Math 21 - Spring 2014
Classnotes, Week 8

This week we will talk about solutions of homogeneous linear differential equations. This material doubles as an introduction to **linear algebra**, which is the subject of the first part of Math 51.

We will also use Taylor series to solve differential equations. This material is covered in a handout, Series Solutions for linear equations, which is posted both under "Resources" and "Course schedule".

8.1 Solutions of homogeneous linear differential equations

We discussed first-order linear differential equations before Exam 2. We will now discuss linear differential equations of arbitrary order.

Definition 8.1. A **linear differential equation of order** n is an equation of the form

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = Q(x),$$

where each P_k and Q is a function of the independent variable x, and as usual $y^{(k)}$ denotes the kth derivative of y with respect to x.

Remark. This is the analogue of the definition we gave in the case of a first-order linear differential equation. In a first-order linear equation, we said that only y and y' can appear, and no functions of y and y', and y and y' cannot be multiplied together. Now that we wish to allow the equation to be of order n, we want that only $y^{(n)}, \ldots, y', y$ appear, and no functions of $y^{(n)}, \ldots, y', y$, and no two or more of these are multiplied together. Any differential equation for which that is true can be put in the form above.

Definition 8.2. A **homogeneous** linear differential equation of order n is an equation of the form

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0.$$

Remark. In other words, "homogeneous" just means that $Q(x) = 0$.

The reason we are interested in solving linear differential equations is simple: they are both interesting (they come up in nature often) and easy enough that we have some hope of solving them. In general, solving differential equations is extremely difficult. Even in the case of first-order equations, there is no method to systematically solve differential equations (in other words, there is no method that always works; we have to rely on tricks that work in specific cases). When we move on to higher order equations, the situation becomes even more hopeless. In the face of insurmountable difficulty, we choose to focus on the narrow class of equations which we have some hope of solving. These are the linear differential equations.

The reason we are interested more specifically in solving homogeneous linear differential equations is that whenever one needs to solve a nonhomogeneous linear differential equation, one must first solve the associated homogeneous differential equation. The reason why this is true is not very complicated and you can read about it online or in a differential equations textbook. This is material covered in Math 53.

The solutions to a homogeneous linear differential equation have a bunch of really great properties:

8.1.1 Multiplication property

If a function f is a solution to the equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

then Cf for any constant C is also a solution to the equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0.$$

Example: Consider the second-order differential equation

$$y'' + 9y = 0.$$

One can check that $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \cos(3x)$ is a solution to this differential equation.

Then any multiple of f is also a solution to this differential equation. For example, $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = 2\cos(3x)$ is also a solution (take a minute to check this!). In fact, for C an arbitrary constant, the function $h\colon \mathbb{R} \to \mathbb{R}$ given by the rule $h(x) = C\cos(3x)$ will always be a solution of the differential equation.

8.1.2 Addition property

If two functions f and g are solutions to the equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

then the function $f + g$ is also a solution to the equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0.$$

Example: Consider again the second-order differential equation

$$y'' + 9y = 0.$$

Again, $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \cos(3x)$ is a solution to this differential equation. It is also true that $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = \sin(3x)$ is a solution to the differential equation.

Then we have also that $h\colon \mathbb{R} \to \mathbb{R}$ given by the rule $h(x) = \cos(3x) + \sin(3x)$ is a solution to the differential equation (again, take a minute to check this!).

8.1.3 Principle of superposition

Combining the two principles above, we have that if $f_1, f_2, \ldots f_k$ are all solutions to the differential equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0,$$

then for any constants $C_1, C_2, \ldots C_k$, the function $C_1f_1 + C_2f_2 + \ldots + C_kf_k$ is also a solution to the equation

$$P_n(x)y^{(n)} + P_{n-1}(x)y^{(n-1)} + \ldots + P_1(x)y' + P_0(x)y = 0.$$

Example: Consider again the second-order differential equation

$$y'' + 9y = 0.$$

Since $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \cos(3x)$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = \sin(3x)$ are both solutions to this differential equation, it is also true that for any constants C_1 and C_2, the function $h\colon \mathbb{R} \to \mathbb{R}$ given by the rule $h(x) = C_1\cos(3x) + C_2\sin(3x)$ is also a solution to the differential equation.

We can now give many solutions to the differential equation

$$y'' + 9y = 0.$$

A few of them are given by the rules $\cos(3x) - \sin(3x)$, $2\cos(3x) + 10\sin(3x)$, $-\pi\cos(3x) + \frac{1}{2}\sin(3x)$. (Pick one of these and check that this is true! This is how you should read math from now on: every time someone says something is true, you should check it for yourself. It's the best way to read math and make sure that you understand what is being discussed. We will stop telling you to check everything now, but you should always do it.)

8.1.4 General solution

The principles above tell us how to find more solutions of a homogeneous linear differential equation once we have one or more solutions. This last principle tells you when you have *all* of the solutions to a homogeneous linear differential equation.

Theorem 8.3. *Given a homogeneous linear differential equation of order n, one can find n* ***linearly independent*** *solutions. Furthermore, these n solutions along with the solutions given by the principle of superposition are* ***all*** *of the solutions of the differential equation.*

We will not go into the definition of linear independence in this class (but it is in the optional section below). Instead, for simplicity we will say that two (or more) functions are linearly independent if they are "different." For example, $\cos x$ and $\sin x$ are linearly independent. Also, e^x and e^{-x} are linearly independent.

However, $\cos x$ and $2\cos x$ are not linearly independent: they are not different from each other; one is just a multiple of the other. Sometimes it can be tricky to tell if two functions are linearly independent or not: For example, $\cos x$ and $\cos(-x)$ look different. But in fact, $\cos(-x) = \cos x$ (remember that $\cos x$ is even), so really $\cos x$ and $\cos(-x)$ are not only linearly dependent, but they are the same! Similarly, $\sin x$ and $\sin(-x)$ are not linearly independent since $\sin(-x) = -\sin x$, which is a multiple of $\sin x$. In Math 53 you will learn how to tell if two or more functions are linearly independent using a mathematical tool called the Wronskian.

Example: Consider once more the second-order differential equation

$$y'' + 9y = 0.$$

This is a homogeneous linear differential equation of order 2. Therefore, if we can find two linearly independent solutions, and use the principle of superposition, we will have all of the solutions of the differential equation.

We already know from above that $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = \cos(3x)$ and $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = \sin(3x)$ are solutions to the differential equation. $\cos(3x)$ and $\sin(3x)$ are linearly independent (they are "different"). In addition, they are 2 functions and the differential equation is of order 2. Therefore, by using the principle of superposition we now have a general solution to the differential equation which we know contains all of the solutions of the differential equation:

$$y = C_1 \cos(3x) + C_2 \sin(3x), \qquad \text{for } C_1 \text{ and } C_2 \text{ arbitrary constants.}$$

Another Example: Consider the third-order homogeneous linear differential equation

$$y''' + y'' - 2y = 0.$$

It is true that $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule $f(x) = e^x$, $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = e^{-x}\cos x$, and $h\colon \mathbb{R} \to \mathbb{R}$ given by the rule $h(x) = e^{-x}\sin x$ are all solutions to this differential equation. Furthermore, f, g, and h are linearly independent. Finally, we have 3 linearly independent solutions and the differential equation is of order 3.

Because of all this, we can say that the general solution to the differential equation is

$$y = C_1 e^x + C_2 e^{-x}\cos x + C_3 e^{-x}\sin x, \qquad \text{for } C_1,\ C_2, \text{ and } C_3 \text{ arbitrary constants,}$$

and this contains all solutions to the differential equation.

Remark. In this class we will not learn how to get the solutions that serve as building blocks for the general solution. In all of the cases presented here, finding the solutions is not very difficult; you could easily read about this online or in a differential equations textbook. This is material covered in Math 53.

Non-example: The first-order differential equation

$$y' = y\left(1 - \frac{y}{4}\right)$$

is *not* linear. (There is a y^2 when we multiply out.)

A solution to this differential equation is $f\colon \mathbb{R} \to \mathbb{R}$ given by the rule

$$f(x) = \frac{4}{1+e^{-x}}.$$

It is *not true* that a multiple of this function is also a solution to the differential equation! (For example, you can check that $g\colon \mathbb{R} \to \mathbb{R}$ given by the rule $g(x) = \frac{8}{1+e^{-x}}$, which is 2 times f, is not a solution to the differential equation.)

From before, we know that the general solution of this differential equation is

$$y = \frac{4}{1+Ce^{-x}}, \qquad \text{for } C \text{ a constant.}$$

(The differential equation is separable and we can solve it using a technique we have learned.) When we vary C, we do not get solutions that are multiples of each other.

Even worse, this general solution does *not* give us *all* of the solutions! You can check that the function $h\colon \mathbb{R} \to \mathbb{R}$ given by the rule $h(x) = 0$ (the stable equilibrium solution) does not belong to this family of functions. (In other words, there is no choice of C in the general solution above which will give us the solution h.)

Hopefully this non-example convinces you that homogeneous linear differential equations are very special. The discussion of when a general solution is a **complete** solution (i.e. gives *all* solutions to a differential equation) is beyond the scope of this class. This is material that is covered in Math 53.

8.2 Initial value problems

When we solved a first-order differential equation, we needed a single initial value to determine the value of the single unknown constant in our general solution.

When we solve a homogeneous linear differential equation of order n, we will have n different constants in our general solution. For this reason, we will need n initial values to find the solution to a given initial value problem.

Example: Consider the initial value problem

$$y'' + 9y = 0, \qquad y(0) = 1, \qquad y'(0) = -6.$$

Using the fact that the general solution to the differential equation is

$$y = C_1\cos(3x) + C_2\sin(3x), \qquad \text{for } C_1 \text{ and } C_2 \text{ arbitrary constants,}$$

find the solution to the initial value problem.

Answer: It is simply a matter of solving for C_1 and C_2.

We first use the first initial value:

$$1 = y(0) = C_1 \cos(0) + C_2 \sin(0) = C_1.$$

So $C_1 = 1$.

To use the second initial value, we must first take a derivative of the general solution:

$$y' = -3C_1 \sin(3x) + 3C_2 \cos(3x).$$

We can now use the second initial value:

$$-6 = y'(0) = -3C_1 \sin(0) + 3C_2 \cos(0) = 3C_2,$$

and $C_2 = -2$.

The solution to the initial value problem is therefore the function $y\colon \mathbb{R} \to \mathbb{R}$ given by the rule $y = \cos(3x) - 2\sin(3x)$.

8.3 Optional material (prep for Math 51)

You will not be tested on the material in this section. However, if you plan to take Math 51 you can read this to get exposed to some of the new concepts presented in Math 51. Alternatively, if you wonder what a football team and a vector space have in common, you should also read on.

At first glance, the beginning of Math 51 will look nothing like Math 21. There will be no discussion of differential equations at all. Despite this, the underlying mathematical structures that are studied in the linear algebra portion of Math 51 are the same as the mathematical structures which we are studying this week. This is the great power of mathematics: by focusing only on certain aspects of the problem, one can leverage knowledge in one area of study to knowledge in another area of study.

8.3.1 Vector spaces

The main object of study of the first part of Math 51 is what mathematicians call linear algebra. A basic structure which is studied in linear algebra is a kind of thing called a **vector space**. A vector space over the real numbers is a bunch of things (called the elements of the vector space) which satisfy the following properties:

- Any two elements in the vector space can be added together to get a third element in the vector space

- Addition is associative (i.e. if f, g, and h are in the vector space, then $(f + g) + h = f + (g + h)$)

- Addition is commutative (i.e. if f and g are in the vector space, then $f + g = g + f$)

- There is a special element of the vector space, usually called 0, which acts as the identity for addition (i.e. for any f in the vector space, $f + 0 = f$)

- Any element of the vector space has an additive inverse (i.e. each element f has a special friend element which we usually call $-f$, such that $f + (-f) = 0$, where 0 is the identity element defined above)
- Any one element in the vector space can be multiplied by a real number to get another element in the vector space
- Multiplication by the real number 0 gives the special element 0 in the vector space (i.e. if f is an element of the vector space, then $0 \cdot f = 0$)
- Multiplication by the real number 1 does not change the element (i.e. if f is an element of the vector space, then $1 \cdot f = f$)
- Multiplication within real numbers is compatible with multiplying an element of the vector space (i.e. if r and s are real numbers and f is an element of the vector space, then $(rs) \cdot f = r \cdot (s \cdot f)$)
- Multiplication by a real number is distributive (i.e. if r is a real number and f and g are elements of the vector space, then $r \cdot (f + g) = r \cdot f + r \cdot g$, and also if r and s are real numbers and f is an element of the vector space, then $(r + s) \cdot f = r \cdot f + s \cdot f$)

An example of a vector space is the space of all solutions to a given homogeneous linear differential equation. The elements of this vector space are the solutions to the differential equation. If you think about it for a little bit, you will see that everything I have written above is true for solutions of a homogeneous linear differential equation (i.e. you can add two solutions and get a third solution, you can multiply a solution by a constant (a real number) and get another solution, etc.).

So this list of properties might seem daunting at first, but it is just a way to write down everything we know to be true about the solutions of homogeneous linear differential equation. In other words, we abstract from some object (the space of solutions of a homogeneous linear differential equation) its main properties, so that we can study any object that has the same properties. As I have said above, this is the great power of mathematics, but it is also just the great power of human beings.

For example, you might have come across football teams in your life. Maybe your high school had a football team. This team had certain properties (they all wear the same jersey, they run towards the end zone, etc.). If you go to visit your cousin in another city, and you see a group of people doing things similar to what your high school's football team does, then you might think that what you are looking at is also a football team. You realize that this is a *different* football team, but it is also, in some ways, the *same* as the football team from your high school. From your knowledge of the football team of your own high school, you already know some things about this new football team. You know there is a special player who is the quarterback, and you know what to expect from this player. (Of course, all players on the team are special. For that matter, all sports and all extracurricular activities are special.) You know that all of the players will wear the same jersey, and they will all run towards the end zone. Your power of abstracting from one football team (the one you know from your high school) to another football team (the one at your cousin's high school) allows you to get a lot of information "for free." The *concept* of a football team is a useful concept.

Similarly, the concept of a vector space is useful to mathematicians. Once we see a set of objects that satisfy certain properties, and recognize this set as a vector space, we know what to expect from these objects. We know what we will be able to do, and certain things which will be true. The vector spaces you will see in Math 51 will not be vector spaces of functions, like ours. However, by learning about those vector spaces you will also be learning certain things about our vector spaces. In Math 53 after you know more about vector spaces this will all be made very explicit.

8.3.2 More rigorous definitions

Definition 8.4. Suppose that $f_1, f_2, \ldots, f_k$ are k elements of a real vector space. We say that these elements are **linearly independent** if whenever $r_1, r_2, \ldots, r_k$ are k real numbers, the equation

$$r_1 f_1 + r_2 f_2 + \ldots + r_k f_k = 0$$

has as its only solution the solution $r_1 = 0$, $r_2 = 0$, ... $r_k = 0$.

So when we say for example that $\cos(3x)$ and $\sin(3x)$ are linearly independent, we are saying that the only way to make sure that

$$r_1 \cos(3x) + r_2 \sin(3x) = 0$$

for every value of x, is to pick $r_1 = 0$ and $r_2 = 0$.

In contrast, $\cos(3x)$ and $2\cos(3x)$ are not linearly independent. If we want to make

$$r_1 \cos(3x) + r_2(2\cos(3x)) = 0,$$

we can choose $r_1 = 2$ and $r_2 = -1$, and the equation will be true for any value of x.

In the discussion above, we said that we could use the principle of superposition to make more solutions. In the context of vector space, this has a fancy name:

Definition 8.5. If $r_1, r_2, \ldots, r_k$ are k real numbers and $f_1, f_2, \ldots, f_k$ are k elements of a real vector space, then we call

$$r_1 f_1 + r_2 f_2 + \ldots + r_k f_k$$

a **linear combination** of the elements $f_1, f_2, \ldots, f_k$.

Rephrasing what we have said above, we can now say that if we have some solutions to a homogeneous linear differential equation, then any linear combination of these solutions is also a solution to the differential equation.

Definition 8.6. If $f_1, f_2, \ldots, f_k$ are k elements of a real vector space, then all of elements that can be obtained from linear combinations of these elements are called the **span** of these elements.

Definition 8.7. Suppose that within a vector space there are n elements $f_1, f_2, \ldots, f_n$ such that

- the n elements are linearly independent
- the whole vector space is spanned by the elements $f_1, f_2, \ldots, f_n$ (i.e. any element in the vector space can be written as a linear combination of the elements $f_1, f_2, \ldots, f_n$).

Then we say that $f_1, f_2, \ldots, f_n$ is a **basis** of the vector space. Furthermore, we say that the vector space has **dimension** n.

Rephrasing what we have said above, we can now say that that the set of solutions of a homogeneous linear differential equation of order n forms a vector space of dimension n, the order of the differential equation. And, as it turns out, this is pretty awesome.