Notes in Introductory Real Analysis

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Contents

Introductory Remarks

These notes were written for an introductory real analysis class, Math 4031, at LSU in the Fall of 2006. In addition to these notes, a set of notes by Professor L. Richardson were used.

There are several different ideologies that would guide the presentation of concepts and proofs in any course in real analysis:

(i) the historical way

(ii) the most natural way

(iii) the most efficient way

(iv) a comprehensive way, explaining the insights from several different approaches

The reality of constraints of time makes (iii) the most convenient approach, and perhaps the best example of this approach is Rudin's Principles of Mathematical Analysis [5]. The downside is that there is little possibility of conveying any insights or intuition.

In studying the notion of completeness, a choice has to be made whether to treat the Cauchy sequence point of view or the existence of suprema as fundamental. I have chosen the latter; it conforms to the classical geometric notion of a positive real number being specified by quantities greater than it and those less than it. This point of view also guides the choice of approach in the treatment of the Riemann integral; the Riemann integral of a function is the unique real number lying between the upper Riemann sums and lower Riemann sums.

The notes here do not include a chapter on continuous functions, for which we followed the Richardson notes.

These notes have not been proof read carefully. I will update them time to time. Comments from many students have helped improve the notes. Among those who deserve thanks are John Tate (in-class comments) and Daniel Donovan (email 2014).

Chapter 1

Ordered Fields and The Real Number System

In this chapter we go over the essential, foundational, facts about the real number system.

Positive real numbers arose from geometry in Greek mathematics, as ratios of magnitudes, such as segments or planar regions or even angles. In the discussion below we focus on segments.

In Euclid's *Elements*, a segment EF is taken to exceed a segment GH, symbolically

$$\text{EF}>\text{GH}$$

if EF is congruent to a segment GK, where K is some point between E and F. An important feature of this order relation is encapsulated in the *archimedean axiom*: *given any two segments, some multiple of any one of them exceeds the other*.

Then aAa pair PQ and RS if for any positive numbers n and m, the segment nAB (which is n copies of AB laid contiguously) exceeds the segment mCD if and only if the segment nPQ exceeds the segment mRS. The ratio

$$\frac{\text{AB}}{\text{CD}}$$

is then essentially the equivalence class of all segment pairs which are in the same ratio as AB is to CD. Euclid also defines the ratio XY/ZW to be *greater* than the ratio PQ/RS :

$$\frac{\text{XY}}{\text{ZW}} > \frac{\text{PQ}}{\text{RS}}$$

if they are unequal and if whenever mZW>nXY then also mRS>nPQ.

A special case is that of *commensurate* segments: if a whole multiple of AB, say nAB, is congruent to a whole multiple of CD, say mCD, then the ratio $\frac{\mathrm{AB}}{\mathrm{CD}}$ is *rational*, and is denoted by

$$\frac{m}{n}.$$

It is readily checked that

$$\frac{m}{n} = \frac{p}{q}$$

if and only if

$$qm = pn.$$

Such ratios are the *rational numbers*. Other ratios are *irrational*. In either case, Euclid's considerations suggest that a ratio of segments may be understood in terms of a set of rational numbers, for example the set of all those rationals which exceed the given ratio.

The axioms of geometry, and the Euclidean construction procedures, show that ratios of segments can be added and multiplied and, when 0 and negatives are included, an algebraic structure called a *field* emerges (this is discussed at length by Hilbert [3]). The maximal such field, respecting the axioms of geometry pertaining to the order relation and congruence, constitutes the *real number system* $\mathbb{R}$.

Leaving aside the historical background, the real number system may be constructed by starting with the empty set, constructing the natural numbers, then the rationals, and then the real numbers by Dedekind's method of identifying a real number with a splitting of the rationals into two disjoint classes with members of one class exceeding those of the other.

Dedekind's method has beed generalized in a striking, and vastly more powerful way, by Conway [1], who shows how the Dedekind cut method can be applied to abstract sets leading to the construction of all real numbers as well as transcendentals and infinitesimals. Knuth's novel [4] is an unusual and entertaining presentation of this construction.

1.1 Ordered Fields

In this section we define and prove simple properties of fields, ordered fields, and absolute values. The reader wishing to move on to properties of the real numbers may skim the contents of the first few subsections, and proceed to subsection 1.1.4.

1.1.1 Fields

A *field* $\mathbb{F}$ is a set along with two binary operations

$$\text{Addition} : \mathbb{F}\times\mathbb{F}\to\mathbb{F} : (a,b)\mapsto a+b \tag{1.1}$$

and

$$\text{Multiplication} : \mathbb{F}\times\mathbb{F}\to\mathbb{F} : (a,b)\mapsto ab \tag{1.2}$$

satisfying the following axioms:

1. The associative law holds for addition

$$a+(b+c)=(a+b)+c \qquad \text{for all } a,b,c\in\mathbb{F} \tag{1.3}$$

2. There is an element $0\in\mathbb{F}$, the *zero* or *additive identity* element, for which

$$a+0=a=0+a \qquad \text{for all } a\in\mathbb{F} \tag{1.4}$$

3. Every element $a\in\mathbb{F}$ has an *additive inverse* $-a$, called the *negative* of a:

$$a+(-a)=0=(-a)+a \tag{1.5}$$

4. The commutative law holds for addition:

$$a+b=b+a \qquad \text{for all } a,b\in\mathbb{F} \tag{1.6}$$

5. The associative law holds for multiplication

6. The associative law holds for addition

$$a(bc)=(ab)c \qquad \text{for all } a,b,c\in\mathbb{F} \tag{1.7}$$

7. There is an element $1\in\mathbb{F}$, the *unit* or *multiplicative identity* element, for which

$$a1=a=1a \qquad \text{for all } a\in\mathbb{F} \tag{1.8}$$

8. Every non-zero element $a\in\mathbb{F}$ has an *multiplicative inverse* a^{-1}, called the *reciprocal* of a:

$$aa^{-1}=1=a^{-1}a \qquad \text{for all non-zero } a\in\mathbb{F} \tag{1.9}$$

9. The commutative law holds for multiplication:

$$ab = ba \qquad \text{for all } a, b \in \mathbb{F} \tag{1.10}$$

10. The *distributive law* holds:

$$a(b+c) = ab + ac, \qquad (b+c)a = ba + ca \quad \text{for all } a, b, c \in \mathbb{F} \tag{1.11}$$

11. The element 1 is not equal to the element 0:

$$1 \neq 0$$

We have not attempted to provide a minimal axiom set, and some of the axioms may be deduced from others. For instance, the commutativity of addition can be deduced from the other axioms.

Because of the associative laws, we will just write

$$a + b + c$$

instead of $a + (b + c)$, and

$$abc$$

instead of abc.

Let us note a few simple consequences:

Theorem 1 *If $x \in \mathbb{F}$ is such that*

$$a + x = a \quad \textit{for some } a \in \mathbb{F}$$

and $y \in \mathbb{F}$ is such that

$$by = b \quad \textit{for some non-zero } b \in \mathbb{F}$$

then

$$x = 0 \qquad \textit{and} \qquad y = 1.$$

In particular, the additive identity and the multiplicative identity are unique. Moreover,

$$-0 = 0 \qquad \textit{and} \qquad 1^{-1} = 1$$

Proof. Adding $-a$ to
$$a+x=a$$
shows that x is 0. Multiplying
$$by=b$$
by b^{-1} shows that y is 1. The other claims follow from
$$0+0=0 \qquad \text{and} \qquad 1\cdot 1=1. \boxed{\text{QED}}$$

Theorem 2 *If $a,b \in \mathbb{F}$, and $b \neq 0$, then*
$$-(-a)=a, \qquad \textit{and} \qquad (b^{-1})^{-1}=b.$$
Moreover,
$$(-a)b=-ab, \qquad \textit{and} \qquad (-a)(-b)=ab.$$

The multiplicative inverse a^{-1} is best written as the reciprocal:
$$\frac{1}{b}=b^{-1},$$
and the product ab^{-1} as
$$\frac{a}{b}=ab^{-1}$$
There are many other easy consequences of the axioms which we will use without comment.

We denote the set of *natural numbers* by $\mathbb{P}$:
$$\mathbb{P}=\{1,2,3,...\}, \tag{1.12}$$
and the set of *integers* by
$$\mathbb{Z}=\{0,1,-1,2,-2,3,-3,...\}, \tag{1.13}$$
We can multiply any element $a \in \mathbb{F}$ by an integer as follows. First define
$$1a=a,$$
where now 1 is the number one in $\mathbb{Z}$. Next,
$$2a \overset{\text{def}}{=} a+a,$$

and, inductively,

$$(n+1)a \stackrel{\text{def}}{=} na + a \tag{1.14}$$

for all $a \in \mathbb{F}$ and all $n \in \mathbb{P}$. Next define negative multiples by

$$(-n)a = -(na) \tag{1.15}$$

for all $n \in \mathbb{P}$ and all $a \in \mathbb{F}$. Finally,

$$0a = a \tag{1.16}$$

where 0 is the integer 0. The following facts can be readily verified:

$$x(ya) = (xy)a \quad \text{for all } x, y \in \mathbb{Z} \text{ and all } a \in \mathbb{F} \tag{1.17}$$

$$x(a+b) = xa + xb \quad \text{for all } x \in \mathbb{Z} \text{ and all } a, b \in \mathbb{F} \tag{1.18}$$

$$(x+y)a = xa + ya \quad \text{for all } x, y \in \mathbb{Z} \text{ and all } a \in \mathbb{F} \tag{1.19}$$

There is a multiplicative analog of this given by powers of elements. If m is a positive integer and $a \in \mathbb{F}$ we define

$$a^1 = a$$

and

$$a^{m+1} = a^m a.$$

We also define

$$a^0 = 1 \qquad \text{for non-zero } a \in \mathbb{F}$$

and, for any positive integer m and non-zero $a \in \mathbb{F}$,

$$a^{-m} = (a^{-1})^n = \frac{1}{a^m}$$

We will use without comment simple facts such as

$$(a^m)^n = a^{mn},$$

valid for suitable $a \in \mathbb{F}$ and $m, n \in \mathbb{Z}$.

The simplest example of a field is the two-element field

$$\mathbb{Z}_2 = \{0, 1\},$$

with addition and multiplication defined modulo 2; for example,

$$1 + 1 = 0 \qquad \text{in } \mathbb{Z}_2$$

We will, however, be concerned with fields which permit a consistent ordering of their elements.

1.1.2 Order Relations

An *order relation* on a set S is a set O of ordered pairs (x,y) of elements of S satisfying the conditions O1 and O2 below. It is convenient to adopt the convention that

$$x < y \text{ means } (x,y) \in O$$

We also write

$$y > x$$

to mean $x < y$. The axioms of order are:

O1. For any $x, y \in \mathbb{F}$ exactly *one* of the following hold:

$$x = y, \quad \text{or} \quad x < y, \qquad \text{or} \quad y < x.$$

O2. If $x < y$ and $y < z$ then $x < z$.

It is also convenient to use the notation:

$$x \leq y \text{ means } x = y \text{ or } x < y$$

and, similarly,

$$x \geq y \text{ means } x = y \text{ or } x > y$$

If T is a subset of an ordered set S then an element $u \in S$ is said to be an *upper bound* of T if

$$t \leq u \quad \text{for all } t \in T \tag{1.20}$$

If there is a least such upper bound then that element is called the *supremum* of T:

$$\sup T = \text{the least upper bound of } T \tag{1.21}$$

We define similarly *lower bounds* and *infimums*:

$$\text{if } l \leq t \text{ for every } t \in T \text{ then } l \text{ is called a lower bound of } T \tag{1.22}$$

and

$$\inf T = \text{the greatest lower bound of } T \tag{1.23}$$

Of course, the sup or the inf might not exist.

1.1.3 Ordered Fields

An *ordered field* is a field $\mathbb{F}$ with an order relation in which, in addition to the field and order axioms stated above, the following hold:

OF1. If $x,y,z \in \mathbb{F}$ and $x < y$ then $x+z < y+z$:

$$x < y \text{ implies } x+z < y+z \text{ for all } x,y,z \in \mathbb{F} \tag{1.24}$$

OF2. If $x,y,z \in \mathbb{F}$ and $x < y$, and if also $z > 0$, then $xz < yz$:

$$x < y \text{ and } z > 0 \text{ imply } xz < yz \text{ for all } x,y,z \in \mathbb{F}. \tag{1.25}$$

If $x > 0$ we say that x is *positive*. If $x < 0$ we say that x is *negative*.
We have the following simple observations for an ordered field:

Theorem 3 *Let $\mathbb{F}$ be an ordered field. Then*

(i) $x > 0$ *if and only if* $-x < 0$

(ii) For any non-zero $x \in \mathbb{F}$ we have $x^2 > 0$

(iii) $1 > 0$

(iv) For any $r \in \mathbb{Z}$ the element $r1 \in \mathbb{F}$ is > 0 if r is a positive integer and is < 0 if r is a negative integer

(v) $x > y$ holds if and only if $x - y > 0$

(vi) If $x \in \mathbb{F}$ and $x > 0$ then $1/x > 0$

(vii) The product of two positive elements is positive

(viii) The product of a positive and negative is negative

(ix) The product of two negative elements is positive

(x) If $x > y$ and $z < 0$ then $xz < yz$.

(xi) If $x > y$ then $-x < -y$

(xii) If $x > y > 0$ then $1/x < 1/y$

Proof. We prove some of the results.

(i) Suppose $x > 0$. Then we have

$$x + (-x) > 0 + (-x),$$

and so

$$0 > -x.$$

Conversely, if $-x < 0$ then adding x to both sides shows that $x > 0$.

(ii) If $x > 0$ then, by OF2, we have

$$x^2 = xx > x0 = 0.$$

On the other hand, if $x < 0$ then we know that $-x > 0$ and so

$$x^2 = (-x)(-x) > (-x)0 = 0.$$

(iii) Since 1 is 1^2, it follows that $1 > 0$.

(vi) Suppose $x > 0$. Since $x(1/x) = 1$, and $1 \neq 0$, it follows that $1/x$ cannot be zero. If $1/x < 0$ then, however, $x(1/x)$ would have to be < 0, but we know that $1 > 0$. Thus, $1/x > 0$.

(xi) If $x > y$ then, adding $-x - y$ shows that $-y < -x$.

(xii) If $x > y > 0$ then $xy > 0$ and hence $1/(xy) > 0$. Multiplying $x > y$ by the positive element $1/(xy)$ gives $1/y > 1/x$. QED

Observe that if $x > 0$ then

$$2x = x + x > x + 0 = x,$$

and

$$3x = 2x + x > 2x + 0 = 2x,$$

and, proceeding inductively, we have

$$0 < x < 2x < 3x < \cdots < nx < (n+1)x < \cdots \qquad \text{for all } n \in \mathbb{P} \text{ and } x > 0 \text{ in } \mathbb{F} \tag{1.26}$$

In particular, inside the ordered field $\mathbb{F}$ we have a copy of the natural numbers $1, 2, 3, \ldots$ on identifying $n \in \mathbb{P}$ with $n1_{\mathbb{F}}$, where $1_{\mathbb{F}}$ is the unit element in $\mathbb{F}$. This then leads to a copy of the integers inside $\mathbb{F}$, and we can assume that

$$\mathbb{Z} \subset \mathbb{F} \tag{1.27}$$

Going further, if $m,n \in \mathbb{Z}$, and $n \neq 0$, we have then the ratio

$$\frac{m}{n} \in \mathbb{F}.$$

The set of all such ratios m/n is the set of *rationals*

$$\mathbb{Q} \subset \mathbb{F} \tag{1.28}$$

and is an ordered subfield of the ordered field $\mathbb{F}$.

1.1.4 The absolute value function

The *absolute value* $|\cdot|$ function in an ordered field $\mathbb{F}$ is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x \leq 0 \end{cases} \tag{1.29}$$

For instance,

$$|1| = 1, \quad \text{and} \quad |-1| = 1.$$

The definition of $|x|$ shows directly that

$$|-x| = |x| \geq 0 \qquad \text{for all } x \in \mathbb{F} \tag{1.30}$$

It is also useful to observe that

$$|x| \text{ is the larger of } x \text{ and } -x \tag{1.31}$$

We think of

$$|a-b|$$

as measuring the *difference* between a and b.

We have then

Theorem 4 *For any $a,b \in \mathbb{F}$ we have:*

(i) the **triangle inequality**

$$|a+b| \leq |a| + |b| \tag{1.32}$$

Equality holds if and only if a and b are both ≥ 0 or both ≤ 0.

(ii) the absolute values differ by at most the difference in a and b:

$$\Big||a|-|b|\Big| \le |a-b| \tag{1.33}$$

(iii) $|ab| = |a||b|$.

Proof. Recall that $|x|$ is the larger of x and $-x$. Therefore, $|a|+|b|$ is greater or equal to $\pm a+(\pm b)$; in particular, it is greater or equal to $a+b$ and $(-a)+(-b)$. Consequently, $|a|+|b|$ is greater or equal to $a+b$ and $-(a+b)$, and so

$$|a|+|b| \ge |a+b|.$$

Equality holds if and only if $|a|=a$ and $|b|=b$ or $|a|=-a$ and $|b|=-b$. Thus, equality holds if and only if a and b are either both ≥ 0 or both ≤ 0.

Next, using the triangle inequality we have

$$|a-b|+|b| \ge |a+b-b| = |a|,$$

and so

$$|a-b| \ge |a|-|b|.$$

Interchanging a and b yields:

$$|b-a| \ge |b|-|a|.$$

Observe that

$$|b-a| = |a-b|.$$

Thus,

$$|a-b| \text{ is } \ge \text{ to both } |a|-|b| \text{ and } -(|a|-|b|).$$

Since the larger of the latter is $\Big||a|-|b|\Big|$, we have proved (1.33).

We know that $|x|$ is x or $-x$, whichever is ≥ 0. Consequently, the product $|a||b|$ is one of the elements

$$ab, (-a)b, a(-b), (-a)(-b),$$

i.e. one of the elements ab and $-ab$. Thus, $|a||b|$ is ab or $-ab$, whichever is ≥ 0, and so it is $|ab|$. QED

1.1.5 The Archimedean Property

An ordered field $\mathbb{F}$ is said to have the *archimedean* property if for any $x, y \in \mathbb{F}$, with $x > 0$, there exists a multiple of x which exceeds y:

$$nx > y \text{ for some } n \in \{1, 2, 3, ...\} \tag{1.34}$$

The field $\mathbb{Q}$ of rationals is clearly archimedean:

Theorem 5 *The ordered field $\mathbb{Q}$ of rationals is acrhimedean.*

Proof. Let $x, y \in \mathbb{Q}$, with $x > 0$. If $y \geq 0$ then we have $1x > y$ and we are done. Now suppose $x, y > 0$. Then

$$x = \frac{p}{q} \quad y = \frac{r}{s},$$

where $p, q, r, s \in \mathbb{P}$. Take

$$n = qr + 1.$$

Then

$$nx = qr(p/q) + p/q > pr \geq \frac{r}{s} = y,$$

and we are done. QED

In an archimedean ordered field there are no infinities, and there are also no infinitesimals:

Theorem 6 *If $\mathbb{F}$ is an archimedean field then for any $w > 0$ in $\mathbb{F}$ there is an $n \in \mathbb{P}$ such that*

$$\frac{1}{n}w < x.$$

Proof. Simply choose n for which nx is $> w$. QED

1.2 The Real Number System $\mathbb{R}$

We shall work with the real number system in an axiomatic way. We will assume that it is an ordered field in which the completeness axiom of completeness holds.

Needless to say, it is essential to actually *construct* such a system so as to be sure that there is no hidden contradiction between the axioms, but we shall not describe a construction in these notes.

1.2.1 Hilbert Maximality and the Completeness Property

As we have mentioned before, the structure of Euclidean geometry, as formalized through the axioms of Hilbert, produces an archimedean ordered field. To complete the story, one can add to these axioms the further requirement that this field is maximal in the sense that it cannot be embedded inside any larger archimedean ordered field. It turns out then that any such ordered field is isomorphic to any other, and thus there is essentially one such ordered field. This ordered field is the real number system $\mathbb{R}$.

A crucial fact about $\mathbb{R}$ is the *completeness* property:

> *If L and U are non-empty subsets of $\mathbb{R}$ such that every element of L is $\leq$ every element of U, then there is a real number m which lies between L and U:*
>
> $$l \leq m \leq u \text{ for all } l \in L \text{ and all } u \in U. \tag{1.35}$$

This property is also often expressed as:

> *If $\mathbb{R}$ is partitioned into two disjoint subsets L and U whose union is $\mathbb{R}$, and if every element of L is $\leq$ every element of U then there is a unique element $x \in \mathbb{R}$ which lies between L and U:*
>
> $$l \leq x \leq u \text{ for all } l \in L \text{ and all } u \in U. \tag{1.36}$$

It is useful to view the real numbers geometrically. Consider a line, with two special points O and I marked on it. For any point P on the line on the same side of O as P we think of the ratio OP/OI as a positive real number. Points on the other side from I correspond to negative real numbers, and the point O itself should be thought of as 0. The completebess property says that there are no 'gaps' in the line.

The completeness property can be formulated equivalently as:

$$\textit{Every non-empty subset of } \mathbb{R} \textit{ which has an upper bound has a least upper bound.} \tag{1.37}$$

The completeness property implies the archimedean peoperty:

Theorem 7 *If in an ordered field the property (1.37) holds then the archimedean property also holds.*

A proof of this is outlined in an exercise below.

The modern understanding and point of view on completeness grew out of the work of Richard Dedekind [2].

1.2.2 Completeness of $\mathbb{R}$ and measurement

Even in Euclid's geometry, a real number was, implicitly, understood in terms of all rationals which exceeded it and all rationals below it. However, the traditional axioms of Euclidean geometry, with requirements on intersections of lines and circles, can work with a field which is larger than the rationals but smaller than $\mathbb{R}$, and completeness is not essential.

The simplest measurement problem is to devise a measure of sets of points in the plane which are made up of a finite collection of segments constructed by Euclidean geometry. Two such sets should have the same measure if they can be decomposed into a finite collection of congruent segments. For this we would not need the full complete system $\mathbb{R}$ of real numbers. Moving up a dimension, with the task of measuring areas of polygonal regions constructed by Euclidean geometry, one could still get away with a less-than-complete system of numbers. However, it was shown by Max Dehn in 1900, in resolving Hilbert's Third Problem, that there are polyedra in three dimensions which have equal volumes (as defined by requirements of 'upper' and 'lower' approximations) which cannot be decomposed into congruent pieces. This, along with, of course, the utility of measuring areas of curved regions even in two dimensions, makes it absolutely essential to work with a notion of measure that goes beyond simply decomposing into geometrically congruent figures. For a truly useful theory of measure, the completeness of the number system is essential.

Capturing a real number between upper approximations and lower approximations proves to be very useful. Archimedes and others computed areas of curved regions by such upper and lower approximations. In modern calculus, this method lives on in the Riemann integral, as we shall see later.

Problem Set 1

1. Prove that in any ordered field, between any two distinct elements there is at least one other element.

2. Prove that in any ordered field, between any two distinct element lie infinitely many elements.

3. If $\mathbb{F}$ is an archimedean ordered field, such as $\mathbb{R}$, show that between any two distinct elements there is a rational element.

4. Let $\mathbb{F}$ be an archimedean ordered field, and let $x, y \in \mathbb{F}$ with $x < y$. If $z \in \mathbb{F}$ show that there is a rational multiple qz of z for which $x < qz < y$.

5. Suppose $x \in \mathbb{R}$ is $\geq$ all elements of a non-empty subset $S \subset \mathbb{R}$. Explain why this implies $x \geq \sup S$.

6. Suppose S is a non-empty subset of $\mathbb{R}$ and $x \in \mathbb{R}$ is not an upper bound of S. Show that there exists $y > x$ which is also not an upper bound of S.

7. Prove that the completeness property (1.37) implies the property (1.36). [Hint: Suppose (1.37) holds. If $S \subset \mathbb{R}$ is a non-empty set which is bounded above, then we can take U to be the set of all upper bounds of S and L to be the complement, i.e. all real numbers which are $<$ some element of S. Let $x \in \mathbb{F}$ be provided by (1.36). Show that x is the least upper bound of S.]

8. Prove that (1.36) implies the completeness property (1.37).

9. Prove that the completeness property implies the archimdean propery. [Hint: Suppose $\mathbb{F}$ is an ordered field in which (1.37) holds. Let $x, y \in \mathbb{F}$, with $x > 0$. Suppose that no positive integral multiple of x exceeds y, and let $S = \{nx : n \in \mathbb{P}\}$. Then y is, by assumption, an upper bound for S. Let $u = \sup S$, the least upper bound of S. Consider the element $u - \frac{1}{2}x$. This, being $< u$, is not an upper bound. Use this to produce an element of S greater than u, thus reaching a contradiction.]

10. Suppose L and U are non-empty subsets of $\mathbb{R}$ such that (i) every element of L is $\leq$ every element of $\mathbb{R}$, and (ii) for any $\varepsilon > 0$ there is an element $l \in L$ and an element $u \in U$ with $u - l < \varepsilon$. Prove that there is a unique real number which is $\leq$ all elements of U and $\geq$ all elements of L.

Chapter 2

The Extended Real Line and Its Topology

In this chapter we study topological concepts in the context of the real line. For technical purposes, it will be convenient to extend the real line $\mathbb{R}$ by adjoining to it a largest element ∞ and a smallest element $-\infty$. No metaphysical meaning need be attached to these infinities. The primary reason for introducing them is to simplify the statements of several theorems.

2.1 The extended real line

The extended real line is obtained by a largest element ∞, and a smallest element $-\infty$, to the real line $\mathbb{R}$:

$$\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\} \tag{2.1}$$

Here ∞ and $-\infty$ are abstract elements. We extend the order relation to $\mathbb{R}$ by declaring that

$$-\infty < x < \infty \qquad \text{for all } x \in \mathbb{R} \tag{2.2}$$

Much of our work will be on $\mathbb{R}^*$, instead of just $\mathbb{R}$.

We define addition on $\mathbb{R}^*$ as follows:

$$\begin{aligned} x + \infty &= \infty = \infty + x \quad \text{for all } x \in \mathbb{R}^* \text{ with } x > -\infty & (2.3) \\ y + (-\infty) &= -\infty = (-\infty) + y \quad \text{for all } y \in \mathbb{R}^* \text{ with } y < \infty. & (2.4) \end{aligned}$$

Note that

$$\infty + (-\infty) \text{ is } \textit{not defined},$$

i.e. there is no useful or consistent definition for it.

The following algebraic facts continue to hold in $\mathbb{R}^*$:

$$x+y=y+x, \qquad x+(y+z)=(x+y)+z, \tag{2.5}$$

whenever either side of these equations holds (i.e. if one side is defined then so is the other and the equality

2.2 Neighborhoods

A *neighborhood* of a point $p \in \mathbb{R}$ is an interval of the form

$$(p-\delta, p+\delta)$$

where $\delta > 0$ is any positive real number. Thus, the neighborhood consists of all points distance less than δ from p:

$$(p-\delta, p+\delta) = \{x \in \mathbb{R} : |x-p| < \delta\}. \tag{2.6}$$

For example,

$$(1.2, 2.8)$$

is a neighborhood of 2.

A typical neighborhood of 0 is of the form

$$(-\varepsilon, \varepsilon)$$

for any positive real number ε.

A *neighborhood of* ∞ in $\mathbb{R}^* = \mathbb{R} \cup \{-\infty, \infty\}$ is a ray of the form

$$(t, \infty] = \{x \in \mathbb{R}^* : x > t\}$$

with t any real number. For example,

$$(5, \infty]$$

is a neigborhood of ∞.

A *neighborhood of* $-\infty$ in $\mathbb{R}^*$ is a ray of the form

$$[-\infty, s) = \{x \in \mathbb{R}^* : x < s\}$$

where $s \in \mathbb{R}$. An example is

$$[-\infty, 4)$$

Observe that *if U and V are neighborhoods of p then $U \cap V$ is also a neighborhood of p.* In fact, either U contains V as a subset of vice versa, and so $U \cap V$ is just the smaller of the two neighborhoods.

Observe also that if N is a neighborhood of a point p, and if $q \in N$ then q had a neighborhood lying entirely inside N. For example, the neighborhood $(2,4)$ of 3 contains 2.5, and we can form the neighborhood $(2,3)$ of 2.5 lying entirely inside $(2,4)$.

Here is a simple but fundamental observation:

Distinct points of $\mathbb{R}^*$ have disjoint neighborhoods. (2.7)

This is called the Hausdorff property of $\mathbb{R}^*$.

For example, 3 and 5 have the neighborhoods

$$(2,4) \qquad \text{and} \qquad (4.5, 5.5)$$

The points 2 and ∞ have disjoint neighborhoods, such as

$$(-1,5) \qquad \text{and} \qquad (12, \infty]$$

Exerise Give examples of disjoint neighborhoods of

(i) 2 and -4

(ii) $-\infty$ and 5

(iii) ∞ and $-\infty$

(iv) 1 and -1

2.3 Types of points for a set

Consider a set

$$S \subset \mathbb{R}^*.$$

A point $p \in \mathbb{R}^*$ is said to be an *interior point* of S if it has a neighborhood U lying entirely inside S, i.e. with

$$U \subset S.$$

For example, for the set

$$E = (-4,5] \cup \{6,8\} \cup [9,\infty],$$

the points $-2,3,11$ are interior points. The point ∞ is also an interior point of E.

A point p is an *exterior point* if it has a neighborhood U lying entirely outside S, i.e.

$$U \subset S^c.$$

For example, for the set E above, points -5, 7, and $-\infty$ are exterior to E.

A point which is neither interior to S nor exterior to S is a *boundary point* of S. Thus p is a boundary point of S if every neighborhood of p intersects both S and S^c.

In the example above, the boundary points of E are

$$-4, 5, 6, 8, 9, \infty.$$

Next consider the set

$$\{3\} \cup (5,\infty)$$

The boundary points are $3,5$, and ∞. It is important to observe that if we work with the real line $\mathbb{R}$ instead of the extended line $\mathbb{R}^*$ then we must exclude ∞ as a boundary point, because it doesn't exist as far as $\mathbb{R}$ is concerned.

Example For the set $A = [-\infty,4) \cup \{5,9\} \cup [6,7)$, decide which of the following are true and which false:

(i) -6 is an interior point (T)

(ii) 6 is an interior point (F)

(iii) 9 is a boundary point (T)

(iv) 5 is an interior point (F)

Exerise For the set $B = [-\infty,-5) \cup \{2,5,8\} \cup [4,7)$, decide which of the following are true and which false:

(i) -6 is an interior point

(ii) -5 is an interior point

(iii) 5 is a boundary point

(iv) 4 is an interior point

(v) 7 is a boundary point.

2.4 Interior, Exterior, and Boundary of a Set

The set of all interior points of a set S is denoted

$$S^0$$

and is called the *interior* of S.

The set of all boundary points of S is denoted

$$\partial S$$

and is called the *boundary* of S.

The set of all points exterior to S is the *exterior* of S, and we shall denote it

$$S^{\text{ext}}.$$

Thus, the whole extended line $\mathbb{R}^*$ is split up into three disjoint pieces:

$$\boxed{\mathbb{R}^* = S^0 \cup \partial S \cup S^{\text{ext}}} \tag{2.8}$$

Recall that a point p is on the boundary of S if every neighborhood of the point intersects both S and S^c. But this is exactly the condition for p to be on the boundary of S^c. Thus

$$\boxed{\partial S = \partial S^c.} \tag{2.9}$$

The interior of the entire extended line $\mathbb{R}^*$ is all of $\mathbb{R}^*$. So

$$\partial \mathbb{R}^* = \emptyset.$$

Example For the set $A = [-\infty, 4) \cup \{5, 9\} \cup [6, 7)$,

(i) $A^0 = [-\infty, 4) \cup (6, 7)$

(ii) $\partial A = \{4, 5, 9, 7, 6\}$

(iii) $A^c = [4, 5) \cup (5, 6) \cup [7, 9) \cup (9, \infty]$

(iv) the interior of the complement A^c is

$$(A^c)^0 = (4, 5) \cup (5, 6) \cup (7, 9) \cup (9, \infty]$$

For the set

$$G = (3, \infty)$$

the boundary of G, when viewed as a subset of $\mathbb{R}^*$, is

$$\partial G = \{3, \infty\}.$$

But if we decide to work only inside $\mathbb{R}$ then the boundary of G is just $\{3\}$.

Exerise For the set $B = \{-4, 8\} \cup [1, 7) \cup [9, \infty)$,

(i) $B^0 =$

(ii) $\partial B =$

(iii) $B^c =$

(iv) the interior of the complement B^c is

$$(B^c)^0 =$$

2.5 Open Sets and Topology

We say that a set is *open* if it does not contain any of its boundary points. For example,

$$(2, 3) \cup (5, 9)$$

is open.

Also

$$(4, \infty]$$

is open.

But

$$(3, 4]$$

is not open, because the point 4 is a boundary point.

The entire extended line $\mathbb{R}^*$ is open, because it has no boundary points.

Moreove, the empty set $\emptyset$ is open, because, again, it doesn't have any boundary points.

Notice then that every point of an open set is an interior point. Thus, a set S is open means that

$$S^0 = S.$$

Thus for an open set S each point has a neighborhood contained entirely inside S. In other words, *S is made up of a union of neighborhoods.*

Viewed in this way, it becomes clear that the *union of open sets is an open set*.

Now consider two open sets A and B. We will show that $A\cap B$ is open. Take any point

$$p \in A\cap B.$$

Then p is in both A and B. Since $p\in A$ and since A is open, there is a neighborhood U of p which is a subset of A:

$$U \subset A.$$

Similarly there is a neighborhood V of p which is a subset of B:

$$V \subset B$$

But then $U\cap V$ is a neighborhood of p which is a subset of both A and B:

$$U\cap V \subset A\cap B.$$

Thus every point in $A\cap B$ has a neighborhood lying inside $A\cap B$. Consequently, $A\cap B$ is open.

Now consider three open sets A,B,C. The intersection

$$A\cap B\cap C$$

can be viewed as

$$(A\cap B)\cap C$$

But here both $A\cap B$ and C are open, and hence so is their intersection. Thus,

$$A\cap B\cap C$$

is open. This type of argument works for any *finite* number of open sets. Thus:

The intersection of a finite number of open sets is open.

Exerise Check that the intersection of the sets $(4,\infty)$ and $(-3 5)$ and $(2,6)$ is open.

The collection of all open subsets of $\mathbb{R}$ is called the *topology* of $\mathbb{R}$.

The set of all open subsets of $\mathbb{R}^*$ is called the *topology* of $\mathbb{R}^*$.

2.6 Closed Sets

A set S is said to be *closed* if it contains all its boundary points.

In other words, S is closed if

$$\partial S \subset S$$

Thus,

$$[4,8] \cup [9,\infty]$$

is closed.

But

$$[4,5)$$

is not closed because the boundary point 5 is not in this set.

The set

$$[3,\infty)$$

is not closed (as a subset of $\mathbb{R}^*$) because the boundary point ∞ is not inside the set. But, viewed as a subset of $\mathbb{R}$ it is closed. So we need to be careful in deciding what is close and what isn't: a set may be closed viewed as a subset of $\mathbb{R}$ but not as a subset of $\mathbb{R}^*$.

The full extended line $\mathbb{R}^*$ is closed.

The empty set $\emptyset$ is also closed.

Note that the sets $\mathbb{R}^*$ and $\emptyset$ are both open and closed.

2.7 Open Sets and Closed Sets

Consider a set $S \subset \mathbb{R}^*$.

If S is open then its boundary points are all outside S:

$$\partial S \subset S^c.$$

But recall that the boundary of S is the same as the boundary of the complement S^c. Thus, for S to be open we must have

$$\partial(S^c) \subset S^c,$$

which means that S^c contains all its boundary points. But this means that S^c is closed.

Thus, *if a set is open then its complement is closed*.

The converse is also true: if a set is closed then its complement is open. Thus,

Theorem 8 *A subset of $\mathbb{R}^*$ is open if and only if its complement is closed.*

Exercise. Consider the open set $(1,5)$. Check that its complement if closed.
Exercise. Consider the closed set $[4,\infty]$. Show that its complement is open.

2.8 Closed sets in $\mathbb{R}$ and in $\mathbb{R}^*$

The set

$$[3,\infty)$$

is closed in $\mathbb{R}$, but is *not closed* in $\mathbb{R}^*$. This is because in $\mathbb{R}$ it has only the boundary point 3, which it contains; in contrast, in $\mathbb{R}^*$ the point ∞ is also a boundary point and is not in the set. Thus, when working with closed sets it is important to bear in mind the distinction between being closed in $\mathbb{R}$ and being closed in $\mathbb{R}^*$. There is no such distinction for open sets.

2.9 Closure of a set

The *closure* of a set S is obtained by throwing in all its boundary points:

$$\boxed{\overline{S} = S \cup \partial S} \tag{2.10}$$

Of course, *if S is closed then its closure is itself.*

For example, the closure of $(3,5)$ is $[3,5]$. The closure of

$$(3,\infty)$$

is $[3,\infty]$. (But, in $\mathbb{R}$ the closure of $(3,\infty)$ is $[3,\infty)$.)

Let us see what the closure of $\mathbf{Q}$ is. Now *every* point in $\mathbb{R}^*$ is a boundary point of $\mathbf{Q}$:

$$\partial\mathbf{Q} = \mathbb{R}^*,$$

because any neighborhood of any point contains both rationals (points in $\mathbf{Q}$) and irrationals (points outside $\mathbf{Q}$).

It is useful to think of the closure

$$\overline{S} = S \cup \partial S$$

in this way:

A point p is in $\overline{S}$ if and only if every neighborhood of p intersects S.

2.10 The closure of a set is closed

Consider the closure $\overline{S}$ of a set S. We will show that $\overline{S}$ is a closed set.

Take any boundary point $p \in \partial\overline{S}$. We have to show that p is actually in $\overline{S}$. Now let N be any neighborhood of p. Then N contains a point q in $\overline{S}$. Choose (as we may) a neighborhood W of q lying entirely inside N. Since $q \in \overline{S}$ it follows that W contains a point of S. Thus, N contains a point of S. So we have seen that every neighborhood of p contains a point of S. This means $p \in \overline{S}$. Thus, we have shown that every boundary point of $\overline{S}$ is in S, and so $\overline{S}$ is closed.

2.11 $\overline{S}$ is the smallest closed set containing S

Consider any closed set K with

$$S \subset K$$

Take any $p \in \overline{S}$. Then every neighborhood of p intersects S, and hence also K. Thus every neighborhood of p intersects K. So, either p is in the interior of K or on its boundary. But K is closed, and so in either case p is in K. Thus,

$$\overline{S} \subset K$$

What we hae shown then is that $\overline{S}$ is the *smallest closed set containing S as a subset.*

2.12 $\mathbb{R}^*$ is compact

There is a special property of $\mathbb{R}^*$ whose full significance is best appreciated at a later stage. However, we have the language and tools to state and prove it now and shall do so.

An *open cover* of $\mathbb{R}^*$ is a collection $\mathcal{U}$ of open sets whose union covers all of $\mathbb{R}^*$. For example,

$$\{[-\infty, 5), (-1, 8), (7, 9) \cup (11, 15), (8, \infty]\}$$

is an open cover of $\mathbb{R}^*$. It is best to draw a picture for yourself to make this clear.

More formaly, an open cover of $\mathbb{R}^*$ is a set $\mathcal{U}$ of open subsets of $\mathbb{R}^*$ such that every point $x \in \mathbb{R}^*$ falls inside some open set U in the collection $\mathcal{U}$, i.e. for each $x \in \mathbb{R}^*$ there exists $U \in \mathcal{U}$ such that

$$x \in U \in \mathcal{U}.$$

Another example of an open cover of $\mathbb{R}^*$ is

$$\{[-\infty,50),(2,100),(8,200)\cup(701,800),(150,750),(760,\infty]\}.$$

The examples of open covers of $\mathbb{R}^*$ given above are both *finite* collections of sets. Let us look at an example of a cover which uses infinitely many sets.

We start with an open cover of $\mathbb{R}$, as opposed to $\mathbb{R}^*$: take all intervals of the form $(a,a+2)$, where a runs over all integers:

$$...,(-8,-6),(-7,-5),...(-2,0),(-1,1),(0,2),(1,3),...$$

In short we are looking at the collection

$$\mathcal{U}' = \{(a,a+2) : a \in \mathbf{Z}\},$$

where, recall that $\mathbf{Z}$ is the set of all integers. This collection fails to include $-\infty$ and ∞.

To obtain an open cover of $\mathbb{R}^*$ from $\mathcal{U}'$ we could, for example, just throw in the open sets $[-\infty,-4)$ and $(5,\infty]$. Thus, this gives us an open cover of $\mathbb{R}^*$:

$$\mathcal{U} = \{[-\infty,-4),(5\infty]\}\cup\{(a,a+2) : a \in \mathbf{Z}\}.$$

If you look at this, however, you realize that even though this collection does contain infinitely many open sets, we don't reall *need* all of them to cover $\mathbb{R}^*$. Indeed, we could just use the sub-collection

$$\mathcal{V} = \{[-\infty,-4),(-3,-1),(-2,0),(-1,1),(0,2),(1,3),(2,4),(3,5),(4,6),(5,\infty]\}$$

and this would cover all of $\mathbb{R}^*$.

This is not just a feature of one particular example. It turns out that

Every open cover of $\mathbb{R}^*$ *has a finite subcover.*

This property is called *compactness*, and so

Theorem 9 $\mathbb{R}^*$ *is compact.*

We can prove the compactness of $\mathbb{R}^*$ using the completeness of the real line.

Proof of compactness of $\mathbb{R}^*$. Let $\mathcal{U}$ be an open cover of $\mathbb{R}^*$. This is a collection of open sets such that every point of $\mathbb{R}^*$ falls inside some set in the collection. In particular, $-\infty$ is in some open set $W \in \mathcal{U}$. Thus W contains a neighborhood of $-\infty$, i.e.

$$[-\infty,t) \subset W$$

for some $t \in \mathbb{R}$. So, for one thing, all the elements of $\mathbb{R}^*$ less than t are covered by just the one set W in $\mathcal{U}$; we would not need any other set from the cover if we are to stick to points to the left of t. Now let

$$S = \{x \in \mathbb{R}^* : \text{finitely many sets in } \mathcal{U} \text{ cover } [-\infty, x]\}$$

This set is not empty because $-\infty \in S$. In fact, $t-1$ is also in S. By completeness, the set S has a supremum $\sup S$. Let

$$m = \sup S.$$

We will argue in three steps:

(i) First we explain that m isn't $-\infty$.

(ii) Next we prove that m is in fact, ∞.

(iii) Finally we prove that $[\infty, m]$, i.e. all of $\mathbb{R}^*$, can be covered by just finitely many sets in $\mathcal{U}$.

For (i), note, as before, that there is a real number t such that $t-1$ is in S, and so m (being an upper bound of S) has to be $\geq t-1$. In particular, m is certainly not $-\infty$.

Now we show that $m = \infty$. Suppose to the contrary that $m < \infty$. Now there is some open set $U \in \mathcal{U}$ such that $m \in U$, because $\mathcal{U}$ covers all points of $\mathbb{R}^*$. Since m is not $-\infty$, and is being assumed to be also not ∞, it is in $\mathbb{R}$ and so there is some neighborhood

$$(m-\varepsilon, m+\varepsilon) \subset U,$$

where ε is a positive real number. Now $m-\varepsilon$ being *less* than the *least* upper bound m of S, there has to be some $x \in S$ which is greater than $m-\varepsilon$. Thus

$$m-\varepsilon < x \leq m \text{ for some } x \in S.$$

So, x being in S, there are finitely many sets, say $U_1, \ldots, U_N$, in $\mathcal{U}$, which cover the ray segment $[-\infty, x]$. The set U covers $(m-\varepsilon, m+\varepsilon)$. Thus, the collection

$$\{U_1, \ldots, U_N, U\}$$

covers all of

$$[-\infty, m+\varepsilon).$$

But this means that, for example, $m+\frac{1}{2}\varepsilon$ is in S, for the ray segment $[-\infty, m+1/2]$ is covered by the finite collection $U_1, \ldots, U_N, U$. But now we have a contradiction, because we have found a number, $m+\varepsilon/2$, greater than m, which is in S. Thus, our original hypothesis concerning m must have been wrong. So $m = \infty$.

Finally we prove that $\mathbb{R}^*$ is covered by finitely many sets in $\mathcal{U}$. The element $\infty \in \mathbb{R}^*$ falls inside some open set V in the collection $\mathcal{U}$. Therefore, there is some 'ray-neighborhood' of ∞

$$(r, \infty] \subset V,$$

for some real number r. Now r being less than $m = \infty$, and the latter being the *least* upper bound of S, there must be some $y \in (r, \infty]$ which is in S. Thus,

$$[-\infty, y]$$

is covered by finitely many open sets, say $V_1, \ldots, V_k$, in $\mathcal{U}$. Note that

$$[-\infty, y] \cup (r, \infty] = \mathbb{R}^*.$$

But then

$$V_1, \ldots, V_k, V$$

cover all of $[-\infty, \infty]$, since V covers the segment $(r, \infty]$. This prove that finitely many sets from $\mathcal{U}$ cover all of $\mathbb{R}^*$.

2.13 Compactness of closed subsets of $\mathbb{R}^*$.

Consider now any closed subset D of $\mathbb{R}^*$. We will prove that it is compact, i.e. that any open cover of D has a finite sub-cover.

Consider any open cover $\mathcal{U}$ of D. This is a collection of open sets such that every point of D is covered by some set in the collection. Now the set D^c, being the complement of a closed set, is an open set. Throw this into the collection, and consider

$$\mathcal{U}' = \mathcal{U} \cup \{D^c\}.$$

This covers *all* of $\mathbb{R}^*$: any point of D would be covered by a set in $\mathcal{U}$ while any point in D^c is, of course, covered by D^c. Then we know that there has to be a finite sub-collection

$$\mathcal{V}' \subset \mathcal{U}'$$

which covers $\mathbb{R}^*$. Now throw out D^c from $\mathcal{V}'$ in case it is there, and consider the collection

$$\mathcal{V} = \mathcal{V}' \setminus \{D^c\}.$$

Of course, this is a finite subcollection of $\mathcal{U}$. Moreover, it covers all points of D, because no point in D could have been covered by the 'rejected' set D^c. Thus, D is covered by a finite sub-collection of sets from $\mathcal{U}$.

Thus, every closed subset of $\mathbb{R}^*$ is compact. The converse is also true, and we can state:

Theorem 10 *A subset of $\mathbb{R}^*$ is compact if and only if it is closed.*

We will leave out the proof of the converse part of this result.

2.14 The Heine-Borel Theorem: Compact subsets of $\mathbb{R}$

A subset B of $\mathbb{R}$ is said to be *bounded* if

$$B \subset [-N, N]$$

for some real number N. Thus, for B to be bounded, there should exist a real number N such that

$$|x| < N \text{ for all } x \in B.$$

Recall that a subset of $\mathbb{R}$ is closed in $\mathbb{R}$ if it contains no boundary point. Equivalently, if a subset of $\mathbb{R}$ is closed if its complement in $\mathbb{R}$ is open.

Consider, for instance, the set

$$[4, \infty) \subset \mathbb{R}.$$

This is a closed subset of $\mathbb{R}$ because, in $\mathbb{R}$, its only boundary point is 4, and this point lies in the set. Note that $[4, \infty)$ is *not* closed when considered as a subset of $\mathbb{R}^*$.

We can now state the **Heine-Borel theorem**:

Theorem 11 *Every closed and bounded subset of $\mathbb{R}$ is compact. Conversely, every compact subset of $\mathbb{R}$ is closed and bounded.*

We shall prove half of this. Suppose $K \subset \mathbb{R}$ is closed and bounded as a subset of $\mathbb{R}$. Boundedness implies that

$$K \subset [-N, N]$$

for some real number N. Then K is closed also as a subset of $\mathbb{R}^*$. Let's check this. We will show that the complement U of K in $\mathbb{R}^*$ is open. Consider any point $p \in U$. If $p \in \mathbb{R}$ then we already know, by closedness of K, that p has a neighborhood lying inside U. If $p = \infty$ then the neighborhood of p given by

$$(N+1, \infty]$$

lies entirely outside K. If $p = -\infty$ then the neighborhood

$$[-\infty, -N-1)$$

is entirely outside K. Thus, in all cases, p has a neighborhood outside K in $\mathbb{R}^*$. So the complement of K in $\mathbb{R}^*$ is open, and hence K is closed in $\mathbb{R}^*$. We know that then K must be compact.

2.15 Sequences

A sequence of elements in a set A is a string

$$a_1, a_2, a_3, \ldots$$

of elements of A.

More precisely, the sequence is actually a *mapping*

$$a : \mathbb{P} \to A : n \mapsto a_n.$$

We will often be concerned with sequences in $\mathbb{R}^*$.

Sometimes out sequence will be specified explicitly as a string of numbers; for example,

$$-5, -3, 5, 6, 9, 12, \ldots$$

Sometimes we may have a formula for the n-th term:

$$a_n = \frac{(-1)^n}{n+1}.$$

Sometimes we have a sequence specified *recursively*. For example, we might know that

$$b_1 = 1, b_2 = 1$$

and then

$$b_n = b_{n-1} + b_{n-2},$$

for all $n \geq 3$. This generates the *Fibonacci numbers*

$$1, 1, 2, 3, 5, 8, 13, 21, ...$$

Sometimes it is better to label a sequence starting with the index 0:

$$a_0, a_1, a_2, ...$$

For example, for a fixed real number r we can form the sequence

$$r^0 = 1,\ r^1 = r,\ r^2,\ r^3,\ ...$$

2.16 Limits points of a sequence

Consider a sequence

$$x_1, x_2, x_3, ...$$

in $\mathbb{R}^*$. A point $p \in \mathbb{R}^*$ is said to be a *limit point* of this sequence if every neighborhood of p is visited infinitely often by the sequence.

For example, for the sequence

$$\frac{1}{2}, 4, \qquad \frac{1}{3}, 4+\frac{1}{3}, \qquad \frac{1}{4}, 4+\frac{1}{4}, \qquad \frac{1}{5}, 4+\frac{1}{5}, \qquad \frac{1}{6}, 4+\frac{1}{6}, ...$$

it is intuitively clear that both 0 and 4 are limit points.

Suppose a sequence (x_n) lies entirely inside a set $S \subset \mathbb{R}$:

$$x_n \in S, \qquad \text{for all } n \in \mathbb{P}$$

Consider then any limit point p of this sequence. Let U be any neighborhood of p. We know that this neighborhood is visited infinitely often by the sequence. Therefore, at least one point of S must be in U. Thus, every neigborhood of p contains a point of S, and so

$$p \in \overline{S}.$$

Thus, *any limit point of a sequence which lies always in a set S must be in the closure of S.*

2.17 Bolzano-Weierstrass Theorem

Consider any sequence in $\mathbb{R}^*$. We shall prove that it must have at least one limit point.

Suppose to the contrary that no point is a limit of the given sequence. Then each point p of $\mathbb{R}^*$ has a neighborhood U_p which is visited only finitely often by the sequence. The neighborhoods U_p form an open cover of all of $\mathbb{R}^*$. Then, by compactness of $\mathbb{R}^*$, there are finitely many of them, say

$$U_{p_1}, \ldots, U_{p_N}$$

which cover all of $\mathbb{R}^*$. But this is impossible: we have covered all of the extended real line $\mathbb{R}^*$ with finitely many sets each of which is visited only finitely many times by the sequence! Thus, we have a contradiction and so there exist a limit of the sequence.

Notice that all we needed above was the compactness of $\mathbb{R}^*$. Thus, we have the more general **Bolzano-Weierstrass theorem**:

Theorem 12 *Every sequence in a compact set has at least one limit point.*

Of course, we have seen that a sequence may well have more than one limit point. For example, the sequence,

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots$$

visits every natural number infinitely often and so every natural number is a limit point of the sequence.

In the next subsection we will show how to actually obtain a limit point of a sequence.

2.18 Limit points and suprema and infima

Consider a sequence

$$x_1, x_2, x_3, \ldots$$

Let p be any limit point of this sequence.

Now let S_1 be the least upper bound of the set $\{x_1, x_2, \ldots\}$:

$$S_1 = \sup\{x_1, x_2, \ldots\}$$

In particular,

$$S_1 \geq x_n \text{ for each } n.$$

Then it is clear that no point 'to the right' of S_1 could possibly be a limit of the sequence: indeed any point $r > S_1$ would have a neighborhood lying entirely to the 'right' of S_1 and this would never be visited by the sequence. Thus, p cannot be $> S_1$. So

$$p \leq S_1$$

Now we can apply the same argument to

$$S_2 = \sup\{x_2, x_3, x_4, ...\}$$

Again, p could not be greater than S_2 for then it would have a neighborhood to the right of S_2 and this would have to be visited infinitely often by $x_1, x_2, ...$ and so at least once by $x_2, x_3, ...$ Thus,

$$p \leq S_2$$

In this way, we can form

$$S_3 = \sup\{x_3, x_4, ...\}$$

and have again

$$p \leq S_3$$

. Thus, p is a *lower bound* for all the S_k's:

$$p \leq S_k \text{ for all } k.$$

Now the inf of a set is the *greatest* lower bound of the set. Therefore,

$$p \leq \inf\{S_1, S_2, S_3, ...\}.$$

The infemum on the right is an important object and is called the *limsup* of the given sequence:

$$\boxed{\limsup_{n\to\infty} x_n \stackrel{\text{def}}{=} \inf\{\sup_{n\geq k} x_n\} : k \in \mathbb{P}\}} \tag{2.11}$$

So we have proved that

Every limit point is $\leq$ the limsup

for any sequence.

Similarly, we have the notion of liminf: it is the supermum of the sequence of infemums:

$$\boxed{\liminf_{n\to\infty} x_n \stackrel{\text{def}}{=} \sup\{\inf_{n\geq k} x_n\} : k \in \mathbb{P}\}} \tag{2.12}$$

In just the same way as before, we can prove

$$\text{Every limit point is} \geq \text{the limsup}$$

Thus, for any limit point p if any sequence (x_n) we have

$$\boxed{\liminf_{n\to\infty} x_n \leq \text{any limit point} \leq \limsup_{n\to\infty} x_n.} \tag{2.13}$$

In fact, the limsup and liminf of the sequence are also limit points, but we won't prove this here.

2.19 Limit of a sequence

A sequence (s_n) is said to lie in a set S *eventually* if after a certain value of n, all the s_n lie in S. Put another way, we say that s_n *lies in* S *for large* n, if $s_n \in S$ for all values of n beyond some value, say n_0.

For example, the sequence

$$-5, -4, -3, -2, -1, 0, 1, 2, 3, \ldots$$

will lie in the set $(20, \infty)$ eventually.

Let us note again: *a sequence* (s_n) *lies in a set* S *eventually if there is an* $n_0 \in \mathbb{P}$ *such that*

$$s_n \in S$$

for all $n \in \mathbb{P}$ *with* $n \geq n_0$.

A sequence (x_n) in $\mathbb{R}^*$ is said to have *limit* $L \in \mathbb{R}^*$ if *for any neighborhood* U *of* L *the sequence lies in this neighborhood eventually.*

We denote this symbolically as

$$x_n \to L, \qquad \text{as } n \to \infty.$$

We shall see later that if a limit exists then it is *unique*; the limit L is denoted

$$\lim_{n\to\infty} x_n.$$

The notion of limit is one of the central notions in mathematics.

For example, the sequence

$$1,2,3,4,....$$

has limit ∞. For, if we take any neighborhood of ∞, say

$$(t,\infty]$$

then eventually n will exceed t (by the archimidean property) and so the sequence will stay in $(t,\infty]$ from the n-th term onwards.

The sequence

$$1,\frac{1}{2},\frac{1}{3},\frac{1}{4},...$$

has limit 0. We will look at this more carefully soon.

2.20 Simple Examples of limits

Let us look at some examples of sequences and try to see what their limits are.

The simplest sequence is a *constant sequence* which keeps repeating the same value. For example,

$$3,3,3,3,....$$

The limit of the this sequence is 3: clearly every neighborhood of 3 is hit eventually by the sequence (indeed it is hit every time, since the sequence is stuck at 3).

The sequence

$$-1,4,5,7,8,8,8,8,8,....$$

which *eventually* stabilizes at the constant value 8 has limit 8. For, again, given any neighborhood of 8 the sequence falls inside this neighborhood eventually and stays there.

In contrast, the sequence

$$1,\ 3,\ 4,\ 1,\ 3,\ 4,\ 1,\ 3,\ 4,\ 1,\ 3,\ 4,...$$

does not have a limit. For example, the point 3 cannot be the limit of the sequence because, for instance,

$$(2.5, 3.5)$$

is a neighborhood of 3, but the sequence keeps falling outside this neighborhood (when it hits 1 or 4).

The sequence

$$1, 3, 5, 7, \ldots$$

has limit ∞. If you take any neighborhood of ∞, an interval of the form

$$(t, \infty]$$

then eventually the sequence falls inside the nighborhood and stays in there.

2.21 The sequence $1/n$

Consider again the sequence

$$1,\ 1/2,\ 1/3,\ 1/4,\ \ldots$$

It is intuitively clear that this sequence has limit 0. But let us *prove* this.

Note first that the n-th term of the sequence is

$$x_n = 1/n.$$

We have to show that given any neighborhood of 0, our sequence will eventually lie inside this neighborhood. So consider any neighborhood of 0:

$$(-\varepsilon, \varepsilon),$$

where ε is a positive real number. We have to show that x_n lies in $(-\varepsilon, \varepsilon)$ for all n beyond some value. Thus we should show that

$$\frac{1}{n} < \varepsilon$$

for all n beyond some initial value. Now the condition $1/n < \varepsilon$ is equivalent to

$$n\varepsilon > 1.$$

The archimedean property guarantees the existence of an $n_0 \in \mathbb{P}$ for which

$$n_0\varepsilon > 1.$$

Therefore, $n\varepsilon > 1$ for all integers $n \geq n_0$. This proves that the sequence does indeed tend to the limit 0:

$$\frac{1}{n} \to 0, \qquad \text{as } n \to \infty$$

2.22 The sequence R^n

Consider the sequence

$$3^0, 3^1, 3^2, 3^3, \ldots$$

It is clear that in this sequence the terms get large very quickly, and intuitively it is clear that the sequence has limit ∞.

Consider now the sequence of powers

$$R^n$$

where $R > 1$.

Let us see by how much each term exceeds the preceding:

$$R^n - R^{n-1} = R^{n-1}(R-1)$$

But R is > 1, and so the multiplier R^{n-1} is ≥ 1 (it is equal to 1 when n is actually 1). Thus,

$$R^n - R^{n-1} \geq R - 1.$$

Let us write x for $R-1$, and note that

$$x > 0$$

and we have

$$R^n - R^{n-1} > x.$$

Now it takes n 'steps' to climb from R^0 to R^n, and so

$$R^n \geq 1 + nx,$$

because each step is at least x. Now it is clear that $R^n \to \infty$ as $n \to \infty$. We just have to show that for any given t,

$$1 + nx > t$$

when n is large enough. But this is just Archimedes again: some multiple $n_0 x$ of $x$exceeds $t - 1$, and then

$$nx > t - 1$$

for all $n \geq n_0$. Note again that it is important that $x > 0$, and this comes from the fact that $R > 1$.

Thus,

$$\lim_{n\to\infty} R^n = \infty \qquad \text{for all } R > 1.$$

Now consider the case $R = 1$. In this case the sequence is just all powers of 1, and so it is

$$1, 1, 1, 1, \ldots.$$

Thus

$$\lim_{n\to\infty} R^n = 1 \qquad \text{if } R = 1.$$

Next consider the case

$$0 < R < 1.$$

As an example, we have $R = 1/3$ and the sequence

$$1, \frac{1}{3}, \frac{1}{3^2}, \frac{1}{3^3}, \ldots$$

We have here a sequence with denomiantor going to ∞ and numerator fixed at 1. It seems then clear that the sequence goes to 0. Indeed, for any real $\varepsilon > 0$ we just have to wait till

$$3^n > 1/\varepsilon$$

and this would ensure that

$$\frac{1}{3^n} < \varepsilon,$$

and so

$$\frac{1}{3^n} \in (-\varepsilon, \varepsilon) \qquad \text{for large } n.$$

Now consider the general case of

$$R^n$$

where $R > 1$. Observe that
$$R = \frac{1}{r},$$
where $r = R^{-1}$ is > 1. Thus
$$R^n = \frac{1}{r^n},$$
where $r > 1$. Now, as we have seen above, the denominator r^n goes to infinity, and so it seems clear that $1/r^n$ should decrease to 0. So we will prove that
$$R^n \to 0$$
in this case. Consider any neighborhood of 0:
$$(-\varepsilon, \varepsilon)$$
where $\varepsilon > 0$ is a positive real number. We want R^n to be in this neighborhood, i.e. R^n should be $< \varepsilon$. But this means we should show that r^n is $> 1/\varepsilon$ for all n large enough. But we know that
$$r^n \to \infty$$
So, after some n_0, we have
$$r^n > \varepsilon^{-1}$$
for all $n \geq n_0$. Consequently,
$$R^n = \frac{1}{r^n} < \varepsilon$$
and so
$$R^n \in (\varepsilon, \varepsilon),$$
for all $n \geq n_0$. Thus
$$\lim_{n\to\infty} R^n = 0 \qquad \text{if } 0 < R < 1.$$
Now it is also clear that
$$\lim_{n\to\infty} R^n = 0 \qquad \text{if } R = 0.$$
Next consider negative values. Suppose
$$-1 < R < 0.$$
For example, for $R = -1/3$ we have the sequence
$$1, -\frac{1}{3}, \frac{1}{3^2}, -\frac{1}{3^3}, \frac{1}{3^4}, \ldots$$

It is clear that this is the same sequence as $1/3^n$ except it swings back and forth between negative and positive values. Thus, this sequence has limit 0.

More generally, if

$$-1 < R < 0$$

we can look at the distance between R^n and 0:

$$|R^n - 0| = |R^n| = |R|^n$$

and

$$|R|^n \to 0 \qquad \text{as } n \to \infty,$$

because

$$0 < |R| < 1.$$

Thus, the conclusion is that

$$\lim_{n\to\infty} R^n = 0 \qquad \text{if } -1 < R < 0.$$

In short,

$$\boxed{\lim_{n\to\infty} R^n = 0 \qquad \text{if } |R| < 1.} \tag{2.14}$$

Lastly, one should look at the case

$$R \leq -1.$$

You should examine a few examples and convince yourself that then there is no limit, for the sequence swings back and forth between widely separated positive and negative values.

Exerise Examine the sequence given by powers of -2:

$$1, -2, 4, -8, 16, \ldots$$

Does this sequence visit every neighborhood of ∞? Does it visit every neighborhood of $-\infty$?

2.23 Monotone Sequences

A sequence such as

$$1, 2, 5, 8, 9, 11, \ldots$$

where each term is $\geq$ the preceding term is said to be *monotone increasing*. Another example is the sequence

$$1, 1.1, 1.11, 1.111, 1.1111, \ldots$$

Thus, a sequence (x_n) is

$$\textit{monotone increasing} \text{ if } x_n \leq x_{n+1} \text{ for all } n \in \mathbb{P} \tag{2.15}$$

We say that (x_n) is

$$\textit{monotone decreasing} \text{ if } x_n \geq x_{n+1} \text{ for all } n \in \mathbb{P} \tag{2.16}$$

An example of a monotone decreasing sequence is

$$1, 1/2, 1/3, 1/4, 1/5, \ldots$$

It is intuitively clear that a monotone increasing sequence (x_n) will tend to the limit $\sup\{x_1, x_2, \ldots\}$. This is indeed true:

If (x_n) is a monotone increasing sequence then

$$\lim_{n\to\infty} x_n = \sup_{n\geq 1} x_n.$$

To prove this let

$$L = \sup_{n\geq 1} x_n.$$

The most extreme case is when L is $-\infty$; as will become clear shortly, we need to take care of this case separately. If $L = -\infty$ each x_n must be $-\infty$. But in this case

$$\lim_{n\to\infty} x_n = -\infty,$$

since x_n is just stuck at $-\infty$. Now consider the other situation: L is not $-\infty$. We will show again that $x_n \to L$, i.e. we will show that for any neigborhood U of L, the sequence (x_n) falls eventually in U and stays there. So consider any neighborhood U of L. Pick a point t of U to the left of L, i.e.

$$t < L \text{ and } t \in U.$$

(Note that there would be no such t if L is $-\infty$.) Since L is the *least* upper bound of $\{x_1, x_2, x_3, \ldots\}$, the number t can't be an upper bound, and so some x_{n_0} is $> t$:

$$x_{n_0} > t.$$

But recall that we are dealing with a monotone *increasing* sequence. Consequently,

$$x_n > t \text{ for all } n \geq n_0.$$

But recall that L is an *upper bound* of $\{x_1, x_2, \ldots\}$; therefore,

$$\text{all the } x_n \text{ are } \leq t.$$

Those x_n which are $> t$ but $\leq L$ are, of course, in the neighborhood U of L. Thus,

$$x_n \in U \text{ for all } n \geq n_0.$$

This proves that L is indeed the limit of the sequence (x_n).

In a similar way, if x_n is a monotone decreasing sequence then

$$\lim_{n\to\infty} x_n = \inf_{n\geq 1} x_n.$$

2.24 The limit of a sequence is unique

Now we shall prove that a sequence can have at most one limit.

Consider a sequence (x_n) and suppose both L and L' are limits of the sequence, and $L \neq L'$. We will arrive at a contradiction. Since L and L' are distinct, they have *disjoint neighborhoods* U and U' respectively. Since $x_n \to L$ we know that $x_n \in U$ eventually. But $x_n \to L'$, and so $x_n \in U'$ eventually. But this is impossible since U and U' are disjoint and thus have no element in common.

Thus, a sequence which has a limit must have a *unique* limit.

2.25 Convergent sequences and Cauchy sequences

A sequence (x_n) in $\mathbb{R}$ is said to be *convergent* if it has a limit and the limit is a real number. If $x_n \to L$, and $L \in \mathbb{R}$, we also say that the sequence x_n *converges* to L.

Note that $x_n \to L \in \mathbb{R}$ if for any real $r > 0$ there is an $n_0 \in \mathbb{P}$ such that

$$|x_n - L| < r$$

for all natural numbers $n > n_0$.

This has a consequence: since the x_n's are all accumulating to L *they are also getting close to each other*. More precisely, for any $\varepsilon > 0$ we can find $N \in \mathbb{P}$ such that

$$|x_n - x_m| < \varepsilon \text{ for all natural numbers } n, m > N.$$

To prove this, observe that there is some $N \in \mathbb{P}$ such that

$$|x_k - L| < \varepsilon/2$$

for all $k \in \mathbb{P}$ with $k > N$. But then for any $n, m \in \mathbb{P}$ with $n, m > N$ we have

$$\begin{aligned} |x_n - x_m| &= |x_n - L + L - x_m| \\ &\leq |x_n - L| + |L - x_m| \\ &= |x_n - L| + |x_m - L| \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon. \end{aligned}$$

A sequence (x_n) in $\mathbb{R}$ which bunches up on itself in the sense above is said to be a *Cauchy sequence*, i.e. if for any $\delta > 0$ there is an $N \in \mathbb{P}$ such that

$$|x_n - x_m| < \varepsilon \text{ for all natural numbers } n, m > N.$$

2.26 Every Cauchy sequence is bounded

Consider a Cauchy sequence (x_n) in $\mathbb{R}$. Then, eventually the points of this sequence are at most distance 1 from each other; in fact, there is an $N \in \mathbb{P}$ such that

$$|x_n - x_m| < 1$$

for all natural numbers $n, m > N$. In particular, fixing a particular $j > N$ we have

$$x_n \in (x_j - 1, x_j + 1),$$

for all $n \geq N$. So

$$x_j - 1 < x_n < x_j + 1$$

for all $n \in \{N+1, N+2, \ldots\}$. Thus, at least from the $(N+1)$-th term on, the sequence *is* bounded. But the terms not counted here,

$$x_1, \ldots, x_N$$

are just finitely many, and so they have a largest B and a smallest A among them. Thus,

$$\text{every } x_n \text{ is} \leq \max\{B, x_j + 1\}$$

and

$$\text{every } x_n \text{ is} \geq \min\{B, x_j - 1\}$$

Thus the entire sequence is bounded.

2.27 Every Cauchy sequence is convergent

The completeness property of the real line $\mathbb{R}$ has an equivalent formulation:

Theorem 13 *Every Cauchy sequence in $\mathbb{R}$ converges.*

Let us prove this.

Consider a Cauchy sequence (x_n) in $\mathbb{R}$. We have seen that it is bounded. Thus,

$$a \leq x_n \leq b \qquad \text{for all } n \in \mathbb{P}$$

for some real numbers a and b.

We know that a sequence always has a limit point in $\mathbb{R}^*$. Let L be a limit point of (x_n). We will prove that L is a real number and $x_n \to L$.

First it is clear that L must also be in $[a, b]$. Therefore, L is not ∞ or $-\infty$, and is a real number.

Take any neighborhood

$$(L - \delta, L + \delta)$$

of L, where $\delta > 0$ is a real number.

The sequence (x_n) visits the neighborhood

$$(L - \frac{\delta}{2}, L + \frac{\delta}{2})$$

infinitely often. Now we also know that eventually, the terms of the sequence vary from each other by $< \delta/2$, i.e. there is some $N \in \mathbb{P}$ such that

$$|x_n - x_m| < \delta/2 \qquad \text{for all } n, m \in \mathbb{P} \text{ with } n, m > N.$$

We can choose some $j > N$ such that

$$x_j \in (L - \frac{\delta}{2}, L + \frac{\delta}{2})$$

because the sequence visits this neighborhood infinitely often. Then

$$\begin{aligned} |x_n - L| &= |x_n - x_j + x_j - L| \\ &\leq |x_n - x_j| + |x_j - L| \\ &< \frac{\delta}{2} + \frac{\delta}{2} \\ &= \delta. \end{aligned}$$

This means that

$$x_n \in (L - \delta, L + \delta)$$

for all $n > N$. Thus,

$$x_n \to L.$$

2.28 The rationals are countable

A set is set to be *countable* if it is finite or if its elements can be enumerated in a sequence. Thus, S is countable if there is a sequence $x_1, x_2, \ldots$ such that

$$S = \{x_1, x_2, x_3, \ldots\}$$

For example, the even numbers form a countable set:

$$\{2, 4, 6, 8, \ldots\}$$

It may seem at first that the set $\mathbf{Z}$ of integers is not countable, but we can certainly lay out all the integers in a sequence:

$$0, 1, -1, 2, -2, 3, -3, \ldots$$

It is much harder to see that the rationals $\mathbf{Q}$ are also countable. This is what we shall prove now.

We will constuct a sequence which enumerates all the rationals, i.e. a sequence $r_1, r_2, \ldots$ such that

$$\{r_1, r_2, r_3, \ldots\} = \mathbf{Q}.$$

Put another way, we will encode each rational by a unique natural number; thus to each natural number n we would associate a rational r_n, and in this way we will cover *all* rationals.

There can be many such encoding schemes. Here is one: Take any rational $x \geq 0$ and write it as $\frac{p}{q}$ where p and q are non-negative integers (of course $q \geq 1$) and q is the smallest possible denominator among all such ways of writing x. (For example, 0.6 can be written as both $6/10$ and $3/5$, but we would take $3/5$ as our representation.) We know that every non-negative rational can be represented in this way uniquely. Associate to this x the natural number

$$f(x) = 2^q \times 3^p$$

For example,

$$f(0) = f(0/1) = 2^1 \times 3^0 = 2$$

$$f(1) = f(1/1) = 2^1 3^1 = 6, \qquad f(4/5) = 2^5 \times 3^4 = 32 \times 81 = 2592.$$

This associates a unique natural number $f(x)$ to each non-negative rational x. If $x \in \mathbf{Q}$ is negative, $x < 0$, let us define $f(x)$ to be just 5 times $f(-x)$:

$$f(x) = f(-x) \times 5 \qquad \text{if } x < 0$$

Now we have labeled each rational by a unique natural number. To enumerate the rationals in a sequence all we have to do then is to run this in reverse: let r_1 be the rational number x with the smallest value for $f(x)$ (so r_1 is in fact 0 because $f(0) = 2$ is the lowest value possible for f); let r_2 be the rational number with the *next* lowest value for $f(x)$, and so on. Thus, r_n is the rational for which $f(r_n)$ is the first value of $f(x)$ greater than $f(r_1), \ldots, f(r_{n-1})$. For example,

$$r_1 = 0$$

and

$$r_2 = 1 \text{ because } f(1) = 2^1 3^1 = 6, \text{ the next value for } f(x) \text{ after } 2$$

Thus we have produced a sequence of *distinct* elements $r_1, r_2, \ldots$ such that

$$\{r_1, r_2, \ldots\} = \mathbf{Q}$$

2.29 The real numbers are uncountable

We will prove that $\mathbb{R}$ is not countable, i.e. there is no sequence which touches on all the real numbers.

In fact we will show that even the real numbers between 0 and 1 cannot be enumerated in a sequence.

The argument used here is the celebrated *diagonal method* due to Georg Cantor (1845-1912). The strategy is to make a list of strings, then take the main diagonal string, and then form a new string by altering each element of the diagonal.

Consider any sequence $x_1, x_2, \ldots$ lying in $(0,1)$. Now each real number y in $(0,1)$ can be expressed uniquely in the form

$$y = 0.y_1y_2y_3\ldots = \frac{y_1}{10} + \frac{y_2}{10^2} + \frac{y_3}{10^3} + \cdots,$$

where $y_1, y_2, \ldots \in \{0,1,2,3,\ldots,9\}$ and we exclude all such representations which use an infinite string of 9's at the end (for example, instead of 0.19999.... we would use 0.20000....). Thus each of the numbers x_1, x_2, also has such an expansion:

$$x_1 = 0.x_{11}x_{12}x_{13}\ldots$$

$$x_2 = 0.x_{21}x_{22}x_{23}\ldots$$

$$x_3 = 0.x_{31}x_{32}x_{33}\ldots$$

$$\vdots = \cdots$$

Now form a number

$$w = 0.w_1w_2w_3\ldots$$

as follows: take w_1 to be any number in $\{1,2,\ldots,8\}$ other than x_{11} ; then choose $w_2 \in \{1,2,\ldots,8\}$ other than x_{22}, and so on. This way we make the n-th decimal place of w different from the n-th decimal place of x_n, for every n. Then w cannot be equal to any of the x_n, and $w \in (0,1)$. Thus the original sequence cannot possibly have covered all of $(0,1)$. [The reason we excluded 0 and 9 from our choices for w_n was to avoid ending up at 0 or 1.]

2.30 Connected Sets

Consider a subset S of $\mathbb{R}^*$. If U is an open set then the part of U in S, i.e. $U \cap S$ is said to be *open in S*.

For example, in $[2,5]$ the set

$$[2,4)$$

is open because, for example,

$$[2,4) = (1,4) \cap [2,5]$$

Put another way, a subset J of S is said to be *open in S* if every $p \in J$ has a neighborhood N such that every point of N which is in S is in fact in J, i.e.

$$N \cap J \subset S$$

A set S is said to be *connected* if it cannot be split into two non-empty disjoint pieces A and B each of which is open in S. The main result for connected sets is:

Theorem 14 *A subset of $\mathbb{R}$, or of $\mathbb{R}^*$, is connected if and only if it is an interval.*

Chapter 3

Integration

Modern integration theory was built from the works of Newton, Riemann, and Lebesgue, but the origins of integration theory lie in the computation of areas and volumes. The idea of measuring area of a curved region by lower and upper bounds is present even Archimedes' study of the area enclosed by a circle.

The area of a rectangle whose sides are 2 units and 3 units is 6 square units, because if you tile the rectangle with unit squares you will need three rows of such squares, each row having 2 tiles. By extension, the area of a rectangle of sides $1/4$ unit and $1/5$ unit should be $1/20$ square units because we would need 20 of these rectangles to cover a unit square. Thus, it is clear that the area enclosed by a rectangle whose sides are a units and b units, where a and b are rational numbers, is ab square units. Consider now a rectangle R whose sides a and b are possibly not rational. Take any rationals a', a'', and b', b'' with

$$a' < a < a'', \quad \text{and} \quad b' < b < b''.$$

Then a rectangle R' of sides a' by b' sits inside R, while a rectangle R'' of sides a'' by b'' contains R. Thus, it makes sense to suppose that

$$\text{area of } R' \leq \text{area of } R \leq \text{area of } R''$$

which is to say:

$$a'b' \leq \text{area of } R \leq a''b''$$

It is intuitively clear, and not hard to prove, that ab is the *unique real number* that lies between all the possible values of $a'b'$ and $a''b''$. Thus,

$$\text{area of } R = ab$$

The rectangle R is made up of two congruent right angled triangles, and so each of these would have area $(ab)/2$. This makes it possible to compute the areas of all kinds of polygonal figures but cutting up these fgures into right angled triangles. However, this strategy fails when we try to compute the area enclosed by a circle. No amount of cutting would turn a disc into a finite number of right angled triangles.

Archimedes computed area enclosed by a circle C by consider polygons P' and P'', where P lies inside the circle C and P'' encloses the circle C; thus

$$\text{area } A' \text{ enclosed by } P' \leq \text{area enclosed by } C \leq \text{area } A'' \text{ enclosed by } P''$$

Some computation shows that for any $\varepsilon > 0$ there are such polygons P' and P'' such that

$$A'' - A' < \varepsilon$$

This implies that there is a *unique real number* A which lies between the area of all the polygons P' and the polygons P''. Clearly then this number should be the area enclosed by the circle C.

Our development of the theory of the Riemann integrals is based on these ideas.

3.1 Approaching the Riemann Integral

Consider a function

$$f : [a,b] \to \mathbb{R}$$

and think of its graph. Assume, for convenience of visualization, that $f \geq 0$. Then f specifies a region which lies below its graph and above the x-axis. In general, this is a region whose upper boundary, given by the graph of f, is curved. The integral

$$\int_a^b f$$

which is also, conveniently, written as

$$\int_a^b f(x)\,dx$$

measures the area of this region.

The strategy used for computing the area is to cut up the region into vertical slices. The area of each such slice is between the area of a larger 'upper' rectangle and a smaller 'lower' rectangle. Thus we should expect the actual area to be the unique real number lying between these 'upper rectangle' areas and the 'lower rectangle' areas.

3.2 Riemann Sums

We will work with functions on an interval

$$[a,b] \subset \mathbb{R}$$

] where $< b$.

A *partition* X of $[a,b]$ is specified by a sequence

$$X = (x_0, x_1, ..., x_N)$$

of points $x_0, x_1, ..., x_N \in [a,b]$ with

$$a = x_0 < x_1 < \cdots < x_N = b$$

Here $N \in \{1,2,3,...\}$.

We will often use the notation

$$\Delta x_j \stackrel{\text{def}}{=} x_j - x_{j-1} \tag{3.1}$$

to denote the length of the j-th interval

$$[x_{j-1}, x_j]$$

marked out by the partition X.

The *width* or *norm* of the partition X is the maximum size of these intervals:

$$\|X\| = \max_{j \in \{1,...,N\}} \Delta x_j \tag{3.2}$$

Consider a function

$$f : [a,b] \to \mathbb{R}$$

on an interval $[a,b] \subset \mathbb{R}$, where $a < b$.

Let

$$M_j(f) = \sup_{x \in [x_{j-1}, x_j]} f(x) \tag{3.3}$$

and

$$m_j(f) = \inf_{x\in[x_{j-1},x_j]} f(x) \tag{3.4}$$

Thus, if A_j were the area of the region under the graph of f over $[x_{j-1}, x_j]$ then

$$m_j(f)\Delta x_j \leq A_j \leq M_j(f)\Delta x_j.$$

Our objective is to specify the actual area A under the graph of f, and this would be the sum of the A_j.

The *upper Riemann sum* $U(f,X)$ is defined to be

$$U(f,X) \stackrel{\text{def}}{=} \sum_{j=1}^{N} M_j(f)\Delta x_j \tag{3.5}$$

and the *lower Riemann sum* $L(f,X)$ is

$$L(f,X) \stackrel{\text{def}}{=} \sum_{j=1}^{N} m_j(f)\Delta x_j \tag{3.6}$$

Note that

$$L(f,X) \leq U(f,X) \tag{3.7}$$

We will show later that in fact every lower sum if less or equal to every upper sum, i.e. $L(f,X)$ is less or equal to $U(f,Y)$ for every partitions X and Y of $[a,b]$.

Now consider a sequence

$$X^* = (x_1^*, ..., x_N^*)$$

obtained by picking a point from each interval $[x_{j-1}, x_j]$::

$$x_j^* \in [x_{j-1}, x_j]$$

We shall indicate this by writing

$$X^* < X$$

The *Riemann sum* $S(f,X,X^*)$ is

$$S(f,X,X^*) = \sum_{j=1}^{N} f(x_j^*)\Delta x_j \tag{3.8}$$

Note that the upper sum is ∞ if and only if one of the M_j is ∞, and this occurs if and only if the superemum of f is ∞:

$$U(f,X) = \infty \qquad \text{if and only if} \qquad \sup_{x\in[a,b]} f(x) = \infty \tag{3.9}$$

However, $U(f,X)$ *cannot be* $-\infty$, because that would mean that at least one of the M_j is $-\infty$ which can only be if f is equal to $-\infty$ on that interval $[x_{j-1}, x_j]$ which contradicts the fact that f is real-valued. (Note that we are working with *non-empty* intervals because $x_{j-1} < x_j$.)

Similarly,

$$L(f,X) = -\infty \qquad \text{if and only if} \qquad \inf_{x\in[a,b]} f(x) = -\infty \tag{3.10}$$

and $L(f,X)$ can never be ∞.

It is useful then to observe that the difference

$$U(f,X) - L(f,X)$$

is always defined, i.e. we never have the $\infty - \infty$ situation.

3.3 Definition of the Riemann Integral

Consider a function

$$f : [a,b] \to \mathbb{R}$$

on an interval $[a,b] \subset \mathbb{R}$, where $a < b$. This function is said to be *Riemann integrable* if there is a unique real number I lying between all the lower sums and all the upper sums:

$$L(f,X) \leq I \leq U(f,X) \tag{3.11}$$

for every partition X of $[a,b]$. This number I is the *integral* of f over $[a,b]$ and denoted

$$\int_a^b f \qquad \text{or} \qquad \int_a^b f(x)\,dx \tag{3.12}$$

The set of all Riemann integrable functions on $[a,b]$ is denote

$$\mathcal{R}[a,b] \tag{3.13}$$

3.4 Refining Partitions

We work with a function

$$f : [a,b] \to \mathbb{R}.$$

We use the notation

$$M(f,[s,t]) = \sup_{x\in[s,t]} f(x), \quad \text{and} \quad m(f,[s,t]) = \inf_{x\in[s,t]} f(x) \tag{3.14}$$

Let

$$X = (x_0, x_1, ..., x_N)$$

be a partition of the interval $[a,b]$ and let X' be a partition obtained by adding one more point x' to X. Let us say that x' lies in the j-th interval

$$x' \in [x_{j-1}, x_j]$$

Thus, X' cuts up $[x_{j-1}, x-j]$ into two intervals

$$[x_{j-1}, x'] \qquad \text{and} \qquad [x', x_j]$$

Let us compare the upper sums $U(f,X)$ and $U(f,X')$. To this end, let us write

$$M_j = M(f,[x_{j-1},x_j]), \quad M'_j = M(f,[x_{j-1},x']), \qquad M''_j = M(f,[x',x_j])$$

It is important to observe that

$$M_j \geq M'_j, \quad \text{and} \quad M_j \geq M''_j \tag{3.15}$$

These sums differ only in the contribution that comes from $[x_{j-1}, x_j]$:

$$\begin{aligned} U(f,X) - U(f,X') &= M_j(x_j - x_{j-1}) \\ &\quad - \left[M'_j(x' - x_{j-1}) + M''_j(x_j - x')\right] \\ &= (M_j - M'_j)(x' - x_{j-1}) + (M_j - M''_j)(x_j - x') \\ &\geq 0. \end{aligned}$$

Thus

$$U(f,X') \leq U(f,X), \tag{3.16}$$

i.e. *adding a point to a partition lowers upper sums*.

If we keep adding points to the initial partition, we keep lower the upper sums.

Arguing similarly, it follows that *adding points to a partition raises lower sums.*

This observation is important enough to state as a theorem:

Theorem 15 *Adding points to a partition lowers upper sums and raises lower sums. Thus if X and X' are partitions of $[a,b]$, with X' containing all the points of X and some more, then*

$$L(f,X) \leq L(f,X') \leq U(f,X') \leq U(f,X) \tag{3.17}$$

for every function $f : [a,b] \to \mathbb{R}$.

Using this we can prove that upper sums always dominate lower sums:

Theorem 16 *If $f : [a,b] \to \mathbb{R}$ is a function on an interval $[a,b] \subset \mathbb{R}$, where $a < b$, and if X and Y are any partitions of $[a,b]$ then*

$$L(f,X) \leq U(f,Y) \tag{3.18}$$

Thus, every lower sum is dominated above by every upper sum.

Proof. Here is a useful trick: we can combine the partitions X and Y to form a partition Z which contains all points of X and all the points of Y as well. Therefore,

$$U(f,Z) \leq U(f,Y)$$

and

$$L(f,X) \leq L(f,Z)$$

Stringing these together we have

$$L(f,X) \leq L(f,Z) \leq U(f,Z) \leq U(f,Y)$$

and this gives the desired result. QED

We have used the fact that adding a point to a partition lowers upper sums and raises lower sums. It will be useful to take a closer look at this and estimate by *how much* the upper and lower sums move as points are added to the partition. Recall the formula

$$U(f,X) - U(f,X') = (M_j - M'_j)(x' - x_{j-1}) + (M_j - M''_j)(x_j - x') \tag{3.19}$$

where X' is the partition obtained from X by adding the single point x' to the j-th interval $[x_{j-1}, x_j]$ of the partition $X = (x_0, \ldots, x_T)$. Now suppose instead of adding just the one point x', we add m distinct points $y_1, \ldots, y_m$ to the interval $[x_{j-1}, x_j]$ to form the new partition X'. We label the new points y_i in increasing order

$$y_1 < \cdots < y_m.$$

For convenience of notation we write y_0 for x_{j-1} and y_{m+1} for x_j. Thus

$$x_{j-1} = y_0 < y_1 < \cdots < y_m < y_{m+1} = x_j$$

The intervals

$$[y_{k-1}, y_k]$$

make up the interval

$$[x_{j-1}, x_j]$$

Then:

$$U(f,X) - U(f,X') = \sum_{k=1}^{m+1} \left[M(f,[x_{j-1},x_j]) - M(f,[y_{k-1},y_k])\right](y_k - y_{k-1}) \quad (3.20)$$

Now

$$0 \leq M(f,[x_{j-1},x_j]) - M(f,[y_{k-1},y_k]) \leq 2\|f\|_{\sup},$$

where $\|f\|_{\sup}$ is supremum of $|f|$ over the full original interval $[a,b]$. The interval lengths $y_k - y_{k-1}$ add up to Δx_j. Consequently,

$$U(f,X) - U(f,X') \leq 2\|f\|_{\sup}\Delta x_j \leq 2\|f\|_{\sup}\|X\| \quad (3.21)$$

This provides an upper bound for how much the upper sum is decreased by addition of points all in one single interval of the original partition X.

If we add points to each of N' intervals to create a new partition X' then

$$U(f,X) - U(f,X') \leq 2N'\|f\|_{\sup}\|X\| \quad (3.22)$$

Similarly,

$$L(f,X') - L(f,X) \leq 2N'\|f\|_{\sup}\|X\| \quad (3.23)$$

Note that

$$N' \leq N,$$

where N is the total number of new points added. Consequently, we have

Lemma 1 *If*

$$X = (x_0, ..., x_T)$$

is a partition of the interval $[a,b] \subset \mathbb{R}$, where $a < b$, and X' is a partition of the same interval containing all the points of X and possibly some more, then

$$U(f,X') - L(f,X') \leq U(f,X) - L(f,X) \quad (3.24)$$

but the decrease in the value of $U - L$ is at most $2N\|f\|_{\sup}\|X\|$:

$$[U(f,X) - L(f,X)] - \left[U(f,X') - L(f,X')\right] \leq 2N\|f\|_{\sup}\|X\| \quad (3.25)$$

where N is the number of new points added.

3.5 The Darboux Criterion

Now that we know that the upper sums dominate the lower sums, it is clear that there will be a unique real number between them if and only if the sup of the lower sums equals the inf of the upper sums:

Theorem 17 *A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if*

$$\sup\{U(f,X) : \textit{all partitions } X \textit{ of } [a,b]\} = \inf\{L(f,X) : \textit{all partitions } X \textit{ of } [a,b]\} \tag{3.26}$$

The common value is $\int_a^b f$.

Thus, the function f is Riemann integrable if and only if there is no 'gap' between the lower sums and upper sums. Thus, another equivalent formulation is the **Darboux criterion**:

Theorem 18 *T:Darbouxcrit A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if for any $\varepsilon > 0$ there is a partition X of $[a,b]$ for which*

$$U(f,X) - L(f,X) < \varepsilon \tag{3.27}$$

This is an extremely useful result: virtually all of our results on integrability will use the Darboux criterion.

Proof. Suppose f is integrable. Then there is a unique real number I which lies between all upper sums and all lower sums. Take any $\varepsilon > 0$. Consider the interval

$$[I, I + \varepsilon/2)$$

All the lower sums lie to the left, i.e. $\leq I$. All upper sums lie to the right ($\geq$) of I. So since I is the only real number lying between the upper sums and lower sums, there must be at least one upper sum which lies in $[I, I+\varepsilon/2)$. Thus, there is a partition Y for which

$$I \leq U(f,Y) < I + \varepsilon/2$$

Similarly, there is a partition Z for which

$$I - \varepsilon/2 < L(f,Z) \leq I.$$

Let X be the partition obtained by pooling together Y and Z. Then

$$I - \varepsilon/2 < L(f,Z) \leq L(f,X) \leq U(f,X) \leq U(f,Z) < I + \varepsilon/2 \tag{3.28}$$

Since both $L(f,X)$ and $U(f,X)$ lie in

$$(I-\varepsilon/2, I+\varepsilon/2)$$

it follows that

$$U(f,X)-L(f,X)<\varepsilon. \tag{3.29}$$

Conversely, suppose that for every $\varepsilon>0$ there is a partition for which (3.29) holds. Suppose that I and I' are distinct real numbers which both lie between all upper sums and all lower sums. Let

$$\varepsilon=|I'-I|$$

We know that there is a partition X satisfying (3.29). Now I and I' both lie between $U(f,X)$ and $L(f,X)$. So the difference between I and I' is $<\varepsilon$:

$$|I'-I|<\varepsilon$$

But this contradicts the deifnition of ε taken above. Thus I and I' must be equal. So there is a unique real number between all the upper sums and all the lower sums. Thus, f is integrable. QED

3.6 Integrable functions are bounded

Suppose $f:[a,b]\to\mathbb{R}$ is integrable. We will prove that f must be bounded.

Theorem 19 *Every Riemann integrable function is bounded.*

Proof. Consider

$$f:[a,b]\to\mathbb{R}$$

where $[a,b]\subset\mathbb{R}$ and $a<b$. Assume that $f\in\mathcal{R}[a,b]$.

Let us suppose that f is unbounded. Then we will reach a contradiction.

Suppose, for example, that f is not bounded above, i.e.

$$\sup_{x\in[a,b]} f(x)=\infty.$$

Consider any partition

$$X=(x_0,\dots,x_N)$$

of $[a,b]$. Then there is some subinterval, say $[x_{j-1},x_j]$, on which f is not bounded above, i.e.

$$M_j = \infty.$$

But then the supper sum for f with this partition is also ∞:

$$U(f,X) = \sum_{k=1}^{N} M_k \Delta x_k = \infty$$

Thus, *every upper sum* for f is ∞. Now let

$$I = \int_a^b f$$

and recall that this is the unique real number lying between all upper sums and all lower sums. Then

$$L(f,X) \leq I < I+1 < \infty = U(f,X) \tag{3.30}$$

for every partition X. But then $I+1$ would also be a real number lying between all upper sums and all lower sums. Thus, we have a contradiction. QED

3.7 Variation of a Function

For any non-empty subset $S \subset \mathbb{R}$ the *diameter* of S is

$$\operatorname{diam}(S) = \sup\{a-b : \, a,b \in S\} \tag{3.31}$$

It is easy to believe and not hard to prove that the diameter of S is the difference between $\sup S$ and $\inf S$:

$$\operatorname{diam}(S) = \sup S - \inf S \tag{3.32}$$

Now consider a function f on an interval $[s,t]$. It will be useful to have a measure of the fluctuation of f over this interval.

The simplest such measure is given by the diameter of the range of f:

$$\operatorname{Var}(f) = \operatorname{diam}(\mathrm{Range\,of\,f}) \tag{3.33}$$

More explicitly, the *variation* of f on the interval $[s,t]$ is:

$$\operatorname{Var}(f,[s,t]) = \sup_{x,y \in [s,t]} (f(x) - f(y)) \tag{3.34}$$

It is also equal to

$$\mathrm{Var}(f,[s,t]) = M(f,[s,t]) - m(f,[s,t]) \tag{3.35}$$

Sometimes we may drop $[s,t]$ and just write $\mathrm{Var}(f)$:

$$\mathrm{Var}(f) = M(f) - m(f) \tag{3.36}$$

We record the following algebraic facts about the variation of functions, which will be very useful in proving corresponding facts about Riemann integration.

Lemma 2 *For any functions f and g on an interval $[a,b] \subset \mathbb{R}$ with $a < b$, we have:*

(i) $\mathrm{Var}(f) \geq 0$;

(ii) $\mathrm{Var}(k) = 0$ if and only if k is constant;

(iii) the variation scales like length, i.e. the variation of a constant times a function is the absolute value of the function time the variation in the function:

$$\mathrm{Var}(kf) = |k|\mathrm{Var}(f) \text{ for any } k \in \mathbb{R}.$$

(iv) the variation satifies the triangle inequality:

$$\mathrm{Var}(f+g) \leq \mathrm{Var}(f) + \mathrm{Var}(g) \tag{3.37}$$

(v) the variation of the product of two functions is bounded above by the sum of their variations, weighted by their sup-norms:

$$\mathrm{Var}(fg) \leq \|f\|_{\sup}\mathrm{Var}(g) + \|g\|_{\sup}\mathrm{Var}(f) \tag{3.38}$$

(vi) the variations in f and in g differ by at most the variation in $|f-g|$:

$$\Big|\mathrm{Var}(f) - \mathrm{Var}(g)\Big| \leq \mathrm{Var}(f-g) \tag{3.39}$$

(vii) if f is not equal to zero anywhere then

$$\mathrm{Var}(1/f) \leq M(|f|^{-1})^2\mathrm{Var}(f) \tag{3.40}$$

(viii) the variation of a function increases montonically with the interval of variation:

$$\mathrm{Var}(f,[s,t]) \leq \mathrm{Var}(f,[a,b]) \text{ if } [s,t] \subset [a,b] \tag{3.41}$$

(ix) The variation in the absolute value of f is bounded by the variation in f:

$$\mathrm{Var}(|f|) \leq \mathrm{Var}(f) \tag{3.42}$$

(x) The variation of f is bounded above by twice $\|f\|_{\sup}$:

$$\mathrm{Var}(f) \leq 2\|f\|_{\sup} \tag{3.43}$$

(xi) If $s,t,u \in [a,b]$ with $s < t < u$ then

$$\mathrm{Var}(f,[s,u]) \leq \mathrm{Var}(f,[s,u]) + \mathrm{Var}(f,[u,t]) \tag{3.44}$$

(xii) If f is monotone on an interval $[s,t]$ then

$$\mathrm{Var}(f,[s,t]) = \begin{cases} f(t) - f(s) & \textit{if } f(t) \geq f(s) \\ f(s) - f(t) & \textit{if } f(t) \leq f(s) \end{cases} \tag{3.45}$$

We also record the following result on variations and partitions for continuous functions:

Lemma 3 *If $f:[a,b] \to \mathbb{R}$ is continuous, where $[a,b] \subset \mathbb{R}$ and $a < b$, then for any $\varepsilon > 0$ there is a partition $X = (x_0,\ldots,x_N)$ of $[a,b]$ such that*

$$\mathrm{Var}(f,[x_{j-1},x_j]) < \varepsilon$$

for every $j \in \{1,\ldots,N\}$.

Proof. Since f is continuous on the compact set $[a,b]$ it is *uniformly* continuous. So for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|f(x) - f(x')| < \varepsilon/2$$

whenever $x,x' \in [a,b]$ with $|x - x'| < \delta$. Take any partition $X = (x_0,\ldots,x_N)$ with all the intervals having length less than δ. Then, for each $j \in \{1,\ldots,N\}$,

$$f(x) - f(x') < \varepsilon/2$$

for every $x, x' \in [x_{j-1}, x_j]$ since these intervals all have length $< \delta$. Consequently,

$$\mathrm{Var}(f, [x_{j-1}, x_j]) \leq \varepsilon/2$$

and we are done. QED

The sup-norm bound on the variation has the following consequence:

Proposition 1 *If $f_1, f_2, \ldots : [a,b] \to \mathbb{R}$ converge uniformly to a function $f : [a,b] \to \mathbb{R}$, then*

$$\lim_{n\to\infty} \mathrm{Var}(f_n) = \mathrm{Var}(f) \tag{3.46}$$

It will be convenient to use the notation

$$\mathrm{Var}_j(f) \stackrel{\text{def}}{=} \mathrm{Var}(f, [x_{j-1}, x_j]) \tag{3.47}$$

The reason why we are interested in the variation is summarized by

Theorem 20 *For any function $f : [a,b] \to \mathbb{R}$, where $[a,b] \subset \mathbb{R}$ and $a < b$, and every partition X of $[a,b]$ we have*

$$U(f,X) - L(f,X) = \sum_{j=1}^{N} \mathrm{Var}_j(f)\Delta x_j \tag{3.48}$$

Proof. Observe that

$$\mathrm{Var}_j(f) = M_j(f) - m_j(f). \tag{3.49}$$

Multiplying this by Δx_j and adding up over all $j \in \{1, \ldots, N\}$ gives the result (3.48). QED

3.8 The algebra $\mathcal{R}[a,b]$

Recall that

$$\mathcal{R}[a,b]$$

is the set of all Riemann integrable functions on $[a,b]$.

Our main objective now is

Theorem 21 *The set $\mathcal{R}[a,b]$ has the following properties:*

(i) Every constant function belongs to $\mathcal{R}[a,b]$

(ii) If $f,g \in \mathcal{R}[a,b]$ then $f+g \in \mathcal{R}[a,b]$

(iii) If $f,g \in \mathcal{R}[a,b]$ then $fg \in \mathcal{R}[a,b]$

(iv) If $f \in \mathcal{R}[a,b]$, and f is never equal to zero and $1/f$ is bounded, then $1/f \in \mathcal{R}[a,b]$.

(v) If $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ and

$$\Big|\int_a^b f\Big| \leq \int_a^b |f| \tag{3.50}$$

Properties (i)-(iii) say that $\mathcal{R}[a,b]$ is an *algebra* under pointwise addition and multiplication of functions. It is important to note that the converse of (v) does not hold, i.e. there are functions which are not Riemann integrable but whose absoulte values are Riemann integrable.

Proof For a constant function k on $[a,b]$ we have

$$L(f,X) = k(b-a) = U(f,X)$$

for every partition X and so $k(b-a)$ is the unique real number lying between all upper sums and all lower sums. Thus

$$\int_a^b k = k(b-a)$$

Now supppose $f,g \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$. By the Darboux condition, there are partitions Y and Z of $[a,b]$ such that

$$U(f,Y) - L(f,Y) < \varepsilon/2 \qquad \text{and} \qquad U(g,Z) - L(g,Z) < \varepsilon/2$$

Let X be the partition obtained by combining Y and Z. Then, because upper sums decrease and lower sums increase when points are added to a partition, we have

$$U(f,X) - L(f,X) < \varepsilon/2 \qquad \text{and} \qquad U(g,X) - L(g,X) < \varepsilon/2$$

Then, with $X = (x_0, \ldots, x_N)$,

$$
\begin{aligned}
U(f+g,X)-L(f+g,X) &= \sum_{j=1}^{N}\left[M_j(f+g)-m_j(f+g)\right]\Delta x_j \\
&= \sum_{j=1}^{N}\mathrm{Var}(f+g,[x_{j-1},x_j])\Delta x_j \\
&\leq \sum_{j=1}^{N}\mathrm{Var}(f,[x_{j-1},x_j])\Delta x_j+\sum_{j=1}^{N}\mathrm{Var}(g,[x_{j-1},x_j])\Delta x_j \\
&\qquad \text{(by Lemma 2 (ii))} \\
&= U(f,X)-L(f,X)+U(g,X)-L(g,X) \qquad \text{by (3.48)} \\
&< \varepsilon/2+\varepsilon/2=\varepsilon.
\end{aligned}
$$

Thus, by the Darbox criterion, $f+g$ is Riemann integrable.

The other results follow in a similar way by applying other parts of Lemma 2. QED

Consider the function g on $[0,1]$ given by

$$
g(x)=\begin{cases}1 & \text{if } x \text{ is rational}\\ -1 & \text{if } x \text{ is irrational}\end{cases}
$$

Then g is *not* Riemann integrable, because no matter what partition X we take of $[0,1]$ the upper sum is always

$$U(g,X)=1$$

and the lower sum is always

$$L(g,X)=-1$$

But the absolute value of g is the constant function 1:

$$|g|=1$$

and so $|g|$ *is* Riemann integrable.

3.9 $C[a,b]\subset\mathcal{R}[a,b]$

Every continuous function on a compact interval is integrable. This is a central result of integration theory.

Theorem 22 *For any interval $[a,b] \subset \mathbb{R}$, with $a < b$, the set $C[a,b]$ of continuous functions on $[a,b]$ is contained in the set $\mathcal{R}[a,b]$ of Riemann integrable functions on $[a,b]$:*

$$C[a,b] \subset \mathcal{R}[a,b]$$

The proof has been presented in class. The key result used is Lemma 3 which says that the variation of a continuous function can be controlled suitably to apply the Darboux criterion for integrability.

It should be noted that discontinuous functions might also be integrable. Indeed, any function which is dicontinuous at only finitely many points is integrable.

3.10 The Integral as a Non-negative Linear Functional

We have seen that the set

$$\mathcal{R}[a,b]$$

of all Riemann integrable functions on a compact interval $[a,b] \subset \mathbb{R}$, with $a < b$, is a *linear space*, i.e. that sums and constant multiples of Riemann integrable functions are again Riemann integrable. Our next objective is to show that the Riemann integral viewed as a function

$$\mathcal{R}[a,b] \to \mathbb{R} : f \mapsto \int_a^b f \tag{3.51}$$

is linear:

Theorem 23 *For interval $[a,b] \subset \mathbb{R}$, where $a < b$, and any $f, g \in \mathcal{R}[a,b]$, we have*

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g \tag{3.52}$$

and, for any $k \in \mathbb{R}$,

$$\int_a^b (kf) = k \int_a^b f \tag{3.53}$$

Thus, the Riemann integral

$$\mathcal{R}[a,b] \to \mathbb{R} : f \mapsto \int_a^b f \tag{3.54}$$

is a linear functional on the linear space $\mathcal{R}[a,b]$.

Proof Let $f, g \in \mathcal{R}[a,b]$, and $k \in \mathbb{R}$. We have already seen, in Theorem 21, that $f+g$ and kf are also in $\mathcal{R}[a,b]$. Now let

$$X = (x_0, \ldots, x_N)$$

be any partition of $[a,b]$. As usual, let $\mathrm{Var}_k(h)$ denote the variation of a function h over the interval $[x_{j-1}, x_j]$:

$$\mathrm{Var}_k(h) = \sup_{s,t \in [x_{j-1}, x_j]} (h(s) - h(t)) = M_j(h) - m_j(h) \tag{3.55}$$

Then

$$\begin{aligned} U(f+g, X) &= \sum_{j=1}^{N} \mathrm{Var}_k(f+g)\Delta_j x \\ &\leq \sum_{j=1}^{N} \mathrm{Var}_k(f)\Delta_j x + \sum_{j=1}^{N} \mathrm{Var}_k(g)\Delta_j x \\ &= U(f,X) + U(g,X) \end{aligned}$$

Similarly,

$$L(f+g, X) \geq L(f,X) + L(g,X) \tag{3.56}$$

Thus, $L(f+g, X)$ and $U(f+g, X)$ are squeezed into the interval

$$[L(f,X) + L(g,X), U(f,X) + U(g,X)] \tag{3.57}$$

Now let $\varepsilon > 0$. By the usual trick of combining partitions, there is a partition X of $[a,b]$ such that

$$U(f,X) - L(f,X) < \varepsilon/2$$

and

$$U(g,X) - L(g,X) < \varepsilon/2$$

So

$$U(f+g,X) - L(f+g,X) \leq U(f,X) + U(g,X) - [L(f,X) + L(g,X)] < \varepsilon \tag{3.58}$$

which, by the Darboux criterion implies that $f+g \in \mathcal{R}[a,b]$. (Of course, we have just repeated the proof of Theorem 21 (ii).)

Let

$$I(f) = \int_a^b f, \quad I(g) = \int_a^b g, \tag{3.59}$$

$$I(f+g) = \int_a^b (f+g) \tag{3.60}$$

Then $I(f)$ lies between $U(f,X)$ and $L(f,X)$, and $I(g)$ lies between $U(g,X)$ and $L(g,X)$. Consequently,

$$L(f,X)+L(g,X) \leq I(f)+I(g) \leq U(f,X)+U(g,X)$$

Moreover,

$$L(f+g,X) \leq I(f+g) \leq U(f+g,X)$$

Putting all this together, we see that $I(f+g)$ and the sum $I(f)+I(g)$ both lie in the interval

$$[L(f,X)+L(g,X), U(f,X)+U(g,X)] \tag{3.61}$$

and the width of this interval is $< \varepsilon$. Therefore, $I(f)+I(g)$ and $I(f+g)$ differ by less than ε. But ε is any positive real number. Therefore,

$$I(f+g) = I(f)+I(g) \tag{3.62}$$

Next, consider the function kf, where $k \in \mathbb{R}$. Let

$$X = (x_0, \ldots, x_N)$$

be a partition of $[a,b]$ such that

$$U(f,X) - L(f,X) < \frac{\varepsilon}{1+|k|} \tag{3.63}$$

(The $1+$ in the denominator is to avoid trouble if k happens to be 0.) Then

$$\begin{aligned}
U(kf,X) - L(kf,X) &= \sum_{j=1}^{N} \mathrm{Var}_j(kf)\Delta x_j \\
&= \sum_{j=1}^{N} |k| \mathrm{Var}_j(f)\Delta x_j \qquad \text{(by Lemma 2 (iii))} \\
&= |k| \left[U(f,X) - L(f,X)\right] \\
&\leq |k| \frac{\varepsilon}{1+|k|} \\
&< \varepsilon.
\end{aligned}$$

Thus, the interval

$$[L(kf,X),U(kf,X)]$$

has width less than ε. The Darboux criterion implies that kf is integrable. Let

$$I(kf) = \int_a^b (kf) \tag{3.64}$$

Now for $k \geq 0$ we have

$$M(kf,[s,t]) = \sup_{x\in[s,t]} kf(x) = k \sup_{x\in[a,b]} f(x) = kM(f,[s,t]) \tag{3.65}$$

and for $k < 0$ we have

$$M(kf,[s,t]) = \sup_{x\in[s,t]} kf(x) = k \inf_{x\in[a,b]} f(x) = km(f,[s,t]), \tag{3.66}$$

because multiplying by a negative number reverses inequalities and transforms sup into inf, and inf into sup. Thus, also

$$m(kf,[s,t]) = km(f,[s,t]) \quad \text{if } k \geq 0 \tag{3.67}$$
$$m(kf,[s,t]) = kM(f,[s,t]) \quad \text{if } k \leq 0 \tag{3.68}$$
$$\tag{3.69}$$

Doing this for each interval $[x_{j-1},x_j]$, and mutiplying everything by Δx_j and adding up, we see that $I(kf)$ lies in the interval

$$[kL(f,X),kU(f,X)] \qquad \text{if } k \geq 0$$

and it lies in

$$[kU(f,X),kL(f,X)] \qquad \text{if } k \leq 0$$

Now $I(f)$ lies between $L(f,X)$ and $U(f,X)$, and so $kI(f)$ lies in the same interval mentioned above as $I(kf)$ does. Consequently,

$$\left|I(kf) - kI(f)\right| \leq |k|\,[U(f,X) - L(f,X)] < \varepsilon, \tag{3.70}$$

as before. Now since $\varepsilon > 0$ is any positive real number we have

$$I(kf) = kI(f).$$

This completes the proof. QED

The Riemann integral is a *non-negative* linear functional in the sense that it carried non-negative functions into non-negative numbers:

Theorem 24 *If $[a,b] \subset \mathbb{R}$ with $a < b$, and $f \in \mathcal{R}[a,b]$ is non-negative, i.e. $f(x) \geq 0$ for all $x \in [a,b]$ then $\int_a^b f \geq 0$:*

$$f \in \mathcal{R}[a,b] \textit{ and } f \geq 0 \textit{ imply that } \int_a^b f \geq 0 \tag{3.71}$$

Consequently, the integral is order-preserving

$$f, \in \mathcal{R}[a,b] \textit{ and } f \geq g \textit{ imply that } \int_a^b f \geq \int_a^b g \tag{3.72}$$

Proof. This is simply because if $f \geq 0$ then all the lower sums are ≥ 0 and so the integral $\int_a^b f$, being $\geq$ all lower sums, is also ≥ 0.

Next, suppose $f, g \in \mathcal{R}[a,b]$ and $f \geq g$. Observe that

$$f - g = f + (-1)g$$

is also in $\mathcal{R}[a,b]$, and is, of course, ≥ 0. Thus,

$$f = (f - g) + g$$

and so, by linearity,

$$\int_a^b f = \int_a^b (f - g) + \int_a^b g$$

Now we have just shown that the first term on the right is ≥ 0. Therefore,

$$\int_a^b f \geq \int_a^b g. \qquad \boxed{\text{QED}}$$

The linear functional given by the Riemann integral is a *bounded linear functional* on $\mathcal{R}[a,b]$ for the sup-norm in the following sense:

Theorem 25 *For any compact interval $[a,b] \subset \mathbb{R}$, with $a < b$, and for any $f \in \mathcal{R}[a,b]$ we have*

$$\left| \int_a^b f \right| \leq \|f\|_{\sup}(b - a) \tag{3.73}$$

Proof We know that if $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$ and

$$\left| \int_a^b f \right| \leq \int_a^b |f| \tag{3.74}$$

Now the function $|f|$ is bounded above by the constant $\|f\|_{\sup}$ (recall from Theorem 19 that this is finite.) Therefore,

$$\int_a^b |f| \leq \int_a^b \|f\|_{\sup} = \|f\|_{\sup}(b-a)$$

and we are done. QED

A useful but simple consequence of this result is:

Theorem 26 *For any compact interval $[a,b] \subset \mathbb{R}$, with $a < b$, if $f_1, f_2, \ldots \in \mathcal{R}[a,b]$ converge uniformly to a function f on $[a,b]$ then $f \in \mathcal{R}[a,b]$ and*

$$\lim_{n\to\infty} \int_a^b f_n = \int_a^b f \tag{3.75}$$

Proof First let us show that $f \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$. By uniform convergence, we have an $n \in \mathbb{P}$ such that

$$\|f_n - f\|_{\sup} < \frac{\varepsilon}{4(b-a)} \tag{3.76}$$

By integrability of f_n we know that there is a partition $X = (x_0, \ldots, x_N)$ of $[a,b]$ such that

$$U(f_n, X) - L(f_n, X) = \sum_{j=1}^{N} \mathrm{Var}_j(f_n)\Delta x_j < \frac{\varepsilon}{2} \tag{3.77}$$

Now

$$\begin{aligned} \Big|\mathrm{Var}_j(f_n) - \mathrm{Var}_j(f)\Big| &\leq \mathrm{Var}_j(f_n - f) \\ &\leq 2\|f_n - f\|_{\sup} \\ &< 2\frac{\varepsilon}{4(b-a)} = \frac{\varepsilon}{2(b-a)} \end{aligned}$$

and so

$$\mathrm{Var}_j(f) \leq \mathrm{Var}_j(f_n) + \frac{\varepsilon}{2(b-a)} \tag{3.78}$$

Therefore,

$$\begin{aligned} U(f,X) - L(f,X) &= \sum_{j=1}^{N} \mathrm{Var}_j(f)\Delta x_j \\ &\leq \sum_{j=1}^{N} \mathrm{Var}_j(f_n)\Delta x_j + \frac{\varepsilon}{2(b-a)} \sum_{j=1}^{N} \Delta x_j \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Thus, $f \in \mathcal{R}[a,b]$.

Next, we have

$$\left|\int f_n - \int f\right| \leq \|f_n - f\|_{\sup}(b-a) \to 0, \quad \text{as } n \to \infty. \qquad \boxed{\text{QED}}$$

3.11 Additivity of the Integral

We will show that the integral of a function over an interval $[a,b]$ is the sum of the integrals of the function over $[a,c]$ and $[c,b]$ for any point $c \in (a,b)$.

If

$$F : S \to U$$

is a function, and T is a non-empty subset of S, then

$$F|T$$

denotes the *restriction* of F to the smaller domain T, i.e. $F|T$ is the function whose domain is T and whse values are given through F:

$$(F|T)(x) = F(x) \qquad \text{for all } x \in T \tag{3.79}$$

Theorem 27 *Let a,c,b be real numbers with $a < c < b$, and consider any function*

$$f : [a,b] \to \mathbb{R}$$

which is intergable over $[a,c]$ and over $[c,b]$. Then $f \in \mathcal{R}[a,b]$, and

$$\int_a^b f = \int_a^c f + \int_c^b f \tag{3.80}$$

<u>Proof</u>. Let $\varepsilon > 0$. By Darboux, there is a partition Y of $[a,c]$ and a partition Z of $[c,b]$ such that

$$U(f|[a,c],Y) - L(f|[a,c],Y) < \varepsilon/2 \tag{3.81}$$

and

$$U(f|[c,b],Z) - L(f|[c,b],Z) < \varepsilon/2 \tag{3.82}$$

Now put together the points of Y and Z. This yields a partition X of the combined interval $[a,b]$. Then,

$$U(f,X) = U(f|[a,c],Y) + U(f|[c,b],Z) \geq \int_a^c f + \int_c^b f \tag{3.83}$$

and

$$L(f,X) = L(f|[a,c],Y) + L(f|[c,b],Z) \leq \int_a^c f + \int_c^b f \tag{3.84}$$

Consequently,

$$\begin{aligned} U(f,X) - L(f,X) &= U(f|[a,c],Y) + U(f|[c,b],Z) - [L(f|[a,c],Y) + L(f|[c,b],Z)] \\ &= U(f|[a,c],Y) - L(f|[a,c],Y) + U(f|[c,b],Z) - L(f|[c,b],Z) \\ &< \varepsilon/2 + \varepsilon/2 \\ &= \varepsilon \end{aligned}$$

Therefore, by Darboux, $f \in \mathcal{R}[a,b]$.

Now from (3.83) and (3.84) it follows that the sum

$$\int_a^c f + \int_c^b f$$

lies between $L(f,X)$ and $U(f,X)$, and, of course, so does $\int_a^b f$. Therefore,

$$\int_a^c f + \int_c^b f \text{ and } \int_a^b f \text{ differ by less than } \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that

$$\int_a^b f = \int_a^c f + \int_c^b f. \quad \boxed{\text{QED}}$$

Now we prove that if a function is ingtergable on an interval then it is integrable on any sub-interval:

Theorem 28 *If $[a,b] \subset \mathbb{R}$, with $a < b$, and if $s,t \in [a,b]$ with $s < t$ then for any function $f \in \mathcal{R}[a,b]$ we have $f|[s,t] \in \mathcal{R}[s,t]$.*

Proof This is, as always, a matter of applying Darboux using one of the properties of Var. Let $\varepsilon > 0$. Since $f \in \mathcal{R}[a,b]$ there is a partition Y of $[a,b]$ such that

$$U(f,Y) - L(f,Y) < \varepsilon.$$

Add to Y the points s and t, in case they are not in Y, to obtain a partition

$$Z = (z_0, \ldots, z_M)$$

of $[a,b]$. We know that this lowers the upper sum and raises the lower sum and so

$$U(f,Z)-L(f,Z)<\varepsilon.$$

Now let X be the partition of $[s,t]$ obtained by taking the points of Z which are in $[s,t]$. Then

$$U(f,Z)-L(f,Z)=U(f|[s,t],X)-L(f|[s,t],X)+\sum_{j\in J}\left[M_j(f)-m_j(f)\right]\Delta x_j \tag{3.85}$$

where J consists of those $j\in\{1,...,M\}$ for which the interval $[z_{j-1},z_j]$ is not contained in $[s,t]$. Therefore,

$$U(f,Z)-L(f,Z)\leq U(f|[s,t],X)-L(f|[s,t],X)<\varepsilon,$$

and we are done. QED

3.12 Monotone Functions are Riemann Integrable

We have made the remark that not every Riemann integrable function is continuous. We will now prove that every monotone function is Riemann integrable on any compact interval.

Let

$$f:[a,b]\to\mathbb{R}$$

be a monotone function, where $[a,b]\subset\mathbb{R}$ and $a<b$. Let

$$X=(x_0,x_1,...,x_N)$$

be any partition of $[a,b]$. Then

$$\begin{aligned} U(f,X)-L(f,X) &= \sum_{j=1}^{N}\mathrm{Var}_j(f)\Delta x_j \\ &\leq \left(\sum_{j=1}^{N}\mathrm{Var}_j(f)\right)\|X\| \end{aligned}$$

where

$$\|X\|=\max_{1\leq j\leq N}\Delta x_j,$$

is the maximum width of the intervals making up the partition.

Suppose for convenience that f is monotone non-decreasing, i.e.

$$f(x) \leq f(y) \text{ for all } x, y \in [a,b] \text{ with } x \leq y$$

Then, by the property of Var for monotone functions given in (3.41) we have

$$\mathrm{Var}_j(f) = f(x_j) - f(x_{j-1}) \qquad \text{for all } j \in \{1, \ldots, N\}$$

Therefore, the sum of the variations over all the intervals is simply the variation over the full interval:

$$\sum_{j=1}^{N} \mathrm{Var}_j(f) = \mathrm{Var}(f, [a,b]) \tag{3.86}$$

The same conclusion holds even if f is monotone non-increasing, i.e. if

$$f(x) \leq f(y) \text{ for all } x, y \in [a,b] \text{ with } x \geq y$$

Thus, in either case, we have

$$U(f,X) - L(f,X) = \mathrm{Var}(f, [a,b]) \|X\| \tag{3.87}$$

To make this less than any chosen $\varepsilon > 0$ all we have to do is take a partition X with all the interval sizes less than

$$\varepsilon / [1 + \mathrm{Var}(f, [a,b])].$$

For example, we could divide $[a,b]$ into N equal pieces, with N chosen large enough that

$$\frac{b-a}{N} < \frac{\varepsilon}{1 + \mathrm{Var}(f, [a,b])}.$$

Thus we have proved:

Theorem 29 *If f is a monotone function on a compact interval $[a,b] \subset \mathbb{R}$, with $a < b$, then $f \in \mathcal{R}[a,b]$.*

3.13 Riemann Sums and the Riemann Integral

We have used the Archimedean strategy of capturing the value of the integral between upper sums and lower sums. This approach led to a smooth development of the central results of the theory. However, this method is not the most intuitive in understanding concepts such as arc length. It is therefore useful to understand the Riemann integral in terms of Riemann sums as well. This method is also amenable to generalizations such as the notion of line integrals. Furthermore, the Riemann sum approach motivates the construction of more advanced notions such as the stochastic integral of Itô.

So consider a function

$$f : [a,b] \to \mathbb{R}$$

where $[a,b] \subset \mathbb{R}$ with $a < b$. Let

$$X = (x_0, ..., x_N)$$

be any partition of $[a,b]$. Recall that the norm or width of X is the length of the largest interval

$$\|X\| = \max_j \Delta x_j$$

Now consider any sequence

$$X^* = (x_1^*, ..., x_N^*)$$

subordinate to X, i.e. with $x_j^* \in [x_{j-1}, x_j]$ for each j. We denote this by

$$X^* < X$$

Recall the Riemann sum

$$S(f, X, X^*) = \sum_{j=1}^{N} f(x_j^*) \Delta x_j \tag{3.88}$$

Now

$$m_j \le f(x_j^*) \le M_j,$$

for each j, and so

$$L(f,X) \le S(f,X,X^*) \le U(f,X) \tag{3.89}$$

If f is integrable, with $I = \int_a^b$, then for any $\varepsilon > 0$ we can choose partition X such that

$$U(f,X) - L(f,X) < \varepsilon.$$

Since both I and $S(f,X,X^*)$ are squeezed in between the upper and lower sums, it follows that

$$|S(f,X,X^*) - I| < \varepsilon \tag{3.90}$$

The following result is is often used to define the Riemann integral in alternative approaches to the theory.

Theorem 30 *A function $f : [a,b] \to \mathbb{R}$ is Riemann integrable if and only if there is a real number I such that for any $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|S(f,X,X^*) - I| < \varepsilon \tag{3.91}$$

for every partition X of norm $< \delta$ and every $X^* < X$. In this case,

$$I = \int_a^b f$$

Proof. Suppose the given condition holds. Then there is a real number I such that for any $\varepsilon > 0$ there is a $\delta > 0$ for which the condition

$$|S(f,X,X^*) - I| < \varepsilon/4 \tag{3.92}$$

holds for all partitions X of $[a,b]$ of width $< \delta$ and all $X^* < X$. Thus,

$$I - \varepsilon/4 < \sum_{j=1}^{N} f(x_j^*)\Delta x_j < I + \varepsilon/4$$

for *every* sequence $X^* < X$. Then, taking the supremum over all possible x_1^* in the first interval $[x_0, x_1]$, we see that

$$M_1 \Delta x_1 + \sum_{j=2}^{N} f(x_j^*)\Delta x_j \leq I + \varepsilon/4$$

and, taking the infimum over all possible x_1^* in $[x_0, x_1]$, we have

$$I - \varepsilon/4 \leq m_1 \Delta x_1 + \sum_{j=2}^{N} f(x_j^*)\Delta x_j$$

Carrying this successively for $j = 2, 3, ..., N$, we conclude that

$$I - \varepsilon/4 \leq L(f,X) \leq U(f,X) \leq I + \varepsilon/4$$

Consequently,

$$U(f,X) - L(f,X) \leq \varepsilon/2 < \varepsilon$$

and so, by Darboux, $f \in \mathcal{R}[a,b]$. Moreover, since both I and the integral $\int_a^b f$ are trapped in the interval $[L(f,X), U(f,X)]$ whose width is ε it follows that I and $\int_a^b f$ differ by less than ε. But, ε is any positive real number. Thus,

$$I = \int_a^b f.$$

For the converse, suppose $f \in \mathcal{R}[a,b]$. Let $\varepsilon > 0$. By Darboux, there is a partition

$$Y = (y_0, \ldots, y_N)$$

of $[a,b]$ such that $U(f,Y)$ and $L(f,Y)$ differ by less than ε:

$$U(f,Y) - L(f,Y) < \varepsilon/2 \tag{3.93}$$

Now let

$$\delta = \frac{\varepsilon/2}{1 + 2N\|f\|_{\sup}}. \tag{3.94}$$

(Where we get this from will be clear later.) Consider any partition

$$X = (x_0, \ldots, x_T)$$

of $[a,b]$ of norm less than δ:

$$\|X\| < \delta$$

We will compare $U - L$ for X with that for Y and conclude that $U - L$ for X is indeed less than ε. Using our standard trick, let Z be the partition of $[a,b]$ obtained by combining X and Y. Then

$$U(f,Z) - L(f,Z) \leq U(f,Y) - L(f,Y) < \varepsilon/2 \tag{3.95}$$

We also know by Lemma 1 that $U - L$ for Z differs from that for X by at most $2N\|f\|_{\sup}\|X\|$, because at most N points were added to X to obtain Z. Thus, the most $U - L$ for X could be is

$$U(f,Z) - L(f,Z) + 2N\|f\|_{\sup}\|X\| \tag{3.96}$$

Thus,

$$U(f,X) - L(f,X) < \varepsilon/2 + 2N\|f\|_{\sup}\delta \tag{3.97}$$

In (3.94) we chose δ just so this right side now works out to ε:

$$U(f,X) - L(f,X) < \varepsilon/2 + \varepsilon/2 = \varepsilon \tag{3.98}$$

Now take any $X^* < X$. Then the Riemann sum $S(f,X,X^*)$ is sandwiched between the lower sum $L(f,X)$ and the upper sum $U(f,X)$, and so it the integral $\int_a^b f$. Therefore, $S(f,X,X^*)$ and $\int_a^b f$ both lie in the interval

$$[L(f,X), U(f,X)]$$

whose width is $< \varepsilon$. Thus,

$$\left| S(f,X,X^*) - \int_a^b f \right| < \varepsilon \tag{3.99}$$

We have shown that for any $\varepsilon > 0$ there is a $\delta > 0$ such that (3.99) holds for any partition X of width $< \delta$ and any $X^* < X$. QED

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Appendix A

Question Bank

<u>Set 1</u>

1. Prove that there is no rational number whose square is 7.

2. Let ε be any positive real number.

   (i) Show that there is an $n_0 \in \mathbb{N}$ such that

   $$\frac{1}{n^2} < \varepsilon$$

   for all $n \in \mathbb{N}$ with $n \geq n_0$.

   (ii) Let a be any real number. Show that there is some $n_1 \in \mathbb{N}$ such that

   $$2a\frac{1}{n} < \varepsilon$$

   for all $n \in \mathbb{N}$ with $n \geq n_1$.

3. Suppose a is a positive real number with $a^2 < 7$. Show that

$$\left(a+\frac{1}{n}\right)^2 < \varepsilon$$

for all $n \in \mathbb{N}$ large enough (i.e. there is some $n' \in \mathbb{N}$ such that the preceding inequality holds for all natural numbers $n \geq n'$).

4. Let

$$S = \{x \in \mathbb{R} : x > 0 \,\text{and}\, x^2 < 7\}.$$

(i) Show that $S \neq \emptyset$.

(ii) Show that S is bounded above in $\mathbb{R}$.

(iii) Let $a = \sup S$. Prove that $a^2 \geq 7$. (Hint: Suppose a^2 were less than 7. Then use the result of Problem 3.)

(iv) Prove that a^2 is, in fact, equal to 7. Thus, there is a real number whose square is 7.

<u>Set 2</u>

1. For the set $B = \{-4, 8\} \cup [1, 7) \cup [9, \infty)$, viewed as a subset of $\mathbb{R}^*$:

   (i) $B^0 =$

   (ii) $\partial B =$

   (iii) $B^c =$

   (iv) the interior of the complement B^c is

   $$(B^c)^0 =$$

   (v) $(B^0)^c =$

   (vi) $\overline{B} =$

   (vii) $\partial \overline{B} =$

   (viii) $\partial B^0 =$

   (ix) The interior of $(B^0)^c$ is:

   (x) The closure of B^c is:

2. Provide brief explanations/answers for the following:

   (i) If U is an open set then $U^0 =$

   (ii) For any set A, the interior of the interior of A is the interior of A, i.e. $(A^0)^0 = A^0$.

   (iii) The set $[0, \infty)$ is closed as a subset of $\mathbb{R}$.

   (iv) If S is closed then its closure $\overline{S}$ is S itself, i.e. $\overline{S} = S$.

   (v) Is there a subset of $\mathbb{R}$ whose boundary in $\mathbb{R}$ is all of $\mathbb{R}$?

3. Give an example of an open cover of $(-1, 1)$ which does not have a finite subcover.

Set 5

1. Consider the partition
$$X = (.25, .5, .75, 1)$$
of $[0,1]$. For the function f on $[0,1]$ given by
$$f(x) = x^2 \qquad \text{for all } x \in [0,1]$$
   (i) Work out the upper sum $U(f,X)$

   (ii) Work out the Riemann sum $S(f,X,X^*)$, where
$$X^* = (0.1, 0.3, 0.6, 0.8)$$

2. Consider the partition

$$X = \left(\frac{1}{N}, \frac{2}{N}, \ldots, \frac{N}{N}\right)$$

of $[0,1]$. For the function f on $[0,1]$ given by

$$f(x) = x^2 \qquad \text{for all } x \in [0,1]$$

(i) Show that

$$U(f,X) = \frac{1}{6}\left(1+\frac{1}{N}\right)\left(2+\frac{1}{N}\right)$$

[Hint: Use the sum formula $1^2+2^2+\cdots+k^2 = k(k+1)(2k+1)/6$.]

(ii) Show that

$$L(f,X) = \frac{1}{6}\left(1-\frac{1}{N}\right)\left(2-\frac{1}{N}\right)$$

(iii) Assuming that f is integrable, prove that

$$\int_0^1 x^2\,dx = \frac{1}{3}$$

using the definition of the Riemann integral and the results of (i) and (ii).

3. Consider the function g on $[0,1]$ given by

$$g(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ -1 & \text{if } x \text{ is irrational} \end{cases}$$

   Prove that g is not Riemann integrable.

4. Prove that, for $a, b \in \mathbb{R}$ with $a < b$, if $f, g \in \mathcal{R}[a,b]$, then $f + g \in \mathcal{R}[a,b]$.

Set 6

1. Write out in a complete and neat way the proof that every continuous function on a compact interval is Riemann integrable.

2. For a function

$$f : [a,b] \to \mathbb{R}$$

we define the *variation* Var(f) by

$$\text{Var}(f) = \sup_{x,y\in[a,b]} (f(x) - f(y)) = M(f) - m(f),$$

where

$$M(f) = \sup_{x\in[a,b]} f(x), \qquad \text{and} m(f) = \inf_{x\in[a,b]} f(x).$$

Prove that for functions f and g on $[a,b]$,

$$\text{Var}(fg) \le M(|f|)\text{Var}(g) + m(|g|)\text{Var}(f)$$

3. Explain the notion of the Riemann integral by clearly stating the definitions of upper sums and lower sums, all properly explained in your own terms, and then stating and explaining the definition of the Riemann integral. Work out a simple integral in such a way as to illustrate the definition of the Riemann integral. Clearly explain, in your own words, the Darboux criterion. (Please note that every piece of notation you bring in must explained.)

<u>Test 1</u>

1. Mark True or False:

   (i) If S is a bounded subset of $\mathbb{R}$ then it contains a largest element

   (ii) If S is a bounded, non-empty subset of $\mathbb{R}$ then it has a least upper bound in $\mathbb{R}$

   (iii) The point ∞ is a boundary point of $[1,\infty]$

   (iv) The point 1 is an interior point of $[0,1]$

   (v) The point 4 is a boundary point of $[0,4)$

   (vi) The set $[0,\infty)$ is a closed subset of $\mathbb{R}^*$

   (vii) The set $[0,\infty)$ is a closed subset of $\mathbb{R}$

   (viii) Every real number is a boundary point of $\mathbb{Q}$

   (ix) The set $\{1,2,3\}$ is compact

   (x) The set $(1,5]$ is compact

2. Prove one of the following statements:

   (i) The union of any collection of open sets is open.

   (ii) The complement of a closed set is open.

   (ii) If S is a non-empty subset of $\mathbb{R}^*$ then $\sup S$ is in the closure $\overline{S}$.

3. State

   (i) the Heine-Borel Theorem

   (ii) the Bolzano-Weierstrass theorem.

4. State the definitions of the following:

   (i) Limit point of a sequence (x_n) in $\mathbb{R}^*$.

   (ii) Limit of a sequence (x_n)in $\mathbb{R}^*$.

   (iii) A Cauchy sequence in $\mathbb{R}$.

<u>Test 2</u>

1. Mark True or False:

   (i) A continuous function on any subset of $\mathbb{R}$ is bounded.

   (ii) A continuous function on any closed subset of $\mathbb{R}$ is bounded.

   (iii) A continuous function on a compact set is bounded.

   (iv) A continuous function on a compact set attains a mimimum value at some point in the set.

   (v) If a sequence of functions converges to a function pointwise then it converges uniformly.

   (vi) If a sequence of functions converges to a function unformly then it converges pointwise.

   (vii) If $f_n : [0,1] \to \mathbb{R}$ is a sequence of functions, and $f_n \to f$, as $n \to \infty$, pointwise, and if each f_n is continuous at a point $p \in [0,1]$, then f is continuous at p.

   (viii) Every uniformly Cauchy sequence of functions converges uniformly.

   (ix) Every uniformly convergent sequence of functions is uniformly Cauchy.

   (x) The uniform limit of a sequence of continuous functions is continuous.

2. Suppose $f : [0,1] \to \mathbb{R}$ is continuous, and $f(p) > 0$ for some $p \in [0,1]$. Show that there is a neighborhood U of p such that $f(x) > 0$ for all $x \in U \cap [0,1]$.

3. State

   (i) the Intermediate Value Theorem

   (ii) the Extreme Value theorem.

4. Show that the equation

$$x^3 + 5x - 2 = 0$$

has a solution in the interval $(0,1)$.

Test 3

1. Consider a function f on an interval $[a,b] \subset \mathbb{R}$, where $a < b$,:

$$f : [a,b] \to \mathbb{R}.$$

(i) Let $X = (a,b)$ be the simplest partition of $[a,b]$. Write down an expression for $L(f,X)$, explaining your notation clearly. (5pts)

(ii) Now consider a partition X' which contains one addtional point t, i.e.

$$X' = (a,t,b),$$

where $a < t < b$. Write down an expression for $L(f,X')$, again explaining the notation clearly. (5pts)

(iii) Show that (5pts)

$$L(f, X') \geq L(f, X).$$

2. Consider a function $f : [a,b] \to \mathbb{R}$, where $[a,b] \subset \mathbb{R}$ and $a < b$. Let X and Y be partitions of $[a,b]$ with Y containing all the points of X and some more. Prove that (5pts)

$$U(f,Y) - L(f,Y) \leq U(f,X) - L(f,X)$$

3. If $f \in \mathcal{R}[a,b]$ and $[c,d] \subset [a,b]$ prove that $f \in \mathcal{R}[c,d]$. (5pts)

4. Mark TRUE or FALSE: (10pts)

   a. For any function $f : [a,b] \to \mathbb{R}$ and any partition $X = (x_0, ..., x_N)$ of $[a,b]$,
   $$U(f,X) - L(f,X) = \sum_{j=1}^{N} \mathrm{Var}_j(f)\Delta x_j,$$
   where $\mathrm{Var}_j(f) = \sup_{x\in[x_{j-1},x_j]} f(x) - \inf_{x\in[x_{j-1},x_j]} f(x)$, and $\Delta x_j = x_j - x_{j-1}$.

   b. Every Riemann integrable function is bounded.

   c. Every bounded function is Riemann integrable.

   d. Every monotone function is Riemann integrable.

   e. Every continuous function is Riemann integrable.

   f. Every Riemann integrable function is continuous.

   g. If $f \in \mathcal{R}[a,b]$ and f is never zero then $1/f \in \mathcal{R}[a,b]$.

   h. For a bounded function on an interval $[a,b] \subset \mathbf{R}$, there is always a real number which is $\geq$ all the lower sums and $\leq$ all the upper sums.

   i. If $|f| \in \mathcal{R}[a,b]$ then $f \in \mathcal{R}[a,b]$.

   j. If $f \in \mathcal{R}[a,b]$ then $|f| \in \mathcal{R}[a,b]$.

Cumulative Question Set

1. Consider the set
$$S = [2,4) \cup (5,6) \cup \{1,8\}.$$
Mark True or False:

   (i) 2 is an interior point of S.

   (ii) 4 is an interior point of S.

   (iii) 4 is a boundary point of S.

   (iv) 9 is an isolated point of S.

   (v) 1 is an isolated point of S.

   (vi) S is an open set.

   (vii) S is a closed set.

   (viii) S is compact.

   (ix) Every sequence in S has a subsequence which is convergent.

   (x) Every sequence in S has a subsequence converging to a point in S.

2. Let S be a non-empty subset of $\mathbb{R}$.

   (i) If t a real number with $t < \sup S$. Explain why there exists $x \in S$ with $x > t$.

   (ii) Show that there is a sequence of elements $s_n \in S$ such that $s_n \to \sup S$. [Hint: Let $U = \sup S$. First assume that S is bounded above; in this case $U \in \mathbb{R}$. Now apply (i) to produce an s_n using $1/n$ for t. If you have time, try also the case $U = \infty$.]

3. Suppose f is continuous at a point p, and $f(p) > 2$. Prove that there is a $\delta > 0$ such that $f(x) > 0$ for all x in the domain of f which lie in the neighborhood $(p-\delta, p+\delta)$.

4. Let f be a continuous function on a closed and bounded set $S \subset \mathbb{R}$. Prove that f reaches its maximum value on S, i.e. there is a point $p \in S$ such that $f(p) = \sup_{x \in S} f(x)$. [Hint: Let $U = \sup_{x \in S} f(x)$. Then choose a sequence of points $s_n \in S$ such that $f(s_n) \to U$. Apply Bolzano-Weierstrass.]

5. Let $a, b \in \mathbb{R}$, with $a < b$. Let (f_n) be a sequence of functions on $[a, b]$. Mark True or False:

   (i) If there is a function f on $[a, b]$ such that $f_n(x) \to f(x)$ for all $x \in [a, b]$ then $f_n \to f$ uniformly.

   (ii) If $f_n \to f$ uniformly then $f_n(x) \to f(x)$ for all $x \in [a, b]$.

   (iii) If (f_n) is Cauchy in sup-norm then $f_n \to f$ uniformly, for some function f on $[a, b]$.

   (iv) If $f_n \to f$ uniformly and each f_n is continuous then f is continuous.

   (v) If $f_n(x) \to f(x)$ for every $x \in [a, b]$, and if each f_n is continuous, then f is continuous.

6. Consider an interval $[a,b] \subset \mathbb{R}$. Mark true or false.

   (i) $C[a,b] \subset R[a,b]$.

   (ii) If $f \in R[a,b]$ then
   $$\left| \int_a^b f \right| \leq \int_a^b |f|$$

   (iii) If $f_n \in R[a,b]$, for $n \in \mathbf{N}$, and $f_n(x) \to f(x)$ for every $x \in [a,b]$ then $\int_a^b f_n \to \int_a^b f$.

   (iv) If X is a partition of $[a,b]$, and $f \in R[a,b]$, then
   $$L(f,X) \leq \int_a^b f$$

   (v) If f is a function on $[a,b]$, and X and Y are partitions of $[a,b]$ with $X \subset Y$ then
   $$U(f,X) < U(f,Y)$$