# Chapter 3

# Kinematics in 2-D (and 3-D)

From *Problems and Solutions in Introductory Mechanics* (Draft version, August 2014)
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## 3.1 Introduction

In this chapter, as in the previous chapter, we won't be concerned with the actual forces that cause an object to move the way it is moving. We will simply take the motion as given, and our goal will be to relate positions, velocities, and accelerations as functions of time. However, since we are now dealing with more general motion in two and three dimensions, we will give one brief mention of forces:

#### Motion in more than one dimension

Newton's second law (for objects with constant mass) is $\mathbf{F} = m\mathbf{a}$, where $\mathbf{a} \equiv d\mathbf{v}/dt$. This law (which is the topic of Chapter 4) is a *vector* equation. (See Appendix A in Section 13.1 for a review of vectors.) So it really stands for three different equations: $F_x = ma_x$, $F_y = ma_y$, and $F_z = ma_z$. In many cases, these three equations are "decoupled," that is, the $x$ equation has nothing to do with what is going on in the $y$ and $z$ equations, etc. In such cases, we simply have three copies of 1-D motion (or two copies if we're dealing with only two dimensions). So we just need to solve for the three *independent* motions along the three coordinate axes.

#### Projectile motion

The classic example of independent motions along different axes is projectile motion. Projectile motion is the combination of two separate linear motions. The horizontal motion doesn't affect the vertical motion, and vice versa. Since there is no acceleration in the horizontal direction (ignoring air resistance), the projectile moves with constant velocity in the $x$ direction. And since there is an acceleration of $-g$ in the vertical direction, we can simply copy the results from the previous chapter (in particular, Eq. (2.3) with $a_y = -g$) for the motion in the $y$ direction. We therefore see that if the initial position is $(X, Y)$ and the initial velocity is $(V_x, V_y)$, then the acceleration components

$$a_x = 0 \qquad \text{and} \qquad a_y = -g \tag{3.1}$$

lead to velocity components

$$v_x(t) = V_x \qquad \text{and} \qquad v_y(t) = V_y - gt \tag{3.2}$$

and position components

$$x(t) = X + V_x t \qquad \text{and} \qquad y(t) = Y + V_y t - \frac{1}{2}gt^2. \tag{3.3}$$

Projectile motion is completely described by these equations for the velocity and position components.

#### Standard projectile results

The initial velocity **V** of a projectile is often described in terms of the initial speed $v_0$ (we'll use a lowercase $v$ here, since it looks a little nicer) and the launch angle $\theta$ with respect to the horizontal. From Fig. 3.1, the initial velocity components are then $V_x = v_0 \cos\theta$ and $V_y = v_0 \sin\theta$, so the velocity components in Eq. (3.2) become

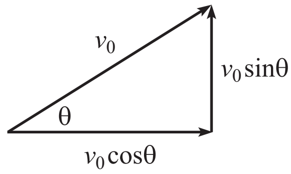


**Figure 3.1**

$$v_x(t) = v_0 \cos\theta \qquad \text{and} \qquad v_y(t) = v_0 \sin\theta - gt, \tag{3.4}$$

and the positions in Eq. (3.3) become (assuming that the projectile is fired from the origin, so that $(X, Y) = (0, 0)$)

$$x(t) = (v_0 \cos\theta)t \qquad \text{and} \qquad y(t) = (v_0 \sin\theta)t - \frac{1}{2}gt^2. \tag{3.5}$$

A few results that follow from these expressions are that the time to the maximum height, the maximum height attained, and the total horizontal distance traveled are given by (see Problem 3.1)

$$t_{\text{top}} = \frac{v_0 \sin\theta}{g}, \qquad y_{\text{max}} = \frac{v_0^2 \sin^2\theta}{2g}, \qquad x_{\text{max}} = \frac{2v_0^2 \sin\theta\cos\theta}{g} = \frac{v_0^2 \sin 2\theta}{g}. \tag{3.6}$$

The last of these results holds only if the ground is level (more precisely, if the projectile returns to the height from which it was fired). As usual, we are ignoring air resistance.

#### Motion along a plane

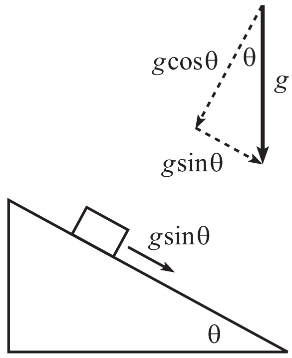


**Figure 3.2**

If an object slides down a frictionless plane inclined at angle $\theta$, the acceleration down the plane is $g \sin\theta$, because the component of **g** (the downward acceleration due to gravity) that points along the plane is $g \sin\theta$; see Fig. 3.2. (There is no acceleration perpendicular to the plane because the normal force from the plane cancels the component of the gravitational force perpendicular to the plane. We'll discuss forces in Chapter 4.) Even though the motion appears to take place in 2-D, we really just have a (tilted) 1-D setup. We effectively have "freefall" motion along the tilted axis, with the acceleration due to gravity being $g \sin\theta$ instead of $g$. If $\theta = 0$, then the $g \sin\theta$ acceleration along the plane equals 0, and if $\theta = 90°$ it equals $g$ (downward), as expected.

More generally, if a projectile flies through the air above an inclined plane, the object's acceleration (which is the downward-pointing vector **g**) can be viewed as the sum of its components along any choice of axes, in particular the $g \sin\theta$ acceleration along the plane and the $g \cos\theta$ acceleration perpendicular to the plane. This way of looking at the downward **g** vector can be very helpful when solving projectile problems involving inclined planes. See Section 13.1.5 in Appendix A for further discussion of vector components.

#### Circular motion

Another type of 2-D motion is circular motion. If an object is moving in a circle of radius $r$ with speed $v$ at a given instant, then the (inward) radial component of the acceleration vector **a** equals (see Problem 3.2(a))

$$a_{\text{r}} = \frac{v^2}{r}. \tag{3.7}$$

This radially inward acceleration is called the *centripetal* acceleration. If additionally the object is speeding up or slowing down as it moves around the circle, then there is also a tangential component of **a** given by (see Problem 3.2(b))

$$a_{\text{t}} = \frac{dv}{dt}. \tag{3.8}$$

This tangential component is the more intuitive of the two components of the acceleration; it comes from the change in the speed $v$, just as in the simple case of 1-D motion. The $a_{\text{r}}$ component is the less intuitive one; it comes from the change in the *direction* of $\mathbf{v}$. Remember that the acceleration $\mathbf{a} \equiv d\mathbf{v}/dt$ involves the rate of change of the entire vector $\mathbf{v}$, not just the magnitude $v \equiv |\mathbf{v}|$. A vector can change because its magnitude changes or because its direction changes (or both). The former change is associated with $a_t$, while the latter is associated with $a_r$.

It is sometimes convenient to work with the *angular frequency* $\omega$ (also often called the *angular speed* or *angular velocity*), which is defined to be the rate at which the angle $\theta$ around the circle (measured in radians) is swept out. That is, $\omega \equiv d\theta/dt$. If we multiply both sides of this equation by the radius $r$, we obtain $r\omega = d(r\theta)/dt$. But $r\theta$ is simply the distance $s$ traveled along the circle,[1] so the right-hand side of this equation is $ds/dt$, which is just the tangential speed $v$. Hence $r\omega = v \implies \omega = v/r$. In terms of $\omega$, the radial acceleration can be written as

$$a_r = \frac{v^2}{r} = \frac{(r\omega)^2}{r} = \omega^2 r. \tag{3.9}$$

Similarly, we can define the *angular acceleration* as $\alpha \equiv d\omega/dt \equiv d^2\theta/dt^2$. If we multiply through by $r$, we obtain $r\alpha = d(r\omega)/dt$. But from the preceding paragraph, $r\omega$ is the tangential speed $v$. Therefore, $r\alpha = dv/dt$. And since the right-hand side of this equation is just the tangential acceleration, we have

$$a_t = r\alpha. \tag{3.10}$$

We can summarize most of the results in the previous two paragraphs by saying that the "linear" quantities (distance $s$, speed $v$, tangential acceleration $a_t$) are related to the angular quantities (angle $\theta$, angular speed $\omega$, angular acceleration $\alpha$) by a factor of $r$:

$$s = r\theta, \qquad v = r\omega, \qquad a_t = r\alpha. \tag{3.11}$$

However, the radial acceleration $a_r$ doesn't fit into this pattern.

## 3.2 Multiple-choice questions

3.1. A bullet is fired horizontally from a gun, and another bullet is simultaneously dropped from the same height. Which bullet hits the ground first? (Ignore air resistance, the curvature of the earth, etc.)

(a) the fired bullet

(b) the dropped bullet

(c) They hit the ground at the same time.

3.2. A projectile is fired at an angle $\theta$ with respect to level ground. Is there a point in the motion where the velocity is perpendicular to the acceleration?

Yes No

3.3. A projectile is fired at an angle $\theta$ with respect to level ground. Does there exist a $\theta$ such that the maximum height attained equals the total horizontal distance traveled?

Yes No

3.4. Is the following reasoning correct? If the launch angle $\theta$ of a projectile is increased (while keeping $v_0$ the same), then the initial $v_y$ velocity component increases, so the time in the air increases, so the total horizontal distance traveled increases.

Yes No

[1] This is true by the definition of a radian. If you take a piece of string with a length of one radius and lay it out along the circumference of a circle, then it subtends an angle of one radian, by definition. So each radian of angle is worth one radius of distance. The total distance $s$ along the circumference is therefore obtained by multiplying the number of radians (that is, the number of "radiuses") by the length of the radius.

3.5. A ball is thrown at an angle $\theta$ with speed $v_0$. A second ball is simultaneously thrown straight upward from the point on the ground directly below the top of the first ball's parabolic motion. How fast should this second ball be thrown if you want it to collide with the first ball?

(a) $v_0/2$ (b) $v_0/\sqrt{2}$ (c) $v_0$ (d) $v_0 \cos\theta$ (e) $v_0 \sin\theta$

3.6. A wall has height $h$ and is a distance $\ell$ away. You wish to throw a ball over the wall with a trajectory such that the ball barely clears the wall at the top of its parabolic motion. What initial speed is required? (Don't solve this from scratch, just check special cases. See Problem 3.10 for a quantitative solution.)

(a) $\sqrt{2gh}$
(b) $\sqrt{4gh}$
(c) $\sqrt{g\ell^2/2h}$
(d) $\sqrt{2gh + g\ell^2/2h}$
(e) $\sqrt{4gh + g\ell^2/2h}$

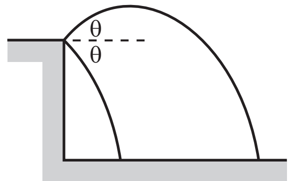


**Figure 3.3**

3.7. Two balls are thrown with the same speed $v_0$ from the top of a cliff. The angles of their initial velocities are $\theta$ above and below the horizontal, as shown in Fig. 3.3. How much farther along the ground does the top ball hit than the bottom ball? *Hint*: The two trajectories have a part in common. No calculations necessary!

(a) $2v_0^2/g$
(b) $2v_0^2 \sin\theta/g$
(c) $2v_0^2 \cos\theta/g$
(d) $2v_0^2 \sin\theta\cos\theta/g$
(e) $2v_0^2 \sin^2\theta\cos^2\theta/g$

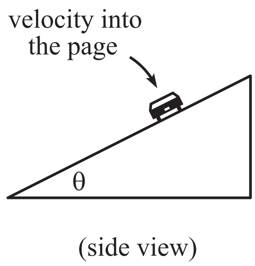


**Figure 3.4**

3.8. A racecar travels in a horizontal circle at constant speed around a circular banked track. A side view is shown in Fig. 3.4. (The triangle is a cross-sectional slice of the track; the car is heading into the page at the instant shown.) The direction of the racecar's acceleration is

(a) horizontal rightward
(b) horizontal leftward
(c) downward along the plane
(d) upward perpendicular to the plane
(e) The acceleration is zero.

3.9. Which one of the following statements is *not* true for *uniform* (constant speed) circular motion?

(a) **v** is perpendicular to **r**.
(b) **v** is perpendicular to **a**.
(c) **v** has magnitude $R\omega$ and points in the **r** direction.
(d) **a** has magnitude $v^2/R$ and points in the negative **r** direction.
(e) **a** has magnitude $\omega^2 R$ and points in the negative **r** direction.

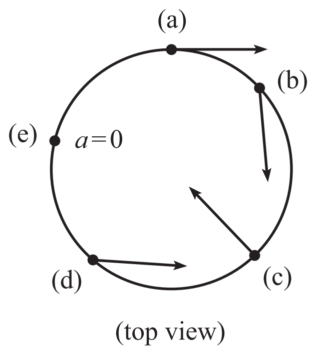


**Figure 3.5**

3.10. A car travels around a horizontal circular track, *not* at constant speed. The acceleration vectors at five different points are shown in Fig. 3.5 (the four nonzero vectors have equal length). At which of these points is the car's speed the largest?

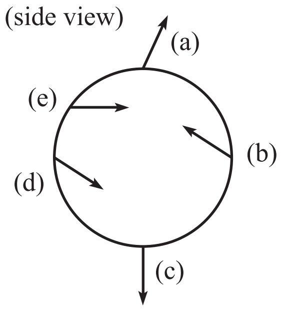
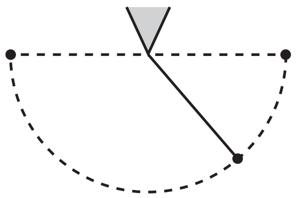



**Figure 3.6**

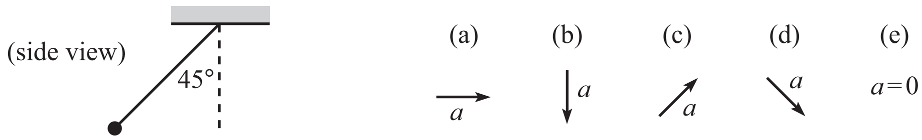


(side view)

**Figure 3.7**

3.11. A bead is given an initial velocity and then circles indefinitely around a frictionless vertical hoop. Only one of the vectors in Fig. 3.6 is a possible acceleration vector at the given point. Which one?

3.12. A pendulum is released from rest at an angle of 45° with respect to the vertical, as shown below. Which vector shows the direction of the initial acceleration?




3.13. A pendulum swings back and forth between the two horizontal positions shown in Fig. 3.7. The acceleration is vertical ($g$ downward) at the highest points, and is also vertical (upward) at the lowest point.

(a) There is at least one additional point where the acceleration is vertical.

(b) There is at least one point where the acceleration is horizontal.

(c) There is at least one point where the acceleration is zero.

(d) None of the above

## 3.3 Problems

*The first three problems are foundational problems.*

3.1. **A few projectile results**

On level ground, a projectile is fired at angle $\theta$ with speed $v_0$. Derive the expressions in Eq. (3.6). That is, find (a) the time to the maximum height, (b) the maximum height attained, and (c) the total horizontal distance traveled.

3.2. **Radial and tangential accelerations**

(a) If an object moves in a circle at constant speed $v$ (uniform circular motion), show that the acceleration points radially inward with magnitude $a_\mathrm{r} = v^2/r$. Do this by drawing the position and velocity vectors at two nearby times and then making use of similar triangles.

(b) If the object speeds up or slows down as it moves around in the circle, then the acceleration also has a tangential component. Show that this component is given by $a_\mathrm{t} = dv/dt$.

3.3. **Radial and tangential accelerations, again**

A particle moves in a circle, not necessarily at constant speed. Its coordinates are given by $(x, y) = (R\cos\theta, R\sin\theta)$, where $\theta \equiv \theta(t)$ is an arbitrary function of $t$. Take two time derivatives of these coordinates to find the acceleration vector, and then explain why the result is consistent with the $a_\mathrm{r}$ and $a_\mathrm{t}$ magnitudes derived in Problem 3.2.

3.4. **Movie replica**

(a) A movie director wants to shoot a certain scene by building a detailed replica of the actual setup. The replica is 1/100 the size of the real thing. In the scene, a person jumps from rest from a tall building (into a net, so it has a happy ending). If the director films a tiny doll being dropped from the replica building, by what factor should the film be sped up or slowed down when played back, so that the falling person looks realistic to someone watching the movie? (Assume that the motion is essentially vertical.)

(b) The director now wants to have a little toy car zoom toward a cliff in the replica (with the same scale factor of 1/100) and then sail over the edge down to the ground below (don't worry, the story has the driver bail out in time). Assume that the goal is to have the movie viewer think that the car is traveling at 50 mph before it goes over the cliff. As in part (a), by what factor should the film be sped up or slowed down when played back? What should the speed of the toy car be as it approaches the cliff in the replica?

3.5. **Doubling gravity**

A ball is thrown with speed $v$ at an angle $\theta$ with respect to the horizontal ground. At the highest point in the motion, the strength of gravity is somehow magically doubled. What is the total horizontal distance traveled by the ball?

3.6. **Ratio of heights**

From the standard $d = gt^2/2$ expression for freefall from rest, we see that if the falling time is doubled, the falling distance is quadrupled. Use this fact to find the ratio of the height of the top of projectile motion (point $A$ in Fig. 3.8) to the height where the projectile would be if gravity were turned off (point $B$ in the figure). Two suggestive distances are drawn.

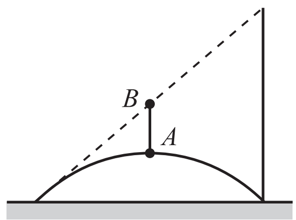



**Figure 3.8**

3.7. **Hitting horizontally**

A ball is thrown with speed $v_0$ at an angle $\theta$ with respect to the horizontal. It is thrown from a point that is a distance $\ell$ from the base of a cliff that has a height also equal to $\ell$. What should $\theta$ and $v_0$ be so that the ball hits the corner of the cliff moving horizontally, as shown in Fig. 3.9?

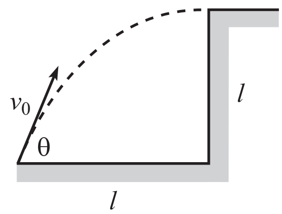



**Figure 3.9**

3.8. **Projectile and tube**

A projectile is fired horizontally with speed $v_0$ from the top of a cliff of height $h$. It immediately enters a fixed tube with length $x$, as shown in Fig. 3.10. There is friction between the projectile and the tube, the effect of which is to make the projectile decelerate with constant acceleration $-a$ ($a$ is a positive quantity here). After the projectile leaves the tube, it undergoes normal projectile motion down to the ground.

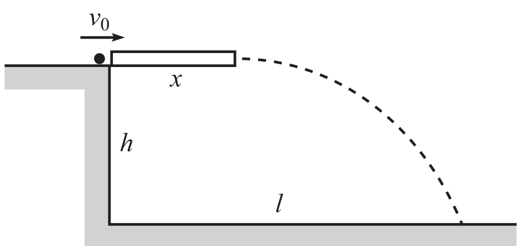



**Figure 3.10**

(a) What is the total horizontal distance (call it $\ell$) that the projectile travels, measured from the base of the cliff? Give your answer in terms of $x$, $h$, $v_0$, $g$, and $a$.

(b) What value of $x$ yields the maximum value of $\ell$?

3.9. **Car in the mud**

A wheel is stuck in the mud, spinning in place. The radius is $R$, and the points on the rim are moving with speed $v$. Bits of the mud depart from the wheel at various random locations. In particular, some bits become unstuck from the rim in the upper left quadrant, as shown in Fig. 3.11. What should $\theta$ be so that the mud reaches the maximum possible height (above the ground) as it flies through the air? What is this maximum height? You may assume $v^2 > gR$.

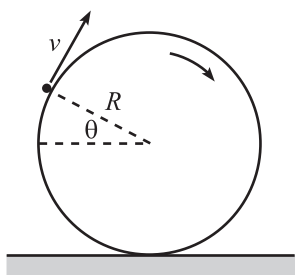



**Figure 3.11**

3.10. **Clearing a wall**

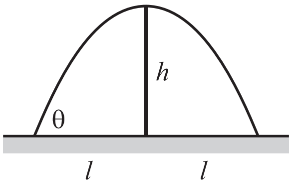


**Figure 3.12**

(a) You wish to throw a ball to a friend who is a distance $2\ell$ away, and you want the ball to just barely clear a wall of height $h$ that is located halfway to your friend, as shown in Fig. 3.12. At what angle $\theta$ should you throw the ball?

(b) What initial speed $v_0$ is required? What value of $h$ (in terms of $\ell$) yields the minimum $v_0$? What is the value of $\theta$ in this minimum case?

3.11. **Bounce throw**

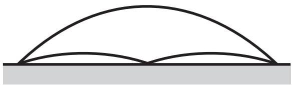


**Figure 3.13**

A person throws a ball with speed $v_0$ at a 45° angle and hits a given target. How much quicker does the ball get to the target if the person instead throws the ball with the same speed $v_0$ but at the angle that makes the trajectory consist of two identical bumps, as shown in Fig. 3.13? (Assume unrealistically that there is no loss in speed at the bounce.)

3.12. **Maximum bounce**

A ball is dropped from rest at height $h$. At height $y$, it bounces elastically (that is, without losing any speed) off a board. The board is inclined at the angle (which happens to be 45°) that makes the ball bounce off horizontally. In terms of $h$, what should $y$ be so that the ball hits the ground as far off to the side as possible? What is the horizontal distance in this optimal case?

3.13. **Falling along a right triangle**

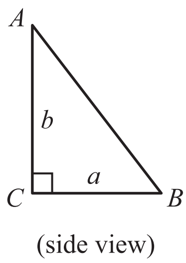


**Figure 3.14**

In the vertical right triangle shown in Fig. 3.14, a particle falls from $A$ to $B$ either along the hypotenuse, or along the two legs (lengths $a$ and $b$) via point $C$. There is no friction anywhere.

(a) What is the time (call it $t_H$) if the particle travels along the hypotenuse?

(b) What is the time (call it $t_L$) if the particle travels along the legs? Assume that at point $C$ there is an infinitesimal curved arc that allows the direction of the particle's motion to change from vertical to horizontal without any change in speed.

(c) Verify that $t_H = t_L$ when $a = 0$.

(d) How do $t_H$ and $t_L$ compare in the limit $b \ll a$?

(e) Excluding the $a = 0$ case, what triangle shape yields $t_H = t_L$?

3.14. **Throwing to a cliff**

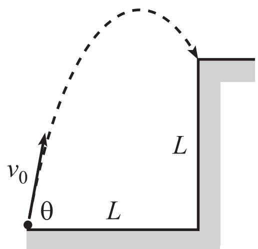


**Figure 3.15**

A ball is thrown at an angle $\theta$ up to the top of a cliff of height $L$, from a point a distance $L$ from the base, as shown in Fig. 3.15.

(a) As a function of $\theta$, what initial speed causes the ball to land right at the edge of the cliff?

(b) There are two special values of $\theta$ for which you can check your result. Check these.

3.15. **Throwing from a cliff**

A ball is thrown with speed $v$ at angle $\theta$ (with respect to horizontal) from the top of a cliff of height $h$. How far from the base of the cliff does the ball land? (The ground is horizontal below the cliff.)

3.16. **Throwing on stairs**

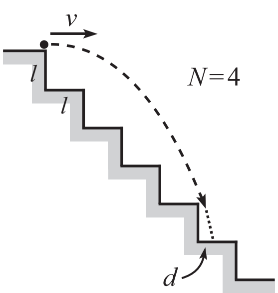


**Figure 3.16**

A ball is thrown horizontally with speed $v$ from the floor at the top of some stairs. The width and height of each step are both equal to $\ell$.

(a) What should $v$ be so that the ball barely clears the corner of the step that is $N$ steps down? Fig. 3.16 shows the case where $N = 4$.

(b) How far along the next step (the distance $d$ in the figure) does the ball hit?

(c) What is $d$ in the limit $N \to \infty$?

(d) Find the components of the ball's velocity when it grazes the corner, and then explain why their ratio is consistent with your answer to part (c).

3.17. **Bullet and sphere**

A bullet is fired horizontally with speed $v_0$ from the top of a fixed sphere with radius $R$, as shown in Fig. 3.17. What is the minimum value of $v_0$ for which the bullet doesn't touch the sphere after it is fired? (*Hint*: Find $y$ as a function of $x$ for the projectile motion, and also find $y$ as a function of $x$ for the sphere near the top where $x$ is small; you'll need to make a Taylor-series approximation. Then compare your two results.) For the $v_0$ you just found, where does the bullet hit the ground?

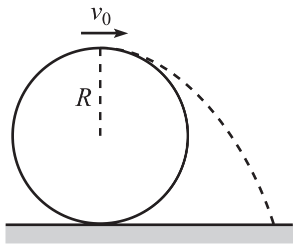



**Figure 3.17**

3.18. **Throwing on an inclined plane**

You throw a ball from a plane inclined at angle $\theta$. The initial velocity is perpendicular to the plane, as shown in Fig. 3.18. Consider the point $P$ on the trajectory that is farthest from the plane. For what angle $\theta$ does $P$ have the same height as the starting point? (For the case shown in the figure, $P$ is higher.) Answer this in two steps:

(a) Give a continuity argument that explains why such a $\theta$ should in fact exist.

(b) Find $\theta$. In getting a handle on where (and when) $P$ is, it is helpful to use a tilted coordinate system and to isolate what is happening in the direction perpendicular to the plane.

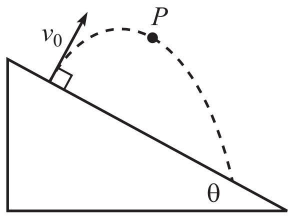



**Figure 3.18**

3.19. **Ball landing on a block**

A block is fired up along a frictionless plane inclined at angle $\beta$, and a ball is simultaneously thrown upward at angle $\theta$ (both $\beta$ and $\theta$ are measured with respect to the horizontal). The objects start at the same location, as shown in Fig. 3.19. What should $\theta$ be in terms of $\beta$ if you want the ball to land on the block at the instant the block reaches its maximum height on the plane? (An implicit equation is fine.) What is $\theta$ if $\beta$ equals 45°? (You might think that we've forgotten to give you information about the initial speeds, but it turns out that you don't need these to solve the problem.)

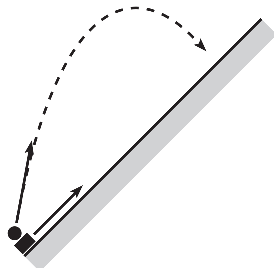


**Figure 3.19**

3.20. ***g*'s in a washer**

A typical front-loading washing machine might have a radius of 0.3 m and a spin cycle of 1000 revolutions per minute. What is the acceleration of a point on the surface of the drum at this spin rate? How many $g$'s is this equivalent to?

3.21. **Acceleration after one revolution**

A car starts from rest on a circular track with radius $R$ and then accelerates with constant tangential acceleration $a_t$. At the moment the car has completed one revolution, what angle does the total acceleration vector make with the radial direction? You should find that your answer doesn't depend on $a_t$ or $R$. Explain why you don't have to actually solve the problem to know this.

3.22. **Equal acceleration components**

An object moves in a circular path of radius $R$. At $t = 0$, it has speed $v_0$. From this point on, the magnitudes of the radial and tangential accelerations are arranged to be equal at all times.

(a) As functions of time, find the speed and the distance traveled.

(b) If the tangential acceleration is positive (that is, if the object is speeding up), there is special value for $t$. What is it, and why is it special?

3.23. **Horizontal acceleration**

A bead is at rest at the top of a fixed frictionless hoop of radius $R$ that lies in a vertical plane. The bead is given an infinitesimal push so that it slides down and around the hoop. Find all the points on the hoop where the bead's acceleration is horizontal. (We haven't covered conservation of energy yet, but use the fact that the bead's speed after it has fallen through a height $h$ is given by $v = \sqrt{2gh}$.)

## 3.4 Multiple-choice answers

3.1. $\boxed{\text{c}}$ This setup is perhaps the most direct example of the independence of horizontal and vertical motions, under the influence of only gravity. The horizontal motion doesn't affect the vertical motion, and vice versa. The gravitational force causes both objects to have the same acceleration $g$ downward, so their heights are both given by $h - gt^2/2$. The fired bullet might travel a mile before it hits the ground, but will still take a time of $t = \sqrt{2h/g}$, just like the dropped ball.

3.2. $\boxed{\text{Yes}}$ At the highest point in the projectile motion, the velocity is sideways, and the acceleration is (always) downward.

Remark: Since $\mathbf{a}$ (which is $\mathbf{g} = -g\hat{\mathbf{y}}$) is perpendicular to $\mathbf{v}$ at the top of the motion, the component of $\mathbf{a}$ in the direction of $\mathbf{v}$ is zero. But this component is what causes a change in the speed (this is just the $a_t = dv/dt$ statement). So $dv/dt = 0$ at the top of the motion. This makes sense because on the way up, the speed decreases (from a tilted $v_0$ to a horizontal $v_0 \cos\theta$); $\mathbf{a}$ has a component in the negative $\mathbf{v}$ direction. And on the way down, the speed increases (from a horizontal $v_0 \cos\theta$ to a tilted $v_0$); $\mathbf{a}$ has a component in the positive $\mathbf{v}$ direction. So at the top of the motion, the speed must be neither increasing nor decreasing. That is, $dv/dt = 0$. On the other hand, the vertical $v_y$ component of the velocity steadily decreases (at a rate of $-g$) during the entire flight, from $v_0 \sin\theta$ to $-v_0 \sin\theta$.

3.3. $\boxed{\text{Yes}}$ If $\theta$ is very small, then the projectile barely climbs above the ground, so the total horizontal distance traveled is much larger than the maximum height. In the other extreme where $\theta$ is close to 90°, the projectile goes nearly straight up and down, so the maximum height is much larger than the total horizontal distance. By continuity, there must exist an intermediate angle for which the maximum height equals the total horizontal distance. As an exercise, you can show that this angle is given by $\tan\theta = 4 \Longrightarrow \theta \approx 76°$.

3.4. $\boxed{\text{No}}$ The reasoning is not valid for all $\theta$. For all $\theta$, the reasoning is correct up until the last "so." The time $t$ in the air does indeed increase as $\theta$ increases (it equals $2v_0 \sin\theta/g$). However, an additional consequence of increasing $\theta$ is that the $v_x$ velocity component (which equals $v_0 \cos\theta$) *decreases.* The total horizontal distance equals $v_x t$, so there are competing effects: increasing $t$ vs. decreasing $v_x$. If we invoke the standard result that the maximum distance is obtained when $\theta = 45°$ (see the solution to Problem 3.1), we see that for $\theta < 45°$, the increase in $t$ wins and the distance increases; but for $\theta > 45°$, the decrease in $v_x$ wins and the distance decreases.

3.5. $\boxed{\text{e}}$ The vertical velocity component of the first ball is $v_0 \sin\theta$. If the second ball is thrown with this speed, then it will always have the same height as a function of time as the first ball. The balls will therefore collide when the first ball's horizontal position coincides with the second ball's (at the top of the parabolic motion).

Remark: The initial location of the second ball on the ground is actually irrelevant. As long as it is thrown simultaneously with speed $v_0 \sin\theta$, it can be thrown from *any* point below the parabolic motion of the first ball, and the balls will still always have the same heights at any moment. They will therefore collide when the first ball's horizontal position coincides with the second ball's. If the collision occurs during the second half of the parabolic motion, the balls will be on their way down.

3.6. $\boxed{\text{d}}$ The answer certainly depends on $\ell$, because the speed must be very large if $\ell$ is very large. So choices (a) and (b) are ruled out. Alternatively, these two choices can be ruled out by noting that in the $h \to 0$ limit, you must throw the ball infinitely fast. This is true because you must throw the ball at a very small angle; so if the speed weren't large, the initial vertical velocity would be very small, which means that the top of the parabolic motion would occur too soon.

In the $\ell \to 0$ limit, you are throwing the ball straight up. And the initial speed in this case is the standard $v = \sqrt{2gh}$. (This can be derived in many ways, for example by using Eq. (2.4).) So the answer must be (d).

3.7. $\boxed{\text{d}}$ When the top ball returns to the initial height (the height of the cliff), its velocity (both magnitude and direction) will be the same as the initial velocity of the bottom ball (speed $v_0$ at an angle $\theta$ below the horizontal). So the trajectory from that point onward will look exactly the same as the entire trajectory of the bottom ball. So the difference in the trajectories is just the symmetric parabola that lies above the initial height. And from Eq. (3.6) we know that the horizontal distance traveled in this part is $2v_0^2 \sin\theta\cos\theta/g$.

REMARK: The first three choices can be eliminated by checking limiting cases. The answer must be zero in both the $\theta = 0$ case (the trajectories are the same) and the $\theta = 90°$ case (both balls travel vertically and hence have the same horizontal distance of zero). So the answer must be (d) or (e). But it takes the above reasoning to show that (d) is correct.

3.8. $\boxed{\text{b}}$ The acceleration has magnitude $v^2/r$ and points toward the center of the circular motion. Since the car is traveling in a horizontal circle, the radial direction is to the left. So the acceleration is horizontal leftward.

REMARK: As long as we are told that the racecar is undergoing uniform (constant speed) circular motion, there is no need to know anything about the various forces acting on the car (which happen to be gravity, normal, and friction; we'll discuss forces in Chapter 4). The acceleration for uniform circular motion, no matter what the cause of the motion, points radially inward with magnitude $v^2/r$, period.

3.9. $\boxed{\text{c}}$ The velocity $\mathbf{v}$ points in the tangential, not radial, direction. The other four statements are all true.

REMARK: If you want to consider *non-uniform* (that is, changing speed) circular motion, then only statement (a) is always true. The acceleration can now have a tangential component, which ruins (b), (d), and (e). And statement (c) is still incorrect.

3.10. $\boxed{\text{c}}$ The radial component of the acceleration has magnitude $a_r = v^2/r$. So the largest speed $v$ corresponds to the $\mathbf{a}$ with the largest (inward) radial component, which is choice (c).

REMARK: Note that the tangential component of the acceleration, which is $a_t = dv/dt$, has nothing to do with the *instantaneous* value of $v$, which is what we're concerned with in this question. A large $a_t$ component (as in choices (a), (b), and (d)) does not imply a large $v$. On the other hand, a zero $a_r$ component (as in choices (a) and (e)) implies a zero $v$.

3.11. $\boxed{\text{d}}$ The acceleration is the vector sum of the radially inward $a_r = v^2/r$ component and the tangentially downward $a_t = g\sin\theta$ component, where $\theta = 0$ corresponds to the top of the hoop. (This is just the component of $g$ that points in the tangential direction.) Only choice (d) satisfies both of these properties. Choice (e) is the trickiest. The acceleration can't be horizontal there, because both $a_r$ and $a_t$ have downward components. There *is* a point in each lower quadrant where the acceleration is horizontal, because in the bottom half of the circle, $a_r$ has an *upward* component which can cancel the downward component of $a_t$ at two particular points.

REMARK: Since the radial $a_r = v^2/r$ component always points radially *inward*, the acceleration vector in any arbitrary circular motion can never have a radially *outward* component. This immediately rules out choices (a) and (c). The borderline case occurs when $a_r = 0$, that is, when $\mathbf{a}$ points tangentially. In this case $v = 0$, so the bead is instantaneously at rest. But any nonzero speed at all will cause an inward $a_r$ component.

3.12. $\boxed{\text{d}}$ The tangential component of the acceleration is $a_t = g \sin 45°$. And the radial component is $a_r = v^2/r = 0$, since $v = 0$ at the start. No matter where the pendulum is released *from rest*, the initial acceleration is always tangential (or zero, if it is "released" when hanging vertically), because $a_r = 0$ when $v = 0$.

3.13. $\boxed{\text{b}}$ Since the acceleration is negative vertical at the highest points and positive vertical at the lowest point, by continuity it must have zero vertical component somewhere in between. That is, it must be horizontal somewhere in between.

The acceleration is never vertical (except at the highest and lowest points), because the $\mathbf{a}_r$ and $\mathbf{a}_t$ vectors either both have rightward components, or both have leftward components, which means that the $x$ component of the total acceleration vector $\mathbf{a}$ is nonzero. This is consistent (if we invoke $F = ma$) with the fact that the tension in the tilted string has a nonzero horizontal component (except at the highest points where the tension is zero and the lowest point where the string is vertical).

## 3.5 Problem solutions

### 3.1. A few projectile results

(a) The components of the velocity and position are given in Eqs. (3.4) and (3.5). At the highest point in the motion, $v_y$ equals zero because the projectile is instantaneously moving horizontally. So Eq. (3.4) gives the time to the highest point as $t_{\text{top}} = v_0 \sin\theta/g$.

(b) First solution: Plugging $t_{\text{top}}$ into Eq. (3.5) gives the maximum height as

$$y_{\text{max}} = v_0 \sin\theta \left(\frac{v_0 \sin\theta}{g}\right) - \frac{1}{2}g\left(\frac{v_0 \sin\theta}{g}\right)^2 = \frac{v_0^2 \sin^2\theta}{2g}. \tag{3.12}$$

This is just the $v_0^2/2g$ result from Problem 2.3, with $v_0$ replaced with the vertical component of the velocity, $v_0 \sin\theta$.

Second solution: We can imagine reversing time (or equivalently, looking at the second half of the motion), in which case the motion is equivalent to (at least as far as the $y$ motion is concerned) an object dropped from rest. We know that the time it takes to reach the ground is $t_{\text{top}} = v_0 \sin\theta/g$, so the distance is $gt_{\text{top}}^2/2 = g(v_0 \sin\theta/g)^2/2 = v_0^2 \sin^2\theta/(2g)$, in agreement with Eq. (3.12).

(c) First solution: Because the ground is level, the up and down parts of the motion are symmetrical, so the total time $t$ in the air is twice the time to the top, that is, $t = 2t_{\text{top}} = 2v_0 \sin\theta/g$. From Eq. (3.5) the total horizontal distance traveled is then

$$x_{\text{max}} = v_0 \cos\theta \left(\frac{2v_0 \sin\theta}{g}\right) = \frac{2v_0^2 \sin\theta \cos\theta}{g}. \tag{3.13}$$

Second solution: The total time in the air can be determined by finding the value of $t$ for which $y(t) = 0$. From Eq. (3.5), we quickly obtain $t = 2v_0 \sin\theta/g$, as we found in the first solution. (A second value of $t$ that makes $y = 0$ in Eq. (3.5) is $t = 0$, of course, because the projectile is on the ground at the start.) Note that if the ground isn't level, then the up and down parts of the motion are *not* symmetrical. So this alternative method of finding the total time (by finding the time for which $y$ takes on a particular value) must be used.

REMARK: If we use the double-angle formula $\sin 2\theta = 2\sin\theta\cos\theta$, the expression for $x_{max}$ in Eq. (3.13) can alternatively be written as $x_{max} = v_0^2 \sin 2\theta/g$. Since $\sin 2\theta$ achieves its maximum value when $2\theta = 90°$, this form makes it immediately clear that for a given speed $v_0$, the maximum horizontal distance is achieved when $\theta = 45°$. This maximum distance is $v_0^2/g$. Note that consideration of units tells us that the maximum distance must be proportional to $v_0^2/g$. But a calculation is necessary to show that the multiplicative factor is 1.

The $v_0^2 \sin 2\theta/g$ form of $x_{max}$ makes it clear (although it is also clear from the $2v_0^2 \sin\theta\cos\theta/g$ form) that the distance is symmetric on either side of 45°. That is, 46° yields the same distance as 44°, and 80° yields the same distance as 10°, etc.

3.2. **Radial and tangential accelerations**

(a) The position and velocity vectors at two nearby times are shown in Fig. 3.20. Their differences, $\Delta\mathbf{r} \equiv \mathbf{r}_2 - \mathbf{r}_1$ and $\Delta\mathbf{v} \equiv \mathbf{v}_2 - \mathbf{v}_1$, are shown in Fig. 3.21. (See Figs. 13.9 and 13.10 in Appendix A for a comment on this $\Delta\mathbf{v}$.) The angle between the $\mathbf{v}$'s is the same as the angle between the $\mathbf{r}$'s, because each $\mathbf{v}$ makes a right angle with the corresponding $\mathbf{r}$. Therefore, the triangles in Fig. 3.21 are similar, and we have

$$\frac{|\Delta\mathbf{v}|}{v} = \frac{|\Delta\mathbf{r}|}{r}, \qquad (3.14)$$

where $v \equiv |\mathbf{v}|$ and $r \equiv |\mathbf{r}|$. Our goal is to obtain an expression for $a \equiv |\mathbf{a}| \equiv |\Delta\mathbf{v}/\Delta t|$. The $\Delta t$ here suggests that we should divide Eq. (3.14) through by $\Delta t$. This gives (using $v \equiv |\mathbf{v}| \equiv |\Delta\mathbf{r}/\Delta t|$)

$$\frac{1}{v}\left|\frac{\Delta\mathbf{v}}{\Delta t}\right| = \frac{1}{r}\left|\frac{\Delta\mathbf{r}}{\Delta t}\right| \implies \frac{|\mathbf{a}|}{v} = \frac{|\mathbf{v}|}{r} \implies a = \frac{v^2}{r}, \qquad (3.15)$$

as desired. We have assumed that $\Delta t$ is infinitesimal here, which allows us to convert the above quotients into derivatives.

The direction of the acceleration vector $\mathbf{a}$ is radially inward, because $\mathbf{a} \equiv d\mathbf{v}/dt$ has the same direction as $d\mathbf{v}$ (or $\Delta\mathbf{v}$), which points radially inward (leftward) in Fig. 3.21 (in the limit where $\theta$ is very small).

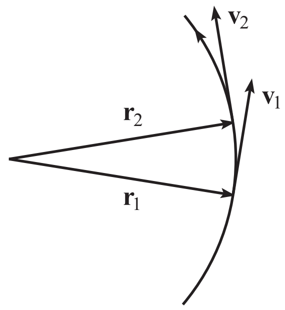


**Figure 3.20**

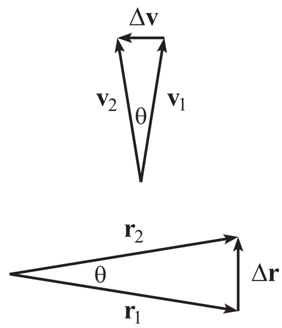


**Figure 3.21**

REMARK: The $a = v^2/r$ result involves the *square* of $v$. That is, $v$ matters twice in $a$. The physical reason for this is the following. The first effect is that the larger $v$ is, the larger the $\Delta\mathbf{v}$ is for a given angle $\theta$ in Fig. 3.21 (because the triangle is larger). The second effect is that the larger $v$ is, the faster the object moves around in the circle, so the larger the angle $\theta$ is (and hence the larger the $\Delta\mathbf{v}$ is) for a given time $\Delta t$. Basically, if $v$ increases, then the $\mathbf{v}$ triangle in Fig. 3.21 gets both taller and wider. Each of these effects is proportional to $v$, so for a given time $\Delta t$ the change in velocity $\Delta\mathbf{v}$ is proportional to $v^2$, as we wanted to show.

(b) If the speed isn't constant, then the radial component $a_r$ still equals $v^2/r$, because the velocity triangle in Fig. 3.21 becomes the triangle shown in Fig. 3.22 (for the case where the speed increases, so that $\mathbf{v}_2$ is longer than $\mathbf{v}_1$). The lower part of this triangle is exactly the same as the triangle in Fig. 3.21 (or at least it would be, if we had drawn $\mathbf{v}_1$ with the same length). So all of the preceding reasoning carries through, leading again to $a_r = v^2/r$. The $v$ here could technically be either $v_1$ or $v_2$. But in the $\Delta t \to 0$ limit, the angle $\theta$ goes to zero, and both $v_1$ and $v_2$ are equal to the instantaneous speed $v$.

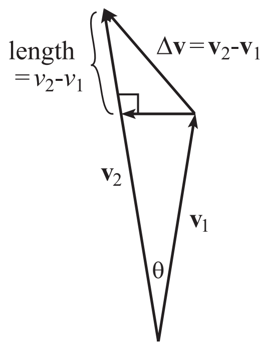


**Figure 3.22**

To obtain the tangential component $a_t$, we can use the upper part of the triangle in Fig. 3.22. This is a right triangle in the $\Delta t \to 0$ limit, and it tells us that $a_t$ (which is the vertical component in Fig. 3.22) is

$$a_t = \frac{v_2 - v_1}{\Delta t} \equiv \frac{\Delta v}{\Delta t} \longrightarrow \frac{dv}{dt}, \qquad (3.16)$$

as desired.

REMARK: A word about the placement of absolute value signs ("| |"): The tangential component of $\mathbf{a}$ is $a_t = dv/dt \equiv d|\mathbf{v}|/dt$, while the complete vector $\mathbf{a}$ is $\mathbf{a} \equiv d\mathbf{v}/dt$, which has magnitude $a = |\mathbf{a}| = |d\mathbf{v}|/dt$ (which equals $a = \sqrt{a_r^2 + a_t^2}$). The placement of the absolute value signs is critical, because $d|\mathbf{v}|$, which is the *change in the magnitude* of the velocity vector, is *not* equal to $|d\mathbf{v}|$, which is the *magnitude of the change* in the velocity vector. The former is associated with the left leg of the right triangle in Fig. 3.22, while the latter is associated with the hypotenuse. The disparity between $d|\mathbf{v}|$ and $|d\mathbf{v}|$ is most obvious in the case of uniform circular motion, where we have $d|\mathbf{v}| = 0$ and $|d\mathbf{v}| \neq 0$; the speed is constant, but the velocity is not. (The one exception to the $d|\mathbf{v}| \neq |d\mathbf{v}|$ statement occurs when the speed is instantaneously zero, so that $a_r = 0$. Since the acceleration is only tangential in this case, we have $d|\mathbf{v}| = |d\mathbf{v}|$.)

3.3. **Radial and tangential accelerations, again**

When taking the time derivatives, we must be careful to use the chain rule and the product rule. Starting with $(x, y) = R(\cos\theta, \sin\theta)$, the velocity is found by taking one time derivative:

$$(\dot{x}, \dot{y}) = R(-\dot{\theta}\sin\theta,\, \dot{\theta}\cos\theta), \tag{3.17}$$

where the $\dot{\theta}$'s come from the chain rule, because $\theta$ is a function of $t$. Another time derivative (using the chain rule again, along with the product rule) yields the acceleration:

$$(\ddot{x}, \ddot{y}) = R(-\ddot{\theta}\sin\theta - \dot{\theta}^2\cos\theta,\ \ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta). \tag{3.18}$$

If we group the $\ddot{\theta}$ terms together, and likewise the $\dot{\theta}^2$ terms, we find

$$(\ddot{x}, \ddot{y}) = R\ddot{\theta}(-\sin\theta,\, \cos\theta) + R\dot{\theta}^2(-\cos\theta,\, -\sin\theta). \tag{3.19}$$

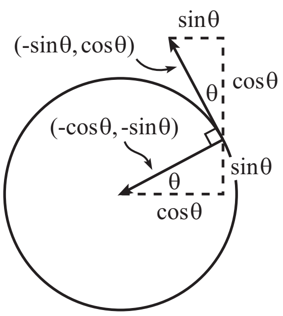


**Figure 3.23**

The first vector here is the tangential acceleration vector, because the magnitude is $R\ddot{\theta} \equiv R\alpha = a_t$, where we have used the fact that $(-\sin\theta, \cos\theta)$ is a unit vector. And this vector $(-\sin\theta, \cos\theta)$ points in the tangential direction, as shown in Fig. 3.23.

The second vector in Eq. (3.19) is the radial acceleration vector, because the magnitude is $R\dot{\theta}^2 \equiv R\omega^2 = R(v/R)^2 = v^2/R = a_r$, where we have used the fact that $(-\cos\theta, -\sin\theta)$ is a unit vector. And this vector $(-\cos\theta, -\sin\theta)$ points radially inward.

UNITS: The units of all of the components in Eq. (3.19) are all correctly $\text{m/s}^2$, because both $\ddot{\theta}$ and $\dot{\theta}^2$ have units of $1/\text{s}^2$. If you forgot to use the chain rule and omitted the $\dot{\theta}$'s, the units wouldn't work out.

LIMITS: If $\ddot{\theta} = 0$ (uniform circular motion), then the first vector in Eq. (3.19) is zero, so the acceleration is only radial; this is correct. And if $\dot{\theta} = 0$ (the object is instantaneously at rest), then the second vector in Eq. (3.19) is zero, so the acceleration is only tangential; this is also correct.

3.4. **Movie replica**

(a) If the movie were played back at normal speed, the person would appear to fall unnaturally fast. To see why, let's say that the doll falls for 1 s in the replica, which means that it falls a distance of 4.9 m (from $d = gt^2/2$, and we're neglecting air resistance, as usual). Since the scale factor is 100, someone watching the movie would think that the person falls 490 m in 1 s. This would look very strange, because it is far too large a distance; an object should fall only 4.9 m in 1 s. An object would fall 490 m in 1 s on a planet that has $g = 980\,\text{m/s}^2$, but not on the earth.

How long does it take something to fall 490 m on the earth? From $d = gt^2/2$, we obtain $t = 10\,\text{s}$. So the answer to this problem is that the movie should be slowed down by a factor of 10 when played back, so that the movie watcher sees a 490 m fall take the correct time of 10 s.

REMARK: The factor of 10 here arises because it is the square root of the scale factor, 100. Mathematically, the 10 comes from the fact that the units of $g$ involve $\text{s}^2$, which leads to the $t^2$ in the expression $d = gt^2/2$. A factor of 10 in the time then leads to the desired factor of

$10^2 = 100$ in the distance. But more intuitively, the factor of 10 arises in the following way. If we don't scale the time at all, then as we saw above, we basically end up on a planet with $g = 980\,\text{m/s}^2$. If we then slow down the movie by a factor of 10, this decreases $g$ by a factor of $10^2$, because it causes the falling object to take *10 times as long* to fall a given distance, at which point it is going only *1/10 as fast*. So the factor of 10 matters *twice*. The rate of change $\Delta v/\Delta t$ of the velocity (that is, the acceleration) is therefore only $(1/10)/10 = 1/100$ of the $g = 980\,\text{m/s}^2$ value that it would have been, so we end up with the desired $g$ value of $9.8\,\text{m/s}^2$.

(b) The vertical motion must satisfy the same conditions as in part (a), because it is unaffected by the horizontal motion. So we again need to slow down the playback by a factor of 10.

We want the speed of the car to appear to be 50 mph. This means that if the movie weren't slowed down by the required factor of 10, then the car's speed in the movie would appear to be 500 mph. Due to the scale factor of 100, this would then require the speed in the replica to be 5 mph. So 5 mph is the desired answer.

Said in an equivalent way, starting from the replica: A speed of 5 mph in the replica translates to a speed of 500 mph in an unmodified movie, due to the scale factor of 100. But then slowing down the movie by a factor of 10 brings the apparent speed down to the desired 50 mph.

3.5. **Doubling gravity**

The horizontal distance traveled during the upward part of the motion is half the total distance of normal projectile motion. Therefore, from Eq. (3.6) the distance traveled during the upward motion is $(v_0^2/g)\sin\theta\cos\theta$. We must now find the horizontal distance traveled during the downward motion. Since the horizontal velocity is constant throughout the entire motion, we just need to get a handle on the time of the downward part.

The time of the upward part is given by $h = gt_\text{u}^2/2$, where $h$ is the maximum height, because we can imagine running time backwards, in which case the ball is dropped from rest, as far as the vertical motion is concerned. (The height $h$ equals $(v_0^2/2g)\sin^2\theta$ from Eq. (3.6), but we won't need to use this.) The time of the downward part is given by $h = (2g)t_\text{d}^2/2$, because gravity is doubled. Therefore $t_\text{d} = t_\text{u}/\sqrt{2}$. So the horizontal distance traveled during the downward motion is $1/\sqrt{2}$ times the horizontal distance traveled during the upward motion. The total distance is therefore

$$d = \frac{v_0^2\sin\theta\cos\theta}{g}\left(1 + \frac{1}{\sqrt{2}}\right). \tag{3.20}$$

3.6. **Ratio of heights**

The given $gt^2/2$ expression holds for an object dropped from rest. But it is also a valid expression for the distance fallen relative to where the object would be if gravity were turned off. Mathematically, this is true because in Eq. (3.3), $Y + V_y t$ is the height of the object in the absence of gravity, and $gt^2/2$ is what is subtracted from this.

The two vertical distances we drew in Fig. 3.8 are the distances fallen relative to the zero-gravity line of motion (the dotted line). Since the time of the entire projectile motion is twice the time to the top, the ratio of these two vertical distances is $2^2 = 4$ (hence the $d$ and $4d$ labels in Fig. 3.24). This figure contains two similar right triangles, with one being twice the size of the other. Comparing the vertical legs of each triangle tells us that $4d/(h + d) = 2$. This yields $h = d$, which means that point $A$ has half the height of point $B$. The answer to the problem is therefore $1/2$.

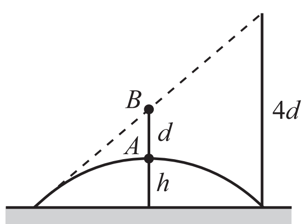



**Figure 3.24**

3.7. **Hitting horizontally**

If $t$ is the time of flight, then we have three unknowns: $t$, $v_0$, and $\theta$. And we have three facts:

- The vertical speed is zero at the top, so $v_0 \sin\theta - gt = 0 \Longrightarrow t = v_0 \sin\theta/g$.
- The horizontal distance is $\ell$, so $(v_0 \cos\theta)t = \ell$.
- The vertical distance is $\ell$, so $(v_0 \sin\theta)t - gt^2/2 = \ell$.

Plugging the value of $t$ from the first fact into the other two gives

$$\frac{v_0^2 \sin\theta\cos\theta}{g} = \ell \qquad \text{and} \qquad \frac{v_0^2 \sin^2\theta}{2g} = \ell. \tag{3.21}$$

Dividing the second of these equations by the first gives

$$\frac{\sin\theta}{2\cos\theta} = 1 \implies \tan\theta = 2 \implies \theta \approx 63.4^\circ. \tag{3.22}$$

(What we've done here is basically find the firing angle of a projectile that makes the maximum height be half of the total range.) This value of $\theta$ corresponds to a 1-2-$\sqrt{5}$ right triangle, which yields $\sin\theta = 2/\sqrt{5}$. The second of the equations in Eq. (3.21) (the first would work just as well) then gives

$$\frac{v_0^2}{2g}\left(\frac{2}{\sqrt{5}}\right)^2 = \ell \implies v_0 = \sqrt{\frac{5g\ell}{2}}. \tag{3.23}$$

3.8. **Projectile and tube**

(a) From the standard $v_f^2 = v_i^2 + 2ad$ formula, the speed of the projectile when it exits the tube is $v = \sqrt{v_0^2 - 2ax}$. You can also obtain this by using $v = v_0 - at$, where $t$ is found by solving $v_0 t - at^2/2 = x$. This quadratic equation has two solutions, of course. You want the "–" root. (What is the meaning of the "+" root?)

Since the projectile motion has zero initial $v_y$, the time to reach the ground is given by $gt^2/2 = h \implies t = \sqrt{2h/g}$. The horizontal distance traveled in the air is $vt$, but we must add on the length of the tube to get the total distance $\ell$. So we have

$$\ell = x + vt = x + \sqrt{v_0^2 - 2ax}\sqrt{\frac{2h}{g}}. \tag{3.24}$$

(b) Looking at the two terms in Eq. (3.24), we see that we have competing effects of $x$. Increasing $x$ increases $\ell$ by having the projectile motion start farther to the right. But increasing $x$ also decreases the projectile motion's initial speed and hence its horizontal distance. Maximizing the $\ell$ in Eq. (3.24) by taking the derivative with respect to $x$ gives

$$\begin{aligned} 0 = \frac{d\ell}{dx} &= 1 + \frac{1}{2}\frac{-2a}{\sqrt{v_0^2 - 2ax}}\sqrt{\frac{2h}{g}} \\ \implies \sqrt{v_0^2 - 2ax} &= a\sqrt{\frac{2h}{g}} \\ \implies x &= \frac{v_0^2}{2a} - \frac{ah}{g}. \end{aligned} \tag{3.25}$$

Both terms here correctly have dimensions of length. This result for the optimal value of $x$ is smaller than $v_0^2/2a$, as it should be, because otherwise the projectile would reach zero speed inside the tube (since $v = \sqrt{v_0^2 - 2ax}$) and never make it out.

However, we aren't quite done, because there are two cases to consider. To see why, note that if $a = v_0\sqrt{g/2h}$, then the $x$ in Eq. (3.25) equals zero. So Eq. (3.25) is applicable only if $a \le v_0\sqrt{g/2h}$. If $a \ge v_0\sqrt{g/2h}$, then Eq. (3.25) yields a negative value of $x$, which isn't physical. The optimal $x$ in the $a \ge v_0\sqrt{g/2h}$ case is therefore simply $x = 0$. Physically, if $a$ is large then any nonzero value of $x$ will hurt $\ell$ (by slowing down the initial projectile speed) more than it will help (by adding on the "head start" distance of $x$). Mathematically, in this case the extremum of $\ell$ occurs at the boundary of the allowed values of $x$ (namely, $x = 0$), as opposed to at a local maximum; a zero derivative therefore isn't relevant.

Limits: If $a \to 0$ then Eq. (3.25) gives $x \to \infty$, which is correct. We can make the tube be very long, and the projectile will keep sliding along; the projectile motion at the end is largely irrelevant. We also see that the optimal $x$ increases with $v_0$ and $g$, and decreases with $a$ and $h$. You should convince yourself that these all make sense.

3.9. **Car in the mud**

At the instant the mud leaves the wheel, it is at height $R + R\sin\theta$ above the ground. It then rises an extra height of $h = (v\cos\theta)^2/2g$ during its projectile motion. This is true because the initial $v_y$ is $v\cos\theta$ (because the initial velocity makes an angle of $90° - \theta$ with respect to the horizontal), so the time to the highest point is $(v\cos\theta)/g$; plugging this into $h = (v\cos\theta)t - (1/2)gt^2$ gives $h = (v\cos\theta)^2/2g$. The total height above the ground at the highest point is therefore

$$H = R + R\sin\theta + \frac{v^2\cos^2\theta}{2g}. \tag{3.26}$$

Maximizing this by taking the derivative with respect to $\theta$ gives

$$R\cos\theta - \frac{v^2}{g}\sin\theta\cos\theta = 0. \tag{3.27}$$

Since $\cos\theta$ is a factor of this equation, we see that there are two solutions. One is $\cos\theta = 0 \Longrightarrow \theta = \pi/2$, which corresponds to the top of the wheel. This is the maximum height if $v^2 < gR$, because in this case the best the mud can do is stay in contact with the wheel the whole time. (After learning about forces in Chapter 4, you can show that if $v^2 < gR$ then the normal force between the wheel and the mud is always nonzero, so the mud will never fly off the wheel.) But we are assuming $v^2 > gR$, so we want the other solution,

$$\sin\theta = \frac{gR}{v^2}. \tag{3.28}$$

Note that $gR/v^2$ is less than 1 if $v^2 > gR$, so such a $\theta$ does indeed exist. (For $v^2 = gR$ we correctly obtain $\theta = \pi/2$.) Plugging Eq. (3.28) into Eq. (3.26) gives a maximum height of

$$\begin{aligned} H_{\text{max}} &= R + R\sin\theta + \frac{v^2}{2g}(1 - \sin^2\theta) \\ &= R + R\left(\frac{gR}{v^2}\right) + \frac{v^2}{2g}\left(1 - \left(\frac{gR}{v^2}\right)^2\right) \\ &= R + \frac{gR^2}{2v^2} + \frac{v^2}{2g}. \end{aligned} \tag{3.29}$$

All three terms here correctly have dimensions of length. This result is valid if $v^2 \ge gR$. If $v^2 \le gR$, then the maximum height (at the top of the wheel) is $2R$.

Limits: If $v$ is large (more precisely, if $v^2 \gg gR$), then Eq. (3.29) gives $H_{\text{max}} \approx R + v^2/2g$. From Eq. (3.28) the mud leaves the wheel at $\theta \approx 0$ (the side point) and travels vertically an extra height of $v^2/2g$, which is the standard height achieved in vertical projectile motion. If $v^2 \approx gR$, then Eq. (3.29) gives $H_{\text{max}} \approx 2R$. The mud barely leaves the wheel at the top, so the maximum height is simply $2R$. It doesn't make sense to take the small-$v$ limit (more precisely, $v^2 \ll gR$) of Eq. (3.29), because this result assumes $v^2 > gR$.

3.10. **Clearing a wall**

(a) The unknowns in this problem are $v_0$ and $\theta$. Problem 3.1(b) gives the maximum height of a projectile, so we want

$$h = \frac{v_0^2 \sin^2\theta}{2g}. \tag{3.30}$$

And Problem 3.1(c) gives the range of a projectile, so we want

$$2\ell = \frac{2v_0^2 \sin\theta\cos\theta}{g}. \tag{3.31}$$

Dividing Eq. (3.30) by Eq. (3.31) gives

$$\frac{h}{\ell} = \frac{\sin\theta}{2\cos\theta} \implies \tan\theta = \frac{2h}{\ell}. \tag{3.32}$$

This means that you should pretend that the wall is twice as tall as it is, and then aim for the top of that imaginary wall.

Limits: If $h \to 0$ then $\theta \to 0$. And if $h \to \infty$ then $\theta \to 90°$. Both of these limits make sense.

(b) First solution: If $\tan\theta = 2h/\ell$, then drawing a right triangle with legs of length $\ell$ and $2h$ tells us that $\sin\theta = 2h/\sqrt{4h^2+\ell^2}$. So Eq. (3.30) gives

$$h = \frac{v_0^2}{2g}\cdot\frac{4h^2}{4h^2+\ell^2} \implies v_0^2 = g\left(\frac{4h^2+\ell^2}{2h}\right). \tag{3.33}$$

(Eq. (3.31) would give the same result.) We want to minimize this function of $h$. Setting the derivative equal to zero gives (ignoring the denominator of the derivative, since we're setting the result equal to zero)

$$0 = 2h(8h) - (4h^2+\ell^2)\cdot 2 \implies 0 = 4h^2 - \ell^2 \implies h = \frac{\ell}{2}. \tag{3.34}$$

If $h$ takes on this value, then Eq. (3.32) gives $\tan\theta = 2(\ell/2)/\ell = 1 \implies \theta = 45°$. You should convince yourself why this result is consistent with the familiar fact that $\theta = 45°$ gives the maximum range of a projectile.

Limits: The $v_0$ in Eq. (3.33) correctly goes to infinity as $h \to \infty$. It also goes to infinity as $h \to 0$. This makes sense because $h \approx 0$ corresponds to a nearly horizontal "line drive." If the speed weren't large, then the ball would quickly hit the ground (since the initial $v_y$ would be very small). Note that since $v_0 \to \infty$ for both $h \to \infty$ and $h \to 0$, a continuity argument implies that $v_0$ achieves a minimum for some intermediate value of $h$. But it takes a little work to show that this $h$ equals $\ell/2$.

Second solution: A somewhat quicker way of obtaining $v_0$ (without first obtaining $\theta$ in Eq. (3.32)) is the following. Let $t$ be the time to the top of the motion. Then the horizontal component of $\mathbf{v}_0$ is $\ell/t$, and the vertical component is $gt$ (because $v_y$ is zero at the top). From the second half of the motion, we also know that $gt^2/2 = h \implies t^2 = 2h/g$. So

$$v_0^2 = v_x^2 + v_y^2 = \frac{\ell^2}{t^2} + g^2t^2 = \frac{\ell^2}{2h/g} + g^2\frac{2h}{g} = g\left(\frac{\ell^2+4h^2}{2h}\right), \tag{3.35}$$

in agreement with Eq. (3.33).

3.11. **Bounce throw**

From Eq. (3.6) we know that the total horizontal distance traveled for a 45° throw is $v_0^2/g$. (As always, we are ignoring air resistance, which is actually a pretty lousy approximation for thrown balls.) For the bounce throw, each of the two bumps has a horizontal span of $v_0^2/2g$. The throwing speed is still $v_0$, so we want the $\sin 2\theta$ factor in Eq. (3.6) to be $1/2$. Hence $2\theta = 30° \implies \theta = 15°$. The (constant) horizontal component of the velocity is therefore $v_0 \cos 15°$ for the entire duration of the bounce throw, as opposed to $v_0 \cos 45°$ for the original throw. Since the time of flight is inversely proportional to the horizontal speed, the total time for the bounce throw is $(v_0 \cos 45°)/(v_0 \cos 15°) \approx 0.73$ as long as the total time for the original throw.

Remarks: Since $\cos 15° = \cos(45° - 30°)$, you can use the trig sum formula for cosine to show that the exact answer to this problem is $2/(\sqrt{3} + 1)$, which can be written as $\sqrt{3} - 1$.

In real life, air resistance makes the trajectory be nonparabolic, and there is also an abrupt decrease in speed at the bounce, due to friction with the ground. But it is still possible for a bounce throw to take less time than the no-bounce throw. This is particularly relevant in baseball games. If a player is making a long throw from the outfield (or even from third base to first base if the player is off balance and the throwing speed is low), then a bounce throw is desirable. The advantage of throwing in a more direct line can outweigh the disadvantage of the loss in speed at the bounce. But the second bump in the throw needs to be relatively small.

3.12. **Maximum bounce**

The time it takes the ball to fall the distance $h - y$ to the board is given by $gt^2/2 = h - y \implies t = \sqrt{2(h - y)/g}$. The speed at this time is $v = gt = \sqrt{2g(h - y)}$. (This is just the standard $v_f = \sqrt{2ad}$ result that follows from Eq. (2.4) when $v_i = 0$.) Since the collision is elastic, this is also the horizontal speed $v_x$ right after the bounce.

The time to fall the remaining distance $y$ to the ground after the horizontal bounce is given by $gt^2/2 = y \implies t = \sqrt{2y/g}$. The horizontal distance traveled is then

$$d = v_x t = \sqrt{2g(h - y)} \sqrt{\frac{2y}{g}} = 2\sqrt{y(h - y)}. \tag{3.36}$$

Our goal is therefore to maximize the function $hy - y^2$. Setting the derivative equal to zero gives $0 = h - 2y \implies y = h/2$. So the board should be at the halfway point. The desired horizontal distance is then $d = 2\sqrt{(h/2)(h - h/2)} = h$.

Limits: The distance $d$ in Eq. (3.36) goes to zero for both $y = 0$ and $y = h$. These limits make sense. In the first case, the board is on the ground, so there is no time after the collision for the ball to travel any horizontal distance. In the second case, the board is located right where the ball is released, so the horizontal speed after the "collision" is zero, and the ball falls straight down.

3.13. **Falling along a right triangle**

(a) If $\theta$ is the inclination angle of the hypotenuse, then the component of the gravitational acceleration along the hypotenuse is $g \sin\theta$, where $\sin\theta = b/\sqrt{a^2 + b^2}$. Using $d = at^2/2$ (this $a$ is the acceleration, not the length of the lower leg!), the time to travel along the hypotenuse is given by

$$\sqrt{a^2 + b^2} = \frac{1}{2}\left(g\frac{b}{\sqrt{a^2 + b^2}}\right)t_H^2 \implies t_H = \sqrt{\frac{2(a^2 + b^2)}{gb}}. \tag{3.37}$$

Limits: If $a \to \infty$ or $b \to \infty$ or $b \to 0$, then $t_H \to \infty$, which makes sense.

(b) The time to fall to $C$ is given by $gt_1^2/2 = b \implies t_1 = \sqrt{2b/g}$. The speed at $C$ is then $gt_1 = \sqrt{2gb}$. The particle then travels along the bottom leg of the triangle at this constant speed, which takes a time of $t_2 = a/\sqrt{2gb}$. The total time is therefore

$$t_L = t_1 + t_2 = \sqrt{\frac{2b}{g}} + \frac{a}{\sqrt{2gb}}. \tag{3.38}$$

LIMITS: Again, if $a \to \infty$ or $b \to \infty$ or $b \to 0$, then $t_L \to \infty$.

(c) If $a = 0$ then both $t_H$ and $t_L$ reduce to $\sqrt{2b/g}$.

(d) If $b \ll a$ then $t_H \approx \sqrt{2}a/\sqrt{gb}$ (the $b^2$ term in Eq. (3.37) is negligible), and $t_L \approx a/\sqrt{2gb}$ (the first term in Eq. (3.38) is negligible). So in this limit we have

$$t_H \approx 2t_L. \tag{3.39}$$

This makes sense for the following reason. In the journey along the legs, the particle is moving at the maximum speed of $\sqrt{2gb}$ for essentially the entire time. But in the journey along the hypotenuse, the particle has the same maximum speed (as you can show with kinematics; this also follows quickly from conservation of energy, discussed in Chapter 5), and the average speed is half of the maximum speed, because the acceleration is constant.

(e) Setting the $t_H$ in Eq. (3.37) equal to the $t_L$ in Eq. (3.38) gives

$$\begin{aligned}
\sqrt{\frac{2(a^2+b^2)}{gb}} = \sqrt{\frac{2b}{g}} + \frac{a}{\sqrt{2gb}} \quad &\Longrightarrow \quad 2\sqrt{a^2+b^2} = 2b + a \\
\Longrightarrow \quad 4(a^2+b^2) = 4b^2 + 4ba + a^2 \quad &\Longrightarrow \quad 3a^2 = 4ba \\
\Longrightarrow \quad a = \frac{4b}{3}.
\end{aligned} \tag{3.40}$$

So the two times are equal if we have a 3-4-5 right triangle, with the bottom leg being the longer one.

REMARKS: Without doing any calculations, the following continuity argument demonstrates that there must exist a triangle shape for which $t_H$ equals $t_L$. We found in part (d) that $t_H > t_L$ (by a factor of 2) when $b \ll a$. But $t_H < t_L$ when $a \ll b$. This can be seen by noting that $t_H = t_L$ when $a = 0$ and that $a$ appears only at second order in the expression for $t_H$ in Eq. (3.37), but at first order in the expression for $t_L$ in Eq. (3.38). When $a$ is small, the second-order $a^2$ term is much smaller than the first-order $a$ term, making $t_H$ smaller than $t_L$. The preceding $t_H > t_L$ and $t_H < t_L$ inequalities imply, by continuity, that there must exist some relation between $a$ and $b$ for which the two times are equal.

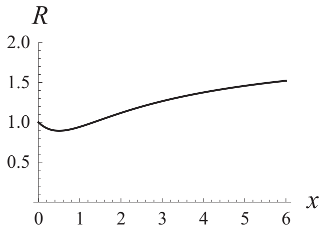



**Figure 3.25**

Note that since the times are equal in both the 3-4-5 case and the $a = 0$ case, the ratio $R \equiv t_H/t_L$ must achieve an extremum (it's a minimum) somewhere in between. As an exercise, you can show that this occurs when $a = b/2$. The associated minimum is $R = 2/\sqrt{5} \approx 0.89$. The plot of $R$ vs. $x \equiv a/b$ is shown in Fig. 3.25. In terms of $x$, you can show that $R(x) = 2\sqrt{1+x^2}/(2+x)$. The value $x \equiv a/b = 0$ corresponds to a tall thin triangle (with $R = 1$), and $x \equiv a/b = \infty$ corresponds to a wide squat triangle (with $R = 2$). You should think physically about the competing effects that make $t_H$ larger than $t_L$ (that is, $R > 1$) in some (most) cases, but smaller in others.

3.14. **Throwing to a cliff**

(a) If $t$ is the time to hit the edge of the cliff, then the standard expressions for the horizontal and vertical positions yield

$$\begin{aligned}
L &= (v_0 \cos\theta)t, \\
L &= (v_0 \sin\theta)t - \frac{gt^2}{2}.
\end{aligned} \tag{3.41}$$

Solving for $t$ in the first equation and plugging the result into the second equation gives

$$L = v_0 \sin\theta \left(\frac{L}{v_0 \cos\theta}\right) - \frac{g}{2}\left(\frac{L}{v_0\cos\theta}\right)^2 = L\tan\theta - \frac{gL^2}{2v_0^2\cos^2\theta}. \tag{3.42}$$

Solving for $v_0$ yields

$$v_0 = \sqrt{\frac{gL}{2\cos\theta(\sin\theta - \cos\theta)}}. \tag{3.43}$$

(b) If $\theta \to 90^\circ$ then $\cos\theta \to 0$, so $v_0 \to \infty$. This makes sense, because the ball is essentially thrown straight up. The horizontal component of the velocity is very small, so the ball needs to spend a very long time in the air. It must therefore have a very large initial speed.

If $\theta \to 45^\circ$ then $\sin\theta \to \cos\theta$, so $v_0 \to \infty$. This also makes sense, because the ball is aimed right at the corner of the cliff, so if it isn't thrown infinitely fast, it will have time to fall down relative to the "zero-gravity" straight-line path and hence hit below the corner.

Remark: Since the speed $v_0$ goes to infinity for both $\theta \to 45^\circ$ and $\theta \to 90^\circ$, by continuity it must achieve a minimum value for some angle in between. As an exercise, you can show that $v_0$ is minimum when $\theta = 3\pi/8 = 67.5^\circ$, which happens to be exactly halfway between $45^\circ$ and $90^\circ$.

3.15. **Throwing from a cliff**

With $y = 0$ taken to correspond to the base of the cliff, the height of the ball as a function of time is $y(t) = h + (v\sin\theta)t - gt^2/2$. The ball hits the ground when this equals zero, which gives

$$\frac{g}{2}t^2 - (v\sin\theta)t - h = 0 \implies t = \frac{v\sin\theta + \sqrt{v^2\sin^2\theta + 2gh}}{g}, \tag{3.44}$$

where we have chosen the "+" root because $t$ must be positive. (The "−" root corresponds to the negative time at which the parabolic motion would hit $y = 0$ if it were extended backward through the cliff.) The desired horizontal position at this time is

$$x = (v\cos\theta)t = \frac{v\cos\theta}{g}\left(v\sin\theta + \sqrt{v^2\sin^2\theta + 2gh}\right). \tag{3.45}$$

Limits: If $\theta = \pi/2$ then $x = 0$, of course, because the ball is thrown straight up. If $\theta = 0$ then $x = v\sqrt{2h/g}$. This is correct because the ball is fired horizontally, so the time to fall the height $h$ is the standard $\sqrt{2h/g}$. And since the horizontal speed is always $v$, the horizontal distance is $v\sqrt{2h/g}$.

If $h = 0$ then $x = (2v^2/g)\sin\theta\cos\theta$, which is the standard projectile range on flat ground. If $h \to \infty$ then we can ignore the $v^2\sin^2\theta$ under the square root, and we end up with

$$x \approx \frac{v^2\sin\theta\cos\theta}{g} + (v\cos\theta)\sqrt{\frac{2h}{g}}. \tag{3.46}$$

The first term here is the horizontal distance traveled by the time the ball reaches the highest point in its motion. The second term is the horizontal distance traveled during the time it takes to fall a height $h$ from the highest point (because $\sqrt{2h/g}$ is the time it takes to fall a height $h$).

However, the reason why Eq. (3.46) isn't exact is that after falling a distance $h$ from the highest point, the ball hasn't quite reached the ground, because there is still an extra distance to fall, corresponding to the initial gain in height from the top of the cliff to the highest point. From Eq. (3.6) this height is $(v^2/2g)\sin^2\theta$. But the ball is traveling so fast (assuming $h$ is large) during this last stage of the motion that it takes a negligible time to fall through the last $(v^2/2g)\sin^2\theta$ interval and hit the ground. The speed is essentially $gt = g\sqrt{2h/g} = \sqrt{2gh}$ at this point, so the additional time is approximately $(v^2/2g)\sin^2\theta/\sqrt{2gh}$. You can show by using a Taylor series that the next term in the approximation to the time in Eq. (3.44) would yield this tiny additional time. (You will want to factor the $2gh$ out of the square root, to put it in the standard $\sqrt{1+\epsilon}$ form.) Further corrections from additional terms in the Taylor series correspond to the fact that the speed isn't constant during this final tiny time.

3.16. **Throwing on stairs**

(a) When the ball reaches the corner of the $N$th step, it has traveled a distance of $N\ell$ both sideways and downward. So if $t$ is the time in the air, then looking at the horizontal distance gives $vt = N\ell$, and looking at the vertical distance gives $gt^2/2 = N\ell$. Solving for $t$ in the first of these equations and plugging the result into the second yields

$$\frac{g}{2}\left(\frac{N\ell}{v}\right)^2 = N\ell \implies v = \sqrt{\frac{N\ell g}{2}}. \tag{3.47}$$

Limits: Big $N$, $\ell$, or $g$ implies big $v$, as expected.

(b) As we noted above, the time needed to fall the distance $N\ell$ to the corner of the $N$th step is given by

$$\frac{1}{2}gt_N^2 = N\ell \implies t_N = \sqrt{\frac{2N\ell}{g}}. \tag{3.48}$$

Likewise, the total time needed to fall to the $(N+1)$st step is $t_{N+1} = \sqrt{2(N+1)\ell/g}$. The difference in these times is the time of flight from the $N$th corner to the $(N+1)$st step. So the horizontal distance along the $(N+1)$st step is (using the result for $v$ from part (a))

$$\begin{aligned} d = v(t_{N+1} - t_N) &= \sqrt{\frac{N\ell g}{2}}\left(\sqrt{\frac{2(N+1)\ell}{g}} - \sqrt{\frac{2N\ell}{g}}\right) \\ &= \ell\left(\sqrt{N(N+1)} - N\right). \end{aligned} \tag{3.49}$$

(c) We need to apply the Taylor series $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$ to the above result for $d$. If we take out a factor of $\sqrt{N^2}$ from the square root, it will take the requisite $\sqrt{1+\epsilon}$ form. We then have

$$d = N\ell\left(\sqrt{1+\frac{1}{N}} - 1\right) \approx N\ell\left(\left(1+\frac{1}{2N}\right) - 1\right) = \frac{\ell}{2}. \tag{3.50}$$

(d) The $x$ component of the velocity is (always) $v_x = v = \sqrt{N\ell g/2}$. The $y$ component of the velocity at the corner of the $N$th step is $v_y = gt_N = g\sqrt{2N\ell/g} = \sqrt{2N\ell g}$. The ratio of these components is $v_y/v_x = 2$.

This is consistent with the result in part (c), because for large $N$, the ball is moving very fast when it grazes the corner, so it travels essentially in a straight line in going to the next step; there's basically no time to accelerate and have the trajectory bend. From part (c), we know that the ball goes down a distance $\ell$, and sideways a distance $\ell/2$. These distances imply (in the straight-line-trajectory approximation) that the ratio of the velocity components is $v_y/v_x = \ell/(\ell/2) = 2$, in agreement with the ratio obtained directly from the components calculated at the corner.

Units: Note that this ratio of 2 is independent of $\ell$, $g$, and $N$. It can't depend on $g$ due to the seconds in $g$'s units. And then it can't depend on $\ell$ due to the meters in $\ell$'s units. The argument that eliminates $N$ is a little tricker, but it is short: The ratio of the velocity components can't depend on $N$, because we could simply turn each step into many little ones. $N$ therefore increases, but the velocity components are the same; the ball has no clue that we subdivided the steps, so the velocity components can't change.

3.17. **Bullet and sphere**

FIRST SOLUTION: If we take the origin to be the center of the sphere, then the projectile motion is given by

$$x(t) = v_0 t \qquad \text{and} \qquad y(t) = R - \frac{1}{2}gt^2. \tag{3.51}$$

Plugging the $t$ from the first equation into the second gives

$$y = R - \frac{gx^2}{2v_0^2}. \tag{3.52}$$

If a point on the sphere (which is a 2-D circle for our purposes) is a distance $x$ to the right of the center, then by the Pythagorean theorem it is a distance $\sqrt{R^2 - x^2}$ above the center. So the $y$ coordinate is $y = \sqrt{R^2 - x^2}$. Taking the $R^2$ out of the square root and using $\sqrt{1+\epsilon} \approx 1 + \epsilon/2$ gives the approximate expression for $y$ as a function of (small) $x$:

$$y = R\sqrt{1 - \frac{x^2}{R^2}} \approx R\left(1 - \frac{x^2}{2R^2}\right) = R - \frac{x^2}{2R}. \tag{3.53}$$

Comparing Eqs. (3.52) and (3.53), we see that the projectile motion matches up with the circle (at least for small $x$) if

$$\frac{g}{2v_0^2} = \frac{1}{2R} \implies v_0 = \sqrt{gR}. \tag{3.54}$$

If $v_0$ takes on this value, you can quickly show that the $y$ value in Eq. (3.52) satisfies $x^2 + y^2 \geq R^2$. That is, the projectile motion always lies *outside* the circle, which is intuitively reasonable. So the bullet does indeed avoid touching the sphere if $v_0 = \sqrt{gR}$.

The bullet hits the ground when $y = -R$ (because we defined the origin to be the center of the sphere). Using the expression for $y$ in Eq. (3.52) with $v_0^2 = gR$, we find the desired distance along the ground to be

$$R - \frac{x^2}{2R} = -R \implies x_g = 2R. \tag{3.55}$$

UNITS: Note that considerations of units tells us that $x_g$ must be proportional to $R$. In general it depends on $v_0$, but we've specifically chosen $v_0$ to equal $\sqrt{gR}$; and $x_g$ can't depend on $g$, due to the seconds in $g$. But it takes a calculation to show that there is a factor of 2 out front.

LIMITS: Large $R$ or large $g$ implies large $v_0$, which makes sense. Large $R$ implies large $x_g$, which also makes sense.

SECOND SOLUTION: A quicker way of finding $v_0$ is to use the fact that the radial acceleration is given by $a = v^2/r$, where $r$ is the radius of the circle that the parabolic trajectory instantaneous matches up with. But the acceleration of the bullet is also of course just $g$, because it is undergoing freefall projectile motion (assuming that it isn't touching the sphere). So if we want the radius of the instantaneous circle to be $R$, then we need

$$\frac{v_0^2}{R} = g \implies v_0 = \sqrt{gR}. \tag{3.56}$$

3.18. **Throwing on an inclined plane**

(a) If $\theta$ is very small (so that the plane is nearly horizontal), then the point $P$ is essentially at the top of the ball's parabolic motion. So $P$ is certainly higher than the starting point.

In the other extreme where $\theta$ is close to 90° (so that the plane is nearly vertical), the ball starts out moving essentially horizontally. So all later points in the motion (including $P$) are lower than the starting point.

Therefore, by continuity there must exist a $\theta$ between 0 and 90° for which $P$ has the same height as the starting point.

(b) We'll calculate the special value of $\theta$ by finding the time at which the ball returns to the starting height, and also the time at which the ball is at point $P$ farthest from the plane. We'll then set these two times equal to each other.

The ball is thrown at an angle of $90° - \theta$ with respect to horizontal, so the vertical component of the initial velocity is $v_0 \cos\theta$ (as opposed to the usual $v_0 \sin\theta$). The time to the top of the parabolic motion is therefore $v_0 \cos\theta/g$. The time to fall back down to the initial height is twice this, or $2v_0 \cos\theta/g$.

The time to reach the point $P$ farthest from the plane can be found in the following quick manner, by using tilted axes. The acceleration components parallel and perpendicular to the plane are $g \sin\theta$ and $g \cos\theta$, respectively. We aren't concerned with the first of these; the motions in the two tilted directions are independent, and we are concerned only with the motion perpendicular to the plane. The $g \cos\theta$ acceleration perpendicular to the plane tells us that the time to reach $P$ equals $v_0/g\cos\theta$. This is true because the velocity component perpendicular to the plane at the start is just $v_0$, while the velocity component perpendicular to the plane at $P$ is zero (by definition, since $P$ is the farthest point from the plane). Basically, someone living in the tilted-axis world would think that the acceleration due to gravity is $g \cos\theta$ "downward" toward the plane, while there is also an additional mysterious force causing an acceleration of $g \sin\theta$ "rightward."

The time to return to the initial height is the same as the time to reach $P$ if

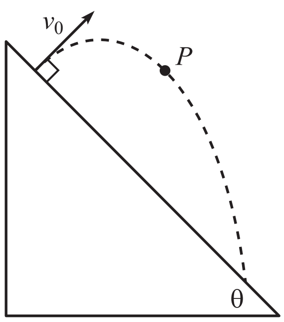


Figure 3.26

$$\frac{2v_0\cos\theta}{g} = \frac{v_0}{g\cos\theta} \implies \cos^2\theta = \frac{1}{2} \implies \theta = 45°. \tag{3.57}$$

The corresponding trajectory is sketched in Fig. 3.26. You can show that $P$ is a distance $v_0^2/\sqrt{2}g$ from the plane in this case.

3.19. **Ball landing on a block**

First solution: Let the initial speed of the block be $u$, and let the initial speed of the ball be $v$. Our strategy will be to (1) equate the times when the ball hits the plane and when the block reaches its maximum height, and then (2) equate the distances along the plane at this time.

Since the slope of the plane is $\tan\beta$, the ball hits the plane when its coordinates satisfy $y/x = \tan\beta$. Using $y = (v\sin\theta)t - gt^2/2$ and $x = (v\cos\theta)t$, this becomes

$$\frac{(v\sin\theta)t - gt^2/2}{(v\cos\theta)t} = \tan\beta \implies t_{\text{hit}} = \frac{2v\cos\theta}{g}(\tan\theta - \tan\beta). \tag{3.58}$$

The block reaches its maximum height at $t_{\text{max}} = u/(g\sin\beta)$, because the acceleration downward along the plane is $g\sin\beta$, so this is the time when the speed is zero. Equating $t_{\text{max}}$ with $t_{\text{hit}}$ gives

$$\frac{u}{g\sin\beta} = \frac{2v\cos\theta}{g}(\tan\theta - \tan\beta) \implies u = 2v\sin\beta\cos\theta(\tan\theta - \tan\beta). \tag{3.59}$$

Now let's demand that the distances along the plane are equal at this time. The ball's distance along the plane is $x/\cos\beta = (v\cos\theta)t/\cos\beta$. And the block's distance is $ut - (g\sin\beta)t^2/2$, because it undergoes motion with constant acceleration $g\sin\theta$ pointing down the plane. Equating these distances, canceling a factor of $t$, and using $t = u/(g\sin\beta)$ from above, we obtain

$$\frac{v\cos\theta}{\cos\beta} = u - \frac{1}{2}(g\sin\beta)\frac{u}{g\sin\beta} \implies v\cos\theta = \frac{u\cos\beta}{2}. \tag{3.60}$$

This makes sense; the constant horizontal speed of the ball (the left-hand side) correctly equals the average horizontal speed of the block (the right-hand side); the block starts with

$u_x = u\cos\theta$ and ends up with $u_x = 0$. Plugging the $u$ from Eq. (3.59) into Eq. (3.60) gives

$$\begin{aligned} v\cos\theta &= v\sin\beta\cos\theta(\tan\theta - \tan\beta)\cdot\cos\beta \\ \Longrightarrow\quad 1 &= \sin\beta\cos\beta(\tan\theta - \tan\beta) \\ \Longrightarrow \tan\theta &= \tan\beta + \frac{1}{\sin\beta\cos\beta}\,. \end{aligned} \tag{3.61}$$

This is the desired implicit equation that gives $\theta$ in terms of $\beta$. If $\beta = 45°$ we have

$$\tan\theta = 1 + \frac{1}{\frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}} = 3 \quad\Longrightarrow\quad \theta \approx 71.6°. \tag{3.62}$$

Limits: If $\beta \to 90°$ then Eq. (3.61) gives $\theta \to 90°$; this makes sense because $\theta$ must be at least as large as $\beta$. If $\beta \to 0$ then $\theta \to 90°$; this makes sense because for a given $u$, the process takes a very long time in the $\beta \to 0$ limit (because the block hardly slows down on the plane). So the ball must spend a very long time in the air. It must therefore be thrown very fast, which means that $\theta$ must be very close to $90°$ so that the horizontal speed is essentially equal to the given speed $u$.

Second solution: In this solution, we'll work with axes parallel and perpendicular to the plane. The objects have the same acceleration along the plane, namely $-g\sin\beta$. Therefore, if their initial velocity components along the plane are equal, then these velocity components will always be equal. And since the objects start at the same place, equal velocity components implies equal positions along the plane. That is, the ball will always be "above" the block (in the tilted reference frame), which implies that the ball will land on the block. Since the angle between the ball's firing angle and the plane is $\theta - \beta$, equating the initial velocity components along the plane gives

$$u = v\cos(\theta - \beta). \tag{3.63}$$

This condition guarantees that the ball will land on the block, but let's now also demand that the landing happens when the block reaches its maximum height on the plane. As in the first solution, the time for the block to reach its maximum height (where it is instantaneously at rest) is $t_{\text{max}} = u/(g\sin\beta)$, because $g\sin\beta$ is the (magnitude of the) acceleration along the plane.

We claim that the time for the ball to return to the plane is $t_{\text{hit}} = 2\cdot v\sin(\theta-\beta)/g\cos\beta$. (You can verify by using the sum formula for sine that this agrees with Eq. (3.58).) This is true because $v\sin(\theta-\beta)$ is the initial velocity perpendicular to the plane, and $g\cos\beta$ is the (magnitude of the) acceleration perpendicular to the plane. We effectively live in a tilted world where gravity has strength $g' = g\cos\beta$ in the "upward" (tilted) direction, so the time to the "top" of the motion is the standard $v_{y'}/g'$, where $v_{y'} = v\sin(\theta-\beta)$ is the initial "upward" velocity. The total time is twice this; hence the above expression for $t_{\text{hit}}$. There is also a mysterious "sideways" acceleration $g\sin\beta$ along the plane in our tilted world, but this doesn't come into play when calculating the time to return to the plane.

We want $t_{\text{max}}$ and $t_{\text{hit}}$ to be equal:

$$t_{\text{max}} = t_{\text{hit}} \quad\Longrightarrow\quad \frac{u}{g\sin\beta} = \frac{2v\sin(\theta-\beta)}{g\cos\beta}\,. \tag{3.64}$$

Substituting the $u$ from Eq. (3.63) into Eq. (3.64) gives

$$\frac{v\cos(\theta-\beta)}{g\sin\beta} = \frac{2v\sin(\theta-\beta)}{g\cos\beta} \quad\Longrightarrow\quad 1 = 2\tan(\theta-\beta)\tan\beta. \tag{3.65}$$

You can use the sum formula for tan to verify that this leads to the same value of $\tan\theta$ we found in Eq. (3.61).

3.20. ***g*'s in a washer**

1000 revolutions per minute equals 16.7 revolutions per second. A point on the surface of the drum moves a distance of $2\pi r = 2\pi(0.3\,\text{m}) = 1.88\,\text{m}$ during one revolution, so the speed of such a point is $v = (1.88\,\text{m})(16.7\,\text{s}^{-1}) \approx 31.4\,\text{m/s}$. The radial acceleration is therefore $a_\text{r} = v^2/r = (31.4\,\text{m/s})^2/(0.3\,\text{m}) = 3300\,\text{m/s}^2$. Since one $g$ is about $10\,\text{m/s}^2$, this acceleration is equivalent to about 330 $g$'s. That's huge!

REMARK: Since 31.4 m/s equals about 70 miles per hour, the results in this problem carry over to the spinning wheels on a car moving at a good highway clip. Note that it is irrelevant that the center of the wheel is moving down the road, as opposed to remaining stationary like the washing machine. In the reference frame moving along with the car, the wheel is simply spinning in place, so we have the same setup as with the washing machine. Adding on the constant velocity of the car to transform to the reference frame of the ground doesn't affect the acceleration of a point on the rim, because the derivative of a constant velocity is zero.

3.21. **Acceleration after one revolution**

The tangential acceleration is always $a_\text{t}$, and the radial acceleration is $v^2/R$. So our goal is to find the value of $v$ after one revolution. The time it takes to complete one revolution is given by $a_\text{t}t^2/2 = 2\pi R \implies t = \sqrt{4\pi R/a_\text{t}}$. The speed after one revolution is therefore $v = a_\text{t}t = \sqrt{4\pi R a_\text{t}}$. (This also follows from the standard $v = \sqrt{2ad}$ kinematic result.) The radial acceleration is then $a_\text{r} = v^2/R = 4\pi a_\text{t}$, which is more than 12 times $a_\text{t}$ (a surprisingly large factor). The angle that the acceleration vector **a** makes with the radial direction is given by $\tan\theta = a_\text{t}/a_\text{r} = a_\text{t}/(4\pi a_\text{t}) = 1/4\pi$. This angle is about 4.5°, which means that **a** points only slightly away from the radial direction.

The angle doesn't depend on $a_\text{t}$ or $R$, because an angle is a dimensionless quantity, and it is impossible to form a dimensionless quantity from $a_\text{t}$ (which has units of $\text{m/s}^2$) and $R$ (which has units of m).

REMARK: The angle $\theta$ that **a** makes with the radial direction starts off at 90° (when $v = 0 \implies a_\text{r} = 0$) and approaches zero after a long time (when $v \to \infty \implies a_\text{r} \to \infty$). From the above reasoning, you can show that the general result for $\theta$ is $\tan\theta = R/2d$, where $d$ is the distance traveled around the circle. The angle that **a** makes with the radial direction therefore equals, for example, 45° when the car has traveled a distance $d = R/2$, which corresponds to 28.6° around the circle (half of a radian).

3.22. **Equal acceleration components**

(a) The acceleration components are $a_\text{r} = v^2/R$ and $a_\text{t} = dv/dt$. We are told that $|dv/dt| = v^2/R$. Let's assume for now that $dv/dt$ is positive (so that the object is speeding up), in which case we can ignore the absolute value operation. Separating variables in $dv/dt = v^2/R$ and integrating gives

$$\int_{v_0}^{v} \frac{dv'}{v'^2} = \int_0^t \frac{dt'}{R} \quad\Longrightarrow\quad -\frac{1}{v}\bigg|_{v_0}^{v} = \frac{t}{R}$$
$$\Longrightarrow\quad \frac{1}{v_0} - \frac{1}{v} = \frac{t}{R} \quad\Longrightarrow\quad v(t) = \frac{1}{\dfrac{1}{v_0} - \dfrac{t}{R}}. \tag{3.66}$$

If $dv/dt$ were negative (so that the object were slowing down), then $|dv/dt|$ would be equal to $-dv/dt$. The negative sign would carry through the above calculation, and we would end up with $v(t) = 1/(1/v_0 + t/R)$.

The distance (arclength) traveled, $s$, equals the integral of $v$. That is, $s = \int v\,dt$.

Assuming that $dv/dt$ is positive, this gives

$$\begin{aligned} s(t) &= \int_0^t \frac{dt}{\frac{1}{v_0} - \frac{t}{R}} = -R \ln\left(\frac{1}{v_0} - \frac{t}{R}\right)\Big|_0^t \\ &= -R\left[\ln\left(\frac{1}{v_0} - \frac{t}{R}\right) - \ln\left(\frac{1}{v_0}\right)\right] \\ &= -R\ln\left(\frac{1/v_0 - t/R}{1/v_0}\right) = -R\ln\left(1 - \frac{v_0 t}{R}\right). \end{aligned} \tag{3.67}$$

If $dv/dt$ is negative, the distance comes out to be $s(t) = R\ln(1 + v_0 t/R)$.

Limits: Using the Taylor approximation $\ln(1-\epsilon) \approx -\epsilon$, we find that if $t$ is small then $s(t)$ for the $dv/dt > 0$ case behaves like $s(t) \approx -R(-v_0 t/R) = v_0 t$. This is correct, because the object hasn't had any time to change its speed. The $s(t)$ for the $dv/dt < 0$ case also correctly reduces to $v_0 t$.

(b) In the case where $dv/dt$ is positive, the special value of $t$ is $T = R/v_0$. At this time, both the $v$ in Eq. (3.66) and the $s$ in Eq. (3.67) go to infinity. After this time, the stated motion is impossible.

In the case where $dv/dt$ is negative, $v$ goes to zero as $t \to \infty$. But it goes to zero slowly enough so that the distance $s$ diverges (slowly, like a log) as $t \to \infty$.

Limits: Small $v_0$ implies large $T$, and large $R$ also implies large $T$. These make intuitive sense.

3.23. **Horizontal acceleration**

Let $\theta$ be the angular position below the horizontal. Then the height fallen is $R + R\sin\theta$, which gives a speed of $v = \sqrt{2gh} = \sqrt{2gR(1+\sin\theta)}$. The radial acceleration is then $a_r = v^2/R = 2g(1+\sin\theta)$. The tangential acceleration comes from the tangential component of gravity, so it is simply $a_t = g\cos\theta$. The total acceleration is horizontal if the vertical components of $a_r$ and $a_t$ cancel, as shown in Fig. 3.27.[2] These two vertical components are $a_r \sin\theta$ upward and $a_t\cos\theta$ downward. So we want

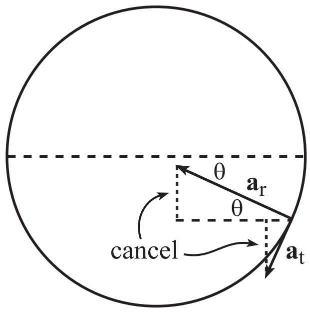



**Figure 3.27**

$$\begin{aligned} a_r \sin\theta &= a_t\cos\theta \\ \Longrightarrow\quad 2g(1+\sin\theta)\cdot\sin\theta &= g\cos\theta\cdot\cos\theta \\ \Longrightarrow\quad 2\sin\theta + 2\sin^2\theta &= \cos^2\theta \\ \Longrightarrow\quad 2\sin\theta + 2\sin^2\theta &= 1 - \sin^2\theta \\ \Longrightarrow\quad 3\sin^2\theta + 2\sin\theta - 1 &= 0 \\ \Longrightarrow\quad (3\sin\theta - 1)(\sin\theta + 1) &= 0 \\ \Longrightarrow\quad \sin\theta &= \frac{1}{3}, \end{aligned} \tag{3.68}$$

which gives $\theta \approx 19.5°$ or $\theta \approx 160.5°$. The positions corresponding to these two angles are shown in Fig. 3.28.

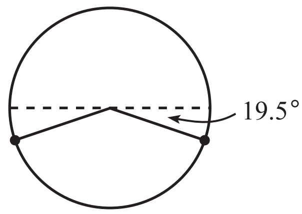



**Figure 3.28**

There is also another root of the above quadratic equation, namely $\sin\theta = -1 \Longrightarrow \theta = -90°$. This corresponds to the top of the hoop, where $a_r = a_t = 0$. So the acceleration does indeed have a zero vertical component there. But it's semantics as to whether or not the zero vector is "horizontal."

You can also solve this problem by considering the forces on the bead; see Problem 5.22.

[2]A common mistake is to say that the vertical component of $a_r$ should cancel the downward acceleration $g$ due to gravity. This isn't correct, because part of the gravitational force (the radial component) is already "included" in the radial acceleration. So you'd be double counting this component of gravity.