0 \tag{4}$$

(iv) Show that

$$\frac{\partial}{\partial t}(u_+ \ln u_+ + u_- \ln u_-) + c\frac{\partial}{\partial x}(u_+ \ln u_+ - u_- \ln u_-) \le 0. \tag{5}$$

$u_+ \ln u_+ + u_- \ln u_-$ has to be interpreted as the negative *entropy density* and $c(u_+ \ln u_+ - u_- \ln u_-)$ as the negative *entropy flux*.
(v) We define

$$S_f := \int (f(u_+) + f(u_-))dx \tag{6}$$

where f is a concave (or convex, respectively) function. In the special case

$$f(s) = -s \ln s$$

S_f is the expression for entropy i.e., $-H$. Show that

$$\frac{dS_f}{dt} \le 0$$

if f is convex. Show that

$$\frac{dS_f}{dt} \ge 0$$

if f is concave.

Solution 41. Simple computation yields

$$\frac{\partial}{\partial t}(f(u_+) + f(u_-)) = \frac{\partial f(u_+)}{\partial u_+}\frac{\partial u_+}{\partial t} + \frac{\partial f(u_-)}{\partial u_-}\frac{\partial u_-}{\partial t}$$

$$= \sigma\left(\frac{\partial f(u_+)}{\partial u_+} - \frac{\partial f(u_-)}{\partial u_-}\right)(u_-^2 - u_+^2) - c\left(\frac{\partial f}{\partial u_+}\frac{\partial u_+}{\partial x} - \frac{\partial f(u_-)}{\partial u_-}\frac{\partial u_-}{\partial x}\right). \tag{7}$$

By integration we arrive at

$$\frac{dS_f}{dt} = \sigma \int \left(\frac{\partial f(u_+)}{\partial u_+} - \frac{\partial f(u_-)}{\partial u_-}\right)(u_+^2 - u_-^2)dx \tag{8}$$

provided that $u_\pm \to 0$ as $x \to \pm\infty$. If f is concave (or convex, respectively), then $\partial f \partial u$ is decreasing (or increasing, respectively). We therefore obtain for the physical solutions ($u_\pm \ge 0$),

$$(\frac{\partial f(u_+)}{\partial u_+} - \frac{\partial f(u_-)}{\partial u_-})(u_-^2 - u_+^2) = (f'(u_+) - f'(u_-))(u_- - u_+)(u_+ + u_-) \ge 0 \tag{9}$$

if f is concave and ≤ 0, if f is convex. And consequently,

$$\frac{dS_f}{dt} \ge 0, \qquad \text{if } f \text{ is concave}$$

$$\frac{dS_f}{dt} \le 0, \qquad \text{if } f \text{ is convex.}$$

In the Carleman model not entropy alone increases with time, but also all functionals S_f (with f being concave) that could be designated "quasi-entropies".

Problem 42. The *Broadwell model* can then be written as

$$\frac{\partial f_+}{\partial t} + \frac{\partial f_+}{\partial x} = \frac{1}{\epsilon}(f_0^2 - f_+ f_-) \tag{1a}$$

$$\frac{\partial f_0}{\partial t} = \frac{1}{\epsilon}(f_+ f_- - f_0^2) \tag{1b}$$

$$\frac{\partial f_-}{\partial t} - \frac{\partial f_-}{\partial x} = \frac{1}{\epsilon}(f_0^2 - f_+ f_-). \tag{1c}$$

The parameter ϵ can qualitatively be understood as the mean free path. This system of equations serves as a model for the *Boltzmann equation*. The limit $\epsilon \to 0$ corresponds to a vanishing mean free parth and the fluid regime, while $\epsilon \to \infty$ approaches free molecular flow. The locally conserved spatial densities

$$\rho := f_+ + 2f_0 + f_-, \qquad \rho u := f_+ - f_- \tag{2}$$

corresponding to mass and x momentum. Show that these are governed by the local conservation laws

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) = 0, \qquad \frac{\partial(\rho u)}{\partial t} + \frac{\partial}{\partial x}(f_+ + f_-) = 0. \tag{3}$$

Solution 42.

Problem 43. Consider the hyperbolic system of conservation laws

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial}{\partial x}\mathbf{f}(\mathbf{u}) = 0 \tag{1}$$

where $\mathbf{f} : \mathbb{R}^m \to \mathbb{R}^m$ is continuously differentiable. A convex function $\eta(\mathbf{u})$ is called an entropy for (1) with entropy flux $q(\mathbf{u})$ if

$$\frac{\partial}{\partial t}\eta(\mathbf{u}) + \frac{\partial}{\partial x}q(\mathbf{u}) = 0 \tag{2}$$

holds identically for any smooth vector field $\mathbf{u}(x,t)$ which staisfies (1).
(i) Show that (2) follows from (1) if

$$\sum_{j=1}^m \frac{\partial \eta}{\partial u_j}\frac{\partial f_j}{\partial u_k} = \frac{\partial q}{\partial u_k}, \qquad k = 1, \ldots, m. \tag{3}$$

(ii) Show that for $m = 1$, every convex function $\eta(n)$ is an entropy for (1) with entropy flux

$$q(u) = \int_0^u \eta'(\omega) df(\omega). \qquad (4)$$

Solution 43.

Problem 44. Consider the so-called *sine-Hilbert equation*

$$H\left(\frac{\partial u}{\partial t}\right) = -\sin u \qquad (1)$$

where the integral operator H is defined by

$$Hu(x,t) := \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{u(y,t)}{y-x} dy. \qquad (2)$$

This is the so-called *Hilbert transform*. P denotes the *Cauchy principal value*. Let f be a continuous function, except at the singularity c. Then the Cauchy principal value is defined by

$$P\int_{-\infty}^{\infty} f(x) := \lim_{\epsilon \to 0}\left(\int_{-\infty}^{c-\epsilon} f(x)dx + \int_{c+\epsilon}^{\infty} f(x)dx\right). \qquad (3)$$

The Cauchy principal value can be found by applying the *residue theorem*. Let

$$u(x,t) := i \ln\left(\frac{f^*(x,t)}{f(x,t)}\right) \qquad (4)$$

where

$$f(x,t) := \prod_{j=1}^{N}(x - x_j(t)) \qquad (5a)$$

$$\Im x_j(t) > 0 \qquad j = 1, 2, \ldots, N, \qquad x_n \neq x_m \quad \text{for } n \neq m. \qquad (5b)$$

Here $x_j(t)$ are complex functions of t and * denotes complex conjugation.
(i) Show that

$$H\left(\frac{\partial u}{\partial t}\right) = -\frac{\partial}{\partial t}(\ln(f^* f)) \qquad (6)$$

follows from (2), (4) and (5).
(ii) Show that (1) is transformed into the form

$$\sum_{j=1}^{N} \frac{1}{x - x_j(t)} \frac{dx_j}{dt} + \sum_{j=1}^{N} \frac{1}{x - x_j^*(t)} \frac{dx_j^*}{dt} = \frac{1}{2i}\left(\frac{\prod_{j=1}^{N}(x - x_j^*(t))}{\prod_{j=1}^{N}(x - x_j(t))} - \frac{\prod_{j=1}^{N}(x - x_j(t))}{\prod_{j=1}^{N}(x - x_j^*(t))}\right). \qquad (7)$$

(iii) Show that (7) is equivalent to the equation

$$\frac{\partial}{\partial t}(f^* f) = \frac{1}{2i}(f^2 - f^{*2}) = \Im(f^2). \tag{8}$$

(iv) Show that by multiplying both sides of (7) by $x - x_n(t)$ and then putting $x = x_n$, we obtain

$$\frac{dx_n}{dt} = \frac{1}{2i}\frac{\prod_{j=1}^{N}(x_n(t) - x_j^*(t))}{\prod_{j=1(j\neq n)}^{N}(x_n(t) - x_j(t))} \qquad n = 1, 2, \ldots, N. \tag{9}$$

Solution 44.

Problem 45. Consider the *Fitzhugh-Nagumo equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u)(u-a) \tag{1}$$

where a is a constant. Without loss of generality we can set $-1 \leq a < 1$. Insert the ansatz

$$u(x,t) = f(x,t)w(z(x,t)) + g(x,t) \tag{2}$$

into (1) and require that $w(z)$ satisfies an ordinary differential equation. This is the so-called *direct method* and z is the so-called *reduced variable*.

Solution 45. Substituting (2) in (1) we obtain

$$\left(f\frac{\partial^2 z}{\partial x^2}\right)\frac{d^2 w}{dz^2} + \left(2\frac{\partial f}{\partial x}\frac{\partial z}{\partial x} + f\frac{\partial^2 z}{\partial x^2} - f\frac{\partial z}{\partial t}\right)\frac{dw}{dz} - f^3 w^3 + ((a+1-3g)f^2)w^2 +$$
$$\left(2(a+1)gf - 3fg^2 - af + \frac{\partial^2 f}{\partial x^2} - \frac{\partial f}{\partial t}\right)w + \left(\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial t} - g(g-a)(g-1)\right) = 0. \tag{3}$$

Now we must require (3) to be an ordinary differential equation for $w(z)$. The procedure using the direct method is to impose that the different relationships among the coefficients of (3) to be a second order ordinary differential equation. However one can equally consider it acceptable to reduce (3) to a first order ordinary differential equation. Setting in (3)

$$g = \frac{\partial z}{\partial x} \tag{4}$$

and demanding

$$3\frac{\partial^2 z}{\partial x^2} - \frac{\partial z}{\partial t} = \pm 2^{1/2}(a+1-3g)\frac{\partial z}{\partial x}, \tag{5a}$$

$$2(a+1)g\frac{\partial z}{\partial x} - 3\frac{\partial z}{\partial x}g^2 - a\frac{\partial z}{\partial x} + \frac{d^3 z}{dx^3} - \frac{\partial^2 z}{\partial x \partial t} = 0, \qquad (5b)$$

$$\frac{\partial^2 g}{\partial x^2} - \frac{\partial g}{\partial t} + g(g-1)(a-g) = 0, \qquad (5c)$$

(3) becomes

$$\frac{d^2 w}{dz^2} - w^3 + \frac{a+1-3g}{\partial z/\partial x}(\pm 2^{1/2}\frac{dw}{dz} + w^2) = 0, \qquad (6)$$

which is satisfied if w verifies the first order ordinary differential equation

$$\pm 2^{1/2}\frac{dw}{dz} + w^2 = 0 \qquad (7)$$

which could be integrated at once yielding

$$w(z) = \frac{\pm 2^{1/2}}{z+z_0}. \qquad (8)$$

By combining (2a), (4) and (8) we can write the solution as

$$u(x,t) = \frac{\pm 2^{1/2}}{z+z_0}\frac{\partial z}{\partial x} + g.$$

Problem 46. Consider the *nonlinear Dirac equations* of the form

$$i\sum_{\mu=0}^{3}\gamma^\mu \frac{\partial}{\partial x_\mu}\psi - M\psi + F(\bar{\psi}\psi)\psi = 0. \qquad (1)$$

The notation is the following

$$\psi : \mathbb{R}^4 \to \mathbb{C}^4, \qquad \psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad (2)$$

M is a positive constant,

$$\bar{\psi}\psi := (\gamma^0\psi, \psi) \equiv \psi_1^*\psi_1 + \psi_2^*\psi_2 - \psi_3^*\psi_3 - \psi_4^*\psi_4 \qquad (3)$$

where $(\cdot,\cdot)$ is the usual scalar product in $\mathbb{C}^4$ and the γ^μ's are the 4×4 matrices of the Pauli-Dirac representation, given by

$$\gamma^0 := \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma^k := \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} \quad \text{for} \quad k = 1,2,3 \qquad (4)$$

where the *Pauli matrices* are given by

$$\sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{5}$$

and $F : \mathbb{R} \to \mathbb{R}$ models the nonlinear interaction. (i) Consider the ansatz (*standing waves*, *stationary states*)

$$\psi(t, \mathbf{x}) = e^{i\omega t}\mathbf{u}(\mathbf{x}) \tag{6}$$

where $x_0 = t$ and $\mathbf{x} = (x_1, x_2, x_3)$. Show that $\mathbf{u} : \mathbb{R}^3 \to \mathbb{C}^4$ satisfies the equation

$$i\sum_{k=1}^{3}\gamma^k \frac{\partial}{\partial x_k}\mathbf{u} - M\mathbf{u} + \omega\gamma^0\mathbf{u} + F(\bar{\mathbf{u}}\mathbf{u})\mathbf{u} = \mathbf{0}. \tag{7}$$

(ii) Let

$$\mathbf{u}(\mathbf{x}) := \begin{pmatrix} v(r)\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ iw(r)\begin{pmatrix} \cos\theta \\ \sin\theta e^{i\phi} \end{pmatrix} \end{pmatrix}. \tag{8}$$

Here $r = |\mathbf{x}|$ and (θ, ϕ) are the angular parameters. Show that w and v satisfy the nonautonomous planar dynamical system

$$\frac{dw}{dr} + \frac{2w}{r} = v(F(v^2 - w^2) - (M - \omega)) \tag{9a}$$

$$\frac{dv}{dr} = w(F(v^2 - w^2) - (M + \omega)). \tag{9b}$$

(ii) Hint. Notice that

$$\bar{\mathbf{u}}\mathbf{u} = u_1^* u_1 + u_2^* u_2 - u_3^* u_3 - u_4^* u_4 \tag{10}$$

Inserting (8) yields

$$\bar{\mathbf{u}}\mathbf{u} = v^2 - w^2. \tag{11}$$

Solution 46.

Problem 47. Consider a one-dimensional system to describe the *electron-beam plasma system*

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e(u_e + V)) = 0 \tag{1a}$$

$$\frac{\partial u_e}{\partial t} + (u_e + V)\frac{\partial u_e}{\partial x} = -\frac{e}{m_e}\mathcal{E} - \frac{1}{m_e n_e}\frac{\partial p_e}{\partial x} \tag{1b}$$

$$\frac{\partial \mathcal{E}}{\partial t} + (u_e + V)\frac{\partial \mathcal{E}}{\partial x} = 4\pi e n_i (u_e + V). \tag{1c}$$

where n_e is the density of the beam-electron fluids, m_e the beam-electron mass, u_e the bulk fluid velocity, $p = K_b T_e n_e$ the particle fluid pressure, e the charge on an electron and $\mathcal{E} = -\nabla\phi$. We study the simplest case of uniform plasma in the absence of an external electromagnetic field, and assume that the positive ions are taken to form a fixed, neutralizing background of uniform density $n_i = N_0 = \text{const}$ throughout the present analysis. The electrons move with a beam drift velocity V corresponding to the ions. Assume that electrostatic perturbation is sinusoidal

$$n_e(x,t) = N_0\left(1 + \frac{n(t)}{N_0}(\sin(kx) + \cos(kx))\right), \qquad n(t) \leq \frac{1}{2}N_0 \tag{2a}$$

$$u_e(x,t) = u(t)(\cos(kx) - \sin(2kx)), \qquad \mathcal{E}(x,t) = E(t)(\cos(kx) - \sin(kx)). \tag{2b}$$

Define

$$X(t) := n(t), \quad Y(t) := \frac{1}{2}ku(t), \quad Z(t) := \frac{e}{m_e p}\frac{k}{2}E(t), \quad q := \frac{K_b T_e}{m_e N_0}\frac{k^2}{2}, \quad p := \frac{4\pi e^2}{m_e}. \tag{3}$$

Show that when (2) and (3) are substituted into (1a)-(1c), we obtain

$$\frac{dX}{dt} = kVX - XY + 2N_0 Y$$
$$\frac{dY}{dt} = -qX + Y^2 - pZ$$
$$\frac{dZ}{dt} = N_0 Y - YZ + kVZ.$$

Solution 47.

Problem 48. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta\frac{\partial^3 u}{\partial x^3} + \frac{\partial^2 u}{\partial x^2} + \frac{\partial^4 u}{\partial x^4} = 0 \tag{1}$$

subject to periodic boundary conditions in the interval $[0, L]$, with initial conditions $u(x,0) = u_0(x)$. We only consider solutions with zero spatial average. We recall that for $L \leq 2\pi$ all initial conditions evolve into $u(x,t) = 0$. We expand the solution for u in the Fourier series

$$u(x,t) = \sum_{n=-\infty}^{\infty} a_n(t)\exp(ik_n x) \tag{2}$$

where $k_n := 2n\pi/L$ and the expansion coefficients satisfy

$$a_{-n}(t) = \bar{a}_n(t). \tag{3}$$

Here $\bar{a}$ denotes the complex conjugate of a. Since we choose solutions with zero average we have $a_0 = 0$. (i) Show that inserting the series expansion (2) into (1) we obtain the following system for the time evolution of the Fourier amplitudes

$$\frac{da_n}{dt} + (k_n^4 - k_n^2 - i\delta k_n^3)a_n + \frac{1}{2}ik_n \sum_{m=0}^{\infty}(a_m a_{n-m} + \bar{a}_m a_{n+m}) = 0. \tag{4}$$

(ii) Show that keeping only the first five modes we obtain the system

$$\frac{da_1}{dt} + (\mu_1 - i\delta k^3)a_1 + ik(\bar{a}_1 a_2 + \bar{a}_2 a_3 + \bar{a}_3 a_4 + \bar{a}_4 a_5) = 0 \tag{5a}$$

$$\frac{da_2}{dt} + (\mu_2 - 8i\delta k^3)a_2 + ik(a_1^2 + 2\bar{a}_1 a_3 + 2\bar{a}_2 a_4 + 2\bar{a}_3 a_5) = 0 \tag{5b}$$

$$\frac{da_3}{dt} + (\mu_3 - 27i\delta k^3)a_3 + 3ik(a_1 a_2 + \bar{a}_1 a_4 + \bar{a}_2 a_5) = 0 \tag{5c}$$

$$\frac{da_4}{dt} + (\mu_4 - 64i\delta k^3)a_4 + 2ik(a_2^2 + 2a_1 a_3 + 2\bar{a}_1 a_5) = 0 \tag{5d}$$

$$\frac{da_5}{dt} + (\mu_5 - 125i\delta k^3)a_5 + 5ik(a_1 a_4 + a_2 a_3) = 0 \tag{5e}$$

where

$$k := 2\frac{\pi}{L}, \qquad \mu_n := k_n^4 - k_n^2. \tag{6}$$

Solution 48. (i) Since

$$\frac{\partial u}{\partial x} = \sum_{m=-\infty}^{\infty} a_m(t) ik_m e^{ik_m x} \tag{7}$$

we have

$$u\frac{\partial u}{\partial x} = \sum_{q=-\infty}^{\infty}\sum_{p=-\infty}^{\infty} a_p(t)a_q(t)ik_q e^{i(k_q+k_p)x}. \tag{8}$$

Furthermore

$$\frac{\partial u}{\partial t} = \sum_{n=-\infty}^{\infty} \frac{\partial a_n}{\partial t} e^{ik_n x} \tag{9}$$

and

$$\frac{\partial^2 u}{\partial x^2} = -\sum_{m=-\infty}^{\infty} a_m(t) k_m^2 e^{ik_m x} \tag{10}$$

$$\frac{\partial^3 u}{\partial x^3} = -i \sum_{m=-\infty}^{\infty} a_m(t) k_m^3 e^{ik_m x} \tag{11}$$

$$\frac{\partial^4 u}{\partial x^4} = \sum_{m=-\infty}^{\infty} a_m(t) k_m^4 e^{ik_m x}. \tag{12}$$

Applying the Kronecker delta $\delta_{m,n}$ to (7), (9), (10), (11), (12) and $\delta_{p+q,n}$ to (8) we obtain (4) where we used (3).

Problem 49. The modified *Boussinesq-Oberbeck equations* are given by

$$\frac{\partial \triangle \psi}{\partial t} = \sigma \frac{\partial \theta}{\partial x} + \sigma \triangle (\triangle \psi) - \left(\frac{\partial \psi}{\partial x} \frac{\partial \triangle \psi}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \triangle \psi}{\partial x} \right)$$

$$\frac{\partial \theta}{\partial t} = R \frac{\partial \psi}{\partial x} + \triangle \theta - \left(\frac{\partial \psi}{\partial x} \frac{\partial \theta}{\partial z} - \frac{\partial \psi}{\partial z} \frac{\partial \theta}{\partial x} \right) + - \frac{\bar{\epsilon}}{T_i} \left[-2R \frac{\partial \theta}{\partial z} + \left(\frac{\partial \theta}{\partial x} \right)^2 + \left(\frac{\partial \theta}{\partial z} \right)^2 \right] \tag{1}$$

where σ is the Prandtl number, R the Rayleigh number, ψ the stream function, and θ a function measuring the difference between the profile of temperature and a profile linearly decreasing with height and time, namely

$$\theta = T - T_0 + \frac{\bar{\epsilon}}{T_i} F^2 t + Rz. \tag{2}$$

Let L be the horizontal extenion of the convection cells, $l = H/L$ and $R_c = \pi^4 (1 + l^2)^3 / l^2$ the critical Rayleigh number for the onset of convection. One sets $r = R/R_c$. Suppose that the temperature T_1 is fixed and that T_0 increases. We define the dimensionless parameter

$$\alpha := \frac{T_0 - T_1}{T_1} \frac{1}{r}. \tag{3}$$

This quantity is a constant once the fluid and the experimental setting are chosen. The dimensionless temperatures appearing in the last term of the second of equations (1) are now expressed by

$$T_0 = R_c \frac{(\alpha r + 1)}{\alpha}, \qquad T_1 = \frac{R_c}{\alpha}. \tag{4}$$

The boundary conditions are

$$\psi = \triangle \psi = 0 \quad \text{at } z = 0,\ H$$

$$\theta = 0, \quad \text{at } z = 0,\ H$$

$$\frac{\partial \psi}{\partial z} = 0 \quad \text{at} \quad x = kL, \quad k \in \mathbb{Z} \tag{5}$$

Show that the equation

$$\frac{dX}{dt} = -\sigma X + \sigma Y, \qquad \frac{dY}{dt} = rX - Y - XZ + \epsilon W\left(1 + \frac{Z}{r}\right)$$

$$\frac{dW}{dt} = -W - \epsilon Y\left(1 + \frac{Z}{r}\right), \qquad \frac{dZ}{dt} = -bZ + XY \tag{6}$$

can be derived from (1) and the boundary condition (5) via a Fourier expansion of the function $\psi(x,z,t)$ and $\theta(x,z,t)$.

Solution 49. The lack of the usual boundary condition stating that no heat flow occurs through the walls of the convection rolls, due to the addition of the sink term in the internal energy equation. Thus in the double Fourier expansion of the functions $\psi(x,z,t)$ and $\theta(x,z,t)$ the unknown Fourier coefficients are

$$\psi(m,n;t) = \Re\psi(m,n;t) =: \psi_1(m,n;t)$$

$$\theta(m,n;t) = \Re\theta(m,n;t) + i\Im\theta(m,n;t) =: \theta_1(m,n;t) + i\theta_2(m,n;t). \tag{7}$$

Linear analysis of system (1) leads us towards the choice of the Lorenz like truncation. The Jacobian matrix of the system in the variables ψ_1, θ_1, θ_2, evaluated at the origin, is block diagonal, with characteristic equation

$$\prod_{m,n} q^3(m,n)(\lambda^3 + (\sigma+2)\lambda^2 + +((\sigma+1) - \sigma(r(m,n)-1)$$

$$+\epsilon^2(m,n))\lambda - \sigma((r(m,n)-1) - \epsilon^2(m,n))) = 0 \tag{8}$$

where we have set

$$q(m,n) = \pi^2(m^2l^2 + n^2), \qquad R_c(m,n) = \frac{q(m,n)^3}{\pi^2 m^2 l^2}$$

$$r(m,n) = \frac{R}{R_c(m,n)}, \qquad \epsilon(m,n) = \tilde{\epsilon}\frac{2R}{T_i}\frac{n\pi}{q}, \quad i = 0,1. \tag{9}$$

The simplest nonlinear system of ordinary differential equations is now obtained by choosing, for fixed (m,n), the four modes

$$\psi_1(m,n), \quad \theta_2(m,n), \quad \theta_1(m,n), \quad \theta_2(0,2n)$$

so that, after a suitable rescaling of the variables and of the time, one obtains (6), with $b = 4n^2/(m^2l^2+n^2)$.

Problem 50. Consider the complex *Ginzburg-Landau equation*

$$\frac{\partial w}{\partial t} = (1 + ic_1)\frac{\partial^2 w}{\partial x^2} + w - (a + ic_2)|w|^2 w. \tag{1}$$

(i) Show that setting $a = -1$ in (1), and writing

$$w(x,t) = R(x,t)\exp(i\Theta(x,t)) \tag{2}$$

one obtains two real equations which, after suitable linear combination and division by c_1^2, can be written as

$$\epsilon^2\frac{\partial R}{\partial t} + \epsilon R\frac{\partial \Theta}{\partial t} = \left((1+\epsilon^2)\left(\frac{\partial^2}{\partial x^2} - \left(\frac{\partial \Theta}{\partial x}\right)^2\right) + \epsilon^2 + (\beta+\epsilon^2)R^2\right)R \tag{3a}$$

$$-\frac{1}{2}\epsilon\frac{\partial}{\partial t}R^2 + \epsilon^2 R^2\frac{\partial \Theta}{\partial t} = (1+\epsilon^2)\frac{\partial}{\partial x}\left(r^2\frac{\partial \Theta}{\partial x}\right) - \epsilon(1+(1-\beta)R^2)R^2. \tag{3b}$$

Here we have introduced $c_2 := -\beta c_1$ and $\epsilon := 1/c_1$. (ii) Make an expansion in ϵ of the form

$$R := R_0 + \epsilon^2 R_2 + \cdots, \qquad \Theta := \epsilon^{-1}(\Theta_{-1} + \epsilon^2\Theta_1 + \cdots) \tag{4}$$

Solution 50. (i) We see that

$$|w|^2 = R^2. \tag{5}$$

Inserting (2) into (1) yields

$$= \tag{6}$$

Separating out the real and imaginary part we obtain

$$= \tag{7}$$

(ii) The expansion (4) becomes meaningful for sufficiently large values of c_1 [$\beta = O(1)$]. This first leads to the orders ϵ^{-2}, where we have

$$R_0\left(\frac{\partial \Theta_{-1}}{\partial x}\right)^2 = 0, \qquad \frac{\partial}{\partial x}\left(R_0^2\frac{\partial \Theta_{-1}}{\partial x}\right) = 0 \tag{8}$$

respectively. Excluding $R_0 = 0$, we obtain $\partial\Theta_{-1}/\partial x = 0$, so that Θ_{-1} only depends on t. Setting

$$\gamma(t) := \frac{\partial \Theta_{-1}}{\partial x} \tag{9}$$

one gets for the next orders

$$0 = \frac{\partial^2}{\partial x^2}R_0 - \gamma R_0 + \beta R_0^3 \quad (\epsilon^0), \tag{10a}$$

$$-\frac{\partial}{\partial x}(R_0^2\Theta_1') = \frac{1}{2}\frac{\partial}{\partial t}R_0^2 - (1+\gamma)(R_0^2 - (1-\beta)R_0^4 \quad (\epsilon^1). \tag{10b}$$

For $\beta, \gamma > 0$, (6a) allows spatially periodic solutions

$$R_0(x,t) = \left[\frac{2\gamma(t)}{[2-m(t)]\beta}\right]^{1/2} \mathrm{dn}\left(\left(\frac{\gamma(t)}{2-m(t)}\right)^{1/2} x, m(t)\right). \qquad (11)$$

Here $\mathrm{dn}(u|m)$ is a Jacobian elliptic function that varies between $(1-m)^{1/2}$ and 1 with period $2K(m)$ [the parameter m is between 0 and 1, and $K(m)$ is the complete elliptic integral of the first kind. For $m \to 1$ the period of dn goes to infinity and (7) degenerates into the pulse

$$(2\gamma/\beta)^{1/2}\mathrm{sech}(\gamma^{1/2}x) \qquad (12)$$

while for $m \to 0$ one has small, harmonic oscillations.

Problem 51. Consider the cubic *nonlinear one-dimensional Schrödinger equation*,

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + Qw|w|^2 = 0 \qquad (1)$$

where Q is a constant. (i) Show that a discretization with the *periodic boundary conditions* $w_{j+N} \equiv w_j$ is given by

$$i\frac{dw_j}{dt} + \frac{w_{j+1} + w_{j-1} - 2w_j}{h^2} + Q|w_j|^2 w_j^{(k)} = 0, \qquad k = 1, 2, \ldots \qquad (2)$$

where

$$\text{(a)} \qquad w_j^{(1)} := w_j \qquad \text{and} \qquad \text{(b)} \quad w_j^{(2)} := \frac{1}{2}(w_{j+1} + w_{j-1}). \qquad (3)$$

(ii) Show that both schemes are of second-order accuracy. (iii) Show that in case (2a) there are first integrals, the L^2 norm,

$$I := \sum_{j=0}^{N-1} |w_j|^2 \qquad (4)$$

and the Hamilton function

$$H = -i\sum_{j=0}^{N-1}\left(\frac{|w_{j+1} - w_j|^2}{h^2} - \frac{1}{2}Q|w_j|^4\right). \qquad (5)$$

The Poisson brackets are the standard ones. Thus when $N = 2$ the system is integrable. This system has been used as a model for a nonlinear dimer. (iv) Show that the Hamilton function of scheme (2b) is given by $(h = 1)$

$$H = -i\sum_{j=0}^{N-1}(w_j^*(w_{j-1} + w_{j+1}) - \frac{4}{Q}\ln(1 + \frac{1}{2}Qw_j w_j^*) \qquad (6)$$

together with the nonstandard Poisson brackets

$$\{q_m, p_n\} := (1 + \frac{1}{2} Q q_n p_n) \delta_{m,n} \tag{7}$$

and

$$\{q_m, q_n\} = \{p_m, p_n\} = 0. \tag{8}$$

Solution 51.

Problem 52. Consider the *sine-Gordon equation* in $3+1$ dimensions

$$\Box \chi = \sin(\chi) \tag{1}$$

where

$$\Box := \frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2} + \frac{\partial}{\partial z^2} - \frac{\partial}{\partial t^2} \tag{2}$$

and $\chi(x, y, z, t)$ is a real valued scalar field. The sinh-Gordon equation is given by

$$\Box \chi = \sinh(\chi) \tag{3}$$

Let

$$\sigma_0 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

be the *Pauli spin matrices* and

$$a := \sigma_0 \frac{\partial}{\partial x} + i\sigma_1 \frac{\partial}{\partial y} + i\sigma_3 \frac{\partial}{\partial z} + \sigma_2 \frac{\partial}{\partial t}. \tag{5}$$

Let $\bar{a}$ denote complex conjugates. Let $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$, $-\infty < \lambda < \infty$ and $-\infty < \tau < \infty$ be arbitrary parameters. We set

$$U := \exp(i\theta\sigma_1 \exp(-i\phi\sigma_2 e^{-\tau\sigma_1})). \tag{6}$$

Assume that α and β are solutions of (1) and (2), respectively. Let

$$u := \alpha - i\beta. \tag{7}$$

The Bäcklund transformation $\hat{B}$ is then given by

$$\alpha \longrightarrow i\beta = \hat{B}(\phi, \theta, \tau)\alpha \tag{8}$$

where $\hat{B}(\phi, \theta, \tau)$ is the Bäcklund transformation operator. The functions α and β are related by

$$\frac{1}{2} a u = \sin\left(\frac{1}{2}\bar{u}\right) U. \tag{9}$$

The Bäcklund transformation works as follows. Let α (respectively β) be a solution of (1) (respectively 3)) then solve (9) for β (respectively α). The solution then solves (3) (respectively (1)).
(i) Show for any U such that

$$\bar{U} = U^{-1} \tag{10}$$

all solutions of (9) must be of the form (i.e. plane travelling wave with speed v less than one)

$$u(x,y,z,t) = f(\eta), \qquad \eta := kx + ly + mz - \omega t \tag{11}$$

where k, l, m, ω are real constants and

$$k^2 + l^2 + m^2 - \omega^2 = 1, \qquad v = \frac{\omega^2}{k^2 + l^2 + m^2}.$$

(ii) Thus show that we have a Bäcklund transformation between the ordinary differential equations

$$\frac{d^2\alpha}{d\eta^2} = \sin\alpha, \qquad \frac{d^2\beta}{d\eta^2} = \sinh\beta$$

defined by

$$\frac{du}{d\eta} = 2e^{i\psi}\sin\left(\frac{1}{2}\bar{u}\right).$$

Solution 52.

Problem 53. The equation which describes small amplitude waves in a dispersive medium with a slight deviation from one-dimensionality is

$$\frac{\partial}{\partial x}\left(4\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\right) \pm 3\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

The + refers to the two-dimensional Korteweg-de Vries equation. The − refers to the two-dimensional Kadomtsev-Petviashvili equation. Let (formulation of the *inverse scattering transform*)

$$u(x,y,t) := 2\frac{\partial}{\partial x}K(x,x;y,t) \tag{2}$$

where

$$K(x,z;y,t) + F(x,z;y,t) + \int_x^\infty K(x,s;y,t)F(s,z;y,t)ds = 0 \tag{3}$$

and F satsifies the system of linear partial differential equations

$$\frac{\partial^3 F}{\partial x^3} + \frac{\partial^3 F}{\partial z^3} + \frac{\partial F}{\partial t} = 0, \qquad \frac{\partial^2 F}{\partial x^2} - \frac{\partial^2 F}{\partial z^2} + \sigma \frac{\partial F}{\partial y} = 0 \tag{4}$$

where $\sigma = 1$ for the two-dimensional Korteweg de Vries equation and $\sigma = i$ for the Kadomtsev Petviashvili equation. Find solutions of the form

$$F(x, z; y, t) = \alpha(x; y, t)\beta(z; y, t) \tag{5}$$

and

$$K(x, z; y, t) = L(x; y, t)\beta(z; y, t). \tag{6}$$

Solution 53. Inserting (5) and (6) into (3) yields

$$L(x; y, t) = -\frac{\alpha(x; y, t)}{\left(1 + \int_x^\infty \alpha(s; y, t)\beta(s; y, t)ds\right)}. \tag{7}$$

From (2) we then have the solution of (1).

$$u(x, y, t) = 2\frac{\partial^2}{\partial x^2} \ln\left(1 + \int_x^\infty \alpha(s; y, t)\beta(s; y, t)ds\right) \tag{8}$$

provided functions α and β can be found. From (4) we obtain

$$\frac{\partial \alpha}{\partial t} + \frac{\partial^3 \alpha}{\partial x^3} = 0, \qquad \frac{\partial \beta}{\partial t} + \frac{\partial^3 \beta}{\partial z^3} = 0 \tag{9}$$

$$\sigma\frac{\partial \alpha}{\partial y} + \frac{\partial^2 \alpha}{\partial x^2} = 0, \qquad \sigma\frac{\partial \beta}{\partial y} - \frac{\partial^2 \beta}{\partial z^2} = 0. \tag{10}$$

Then α and β admit the solution

$$\alpha(x, y, t) = \exp(-lx - (l^2/\sigma)y + l^3 t + \delta) \tag{11}$$

and similarly

$$\beta(x, y, t) = \exp(-Lz + (L^2/\sigma)y + L^3 t + \Delta) \tag{12}$$

where δ, Δ are arbitrary shifts and l, L are constants. This form gives the oblique solitary wave solution of the two-dimensional Korteweg de Vries equation,

$$u(x, y, t) = 2a^2 \text{sech}^2(a(x + 2my - (a^2 + 3m^2)t)) \tag{13}$$

where

$$a = \frac{1}{2}(l + L), \qquad m = \frac{1}{2}(l - L). \tag{14}$$

For the Kadomtsev-Petviashvili equation we have $\sigma = i$, and so if we regard l, L as complex constants a real solution is just (13) with $m \to -im$ and $l = \bar{L}$.

Problem 54. Show that Hirota's operators $D_x^n(f \cdot g)$ and $D_x^m(f \cdot g)$ given in (1) and (2) can be written as

$$D_x^n(f \cdot g) = \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^j f}{\partial x^j} \frac{\partial^{n-j} g}{\partial x^{n-j}}, \tag{3}$$

$$D_x^m D_t^n(f \cdot g) = \sum_{j=0}^{m} \sum_{i=0}^{n} \frac{(-1)^{(m+n-j-i)} m!}{j!(m-j)!} \frac{n!}{i!(n-i)!} \frac{\partial^{i+j} f}{\partial t^i \partial x^j} \frac{\partial^{n+m-i-j} g}{\partial t^{n-i} \partial x^{m-j}}. \tag{4}$$

Solution 54. We prove (3) by mathematical induction. The formula (4) can be proven in a similar way. We first try to show that

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n (f(x) \cdot g(x')) = \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^j f(x)}{\partial x^j} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}. \tag{5}$$

For $n = 1$,

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) (f(x) \cdot g(x')) = \frac{\partial f(x)}{\partial x} \cdot g(x') - f(x) \cdot \frac{\partial g(x')}{\partial x'}$$

$$= \sum_{j=0}^{1} \frac{(-1)^{(1-j)} 1!}{j!(1-j)!} \frac{\partial^j f(x)}{\partial x^j} \frac{\partial^{1-j} g(x')}{\partial x'^{1-j}}, \tag{6}$$

(5) obviously holds. If we assume (5) to be true for $n-1$, then

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n-1} (f(x) \cdot g(x')) = \sum_{j=0}^{n-1} \frac{(-1)^{(n-1-j)} (n-1)!}{j!(n-1-j)!} \frac{\partial^j f(x)}{\partial x^j} \frac{\partial^{n-1-j} g(x')}{\partial x'^{n-1-j}}. \tag{7}$$

Using (7), we have

$$\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^n (f(x) \cdot g(x')) = \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right)^{n-1} (f(x) \cdot g(x'))$$

$$= \left(\frac{\partial}{\partial x} - \frac{\partial}{\partial x'}\right) \sum_{j=0}^{n-1} \frac{(-1)^{(n-1-j)} (n-1)!}{j!(n-1-j)!} \frac{\partial^j f(x)}{\partial x^j} \frac{\partial^{n-1-j} g(x')}{\partial x'^{n-1-j}}$$

$$= \sum_{j=0}^{n-1} \frac{(-1)^{(n-1-j)} (n-1)!}{j!(n-1-j)!} \left(\frac{\partial^{j+1} f(x)}{\partial x^{j+1}} \frac{\partial^{n-1-j} g(x')}{\partial x'^{n-1-j}} - \frac{\partial^j f(x)}{\partial x^j} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}\right)$$

$$= \sum_{j=0}^{n-1} \frac{(-1)^{(n-1-j)}(n-1)!}{j!(n-1-j)!} \frac{\partial^{j+1} f(x)}{\partial x^{j+1}} \frac{\partial^{n-1-j} g(x')}{\partial x'^{n-1-j}}$$

$$- \sum_{j=0}^{n-1} \frac{(-1)^{(n-1-j)}(n-1)!}{j!(n-1-j)!} \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}$$

$$= \sum_{j=1}^{n} \frac{(-1)^{(n-j)}(n-1)!}{j-1)!(n-j)!} \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}$$

$$+ \sum_{j=0}^{n-1} \frac{(-1)^{(n-j)}(n-1)!}{j!(n-1-j)!} \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}$$

$$= \sum_{j=1}^{n-1} \left(\frac{(-1)^{(n-j)}(n-1)!}{j-1)!(n-j)!} + \frac{(-1)^{n-j)}(n-1)!}{j!(n-1-j)!} \right) \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}$$

$$+ \frac{\partial^n f(x)}{\partial x^n} + (-1)^n f(x) \frac{\partial^n g(x')}{\partial x'^n}$$

$$= \sum_{j=1}^{n-1} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x')}{\partial x'^{m-j}}$$

$$+ \frac{\partial^n f(x)}{\partial x^n} + (-1)^n f(x) \frac{\partial^n g(x')}{\partial x'^n}$$

$$= \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x')}{\partial x'^{n-j}}.$$

Thus, (7) is true for all $n = 1, 2, \ldots$. Setting $x' = x$ we have

$$D_x^n (f \cdot g) = \sum_{j=0}^{n} \frac{(-1)^{(n-j)} n!}{j!(n-j)!} \frac{\partial^{j} f(x)}{\partial x^{j}} \frac{\partial^{n-j} g(x)}{\partial x^{n-j}}.$$

Problem 55. Consider the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

(i) Consider the dependent variable transformation

$$u(x,t) = 2 \frac{\partial^2 \ln f(x,t)}{\partial x^2}. \tag{2}$$

Show that f satisfies the differential equation

$$f \frac{\partial^2 f}{\partial x \partial t} - \frac{\partial f}{\partial x} \frac{\partial f}{\partial t} + f \frac{\partial^4 f}{\partial x^4} - 4 \frac{\partial f}{\partial x} \frac{\partial^3 f}{\partial x^3} + 3 \left(\frac{\partial^2 f}{\partial x^2} \right)^2 = 0. \tag{3}$$

(ii) Show that this equation in f can be written in bilinear form

$$(D_x D_t + D_x^4)(f \cdot f) = 0. \tag{4}$$

Solution 55.

Problem 56. The *Sawada-Kotera equation* is given by

$$\frac{\partial u}{\partial t} + 45u^2\frac{\partial u}{\partial x} + 15\frac{\partial u}{\partial x}\frac{\partial^2 u}{\partial x^2} + 15u\frac{\partial^3 u}{\partial x^3} + \frac{\partial^5 u}{\partial x^5} = 0. \tag{1}$$

Show that using the dependent variable transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \tag{2}$$

and integrating once with respect to x we arrive at

$$(D_x D_t + D_x^6)(f \cdot f) = 0. \tag{3}$$

Solution 56.

Problem 57. The *Kadomtsev-Petviashvili equation* is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) + 3\frac{\partial^2 u}{\partial y^2} = 0. \tag{1}$$

Show that using the transformation

$$u(x,t) = 2\frac{\partial^2 \ln f(x,t)}{\partial x^2} \tag{2}$$

the Kadomtsev-Petviashvili equation takes the form

$$(D_x D_t + D_x^4 + 3D_y^2)(f \cdot f) = 0. \tag{3}$$

Solution 57.

Problem 58. Consider the system of partial differential equations

$$\begin{aligned}
\frac{\partial u}{\partial t} &= \alpha u\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \\
\frac{\partial v}{\partial t} &= \beta v\frac{\partial u}{\partial x} + cu\frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \\
\frac{\partial w}{\partial t} &= \gamma w\frac{\partial u}{\partial x} + \delta u\frac{\partial w}{\partial x}
\end{aligned}$$

where $\alpha, \beta, \gamma, c, \delta$ are constants. Show that the system admits the converved densities

$$\begin{aligned}
H_0 &= u \\
H_1 &= v + \frac{1}{2}(\beta - c)u^2 \\
H_2 &= uv + \frac{1}{\gamma - \delta}w + \left(\frac{\alpha+\beta}{2} - c\right)\frac{u^3}{3}, \qquad \gamma \neq \delta \\
H_3 &= w, \qquad \gamma = \delta.
\end{aligned}$$

Solution 58. The conserved density H_0 is obvious since the first partial differential equation can be written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{2}\alpha u^2 + v\right).$$

Problem 59. The *polytropic gas dynamics* in $1+1$ dimensions is of the form

$$\begin{aligned}
\frac{\partial u}{\partial t} &= u\frac{\partial u}{\partial x} + C\rho^{\Gamma}\frac{\partial \rho}{\partial x} \\
\frac{\partial \rho}{\partial t} &= \frac{\partial}{\partial x}(\rho u)
\end{aligned}$$

where x is the space coordinate, t is the (minus physical) time coordinate, u is the velocity, ρ the density and $\Gamma = \gamma - 2$. Here γ is the polytropic exponent. The constant C can be removed by a rescaling of ρ. Express this system of partial differential equations applying the *Riemann invariants*

$$r_{1,2}(x,t) = u(x,t) \pm \frac{2}{\Gamma+1}\rho^{(\Gamma+1)/2}(x,t). \qquad \Gamma \neq -1.$$

Solution 59. We obtain (check second equation r_2)

$$\begin{aligned}
\frac{\partial r_1}{\partial t} &= \left(\left(\Gamma + \frac{3}{2}\right)r_1 - \left(\Gamma + \frac{1}{2}\right)r_2\right)\frac{\partial r_1}{\partial x} \\
\frac{\partial r_2}{\partial t} &= \left(-\left(\Gamma + \frac{1}{2}\right)r_2 + \left(\Gamma + \frac{3}{2}\right)r_2\right)\frac{\partial r_2}{\partial x}.
\end{aligned}$$

Problem 60. The Korteweg-de Vries equation is given by

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

For a steady-state pulse solution we make the ansatz

$$u(x,t) = -\frac{b}{2}\text{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right)$$

where

$$\text{sech}(y) := \frac{1}{\cosh(y)}.$$

Find the condition on b and c such that this ansatz is a solution of the Korteweg-de Vries equation.

Solution 60. We find $b = c$.

Problem 61. Consider the Korteweg-de Vries equation and its solution given in the previous problem. Show that

$$\int_{-\infty}^{\infty} \sqrt{|u(x,t)|}dx = \pi.$$

Hint. We have

$$\int \text{sech}(s)ds \equiv \int \frac{1}{\cosh(s)}ds = 2\arctan(e^s).$$

Solution 61.

Problem 62. Consider the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Show that

$$u(x,t) = -12\frac{4\cosh(2x-8t)+\cosh(4x-64t)+3}{(3\cosh(x-28t)+\cosh(3x-36t))^2}$$

is a solution of the Korteweg-de Vries equation. This is a so-called two soliton solution.

Solution 62.

Problem 63. Consider the one-dimensional Euler equations

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \frac{1}{\rho}\frac{\partial p}{\partial x} = 0$$
$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + \rho\frac{\partial u}{\partial x} = 0$$
$$\frac{\partial p}{\partial t} + \gamma p\frac{\partial u}{\partial x} + u\frac{\partial p}{\partial x} = 0$$

where $u(x,t)$ is the velocity field, $\rho(x,t)$ is the density field and $p(x,t)$ is the pressure field. Here t is the time, x is the space coordinate and γ is the ratio of specific heats. Find the linearized equation around $\widetilde{u}$, $\widetilde{p}$, $\widetilde{\rho}$.

Solution 63. We set

$$u(x,t) = \widetilde{u} + \epsilon u'(x,t), \quad \rho(x,t) = \widetilde{\rho} + \epsilon \rho'(x,t), \quad p(x,t) = \widetilde{p} + \epsilon p'(x,t).$$

Inserting this ansatz into the one-dimensional Euler equations and neglecting higher order terms of $O(\epsilon^2)$ yields the linearized system

$$\frac{\partial u'}{\partial t} + \widetilde{u}\frac{\partial u'}{\partial x} + \frac{1}{\widetilde{\rho}}\frac{\partial p'}{\partial x} = 0$$
$$\frac{\partial \rho'}{\partial t} + \widetilde{u}\frac{\partial \rho'}{\partial x} + \widetilde{\rho}\frac{\partial u'}{\partial x} = 0$$
$$\frac{\partial p'}{\partial t} + \gamma\widetilde{p}\frac{\partial u'}{\partial x} + \widetilde{u}\frac{\partial p'}{\partial x} = 0.$$

The linearized Euler equations are often used to model sound propagation.

Problem 64. Consider the classical it Heisenberg ferromagnetic equation

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2}$$

where $\mathbf{S} = (S_1, S_2, S_3)^T$, $S_1^2 + S_2^2 + S_3^2 = 1$ and $\times$ denotes the vector product The natural boundary conditions are $\mathbf{S}(x,t) \to (0,0,1)$ as $|x| \to \infty$.
(i) Find partial differential equation under the stereographic projection

$$S_1 = \frac{2u}{Q}, \quad S_2 = \frac{2v}{Q}, \quad S_3 = \frac{-1 + u^2 + v^2}{Q}$$

where $Q = 1 + u^2 + v^2$.
(ii) Perform a Painlevé test.
(iii) The Heisenberg ferromagnetic equation in the form given for u, v is gauge equivalent to the one-dimensional Schrödinger equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 v}{\partial x^2} + 2(u^2 + v^2)v = 0$$
$$\frac{\partial v}{\partial t} - \frac{\partial^2 u}{\partial x^2} - 2(u^2 + v^2)u = 0.$$

Both systems of differential equations arise as consistency conditions of a system of linear partial differential equations

$$\frac{\partial \psi}{\partial x} = U\psi, \qquad \frac{\partial \psi}{\partial t} = V\psi$$

where $\psi = (\psi_1, \psi_2)^T$ and U and V are 2×2 matrices. The consistency condition is given by

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0.$$

Two systems of nonlinear partial differential equations that are integrable if there is an invertible 2×2 matrix g which depends on x and t such that

$$U_1 = gU_2g^{-1} + \frac{\partial g}{\partial x}g^{-1}, \qquad V_1 = gV_2g^{-1} + \frac{\partial g}{\partial t}g^{-1}.$$

Are the resonances of two gauge equivalent systems the same?

Solution 64. (i) We obtain

$$Q\frac{\partial v}{\partial t} + Q\frac{\partial^2 u}{\partial x^2} - 2\left(\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2\right)u - 4v\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0$$

$$Q\frac{\partial u}{\partial t} - Q\frac{\partial^2 v}{\partial x^2} - 2\left(\left(\frac{\partial u}{\partial x}\right)^2 - \left(\frac{\partial v}{\partial x}\right)^2\right)v + 4u\frac{\partial u}{\partial x}\frac{\partial v}{\partial x} = 0.$$

(ii) Inserting the ansatz

$$u = \phi^n \sum_{j=0}^{\infty} u_j \phi^j, \qquad v = \phi^m \sum_{j=0}^{\infty} v_j \phi^j$$

into the system of partial differential equations we obtain $n = m = -1$. The resonances are $r_1 = -1$ (twice) and $r_2 = 0$ (twice). Resonances are those values of j at which it is possible to introduce arbitrary functions into the expansions. For each nontrivial resonance there appears a compatibility condition that must be satisfied if the solution is to have single-valued expansions. At $r_2 = 0$ we find that u_0 and v_0 can be chosen arbitrarily.
(iii) The dominat behaviour is the same, i.e. $n = m = -1$. However, the resonances of the one-dimensional nonlinear Schrödinger equation are $r_1 = -1$, $r_2 = 0$, $r_3 = 3$, $r_4 = 4$.

Problem 65. The system of partial differential equations of the system of *chiral field equations* can be written as

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x} - \mathbf{u} \times (J\mathbf{v}) = \mathbf{0}$$

$$\frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{v}}{\partial x} - \mathbf{v} \times (J\mathbf{u}) = \mathbf{0}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{v} = (v_1, v_2, v_3)$, $\mathbf{u}^2 = \mathbf{v}^2 = 1$, $J = \mathrm{diag}(j_1, j_2, j_3)$ is a 3×3 diagonal matrix and $\times$ denotes the vector product. Consider the linear mapping $M : \mathbb{R}^3 \to so(3)$

$$M(\mathbf{u}) = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}$$

where $so(3)$ is the simple Lie algebra of the 3×3 skew-symmetric matrices. Rewrite the system of partial differential equations using $M(\mathbf{u})$.

Solution 65. First we note that

$$[M(\mathbf{u}), M(\mathbf{v})] = -M(\mathbf{u} \times \mathbf{v}).$$

Thus

$$\frac{\partial M(\mathbf{u})}{\partial t} + \frac{\partial M(\mathbf{u})}{\partial x} + [M(\mathbf{u}), M(J\mathbf{v})] = 0$$
$$\frac{\partial M(\mathbf{v})}{\partial t} - \frac{\partial M(\mathbf{v})}{\partial x} + [M(\mathbf{v}, M(J\mathbf{u})] = 0$$

with $\mathbf{u}^2 = \mathbf{v}^2 = 1$.

Problem 66. The *Landau-Lifshitz equation*

$$\frac{\partial \mathbf{u}}{\partial t} = \mathbf{u} \times \frac{\partial^2 \mathbf{u}}{\partial x^2} + \mathbf{u} \times (K\mathbf{v})$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{u}^2 = 1$, $K = \text{diag}(k_1, k_2, k_3)$ is a 3×3 diagonal matrix and $\times$ denotes the vector product. Consider the linear mapping $M : \mathbb{R}^3 \to so(3)$

$$M(\mathbf{u}) = \begin{pmatrix} 0 & u_3 & -u_2 \\ -u_3 & 0 & u_1 \\ u_2 & -u_1 & 0 \end{pmatrix}$$

where $so(3)$ is the simple Lie algebra of the 3×3 skew-symmetric matrices. Rewrite the system using $M(\mathbf{u})$.

Solution 66. First we note that

$$[M(\mathbf{u}), M(\mathbf{v})] = -M(\mathbf{u} \times \mathbf{v}).$$

Thus we find

$$\frac{\partial M(\mathbf{u})}{\partial t} + [M(\mathbf{u}, M(\mathbf{u})_{xx}] + [M(\mathbf{u}), M(K\mathbf{u})] = 0$$

with $\mathbf{u}^2 = 1$ and $M(\mathbf{u})_{xx} \equiv \partial^2 M/\partial x^2$.

Problem 67. Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}.$$

Let

$$u(x,t) = -\frac{\phi_t}{\phi_x}.$$

Find the partial differential equation for ϕ.

Solution 67. We obtain

$$\phi_t^2 \frac{\partial^2 \phi}{\partial x^2} - \phi_x^2 \frac{\partial^2 \phi}{\partial t^2} = 0.$$

Problem 68. (i) Show that the *Burgers equation*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}$$

admits the Lax reprsentation

$$\left(\frac{\partial}{\partial x} + \frac{u}{2}\right)\psi = \lambda\psi, \qquad \frac{\partial \psi}{\partial t} = \frac{\partial^2 \psi}{\partial x^2} + u\frac{\partial \psi}{\partial x}$$

where u is a smooth function of x and t.
(ii) Show that

$$[T, K]\psi = 0$$

also provides the Burgers equation, where

$$T := \frac{\partial^2}{\partial x^2} + u\frac{\partial}{\partial x} - \frac{\partial}{\partial t}, \qquad K := \frac{\partial}{\partial x} - \lambda + \frac{u}{2}.$$

Solution 68.

Problem 69. Consider the Burgers equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u\frac{\partial u}{\partial x}.$$

Consider the operator (so-called *recursion operator*)

$$R := D + \frac{1}{2}\frac{\partial u}{\partial x}D^{-1} + \frac{u}{2}$$

where

$$D := \frac{\partial}{\partial x}, \qquad D^{-1} f(x) := \int^x f(s) ds.$$

(i) Show that applying the recursion operator R to the right-hand side of the Burgers equation results in the partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + \frac{3}{2} u \frac{\partial^2 u}{\partial x^2} + \frac{3}{4} u^2 \frac{\partial^2 u}{\partial x}.$$

(ii) Show that this partial differential equation can also be derived from the linear partial differential equation

$$\frac{\partial \phi}{\partial t} = \frac{\partial^3 \phi}{\partial x^3}$$

and the transformation

$$\phi(x,t) = \exp\left(\frac{1}{2} \int^x u(s,t) ds \right).$$

Note that

$$D^{-1} \left(\frac{\partial u}{\partial x} \right) = u.$$

Solution 69.

Problem 70. Find the solution of the system of partial differential equations

$$\frac{\partial f}{\partial x} + f^2 = 0, \quad \frac{\partial f}{\partial t} + fg = 0$$
$$\frac{\partial g}{\partial x} + fg = 0, \quad \frac{\partial g}{\partial t} + g^2 = 0.$$

Solution 70. We obtain

$$f(x,t) = \frac{c_1}{t + c_1 x + c_2}, \qquad g(x,t) = \frac{1}{t + c_1 x + c_2}.$$

Problem 71. Consider the Kortweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Show that

$$u(x,t) = -2 \frac{\partial^2}{\partial x^2} \log \Delta(x,t)$$

is a solution of the Kortweg-de Vries equation, where

$$\Delta(x,t) = \det\left(\delta_{jk} + \frac{c_j c_k}{\eta_j + \eta_k}\exp(-(\eta_j^3 + \eta_k^3)t - (\eta_j + \eta_k)x)\right)_{j,k=1,\ldots,N}.$$

This is the so-called N-soliton solution.

Solution 71.

$$=$$

$$=$$

Problem 72. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + \left(\frac{\partial u}{\partial x}\right)^2 = 0.$$

Let $v(x,t) = \exp(u(x,t))$. Find the partial differential equation for v.

Solution 72. We obtain

$$\frac{\partial v}{\partial t} + \frac{\partial^2 v}{\partial x^2} = 0.$$

Problem 73. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial \psi}{\partial t} + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2} + |\psi|^2\psi = 0.$$

Show that

$$\psi(x,t) = 2\nu \operatorname{sech}(s)\exp(i\phi(s,t))$$

with

$$s := 2\nu(x - \zeta(t)), \quad \phi(s,t) := \frac{\mu}{\nu}s + \delta(t)$$

$$\zeta(t) = 2\mu t + \zeta_0, \quad \delta(t) = 4(\mu^2 + \nu^2)t + \delta_0.$$

is a solution (so-called one-soliton solution) of the nonlinear Schrördinger equation.

Solution 73.

$$=$$

Problem 74. Consider the system of nonlinear partial differential equations

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} - u + u(u^2 + v^2) = 0, \quad \frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - \left(1 - \frac{\kappa}{2}\right)v + v(u^2 + v^2) = 0.$$

(i) Show that

$$u(x,t) = \pm\tanh(s/\sqrt{2}), \quad v(x,t) = 0$$

is a solution, where $s := \gamma(x - ct)$.
(ii) Show that

$$u(x,t) = \pm\tanh(\sqrt{\kappa/2}s), \qquad v(x,t) = (1-\kappa)^{1/2}\text{sech}(\sqrt{\kappa/2}s)$$

is a solution, where $s := \gamma(x - ct)$.
(iii) Show that

$$u(x,t) = \pm\tanh(\sqrt{\kappa/2}s), \qquad v(x,t) = -(1-\kappa)^{1/2}\text{sech}(\sqrt{\kappa/2}s)$$

is a solution, where $s := \gamma(x - ct)$.

Solution 74.

$$=$$

Problem 75. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2|u|^2 u = 0, \qquad -\infty < x < \infty$$

Show that

$$u(x,t) = 2\eta\exp(i\phi(x,t))\text{sech}(\psi(x,t))$$

where

$$\phi(x,t) = -2(\xi x + 2(\xi^2 - \eta^2)t) + \phi_0, \quad \psi(x,t) = 2\eta(x + 4\xi t) + \psi_0$$

is a solution of the one-dimensional nonlinear Schrödinger equation.

Solution 75.

Problem 76. The Landau-Lifshitz equation describing nonlinear spin waves in a ferromagnet is given by

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2} + \mathbf{S} \times J\mathbf{S}$$

where

$$\mathbf{S} = (S_1, S_2, S_3)^T, \qquad S_1^2 + S_2^2 + S_3^2 = 1$$

and $J = \text{diag}(J_1, J_2, J_3)$ is a constant 3×3 diagonal matrix. Show that

$$\frac{\partial \mathbf{w}}{\partial x_1} = L\mathbf{w}, \qquad \frac{\partial \mathbf{w}}{\partial x_2} = M\mathbf{w}$$

with $x_1 = x$, $x_2 = -it$ and

$$L = \sum_{\alpha=1}^{3} z_\alpha S_\alpha \sigma_\alpha$$

$$M = i \sum_{\alpha,\beta,\gamma=1}^{3} z_\alpha \sigma_\alpha S_\beta \frac{\partial S_\gamma}{\partial x} \epsilon^{\alpha\beta\gamma} + 2z_1 z_2 z_3 \sum_{\alpha=1}^{3} z_\alpha^{-1} S_\alpha \sigma_\alpha$$

provide a Lax pair. Here σ_1, σ_2, σ_3 are the Pauli spin matrices and the spectral parameters (z_1, z_2, z_3) constitue an algebraic coordinate of an elliptic curve defined by

$$z_\alpha^2 - z_\beta^2 = \frac{1}{4}(J_\alpha - J_\beta), \qquad \alpha, \beta = 1, 2, 3.$$

Solution 76.

Problem 77. Consider the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u \frac{\partial u}{\partial x}.$$

Consider the generalized Hopf-Cole transformation

$$v(t,x) = w(t)u(t,x) \exp\left(\int_0^s dsu(s,t)\right)$$

with

$$u(t,x) = \frac{v(t,x)}{w(t) + \int_0^s dsv(s,t)}.$$

Find the differential equations for v and w.

Solution 77. We obtain

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \qquad \frac{dw}{dt} = \frac{\partial v(0,t)}{\partial x}.$$

Problem 78. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} + 2c|u|^2 = 0$$

where $x \in \mathbb{R}$ and $c = \pm 1$. Show that it admits the solution

$$u(x,t) = \frac{A}{\sqrt{t}} \exp\left(it\left(\frac{1}{4}\left(\frac{x}{t}\right)^2 + 2cA^2\frac{\ln(t)}{t} + \frac{\phi}{t}\right)\right).$$

Solution 78.

Problem 79. Let α be a positive constant. The *Kadomtsev Petviashvilli equation* is given by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}+6u\frac{\partial u}{\partial x}+\frac{\partial^3 u}{\partial x^3}\right)+\alpha\frac{\partial^2 u}{\partial y^2}=0.$$

Consider the one-soliton solution

$$u(x,y,t)=2k_1^2\mathrm{sech}^2(k_1x+k_2y-\omega t)$$

where $\mathrm{sech}(x)\equiv 1/\cosh(x)\equiv 2/(e^x+e^{-x})$. Find the *dispersion relation* $\omega(k_1,k_2)$.

Solution 79. Inserting the one-soliton solution into the Kadomtsev Petviashvilli equation yields

$$=$$

Thus the dispersion relation is (check)

$$\omega(k_1,k_2)=4k_1^2+\alpha\frac{k_2^2}{k_1}.$$

Problem 80. Find the partial differential equation given by the condition

$$\det\begin{pmatrix} u & \partial u/\partial z_1 \\ \partial u/\partial z_2 & \partial^2 u/\partial z_1\partial z_2\end{pmatrix}=0.$$

Find a solution of the partial differential equation.

Solution 80. We find

$$=0$$

This is the Monge-Ampére equation. A solution is

$$=$$

Problem 81. A one-dimensional Schrödinger equation with cubic nonlinearity is given by

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2}-g\rho\psi,\qquad \rho:=\psi^*\psi$$

where $g>0$.

(i) Show that the partial differential equation admits the (soliton) solution

$$\psi_s(x,t) = \pm \exp\left(i\frac{mv}{\hbar}(x-ut)\right)\frac{\hbar}{\sqrt{gm}}\frac{\gamma}{\cosh(\gamma(x-vt))}, \qquad \gamma^2 = \frac{m^2v^2}{\hbar^2}\left(1-\frac{2u}{v}\right).$$

The soliton moves with group velocity v. The phase velocity u must be $u < v/2$.
(ii) Show that the partial differential equation is Galileo-invariant. This means that any solution of partial differential equation can be mapped into another solution via the Galileo boost

$$x \to x - Vt, \qquad \psi(x,t) \to \exp\left(\frac{i}{\hbar}mV\left(x - \frac{1}{2}Vt\right)\right)\psi(t, x-Vt).$$

Show that the soliton can be brought to rest.

Solution 81. (ii) Applying the Galileo boost with $V = -v$ provides

$$\psi_s(t,x) \to \pm \exp\left(i\frac{\hbar\gamma^2}{2m}t\right)\frac{\hbar}{\sqrt{gm}}\frac{\gamma}{\cosh(\gamma x)}.$$

It solves the partial differential equation and is the soliton at rest.

Problem 82. The KP-equations are given by

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3}\right) = -3\alpha^2\frac{\partial^2 u}{\partial y^2}$$

with $\alpha = i$ and $\alpha = -1$. Show that this equation is an integrability condition on

$$L\psi \equiv \alpha\frac{\partial \psi}{\partial y} + \frac{\partial^2 \psi}{\partial x^2} + u\psi = 0$$

$$M\psi \equiv \frac{\partial \psi}{\partial t} + 4\frac{\partial^3 \psi}{\partial x^3} + 6u\frac{\partial \psi}{\partial x} + 3\left(\frac{\partial u}{\partial x} - \alpha\int_{-\infty}^{x}\frac{\partial u(x',y)}{\partial y}dx'\right) + \beta\psi = 0.$$

Solution 82.

$$=$$

Problem 83. Show that the nonlinear partial differential equation

$$\frac{\partial u}{\partial t} = (1+u^2)\frac{\partial^2 u}{\partial x^2} - u\left(\frac{\partial u}{\partial x}\right)^2$$

is transformated under the transformation $u \mapsto u/\sqrt{1+u^2}$ into the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial x^2}(\tanh^{-1} u).$$

Solution 83.

Problem 84. Consider the metric tensor field

$$g = g_{11}(x_1,x_2)dx_1 \otimes dx_1 + g_{12}(x_1,x_2)dx_1 \otimes dx_2 + g_{21}(x_1,x_2)dx_2 \otimes dx_1 + g_{22}(x_1,x_2)dx_2 \otimes dx_2$$

with $g_{12} = g_{21}$ and the g_{jk} are smooth functions of x_1, x_2. Let $\det(g) \equiv g_{11}g_{22} - g_{12}g_{21} \neq 0$ and

$$R = \frac{2}{(\det(g))^2} \det \begin{pmatrix} g_{11} & g_{22} & g_{12} \\ \partial g_{11}/\partial x_1 & \partial g_{22}/\partial x_1 & \partial g_{12}/\partial x_1 \\ \partial g_{11}/\partial x_2 & \partial g_{22}/\partial x_2 & \partial g_{12}/\partial x_2 \end{pmatrix}.$$

Find solutions of the partial differential equation $R = 0$. Find solutions of the partial differential equation $R = 1$.

Solution 84.

Problem 85. Consider the one-dimensional nonlinear Schrödinger equation

$$i\frac{\partial w}{\partial t} + \frac{\partial^2 w}{\partial x^2} + 2|w|^2 w = 0.$$

Consider the ansatz

$$w(x,t) = \exp(i\omega t)u(x).$$

Find the ordinary differential equation for u.

Solution 85. We obtain

$$\frac{d^2u}{dx^2} - \omega u + 2u^3 = 0.$$

Problem 86. Consider the system of partial differential differential equations

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2} + \mathbf{S} \times (J\mathbf{S}), \quad J = \begin{pmatrix} j_1 & 0 & 0 \\ 0 & j_2 & 0 \\ 0 & 0 & j_3 \end{pmatrix}$$

where $\mathbf{S} = (S_1, S_2, S_3)^T$ and $S_1^2 + S_2^2 + S_3^2 = 1$. Express the partial differental equation using $p(x,t)$ and $p(x,t)$ given by

$$S_1 = \sqrt{1-p^2}\cos(q), \quad S_2 = \sqrt{1-p^2}\sin(q), \quad S_3 = p.$$

Solution 86. We obtain

$$\frac{\partial q}{\partial t} = -\frac{1}{1-p^2}\frac{\partial^2 p}{\partial x^2} -$$
$$\frac{\partial p}{\partial t} = (1-p^2)\frac{\partial^2 q}{\partial x^2} - \dots.$$

Problem 87. Consider the two-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} - \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \sin(u).$$

Let

$$u(x_1, x_2, t) = 4\arctan(v(x_1, x_2, t)).$$

Find the partial differential equation for v. Separate this partial differential equation into a linear part and nonlinear part. Solve these partial differential equations to find solutions for the two-dimensional sine-Gordon equation. Note that

$$\sin(4\alpha) \equiv 4\sin(\alpha)\cos(\alpha) - 8\sin^3(\alpha)\cos(\alpha)$$

$$\sin(\arctan(\alpha)) = \frac{\alpha}{\sqrt{1+\alpha^2}}, \quad \cos(\arctan(\alpha)) = \frac{1}{\sqrt{1+\alpha^2}}$$

and therefore

$$\sin(4\arctan(v)) = \frac{4v(1-v^2)}{(1+v^2)^2}.$$

Furthermore

$$\frac{\partial^2}{\partial x_1^2}\arctan(v) = \frac{-2v}{(1+v^2)^2}\left(\frac{\partial v}{\partial x_1}\right)^2 + \frac{1}{1+v^2}\frac{\partial^2 v}{\partial x_1^2}.$$

Solution 87. We obtain

$$(1+v^2)\left(\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} - \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2}\right) - 2v\left(\left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 - \frac{1}{c^2}\left(\frac{\partial v}{\partial t}\right)^2\right) = v(1-v^2).$$

Thus we can separate the partial differental equation as

$$\frac{\partial^2 v}{\partial x_1^2} + \frac{\partial^2 v}{\partial x_2^2} - \frac{1}{c^2}\frac{\partial^2 v}{\partial t^2} = v, \qquad \left(\frac{\partial v}{\partial x_1}\right)^2 + \left(\frac{\partial v}{\partial x_2}\right)^2 - \frac{1}{c^2}\left(\frac{\partial v}{\partial t}\right)^2 = v^2.$$

The first equation is a linear wave equation. The second equation can be separated in two equations

$$\left(\frac{\partial v}{\partial x_1}\right)^2 - \frac{1}{c^2}\left(\frac{\partial v}{\partial t}\right)^2 = 0, \qquad \left(\frac{\partial v}{\partial x_2}\right)^2 = v^2.$$

Consequently

$$\frac{\partial v}{\partial x_1} = s_1 \frac{1}{c}\frac{\partial v}{\partial t}, \qquad \frac{\partial v}{\partial x_2} = s_2 v$$

where $s_1 = \pm 1$ and $s_2 = \pm 1$ and

$$v(x_1, x_2, t) = f(x_1, t) e^{s_2 x_2}.$$

This leads to $\partial f/\partial x_1 = (s_1/c)\partial f/\partial t$ with a smooth function $f(x_1, t) = f(x_1 - s_1 ct)$. As final solution we obtain

$$v(x_1, x_2, t) = 4\arctan(f(x_1 - s_1 ct)e^{s_2 x_2}).$$

Problem 88. (i) Find a non-zero vector field in $\mathbb{R}^3$ such that

$$V \cdot \mathrm{curl} V = 0.$$

(ii) Find a non-zero vector field in $\mathbb{R}^3$ such that

$$V \times \mathrm{curl}(V) = \mathbf{0}.$$

Solution 88.

$$=$$

Problem 89. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Consider the partial differential equation

$$\frac{\partial u}{\partial t} + \frac{\partial f(u(x))}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Find solutions of the form $u(x, t) = \phi(x - ct)$ (traveling wave solutions) where ϕ is a smooth function. Integrate the obtained ordinary differential equation.

Solution 89. We set $s = x - ct$. Then we obtain the ordinary differential equation

$$-c\frac{d\phi}{ds} + \frac{d}{ds}(f(\phi(s)) + \frac{d^3\phi}{ds^3} = 0.$$

Integrating once we arrive at

$$-c\phi + f(\phi(s)) + \frac{d^2\phi}{ds^2} = C_1$$

where C_1 is a constant of integration. One more integration yields

$$\frac{1}{2}\left(\frac{d\phi}{ds}\right)^2 = C_2 + C_1\phi + \frac{c}{2}\phi^2 - F(\phi(s)), \quad F(\phi) = \int_0^\phi f(y)dy$$

where C_2 is another constant of integration.

Problem 90. Consider the partial differential equation (Thomas equation)

$$\frac{\partial^2 u}{\partial x \partial t} + a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial t} = 0$$

where a, b are constants. Show that the equation can be linearized with the transformation

$$u(x,t) = -bx - at + \ln(v(x,t).$$

Solution 90. We obtain the linear hyperbolic equation

$$\frac{\partial^2 v}{\partial x \partial t} = abv.$$

Problem 91. The Korteweg-de Vries equation is given by

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

Setting $u = \partial v/\partial x$ we obtain the equation

$$\frac{\partial^2 v}{\partial x \partial t} - 6\frac{\partial v}{\partial x}\frac{\partial^2 v}{\partial x^2} + \frac{\partial^4 v}{\partial x^4} = 0.$$

(i) Show that this equation can be dervied from the Lagrangian density

$$\mathcal{L} = -\frac{1}{2}v_x v_t + (v_x)^3 + \frac{1}{2}(v_{xx})^2$$

where the Lagrange equation is given by

$$\frac{\partial}{\partial t}\left(\frac{\partial \mathcal{L}}{\partial v_t}\right) + \frac{\partial}{\partial x}\left(\frac{\partial \mathcal{L}}{\partial v_x}\right) - \frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathcal{L}}{\partial v_{xx}}\right) - \frac{\partial \mathcal{L}}{\partial v} = 0.$$

(ii) Show the Hamiltonian density is given by

$$\mathcal{H} = v_t\frac{\partial \mathcal{L}}{\partial v_t} - \mathcal{L} = -(v_x)^3 - \frac{1}{2}(v_{xx})^2.$$

Solution 91.

Problem 92. Show that the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

admits the solution

$$u(x,t) = -\frac{c}{2}\mathrm{sech}^2\left(\frac{\sqrt{c}}{2}(x-ct)\right).$$

Show that

$$\int_{-\infty}^{\infty} dx\sqrt{|u(x,t)|} = \pi.$$

Solution 92.

Problem 93. Consider the one-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - c^2\frac{\partial^2 u}{\partial x^2} + \omega^2\sin(u) = 0.$$

Show that

$$u_+(x,t) = 4\arctan(\exp(d^{-1}(x-vt-X)\cosh(\alpha))$$
$$u_-(x,t) = 4\arctan(\exp(-d^{-1}(x-vt-X)\cosh(\alpha))$$

are solutions of the one-dimensional sine-Gordon equation, where $v = c\tanh(\alpha)$, $d = c/\omega$. Discuss.

Solution 93.

$$=$$

Problem 94. Let $\kappa(s,t)$ be the curvature and $\tau(s,t)$ be the torsion with s and t being the arclength and time, respectively. Consider the complex valued function

$$w(s,t) = \kappa(s,t)\exp(i\int_0^s ds'\tau(s',t)).$$

Show that if the motion is described by

$$\frac{\partial}{\partial t}\mathbf{r} - \frac{\partial}{\partial s}\mathbf{r}\times\frac{\partial^2}{\partial s^2}\mathbf{r} = \kappa\mathbf{b}$$

where $\mathbf{b}$ is the binormal unit vector, then $w(s,t)$ satisfies the nonlinear Schrödinger equation

$$i\frac{\partial}{\partial t}w + \frac{\partial^2}{\partial s^2}w + \frac{1}{2}|w|^2 w = 0.$$

Solution 94.

$$=$$

Problem 95. Consider the first order system of partial differential equation

$$\frac{\partial \mathbf{u}}{\partial t} + A\frac{\partial \mathbf{u}}{\partial x} = \mathbf{0}$$

where $\mathbf{u} = (u_1, u_2, u_3)^T$ and the 3×3 matrix is given by

$$\begin{pmatrix} \Gamma u_1 & u_2 & u_3 \\ u_2 & u_1 - \Delta & 0 \\ u_3 & 0 & u_1 - \Delta \end{pmatrix}$$

where Γ and Δ are constants. Along a characteristic curve $C : x = x(s), t = t(s)$ for the system of partial differential equation one has

$$\det(A - \lambda I_3) = 0, \qquad \lambda = \frac{x'(s)}{t'(s)}.$$

Find the Riemann invariants.

Solution 95. The Riemann invariants of the systems of partial differential equations are functions $R(u_1, u_2, u_3)$ that are constant along the characteristic. Consequently $\nabla_{\mathbf{u}} R = (R_{u_1}, R_{u_2}, R_{u_3})$ is a left eigenvector of the matrix

$$\nabla_{\mathbf{u}} R \cdot (A - \lambda I_3) = 0.$$

From $\det(A - \lambda I_3) = 0$ the eigenvalue equation is

$$-(\lambda - (u_1 - \Delta))(\lambda^2 + (\Delta - (\Gamma + 1)u_1)\lambda + \Gamma u_1(u_1 - \Delta) - r^2) = 0$$

where $r^2 := u_2^2 + u_3^2$. One finds the three eigenvalues

$$\lambda = u_1 - \Delta, \quad \lambda_+ = \frac{1}{2}((\Gamma+1)u_1 - \Delta + D^{1/2}), \quad \lambda_- = \frac{1}{2}((\Gamma+1)u_1 - \Delta - D^{1/2})$$

where $D = ((\Gamma - 1)u_1 + \Delta)^2 + 4r^2$. The system of partial differential equations $\nabla_{\mathbf{u}} R \cdot (A - \lambda I_3) = 0$ for the Riemann invariants is given by

$$\begin{aligned}(\Gamma u_1 - \lambda)\frac{\partial R}{\partial u_1} + u_2\frac{\partial R}{\partial u_2} + u_3\frac{\partial R}{\partial u_3} &= 0 \\ u_2\frac{\partial R}{\partial u_1} + (u_1 - \Delta - \lambda)\frac{\partial R}{\partial u_2} &= 0 \\ u_3\frac{\partial R}{\partial u_1} + (u_1 - \Delta - \lambda)\frac{\partial R}{\partial u_3} &= 0.\end{aligned}$$

With polar coordinates $u_2 = r\cos\theta$, $u_3 = r\sin\theta$ we obtain

$$\begin{aligned}(\Gamma u_1 - \lambda)\frac{\partial R}{\partial u_1} + r\frac{\partial R}{\partial r} &= 0 \\ (u_1 - \Delta - \lambda)\frac{\partial R}{\partial \theta} &= 0.\end{aligned}$$

For $\lambda = u_1 - \Delta$ a solution is $R = u_3/u_2$.

Problem 96. Consider the system of partial differential equations

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} = Z^2 - VW$$
$$\frac{\partial W}{\partial t} - \frac{\partial W}{\partial x} = Z^2 - VW$$
$$\frac{\partial Z}{\partial t} = -\frac{1}{2}Z^2 + \frac{1}{2}VW$$

Let $N := V + W + 4Z$ and $J := V - W$. Find $\partial N/\partial t + \partial J/\partial x$ and $\partial J/\partial t + \partial V/\partial x + \partial W/\partial x$.

Solution 96. Straightforward calculation provides

$$\frac{\partial N}{\partial t} + \frac{\partial J}{\partial x} = 0, \qquad \frac{\partial J}{\partial t} + \frac{\partial V}{\partial x} + \frac{\partial W}{\partial x} = 0.$$

These are conservation laws.

Problem 97. Consider the Navier-Stokes equations

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v}.$$

(i) Show that in the limit $\nu \to 0$ the Navier-Stokes equation are invariant under the scaling transformation ($\lambda > 0$)

$$r \to \lambda r, \quad \mathbf{v} \to \lambda^h \mathbf{v}, \quad t \to \lambda^{1-h} t.$$

(ii) Show that for finite ν one finds invariance of the Navier-Stokes equation if $\nu \to \lambda^{1+h}\nu$.

Solution 97.

Problem 98. (i) Show that

$$u(x,t) = \frac{k^2/2}{(\cosh((kx - \omega t)/2))^2}, \quad k^3 = \omega$$

is a solution (solitary wave solution) of the Korteweg de Vries equation

$$\frac{\partial u}{\partial t} - 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0.$$

(ii) Let

$$u(x,t) = 2\frac{\partial^2}{\partial x^2}\ln(F(x,t)).$$

Show that $F(x,t) = 1 + e^{kx-\omega t}$.

Solution 98.

Problem 99. Let $L > 0$. Consider the one-dimensional sine-Gordon equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} + \sin(u) = 0$$

with periodic boundary conditions $u(x+L,t) = u(x,t)$. Assume that

$$u(x,t) = \pi + v(x,t), \quad |v(x,t)| \ll 1$$

and show that up to first order one finds

$$\frac{\partial^2 v}{\partial t^2} - \frac{\partial^2 v}{\partial x^2} - v = 0$$

and for $v(x,t) = \hat{v}(t)e^{ik_n x}$ one obtains

$$\frac{d^2\hat{v}}{dt^2} + (k_n^2 - 1)v_n = 0$$

with $k_n = 2\pi n/L$. Show that all modes with $0 < k_n^2 < 1$ are unstable.

Solution 99.

Problem 100. Study the nonlinear partial differential equation

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}\right) u_j = -4\sum_{k=1}^{2} K_{jk} \exp(u_k), \quad j = 1,2$$

where K is the Cartan matrix

$$K = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

Solution 100.

Problem 101. Two systems of nonlinear differential equations that are integrable by the inverse scattering method are said to be *gauge equivalent* if the corresponding flat connections U_j, V_j, $j = 1,2$, are defined in the same fibre bundle and obtained from each other by a λ-independent gauge transformation, i.e. if

$$U_1 = gU_2g^{-1} + \frac{\partial g}{\partial x}g^{-1}, \qquad V_1 = gV_2g^{-1} + \frac{\partial g}{\partial t}g^{-1} \tag{1}$$

where $g(x,t) \in GL(n,\mathbb{R})$. We have

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} + [U_1, V_1] = 0. \tag{2}$$

Show that

$$\frac{\partial U_2}{\partial t} - \frac{\partial V_2}{\partial x} + [U_2, V_2] = 0. \tag{3}$$

Solution 101. First we notice that

$$\frac{\partial g^{-1}}{\partial x} = -g^{-1}\frac{\partial g}{\partial x}g^{-1}, \qquad \frac{\partial g^{-1}}{\partial t} = -g^{-1}\frac{\partial g}{\partial t}g^{-1} \tag{4}$$

which follows from $gg^{-1} = I$, where I is the $n \times n$ identity matrix. From (1) it follows that

$$\frac{\partial U_1}{\partial t} = \frac{\partial g}{\partial t}U_2g^{-1} + gU_{2t}g^{-1} - gU_2g^{-1}\frac{\partial g}{\partial t}g^{-1} + \frac{\partial^2 g}{\partial x \partial t}g^{-1} - \frac{\partial g}{\partial x}g^{-1}\frac{\partial g}{\partial t}g^{-1} \tag{5}$$

$$\frac{\partial V_1}{\partial x} = \frac{\partial g}{\partial x}V_2g^{-1} + gV_{2t}g^{-1} - gV_2g^{-1}\frac{\partial g}{\partial x}g^{-1} + \frac{\partial^2 g}{\partial x \partial t}g^{-1} - \frac{\partial g}{\partial t}g^{-1}\frac{\partial g}{\partial x}g^{-1} \tag{6}$$

where we have used (3). From (4) and (5) we obtain

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} = g\frac{\partial U_2}{\partial t}g^{-1} - g\frac{\partial V_2}{g}^{-1} = g\left(\frac{\partial U_2}{\partial t} - \frac{\partial V_1}{\partial x}\right)g^{-1} \tag{8}$$

The commutator yields

$$\begin{aligned}
[U_1, V_1] &= [gU_2g^{-1} + \frac{\partial g}{\partial x}g^{-1}, gV_2g^{-1} + \frac{\partial g}{\partial t}g^{-1}] \\
&= [gU_2g^{-1}, gV_2g^{-1}] + [gU_2g^{-1}, g_tg^{-1}] + [,] + [,] \\
&= gU_2V_2g^{-1} - gV_2U_2g^{-1} = g([U_2, V_2])g^{-1}
\end{aligned}$$

Adding (6) and (7) results in

$$g\left(\frac{\partial U_2}{\partial t} - \frac{\partial V_2}{\partial x} + [U_2, V_2]\right)g^{-1} = 0\,.$$

Thus (3) follows.

Both equations are integrable by the inverse scattering method. Both arise as consistency condition of a system of linear differential equations

$$\frac{\partial \phi}{\partial t} = U(x,t,\lambda)\phi, \qquad \frac{\partial \phi}{\partial x} = V(x,t,\lambda)\phi$$

where λ is a complex parameter. The consistency conditions have the form

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0.$$

Let

$$U_1 := gU_2g^{-1} + \frac{\partial g}{\partial x}g^{-1}, \qquad V_1 := gV_2g^{-1} + \frac{\partial g}{\partial t}g^{-1}$$

and

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} + [U_1, V_1] = 0.$$

Problem 102. Consider the *nonlinear Schrödinger equation* in one space dimension

$$i\frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2\psi = 0 \qquad (1)$$

and the *Heisenberg ferromagnet equation* in one space dimension

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2}, \qquad \mathbf{S}^2 = 1 \qquad (2)$$

where $\mathbf{S} = (S_1, S_2, S_3)^T$. Both equations are integrable by the inverse scattering method. Both arise as the consistency condition of a system of linear differential equations

$$\frac{\partial \Phi}{\partial t} = U(x, t, \lambda)\Phi, \qquad \frac{\partial \phi}{\partial x} = V(x, t, \lambda)\Phi \qquad (3)$$

where λ is a complex parameter. The consistency conditions have the form

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0 \qquad (4)$$

(i) Show that $\phi_1 = g\phi_2$.
(ii) Show that (1) and (2) are gauge equivalent.

Solution 102.

Problem 103. The study of certain questions in the theory of $SU(2)$ gauge fields reduced to the construction of exact solutions of the following nonlinear system of partial differential equations

$$u\left(\frac{\partial^2 u}{\partial y \partial \bar{y}} + \frac{\partial^2 u}{\partial z \partial \bar{z}}\right) - \frac{\partial u}{\partial y}\frac{\partial u}{\partial \bar{y}} - \frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial v}{\partial z}\frac{\partial \bar{v}}{\partial \bar{z}} = 0.$$

$$u\left(\frac{\partial^2 v}{\partial y \partial \bar{y}} + \frac{\partial^2 v}{\partial z \partial \bar{z}}\right) - 2\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial z}\frac{\partial u}{\partial \bar{z}}\right) = 0$$

$$u\left(\frac{\partial^2\bar{v}}{\partial\bar{y}\partial y}+\frac{\partial^2\bar{v}}{\partial\bar{z}\partial z}\right)-2\left(\frac{\partial\bar{v}}{\partial\bar{y}}\frac{\partial u}{\partial y}+\frac{\partial\bar{v}}{\partial\bar{z}}\frac{\partial u}{\partial z}\right)=0, \tag{1}$$

where u is a real function and v and $\bar{v}$ are complex unknown functions of the real variables $x_1,\ldots,x_4$. The quantities y and z are complex variables expressed in terms of $x_1,\ldots,x_4$ by the formulas

$$\sqrt{2}y := x_1+ix_2, \qquad \sqrt{2}z := x_3-ix_4 \tag{2}$$

and the bar over letters indicates the operation of complex conjugations.
(i) Show that a class of exact solutions of the system (1) can be constructed, namely solutions for the linear system

$$\frac{\partial v}{\partial y}-\frac{\partial u}{\partial\bar{z}}=0, \qquad \frac{\partial v}{\partial z}-\frac{\partial u}{\partial\bar{y}}=0 \tag{3}$$

where we assume that u, v, and $\bar{v}$ are functions of the variables

$$r := (2y\bar{y})^{1/2} = (x_1^2+x_2^2)^{1/2} \tag{4}$$

and x_3, i.e., for the stationary, axially symmetric case. (ii) Show that a class of exact solutions of (1) can be given, where

$$u=u(w), \qquad v=v(w), \qquad \bar{v}=\bar{v}(w) \tag{5}$$

where w is a solution of the Laplace equation in complex notation

$$\frac{\partial^2 u}{\partial y\partial\bar{y}}+\frac{\partial^2 u}{\partial z\partial\bar{z}}=0, \tag{6}$$

and u, v and $\bar{v}$ satisfy

$$u\frac{d^2u}{dw^2}-\left(\frac{du}{dw}\right)^2+\frac{dv}{dw}\frac{d\bar{v}}{dw}=0, \qquad u\frac{d^2v}{dw^2}-2\frac{dv}{dw}\frac{du}{dw}=0. \tag{7}$$

Hint. Let $z=x+iy$, where $x,y\in\mathbb{R}$. Then

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x}-i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial\bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x}+i\frac{\partial}{\partial y}\right) \tag{8}$$

Solution 103.

Problem 104. The spherically symmetric SU(2) Yang-Mills equations can be written as

$$\frac{\partial\varphi_1}{\partial t}-\frac{\partial\varphi_2}{\partial r}=-A_0\varphi_2-A_1\varphi_1 \tag{1a}$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{\partial \varphi_1}{\partial r} = -A_1 \varphi_2 + A_0 \varphi_1 \tag{1b}$$

$$r^2 \left(\frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial r} \right) = 1 - (\varphi_1^2 + \varphi_2^2) \tag{1c}$$

where r is the spatial radius-vector and t is the time. To find partial solutions of these equations, two methods can be used. The first method is the inverse scattering theory technique, where the $[L, A]$-pair is found, and the second method is based on Bäcklund transformations.
(ii) Show that system (1) can be reduced to the classical Liouville equation, and its general solution can be obtained for any gauge condition.

Solution 104. Introducing the complex function

$$z(r,t) := \varphi_1(r,t) + i\varphi_2(r,t) \tag{2}$$

we have from the first two equations of (1)

$$\frac{\partial z}{\partial t} + i\frac{\partial z}{\partial r} = (A_0 i - A_1)z. \tag{3}$$

Now we use the new variables

$$z(r,t) := R(r,t)\exp(i\theta(r,t)). \tag{4}$$

Separating the imaginary and real parts we obtain from (3)

$$A_0 = \frac{\partial \theta}{\partial t} + \frac{\partial \ln R}{\partial r}, \qquad A_1 = \frac{\partial \theta}{\partial r} - \frac{\partial \ln R}{\partial t}. \tag{5}$$

Substitution of (5) into (1c) yields

$$r^2 L(\ln R) = R^2 - 1, \qquad L := \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial t^2}. \tag{6}$$

Changing the variables

$$R(r,t) = r\exp(g(r,t)) \tag{7}$$

we come to the classical *Liouville equation*

$$Lg = \exp(2g) \tag{8}$$

which has the general solution in terms of two harmonic functions $a(r,t)$ and $b(r,t)$ related by the *Cauchy-Riemann conditions*

$$\exp(2g) = 4\left(\left(\frac{\partial a}{\partial r}\right)^2 + \left(\frac{\partial a}{\partial t}\right)^2 \right)(1 - a^2 - b^2)^{-2}, \tag{9a}$$

$$La = Lb = 0, \qquad \frac{\partial a}{\partial r} = \frac{\partial b}{\partial t}, \qquad \frac{\partial a}{\partial t} = -\frac{\partial b}{\partial r}. \tag{9b}$$

Equation (8) was obtained with an arbitrary function $\theta(r,t)$. Thus (5) and (9) give the general solutions of problem (1) with an arbitrary function $\theta(r,t)$ and harmonic functions $a(r,t)$ and $b(r,t)$, with

$$R = \pm 2r \left(\left(\frac{\partial a}{\partial r} \right)^2 + \left(\frac{\partial a}{\partial t} \right)^2 \right)^{1/2} (a^2 + b^2 - 1)^{-1}. \tag{10}$$

Problem 105. We consider the *Georgi-Glashow model* with gauge group $SU(2)$ broken down to $U(1)$ by *Higgs triplets*. The Lagrangian of the model is

$$\mathcal{L} := -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a} + \frac{1}{2} D_\mu \phi^a D^\mu \phi^a - V(\phi) \tag{1}$$

where

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc} A^b_\mu A^c_\nu \tag{2}$$

$$D_\mu \phi_a := \partial_\mu \phi_a + g\epsilon_{abc} A^b_\mu \phi_c \tag{3}$$

and

$$V(\phi) := -\frac{\lambda}{4} \left(\phi^2 - \frac{m^2}{\lambda} \right)^2. \tag{4}$$

(i) Show that the equations of motion are

$$D_\nu F^{\mu\nu a} = -g\epsilon_{abc}(D^\mu \phi_b)\phi_c, \qquad D_\mu D^\mu \phi_a = (m^2 - \lambda\phi^2)\phi_a. \tag{5}$$

(ii) Show that the vacuum expectation value of the scalar field and Higgs boson mass are

$$\langle \phi^2 \rangle = F^2 = \frac{m^2}{\lambda} \tag{6}$$

and

$$M_H = \sqrt{2\lambda} F,$$

respectively. Mass of the gauge boson is $M_w = gF$.
(iii) Using the time-dependent t' Hooft-Polyakov ansatz

$$A^a_0(r,t) = 0, \qquad A^a_i(r,t) = -\epsilon_{ain} r_n \frac{1 - K(r,t)}{r^2}, \qquad \phi_a(r,t) = \frac{1}{g} r_a \frac{H(r,t)}{r^2} \tag{7}$$

where $r_n = x_n$ and r is the radial variable. Show that the equations of motion (5) can be written as

$$r^2 \left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2} \right) K = (K^2 + H^2 - 1) \tag{8a}$$

$$r^2\left(\frac{\partial^2}{\partial r^2}-\frac{\partial^2}{\partial t^2}\right)H = H\left(2K^2 - m^2r^2 + \frac{\lambda H^2}{g^2}\right). \qquad (8b)$$

(iv) Show that with

$$\beta := \frac{\lambda}{g^2} = \frac{M_H^2}{2M_w^2}$$

and introducing the variables $\xi := M_w r$ and $\tau := M_w t$, system (8) becomes

$$\left(\frac{\partial^2}{\partial \xi^2}-\frac{\partial^2}{\partial \tau^2}\right)K = \frac{K(K^2+H^2-1)}{\xi^2} \qquad (10a)$$

$$\left(\frac{\partial^2}{\partial \xi^2}-\frac{\partial^2}{\partial \tau^2}\right)H = \frac{H(2K^2+\beta(H^2-\xi^2))}{\xi^2}. \qquad (10b)$$

(v) The total energy of the system E is given by

$$C(\beta) = \frac{g^2E}{4\pi M_w} =$$

$$\int_0^\infty \left(K_\tau^2 + \frac{H_\tau^2}{2} + K_\xi^2 + \frac{1}{2}\left(\frac{\partial H}{\partial \xi}-\frac{H}{\xi}\right)^2 + \frac{1}{2\xi^2}(K^2-1)^2 + \frac{K^2H^2}{\xi^2} + \frac{\beta}{4\xi^2}(H^2-\xi^2)^2\right)d\xi. \qquad (10)$$

As time-independent version of the ansatz (3) gives the 't Hooft-Polyakov monopole solution with winding number 1. Show that for finiteness of energy the field variables should satisfy the following conditions

$$H \to 0, \qquad K \to 1 \quad \text{as} \quad \xi \to 0 \qquad (11)$$

and

$$H \to \xi, \qquad K \to 0 \quad \text{as} \quad \xi \to \infty. \qquad (12)$$

The 't Hooft-Polyakov monopole is more realistic than the Wu-Yang monopole; it is non-singular and has finite energy.
(vi) Show that in the limit $\beta \to 0$, known as the Prasad-Somerfeld limit, we have the static solutions,

$$K(\xi) = \frac{\xi}{\sinh\xi}, \qquad H(\xi) = \xi\coth\xi - 1. \qquad (13)$$

Solution 105.

Problem 106. Consider the linear operators L and M defined by

$$L\psi(x,t,\lambda) := \left(i\frac{\partial}{\partial x} + U(x,t,\lambda)\right)\psi(x,t,\lambda)$$

$$M\psi(x,t,\lambda) := \left(i\frac{\partial}{\partial t} + V(x,t,\lambda)\right)\psi(x,t,\lambda).$$

Find the condition on L and M such that $[L,M]=0$, where $[\,,\,]$ denotes the commutator. The potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are typicaly chosen as elements of some semisimple Lie algebra.

Solution 106. We obtain

$$i\frac{\partial}{\partial x}V - i\frac{\partial}{\partial t}U + [U,V] = 0.$$

Problem 107. Let $c>0$. Give solutions to the nonlinear partial differemtial equations

$$\left(\frac{\partial u}{\partial x_1}\right)^2 + \left(\frac{\partial u}{\partial x_2}\right)^2 = c.$$

Solution 107. A solution is

$$u(x_1,x_2) = a_1x_1 + a_2x_2 + a_3$$

with $a_1^2 + a_2^2 = c$.

Problem 108. Two systems of nonlinear differential equations that are integrable by the inverse scattering method are said to be *gauge equivalent* if the corresponding flat connections U_j, V_j, $j=1,2$, are defined in the same fibre bundle and obtained from each other by a λ-independent gauge transformation, i.e. if

$$U_1 = gU_2g^{-1} + \frac{\partial g}{\partial x}g^{-1}, \qquad V_1 = gV_2g^{-1} + \frac{\partial g}{\partial t}g^{-1} \tag{1}$$

where $g(x,t) \in GL(n,\mathbb{R})$. We have

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} + [U_1,V_1] = 0. \tag{2}$$

Show that

$$\frac{\partial U_2}{\partial t} - \frac{\partial V_2}{\partial x} + [U_2,V_2] = 0. \tag{3}$$

Solution 108. First we notice that

$$\frac{\partial g^{-1}}{\partial x} = -g^{-1}\frac{\partial g}{\partial x}g^{-1}, \qquad \frac{\partial g^{-1}}{\partial t} = -g^{-1}\frac{\partial g}{\partial t}g^{-1} \tag{4}$$

which follows from $gg^{-1} = I$, where I is the $n \times n$ identity matrix. From (1) it follows that

$$\frac{\partial U_1}{\partial t} = \frac{\partial g}{\partial t} U_2 g^{-1} + g U_{2t} g^{-1} - g U_2 g^{-1} \frac{\partial g}{\partial t} g^{-1} + \frac{\partial^2 g}{\partial x \partial t} g^{-1} - \frac{\partial g}{\partial x} g^{-1} \frac{\partial g}{\partial t} g^{-1} \quad (5)$$

$$\frac{\partial V_1}{\partial x} = \frac{\partial g}{\partial x} V_2 g^{-1} + g V_{2t} g^{-1} - g V_2 g^{-1} \frac{\partial g}{\partial x} g^{-1} + \frac{\partial^2 g}{\partial x \partial t} g^{-1} - \frac{\partial g}{\partial t} g^{-1} \frac{\partial g}{\partial x} g^{-1} \quad (6)$$

where we have used (3). From (4) and (5) we obtain

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} = g \frac{\partial U_2}{\partial t} g^{-1} - g \frac{\partial V_2}{g}^{-1} = g \left(\frac{\partial U_2}{\partial t} - \frac{\partial V_1}{\partial x} \right) g^{-1} \quad (8)$$

The commutator yields

$$\begin{aligned} [U_1, V_1] &= [g U_2 g^{-1} + \frac{\partial g}{\partial x} g^{-1}, g V_2 g^{-1} + \frac{\partial g}{\partial t} g^{-1}] \\ &= [g U_2 g^{-1}, g V_2 g^{-1}] + [g U_2 g^{-1}, g_t g^{-1}] + [,] + [,] \\ &= g U_2 V_2 g^{-1} - g V_2 U_2 g^{-1} = g([U_2, V_2]) g^{-1} \end{aligned}$$

Adding (6) and (7) results in

$$g \left(\frac{\partial U_2}{\partial t} - \frac{\partial V_2}{\partial x} + [U_2, V_2] \right) g^{-1} = 0\,.$$

Thus (3) follows.

Both equations are integrable by the inverse scattering method. Both arise as consistency condition of a system of linear differential equations

$$\frac{\partial \phi}{\partial t} = U(x, t, \lambda) \phi, \qquad \frac{\partial \phi}{\partial x} = V(x, t, \lambda) \phi$$

where λ is a complex parameter. The consistency conditions have the form

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U, V] = 0.$$

Let

$$U_1 := g U_2 g^{-1} + \frac{\partial g}{\partial x} g^{-1}, \qquad V_1 := g V_2 g^{-1} + \frac{\partial g}{\partial t} g^{-1}$$

and

$$\frac{\partial U_1}{\partial t} - \frac{\partial V_1}{\partial x} + [U_1, V_1] = 0.$$

Problem 109. Consider the *nonlinear Schrödinger equation* in one space dimension

$$i \frac{\partial \psi}{\partial t} + \frac{\partial^2 \psi}{\partial x^2} + 2|\psi|^2 \psi = 0 \quad (1)$$

and the *Heisenberg ferromagnet equation* in one space dimension

$$\frac{\partial \mathbf{S}}{\partial t} = \mathbf{S} \times \frac{\partial^2 \mathbf{S}}{\partial x^2}, \qquad \mathbf{S}^2 = 1 \tag{2}$$

where $\mathbf{S} = (S_1, S_2, S_3)^T$. Both equations are integrable by the inverse scattering method. Both arise as the consistency condition of a system of linear differential equations

$$\frac{\partial \Phi}{\partial t} = U(x,t,\lambda)\Phi, \qquad \frac{\partial \phi}{\partial x} = V(x,t,\lambda)\Phi \tag{3}$$

where λ is a complex parameter. The consistency conditions have the form

$$\frac{\partial U}{\partial t} - \frac{\partial V}{\partial x} + [U,V] = 0 \tag{4}$$

(i) Show that $\phi_1 = g\phi_2$.
(ii) Show that (1) and (2) are gauge equivalent.

Solution 109.

Problem 110. The study of certain questions in the theory of $SU(2)$ gauge fields reduced to the construction of exact solutions of the following nonlinear system of partial differential equations

$$u\left(\frac{\partial^2 u}{\partial y \partial \bar{y}} + \frac{\partial^2 u}{\partial z \partial \bar{z}}\right) - \frac{\partial u}{\partial y}\frac{\partial u}{\partial \bar{y}} - \frac{\partial u}{\partial z}\frac{\partial u}{\partial \bar{z}} + \frac{\partial v}{\partial y}\frac{\partial \bar{v}}{\partial \bar{y}} + \frac{\partial v}{\partial z}\frac{\partial \bar{v}}{\partial \bar{z}} = 0.$$

$$u\left(\frac{\partial^2 v}{\partial y \partial \bar{y}} + \frac{\partial^2 v}{\partial z \partial \bar{z}}\right) - 2\left(\frac{\partial v}{\partial y}\frac{\partial u}{\partial \bar{y}} + \frac{\partial v}{\partial z}\frac{\partial u}{\partial \bar{z}}\right) = 0$$

$$u\left(\frac{\partial^2 \bar{v}}{\partial \bar{y} \partial y} + \frac{\partial^2 \bar{v}}{\partial \bar{z} \partial z}\right) - 2\left(\frac{\partial \bar{v}}{\partial \bar{y}}\frac{\partial u}{\partial y} + \frac{\partial \bar{v}}{\partial \bar{z}}\frac{\partial u}{\partial z}\right) = 0, \tag{1}$$

where u is a real function and v and $\bar{v}$ are complex unknown functions of the real variables $x_1, \ldots, x_4$. The quantities y and z are complex variables expressed in terms of $x_1, \ldots, x_4$ by the formulas

$$\sqrt{2}y := x_1 + ix_2, \qquad \sqrt{2}z := x_3 - ix_4 \tag{2}$$

and the bar over letters indicates the operation of complex conjugations.
(i) Show that a class of exact solutions of the system (1) can be constructed, namely solutions for the linear system

$$\frac{\partial v}{\partial y} - \frac{\partial u}{\partial \bar{z}} = 0, \qquad \frac{\partial v}{\partial z} - \frac{\partial u}{\partial \bar{y}} = 0 \tag{3}$$

where we assume that u, v, and $\bar{v}$ are functions of the variables

$$r := (2y\bar{y})^{1/2} = (x_1^2 + x_2^2)^{1/2} \tag{4}$$

and x_3, i.e., for the stationary, axially symmetric case. (ii) Show that a class of exact solutions of (1) can be given, where

$$u = u(w), \qquad v = v(w), \qquad \bar{v} = \bar{v}(w) \tag{5}$$

where w is a solution of the Laplace equation in complex notation

$$\frac{\partial^2 u}{\partial y \partial \bar{y}} + \frac{\partial^2 u}{\partial z \partial \bar{z}} = 0, \tag{6}$$

and u, v and $\bar{v}$ satisfy

$$u\frac{d^2u}{dw^2} - \left(\frac{du}{dw}\right)^2 + \frac{dv}{dw}\frac{d\bar{v}}{dw} = 0, \qquad u\frac{d^2v}{dw^2} - 2\frac{dv}{dw}\frac{du}{dw} = 0. \tag{7}$$

Hint. Let $z = x + iy$, where $x, y \in \mathbb{R}$. Then

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \qquad \frac{\partial}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right) \tag{8}$$

Solution 110.

Problem 111. The spherically symmetric SU(2) Yang-Mills equations can be written as

$$\frac{\partial \varphi_1}{\partial t} - \frac{\partial \varphi_2}{\partial r} = -A_0\varphi_2 - A_1\varphi_1 \tag{1a}$$

$$\frac{\partial \varphi_2}{\partial t} + \frac{\partial \varphi_1}{\partial r} = -A_1\varphi_2 + A_0\varphi_1 \tag{1b}$$

$$r^2\left(\frac{\partial A_1}{\partial t} - \frac{\partial A_0}{\partial r}\right) = 1 - (\varphi_1^2 + \varphi_2^2) \tag{1c}$$

where r is the spatial radius-vector and t is the time. To find partial solutions of these equations, two methods can be used. The first method is the inverse scattering theory technique, where the $[L, A]$-pair is found, and the second method is based on Bäcklund transformations.
(ii) Show that system (1) can be reduced to the classical Liouville equation, and its general solution can be obtained for any gauge condition.

Solution 111. Introducing the complex function

$$z(r, t) := \varphi_1(r, t) + i\varphi_2(r, t) \tag{2}$$

we have from the first two equations of (1)

$$\frac{\partial z}{\partial t} + i\frac{\partial z}{\partial r} = (A_0 i - A_1)z. \qquad (3)$$

Now we use the new variables

$$z(r,t) := R(r,t)\exp(i\theta(r,t)). \qquad (4)$$

Separating the imaginary and real parts we obtain from (3)

$$A_0 = \frac{\partial \theta}{\partial t} + \frac{\partial \ln R}{\partial r}, \qquad A_1 = \frac{\partial \theta}{\partial r} - \frac{\partial \ln R}{\partial t}. \qquad (5)$$

Substitution of (5) into (1c) yields

$$r^2 L(\ln R) = R^2 - 1, \qquad L := \frac{\partial^2}{\partial r^2} + \frac{\partial^2}{\partial t^2}. \qquad (6)$$

Changing the variables

$$R(r,t) = r\exp(g(r,t)) \qquad (7)$$

we come to the classical *Liouville equation*

$$Lg = \exp(2g) \qquad (8)$$

which has the general solution in terms of two harmonic functions $a(r,t)$ and $b(r,t)$ related by the *Cauchy-Riemann conditions*

$$\exp(2g) = 4\left(\left(\frac{\partial a}{\partial r}\right)^2 + \left(\frac{\partial a}{\partial t}\right)^2\right)(1 - a^2 - b^2)^{-2}, \qquad (9a)$$

$$La = Lb = 0, \qquad \frac{\partial a}{\partial r} = \frac{\partial b}{\partial t}, \qquad \frac{\partial a}{\partial t} = -\frac{\partial b}{\partial r}. \qquad (9b)$$

Equation (8) was obtained with an arbitrary function $\theta(r,t)$. Thus (5) and (9) give the general solutions of problem (1) with an arbitrary function $\theta(r,t)$ and harmonic functions $a(r,t)$ and $b(r,t)$, with

$$R = \pm 2r\left(\left(\frac{\partial a}{\partial r}\right)^2 + \left(\frac{\partial a}{\partial t}\right)^2\right)^{1/2}(a^2 + b^2 - 1)^{-1}. \qquad (10)$$

Problem 112. We consider the *Georgi-Glashow model* with gauge group $SU(2)$ broken down to $U(1)$ by *Higgs triplets*. The Lagrangian of the model is

$$\mathcal{L} := -\frac{1}{4}F^a_{\mu\nu}F^{\mu\nu a} + \frac{1}{2}D_\mu\phi^a D^\mu\phi^a - V(\phi) \qquad (1)$$

where

$$F^a_{\mu\nu} := \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g\epsilon_{abc}A^b_\mu A^c_\nu \tag{2}$$

$$D_\mu \phi_a := \partial_\mu \phi_a + g\epsilon_{abc}A^b_\mu \phi_c \tag{3}$$

and

$$V(\phi) := -\frac{\lambda}{4}\left(\phi^2 - \frac{m^2}{\lambda}\right)^2 . \tag{4}$$

(i) Show that the equations of motion are

$$D_\nu F^{\mu\nu a} = -g\epsilon_{abc}(D^\mu \phi_b)\phi_c, \qquad D_\mu D^\mu \phi_a = (m^2 - \lambda\phi^2)\phi_a. \tag{5}$$

(ii) Show that the vacuum expectation value of the scalar field and Higgs boson mass are

$$\langle \phi^2 \rangle = F^2 = \frac{m^2}{\lambda} \tag{6}$$

and

$$M_H = \sqrt{2\lambda}F,$$

respectively. Mass of the gauge boson is $M_w = gF$.
(iii) Using the time-dependent t' Hooft-Polyakov ansatz

$$A^a_0(r,t) = 0, \qquad A^a_i(r,t) = -\epsilon_{ain}r_n\frac{1-K(r,t)}{r^2}, \qquad \phi_a(r,t) = \frac{1}{g}r_a\frac{H(r,t)}{r^2} \tag{7}$$

where $r_n = x_n$ and r is the radial variable. Show that the equations of motion (5) can be written as

$$r^2\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2}\right)K = (K^2 + H^2 - 1) \tag{8a}$$

$$r^2\left(\frac{\partial^2}{\partial r^2} - \frac{\partial^2}{\partial t^2}\right)H = H\left(2K^2 - m^2r^2 + \frac{\lambda H^2}{g^2}\right). \tag{8b}$$

(iv) Show that with

$$\beta := \frac{\lambda}{g^2} = \frac{M_H^2}{2M_w^2}$$

and introducing the variables $\xi := M_w r$ and $\tau := M_w t$, system (8) becomes

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}\right)K = \frac{K(K^2 + H^2 - 1)}{\xi^2} \tag{10a}$$

$$\left(\frac{\partial^2}{\partial \xi^2} - \frac{\partial^2}{\partial \tau^2}\right)H = \frac{H(2K^2 + \beta(H^2 - \xi^2))}{\xi^2}. \tag{10b}$$

(v) The total energy of the system E is given by

$$C(\beta) = \frac{g^2 E}{4\pi M_w} =$$

$$\int_0^\infty \left(K_\tau^2 + \frac{H_\tau^2}{2} + K_\xi^2 + \frac{1}{2}\left(\frac{\partial H}{\partial \xi} - \frac{H}{\xi}\right)^2 + \frac{1}{2\xi^2}(K^2-1)^2 + \frac{K^2H^2}{\xi^2} + \frac{\beta}{4\xi^2}(H^2-\xi^2)^2 \right) d\xi. \tag{10}$$

As time-independent version of the ansatz (3) gives the 't Hooft-Polyakov monopole solution with winding number 1. Show that for finiteness of energy the field variables should satisfy the following conditions

$$H \to 0, \qquad K \to 1 \quad \text{as} \quad \xi \to 0 \tag{11}$$

and

$$H \to \xi, \qquad K \to 0 \quad \text{as} \quad \xi \to \infty. \tag{12}$$

The 't Hooft-Polyakov monopole is more realistic than the Wu-Yang monopole; it is non-singular and has finite energy.
(vi) Show that in the limit $\beta \to 0$, known as the Prasad-Somerfeld limit, we have the static solutions,

$$K(\xi) = \frac{\xi}{\sinh \xi}, \qquad H(\xi) = \xi \coth \xi - 1. \tag{13}$$

Solution 112.

Problem 113. Consider the linear operators L and M defined by

$$L\psi(x,t,\lambda) := \left(i\frac{\partial}{\partial x} + U(x,t,\lambda) \right) \psi(x,t,\lambda)$$

$$M\psi(x,t,\lambda) := \left(i\frac{\partial}{\partial t} + V(x,t,\lambda) \right) \psi(x,t,\lambda).$$

Find the condition on L and M such that $[L, M] = 0$, where $[\,,\,]$ denotes the commutator. The potentials $U(x,t,\lambda)$ and $V(x,t,\lambda)$ are typicaly chosen as elements of some semisimple Lie algebra.

Solution 113. We obtain

$$i\frac{\partial}{\partial x}V - i\frac{\partial}{\partial t}U + [U, V] = 0.$$

Problem 114. Show that the Korteweg-de Vries and nonlinear Schrödinger equations are reductions of the self-dual Yang-Mills equations. We work on $\mathbb{R}^4$ with coordinates $x^a = (x, y, u, t)$ and metric tensor field

$$g = dx \otimes dx - dy \otimes dy + du \otimes dt - dt \otimes du$$

of signature (2,2) and a totally skew orientation tensor $\epsilon_{abcd} = \epsilon_{[abcd]}$. We consider a Yang-Mills connection $D_a := \partial_a - A_a$ where the A_a where the A_a are, for the moment, elements of the Lie algebra of $SL(2,\mathbb{C})$. The A_a are defined up to gauge transformations

$$A_a \to hA_ah^{-1} - (\partial_a h)h^{-1}$$

where $h(x_a) \in SL(2,\mathbb{C})$. The connection is said to be self-dual when (summation convention)

$$\frac{1}{2}\epsilon_{ab}^{cd}[D_c, D_d] = [D_a, D_b]. \tag{3}$$

Solution 114. This is equivalent to the following three commutator equations

$$[D_x + D_y, D_u] = 0$$
$$[D_x - D_y, D_x + D_y] + [D_u, D_t] = 0$$
$$[D_x - D_y, D_t] = 0.$$

These follow from the integrability condition on the following linear system of equations

$$L_0 s = (D_x - D_y + \lambda D_u)s = 0, \qquad L_1 s = (D_t + \lambda(D_x + D_y))s = 0$$

where λ is an affine complex coordinate on the Riemann sphere $\mathbf{CP}^1$ (the "spectral parameter") and s is a two component column vector. We put

$$D_x := \partial_x - A, \qquad D_u := \partial_u - B$$

$$D_t := \partial_t - C, \qquad D_y := \partial_y - D.$$

Now require that the bundle and its connection possess two commuting symmetries which project to a pair of orthogonal spacetime translations one timelike and one null. In our coordinates these are along $\partial/\partial y$ and $\partial/\partial u$. We now restrict ourselves to gauges in which the components of the connection, (A, B, C, D) are independent of u and y. We also impose the gauge condition $A + D = 0$. The gauge transformations are now restricted to $SL(2,\mathbb{C})$ valued functions of t alone under which A and B transform by conjugation, $B \to hBh^{-1}$, etc. The equations now reduce to

$$\partial_x B = 0$$

$$[\partial_x - 2A, \partial_t - C] = 0$$
$$2\partial_x A - [B, C] = \partial_t B.$$

These equations follow from the integrability conditions on the reduction of the linear system

$$L_0 s = (\partial_x - 2A + \lambda B)s = 0$$
$$L_1 s = (\partial_t - C + \lambda \partial_x)s = 0.$$

When (2a) holds, B depends only on the variable t, so the gauge freedom may be used to reduce B to a normal form. The reductions are partially classified by the available normal forms. When B vanishes, the equations are trivially solveable, with the result that the connection may be put in the form $A_a dx^a = A(t)d(x+y)$. Equation (1) is then satisfied. Thus we assume that B is everywhere non-vanishing. The matrix B then has just two normal forms

$$(\alpha) B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$(\beta) B = \kappa \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We assume that the type of B is constant. In these case of type β, as t varies, κ becomes a non-zero function of t. When B is in the Lie algebra of $SU(2)$, $SU(1,1)$ or $SL(2,\mathbb{R})$, κ is non-zero and is either real or pure imaginary. Case (α) leads to the Korteweg-de Vries equation, and case (β) leads to the nonlinear Schrödinger equation.

A detailed analysis of the further reduction of the remaining equations leads to the following.

The self-dual Yang-Mills equations are solved with B of type α by the ansatz

$$2A = \begin{pmatrix} q & 1 \\ q_x - q^2 & -q \end{pmatrix}$$
$$2C = \begin{pmatrix} (q_x - q^2)_x & -2q_x \\ 2w & -(q_x - q^2)_x \end{pmatrix}$$

where

$$4w = q_{xxx} - 4qq_{xx} - 2q_x^2 + rq^2 q_x$$

a subscript x or t denotes differentiation with respect to that variable, and provided that q satisfies

$$4\frac{\partial q}{\partial t} = \frac{\partial^3 q}{\partial x^3} - 6\left(\frac{\partial q}{\partial x}\right)^2.$$

With the definition $u = -q_x = \mathrm{tr}(BC)$ we obtain the Korteweg-de Vries equation

$$4\frac{\partial u}{\partial t} = \frac{\partial^3 u}{\partial x^3} + 12u\frac{\partial u}{\partial x}.$$

Conversely, every solution of the equations have type α, with trace(AB) everywhere non-zero may be reduce to this form, at worst after suitable co-ordinate and gauge transformations.

When $\mathrm{tr}(AB)$ is identically zero, the equations are explicitly solveable.

The self-dual Yang-Mills equations are solved, with B of type β, by the ansatz

$$2A = \begin{pmatrix} 0 & \psi \\ -\widetilde{\psi} & 0 \end{pmatrix}, \quad 2\kappa C = \begin{pmatrix} \psi\widetilde{\psi} & \psi_x \\ \widetilde{\psi}_x & -\psi\widetilde{\psi} \end{pmatrix}$$

provided ψ and $\widetilde{\psi}$

$$2\kappa\psi_t = \psi_{xx} + 2\psi^2\widetilde{\psi}, \qquad 2\kappa\widetilde{\psi}_t = -\widetilde{\psi}_{xx} - 2\widetilde{\psi}^2\psi$$

and $2\kappa = 1$ or $-i$. Conversely, every solution of the equations for type β may be reduced to this form.

Chapter 3

Lie Symmetry Methods

Problem 1. Show that the partial differential equation

$$\frac{\partial u}{\partial x_2} = u \frac{\partial u}{\partial x_1} \tag{1}$$

is invariant under the transformation

$$\bar{x}_1(x_1, x_2, \epsilon) = x_1, \qquad \bar{x}_2(x_1, x_2, \epsilon) = \epsilon x_1 + x_2 \tag{2a}$$

$$\bar{u}(\bar{x}_1(x_1, x_2), \bar{x}_2(x_1, x_2), \epsilon) = \frac{u(x_1, x_2)}{1 - \epsilon u(x_1, x_2)} \tag{2b}$$

where ϵ is a real parameter.

Solution 1. We have

$$\frac{\partial \bar{u}}{\partial x_1} = \frac{\partial \bar{u}}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_1} + \frac{\partial \bar{u}}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_1} = \frac{\partial \bar{u}}{\partial \bar{x}_1} + \frac{\partial \bar{u}}{\partial \bar{x}_2}\epsilon = \frac{1}{(1-\epsilon)^2}\left(\frac{\partial u}{\partial x_1}(1 - \epsilon u) + \epsilon u \frac{\partial u}{\partial u}\partial x_1\right). \tag{3}$$

Analogously

$$\frac{\partial \bar{u}}{\partial x_2} = \frac{\partial \bar{u}}{\partial \bar{x}_1}\frac{\partial \bar{x}_1}{\partial x_2} + \frac{\partial \bar{u}}{\partial \bar{x}_2}\frac{\partial \bar{x}_2}{\partial x_2} = \frac{\partial \bar{u}}{\partial \bar{x}_1}. \tag{4}$$

Therefore

$$\frac{\partial \bar{u}}{\partial \bar{x}_2} = \frac{1}{(1-\epsilon)^2}\left(\frac{\partial u}{\partial x_2}(1 - \epsilon u) + \epsilon u \frac{\partial u}{\partial x_2}\right). \tag{5}$$

From (3) and (5) it follows that

$$\frac{\partial \bar{u}}{\partial \bar{x}_1} = \frac{1}{(1-\epsilon)^2}\left(\frac{\partial u}{\partial x_1}(1 - \epsilon u) + \epsilon u \frac{\partial u}{\partial x_1} - \epsilon(1-\epsilon u)\frac{\partial u}{\partial x_2} - \epsilon^2 u \frac{\partial u}{\partial x_2}\right) \tag{6}$$

Inserting (2b), (3) and (6) into

$$\frac{\partial \bar{u}}{\partial \bar{x}_2} = \bar{u}\frac{\partial \bar{u}}{\partial \bar{x}_1} \tag{7}$$

yields (1).

Problem 2. Consider the Korteweg-de Vries equation

$$\frac{\partial u}{\partial t} + 12u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0\,.$$

Show that the scaling

$$x' = cx, \quad t' = c^3 t, \quad u(x,t) = c^2 u'(x',t')$$

leaves the Korteweg-de Vries equation invariant.

Solution 2.

Problem 3. Consider the partial differential equation

$$\Phi(\Box u, (\nabla u)^2, u) = 0$$

where Φ is an analytic function, u depends on x_0, x_1, $\ldots$, x_n and

$$\Box u := \frac{\partial u}{\partial x_0} - \frac{\partial u}{\partial x_1} - \cdots - \frac{\partial u}{\partial x_n}, \quad (\nabla u)^2 := \left(\frac{\partial u}{\partial x_0}\right)^2 - \left(\frac{\partial u}{\partial x_1}\right)^2 - \cdots - \left(\frac{\partial u}{\partial x_n}\right)^2 .$$

Show that the equation is invariant under the Poincaré algebra

$$\frac{\partial}{\partial x_0}, \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}$$

$$x_0\frac{\partial}{\partial x_1} + x_1\frac{\partial}{\partial x_0}, x_0\frac{\partial}{\partial x_2} + x_2\frac{\partial}{\partial x_0}, \ldots, x_0\frac{\partial}{\partial x_n} + x_n\frac{\partial}{\partial x_0}$$

$$x_j\frac{\partial}{\partial x_k} - x_k\frac{\partial}{\partial x_k}, \quad j,k = 1,2,\ldots,n \quad j \neq k\,.$$

Solution 3.

Problem 4. Consider the nonlinear partial differential equation (*Born-Infeld equation*)

$$\left(1 - \left(\frac{\partial u}{\partial t}\right)^2\right)\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial u}{\partial x}\frac{\partial u}{\partial t}\frac{\partial^2 u}{\partial x \partial t} - \left(1 + \left(\frac{\partial u}{\partial x}\right)^2\right)\frac{\partial^2 u}{\partial t^2} = 0. \tag{1}$$

Show that this equation admits the following seven Lie symmetry generators

$$Z_1 = \frac{\partial}{\partial x}, \qquad Z_2 = \frac{\partial}{\partial t}, \qquad Z_3 = \frac{\partial}{\partial u}$$

$$Z_4 = x\frac{\partial}{\partial t} + t\frac{\partial}{\partial x}, \qquad Z_5 = u\frac{\partial}{\partial x} - x\frac{\partial}{\partial u}$$

$$Z_6 = u\frac{\partial}{\partial t} + t\frac{\partial}{\partial u}, \qquad Z_7 = x\frac{\partial}{\partial x} + t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}\,. \tag{2}$$

Solution 4.

Problem 5. Consider the *Harry-Dym equation*

$$\frac{\partial u}{\partial t} - u^3\frac{\partial^3 u}{\partial x^3} = 0. \tag{1}$$

(i) Find the Lie symmetry vector fields.
(ii) Compute the flow for one of the Lie symmetry vector fields.

Solution 5. (i) The Lie symmetry vector field is given by

$$V = \eta_x(x,t,u)\frac{\partial}{\partial x} + \eta_t(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u}. \tag{2}$$

There are eight determining equations,

$$\frac{\partial \eta_t}{\partial u} = 0, \qquad \frac{\partial \eta_t}{\partial x} = 0, \qquad \frac{\partial \eta_x}{\partial u} = 0, \qquad \frac{\partial^2 \phi}{\partial u^2} = 0,$$

$$\frac{\partial^2 \phi}{\partial u \partial x} - \frac{\partial^2 \eta_x}{\partial x^2} = 0, \qquad \frac{\partial \phi}{\partial t} - u^3\frac{\partial^3 \phi}{\partial x^3} = 0$$

$$3u^3\frac{\partial^3 \phi}{\partial u \partial x^2} + \frac{\partial \eta_x}{\partial t} - u^3 - \frac{\partial^3 \eta_x}{\partial x^3} = 0, \qquad u\frac{\partial \eta_t}{\partial t} - 3u\frac{\partial \eta_x}{\partial x} + 3\phi = 0. \tag{3}$$

These determining equations can easily be solved explicitly. The general solution is

$$\eta_x = k_1 + k_3 x + k_5 x^2, \quad \eta_t = k_2 - 3k_4 t, \quad \phi = (k_3 + k_4 + 2k_5 x)u \tag{4}$$

where $k_1, ..., k_5$ are arbitrary constants. The five infinitesimal generators then are

$$G_1 = \frac{\partial}{\partial x}, \qquad G_2 = \frac{\partial}{\partial t}$$

$$G_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \qquad G_4 = -3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \qquad G_5 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}. \tag{5}$$

Thus (1) is invariant under translations (G_1 and G_2) and scaling (G_3 and G_4). The flow corresponding to each of the infinitesimal generators can be obtained via simple integration.
(ii) Let us compute the flow corresponding to G_5. This requires integration of the first order system

$$\frac{d\bar{x}}{d\epsilon} = \bar{x}^2, \qquad \frac{d\bar{t}}{d\epsilon} = 0, \qquad \frac{d\bar{u}}{d\epsilon} = 2\bar{x}\bar{u} \tag{6}$$

together with the initial conditions

$$x(0) = x, \qquad \bar{t}(0) = t, \qquad \bar{u}(0) = u \tag{7}$$

where ϵ is the parameter of the transformation group. One obtains

$$\bar{x}(\epsilon) = \frac{x}{(1-\epsilon x)}, \qquad \bar{t}(\epsilon) = t, \qquad \bar{u}(\epsilon) = \frac{u}{(1-\epsilon x)^2}. \tag{8}$$

We therefore conclude that for any solution $u = f(x,t)$ of (1), the transformed solution

$$\bar{u}(\bar{x},\bar{t}) = (1+\epsilon\bar{x})^2 f\left(\frac{\bar{x}}{1+\epsilon\bar{x}}, \bar{t}\right) \tag{9}$$

will solve

$$\frac{\partial\bar{u}}{\partial\bar{t}} - \bar{u}^3\frac{\partial^3\bar{u}}{\partial\bar{x}^3} = 0. \tag{10}$$

Problem 6. Consider the *Magneto-Hydro-Dynamics equations* and carry out the Lie symmetry analysis. We neglect dissipative effects, and thus restrict the analysis to the ideal case. The equations are given by

$$\frac{\partial\rho}{\partial t} + (\mathbf{v}\cdot\nabla)\rho + \rho\nabla\cdot\mathbf{v} = 0 \tag{1a}$$

$$\rho\left(\frac{\partial\mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right) + \nabla(p + \frac{1}{2}\mathbf{H}^2) - (\mathbf{H}\cdot\nabla)\mathbf{H} = \mathbf{0} \tag{1b}$$

$$\frac{\partial\mathbf{H}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{H} + \mathbf{H}\nabla\cdot\mathbf{v} - (\mathbf{H}\cdot\nabla)\mathbf{v} = \mathbf{0} \tag{1c}$$

$$\nabla\cdot\mathbf{H} = 0 \tag{1d}$$

$$\frac{\partial}{\partial t}\left(\frac{p}{\rho^\kappa}\right) + (\mathbf{v}\cdot\nabla)\left(\frac{p}{\rho^\kappa}\right) = 0 \tag{1e}$$

with pressure p, mass density ρ, coefficient of viscosity κ, fluid velocity $\mathbf{v}$ and magnetic field $\mathbf{H}$.

Solution 6. Using the first equation, we eliminate ρ from the last equation, and replace it by

$$\frac{\partial p}{\partial t} + \kappa p(\nabla \cdot \mathbf{v}) + (\mathbf{v} \cdot \nabla)p = 0. \tag{2}$$

If we split the vector equations in scalar equations for the vector components, we have a system of $m = 9$ equations, with $p = 4$ independent variables and $q = 8$ dependent variables. For convenience, we denote the components of the vector $\vec{v}$ by v_x, v_y and v_z, not to be confused with partial derivatives of v. The variables to be eliminated are selected as follows: for the first 7 variables and the ninth variable we pick the partial derivatives with respect to t of ρ, v_x, v_y, v_z, H_x, H_y, H_z and p. From the eighth equation we select $\partial H_x/\partial x$ for elimination. We consider the case where $k \neq 0$. We find 222 determining equations for the coefficients of the vector field

$$V = \eta^x \frac{\partial}{\partial x} + \eta^y \frac{\partial}{\partial y} + \eta^z \frac{\partial}{\partial z} + \eta^t \frac{\partial}{\partial t} + \phi^\rho \frac{\partial}{\partial \rho} + \phi^p \frac{\partial}{\partial p}$$

$$+\phi^{v_x} \frac{\partial}{\partial v_x} + \phi^{v_y} \frac{\partial}{\partial v_y} + \phi^{v_z} \frac{\partial}{\partial v_z} + \phi^{H_x} \frac{\partial}{\partial H_x} + \phi^{H_y} \frac{\partial}{\partial H_y} + \phi^{H_z} \frac{\partial}{\partial H_z}. \tag{3}$$

The the components of the Lie symmetry vector fields are given by

$$\begin{aligned}
\eta^x &= k_2 + k_5 t - k_8 y - k_9 z + k_{11} x \\
\eta^y &= k_3 + k_6 t + k_8 x - k_{10} z + k_{11} y \\
\eta^z &= k_4 + k_7 t + k_9 x + k_{10} y + k_{11} z \\
\eta^t &= k_1 + k_{12} t \\
\phi^\rho &= -2(k_{11} - k_{12} - k_{13})\rho \\
\phi^p &= 2k_{13} p \\
\phi^{v_x} &= k_5 - k_8 v_y - k_9 v_z + (k_{11} - k_{12}) v_x \\
\phi^{v_y} &= k_6 + k_8 v_x - k_{10} v_z + (k_{11} - k_{12}) v_y \\
\phi^{v_z} &= k_7 + k_9 v_x + k_{10} v_y + (k_{11} - k_{12}) v_z \\
\phi^{H_x} &= k_{13} H_x - k_8 H_y - k_9 H_z \\
\phi^{H_y} &= k_{13} H_y + k_8 H_x - k_{10} H_z \\
\phi^{H_z} &= k_{13} H_z + k_9 H_x + k_{10} H_y.
\end{aligned}$$

There is a thirteen dimensional Lie algebra spanned by the generators

$$G_1 = \frac{\partial}{\partial t}, \qquad G_2 = \frac{\partial}{\partial x}, \qquad G_3 = \frac{\partial}{\partial y}, \qquad G_4 = \frac{\partial}{\partial z}$$

$$G_5 = t\frac{\partial}{\partial x} + \frac{\partial}{\partial v_x}, \qquad G_6 = t\frac{\partial}{\partial y} + \frac{\partial}{\partial v_y}, \qquad G_7 = t\frac{\partial}{\partial z} + \frac{\partial}{\partial v_z}$$

$$G_8 = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} + v_x\frac{\partial}{\partial v_y} - v_y\frac{\partial}{\partial v_x} + H_x\frac{\partial}{\partial H_y} - H_y\frac{\partial}{\partial H_x}$$

$$G_9 = y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} + v_y\frac{\partial}{\partial v_z} - v_z\frac{\partial}{\partial v_y} + H_y\frac{\partial}{\partial H_z} - H_z\frac{\partial}{\partial H_y}$$

$$G_{10} = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + v_z\frac{\partial}{\partial v_z} - v_x\frac{\partial}{\partial v_z} + H_z\frac{\partial}{\partial H_x} - H_x\frac{\partial}{\partial H_z}$$

$$G_{11} = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} - 2\rho\frac{\partial}{\partial \rho} + v_x\frac{\partial}{\partial v_x} + v_y\frac{\partial}{\partial v_y} + v_z\frac{\partial}{\partial v_z}$$

$$G_{12} = t\frac{\partial}{\partial t} + 2\rho\frac{\partial}{\partial \rho} - (v_x\frac{\partial}{\partial v_x} + v_y\frac{\partial}{\partial v_y} + v_z\frac{\partial}{\partial v_z})$$

$$G_{13} = 2\rho\frac{\partial}{\partial \rho} + 2p\frac{\partial}{\partial p} + H_x\frac{\partial}{\partial H_x} + H_y\frac{\partial}{\partial H_y} + H_z\frac{\partial}{\partial H_z}. \qquad (5)$$

The Magneto-Hydro-Dynamic equations are invariant under translations G_1 through G_4, Galilean boosts G_5 through G_7, rotations G_8 through G_{10}, and dilations G_{11} through G_{13}.

Problem 7. The stimulated *Raman scattering equations* in a symmetric form are given by

$$\frac{\partial v_1}{\partial x} = ia_1 v_2^* v_3^*, \qquad \frac{\partial v_2}{\partial x} = ia_2 v_3^* v_1^*, \qquad \frac{\partial v_3}{\partial t} = ia_3 v_1^* v_2^*. \qquad (1)$$

The a_i are real coupling constants that can be normalized to ± 1 and we have set $v_1 = iA_1^*$, $v_2 = A_2$, $v_3 = X$, where A_1, A_2 and X are the complex pump, Stokes and material excitation wave envelopes, respectively. The stars denote complex conjugation. Equations (1) are actually a special degenerate case of the full three wave resonant interaction equations. Find the similarity solutions using the similarity ansatz

$$s(x,t) := xt^{-\epsilon} \qquad \text{similarity variable} \qquad (2)$$

and

$$v_1(x,t) = t^{-\frac{1+\epsilon}{2}}\rho_1(s)\exp(i\phi_1(s)), \qquad v_2(x,t) = t^{-\frac{1+\epsilon}{2}}\exp(-i\epsilon alnt)\rho_2(s)\exp(i\phi_2(s))$$

$$v_3(x,t) = \frac{1}{x}e^{ialnx}\rho_3(s)\exp(i\phi_3(s)) \qquad (3)$$

where $\rho_i \geq 0$, $0 \leq \phi_i < 2\pi$ and $\epsilon = \pm 1$.

Solution 7. We substitute (3) into (1), introduce a phase

$$\phi := \phi_1 + \phi_2 + \phi_3 + alns \qquad (4)$$

separate out the real and imaginary parts of the equations and obtain a system of six real ordinary differential equations

$$\frac{d\rho_1}{ds} = -\frac{1}{s}\rho_2\rho_3 \sin\phi, \qquad \rho_1\frac{d\phi_1}{dt} = -\frac{1}{s}\rho_2\rho_3\cos\phi$$

$$\frac{d\rho_2}{ds} = \frac{1}{s}\rho_3\rho_1 \sin\phi, \qquad \rho_2\frac{d\phi_2}{dt} = \frac{1}{s}\rho_3\rho_1\cos\phi$$

$$\frac{d\rho_3}{ds} = -\epsilon\rho_1\rho_2 \sin\phi, \qquad \rho_3\frac{d\phi_3}{dt} = -\epsilon\rho_1\rho_2\cos\phi. \tag{5}$$

System (5) allows two first integrals

$$I_1 = \rho_1^2 + \rho_2^2, \qquad I_2 = \rho_1\rho_2\rho_3\cos\phi - \frac{a}{2}\rho_1^2. \tag{6}$$

These two first integrals can be used to decouple the equations. In terms of ρ_1 we have

$$\rho_2 = (I_1 - \rho_1^2)^{1/2}, \qquad \rho_3\sin\phi = -s\frac{d\rho_1}{dt}(I_1-\rho_1^2)^{-1/2}$$

$$\rho_3\cos(\phi) = -\frac{1}{\rho_1}(I_2 + \frac{a}{2}\rho_1^2)(I_1-\rho_1^2)^{-1/2}. \tag{7}$$

The amplitude ρ_1 then satisfies a second order nonlinear ordinary differential equation

$$s\frac{d^2\rho_1}{ds^2}+\frac{d\rho_1}{ds} = -s\frac{\rho_1}{I_1-\rho_1^2}\left(\frac{d\rho_1}{ds}\right)^2-\frac{(I_2+\frac{a}{2}\rho_1^2)^2}{s\rho_1(I_1-\rho_1^2)}+\left(I_1\epsilon-\frac{a^2}{4s}\right)\rho_1-\epsilon\rho_1^3+\frac{I_2^2}{s\rho_1^3}. \tag{8}$$

Under the transformation

$$\rho_1(s) = \sqrt{I_1}\left(\frac{W(s)}{W(s)-1}\right)^{1/2} \tag{9}$$

(8) takes the form

$$\frac{d^2W}{ds^2} = \left(\frac{1}{2W}+\frac{1}{W-1}\right)\left(\frac{dW}{ds}\right)^2-\frac{1}{s}\frac{dW}{ds}+\frac{(W-1)^2}{s^2}\left(\alpha W+\frac{\beta}{W}\right)+\frac{\gamma}{s}W+\frac{\delta W(W+1)}{W-1} \tag{10}$$

where the constants are

$$\alpha := \frac{1}{2}\left(\frac{2I_2}{I_1}+a\right)^2, \qquad \beta := \frac{2I_2^2}{I_1^2}, \qquad \gamma := 2\epsilon I_1, \qquad \delta = 0. \tag{11}$$

Equation (11) is one of the six irreducible *Painlevé equations*, namely the one defining the Painlevé transcendent $P_V(s;\alpha,\beta,\gamma,\delta)$. For $\gamma = \delta = 0$ this transcendent can be expressed in terms of elementary functions. When

$\gamma \neq 0$, $\delta = 0$, $P_V(s;\alpha,\beta,\gamma,0)$ can be expressed in terms of the third Painleé transcendent P_{III}.

Problem 8. The motion of an inviscid, incompressible-ideal fluid is goverened by the system of equations

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\Delta\mathbf{u} = -\Delta p \qquad (1)$$

$$\Delta\cdot\mathbf{u} = 0 \qquad (2)$$

first obtained by Euler. Here $\mathbf{u} = (u,v,w)$ are the components of the three-dimensional velocity field and p the pressure of the fluid at a position $\mathbf{x} = (x,y,z)$.
(i) Show that the Lie symmetry group of the Euler equations in three dimensions is generated by the vector fields

$$_a = a\frac{\partial}{\partial x} + a'\frac{\partial}{\partial u} - a''x\frac{\partial}{\partial p}, \qquad \mathbf{v}_b = b\frac{\partial}{\partial y} + b'\frac{\partial}{\partial v} - b''y\frac{\partial}{\partial p}$$

$$\mathbf{v}_c = c\frac{\partial}{\partial z} + c'\frac{\partial}{\partial w} - c''z\frac{\partial}{\partial p}, \qquad \mathbf{v}_0 = \frac{\partial}{\partial t}$$

$$\mathbf{s}_1 = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z} + t\frac{\partial}{\partial t}, \qquad \mathbf{s}_2 = t\frac{\partial}{\partial t} - u\frac{\partial}{\partial u} - v\frac{\partial}{\partial v} - w\frac{\partial}{\partial w} - 2p\frac{\partial}{\partial p}$$

$$\mathbf{r}_2 = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + v\frac{\partial}{\partial u} - u\frac{\partial}{\partial v}, \qquad \mathbf{r}_y = z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} + w\frac{\partial}{\partial u} - u\frac{\partial}{\partial w}$$

$$\mathbf{r}_x = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z} + w\frac{\partial}{\partial v} - v\frac{\partial}{\partial w}, \qquad \mathbf{v}_q = q\frac{\partial}{\partial p} \qquad (4)$$

where a,b,c,q are arbitrary functions of t.
(ii) Show that these vector fields exponentiate to familiar one-parameter symmetry groups of the Euler equations. For instance, a linear combination of the first three fields $\mathbf{v}_a + \mathbf{v}_b + \mathbf{v}_c$ generates the group transformations

$$(\mathbf{x}, t, \mathbf{u}p) \to (\mathbf{x} + \epsilon\mathbf{a}, t, \mathbf{u} + \epsilon\mathbf{a}', p - \epsilon\mathbf{x}\cdot\mathbf{a}'' + \frac{1}{2}\epsilon^2\mathbf{a}\cdot\mathbf{a}'')$$

where ϵ is the group parameter, and $\mathbf{a} := (a,b,c)$. These represent changes to arbitrarily moving coordinate systems, and have the interesting consequence that for a fluid with no free surfaces, the only essential effect of changing to a moving coordinate frame is to add an extra component, namely,

$$-\epsilon\mathbf{x}\cdot\mathbf{a} + \frac{1}{2}\epsilon^2\mathbf{a}\cdot\mathbf{a}''$$

to the resulting pressure.

(iii) Show that the group generated by $\mathbf{v}_0$ is that of time translations, reflecting the time-independence of the system. (iv) Show that the next two groups are scaling transformations

$$\mathbf{s}_1 : (\mathbf{x}, t\mathbf{u}, p) \to \epsilon\mathbf{x}, \epsilon t, \mathbf{u}, p)$$

$$\mathbf{s}_2 : (\mathbf{x}, t, \mathbf{u}, p) \to (\mathbf{x}, \epsilon t, \epsilon^{-1}\mathbf{u}, \epsilon^{-2}p).$$

The vector fields $\mathbf{r}_x$, $\mathbf{r}_y$, $\mathbf{r}_2$ generate the orthogonal group $SO(3)$ of simultaneous rotations of space and associated velocity field; e.g., $\mathbf{r}_x$ is just an infinitesimal rotation around the x axis. The final group indicates that arbitrary functions of t can be added to the pressure.

Solution 8.

Problem 9. Consider the nonlinear diffusion equation

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{1}{u^2}\frac{\partial u}{\partial x}\right). \tag{1}$$

Within the jet bundle formalism we consider instead of (1) the submanifold

$$F(x, t, u, u_x, u_{xx}) \equiv u_t - \frac{u_{xx}}{u^2} + 2\frac{u_x^2}{u^3} = 0 \tag{2}$$

and its differential consequences

$$F_x(x, t, u, u_x, \cdots) \equiv u_{tx} - \frac{u_{xxx}}{u^2} + \frac{6u_x u_{xx}}{u^3} - \frac{6u_x^3}{u^4} = 0, \ldots \tag{3}$$

with the contact forms

$$\theta = du - u_x dx - u_t dt, \qquad \theta_x = du_x - u_{xx}dx - u_{tx}dt, \ldots \tag{4}$$

The non-linear partial differential equation (1) admits the Lie point symmetries (infinitesimal generator)

$$X_v = -u_x\frac{\partial}{\partial u}, \qquad T_v = u_t\frac{\partial}{\partial u}$$

$$S_v = (-xu_x - 2tu_t)\frac{\partial}{\partial u}, \qquad V_v = (xu_x + u)\frac{\partial}{\partial u}. \tag{5}$$

The subscript v denotes the vertical vector fields. The non-linear partial differential equation (1) also admits *Lie-Bäcklund vector fields*. The first in the hierarchy is

$$U = \left(\frac{u_{xxx}}{u^3} - \frac{9u_x u_{xx}}{u^4} + \frac{12u_x^3}{u^5}\right)\frac{\partial}{\partial u}. \tag{6}$$

Find a similarity solution using a linear combination of the Lie symmetry vector field T_v and U.

Solution 9. For reducing (1) we consider a linear combination of the vector fields T_v and U, i.e. $aT_v + U (a \in \mathbb{R})$. The equation

$$(aT_v + U)\rfloor\theta = 0 \tag{7}$$

where $\rfloor$ denotes the contraction, leads to the submanifold

$$-au_t + \frac{u_{xxx}}{u^3} - \frac{9u_x u_{xx}}{u^4} + \frac{12u_x^3}{u^5} = 0 \tag{8}$$

Consequently, it follows that

$$js^*\left[-a\left(\frac{u_{xx}}{u^2} - \frac{2u_x^2}{u^3}\right) + \frac{u_{xxx}}{u^3} - \frac{pu_x u_{xx}}{u^4} + \frac{12u_x^3}{u^5}\right] \equiv \frac{\partial^3}{\partial x^3}\left(\frac{1}{2u^2}\right) - a\frac{\partial^2}{\partial x^2}\left(\frac{1}{u}\right) = 0 \tag{9}$$

where s is a cross section $s(x,t) = (x, t, u(x,t))$ with $js^*\theta = 0$, $js^*\theta_x = 0, \cdots$. Here js is the extension of s up to infinite order. For deriving (9) we have taken into account the identity

$$\frac{1}{u^3}\frac{\partial^3 u}{\partial x^3} - \frac{9}{u^4}\frac{\partial u \partial^2 u}{\partial x \partial x^2} + \frac{12}{u^5}\left(\frac{\partial u}{\partial x}\right)^3 \equiv -\frac{\partial^3}{\partial x^3}\left(\frac{1}{2u^2}\right) \tag{10}$$

and the identity given by (1). Since derivates of u with respect to t do not appear in (9) we are able to consider (9) as an ordinary differential equation of second order

$$\frac{d^3}{dx^3}\left(\frac{1}{2u^2}\right) - a\frac{d^2}{dx^2}\frac{1}{u} = 0 \tag{11}$$

where t plays the role of a parameter and occurs in the constant of integration. The integration of (11) yields

$$\frac{du}{dx} + au^2 = (C_1(t)x + C_2(t))u^3. \tag{12}$$

In order to determine the constants of integration C_1 and C_2 we must first solve the ordinary differential equation (12), where a new constant of integration appears which also depends on time. Then we insert the solution into the partial differential equation (1) and determine the quantities C_1, C_2 and C_3. Equation (12) is a special case of Abel's equation. In order to solve Abel's equation we set $C_1(t) = 2a^2/9$. To simplify the calculations further we set $C_2(t) = 0$. Now $C_3(t)$ is the constant of integration of the simplified Abel equation. Imposing the condition that $u(x,t)$ must be a solution of (1) yields the following linear differential equation for C_3

$$\frac{dC_3}{dt} - \frac{2}{9}a^2 C_3(t) = 0 \tag{13}$$

with the solution

$$C_3(t) = k_1 e^{(2/9)a^2 t}$$

where k_1 is the constant of integration. Consequently, we find a similarity solution of the diffusion equation (1)

$$u(x,t) = \frac{3[81 - 4a^4 k_1 x e^{(2/9)a^2 t}]^{1/2} + 27}{2ax[81 - a^4 k_1 x e^{(2/9)a^2 t}]^{1/2}}.$$

If k_1 and x are positive (or both are negative), then the solution exists only for a finite time. If $k_1 = 0$ we arrive at the time-independent solution $u(x,t) = K/x$, where K is a constant.

Problem 10. Consider the nonlinear partial differential equation

$$\frac{\partial^3 u}{\partial x^3} + u\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

where c is a constant. Show that this partial differential equation admits the Lie symmetry vector field

$$V = t\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

First solve the first order autonomous system of differential equations (initial value problem) which belongs to the vector field V, i.e.

$$\frac{dt'}{d\epsilon} = t', \qquad \frac{dx'}{d\epsilon} = ct', \qquad \frac{du'}{d\epsilon} = u'.$$

From this solution of the initial value problem find the transformation between the prime and unprime system. Then using differentiation and the chain rule show that the prime and unprime system have the same form.

Solution 10. The solution of the initial value problem of the autonomous system of first order is given by

$$t'(\epsilon) = e^\epsilon t, \qquad x'(\epsilon) = x + ct(e^\epsilon - 1), \qquad u'(\epsilon) = e^\epsilon u.$$

Then the transformation between the prime and unprime system is

$$\begin{aligned} t'(t,x) &= e^\epsilon t \\ x'(t,x) &= x + ct(e^\epsilon - 1) \\ u'(x'(x,t), t'(x,t)) &= e^\epsilon u(x,t). \end{aligned}$$

Since $\partial t'/\partial t = e^\epsilon$, $\partial t'/\partial x = 0$, $\partial x'/\partial t = c(e^\epsilon - 1)$, $\partial x'/\partial x = 1$ we have

$$\frac{\partial u'}{\partial t} = \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial t} + \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial t} = e^\epsilon \frac{\partial u}{\partial t}.$$

Thus

$$\frac{\partial u'}{\partial x'}c(e^{\epsilon}-1)+\frac{\partial u'}{\partial t'}e^{\epsilon}=e^{\epsilon}\frac{\partial u}{\partial t}\,.$$

For the first space derivative we have

$$\frac{\partial u'}{\partial x}=\frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial x}+\frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial x}=e^{\epsilon}\frac{\partial u}{\partial x}\,.$$

Thus

$$\frac{\partial u'}{\partial x'}=e^{\epsilon}\frac{\partial u}{\partial x}\,.$$

For the second space derivative we have

$$\frac{\partial^2 u'}{\partial x^2}=\frac{\partial^2 u'}{\partial x'^2}\frac{\partial x'}{\partial x}\frac{\partial x'}{\partial x}+\frac{\partial u'}{\partial x'}\frac{\partial^2 x'}{\partial x^2}+\frac{\partial^2 u'}{\partial x'\partial t'}\frac{\partial t'}{\partial x}\frac{\partial x'}{\partial x}=e^{\epsilon}\frac{\partial^2 u}{\partial x^2}\,.$$

Analogously

$$\frac{\partial^3 u'}{\partial x'^3}=e^{\epsilon}\frac{\partial^3 u}{\partial x^3}\,.$$

Inserting these equations into the partial differential equation yields

$$e^{-\epsilon}\frac{\partial^3 u'}{\partial x'^3}+e^{-\epsilon}u'\left(e^{-\epsilon}c(e^{\epsilon}-1)+\frac{\partial u'}{\partial t'}\right)+e^{-\epsilon}cu'e^{-\epsilon}\frac{\partial u'}{\partial x'}=0$$

or

$$\frac{\partial^3 u'}{\partial x'^3}+u'\left(e^{-\epsilon}c(e^{\epsilon}-1)+\frac{\partial u'}{\partial t'}\right)+cu'e^{-\epsilon}\frac{\partial u'}{\partial x'}=0$$

Thus

$$\frac{\partial^3 u'}{\partial x'^3}+u'\left(\frac{\partial u'}{\partial t'}+cu'\frac{\partial u'}{\partial x'}\right)=0$$

Problem 11. Consider the nonlinear Schrödinger equation

$$i\frac{\partial w}{\partial t}+\Delta w=F(w)$$

with a nonlinearity $F:\mathbb{C}\to\mathbb{C}$ and a complex valued $w(t,x_1,\ldots,x_n)$. We assume that for the given nonlinearity the energy remains bounded. We also assume that there exists a C^1-function $G:\mathbb{R}^+\to\mathbb{R}$ with $G(0)=0$ such that $F(z)=G'(|z|^2)z$ for all $z\in\mathbb{C}$.
(i) Show that the nonlinear Schrödinger is translation invariant.
(ii) Show that it is phase invariant, i.e. $w\to e^{i\alpha}w$.
(iii) Show that it is Galilean invariant

$$w(t,\mathbf{x})\to e^{i\mathbf{v}\cdot\mathbf{x}/2}e^{-i|\mathbf{v}|^2t/4}w(t,\mathbf{x}-\mathbf{v}t)$$

for any velocity $\mathbf{v}$.
(iv) Show that the mass defined by

$$M(w) = \int_{\mathbb{R}^n} |w(t,\mathbf{x})|^2 d\mathbf{x}$$

is a conserved quantity.
(v) Show that the Hamilton density

$$H(w) = \int_{\mathbb{R}^n} \frac{1}{2}|\nabla w(t,\mathbf{x})|^2 + \frac{1}{2}G(|w(t,\mathbf{x})|^2) d\mathbf{x}$$

is a conserved quantity.

Solution 11. For the given problem we can work with a Hilbert space using the inner product

$$\langle w, v\rangle := \int_{\mathbb{R}^n} w(\mathbf{x})\overline{v(\mathbf{x})} + \nabla w(\mathbf{x}) \cdot \overline{\nabla v(\mathbf{x})} d\mathbf{x}\,.$$

Problem 12. For a barotropic fluid of index γ the *Navier-Stokes equation* read, in one space dimension

$$\frac{\partial c}{\partial t} + v\frac{\partial c}{\partial x} + \frac{(\gamma-1)}{2}c\frac{\partial v}{\partial x} = 0 \qquad \text{continuity equation} \qquad (1a)$$

$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} + \frac{2}{(\gamma-1)}c\frac{\partial c}{\partial x} - \frac{\partial^2 v}{\partial x^2} = 0 \qquad \text{Euler's equation} \qquad (1b)$$

where v represents the fluid's velocity and c the sound speed. For the class of solutions characterized by a vanishing presure (i.e., $c = 0$), the above system reduces to the Burgers equation. We assume for simplicity the value $\gamma = 3$ in what follows.
(i) Show that the *velocity potential* Φ exists, and it is given by the following pair of equations

$$\frac{\partial \Phi}{\partial x} = v, \qquad \frac{\partial \Phi}{\partial t} = \frac{\partial v}{\partial x} - \frac{1}{2}(v^2 + c^2). \qquad (2)$$

Hint. From (2) with

$$\frac{\partial^2 \Phi}{\partial t \partial x} = \frac{\partial^2 \Phi}{\partial x \partial t}$$

the Euler equation (1b) follows. Thus the condition of integrability of Φ precisely coincides with the Euler equation (1b).
(ii) Consider the similarity ansatz

$$v(x,t) = f(s)\frac{x}{t}, \qquad c(x,t) = g(s)\frac{x}{t} \qquad (3)$$

where the similarity variable s is given by

$$s(x,t) := \frac{x}{\sqrt{t}}\,. \tag{4}$$

Show that the Navier-Stokes equation yields the following system of ordinary differential equations

$$s\frac{df}{ds}g + s\left(f - \frac{1}{2}\right)\frac{dg}{ds} + g(2f-1) = 0 \tag{5a}$$

$$2\frac{d^2 f}{ds^2} + \frac{df}{ds}\left(\frac{4}{s} + s(1-2f)\right) + 2f(1-f) = 2g\left(s\frac{dg}{ds} + g\right). \tag{5b}$$

(iii) Show that the continuity equation (1a) admits of a first integral, expressing the law of mass conservation, namely

$$s^2 g(2f-1) = C \tag{6}$$

where C is a constant. (iv) Show that g can be eliminated, and we obtain a second-order ordinary differential equation for the function f.

Solution 12. (ii) From (3) we obtain

$$\frac{\partial v}{\partial t} = \frac{df}{ds}\frac{ds}{dt}\frac{x}{t} - f\frac{x}{t^2}, \qquad \frac{\partial v}{\partial x} = \frac{df}{ds}\frac{ds}{dx}\frac{x}{t} + f\frac{1}{t} \tag{7a}$$

$$\frac{\partial c}{\partial t} = \frac{dg}{ds}\frac{ds}{dt}\frac{x}{t} - g\frac{x}{t^2}, \qquad \frac{\partial c}{\partial x} = \frac{dg}{ds}\frac{ds}{dx}\frac{x}{t} + g\frac{1}{t} \tag{7b}$$

and

$$\frac{\partial^2 v}{\partial x^2} = \frac{d^2 f}{ds^2}\left(\frac{ds}{dx}\right)^2\frac{x}{t} + \frac{df}{ds}\frac{d^2 s}{dx^2}\frac{x}{t} + \frac{df}{ds}\frac{ds}{dx}\frac{1}{t} + \frac{df}{ds}\frac{ds}{dx}\frac{1}{t}. \tag{8}$$

Since

$$\frac{ds}{dx} = \frac{1}{\sqrt{t}}, \qquad \frac{ds}{dt} = -\frac{1}{2}xt^{-3/2} \tag{9}$$

(5a) and (5b) follow.

Problem 13. The nonlinear partial differential equation

$$\frac{\partial^2 u}{\partial x_1 \partial x_2} + \frac{\partial u}{\partial x_1} + u^2 = 0 \tag{1}$$

describes the relaxation to a Maxwell distribution. The symmetry vector fields are given by

$$Z_1 = \frac{\partial}{\partial x_1}, \quad Z_2 = \frac{\partial}{\partial x_2}, \quad Z_3 = -x_1\frac{\partial}{\partial x_1} + u\frac{\partial}{\partial u}, \quad Z_4 = e^{x_2}\frac{\partial}{\partial x_2} - e^{x_2}u\frac{\partial}{\partial u}\,. \tag{2}$$

Construct a similarity ansatz from the symmetry vector field

$$Z = c_1 \frac{\partial}{\partial x_1} + c_2 \frac{\partial}{\partial x_2} + c_3 \left(-x_1 \frac{\partial}{\partial x_1} + u \frac{\partial}{\partial u} \right) \tag{3}$$

and find the corresponding ordinary differential equation. Here $c_1, c_2, c_3 \in \mathbb{R}$.

Solution 13. The corresponding initial value problem of the symmetry vector field Z is given by

$$\frac{dx_1'}{d\epsilon} = c_1 - c_3 x_1', \quad \frac{dx_2'}{d\epsilon} = c_2, \quad \frac{du'}{d\epsilon} = c_3 u' \,. \tag{4}$$

The solution to this system provides the transformation group

$$x_1'(\mathbf{x}, u, \epsilon) = \frac{c_1}{c_3} - \frac{c_1 - c_3 x_1}{c_3} e^{-c_3 \epsilon}, \quad x_2'(\mathbf{x}, u, \epsilon) = c_2 \epsilon + x_2, \quad x_3'(\mathbf{x}, u, \epsilon) = u e^{c_3 \epsilon} \tag{5}$$

where $c_3 \neq 0$. Now let $x_2 = s/c$ and $x_1 = 1$ with the constant $c \neq 0$. The *similarity variable* s follows as

$$s = c x_2' + \frac{c_2 c}{c_3} \ln \frac{c_3 x_1' - c_1}{c_3 - c_1} \tag{6}$$

and the *similarity ansatz* is

$$u'(x_1', x_2') = v(s) \frac{c_1 - c_3}{c_1 - c_3 x_1'} \,. \tag{7}$$

Inserting (7) into

$$\frac{\partial^2 u'}{\partial x_1' \partial x_2'} + \frac{\partial u'}{\partial x_1'} + u'^2 = 0 \tag{8}$$

leads to the ordinary differential equation

$$c \frac{d^2 v}{ds^2} + (1 - c) \frac{dv}{ds} - (1 - v) v = 0. \tag{9}$$

Problem 14. Consider the Kortweg de Vries equation

$$\frac{\partial u}{\partial t} - 6u \frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0 \,.$$

Insert the similarity solution

$$u(x,t) = \frac{w(s)}{(3t)^{2/3}}, \qquad s = \frac{x}{(3t)^{1/3}}$$

where s the similarity variable and find the ordinary differential equation.

Solution 14. We obtain

$$\frac{d^3w}{ds^3} - 6w\frac{dw}{ds} - 2w - s\frac{dw}{ds} = 0\,.$$

Problem 15. Let $\mathbf{u}$ be the velocity field and p the pressure. Show that the Navier-Stokes equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \Delta\mathbf{u} + \nabla p = \mathbf{0}, \quad \mathrm{div}\mathbf{u} = 0$$

admits the Lie symmetry vector fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x_1}, \quad \frac{\partial}{\partial x_2}, \quad \frac{\partial}{\partial x_3}$$

$$t\frac{\partial}{\partial x_1} + \frac{\partial}{\partial u_1}, \quad t\frac{\partial}{\partial x_2} + \frac{\partial}{\partial u_2}, \quad t\frac{\partial}{\partial x_3} + \frac{\partial}{\partial u_3}$$

$$x_1\frac{\partial}{\partial x_2} - x_2\frac{\partial}{\partial x_1} + u_1\frac{\partial}{\partial u_2} - u_2\frac{\partial}{\partial u_1},$$

$$x_2\frac{\partial}{\partial x_3} - x_3\frac{\partial}{\partial x_2} + u_2\frac{\partial}{\partial u_3} - u_3\frac{\partial}{\partial u_2},$$

$$x_3\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_3} + u_3\frac{\partial}{\partial u_1} - u_1\frac{\partial}{\partial u_3}$$

$$2t\frac{\partial}{\partial t} + x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2} + x_3\frac{\partial}{\partial x_3} - u_1\frac{\partial}{\partial u_1} - u_2\frac{\partial}{\partial u_2} - u_3\frac{\partial}{\partial u_3} - 2p\frac{\partial}{\partial p}\,.$$

Find the commutators of the vector fields and thus show that we have a basis of a Lie algebra. Show that $\sqrt{(x_1^2+x_2^2)}/x_3$ is a similarity variable. Do which vector fields does it belong.

Solution 15.

$$=$$

Problem 16. The AB system in its canonical form is given by the coupled system of partial differential equations

$$\frac{\partial^2 Q}{\partial\xi\partial\eta} = QS, \quad \frac{\partial S}{\partial \xi} = -\frac{1}{2}\frac{\partial |Q|^2}{\partial \eta}$$

where ξ and η are semi-characteristic coordinates, Q (complex valued) and S are the wave amplitudes satisfying the normalization condition

$$\left|\frac{\partial Q}{\partial \eta}\right|^2 + S^2 = 1\,.$$

Show that these system of partial differential equations can be written as a compatibility condition

$$\frac{\partial^2 \psi_{1,2}}{\partial \xi \partial \eta} = \frac{\partial^2 \psi_{1,2}}{\partial \eta \partial \xi}$$

of two linear systems of partial differential equations

$$\frac{\partial \psi_1}{\partial \xi} = F\psi_1 + G\psi_2, \quad \frac{\partial \psi_1}{\partial \eta} = A\psi_1 + B\psi_2$$

$$\frac{\partial \psi_2}{\partial \xi} = H\psi_1 - F\psi_2, \quad \frac{\partial \psi_2}{\partial \eta} = C\psi_1 - A\psi_2\,.$$

Solution 16. From the compatibility condition we obtain the set of partial differential equation

$$\frac{\partial F}{\partial \eta} - \frac{\partial A}{\partial \xi} + CG - BH = 0$$
$$\frac{\partial G}{\partial \eta} - \frac{\partial B}{\partial \xi} + 2(BF - AG) = 0$$
$$\frac{\partial H}{\partial \eta} - \frac{\partial C}{\partial \xi} + 2(AH - CF) = 0\,.$$

If we set

$$A = -\frac{1}{4i\mu}, \quad B = \frac{1}{4i\mu}\frac{\partial Q}{\partial \eta}, \quad C = \frac{1}{4i\mu}\frac{\partial Q^*}{\partial \eta}$$

$$F = -i\mu, \quad G = \frac{1}{2}Q, \quad H = -\frac{1}{2}Q^*$$

where μ is the spectral parameter.

Problem 17. Show that the system of partial differential equations

$$\frac{\partial u_1}{\partial t} = \frac{\partial u_2}{\partial y}$$
$$\frac{\partial u_2}{\partial t} = \frac{1}{3}\frac{\partial^3 u_1}{\partial y^3} + \frac{8}{3}u_1\frac{\partial u_1}{\partial y}$$

admits the (scaling) Lie symmetry vector field

$$S = -2t\frac{\partial}{\partial t} - y\frac{\partial}{\partial y} + 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2}\,.$$

Solution 17.

Problem 18. Show that the partial differential equation

$$\frac{\partial u}{\partial t}\left(\frac{\partial u}{\partial x}\right)^2 = \frac{\partial^2 u}{\partial x^2}$$

admits the Lie symmetry vector field

$$V = -\frac{\partial}{\partial t} - \frac{1}{2}xu\frac{\partial}{\partial x} + t\frac{\partial}{\partial u}\,.$$

Solution 18.

Problem 19. Consider the Kuramoto-Sivashinsky equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + a\frac{\partial^2 u}{\partial x^2} + b\frac{\partial^4 u}{\partial x^4} = 0\,.$$

Show that the equation is invariant under the *Galilean transformation*

$$(u, x, t) \mapsto (u + c, x - ct, t)\,.$$

Solution 19.

Problem 20. Consider the nonlinear partial differential equation

$$\frac{\partial^3 u}{\partial x^3} + u\left(\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x}\right) = 0$$

where c is a constant. Show that the partial differential equation admits the Lie symmetry vector fields

$$V_1 = \frac{\partial}{\partial t}, \quad V_2 = \frac{\partial}{\partial x},$$

$$V_3 = 3t\frac{\partial}{\partial t} + (x + 2ct)\frac{\partial}{\partial x}, \quad V_4 = t\frac{\partial}{\partial t} + ct\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}.$$

Solution 20. Consider the vector field V_4. The autonomous system of first order differential equation corresponding to the vector field is

$$\frac{dt'}{d\epsilon} = t', \quad \frac{dx'}{d\epsilon} = ct', \quad \frac{du'}{d\epsilon} = u'.$$

The solution of the initial value problem is

$$\begin{aligned} t'(\epsilon) &= e^{\epsilon} t'(0) \\ x'(\epsilon) &= (e^{\epsilon} - 1)ct'(0) + x'(0) \\ u'(\epsilon) &= e^{\epsilon} u'(0). \end{aligned}$$

This leads to the transformation

$$\begin{aligned} t'(x,t) &= e^{\epsilon} t \\ x'(x,t) &= (e^{\epsilon} - 1)ct + x \\ u'(x'(x,t), t'(x,t)) &= e^{\epsilon} u(x,t) \end{aligned}$$

Then since $\partial t'/\partial x = 0$, $\partial x'/\partial x = 1$ we have

$$\begin{aligned} \frac{\partial u'}{\partial x} &= \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial x} = e^{\epsilon}\frac{\partial u}{\partial x} \\ &= \frac{\partial u'}{\partial x'} = e^{\epsilon}\frac{\partial u}{\partial x} \end{aligned}$$

$$\begin{aligned} \frac{\partial u'}{\partial t} &= \frac{\partial u'}{\partial t'}\frac{\partial t'}{\partial t} + \frac{\partial u'}{\partial x'}\frac{\partial x'}{\partial t} = e^{\epsilon}\frac{\partial u}{\partial t} \\ &= \frac{\partial u'}{\partial t'}e^{\epsilon} + \frac{\partial u'}{\partial x'}(e^{\epsilon} - 1)c = e^{\epsilon}\frac{\partial u}{\partial t} \end{aligned}$$

$$\begin{aligned} \frac{\partial^2 u'}{\partial x^2} &= \frac{\partial^2 u'}{\partial x'^2}\frac{\partial x'}{\partial x} + \frac{\partial^2 u'}{\partial t' \partial x'}\frac{\partial t'}{\partial x} = e^{\epsilon}\frac{\partial^2 u}{\partial x^2} \\ &= \frac{\partial^2 u'}{\partial x'^2} = e^{\epsilon}\frac{\partial^2 u}{\partial x^2} \end{aligned}$$

$$\begin{aligned} \frac{\partial^3 u'}{\partial x^3} &= \frac{\partial^3 u'}{\partial x'^3}\frac{\partial x'}{\partial x} = e^{\epsilon}\frac{\partial^3 u}{\partial x^3} \\ &= \frac{\partial^3 u'}{\partial x'^3} = e^{\epsilon}\frac{\partial^3 u}{\partial x^3}. \end{aligned}$$

Inserting these expression into the partial differential equation yields

$$e^{-\epsilon}\frac{\partial^3 u'}{\partial x'^3} + e^{-\epsilon}u'\left(\frac{\partial u'}{\partial t'} + e^{-\epsilon}\frac{\partial u'}{\partial x'}(e^{\epsilon} - 1)c + ce^{-\epsilon}\frac{\partial u'}{\partial x'}\right) = 0.$$

Thus

$$e^{-\epsilon}\frac{\partial^3 u'}{\partial x'^3} + e^{-\epsilon}u'\left(\frac{\partial u'}{\partial t'} + c\frac{\partial u'}{\partial x'}\right) = 0$$

with $e^{-\epsilon} \neq 0$.

Problem 21. Consider the stationary incompressible *Prandtl boundary layer equation*

$$\frac{\partial^3 u}{\partial \eta^3} = \frac{\partial u}{\partial \eta}\frac{\partial^2 u}{\partial \eta \partial \xi} - \frac{\partial u}{\partial \xi}\frac{\partial^2 u}{\partial \eta \partial \xi}.$$

Using the classical Lie method we obtain the similarity reduction

$$u(\xi, \eta) = \xi^\beta y(x), \qquad x = \eta\xi^{\beta-1} + f(\xi)$$

where f is an arbitrary differentiable function of ξ. Find the ordinary differential equation for y.

Solution 21. Differentiation and applying the chain rule provides

$$\frac{d^3y}{dx^3} = \beta y\frac{d^2y}{dx^2} - (2\beta - 1)\left(\frac{dy}{dx}\right)^2$$

which, in the special case $\beta = 2$, is the Chazy equation.

Problem 22. Show that the *Chazy equation*

$$\frac{d^3y}{dx^3} = 2y\frac{d^2y}{dx^2} - 3\left(\frac{dy}{dx}\right)^2$$

admits the vector fields

$$\frac{\partial}{\partial x}, \qquad x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}, \qquad x^2\frac{\partial}{\partial x} - (2xy + 6)\frac{\partial}{\partial y}$$

as symmetry vector fields. Show that the first two symmetry vector fields can be used to reduce the Chazy equation to a first order equation. Find the commutators and classify the Lie algebra.

Solution 22.

Problem 23. Show that the *Laplace equation*

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)u = 0$$

admits the Lie symmetries

$$P_x = \frac{\partial}{\partial x}, \quad P_y = \frac{\partial}{\partial y}, \quad P_z = \frac{\partial}{\partial z}$$

$$M_{yx} = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}, \quad M_{xz} = x\frac{\partial}{\partial z} - z\frac{\partial}{\partial x}, \quad M_{zy} = z\frac{\partial}{\partial y} - y\frac{\partial}{\partial z}$$

$$D = -\left(\frac{1}{2} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)$$

$$K_x = -2xD - r^2\frac{\partial}{\partial x}, \quad K_y = -2yD - r^2\frac{\partial}{\partial y}, \quad K_z = -2xD - r^2\frac{\partial}{\partial z}$$

where $r^2 := x^2 + y^2 + z^2$.

Solution 23.

Problem 24. Find the Lie symmetry vector fields for the Monge-Ampére equations

$$\frac{\partial^2 u}{\partial t^2}\frac{\partial^2 u}{\partial x^2} - \left(\frac{\partial^2 u}{\partial x \partial t}\right)^2 = -K$$

where $K = +1$, $K = 0$, $K = -1$.

Solution 24.

Problem 25. Let $\mathbf{u} = (u_1, u_2, u_3)^T$, $\mathbf{v} = (v_1, v_2, v_3)^T$ and $u_1^2+u_2^2+u_3^2 = 1$, $v_1^2 + v_2^2 + v_3^2 = 1$. Let $\times$ be the vector product. The $O(3)$ chiral field equations are the system of partial differential equations

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{\partial \mathbf{u}}{\partial x} + \mathbf{u} \times R\mathbf{v} = \mathbf{0}, \qquad \frac{\partial \mathbf{v}}{\partial t} - \frac{\partial \mathbf{v}}{\partial x} + \mathbf{v} \times R\mathbf{u} = \mathbf{0}$$

where R is a 3×3 diagonal matrix with non-negative entries. Show that this system of partial differential equations admits a Lax pair (as 4×4 matrices) L, M, i.e. $[L, M] = 0$.

Solution 25.

$$=$$

Problem 26. Consider the Schrödinger-Newton equation (MKSA-system)

$$-\frac{\hbar^2}{2m}\Delta\psi + U\psi = \mu\psi$$

$$\Delta U = 4\pi G m^2 |\psi|^2$$

where m is the mass, G the gravitational constant and μ the energy eigenvalues. The normalization consition is

$$\int_{\mathbb{R}^3} |\psi(x_1, x_2, x_3)|^2 dx_1 dx_2 dx_3 = 1\,.$$

Write the Schrödinger-Newton equation in dimension less form. Then find the Lie symmetries of this system of partial differential equation.

Solution 26.

Problem 27. Show that the one-dimensional wave equation

$$\frac{1}{c^2}\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

is invariant under

$$\begin{pmatrix} x'(x,t) \\ ct'(x,t) \end{pmatrix} = \begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix} \begin{pmatrix} x \\ ct \end{pmatrix}$$

$$u'(x'(x,t), t'(x,t)) = u(x,t).$$

Solution 27.

Problem 28. Study the system of nonlinear partial differential equation

$$\left(\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial s^2}\right) u_j = -4 \sum_{\ell=1}^{2} K_{j\ell} \exp(u_\ell)$$

with $j = 1, 2$ and K is the Cartan matrix

$$K = \begin{pmatrix} 2 & -1 \\ 1 & 2 \end{pmatrix}.$$

Solution 28.

Problem 29. The partial differential equation

$$\frac{\partial^2 u}{\partial x \partial y} = e^u - e^{-2u}$$

is called the Tzitzeica equation. Applying $v(x,y) = e^{u(x,t)}$ show that the equation takes the form

$$v\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial v}{\partial x}\frac{\partial v}{\partial y} = v^3 - 1.$$

Find the Lie symmetries of both equations.

Solution 29.

Problem 30. Find the Lie symmetry vector fields for the one-dimensional telegraph equation

$$\frac{\partial^2 u}{\partial t^2} + (\alpha + \beta)\frac{\partial u}{\partial t} + \alpha\beta u = c^2 \frac{\partial^2 u}{\partial x^2}$$

where $c^2 = 1/(LC)$, $\alpha = G/C$, $\beta = R/L$ and G is the conductance.

Problem 31. Find the Lie symmetry vector fields of the Thomas equation

$$\frac{\partial^2 u}{\partial x \partial t} + \alpha \frac{\partial u}{\partial t} + \beta \frac{\partial u}{\partial x} + \gamma \frac{\partial u}{\partial x}\frac{\partial u}{\partial t} = 0.$$

Problem 32. The *Harry Dym equation* is given by

$$\frac{\partial u}{\partial t} - u^3 \frac{\partial^3 u}{\partial x^3} = 0.$$

Show that it admits the Lie symmetry vector fields

$$V_1 = \frac{\partial}{\partial x}, \qquad V_2 = \frac{\partial}{\partial t}$$

$$V_3 = x\frac{\partial}{\partial x} + u\frac{\partial}{\partial u}, \quad V_4 = -3t\frac{\partial}{\partial t} + u\frac{\partial}{\partial u}, \quad V_5 = x^2\frac{\partial}{\partial x} + 2xu\frac{\partial}{\partial u}.$$

Is the Lie algebra spanned by these generators semi-simple?

Bibliography

Books

Courant R. and Hilbert D.
Methods of Mathematical Physics, Volume 1, Wiley-VCH Verlag (2004)

Courant R. and Hilbert D.
Methods of Mathematical Physics, Volume 2, Wiley-VCH Verlag (2008)

DiBenedetto E.
Partial Differential Equations, second edition, Springer (2010)

John F.
Partial Differential Equations, Volume I, Springer 1982

Sneddon I.
Elements of Partial Differential Equations, McGraw-Hill (1957)

Steeb W.-H.
The Nonlinear Workbook, fourth edition, World Scientific Publishing, Singapore, 2008

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, Second Edition, Volume I: Introductory Level
World Scientific Publishing, Singapore (2003)

Steeb W.-H.
Problems and Solutions in Theoretical and Mathematical Physics, Second Edition, Volume II: Adanced Level
World Scientific Publishing, Singapore (2003)

Papers

Ouroushev D., Martinov N. and Grigorov A., "An approach for solving the two-dimensional sine-Gordon equation", J. Phys. A: Math. Gen. **24**, L527-528 (1991)

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