Hydraulics 1: Course notes

Staff

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Course Outline

Hydraulics I

A   Fluid properties
   A1 Introduction: Fluids, continuum and density
   A2 Viscosity, surface tension and pressure
   A3 Tutorial: fluid properties

B   Hydrostatics
   B1 Hydrostatic pressure and the hydrostatic equation
   B2 Pressure measurement
   B3 Hydrostatic force on a plane surface
   B4 Buoyancy and Archimedes Principle
   B5 Hydrostatic force on a curved surface
   B6 Tutorial: Hydrostatics

C   Kinematics and continuity
   C1 Kinematics
   C2 Conservation of mass: continuity
   C3 Tutorial: Kinematics and continuity

D   Energy and momentum: Principles
   D1 Conservation of energy: Bernoulli's Equation
   D2 Bernoulli's Equation: Applications to flow measurement
   D3 Momentum principle: control volumes
   D4 Momentum principle: open channel flow
   D5 Tutorial: forces and hydraulic jumps

E   Pipeflow
   E1 Reynolds Experiment: Laminar and turbulent flow
   E2 Pipeflow: laminar flow
   E3 Flow from static reservoir (no energy losses)
   E4 Turbulent flow and head loss
   E5 Pipeflow: other head losses
   E6 Tutorial: Pipeflow

F   Energy and momentum: further applications
   F1 Sharp expansions and orifice meters
   F2 Momentum principle: effects of gravity
   F3 Tutorial: Gravity and flow measurement

Assessment

Coursework (laboratory work and problems):

Fluid properties and hydrostatics	5%
Momentum and energy	5%
Pipeflow network design project	10%

Exam:	Four out of six questions (free choice)	80%

Books

Massey BS (revised Ward-Smith), Mechanics of Fluids. 532

Chadwick AJ and Morfett JC (and Borthwick M, for later editions), Hydraulics in Civil and Environmental Engineering. 627

Hamill L, Understanding Hydraulics. 627

White FM, Fluid Mechanics. 532

Douglas JF and Matthew RD, Solving Problems in Fluid Mechanics. 532

Featherstone RE and Nalluri, Civil Engineering Hydraulics. 627

Hydraulics I

A   Fluid properties

A1 Introduction: Fluids, continuum and density

Definition of a fluid

A fluid is a substance that flows. A fluid deforms continuously under the influence of an applied force, whereas a solid deforms a finite amount and then resists further deformation. This is because a solid can sustain a static internal shear stress whereas a fluid cannot. The molecules of a fluid are not held in a fixed arrangement and can move past each other.

Fluids include liquids and gases, and for civil engineers the most important fluids are water and air. As civil engineers, you need to understand the behaviour of fluids in both the built and natural environment. Engineers study flow in reservoirs, pipes, water and waste water treatment, and building ventilation, and also river flooding, groundwater flows, waves and wind loading on structures.

Continuum Hypothesis

Although a fluid is made up of individual molecules, at any scale significantly larger than the separation between molecules we can approximate the fluid as effectively continuous. This means we can regard properties such as temperature, pressure, density and velocity as functions of positions in space, and apply differential calculus to derive equations. There are applications where this approach is not valid: these include nanotechnology (operating at scales comparable to those of the fluid molecules) or in environments where the molecular separation becomes significant compared with the scale of motion we are interested in (e.g. aerospace applications at the edge of the atmosphere).

Density

Density is defined as mass per unit volume. It is usually denoted by the Greek letter ρ. In SI units density is given in kg m^{-3}. In general the density can vary at different points in space and at different times so $\rho(\mathbf{x},t)$, although we will generally deal with uniform fixed density.

The typical density of (fresh) water is approximately ρ_{water} = 1000 kg m^{-3}, and changes relatively little at normal temperatures and pressures. However, we will also consider problems where we have seawater rather than fresh water. The salt in seawater increases the density, so that a typical seawater density is $\rho_{seawater}$ ~ 1026 to 1028 kg m^{-3} (depending on the salt concentration), while water in estuaries will lie somewhere between the fresh water and seawater values.

A typical air density is ρ_{air} ~1.2 kg m^{-3}, but like all gases, the density of air varies more with pressure and temperature than the density of water. This is because gases are more easily compressed than liquids. We will discuss this in more detail later.

Specific weight

Whereas density is mass per unit volume, specific weight is weight per unit volume: $\gamma = \rho g$.

Relative density or specific gravity

The relative density is the density relative to some standard or reference density. Generally only used for liquids rather than gases, and so the reference density is often the density of water. The relative density is sometimes referred to as the specific gravity (s.g.). Thus if a manometer fluid has a specific gravity of 0.75, then it has a density of 750 kg m^{-3}. Note that relative density has no units ("dimensionless").

A2 Viscosity, surface tension and pressure

Viscosity
As the fluid molecules move past each other there is a resistance to this relative motion. While the stress in a solid depends on the amount of deformation (strain) of the solid, for a fluid it is the <u>rate</u> of deformation (strain rate) that is related to the stress.

The velocity profile in a fluid near a solid boundary typically has the form shown below:
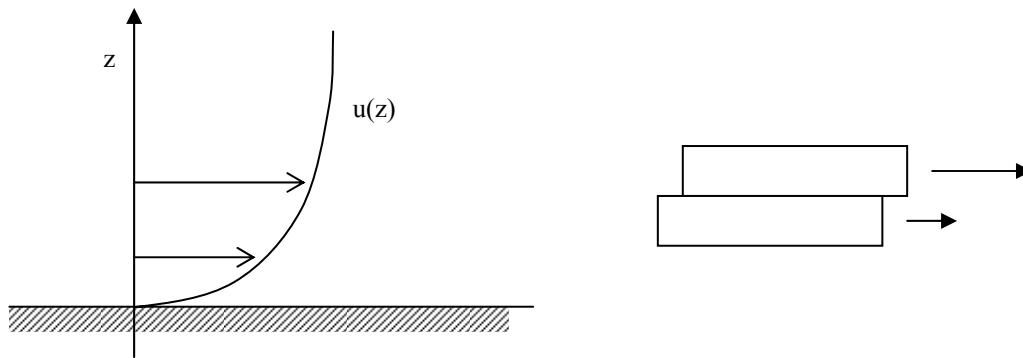
Fluid further from the boundary is moving faster, and the relative motion of one layer past another is resisted by intermolecular forces in the fluid. The resistance is in the form of a stress (a force per unit area), and is generally found to be linearly dependent on the strain rate (shear or velocity gradient). Thus the stress, τ, is given by

$$\tau = \mu \frac{du}{dz}.$$

The constant of proportionality is known as the viscosity (or coefficient of viscosity or dynamic viscosity) and is usually denoted by μ. Stress has units of force/area: $N\ m^{-2}$ or Pa, while shear has units of s^{-1}. Thus the units of viscosity can be written as Pa s, or alternatively as $kg\ m^{-1}\ s^{-1}$. Fluids for which there is a linear relation between stress and shear are known as **Newtonian fluids**, and include water and air.

Typical values for viscosity are:
$\mu_{air} = 1.8 \times 10^{-5}\ kg\ m^{-1}\ s^{-1}$, $\mu_{water} = 10^{-3}\ kg\ m^{-1}\ s^{-1}$, while $\mu_{glycerin} = 1.5\ kg\ m^{-1}\ s^{-1}$.
However, viscosity depends strongly on temperature.

The force acts to oppose the motion, so in the example above there is a force by the fluid on the boundary tending to move the boundary to the right (and an equal and opposite force on the fluid slowing it down). The fluid in contact with the boundary cannot move (u = 0) otherwise there would be an infinite viscous force. This is known as the **no-slip condition**.
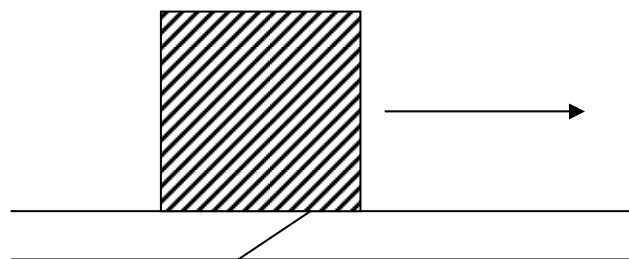
<u>Example</u>
A square block (of side 0.3 m) moving at 2 m/s on a thin (2mm) layer of water. Assuming a linear velocity profile what is the force on block?

$Shear = \dfrac{du}{dz} = 2/0.002 = 1000\ s^{-1}$

Stress
$\tau = \mu \dfrac{du}{dz} = 1000 \times 10^{-3} = 1\ N\ m^{-2}$

Force $= \tau \times$ area $= 1 \times 0.3 \times 0.3 = 0.09$ N.

Surface tension

Any free surface (or interface between immiscible fluids) acts as if it were a sheet under uniform tension, known as surface tension. The surface tension is usually denoted by σ and has units of force per unit length N m^{-1}. (It can also be regarded as an energy per unit surface area.) Across any line drawn on the surface there is a force of magnitude σ per unit length in a direction normal to the line and tangential to the interface. Where a surface is curved (e.g. in a drop) there must be a change in pressure across the surface to balance the surface tension. If the principle radii of curvature of the surface are R_1 and R_2, then

$$\Delta p = \sigma \left(\frac{1}{R_1} + \frac{1}{R_2} \right)$$

The surface tension for the interface between clean water and air at 20 °C is σ = 0.073 N m^{-1}. Surface tension is strongly affected by the presence of surfactants (most impurities have an effect on surface tension).

E.g. what is the pressure (compared to atmospheric pressure) inside a spherical water drop of diameter 1mm? Here $R_1 = R_2 = 1$ mm, $\Delta p = 2\sigma/R = 146$ Pa.

Contact angle

Where a liquid interface intersects a solid boundary, another important property is the angle the interface makes with the boundary, known as the contact angle. This angle is highly dependent on the surface chemistry and physics of the fluids and solid.
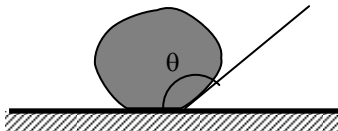
θ < 90° liquid "wets" surface,
θ > 90° (as in example) liquid is non-wetting

Pressure

Any plane surface within a fluid experiences a force per unit area (stress) normal to that surface called the fluid pressure (units Pa or N m^{-2}). The pressure acts equally in all directions, and is thus a scalar quantity.

Perfect gases

For a gas that can be considered as made up of infinitesimally small molecules colliding with each and which does not undergo any change of state (e.g. condensation) there is a simple relationship between pressure, temperature and density known as the **equation of state**:

$$p = R\rho T$$

where T is the temperature in Kelvin (i.e. above absolute zero: 0 °C = 273.15 K) and R is the specific gas constant. For dry air R = 287 J kg^{-1} K^{-1}. From this equation we can see that as the pressure increases, so does the density as the same amount of gas is compressed into a smaller volume. Liquids such as water, on the hand, can often be regarded as incompressible because even pressures several thousand times larger than atmospheric pressure result in a reduction in volume of only a few per cent.

Example

Estimate the density of air in a laboratory when the temperature is 18° C and the pressure is 1010 mbar.

T = 273 + 18 = 291 K,   p = 1010 mbar = 101 kPa $\rho = \dfrac{p}{RT}$

$$\rho = 1.21 \text{ kg m}^{-3}$$

[Air pressure is often measured in millibar (mbar), where typical atmospheric pressure is approximately 1000 mbar or 1 bar. 1 mbar = 0.1 kPa, so typical atmospheric pressure is 100 kPa (standard atmospheric pressure is 101.325 kPa)]

## A3 Tutorial: fluid properties

Viscous shear

Capillary tube

B   Hydrostatics

B1 Hydrostatic pressure and the hydrostatic equation

Hydrostatics:
Hydrostatics deals with the study of fluids at rest and in equilibrium (like statics in mechanics). Net forces are zero, and there is no flow.

Consider a small static cylinder of fluid, with axis of the cylinder (s-axis) tilted at an angle θ to the vertical, z-axis.
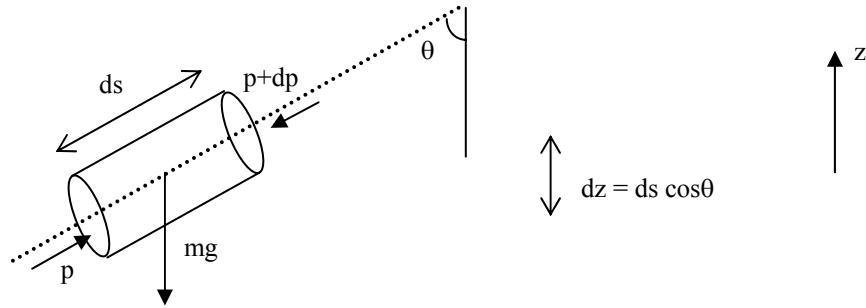
Volume of fluid element = dA ds
Mass of fluid element $m = \rho$ dA ds

Resolving forces in the s-direction

Gravitational force = -mg = -ρg dA ds (downwards).

Pressure force (s-direction) = p dA - (p+dp) dA = - dp dA

Gravitational force (s-direction) = -mg $\cos\theta$ = -ρg dA ds $\cos\theta$ = -ρg dA dz

So total force in s-direction
$$- dp\,dA - \rho g\,dA\,dz = 0.$$

Rearranging this gives us:

Hydrostatic equation

$$\frac{dp}{dz} = - \rho g$$

Note the pressure increases as z decreases (or depth increases). Integrating downwards (in the negative z-direction) from a point z_0 of known pressure p_0 to a point $z < z_0$

$$p(z) = p_0 - \int_{z_0}^{z} \rho g\ dz = p_0 + \int_{z}^{z_0} \rho g\ dz$$

Thus the pressure (relative to the reference pressure) is given by the weight of fluid per unit area above that point.

Pressure underwater

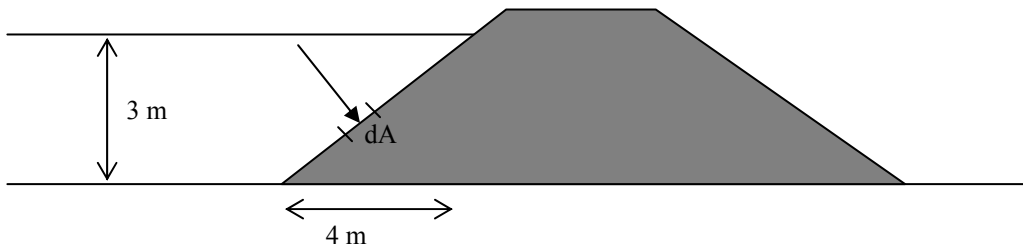We will usually deal with simple uniform constant pressure. E.g. the pressure under water at a depth h below the free surface (where the pressure is p_{atm}) is given by

$$p = p_{atm} + \rho gh$$

Example: What is the pressure (relative to atmospheric pressure) at a depth of 10 m and 1000 m underwater?   [98.1 kPa, 9.81 MPa]

Example: vertical force on a dam.

Dam length 40 m.

The pressure force will act on the surface of the dam in a direction normal to the face of the dam. Thus to find the total force we would need to integrate the pressure force (which varies with depth) over the sloping dam face. However, the vertical component of this force is the local pressure multiplied by the plan projection of the surface:
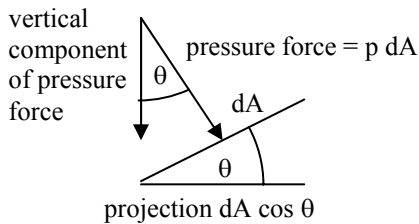
$$(p \, dA) \cos \theta = p \, (dA \cos \theta)$$
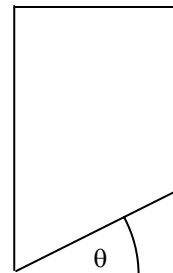
Since the pressure is equal to the total weight per unit area of the water above that point, the downward component of the force on the element dA is equal to the total weight of water above dA. Thus the total vertical force on the dam is simply the weight of water above the dam [= 2.354 MN].

Absolute, gauge and vacuum pressure

Absolute pressure is the pressure measured relative to a total vacuum, and thus is always positive.

We often measure pressures relative to the local atmospheric pressure (e.g. the pressure underwater above) and this is known as the gauge pressure   Since pressure differences are what gives net forces, subtracting a constant reference pressure is not generally important in fluid dynamics (but we need the absolute pressure for the gas equation, for example).

If the pressure is less than atmospheric pressure but given as relative to atmospheric pressure this is sometimes referred to as a vacuum pressure (e.g. a vacuum pressure of 10 kN/m^2 is a pressure of 10 kN/m^2 below atmospheric pressure, or a gauge pressure of -10kN/m^2).

B2 Pressure measurement

Piezometer

The pressure in a liquid can be measured (relative to atmospheric pressure) using a piezometer tube, e.g. by tapping a tube into a pipe:
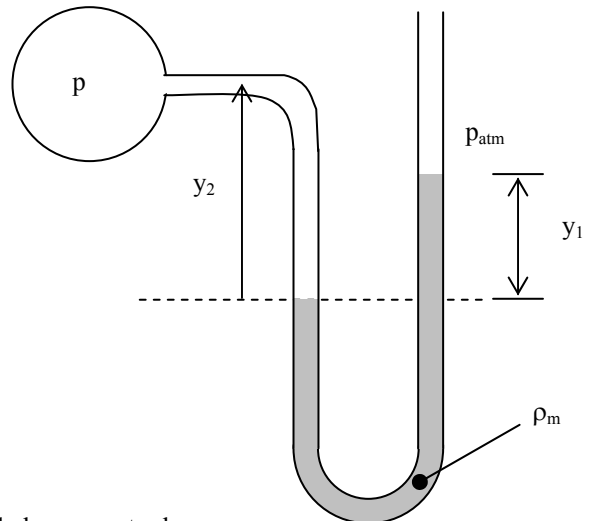

$$p - p_{atm} = \rho g h$$
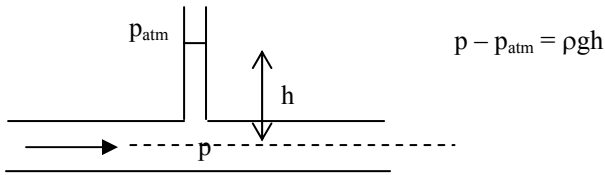
Manometer

A U-tube manometer can measure the pressure difference between two points or relative to atmospheric pressure (as shown).

The pressure on the horizontal line must be the same in both arms of the tube:

$$p + \rho g y_2 = p_{atm} + \rho_m g y_1$$

thus

$$p - p_{atm} = \rho_m g y_1 - \rho g y_2$$

E.g. what is p (relative to atmospheric) if a water filled manometer has y_1 = 4cm and y_2 = 10 cm? (assume ρ = 1.2 kg m^{-3}.)

[answer 391 Pa]

Inclined manometer

When measuring small pressures using a manometer, the height moved by the fluid can be small. To make measurement easier an inclined manometer can be used:
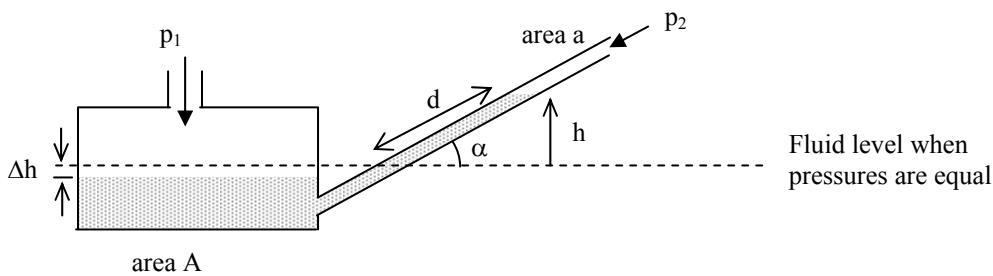
$$p_1 = p_2 + \rho g(h + \Delta h) \qquad h = d \sin \alpha \qquad A\Delta h = a\, d$$
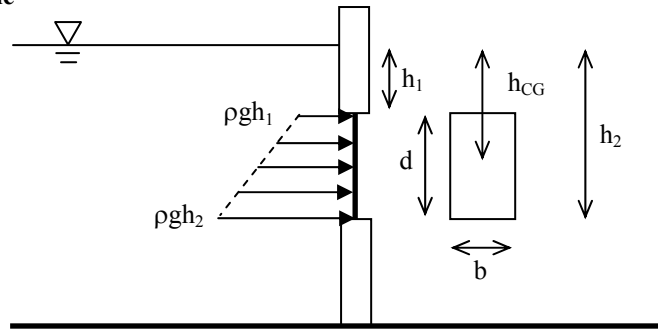
$$p_1 - p_2 = \rho g d(\sin \alpha + a/A)$$

If a << A, then $\qquad\qquad p_1 - p_2 = \rho g d \sin \alpha$

B3 Hydrostatic force on a plane surface

The force on a submerged surface is always normal to surface, whatever its orientation or shape.
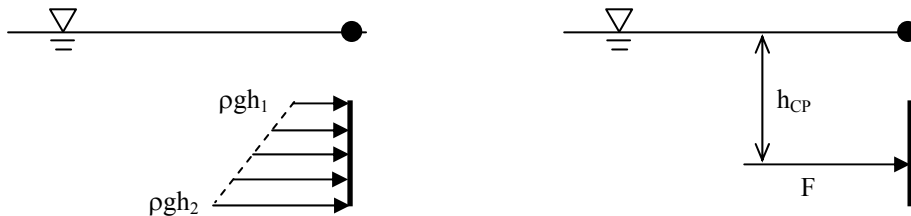
Vertical rectangle

The magnitude of the total force is given by the area under the pressure diagram times the width (b).

$$F = \tfrac{1}{2}\,(\rho g h_1 + \rho g h_2)\, d\, b = \rho g h_{CG}\, d\, b$$

The force is the average pressure multiplied by the area (A), and the average pressure is the pressure at the centroid of the submerged shape (here just the centre of the rectangle).

$$F = p_{CG}\, A = \rho g h_{CG}\, A$$

Although the force can be found from the pressure at the centroid of the rectangle, the force does not act through this point. Instead it acts through a point known as the centre of pressure, which is below the centroid as the pressures are higher with depth. We can find the centre of pressure by considering moments of the forces about an axis at the surface.

$$\int h\, p\, dA = \int h\, \rho g h\; dA \qquad = \qquad h_{CP}\, F = h_{CP}\, \rho g h_{CG}\, A$$

$$h_{CP} = \frac{\int h^2 dA}{A h_{CG}}$$

Inclined rectangle
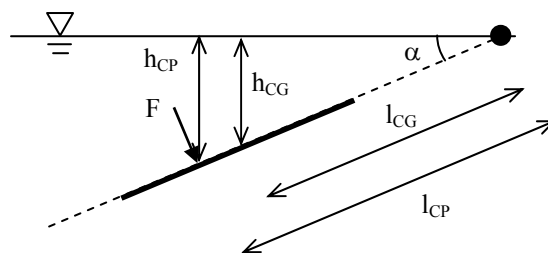
$$F = \rho g h_{CG}\, A = \rho g l_{CG}\, \sin\alpha\, A$$

$$l_{CP} = \frac{\int l^2 dA}{A l_{CG}}$$

$$h_{CP} = l_{CP}\, \sin\alpha$$

Examples: rectangles
Vertical rectangle, height 2m, width 1.5 m, with the centre of the rectangle 3m below the surface.

[Total force = 88.3 kN (in horizontal direction), acting at a point 3.11 m below the surface.]

Inclined rectangle: same as before but this time inclined with $\alpha = 60°$.

[Total force is still 88.3 kN, but this time acting 30° below the horizontal, at a point 3.08 m below the surface (a distance 9.6 cm along the rectangle from its centre).]

General plane shape
For any submerged inclined plane shape, we find the same relationships as for the inclined rectangle.

$$F = \rho g h_{CG} A = \rho g l_{CG} \sin\alpha\, A \qquad\qquad l_{CP} = \frac{\int l^2 dA}{A l_{CG}} \qquad\qquad h_{CP} = l_{CP} \sin\alpha$$

$\int l^2\, dA$ is the second moment of area for the shape about the axis on the water surface as shown above.

<u>Parallel axis theorem</u>
There are standard results for the second moments of area for various shapes but generally about an axis through their centre, I_{CG}. We can use the parallel axis theorem to find second moments about other axes, such as at the surface:

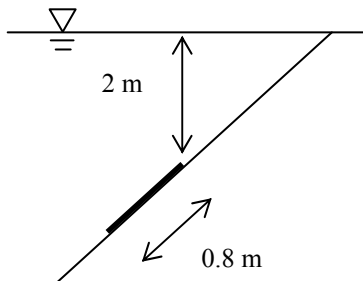$$\int l^2\, dA \;=\; I_{CG} + A\, l_{CG}{}^2$$

Note we are measuring the lengths l on the inclined surface (the pressure force acts normal to this surface). We can use this result to find the **centre of pressure**:

$$l_{CP} = \frac{I_{CG}}{A\, l_{CG}} + l_{CG}$$

$$I_{CG} = \frac{1}{12} b\, d^3 \qquad\qquad\qquad I_{CG} = \frac{1}{64}\pi\, d^4$$
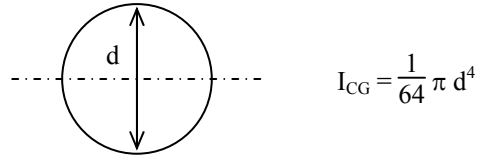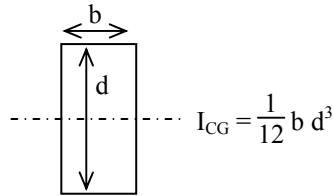
Example: Circular gate
What is the force on a circular gate of diameter 0.8m mounted on a sloping dam face of angle 45°, and where does it act?

Distance along the slope to the top of the gate is 2.828 m, so $l_{CG} = 2.828 + 0.4 = 3.228$ m. Thus $h_{CG} = 2.283$ m.

$$A = 0.503 \text{ m}^2, \qquad\qquad F = 11.27 \text{ kN}$$
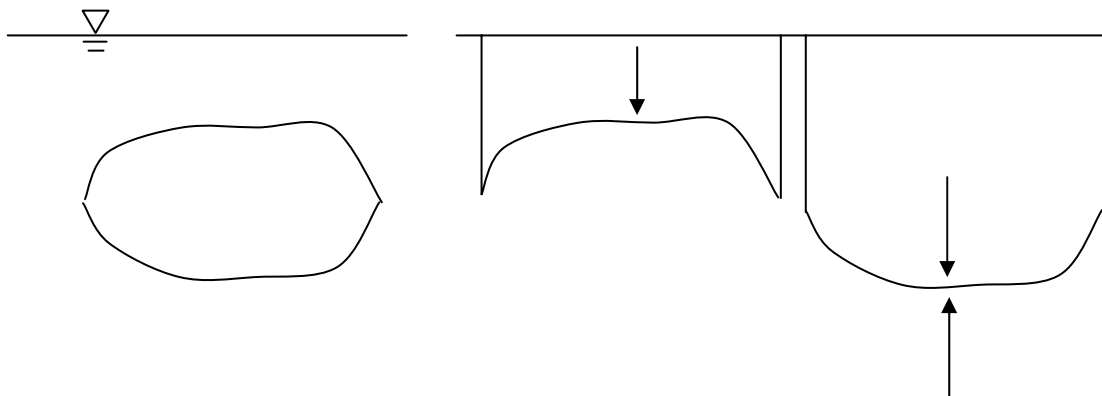
$$I_{CG} = 0.02011 \text{ m}^4$$

$$l_{CP} = 0.0124 \text{ m} + l_{CG} = 3.240 \text{ m}$$

This is a distance of 0.412 m along the slope from the top of the gate, so, for example, the moment of the force about the top of the gate is 4.64 kN m.

B4 Buoyancy and Archimedes Principle

Forces on a submerged object

The vertical force on a submerged object can be thought of as a downward force on the upper surface of the object together with an upward force on the lower surface. The force on the upper surface is the weight of the water above this surface. The force on the lower surface is of the same magnitude (but opposite direction) as the weight of water that would be above this surface if the object were not present. The difference is an upward force equal to the weight of water displaced by the object (Archimedes Principle).

Alternatively consider the forces acting on a submerged object of volume V and density ρ_O. The gravitational force on the object is $mg = \rho_O Vg$ downwards. If the object is denser than water we expect the object to sink (net force downwards), while if it is less dense we expect it to float (net force upwards). If the object has the same density as water ($\rho_O = \rho$) we expect the net force to be zero, so the buoyancy force must balance the gravitational force. Thus $B = mg = \rho Vg$, or again the buoyancy force is equal to the weight of water of the same volume as the object.

Example: Forces on a cuboid

What are the hydrostatic forces on the cuboid shown, if the other dimension is 3 m?
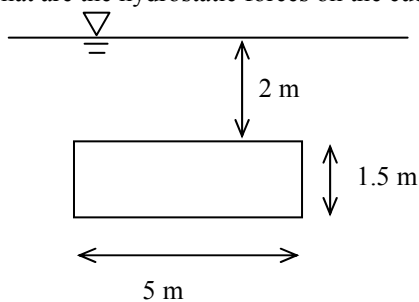
Pressure on top face = 19.62 kPa

Force on top face = 294.3 kN (downwards)

Pressure on bottom face = 34.34 kPa

Force on bottom face = 515.0 kN (upwards)

Thus total force = 220.7 kN (upwards)

Volume of cuboid = 22.5 m^3, weight of water displaced = 220.7 kN.

B5 Hydrostatic force on a curved surface

The hydrostatic pressure force is always normal to the surface, so for a simple curved surface of uniform curvature, the total force must pass through centre of curvature.

It is generally easiest to calculate the vertical and horizontal components of the total force separately, and then the point of application of the force can be found by ensuring the total force passes through the centre of curvature.
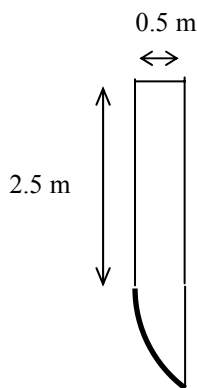
Width: 5.5 m

2.5 m

1 m

60 °
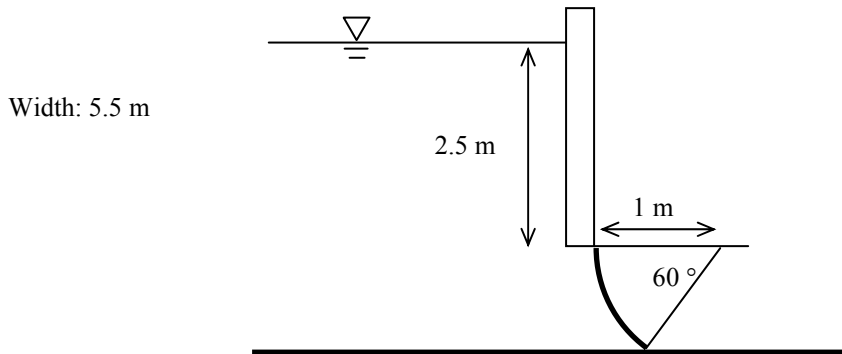
Example: forces on a curved gate
First, some geometry.

1 m

0.866 m

A
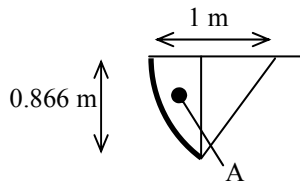
Area of sector = 0.524 m².
Area of triangle = 0.217 m².
Area A = 0.307 m².

<u>Vertical force</u>

0.5 m

2.5 m
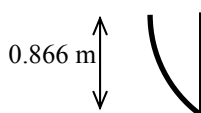
Vertical force equal to weight of fluid displaced by following shape:

Area of shape = $0.5 \times 2.5 + A = 1.557$ m²
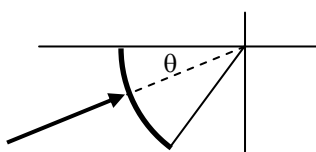
Weight = 84.01 kN

<u>Horizontal force</u>

0.866 m

Horizontal force is the same as that on a vertical surface of the same height. Average pressure = $(2.5 + \frac{1}{2} \times 0.866) \rho g = 28.77$ kPa

Force = $28.77 \times 0.866 \times w = 137.0$ kN

<u>Total force</u>

θ

magnitude = 160.7 kN, θ = 31.5°

B6 Tutorial: Hydrostatics

Rectangular gate

Curved gate

C   Kinematics and continuity

C1 Kinematics

Velocity of a fluid is a vector function of position and time.

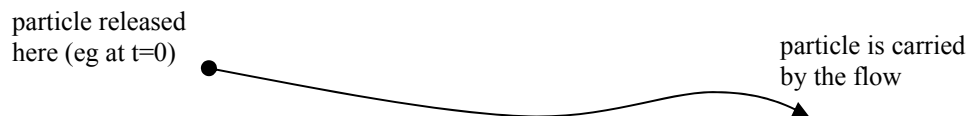$$\mathbf{u}(\mathbf{x}, t) = (u, v, w) \text{ or } (u_x, u_y, u_z) \text{ where } \mathbf{x} = (x, y, z).$$

Typically, velocity is measured by an instrument placed at a particular position **x** in the flow. For example: hot wire probes, pitot tubes, anemometers, current meters, LDA, acoustic Doppler. This way of thinking of the velocity as a function of position is called **Eulerian** representation. The disadvantage of the Eulerian representation is that you're not measuring the same piece of fluid.

Alternatively, flow can be measured by following the motion of particular fluid elements (e.g. using tracers, dyes, floats, bubbles, or small particles). Thinking of the flow in this way is called the **Lagrangian** representation.

**Pressure** is a scalar function of position and time. $p(\mathbf{x}, t)$

Path line or particle path
The line followed by a particle in the fluid released at some point in the flow (what you would see with a long exposure photograph of the flow).
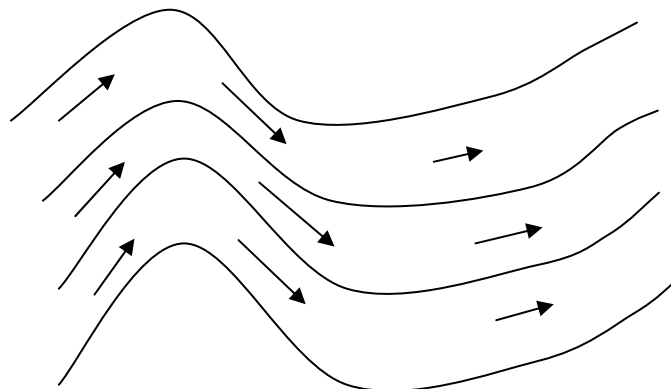
particle released
here (eg at t=0)   particle is carried
                                                                                     by the flow

Streakline or dyeline
The line of dye resulting by continuously injecting dye at a particular point in the flow.

dye continuously
injected here
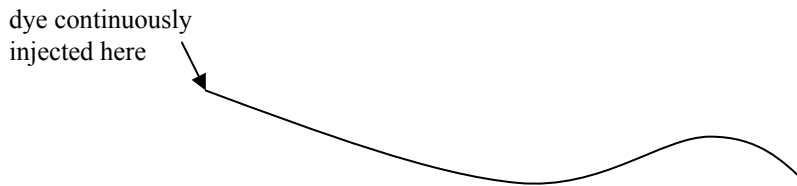
Streamlines
Lines everywhere tangent to the flow direction. This gives an overall picture of the flow field (at a particular instant in time).

Steady flow

Flow that doesn't change in time (though the velocity may be different in different parts of the fluid).

$$\frac{\partial \mathbf{u}}{\partial t} = 0 \ \ \text{(at constant } \mathbf{x})$$

For steady flow, streamlines, streaklines and particle-paths are all the same.

Acceleration

Just because the velocity at a point isn't changing, this doesn't mean that fluid passing that point isn't accelerating.

Example: steady flow along a contracting pipe:
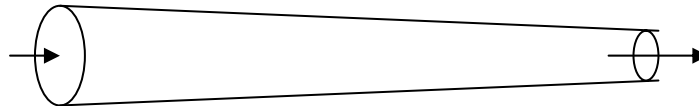
Flow in = flow out, so the velocity increases.

Following a fluid element (Lagrangian) in steady flow,

$$a = \frac{du}{dt} = \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial x}u$$

acceleration = velocity × (velocity gradient)
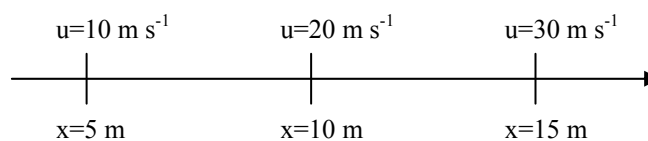
In general,

$$a = u\frac{\partial u}{\partial x} + \frac{\partial u}{\partial t}$$

$$\left(\text{In 3D, we write } \frac{D\mathbf{u}}{Dt} = \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \bullet \nabla \mathbf{u} \right)$$

Note, $a = u\dfrac{\partial u}{\partial x} = \dfrac{1}{2}\dfrac{\partial\left(u^2\right)}{\partial x}$, which (for constant acceleration) leads to $v^2 - u^2 = 2as$.

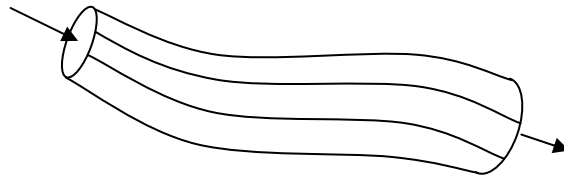Example

u = kx (with k=2 s^{-1})

Velocity gradient = $\dfrac{\partial u}{\partial x}$ = 2 s^{-1} (units are [m/s] / [m]).

Acceleration = u $\dfrac{\partial u}{\partial x}$. At x = 5 m, the acceleration is 10×2 = 20 m s^{-2}, while at x = 10 m, a = 40 m s^{-2}.
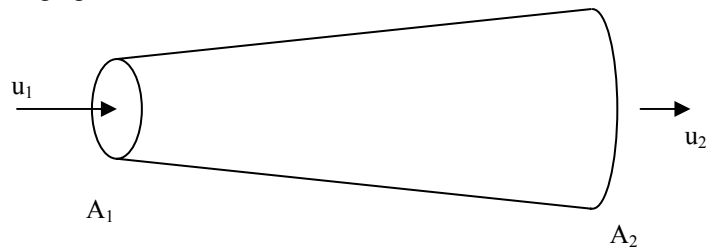
C2 Conservation of mass: continuity

Streamtube
Bounded by a set of streamlines, so fluid remains within the tube (a "virtual pipe").

Pipes
A pipe with changing cross-section:

If u_1 is the average velocity where the cross-sectional area is A_1, and similarly for u_2 and A_2, then (for incompressible flow),

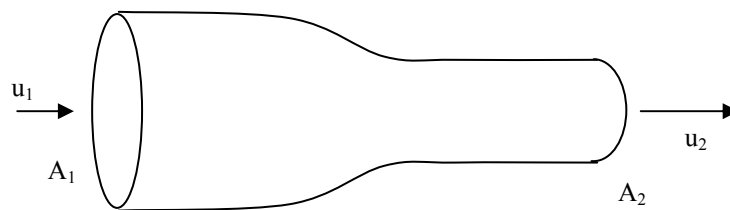$$\text{Mass flow in = mass flow out}$$

$$\rho\, u_1\, A_1 = \rho\, u_2\, A_2 \ \ (\text{kg s}^{-1})$$

The volume flow rate or discharge $Q = u_1\, A_1 = u_2\, A_2\ \ (\text{m}^3\ \text{s}^{-1})$.

Thus $u_2 = u_1(A_1/A_2)$.

Example
A pipe of internal diameter 10mm is connected to a pipe of internal diameter 5mm. If the fluid speed entering the larger diameter pipe is 1 m s^{-1}, what is the speed of the fluid as it flows through the smaller pipe?
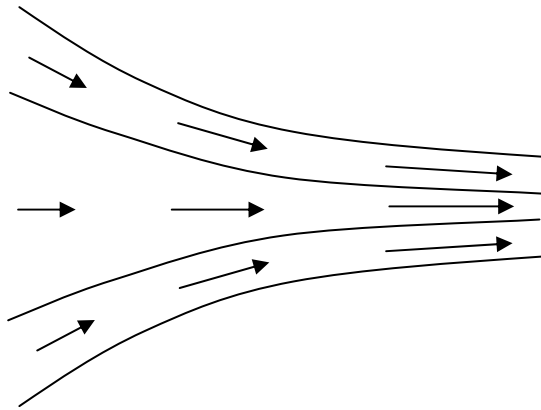

$u_1 = 1$ m s^{-1}, $A_1 = \pi \times 0.01^2/4$ m^2, $A_2 = \pi \times 0.005^2/4$ m^2.

$u_2 = u_1(A_1/A_2) = 1\ (0.01^2/0.005^2) = 4$ m s^{-1}.

Going back to streamlines
Flow speed increases where streamlines converge.

[u_x increases in x direction,
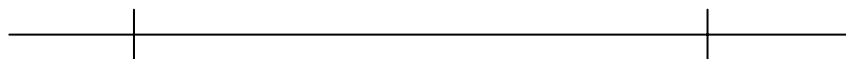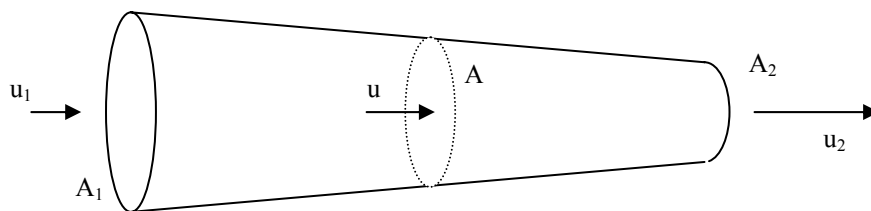u_y decreases in y direction.
In general,

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = 0$$

or $\qquad \nabla \bullet \mathbf{u} = 0$

for incompressible flows.]

Acceleration example
In the example of the contracting flow ($u_1 = 1$ m s^{-1},), if the diameter changes linearly from 10 mm to 5mm over a distance of 100 mm, what is the velocity and acceleration as a function of distance along the contraction?

x = 0 m

d = 0.01 m

x = 0.1 m

d = 0.005 m

d = 0.01 - 0.05x, $\qquad$ area = $(\pi/4)d^2$, $\quad \Rightarrow \quad$ area = $(\pi/4) (0.01 - 0.05x)^2$.

$$Q = u\,A = u_1\,A_1 = 1 \times (\pi/4) \times 0.01^2, \text{ constant.}$$

So $u = u_1 (A_1/A) = 1 \times 0.01^2/(0.01 - 0.05x)^2 = 1/(1 - 5x)^2 = (1 - 5x)^{-2}$

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(1 - 5x)^{-2} = 10(1 - 5x)^{-3}$$

$$a = u\frac{\partial u}{\partial x} = 10(1 - 5x)^{-5}$$

At x = 0, 0.05, 0.1 m, u = 1, 1.78, 4 m s^{-1}, a = 10, 42.1, 320 m s^{-2}.

C3 Tutorial: Kinematics and continuity

Streamlines, particles paths and streaklines in unsteady flow.

Example

$u_x = V_0\cos(2\pi t/T)$ $\qquad\qquad$ $u_y = V_0\sin(2\pi t/T)$

What do streamlines look like (at a particular value of t)?
Release particle at $(x,y) = (0,0)$ at $t = 0$, what is the path?
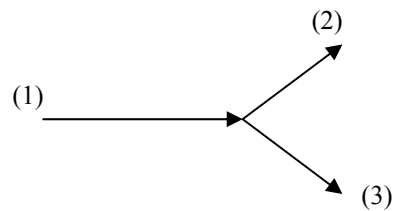Release dye at $(0,0)$ from $t = 0$ to $t = T$, what is the resulting dyeline?

Continuity in pipe networks

Flow in = flow out.

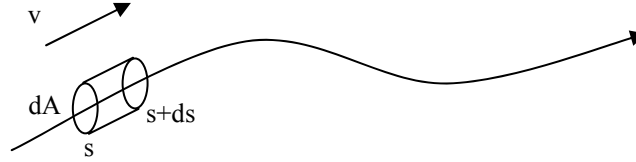Flow in (1) 10mm ID, 0.1 m s^{-1}.

Flow out (2) 5mm ID, 0.05 m s^{-1}.

Flow out (3) 7mm ID, what is u_3?

D   Energy and momentum: Principles

D1 Conservation of energy: Bernoulli's Equation

Steady flow, no friction

Fluid element moves along a fixed streamline (like a bead on a wire). Distance measured along streamline denoted by s, velocity along the streamline by v (=ds/dt).
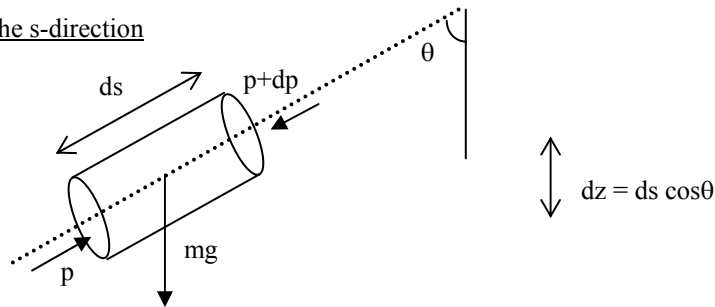
Volume of fluid element = dA ds
Mass of fluid element m = ρ dA ds

Gravitational force = -mg = -ρg dA ds (downwards).

Resolving forces in the s-direction

Pressure force (s-direction) = p dA - (p+dp) dA = - dp dA

Gravitational force (s-direction) = -mg cosθ = -ρg dA ds cosθ = -ρg dA dz

$F = ma$, and $a = \dfrac{dv}{dt}$ so,

$$\rho \, dA \, ds \, \frac{dv}{dt} = \text{- dp dA - } \rho g \, dA \, dz.$$

$$\frac{dv}{dt} = -\frac{1}{\rho} \frac{dp}{ds} - g \frac{dz}{ds}.$$

Now, $\dfrac{dv}{dt} = \dfrac{ds}{dt} \dfrac{dv}{ds} = v \dfrac{dv}{ds} = \dfrac{1}{2} \dfrac{d(v^2)}{ds}$, so we have,

$$\frac{1}{2} \frac{d(v^2)}{ds} + \frac{1}{\rho} \frac{dp}{ds} + g \frac{dz}{ds} = 0.$$

Integrate ∫ds (along streamline)

$$\frac{1}{2} v^2 + \frac{p}{\rho} + gz = \text{constant}$$

$$\frac{1}{2} v^2 + \frac{p}{\rho} + gz = gH$$

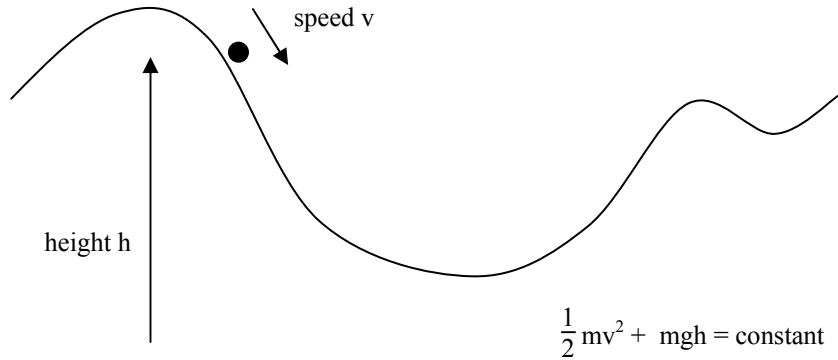<div style="text-align:right">Bernoulli's Equation
(along streamline)</div>

Kinetic energy + potential energy = constant (no friction)

(This form is energy per unit mass.)

Analogy
Particle on a smooth surface

speed v

height h

$$\frac{1}{2} mv^2 + mgh = \text{constant}$$

Head
The constant H in Bernoulli's Equation is known as the **total head** (relative to some datum z=0).

Dividing the equation above by g gives Bernoulli's Equation in the form of energy per unit weight or "head" (the form most commonly used by engineers):

$$\frac{v^2}{2g} + \frac{p}{\rho g} + z = H$$

The kinetic energy term $\frac{v^2}{2g}$ is known as the **dynamic head**.

The sum of the potential energy terms $\left(\frac{p}{\rho g} + z\right)$ is known as the **piezometric head**, while the pressure term on its own $\frac{p}{\rho g}$ is known as the **pressure energy, pressure head or static head**.

At constant z, high pressure corresponds to low speeds and low pressure to high speeds.

Power
The maximum available power = energy per unit time = $\rho g H Q$.

E.g. a static reservoir 10 m above a hydroelectric power station with a flow rate of 20 m^3 s^{-1} has a maximum available power of $1000 \times 9.81 \times 10 \times 20 = 1.96$ MW (mega watts).

D2 Bernoulli's Equation: Applications to flow measurement

Pitot tube
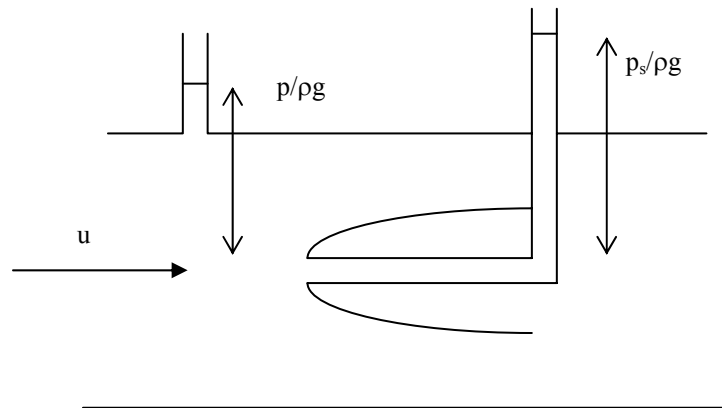
At the tip of the probe there is a stagnation point and the flow speed is zero. The height z is the same for both measurements, so we have,

$$\frac{u^2}{2g} + \frac{p}{\rho g} = H = \frac{0^2}{2g} + \frac{p_s}{\rho g}$$

$$u = \sqrt{\frac{2(p_s - p)}{\rho}}$$

Venturimeter

Continuity (conservation of mass)

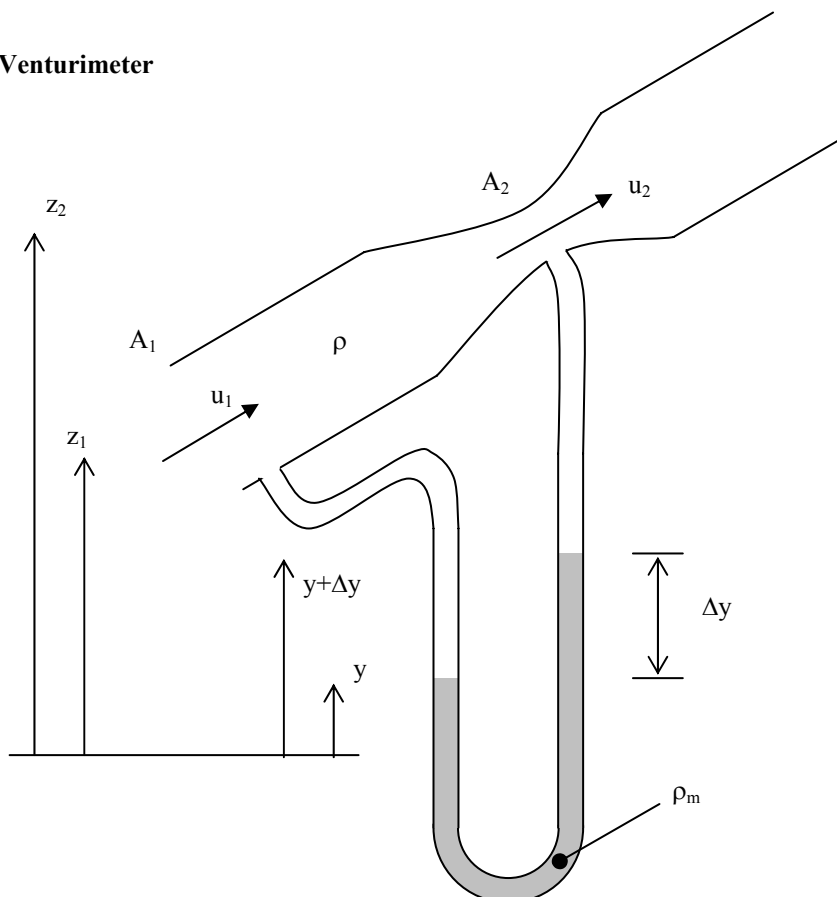$$Q = u_1 A_1 = u_2 A_2, \qquad \Rightarrow \qquad u_2 = u_1 (A_1/A_2).$$

Bernoulli's Equation (conservation of energy)

$$\frac{u_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = H = \frac{u_2^2}{2g} + \frac{p_2}{\rho g} + z_2$$

or,

$$\frac{u_2^2}{2g} - \frac{u_1^2}{2g} = \left(\frac{p_1}{\rho g} + z_1 \right) - \left(\frac{p_2}{\rho g} + z_2 \right) \equiv \Delta h$$

Pressures in the manometer
Write p_0 for the pressure in the manometer at z=0.

$$p_0 = p_1 + \rho g(z_1 - y) + \rho_m g y \qquad \text{and} \qquad p_0 = p_2 + \rho g(z_2 - (y+\Delta y)) + \rho_m g(y+\Delta y)$$

$$(p_1 + \rho g z_1) - (p_2 + \rho g z_2) = + \rho g y - \rho_m g y - \rho g(y+\Delta y) + \rho_m g(y+\Delta y) = \Delta y(\rho_m - \rho)g$$

$$\Delta h = \Delta y(\rho_m - \rho)/\rho = \Delta y(\rho_m/\rho - 1).$$

Now,

$$u_2^2 - u_1^2 = 2g\Delta h,$$

using continuity gives,

$$u_1^2 \left(\frac{A_1^2}{A_2^2} - 1 \right) = 2g\Delta h,$$

and so,

$$u_1 = \sqrt{\frac{2g\Delta h}{\left(\dfrac{A_1^2}{A_2^2} - 1 \right)}}$$

and $Q = u_1 A_1$. However, this assumes no energy loss. In practice we find,

$$Q = A_1 C_d \sqrt{\frac{2g\Delta h}{\left(\dfrac{A_1^2}{A_2^2} - 1 \right)}},$$

where $C_d \approx 0.95$ to 0.99 for a smooth contraction.

C_d is known as the **Discharge Coefficient**.

D3 Momentum principle: control volumes

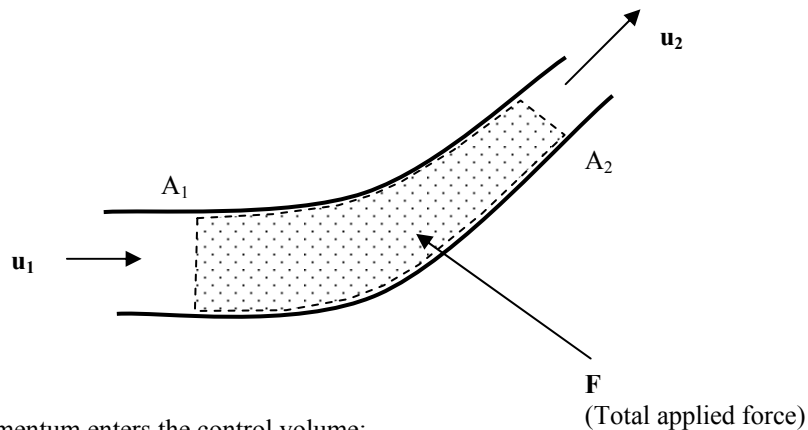**Control volume**
Identify a region of the flow.

Identify all the forces acting <u>on</u> the fluid in this region.

Calculate the rate of change of momentum of the fluid passing through this region (out - in).

Important: velocity, momentum and force are all **vector** quantities. On the other hand, energy equations (like the Bernoulli Equation) and conservation of volume (in pipe flows) can only give speeds and not directions.

<u>Example</u>
Flow in a pipe

F
(Total applied force)

Rate at which momentum enters the control volume:

$$\rho_1 \mathbf{u_1} Q_1$$

Rate at which momentum leaves the control volume

$$\rho_2 \mathbf{u_2} Q_2$$

For incompressible flows $\rho_1 = \rho_2$ and $Q_1 = Q_2 = Q$, so

$$(\mathbf{u_2} - \mathbf{u_1})\, \rho Q = \mathbf{F}$$

(rate of change of momentum = total force)

<u>The total force includes:</u>
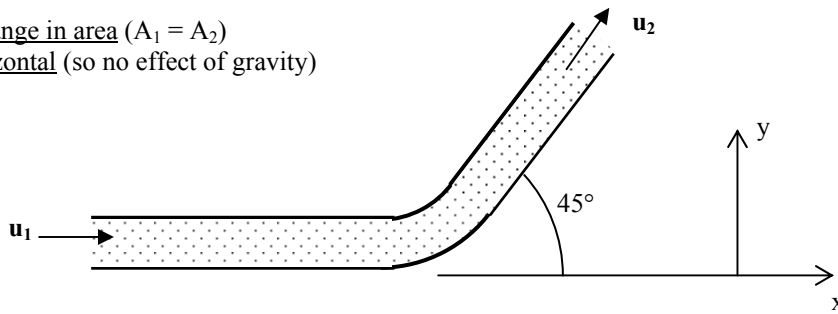
Fluid pressure acting at each end of the control volume

$p_2 A_2$

$p_1 A_1$

m**g**
gravity
(body forces)
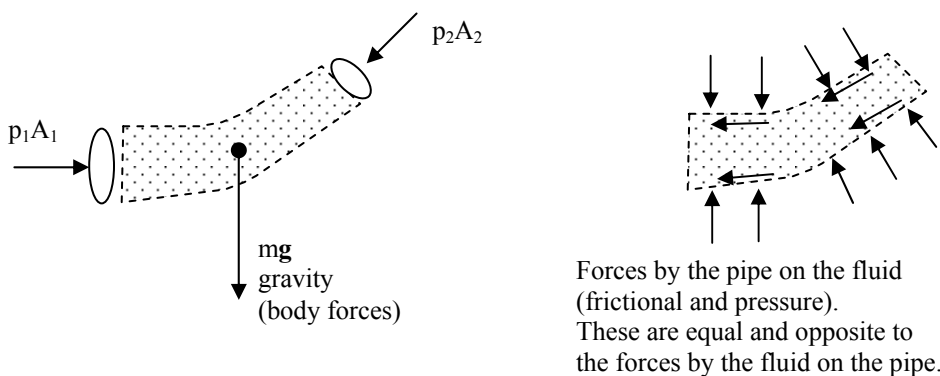
Forces by the pipe on the fluid
(frictional and pressure).
These are equal and opposite to
the forces by the fluid on the pipe.

Flow around a pipe bend

a) <u>no change in area</u> ($A_1 = A_2$)
<u>and horizontal</u> (so no effect of gravity)

Assume no change in pressure ($p_1 = p_2 = p$).

Rate of change of momentum

$$(\mathbf{u_2} - \mathbf{u_1})\,\rho Q = (\mathbf{u_2} - \mathbf{u_1})\,\rho A u$$

Where u is the speed ($u = |\mathbf{u_2}| = |\mathbf{u_1}|$), and so $\mathbf{u_1} = (u, 0)$ and $\mathbf{u_2} = (u\cos45°, u\sin45°)$

So the x-component of the total force on the fluid is:

$$F_x = (u\cos45° - u)\,\rho A u = -(1 - \cos45°)\,\rho A u^2,$$

and the y-component is

$$F_y = u\sin45°\,\rho A u = \sin45°\,\rho A\, u^2.$$

The sum of the pressure forces at the two ends is:

$$(pA, 0) + (-pA\cos45°, -pA\sin45°) = pA(1-\cos45°, -\sin45°)$$

total force on fluid = pressure force at ends + force by pipe on fluid

force by fluid on pipe = -force by pipe on fluid = pressure force - total force

$$= pA(1-\cos45°, -\sin45°) - (F_x, F_y)$$

$$= (pA + \rho A\, u^2)(1 - \cos45°, -\sin45°)$$

<u>Example</u>
Pipe carrying water, diameter 10 mm, pressure 10^4 Pa, flow rate $Q = 1.20\times10^{-4}$ m^3 s^{-1}.

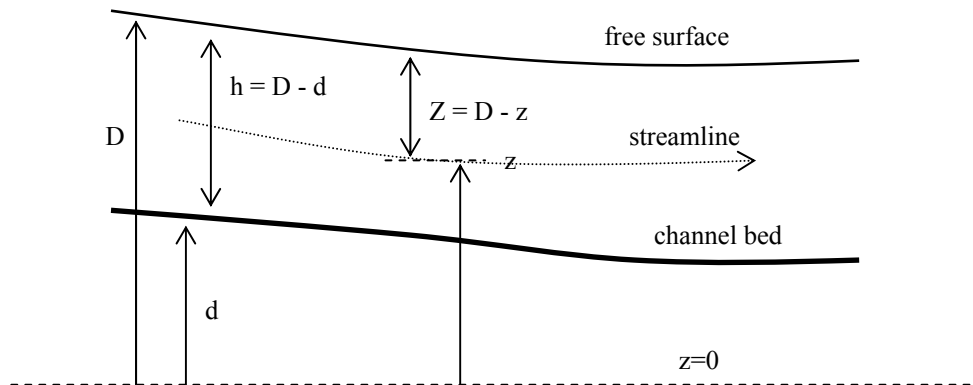$$A = (\pi/4)(0.01)^2 = 7.85\times10^{-5} \text{ m}^2. \qquad u = Q/A = 1.53 \text{ m s}^{-1}.$$

$$(pA + \rho A\, u^2) = 7.85\times10^{-5} \times (10^4 + 1000\times1.53^2) = 0.968 \text{ N}.$$

Force on pipe:

$$(0.28, -0.68) \text{ N}$$

If there is a change in area and if we can ignore energy losses, then we can use Bernoulli's Equation to find the relation between changes in pressure and changes in speed.

D4 Momentum principle: open channel flow
Conservation of energy (Bernoulli's Equation) - steady flow

If the channel bed and the flow is slowly-varying, so that there are no strong vertical velocities or accelerations, then the pressure is approximately hydrostatic:

$$p = \rho g Z = \rho g(D - z)$$

Thus the total head (for steady flow) on a streamline through z is given by,

$$H = \frac{u^2}{2g} + \frac{p}{\rho g} + z = \frac{u^2}{2g} + \frac{\rho g(D-z)}{\rho g} + z = \frac{u^2}{2g} + D \ .$$

Note this depends on the height of the free-surface, but not the actual height of the streamline.

If the flow is uniform (the same at all depths in the fluid), we simply have

$$H = \frac{u^2}{2g} + D \ ,$$

for all positions along the channel. (In practice there are frictional losses along the channel.)

Momentum: Hydraulic jumps

Control volume
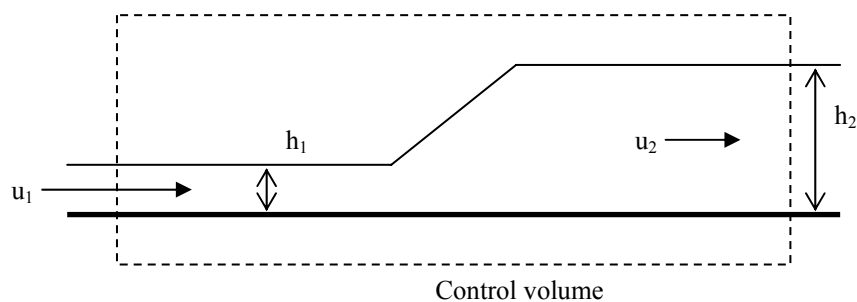
Assume the width, w, is constant.

$$Q = wu_1 h_1 = wu_2 h_2, \qquad \text{and so,} \qquad u_2 = u_1 h_1/h_2.$$

Pressure forces: hydrostatic pressure giving a horizontal force at each end.

Total force (per unit width),

$$F = \int_0^{h_1} \rho g Z \, dZ \; - \int_0^{h_2} \rho g Z \, dZ \; = \tfrac{1}{2} \, \rho g (h_1^2 - h_2^2) = u_1 h_1 \rho \, (u_2 - u_1)$$

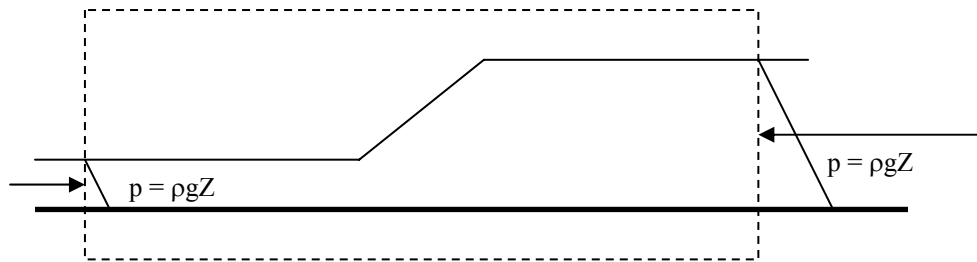$$\tfrac{1}{2} \, g(h_1^2 - h_2^2) = u_1^2 h_1 (h_1/h_2 - 1)$$

$$u_1 = \sqrt{gh_1} \, \sqrt{\tfrac{1}{2} r(1+r)} \;,$$

where $r = h_2/h_1$ is the size of the jump.
Note that for a jump ($r>1$), we need $u_1 > \sqrt{gh_1}$ ("supercritical flow").

Energy loss

$$H_1 = u_1^2/(2g) + h_1, \qquad H_2 = u_2^2/(2g) + h_2.$$

This gives the head loss:

$$H_1 - H_2 = (h_1/4r)\,(r - 1)^3.$$

Power loss is given by,

$$\rho g \, (H_1 - H_2)Q.$$

Froude number
The ratio $Fr = u/\sqrt{gh}$ is known as the Froude number. Since $\sqrt{gh}$ is the speed of surface waves, the Froude number can be thought of as analogous to the Mach number, in that it is a ratio of a flow speed to a wave speed.

$$Fr > 1 \quad \text{Supercritical flow}$$
$$Fr < 1 \quad \text{Subcritical flow}$$
$$Fr = 1 \quad \text{Critical flow}$$

D5 Tutorial: forces and hydraulic jumps

(a) Flow in a bend (no energy losses)

Water flows at a rate of 0.1 m^3 s^{-1} through a contracting U-tube (in the horizontal plane) into the atmosphere as shown in the figure. What is the force on the tube? (You may ignore energy losses.)

Outlet diameter 50 mm

Inlet diameter 75 mm
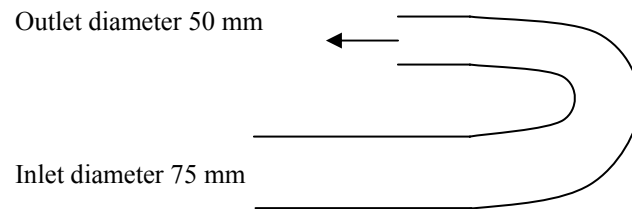
(b) Hydraulic jump

Flow in a 150 mm wide channel jumps from a depth of 50 mm to 150 mm. What is the flow rate and what is the rate of energy loss at the jump?

E   Pipeflow

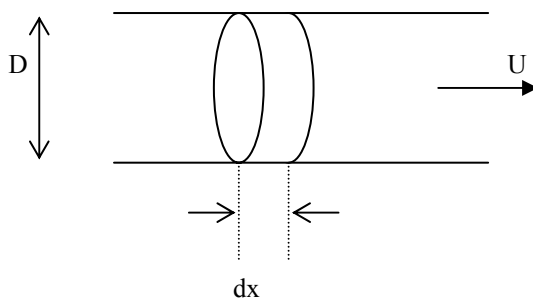## E1 Reynolds Experiment: Laminar and turbulent flow

Osborne Reynolds (1842-1912), Professor at Manchester University
Derived an expression for the relative size of viscous and other accelerations (Reynolds number: see below) and demonstrated the importance of this relation by examining the nature of flow in a pipe.

dye
injected

"direct" (laminar) motion

dye
injected

"sinuous" (turbulent) motion
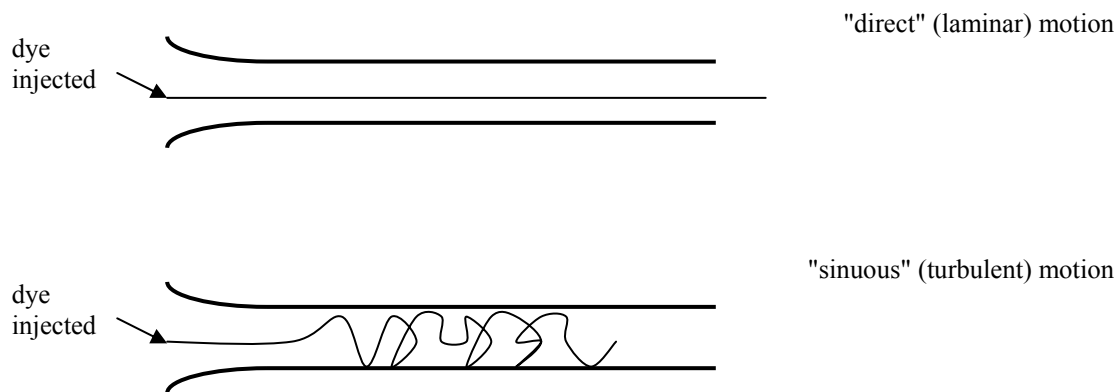
Reynolds Number (Re)
Consider flow in a pipe of diameter D with typical speed U.

D

U

dx

Volume of fluid element

$$V = (\pi/4)D^2 \, dx.$$

So mass of fluid element

$$m = \rho(\pi/4)D^2 \, dx.$$

Area of fluid element in contact with pipe wall

$$\pi D \, dx.$$

Typical size of velocity gradient

$$\frac{\partial u}{\partial r} \sim \frac{U}{D}$$

So typical size of viscous force on fluid element (ignore π)

$$F_V \sim \mu \frac{U}{D} D \, dx \sim \mu \, U \, dx$$

Thus accelerations due to viscous forces (again ignoring constants)

$$a_V = F_V/m \sim \frac{\mu \, U \, dx}{\rho D^2 \, dx} = \frac{\nu \, U}{D^2},$$

where $\nu = \frac{\mu}{\rho}$ is the kinematic viscosity.

But we can find another scale for accelerations

$$a \sim u\frac{\partial u}{\partial x} \sim U\frac{U}{D} = \frac{U^2}{D}.$$

The ratio of this to the viscous acceleration is given by

$$\frac{U^2}{D}\frac{D^2}{\nu U} = \frac{U D}{\nu} \equiv Re$$

The <u>Reynolds number</u> (usually denoted by Re) is the ratio of inertial to viscous accelerations and is equal to a velocity scale × length scale / kinematic viscosity. Note that the dimensions cancel out (ν has units $m^2 \, s^{-1}$), so that the Reynolds number is a dimensionless number and will be the same whatever units are used to measure lengths and times.

High Re - Turbulent flow (Re > 4000 for typical pipe flows). Inertial forces dominate.

Low Re - Laminar flow (Re < 2000 for typical pipe flows). Viscous forces dominate.

At intermediate Re the flow is "transitional" and may have intermittent "bursts" of turbulence in an otherwise smooth flow.

For flow in a pipe, the nature of the flow doesn't depend on the individual parameters (U, D, ν) but on the combination Re (the velocity scale used is usually the average velocity U=Q/A).

<u>Examples</u>
Flow rate Q = 0.01 $m^3 \, s^{-1}$ of water ($\nu = 10^{-6} \, m^2 \, s^{-1}$), in a pipe of diameter D = 100 mm. What is the Reynolds number and the expected type of flow?

$$A = (\pi/4) \, 0.1^2 = 7.85 \times 10^{-3} \, m^2. \qquad\qquad U = Q/A = 1.27 \, m \, s^{-1}.$$

Re = UD/ν = 1.27×0.1/10^{-6} = 127 000, so the flow will be turbulent.

Flow rate Q = $10^{-6} \, m^3 \, s^{-1}$ (= 1 $cm^3 \, s^{-1}$) of olive oil ($\nu = 10^{-4} \, m^2 \, s^{-1}$), in a pipe of diameter D = 5 mm. What is the Reynolds number and the expected type of flow?

$$A = (\pi/4) \, 0.005^2 = 1.96 \times 10^{-5} \, m^2. \qquad\qquad U = Q/A = 0.051 \, m \, s^{-1}.$$

Re = UD/ν = 0.051×0.005/10^{-4} = 2.6, so the flow will be laminar.

Most fluid flows in civil engineering are at high Reynolds number (an exception is flows through porous media, e.g. groundwater, water filtration).

| Flow	U (m s^{-1})	L (m)	ν (m^2 s^{-1})	Re
Water in a domestic pipe	2	0.01	10^{-6}	2×10^4
Air flow past a building	5	10	1.5×10^{-5}	3.3×10^6
River flow	2	5	10^{-6}	10^7
Air flow through a doorway	0.5	2	1.5×10^{-5}	6.7×10^4
Water percolating through sand	6×10^{-5}	10^{-4}	10^{-6}	6×10^{-3}

E2 Pipeflow: laminar flow

Low Re, viscous forces are important.
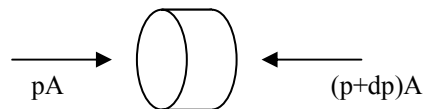
Laminar flow in a pipe (Poiseuille Flow)
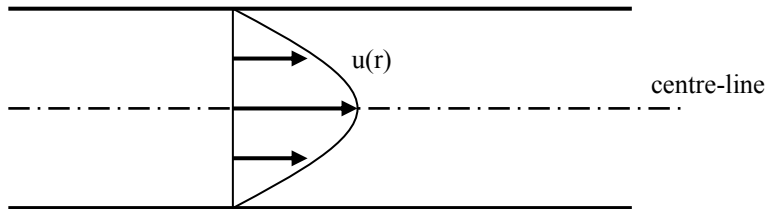
Smooth, steady flow. Expect velocity u to be a function of radius r, with u = 0 at r = R (no flow at the pipe wall) but do not expect the speed to vary along the pipe (at a given radius).

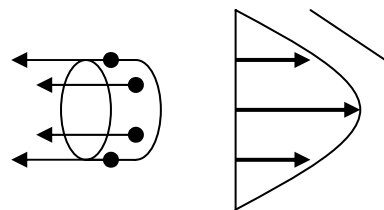Consider the forces on a cylindrical fluid element within the pipe:

Area of flat faces

$$A = \pi r^2$$

Area of curved surface

$$S = 2\pi r\, dx$$

Volume
$$V = \pi r^2\, dx$$
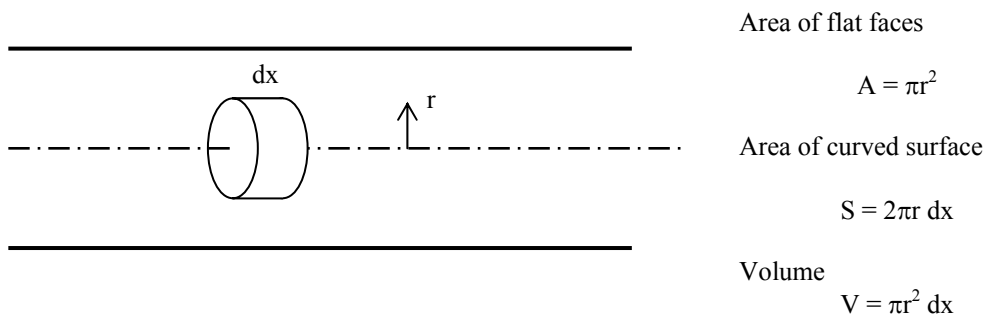
<u>Pressure forces</u>

Total pressure force = -dp A = -dp πr^2.

<u>Viscous forces</u>

The viscous force depends on the shear at that radius.

Force per unit area (viscous stress)

$$= \mu \frac{\partial u}{\partial r}$$

So total force = $\mu \dfrac{\partial u}{\partial r}$ S = $\mu \dfrac{\partial u}{\partial r} \times 2\pi r\, dx$

Here velocity decreases as r increases, so this is negative - the viscous forces tend to slow the fluid down.

For steady flow which isn't changing along the pipe (du/dx = 0), the sum of the viscous and pressure forces must be zero.

$$-dp\, \pi r^2 + \mu \frac{\partial u}{\partial r} \times 2\pi r\, dx = 0$$

Rearranging gives,

$$2\mu \frac{\partial u}{\partial r} = r\frac{dp}{dx}$$

If the pressure gradient driving the flow along the pipe is constant $G = \frac{dp}{dx}$, then,

$$\frac{\partial u}{\partial r} = \frac{Gr}{2\mu}.$$

If the flow is being driven in the x-direction we expect the pressure to decrease as x increases, so G would be negative.

Integrating this up we get $u = \frac{Gr^2}{4\mu}$ + constant. We find the constant by using $u(R) = 0$:

$$u = -\frac{G}{4\mu}(R^2 - r^2)$$

The velocity profile for laminar flow is parabolic. This flow is known as Poiseuille flow. The total flow rate is given by,

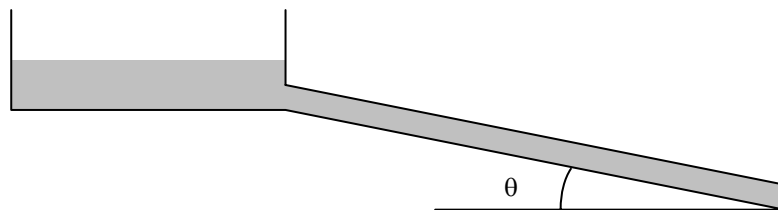$$Q = \int_0^R u\, 2\pi r\, dr = -\frac{\pi G R^4}{8\mu}.$$

The average flow speed $U = Q/A$ is

$$U = -\frac{GR^2}{8\mu} = \frac{1}{2}U_{max}.$$

For viscous flows, velocity and flow rates are proportional to the pressure gradient.

Note that energy is lost and Bernoulli's Equation no longer applies (or has to be adjusted to include energy losses).

Example: viscous flow down inclined tube

Here there is no significant pressure gradient driving the flow, but the component of gravity along the tube has the same effect:

$$\rho g \sin \theta = -G$$
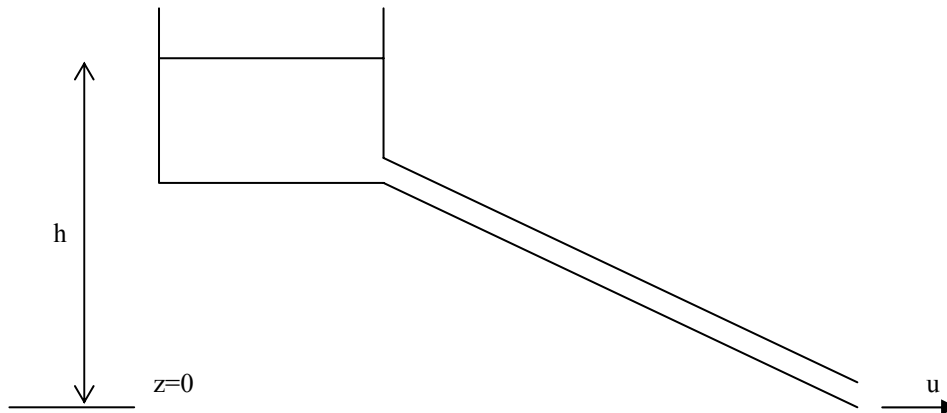
For $d = 3$ mm, $\theta = 10°$, with water we get,

$$U = -\frac{GR^2}{8\mu} = \frac{\rho g \sin\theta\, d^2}{32\mu} = \frac{g \sin\theta\, d^2}{32\nu} = \frac{9.81 \times \sin10°\ 0.003^2}{32\text{x}10^{-6}} = 0.47 \text{ m s}^{-1}.$$

Check Reynolds number:

$$Re = UD/\nu = 1410 \text{ (just about laminar)}.$$

E3 Flow from static reservoir (no energy losses)

Now consider the case where we can ignore viscosity and energy losses.

If the water is discharged into the atmosphere, then the pressure there is atmospheric (approximately the same as at the surface of the reservoir). Bernoulli's Equation reduces to,

$$h = \frac{u^2}{2g}, \text{ so } u = \sqrt{2gh}. \text{ (This is the speed a body falling from height h would have.)}$$

The total head is given by the height of the water surface in the static reservoir, i.e. $H = h$.
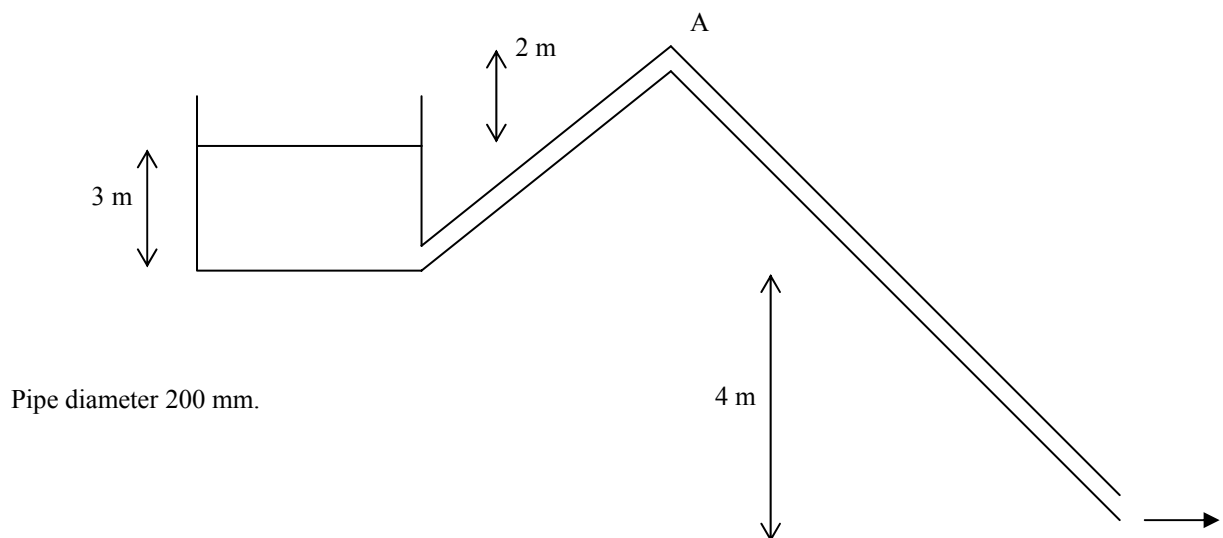
$$H = \frac{u^2}{2g} + \frac{p}{\rho g} + z,$$

where p is the gauge pressure (relative to atmospheric).

Thus in the pipe, the pressure is simply given by

$$p = -\rho g z.$$

Example

Pipe diameter 200 mm.

Height of surface above outlet H = 3 + 4 = 7 m.

Speed of flow $u = \sqrt{2 \times g \times 7} = 11.7$ m s^{-1}. Area A = $(\pi/4)d^2 = 0.314$ m^2.

Thus flow rate Q = 11.7×0.314 = 3.67 m^3 s^{-1} (= 3670 litres/s).

Pressure at A (relative to atmospheric)

p = - ρgz = -1000×9.81×(4+3+2) = -88.3 kN m^{-2} = -88.3 kPa.

(Note: for total heights over about 10 m, predicted pressures would be less than absolute zero, which is not possible.)

Example
An open rectangular tank 2m×3m×1m high is full of water. It is emptied through a tube of diameter 15mm, discharging to the atmosphere 2m below the bottom of the tank. How long does it take to empty?
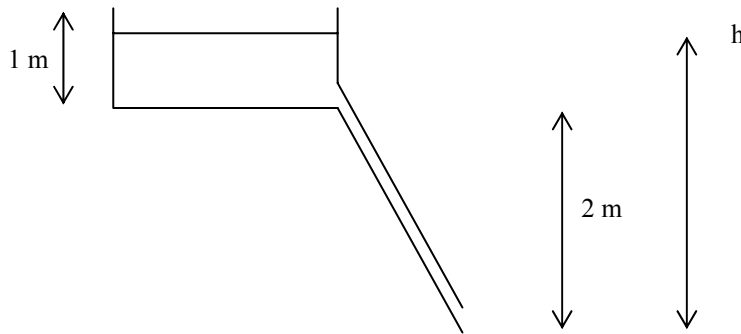
When the depth of water in the tank is h, the speed of the water coming out of the tube is given by,

$$u = \sqrt{2gh} \,,$$

(assuming the vertical speed of the water surface is negligible and ignoring any energy losses).

The flow rate is $Q = ua = \sqrt{2gh}\ \frac{\pi}{4} d^2$, where d = 0.015 m and a is the tube area.

The rate at which the water level descends is Q/A where A is the area of the tank (A = 6 m^2).

$$\frac{dh}{dt} = -\frac{Q}{A} = -\frac{\sqrt{2gh}}{6}\left(\frac{\pi}{4}\right)0.015^2 \quad \text{or} \quad \frac{dh}{dt} = -1.305 \times 10^{-4}\sqrt{h}\,.$$

So

$$\int_{h=3}^{h=2} h^{-1/2}dh = -\int_{t=0}^{t=t_e} 1.305 \times 10^{-4}\,dt\,.$$

Thus 2×(3$^{1/2}$ - 2$^{1/2}$) = 1.305×10^{-4} t$_e$ or t$_e$ = 4870 s (approximately one hour twenty minutes).

E4 Turbulent flow and head loss

Head loss in pipes: Darcy's formula
In Civil Engineering applications, flows are usually at high Reynolds numbers. Conservation of energy (Bernoulli's Equation) gave

$$H = \frac{u^2}{2g} + \frac{p}{\rho g} + z,$$

with H a constant.

For flow in a pipe of constant diameter u is constant and so the dynamic head is constant. If Bernoulli's Equation applied we would expect the piezometric head to also remain constant. However, experiments show that the piezometric head falls, with the fall proportional to the length of the tube and the square of the speed (at high Reynolds number).

This can be thought of as a drop in the total head. For a pipe of length l and diameter d, the drop is found to be,

$$h_f = \left(\frac{\lambda l}{d}\right) \left(\frac{u^2}{2g}\right),$$

where λ is a constant, known as the **friction factor**. This expression is know as Darcy's formula. (It is convenient to keep it as a multiple of the dynamic head.) The constant λ depends on the relative roughness of the pipe (k_S/d, where k_S, or just k, is the size of the roughness - "bumps") and on the Reynolds number of the flow. For very large values of Re, λ becomes independent of Re and is approximately given by,

$$\lambda = (2 \log_{10}(3.7 \, d/k_S))^{-2}$$

Darcy's formula is sometimes expressed as

$$h_f = \left(\frac{4fl}{d}\right) \left(\frac{u^2}{2g}\right),$$

where $f=\lambda/4$. WARNING: Sometimes (especially in the USA) the symbol f is used for λ, giving a different f from that we've defined (by a factor of 4). When using published graphs and tables it is important to check which version of f or λ is used.

Example
For galvanized steel, the roughness scale k_S = 0.15 mm. For a pipe of diameter 100 mm, d/k_S = 667.
Thus the friction factor (for high Re) is λ = 0.0217.

Typical roughness scales k_S (mm)	
Riveted steel	1 - 10
Concrete	0.3 - 3
Wood stave	0.2 - 1
Cast iron	0.25
Galvanized steel	0.15
Steel or wrought iron	0.045
Drawn tubing	0.0015

Roughness tends to increase with age because of deposits and corrosion.

Friction factor for laminar flow
If there is a head loss of h_f, then the corresponding pressure change is simply

$$\Delta p = h_f \, \rho g.$$

This gives a pressure gradient (along a pipe of length l) of

$$G = -\frac{\Delta p}{l} = -\frac{h_f \, \rho g}{l}.$$

For low Re, laminar flow, we found the average velocity was given by

$$U = \frac{Gd^2}{32\mu} = \frac{h_f\, \rho g\, d^2}{32\, \mu\, l}.$$

Rearranging gives

$$h_f = \frac{32\mu l U}{\rho g d^2} = \left(\frac{l}{d}\right)\left(\frac{64\nu}{Ud}\right)\left(\frac{U^2}{2g}\right) = \left(\frac{l}{d}\right)\left(\frac{64}{Re}\right)\left(\frac{U^2}{2g}\right)$$

Which is equivalent to Darcy's formula with $\lambda = 64/Re$.

Moody's Diagram
At low Re (laminar flow) the pressure drop (and head loss) along a pipe is proportional to the velocity, while at very large Re the pressure drop is proportional to the square of the velocity. For a given relative roughness (k_S/d), a plot of friction factor λ as a function of Reynolds number will have $\lambda = 64/Re$ at low Re, tending towards a constant $\lambda = (2\log_{10}(3.7\, d/k_S))^{-2}$ at high Re. At intermediate values of Re there is a transition between the laminar and very turbulent flows. For very smooth pipes, where the boundary layer is larger than the roughness elements, $\lambda = 0.316 Re^{-1/4}$ gives good agreement with experiment up to Re = 10^5. A Moody Diagram shows the relationship between λ and Re for various relative roughnesses (k_S/d). The plot is usually on logarithmic axes, so that the laminar formula $\lambda = 64/Re$ is a straight line.

(WARNING: If you see a plot where the laminar formula is friction factor $= 16/Re$, then this is a plot of $f = \lambda/4$, as described earlier. Older exam papers also give f instead of λ, and you will need to use $\lambda = 4f$ to get the correct results.)

For the present course, we will assume that the flow can be described by a constant λ (rough pipe flow), or by laminar flow. At intermediate values calculations become more complicated because the flow depends on λ, but λ depends on Re which in turn depends on the flow. Thus an iterative procedure is usually needed to calculate the flow. Formulae that match the smooth pipe flow equation to the high Re (constant λ) values and the methods of using them will be dealt with in later courses.
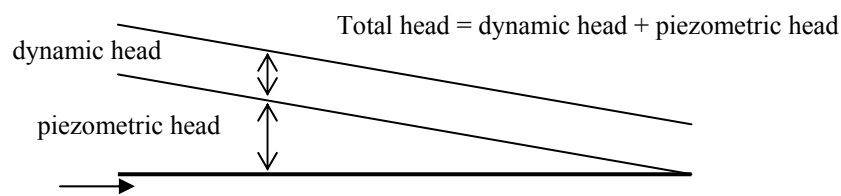
Example
A flow Q = 0.1 m^3 s^{-1} is flowing through a galvanized steel pipe of diameter 100 mm. What is the head loss over a distance of 15 m?

$$A = (\pi/4)\, 0.1^2 = 7.85\times10^{-3} \text{ m}^2, \qquad k_S = 0.15 \text{ mm}, \quad \lambda = 0.0217. \qquad u = Q/A = 12.7 \text{ m s}^{-1}$$

$$h_f = \left(\frac{\lambda l}{d}\right)\left(\frac{u^2}{2g}\right) = \left(\frac{0.0217\times15}{0.1}\right)\left(\frac{12.7^2}{2\times9.81}\right) = 26.8 \text{ m}$$

Power loss = Q $\rho g\, h_f$ = 26.3 kW.

Total head = dynamic head + piezometric head

dynamic head

piezometric head

E5 Pipeflow: other head losses

Head loss coefficient
We've already seen that the head loss along a pipe of length l is given by

$$h_f = \left(\frac{\lambda l}{d}\right)\left(\frac{u^2}{2g}\right).$$

There are other head losses (energy losses) caused by enlargements, contractions, bends, valves and other fittings. These are generally expressed in the form

$$h_l = k\left(\frac{u^2}{2g}\right).$$

In principle, the head loss coefficient, k, varies with Re but at large Re the head loss coefficient is effectively constant. (**Warning**: don't confuse the non-dimensional head loss coefficient k with the roughness length also unfortunately often denoted by k or k_S.)
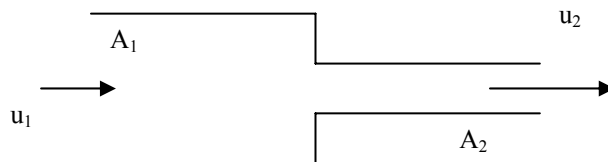
Abrupt enlargement
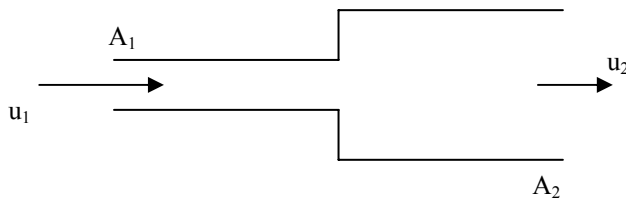
$$h_l = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{u_1^2}{2g} = \left(\frac{A_2}{A_1} - 1\right)^2 \frac{u_2^2}{2g}$$

(Proof later in F1.) As $A_2 \to \infty$, the **"exit loss"** (e.g. from a pipe into a reservoir) $h_l \to \dfrac{u_1^2}{2g}$.
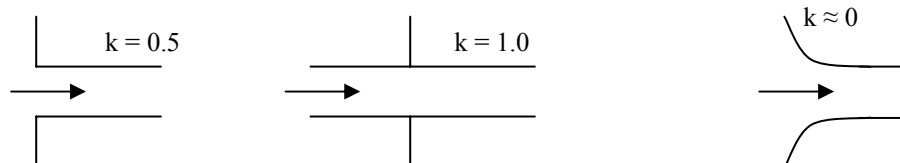
Abrupt contraction

$h_l = k\dfrac{u_2^2}{2g}$. As $A_1 \to \infty$, the **"entry loss"** (e.g. from a reservoir into a pipe) $h_l \to 0.5\dfrac{u_2^2}{2g}$

d_2/d_1 (diameter ratio)	0	0.2	0.4	0.6	0.8
k	0.5	0.45	0.38	0.28	0.14

Other entry losses

Pipe fittings, typical losses

Fitting	90° bend	90° corner	45° bend	T (in-line)	T (side)
k	0.9	1.1	0.4	0.4	1.2

Example

A reservoir is drained through a 150 mm diameter pipe, 500 m long, with an exit to the atmosphere 10 m below the water surface in the reservoir. What is the discharge? (The friction factor for the pipe is $\lambda = 0.020$.)

10 m
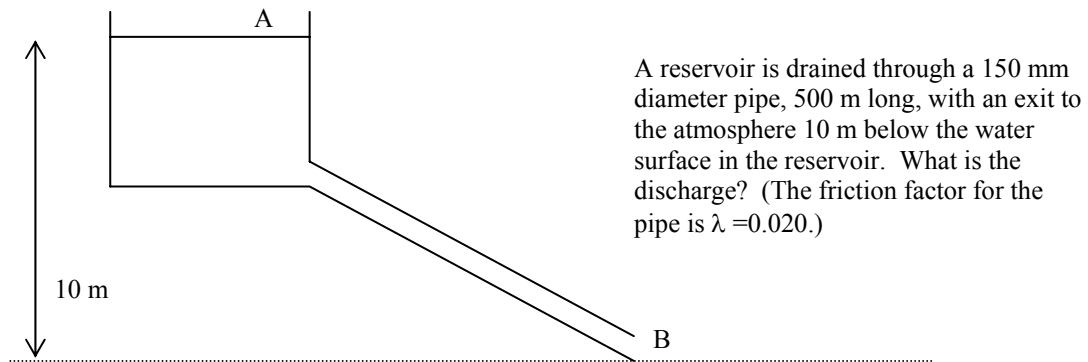
Energy at A = Energy at B + Energy losses from A to B

$$H_A = H_B + H_{losses}$$

H_{losses} = entry loss + pipe loss (assume no significant loss on exit to atmosphere)

$$10 = \frac{u^2}{2g} + k_{entry}\frac{u^2}{2g} + \left(\frac{\lambda l}{d}\right)\frac{u^2}{2g}$$

$$10 = \frac{u^2}{2g} + 0.5\frac{u^2}{2g} + \left(\frac{0.020\times500}{0.15}\right)\frac{u^2}{2g}$$

$$10 = (1 + 0.5 + 66.7)\frac{u^2}{2g}$$

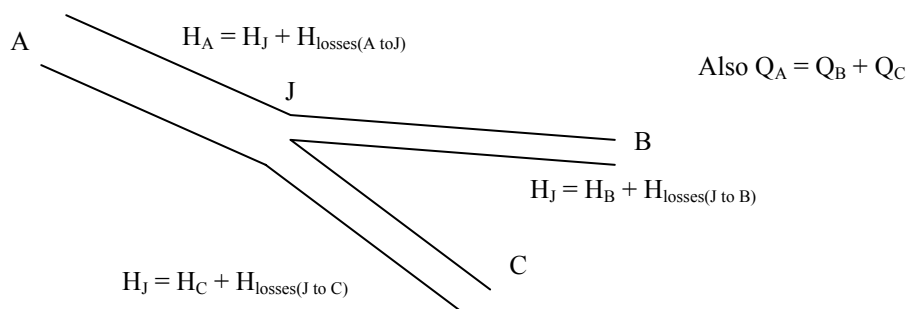$$\frac{u^2}{2g} = 10/68.2 = 0.147, \qquad \text{so } u = 1.70 \text{ m s}^{-1}.$$

This gives $Q = 0.030$ m^3 s^{-1}.

(Check Re = $ud/\nu = 2.6\times10^5$.)

Note that most of the losses come from flow along the pipe. In practice ignoring entry and exit losses, and losses at bends and fittings does not usually lead to significant errors in most civil engineering applications.

Junctions

$H_A = H_J + H_{losses(A\ to\ J)}$

Also $Q_A = Q_B + Q_C$

$H_J = H_B + H_{losses(J\ to\ B)}$

$H_J = H_C + H_{losses(J\ to\ C)}$

E6 Tutorial: Pipeflow

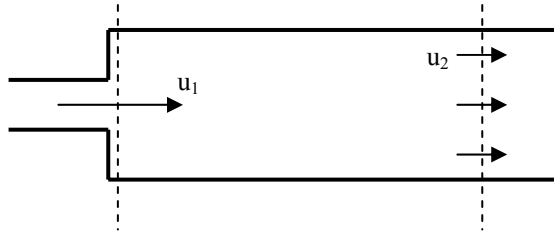Calculate the flow between the two reservoirs connected as shown.
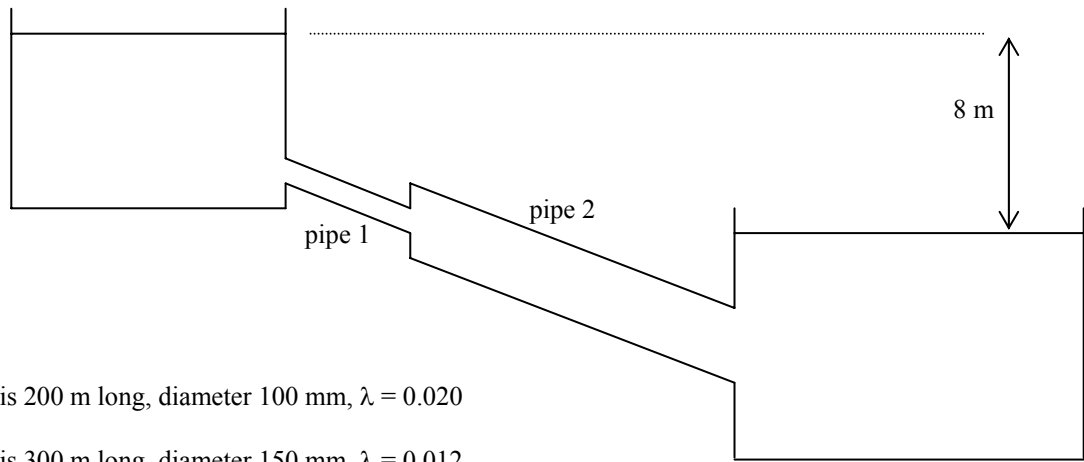
Pipe 1 is 200 m long, diameter 100 mm, $\lambda = 0.020$

Pipe 2 is 300 m long, diameter 150 mm, $\lambda = 0.012$

F   *Energy and momentum: further applications*

F1 Sharp expansions and orifice meters

<u>Sharp expansion</u>

The pressure on the face of the vertical surface just downstream of the expansion is found to be close to p_1. (Weak radial accelerations there.)

If we assumed there was no energy loss we would find,

$$(p_1 - p_2) = \rho \, (u_2{}^2 - u_1{}^2)/2,$$

$$= \rho \, \bar{u} \, (u_2 - u_1),$$

where $\bar{u} = (u_2 + u_1)/2$ is the average of u_2 and u_1.

However, applying the momentum principle to a control volume as marked on the diagram (and using the observation that the pressure just downstream of the expansion is close to p_1 over the face of the vertical surface), we find

$$(p_1 - p_2) \, A_2 = \rho Q \, (u_2 - u_1)$$

so,

$$(p_1 - p_2) = \rho \, u_2 \, (u_2 - u_1).$$

If the head loss is h_l,

$$H_1 = H_2 + h_l$$

$$\frac{u_1^2}{2g} + \frac{p_1}{\rho g} = \frac{u_2^2}{2g} + \frac{p_2}{\rho g} + h_l$$

$$h_l = \frac{u_1^2 - u_2^2}{2g} + \frac{p_1 - p_2}{\rho g}$$

$$h_l = \frac{u_1^2 - u_2^2}{2g} + \frac{\rho u_2 \left(u_2 - u_1 \right)}{\rho g}$$

$$h_l = \frac{u_1^2 - u_2^2 + 2u_2^2 - 2u_2 u_1}{2g}$$

$$h_l = \frac{\left(u_1 - u_2 \right)^2}{2g}$$

But

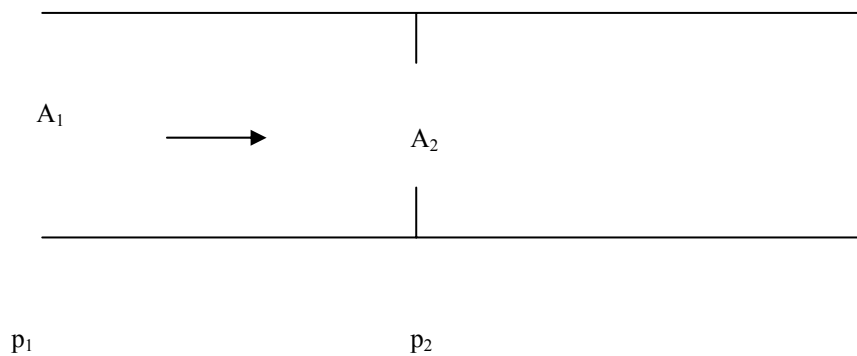$$Q = u_1 \, A_1 = u_2 \, A_2, \qquad \Rightarrow \qquad u_2 = u_1 \, (A_1/A_2).$$

so,

$$h_l = \frac{(u_1 - u_1(A_1/A_2))^2}{2g},$$

or

$$h_l = \left(1 - \frac{A_1}{A_2}\right)^2 \frac{u_1^2}{2g} = \left(\frac{A_2}{A_1} - 1\right)^2 \frac{u_2^2}{2g}$$

(as given in E5).

Orifice meter

A_1

A_2

p_1 p_2

Similar to a venturimeter (see D2) but much more energy is lost with this type of flow. We have a sharp contraction followed by a sharp expansion. We can use a similar formula as for the venturimeter,
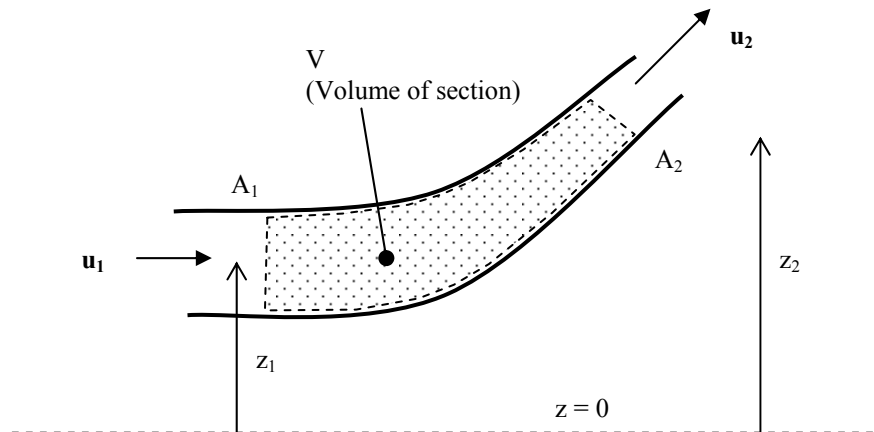
$$Q = A_1 C_d \sqrt{\frac{2g\Delta h}{\left(\dfrac{A_1^2}{A_2^2} - 1\right)}},$$

but now $C_d \approx 0.6$. While it is possible to estimate C_d theoretically, in practice for accurate measurements the orifice meter will be calibrated to find C_d.

F2 Momentum principle: effects of gravity
Flow around a pipe bend (continuation of D3)

b) change in area ($A_1 \neq A_2$) and now include gravity

Conservation of mass

$$Q = uA = u_1A_1 = u_2A_2$$

If there are no (or negligible) energy losses

$$\frac{u_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = H = \frac{u_2^2}{2g} + \frac{p_2}{\rho g} + z_2$$

The gravitational force is ρVg (downwards).

Example
Water is flowing at 0.05 m³ s⁻¹ through a 90° reducing elbow turning from horizontal to vertical, emerging into the atmosphere with the exit 0.25 m above the large pipe. If the diameter reduces from 100 mm to 50 mm, and the volume of the fitting is given as 1.6×10^{-3} m³, what force is required to keep the fitting in place?
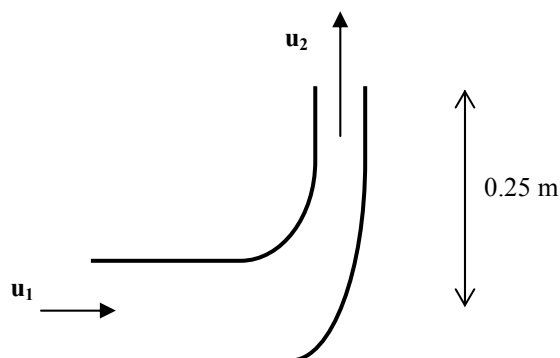
(For the force on the elbow, we need the difference in pressure from atmospheric, so take $p_2 = 0$.)

Conservation of mass

$Q = 0.05$ m^3 s^{-1},　　　　$A_1 = (\pi/4)\, 0.1^2 = 7.85{\times}10^{-3}$ m^2;　　$A_2 = (\pi/4)\, 0.05^2 = 1.96{\times}10^{-3}$ m^2;

and thus　　$u_1 = Q/A_1 = 6.37$ m s^{-1};　　$u_2 = Q/A_2 = 25.5$ m s^{-1};

giving velocities　　$\mathbf{u_1} = (6.37, 0.0)$ m s^{-1};　　$\mathbf{u_2} = (0.0, 25.5)$ m s^{-1}.

Assuming no energy loss in the bend,

$$\frac{u_1^2}{2g} + \frac{p_1}{\rho g} + z_1 = H = \frac{u_2^2}{2g} + \frac{p_2}{\rho g} + z_2 ,$$

$$\frac{6.37^2}{2\times 9.81} + \frac{p_1}{9.81\times 1000} + 0 = \frac{25.5^2}{2\times 9.81} + \frac{0}{9.81\times 1000} + 0.25 ,$$

so,

$$p_1 = 3.07{\times}10^5 \text{ N m}^{-2}.$$

(In practice there are likely to be energy losses.)
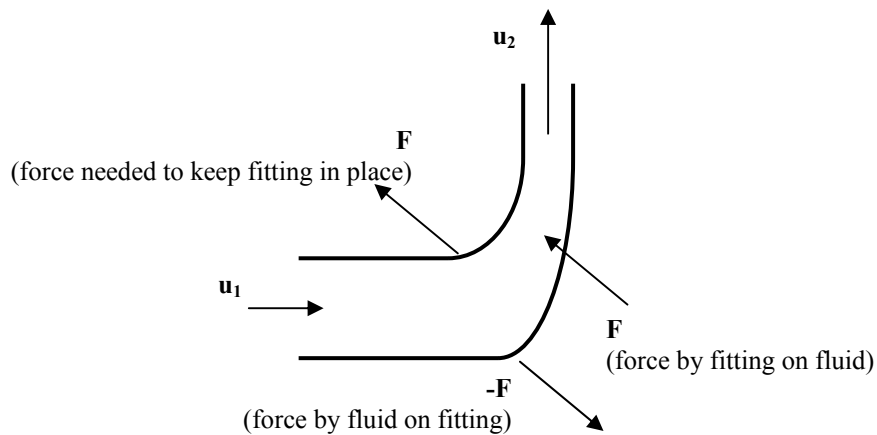
Horizontal forces (acting on fluid)　　　　　vertical forces (acting on fluid)

$p_1 A_1 + F_x = 0 - \rho Q u_1$　　　　　　$-p_2 A_2 + F_z - \rho V g = \rho Q u_2 - 0.$

Where the force by the fitting <u>on</u> the fluid is $\mathbf{F} = (F_x, F_z)$.

Thus $\mathbf{F} = (-2730, 1290)$ N, and the force by the fluid on the fitting is $-\mathbf{F} = (2730, -1290)$ N. Thus the force needed to keep the fitting in place is $\mathbf{F}$.

F3 Tutorial: Gravity and flow measurement

Flow in a bend with energy loss

What would be the result to the previous example (flow in a bend with gravity) where the applied pressure p_1 is the same but you assume a head loss in the bend given by

$$h_l = k \frac{u_1^2}{2g}, \text{ with k=0.9?}$$

Orifice meter

Air flowing through a tube of diameter 10cm, with a sharp orifice of diameter 5 cm.

The pressure drop is measured with a water manometer giving a change in level of 10 cm. What is the flow rate? (Assume $C_d = 0.6$.)