Some notes on acoustics

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Being to treat of the Doctrine of Sounds, *I hold it convenient to premise something in the general concerning this Theory; which may serve at once to engage your attention, and excuse my pains, when I shall have recommended them, as bestow'd on a subject not altogether useless and unfruitful.*

Narcissus Marsh, 1683/4, *Phil. Trans. Roy. Soc. Lond.*, **156**:472–486.

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Chapter 1

What is sound?

Acoustics is a branch of physics and, as such, anything it tells you about the world has to make sense. If it tells you something you don't believe then either it's wrong or you are. To start, it's worth looking at the things you already know about acoustics from your daily life. These are fundamental facts which also happen to be correct.

The first example we can consider is that of a lecturer droning on at a class. Everyone in the class hears the lecturer say the same thing at the same pitch: we don't have one part of the class hearing the lecturer speak with a squeaky voice while another part hears her speak in a deep bass. Furthermore, everyone hears the lecturer speak at the same speed with the words in the same order. This tells us that

$$\boxed{\text{sound travels undistorted}}$$

so, no matter where we are, as long as we can hear the speaker, we hear the same words at the same pitch and at the same rate.

Ponder now the forces of nature: the next time you are caught in a thunderstorm note the relationship between thunder and lightning. You will notice, if you have not already done so, that there is a delay between seeing the flash of the lightning and hearing the thunder:

$$\boxed{\text{sound travels with some time delay}}$$

so that we do not hear sound from a source immediately but have to wait for it to travel over the space between it and us.

Finally, bored by the lecture and soaked by the storm, you go to a concert. For my purposes, I assume that you are a fan of a singer armed with a guitar. If you listen to the singer and the guitar, you will be able to distinguish the singer's voice from the sound of the guitar:

$$\boxed{\text{sound from different sources travels independently}}$$

or in other words, the sound coming from the singer does not influence the sound from the guitar—you simply hear both of them added together.

1.1 Sound in time and space

We need some way to describe sound. The first obvious way to think physically about sound is as a signal measured at some position, our ears or a microphone, say. If we measure pressure, this signal can be written $p(t)$. It changes over time and, if we want, we can record it. On the other hand, at any given time, two people can measure sound at two different positions. We could also say that sound is a function of position and write $p(\mathbf{x})$. Clearly, sound depends on both time and position, so the correct thing to do is write $p(\mathbf{x}, t)$.

If we wanted to, we could leave the matter there. On the other hand, we know that there has to be some connection between the pressure measured at one point and the pressure measured at another: sound cannot vary independently in time and in space. What is this connection? From the statements at the start of the

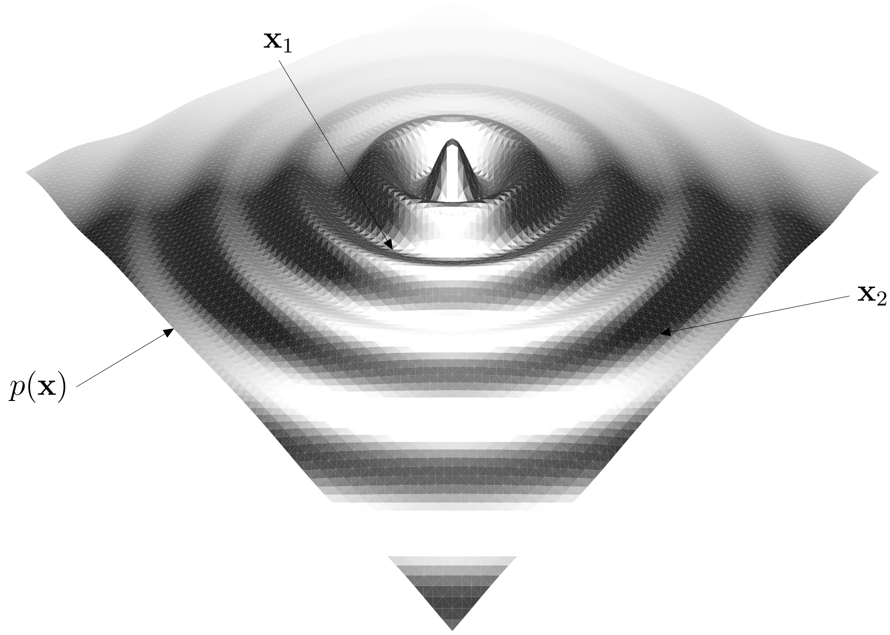


Figure 1.1: Sound pressure p at a fixed time

chapter, we know that the sound heard at one point is the same as the sound heard at another, although they might not be heard at the same time.

Figure 1.1 shows a snapshot of a wave radiating from some point, found by plotting pressure $p(\mathbf{x})$ at some fixed time t. If we pick two points $\mathbf{x}_1$ and $\mathbf{x}_2$ and look at the sound at those two points $p(t)$ and $q(t)$, say, we know that the two sounds are different. On the other hand they must be connected: one point cannot be hearing Mozart while the other hears a pneumatic drill. So, we know that the two sounds are the same with the possible exception of some time difference:

$$p(t) = q(t + \Delta t),$$

where Δt is a time difference. If we assume that sound 'travels' at some speed c (we will prove this is true later on), we could say that $\Delta t = R/c$ where R is some distance. Then we can write:

$$p(t) = q(t \pm R/c),$$

so that the time difference between the two signals is related to some distance over which sound has to travel. In the next section we will show that this kind of solution arises from the standard equations of fluid dynamics.

1.2 The wave equation

From a physical or mathematical point of view, acoustics can be viewed as the study of solutions of the wave equation for a fluid. The linear wave equation, which we will derive presently, is the equation governing the propagation of small (linear) disturbances in a compressible medium. The wave equation can be applied to many different systems with different governing equations: here we apply it to fluids governed by the Navier–Stokes equations.

The equations of continuity and momentum for an inviscid fluid are:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0, \tag{1.1a}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \nabla p + \rho \mathbf{v} \nabla \mathbf{v} = 0. \tag{1.1b}$$

These equations tell us, first, that matter is conserved and, second, that Newton's laws apply to a fluid as well as to solid particles. The first thing we do in deriving a wave equation is introduce the assumption that the fluctuations in the fluid dynamical quantities are small. This means that we write quantities as the sum of a mean part and a small fluctuation. These fluctuating parts are so small that their products can be neglected. Decomposing the quantities:

$$\begin{aligned} \rho &= \rho_0 + \rho'(t), \\ \mathbf{v} &= \mathbf{v}'(t), \\ p &= p_0 + p'(t), \end{aligned}$$

where 0 indicates a mean value and a prime symbol a fluctuation.

Applying this assumption to the equations of continuity and momentum and neglecting second order terms (products of small quantities), we find the linearized Euler equations:

$$\frac{\partial \rho'}{\partial t} + \rho_0 \nabla . \mathbf{v}' = 0, \tag{1.2a}$$

$$\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p' = 0. \tag{1.2b}$$

To make life easier, we can eliminate the velocity $\mathbf{v}'$ to give us a single equation:

$$\begin{aligned} &\frac{\partial}{\partial t}\left(\frac{\partial \rho'}{\partial t} + \rho_0 \nabla . \mathbf{v}'\right) - \nabla \left(\rho_0 \frac{\partial \mathbf{v}'}{\partial t} + \nabla p'\right) \\ =&\frac{\partial^2 \rho'}{\partial t^2} - \nabla^2 p' = 0. \end{aligned} \tag{1.3}$$

This is almost the wave equation except that it contains both pressure and density and we would like to deal with only one quantity at a time. To eliminate the density, we need a relationship between it and pressure. This depends on the thermodynamical properties of the fluid, as we will see below. Since we have linearized everything else, we can linearize the pressure–density relationship as well:

$$\begin{aligned} p &= p_0 + \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0) + \frac{1}{2} \left.\frac{\partial^2 p}{\partial \rho^2}\right|_{\rho=\rho_0} (\rho - \rho_0)^2 + \ldots, \\ p' &= p - p_0 \approx \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0) = c^2 \rho', \\ c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}. \end{aligned}$$

The constant is written c^2 because it is always positive (why?). Substituting this relationship into equation 1.3, we find a wave equation for the acoustic pressure:

$$\boxed{\frac{1}{c^2} \frac{\partial^2 p}{\partial t^2} - \nabla^2 p = 0} \tag{1.4}$$

This is the most fundamental equation in acoustics. It describes the properties of a sound field in space and time and how those properties evolve. It is quite unlike the incompressible flow equations to which you may be accustomed because it describes very weak processes which happen over large distances. The most fundamental obvious property of the wave equation is that it is *linear*. This means that the sum of two solutions of the wave equation is also itself a solution, which is why we can tell a singer from an instrument.

When we come to solve the wave equation, we will find that c is the speed of sound, the speed at which a small disturbance propagates through a fluid. It depends on the thermodynamical properties of the fluid and is calculated on the assumption that sound propagation is *adiabatic*. For an adiabatic process in a gas:

$$p = k \rho^{\gamma},$$

where γ is the ratio of the specific heats. Then

$$\begin{aligned} c^2 &= \left.\frac{\partial p}{\partial \rho}\right|_{\rho=\rho_0}, \\ &= \gamma k \rho^{\gamma-1} = \frac{\gamma p}{\rho}, \\ p &= \rho R T \end{aligned}$$

so that

$$\boxed{c^2 = \gamma R T}.$$

The speed of sound in air at STP is $343\mathrm{m/s}$. The validity of the adiabatic assumption depends on the frequency of the sound. For low-frequency sound, there is no appreciable heat generation by conduction in the fluid and the assumption is a good one. For air, ‘low frequency’ means ‘less than $1\mathrm{GHz}$’.

Note that if $c \to \infty$, the wave equation becomes $\nabla^2 p = 0$, the equation of incompressible flow. Saying $c \to \infty$ is the same as saying that density is independent of pressure, i.e. that the flow is incompressible. Since c is the speed at which disturbances propagate in a fluid, this is equivalent to the statement that disturbances propagate instantaneously in an incompressible flow.

1.3 Single frequency waves

If we write $p = P\exp[-\mathrm{j}\omega t]$ where ω is the radian frequency, the wave equation becomes the *Helmholtz* equation:

$$\boxed{\nabla^2 P + k^2 P = 0.} \tag{1.5}$$

Note that t has disappeared, reducing the order of the equation by one. The *wavenumber* $k = \omega/c$.

When we are dealing with waves of constant frequency, the sound field is a sinusoidal pattern which propagates in space.

1.4 Quantifying sound

Before going any further, you will need to know how to describe a sound or sound field. We characterize noise by its pitch (frequency) and its ‘volume’ (amplitude). To describe the amplitude of a sound we usually use the root mean square (rms) pressure:

$$p_{\mathrm{rms}} = \left(\overline{p^2}\right)^{1/2}$$

where the bar denotes ‘time average’. This is a useful measure but suffers from the problem that acoustic pressures of interest vary over a huge range. The threshold of human hearing is at $p_{\mathrm{rms}} = 20\mu\mathrm{Pa}$ while the threshold of pain and the onset of hearing damage are at $p_{\mathrm{rms}} \approx 200\mathrm{mPa}$, a range of seven orders of magnitude. To keep the numbers manageable, we use a logarithmic scale. On this scale, the ‘difference’ in *sound pressure level* between two pressures p_1 and p_2 is:

$$\Delta_{\mathrm{SPL}} = 10\log_{10}\frac{\overline{p_1^2}}{\overline{p_2^2}}.$$

When we want to talk about only one signal, we use a standard reference pressure. Then the sound pressure level is

$$\mathrm{SPL} = 10\log_{10}\frac{\overline{p^2}}{p_{\mathrm{ref}}^2}. \tag{1.6}$$

Level/dB	Example
140	3m from a jet engine
130	Threshold of pain
120	Rock concert
110	Accelerating motorcycle at 5m
80	Vacumn cleaner
60	Two people talking
10	3m from human breathing

Table 1.1: Some sample approximate noise levels

The reference level is the nominal threshold of human hearing $20\mu\mathrm{Pa}$. The 'units' of SPL are decibels, dB.

Table 1.1 shows levels for some typical noises. A good rule of thumb is that if you have to raise your voice to speak, the noise level is greater than $80\mathrm{dB}$, and if you have to shout, the noise level is greater than $85\mathrm{dB}$ and you risk hearing damage.

1.5 Solutions of the wave equation in one dimension: Plane waves

To illustrate some aspects of the solution of the wave equation, we look first at waves in one dimension. This corresponds to sound propagating in a pipe, for example. If we take x as the coordinate along the pipe, the wave properties are independent of y and z and the wave equation becomes:

$$\frac{1}{c^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = 0. \tag{1.7}$$

You can show quite easily that solutions of the form $p = f(x \pm ct)$ satisfy equation 1.7. This means that disturbances propagate as fixed shapes which shift along the x-axis at speed c. Figure 1.2 is a simple example, showing both solutions $x \pm ct$.

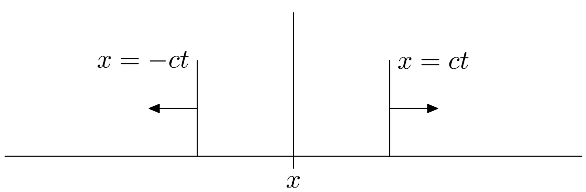


Figure 1.2: Wave propagation: right propagating wave with $x = ct$ and left propagating wave with $x = -ct$.

A pulse starts at a point $x = 0$ at time $t = 0$ so that $x \pm ct = 0$. At a later time, the wave will have moved left to a point $x = -ct$, still satisfying $x + ct = 0$ and right to a point $x = ct$, satisfying $x - ct = 0$. In both cases, the value of p will be the same as at time $t = 0$. As we might expect, the wave travels to the left or right at speed c, which is why c is called the speed of sound.

When waves propagate like this, they are called *plane waves* because their properties are constant over planes of constant x. Waves can be modelled as planar when they propagate at low frequency in pipes or ducts, such as long pipelines or engine exhaust systems. Plane waves also occur in other situations and are very useful in analyzing general problems. If a plane wave propagates in a general direction, we can write it as $f(t - \mathbf{x}.\mathbf{n})$ where $\mathbf{n}$ is the direction of propagation or normal to the wave.

1.6 Solutions of the wave equation in three dimensions

Naturally, one-dimensional waves are of little interest to rounded personalities such as ourselves and we must eventually face reality in all of its three dimensions. Solving the wave equation in three dimensions is not much more difficult than doing so in one dimension. The most convenient approach is to work in spherical polar coordinates, §7.2. In this coordinate system:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\phi^2}.$$

We simplify this by considering the case of sound propagating in free space in a uniform medium. Then, by symmetry, p' is independent of ϕ and θ, so that:

$$\begin{aligned}\nabla^2 p &= \frac{\partial^2 p}{\partial r^2} + \frac{2}{r}\frac{\partial p}{\partial r}\\ &= \frac{1}{r}\frac{\partial^2}{\partial r^2}(rp)\end{aligned} \tag{1.8}$$

and the wave equation now reads

$$\frac{1}{c^2}\frac{\partial^2}{\partial t^2}(rp) - \frac{\partial^2}{\partial r^2}(rp) = 0, \tag{1.9}$$

which is identical in form to equation 1.7. Using the solution of that equation, $rp = f(r \pm ct)$, we find

$$p = \frac{f(t - r/c)}{r}. \tag{1.10}$$

For reasons of *causality* (things cannot happen before they have been caused), we reject the solution $rp = f(r + ct)$.

This solution contains three useful pieces of information. The first, as in the one-dimensional case, is that the sound at time t depends on what happened at time $t - r/c$, the *emission time* or *retarded time*. The second, again similarly to the one dimensional case, is that the shape of the wave $f(\cdot)$ does not change. The big difference between one and three dimensional waves, however, is that the magnitude of the pressure perturbation (though not its shape) reduces as it propagates.

1.7 Acoustic velocity and intensity

When we derived the wave equation, we chose to eliminate velocity and density and concentrated on pressure as our dependent variable. There are two main reasons for doing this: the first is that pressure is a scalar and so is conceptually easier to work with than velocity. In practice, given that we could use a velocity potential, this is not a huge advantage. The second, and more important, reason is that pressure is what we hear and what we measure. Our ears and the microphones we use to measure sound are sensitive to pressure fluctuations, so that is what we choose as our main quantity.

There are times, however, when we will need to use some other quantity. The fundamental theory of aerodynamically generated noise is actually based on density fluctuations (which are usually converted to pressure variations using a linear relationship). A more important relationship is that between pressure and velocity because the acoustic velocity is often used as a boundary condition in calculations involving solid bodies. Remember that acoustics is a branch of fluid dynamics and it is a fluid-dynamical boundary condition that must be satisfied, i.e. usually a velocity.

The linearized momentum equation (1.2b) gives us the relationship we need:

$$\frac{\partial \mathbf{v}'}{\partial t} = -\frac{\nabla p'}{\rho_0},$$

in other words, the acoustic velocity is proportional to the pressure gradient. If we write the solution of the wave equation in terms of a velocity potential $\phi = f(t - R/c)$, the pressure and radial velocity are related via:

$$\begin{aligned} p &= -\rho_0\frac{\partial\phi}{\partial t}, \quad \mathbf{v} = \nabla\phi,\\ v &= \frac{p}{\rho_0 c} + \frac{f(t - R/c)}{\rho_0 R^2}.\end{aligned} \tag{1.11}$$

For a wave of constant frequency, the acoustic velocity amplitude V is related to the acoustic pressure by

$$V = -\mathrm{j}\frac{\nabla P}{\rho_0 \omega}. \tag{1.12}$$

For a plane wave $\nabla \rightarrow \partial/\partial x$ and $V = P/\rho_0 c$. For large R, the pressure–velocity relationship for a spherical wave reduces to this form, as seen in equation 1.11.

A basic characteristic of a source is the rate at which it transfers energy. If we multiply equation 1.2a by $c^2\rho'$,

$$c^2\rho'\frac{\partial \rho'}{\partial t} + \rho_0 c^2 \rho' \frac{\partial v}{\partial x} = 0 \tag{1.13}$$

and note that $\rho'\partial\rho'/\partial t = \frac{1}{2}(\partial/\partial t){\rho'}^2$ and that $c^2\rho' = p'$,

$$\frac{c^2}{\rho_0}\frac{1}{2}\frac{\partial}{\partial t}{\rho'}^2 + p'\frac{\partial v}{\partial x} = 0.$$

Multiplying the momentum equation 1.2b by v gives

$$\rho_0 v \frac{\partial v}{\partial t} + v\frac{\partial p'}{\partial x} = 0,$$

which can be rearranged:

$$\frac{1}{2}\rho_0\frac{\partial}{\partial t}v^2 + v\frac{\partial p'}{\partial x} = 0. \tag{1.14}$$

Adding equations 1.13 and 1.14 gives a result for the energy transport in the sound field:

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho_0 v^2 + \frac{1}{2}\frac{c^2}{\rho_0}{\rho'}^2\right) + \frac{\partial}{\partial x}(p'v) = 0. \tag{1.15}$$

In equation 1.15, $\rho_0 v^2/2$ is the *kinetic energy* per unit volume, $c^2/\rho_0{\rho'}^2/2$ is the *potential energy* per unit volume and $p'v$ is the *acoustic intensity* I which is the rate of energy transport across unit area. Equation 1.15 is a statement of energy conservation for the system and says that the rate of change of energy in a region is equal to the net rate at which energy is carried into the region.

If insert the relationship between pressure and velocity, equation 1.11, the acoustic intensity is

$$I = \frac{p^2}{\rho c} + \frac{\partial}{\partial t}\left(\frac{f^2(t - R/c)}{2\rho R^3}\right).$$

If we average I over time for a periodic wave, the second term has a mean value of zero and the resulting mean intensity is:

$$\bar{I} = \frac{\overline{p^2}}{\rho c}. \tag{1.16}$$

Example: Acoustic displacement

The threshold of human hearing is nominally $0\mathrm{dB}$. Knowing that this corresponds to a particular pressure ($2\times10^{-5}\mathrm{Pa}$), we can calculate an acoustic velocity and from this an acoustic displacement. If we assume that we are listening to sound at 1kHz (where the human ear is most sensitive), we can calculate the velocity amplitude corresponding to this pressure from Equation 1.12:

$$V = \frac{P}{\rho c} = \frac{2 \times 10^{-5}}{1.225 \times 343} = 4.76 \times 10^{-8}\mathrm{m/s}.$$

Since we also know that the amplitude of displacement X is related to the velocity via:

$$V = \omega X,$$

we can work out the displacement of the eardrum when you hear a sound of $1\mathrm{kHz}$ at the threshold of human hearing:

$$X = \frac{4.76 \times 10^{-8}}{2\pi \times 1000} = 0.76 \times 10^{-11}\mathrm{m},$$

or something like the diameter of a hydrogen atom.

1.8 Questions

1. Show that $f(x \pm ct)$ is a solution of the one-dimensional wave equation.

2. The sound from a point source $q(t)$ is $q(t - R/c)/4\pi R$. If the source is sinusoidal with frequency ω, write down an expression for the sound from the source.

3. To reduce noise in aircraft, we can use loudspeakers inside the aircraft to generate ‘anti-noise’. If we assume the noise at head level in business class is generated by a point source of strength q and frequency ω at a position $\mathbf{x_1}$, what strength should a source (loudspeaker) at a position $\mathbf{x}_2$ have to cancel the noise?

4. If a jet engine generates a noise of SPL 140dB at 3m, how far away do you need to move to reach a safe position?

Chapter 2

Making sound

2.1 Pulsating sphere

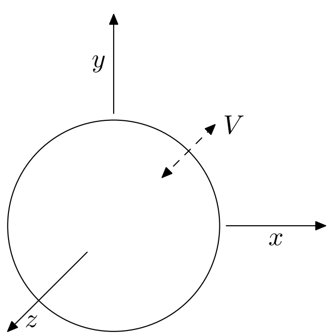


Figure 2.1: A pulsating spherical surface

The simplest three-dimensional problem we can solve is that of sound radiated by a pulsating sphere. This sphere could be, for example, a bubble, a varying heat source or an approximation to a body of varying volume. The sphere has radius a and oscillates with velocity amplitude V at frequency ω. From the linearized momentum equation (1.2b), we can find a relationship between acceleration and pressure gradient:

$$\nabla p = -\rho_0 \frac{\partial \mathbf{v}}{\partial t}. \tag{2.1}$$

Writing the radial velocity of the sphere surface as $v = V \exp[-\mathrm{j}\omega t]$, we can see that p must also have frequency ω so that we can write it as $p = P \exp[-\mathrm{j}\omega t]$ and:

$$\nabla P \mathrm{e}^{-\mathrm{j}\omega t} = \mathrm{j}\omega \rho_0 V \mathrm{e}^{-\mathrm{j}\omega t}. \tag{2.2}$$

Since p is a solution of the wave equation, we know from §1.6 that

$$p = \frac{f(t - r/c)}{r} = \frac{A\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}}{r}, \tag{2.3}$$

where A is to be found from the boundary condition at a, the sphere surface. Writing out the pressure gradient:

$$\nabla p = \frac{A}{r^2}\left[\frac{\mathrm{j}\omega r}{c} - 1\right]\mathrm{e}^{-\mathrm{j}\omega(t-r/c)}, \tag{2.4}$$

and applying the boundary condition:

$$\frac{A}{a^2}\left[\frac{\mathrm{j}\omega a}{c} - 1\right]\mathrm{e}^{-\mathrm{j}\omega(t-a/c)} = \mathrm{j}\omega\rho_0 V \mathrm{e}^{-\mathrm{j}\omega t}, \tag{2.5}$$

we can fix the constant A:

$$A = \frac{(ka)(ka - \mathrm{j})\rho_0 V ca}{(ka)^2 + 1}\mathrm{e}^{-\mathrm{j}ka}, \tag{2.6}$$

where $k = \omega/c$ is the *wavenumber*. The solution for the pressure is then:

$$p = \frac{ka}{r}\frac{ka - \mathrm{j}}{(ka)^2 + 1}(\rho_0 V ca)\mathrm{e}^{-\mathrm{j}k(r-a)}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{2.7}$$

There are two approximations we can make which simplify this formula. When $ka \ll 1$ (i.e. when the sphere is small or it vibrates at low frequency), (2.7) can be written:

$$p \approx -\mathrm{j}\frac{\rho_0 cka^2}{r} V \mathrm{e}^{\mathrm{j}kr}\mathrm{e}^{-\mathrm{j}\omega t}; \tag{2.8}$$

when $ka \gg 1$ (i.e. when the sphere is large or vibrating at high frequency):

$$p \approx \frac{\rho_0 V ca}{r}\mathrm{e}^{-\mathrm{j}k(r-a)}\mathrm{e}^{-\mathrm{j}\omega t}. \tag{2.9}$$

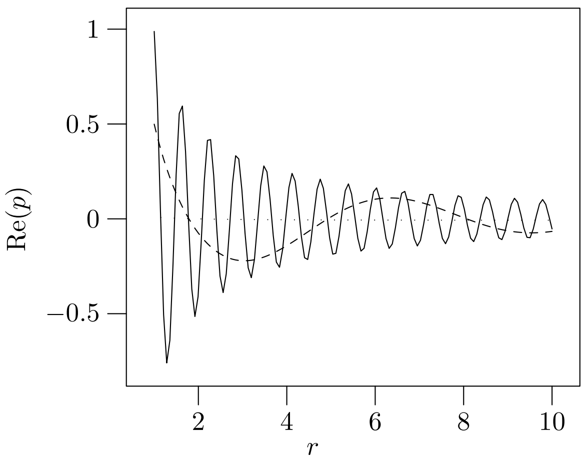


Figure 2.2: Sound field around a pulsating sphere: dotted $k = 0.1$; dashed $k = 1$; solid $k = 10$.

The parameter ka, a non-dimensional combination of wavelength and a characteristic dimension of the body, is an important parameter in characterizing sources and is called the *compactness*. When ka is small, the source is point-like and can be treated as a simple source; when it is large, the acoustic field becomes more complicated, as in figure 2.2.

2.2 Point sources

When we look at sound production by real systems, we cannot usually model them with simple shapes such as spheres. The solution for a sphere is useful, however, because we can use it to work out the noise radiated by a *point source*, an idealized solution for the sound radiated by an infinitesimal element of a real system.

We start with equation 2.8, the result for a small oscillating sphere. We want to write this in terms of some "source strength". When the sphere oscillates, it is injecting momentum into the fluid. A sphere of radius a has surface area $4\pi a^2$ and if it oscillates with velocity $V\exp[-\mathrm{j}\omega t]$, the momentum being injected at the surface of the sphere is:

$$M = \rho_0 4\pi a^2 V \mathrm{e}^{-\mathrm{j}\omega t} \tag{2.10}$$

and the rate of change of momentum is:

$$\frac{\partial M}{\partial t} = -\mathrm{j}\rho_0\omega 4\pi a^2 V \mathrm{e}^{-\mathrm{j}\omega t}. \tag{2.11}$$

Noting that $\omega = kc$, we can compare equation 2.11 to equation 2.8 and find that:

$$p = \frac{1}{4\pi}\frac{\partial M}{\partial t}\frac{\mathrm{e}^{\mathrm{j}kr}}{r}, \tag{2.12}$$

so that sound is generated by fluctuations in momentum. If write this in terms of a source strength $q = \rho_0 v(t)$, this equation can also be written:

$$p = \frac{\partial}{\partial t}\frac{q(t-R/c)}{4\pi R}, \tag{2.13}$$

which is the result for sound radiated by an infinitesimal point source. In a real problem, we can work out the sound from a source as a sum of contributions from point sources. This sum becomes an integral if we look at a smooth distribution of sources over a volume V:

$$p(\mathbf{x}, t) = \frac{\partial}{\partial t}\int_V \frac{q(\mathbf{y}, t-R/c)}{4\pi R}\,\mathrm{d}V. \tag{2.14}$$

We can write this in a form which will be useful to us later:

$$p(\mathbf{x}, t) = \frac{\partial}{\partial t} \int_V G(\mathbf{x}, t; \mathbf{y}, \tau) q(\tau) \, \mathrm{d}V, \tag{2.15}$$

where G is the Green's function for the problem. A Green's function is a fundamental solution, in this case the response due to a point source "firing" instantaneously. We can write the Green's function using the Dirac delta function $\delta(\,\cdot\,)$:

$$G(\mathbf{x}, t; \mathbf{y}, \tau) = \frac{\delta(t - \tau + R/c)}{4\pi R}, \tag{2.16}$$
$$R = |\mathbf{x} - \mathbf{y}|.$$

The delta function is a curious beast which is zero everywhere except at zero, where it jumps to an infinite value. The area under the delta function, however, is one. It has the property that:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0) \, \mathrm{d}x = f(x_0),$$

called the "sifting property". In the case of equation 2.16, this means that $t - \tau + R/c$ or, $\tau = t - R/c$. Here τ, the *retarded time* is the time when sound leaves the source and t is the time when it arrives, so that R/c is the time delay between sound leaving a source and sound arriving at some point, which should be no surprise by now.

2.3 Loudspeakers

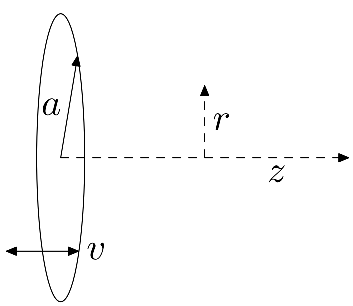


Figure 2.3: A rigid piston vibrating in a rigid wall.

Taking a step up in difficulty (and realism), we now look at the sound radiated by a rigid piston embedded in a wall. This is a basic model of a loudspeaker and is related to a number of other problems in the acoustics of sound generation by moving surfaces. Figure 2.3 shows a rigid circular piston of radius a which vibrates periodically at frequency ω and velocity amplitude v so that its velocity is $v\exp[-\mathrm{j}\omega t]$. From equation 2.15:

$$p\mathrm{e}^{-\mathrm{j}\omega t} = 2\frac{\partial}{\partial t} \iint_S \frac{q(\mathbf{y}, \tau)}{4\pi R} \, \mathrm{d}S,$$

where the factor 2 has been included to account for the image source in the wall and the integration is performed over the surface S of the piston. Given the velocity, the source $q = \rho_0 v \exp[-\mathrm{j}\omega t]$ so that the resulting integral for the radiated sound is:

$$p(\omega) = -\mathrm{j}\frac{\omega\rho_0}{2\pi} \iint_S \frac{\mathrm{e}^{\mathrm{j}kR}}{R} v \, \mathrm{d}S.$$

To evaluate the integral, we switch to cylindrical coordinates (r, θ, z):

$$x = r\cos\theta, \quad y = r\sin\theta.$$

We assume that the observer is at $\theta = 0$ and the integral to be evaluated is:

$$p(\omega) = -\mathrm{j}\frac{\omega\rho_0 v}{2\pi} \int_0^{2\pi} \int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R} r_1 \, \mathrm{d}r_1 \, \mathrm{d}\theta_1,$$
$$R = (r^2 + r_1^2 - 2rr_1\cos\theta_1 + z^2)^{1/2},$$

where (r_1, θ_1) indicates a point on the piston surface.

This integral cannot be evaluated exactly for a general observer position but we can restrict it to the case where the observer is on the axis of the piston. Then $r = 0$ and $R = (r_1^2 + z^2)^{1/2}$:

$$p = -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R} r_1\,\mathrm{d}r_1\,\mathrm{d}\theta_1,$$
$$= -\mathrm{j}\omega\rho_0 v\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R} r_1\,\mathrm{d}r_1,$$

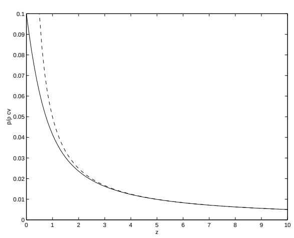
and making the transformation $r_1 \to R$,

$$p = -\mathrm{j}\omega\rho_0 v\int_{R_0}^{R_a} \mathrm{e}^{\mathrm{j}kR}\,\mathrm{d}R.$$

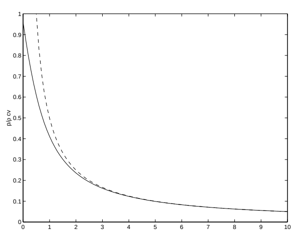
Here, $R_0 = z$ is the distance from the observer to the centre of the piston and $R_a = (a^2 + z^2)^{1/2}$ is the distance to the rim of the piston. The solution is then:

$$p = -\rho_0 c v(\mathrm{e}^{\mathrm{j}kR_a} - \mathrm{e}^{\mathrm{j}kz}). \tag{2.17}$$

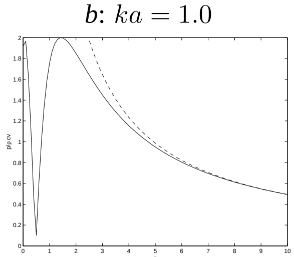
If we examine the acoustic field defined by equation 2.17 as a function of frequency, we can see that it changes quite rapidly as ka is increased. Figure 2.4 shows the absolute value of the non-dimensional pressure $|p/\rho_0 cv|$ for different values of ka. For comparison, the curve $1/R_0 = 1/z$ is also shown. The results for $ka = 0.1$ and $ka = 1$ are similar with a smooth $1/R_0$ decay but the $ka = 10$ curve is quite different, having a sharp drop before it begins to follow a $1/R_0$ curve. This is a result of interference between sound from different parts of the piston. When a body is large compared to the wavelength of the sound it generates, interference between different parts of the body gives rise to a complicated sound pattern, especially in the region near the body. When the body is small on a wavelength scale (or, equivalently, vibrates at low frequency), the phase difference between different parts of the source is not enough to give rise to much interference and the body radiates like a point source. The 'size' of the body at a given frequency is called its *compactness* and is characterized by the parameter ka where a is a characteristic dimension, or by the ratio of characteristic dimension to wavelength a/λ. A compact source, one with $ka \ll 1$, radiates like a point source, while non-compact bodies must be treated in more detail, as we saw in the case of a sphere in §2.1.



a: $ka = 0.1$



b: $ka = 1.0$



c: $ka = 10.0$

Figure 2.4: Acoustic field (absolute value of p) along the axis of a vibrating piston. The dashed line shows the $1/z$ fit.

Example: Noise from aircraft engines

The formula for sound radiated from an oscillating piston can also be used as an approximation for low frequency noise from flanged pipes. If we slightly abuse the formula, we can use it to make a guess at the noise from the end of a duct, such as an aircraft engine intake (or a cooling tower or all sorts of other things). The internal processes in an engine, such as the rotation of the fan, generate an oscillating velocity at the intake. We can pretend that this is a piston spanning the face of the intake and calculate the radiated noise using the formula derived above.

2.4 Combustion noise

Another important application of one-dimensional acoustics is in combustion instability in engines. In order to model such a problem, we need to look at the *thermodynamics* of the system in order to model the effects of heat release. When we derived the wave equation in §1.2, we assumed that the system was adiabatic—no heat was added or removed. Obviously, if we want to look at a problem involving heat addition, this assumption is wrong so we have to include some extra information.

From thermodynamics, we know that:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2}\frac{\mathrm{D}p}{\mathrm{D}t} + \left.\frac{\partial\rho}{\partial s}\right|_p \frac{\mathrm{D}s}{\mathrm{D}t}, \tag{2.18}$$

which is what we derived in §1.2 but we now include a term which depends on s the *entropy* of the fluid. When, as we assumed previously, the flow is isentropic, the second term disappears. When we include heat release in the problem, however, we cannot ignore the entropy variations.

When we ignore viscosity and heat conduction, the heat input q per unit volume is given by

$$q(\mathbf{x}, t) = \rho T \frac{\mathrm{D}s}{\mathrm{D}t}.$$

For a perfect gas,

$$\left.\frac{\partial\rho}{\partial s}\right|_p = -\frac{\rho}{c_p} = -\frac{\rho T(\gamma - 1)}{c^2},$$

where c_p is the specific heat at constant pressure and γ the ratio of the specific heats. We can substitute this relation into equation 2.18:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c^2}\left[\frac{\mathrm{D}p}{\mathrm{D}t} - (\gamma - 1)q\right]. \tag{2.19}$$

If we assume that perturbations are small and that there is no mean heat addition (otherwise the speed of sound and other thermodynamic properties would change), we can linearize this equation:

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = \frac{1}{c_0^2}\left[\frac{\partial p'}{\partial t} - (\gamma - 1)q\right], \tag{2.20}$$

where c_0 is the mean speed of sound. If we now return to equation 1.3,

$$\frac{\partial^2\rho'}{\partial t^2} - \nabla^2 p' = 0,$$

we can insert this new relationship between p' and ρ' to find:

$$\frac{1}{c_0^2}\frac{\partial^2 p'}{\partial t^2} - \nabla^2 p' = \frac{\gamma - 1}{c_0^2}\frac{\partial q}{\partial t}, \tag{2.21}$$

and we end up with a linear wave equation with a source term on the right hand side which is related to the heat input per unit volume. If we reduce this to the one-dimensional case,

$$\frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} - \frac{\partial^2 p}{\partial x^2} = \frac{\gamma - 1}{c_0^2}\frac{\partial q}{\partial t}, \tag{2.22}$$

we can look at some simple problems related to combustion.

If we think of combustion happening in a tube of length L open at both ends, the pressure inside the tube has to be of the form

$$p(x, t) = P(t)\sin\frac{n\pi x}{L}$$

and the wave equation becomes

$$\left[\frac{\ddot{P}}{c_0^2} + \frac{n^2\pi^2}{L^2}P\right]\sin\frac{n\pi x}{L} = \frac{\gamma - 1}{c_0^2}\frac{\partial q}{\partial t}.$$

If we now assume that the unsteady heat release is related to the unsteady pressure, we can see how it affects the acoustics.

The first simple assumption is that the heat release is proportional to pressure,

$$q = \frac{-\alpha c_0^2 p'}{\gamma - 1},$$

which leads to the equation for pressure amplitude,

$$\frac{\ddot{P}}{c_0^2} + \alpha\dot{P} + \frac{n^2\pi^2}{L^2}P = 0,$$

which is the equation for a damped oscillator (think of the spring-mass-dashpot system you saw in mechanics). If α is positive, the response P decays with time. If, however, α is negative, the response grows over time: the combustion is unstable. The case where α is positive corresponds to heat addition 180° out of phase with the pressure; negative α means that the heat addition is in phase with the pressure. This is *Rayleigh's criterion*: heat must be added in phase with pressure if energy is to be transferred into the acoustic waves. Remember that the heat release is proportional to the pressure, so if the pressure is unstable, so is the heat release and your engine blows up.

This is a very simple example which ignores the mechanism of heat addition—the combustion of fuel—but it illustrates how the combustion depends on the relationship between the acoustics and the heat generated in the system.

2.5 Questions

1. Write down the solution to the following integrals:
$\int_{-\infty}^{\infty} \delta(x)\,\mathrm{d}x$; $\int_{-\infty}^{\infty} x^2\delta(x-3)\,\mathrm{d}x$; $\int_{-\infty}^{\infty} \cos x\delta(x+\pi)\,\mathrm{d}x$.

2. A circular piston of radius a is started impulsively from rest. An observer at position (r, z) hears the sound generated by the impulsive motion. Calculate:

   a) the time of arrival of the start of the pulse.

   b) the time of arrival of the end of the pulse.

   c) the duration of the signal heard by the observer.

   What is the maximum pulse length generated? What is the minimum pulse length?

3. At low frequencies, the noise radiated from the intake of an aircraft engine can be approximated as that due to a piston set in the intake. On this approximation, estimate the SPL $20\mathrm{m}$ from an engine with intake diameter $3\mathrm{m}$, subject to a velocity fluctuation of frequency $80\mathrm{Hz}$ and amplitude $0.02\mathrm{m/s}$.

4. In the far field, $R \gg a$, $R \gg ka$, we can estimate the sound radiated off-axis by a piston, using the following approximations:

$$\frac{1}{R} \approx \frac{1}{R_0},$$
$$R \approx R_0 - r_1 \sin\phi\cos\theta_1$$

   where $\phi = \tan^{-1} r/z$ and $R_0 = [r^2 + z^2]^{1/2}$. Given that the *Bessel function* of zero order is:

$$J_0(x) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{e}^{-\mathrm{j}x\cos\theta_1}\,\mathrm{d}\theta_1,$$

and that:

$$\int xJ_0(x)\,\mathrm{d}x = xJ_1(x),$$

where $J_1(x)$ is the Bessel function of first order, derive an approximate formula for the far field noise radiated by a piston.

Chapter 3

Modifying sound

3.1 Reflection by a hard wall

The simplest realistic problem of interest involving the effect of a boundary on a sound field is that of the interaction of the field from a point source with a plane wall, figure 3.1. The problem is, given a source at a point $\mathbf{x}$, near a rigid plane, to calculate the resulting overall sound field. If the wall were not present, we know that the sound field at a frequency ω would have the form:

$$p_{\mathrm{i}}\mathrm{e}^{-\mathrm{j}\omega t} = \frac{\mathrm{e}^{-\mathrm{j}\omega(t-R/c)}}{4\pi R},$$

where p_{i} is the *incident* sound field.

We will drop the factor $\exp[-\mathrm{j}\omega t]$ because it is the same for all sound fields in the problem and write:

$$p_{\mathrm{i}} = \frac{\mathrm{e}^{\mathrm{j}kR}}{4\pi R}.$$

Our problem now is to find a second acoustic field p_{s} (the 'scattered' field), such that the total field $p_{\mathrm{t}} = p_{\mathrm{i}} + p_{\mathrm{s}}$ satisfies the wave equation and the boundary conditions on the wall. By linearity, §1.2, this means that p_{s} must be a valid solution of the wave equation, since the sum of two solutions is itself a solution. Now we need to decide what boundary condition to apply. As in inviscid fluid dynamics, the boundary condition is that the total velocity normal to the wall must be zero. We know that the acoustic velocity is proportional to the pressure gradient, §1.7, so this boundary condition is equivalent to

$$\left.\frac{\partial p_{\mathrm{t}}}{\partial x}\right|_{x=0} \equiv 0,$$

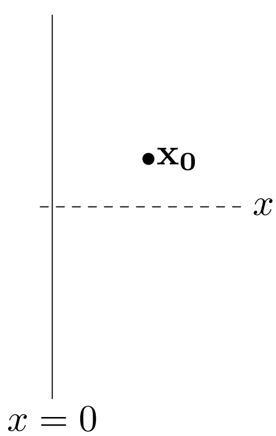


Figure 3.1: A point source near a wall

or, in terms of the incident and scattered fields,

$$\left.\frac{\partial p_{\mathrm{s}}}{\partial x}\right|_{x=0} \equiv -\left.\frac{\partial p_{\mathrm{i}}}{\partial x}\right|_{x=0}.$$

For a source at $\mathbf{x}_0 = (x_0, y_0, z_0)$,

$$\frac{\partial p_{\mathrm{i}}}{\partial x} = \frac{x - x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right),$$

and at $x = 0$,

$$\left.\frac{\partial p_{\mathrm{i}}}{\partial x}\right|_{x=0} = -\frac{x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right),$$
$$R = [x_0^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.$$

The solution of our problem is an acoustic field p_s with

$$\left.\frac{\partial p_\mathrm{s}}{\partial x}\right|_{x=0} = \frac{x_0}{4\pi}\frac{\mathrm{e}^{\mathrm{j}kR}}{R^3}\left(\mathrm{j}kR - 1\right).$$

A source positioned at $\mathbf{x}_- = (-x_0, y_0, z_0)$ gives just such a field so a valid solution to the problem can be found using an *image source*, the reflection of our orginal source in the rigid wall. The total field is then

$$\begin{aligned}
p_\mathrm{t} &= p_\mathrm{i} + p_\mathrm{s},\\
p_\mathrm{i} &= \frac{\mathrm{e}^{\mathrm{j}kR_+}}{4\pi R_+},\\
p_\mathrm{s} &= \frac{\mathrm{e}^{\mathrm{j}kR_-}}{4\pi R_-},\\
R_\pm &= [(x \mp x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{1/2}.
\end{aligned}$$

One immediate result of this analysis is that the pressure generated on the wall by a source is twice that which would be generated if the wall were not present. This has two immediate applications: the first is that excessive noise in confined spaces (discotheques and clubs, for example) can be extremely damaging to hearing; the second is where the ‘wall’ is the ground and we want to know how noise propagates across a landscape.

You should repeat this calculation for the boundary condition $p = 0$, the so-called pressure-release surface which applies to underwater noise problems.

3.2 Reflection by a soft wall

A concept which is very useful and we will need later on is that of *acoustic impedance*. This is like the impedance we see in mechanical systems and is defined as the ratio of acoustic pressure to acoustic velocity:

$$Z = \frac{P}{V}. \tag{3.1}$$

The acoustic impedance of a material (including gases and liquids) is a property of the material and of frequency. We usually work in terms of *specific acoustic impedance* which is simply Z/A where A is the area of material.

For a hard wall, $V = 0$ and the impedance is infinite. For a substance which is porous, the effect of flow into the pores of the material must be taken into account. We can model this by lumping the material properties together into a single impedance, which means that we do not need to know very much else about a material. Note that, in general, Z is a function of frequency.

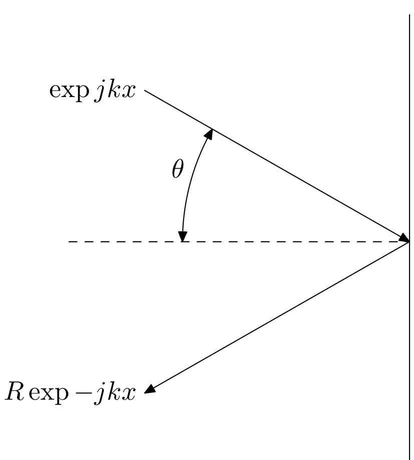


Figure 3.2: Reflection from a finite impedance wall

If we examine reflection of a plane wave from a wall with some finite impedance, we can look at the problem of acoustic treatment of rooms. In order to line a room to stop reflections (for music recording or performances, say), we want to minimize reflections or echos so we need to know how much sound is reflected from a wall for a given impedance. Figure 3.2 shows the incoming and reflected waves. The pressure and velocity are given by:

$$P = \mathrm{e}^{\mathrm{j}k_y y}\left(\mathrm{e}^{\mathrm{j}k_x x} + R\mathrm{e}^{-\mathrm{j}k_x x}\right), \tag{3.2}$$

$$V = \frac{\mathrm{e}^{\mathrm{j}k_y y}}{\rho c}\left(\mathrm{e}^{\mathrm{j}k_x x} - R\mathrm{e}^{-\mathrm{j}k_x x}\right)\cos\theta, \tag{3.3}$$

where the $\cos\theta$ is needed to extract the component of velocity normal to the wall—sound propagating parallel to the wall will not be affected by the impedance. The boundary condition on the wall is that $Z = P/V$ so we can write:

$$R = \frac{Z\cos\theta - \rho c}{Z\cos\theta + \rho c}. \tag{3.4}$$

Example: How to bug an embassy

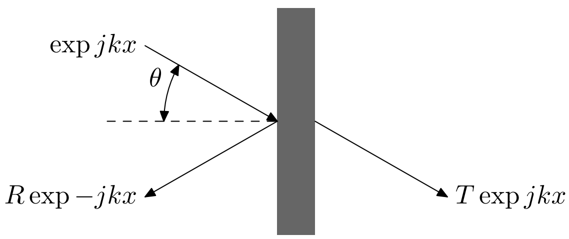


Figure 3.3: A slab of material under acoustic excitation

One type of 'soft' wall is a slab of material which vibrates in response to acoustic pressure. Figure 3.3 shows the arrangement: a slab or sheet of material is subject to a plane wave. We want to know the complex amplitude R of the reflected wave and the amplitude T of the wave transmitted out the other side of the material. For a thin, non-deforming slab, we can assume that the velocities on each side of the slab are equal:

$$v_i = v_t, \tag{3.5}$$

and we know from the definition of impedance that:

$$P_i - P_t = Z_{\text{sl}} v_i = Z_{\text{sl}} v_t. \tag{3.6}$$

The reflection coefficient on the incoming wave side is (from Equation 3.4):

$$R = \frac{Z_i - Z_1}{Z_i + Z_1}, \tag{3.7}$$

where the local impedance $Z_1 = \rho c/\cos\theta$. This means that the velocity on side 1 is:

$$v_1 = \frac{p_1}{Z_1}(1 - R), \tag{3.8}$$

$$= \frac{2P_i}{2Z_1 + Z_{\text{sl}}}. \tag{3.9}$$

Given that the normal velocity is equal on both sides, we can work out the amplitude of the transmitted wave:

$$T = Z_1 V_2 = \frac{2\rho c/\cos\theta}{Z_{\text{sl}} + 2\rho c/\cos\theta}. \tag{3.10}$$

In 1987, *Time* reported that the Soviet Union might be using lasers to measure the vibrations of the windows of the US embassy in Moscow as a way of listening to conversations inside[1]. A modern laser vibrometer can measure velocities to a resolution of about $0.01\mu\text{m/s}$. If a window pane is 5mm thick, what is the quietest conversation we can listen to?

A simple assumption is that the glass acts as a *limp plate* and the only resistance to motion is the slab inertia. Then, for a plate of mass per unit area m moving at a frequency ω

$$-\text{j}\omega V = P_i - P_t \tag{3.11}$$

and $Z_{\text{sl}} = -\text{j}\omega m$. The transmitted wave then has amplitude:

$$|T| = \left[1 + \left(\frac{\omega m}{2\rho c}\right)^2 \cos^2\theta\right]^{-1/2}.$$

[1]The article is available online at: `http://www.bugsweeps.com/info/hitech_snooping.html`

From equation 3.8, and assuming $\theta = 0$,

$$v = \frac{2P_i}{2\rho c - \mathrm{j}\omega m}.$$

If we are interested in sound at around $3\mathrm{kHz}$(roughly in the middle of the range of human speech), given that the density of glass is about $2500\mathrm{kg/m}^3$, $m = 12.5\mathrm{kg/m}^2$ and:

$$v = \frac{2}{1.2 \times 340 - \mathrm{j}2\pi \times 3000 \times 12.5} P_i = \frac{1}{204 - \mathrm{j}1.178 \times 10^5} P_i$$

and

$$|v| = |P_i|/1.178 \times 10^5.$$

If we assume we can measure the velocity over a range of $1\mu\mathrm{m/s}$,

$$|P_i| = 1.178 \times 10^5 \times 10^{-6}\mathrm{Pa} = 75\mathrm{dB}.$$

For comparison, the *sound* transmitted on the other side of the window would be TP_i which has magnitude:

$$\begin{aligned} |TP_i| &= \left[1 + \left(\frac{\omega m}{2\rho c}\right)^2 \cos^2\theta\right]^{-1/2} P_i, \\ &= 1.178 \times 10^5 \times 10^{-6}/289\mathrm{Pa} \qquad = 26.2\mathrm{dB}. \end{aligned}$$

It might be possible to measure this signal very close to the window, but at a distance of $100\mathrm{m}$ it would be impossible. A sophisticated laser system, however, could measure the window's vibrations from a distance of hundreds of meters. It is interesting to know that the Russian embassy in Washington is on high ground looking down onto a number of important buildings, including the White House.

3.3 Ducts and silencers

Figure 3.4 shows a simple example of propagation along a duct whose section changes suddenly. If a wave of the form $\exp(\mathrm{j}kx)$ propagates to the right and hits the change in section, there is a reflected wave $R\exp(-\mathrm{j}kx)$ which propagates to the left and a transmitted wave $T\exp(\mathrm{j}kx)$ which carries on to the right past the change in section.

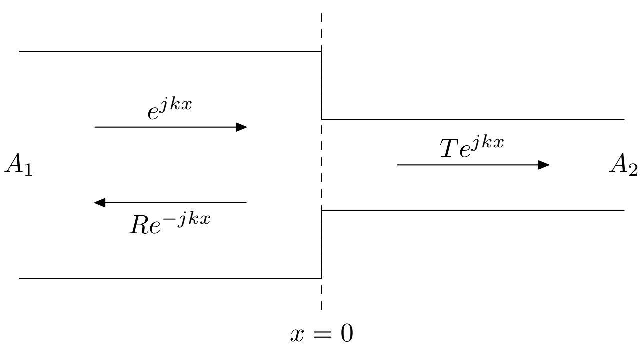


Figure 3.4: Change in duct section

For low-frequency applications, we can assume that the only thing that matters is the change in area going from one section to the next. If the initial part of the duct has area A_1 and the second part area A_2, the boundary conditions at the change in section $x = 0$ are continuity of pressure and conservation of mass. The first of these conditions is simple; the second requires that the volume flow rate be conserved across the interface, so that $A_1U_1 = A_2U_2$ where U is acoustic velocity, which we can relate to the acoustic pressure using equation 1.12. Setting $x = 0$, the boundary conditions are then:

$$1 + R = T, \tag{3.12a}$$

$$A_1(1 - R) = A_2T. \tag{3.12b}$$

Solving for R and T, we find that:

$$R = \frac{A_1 - A_2}{A_1 + A_2}, \tag{3.13a}$$

$$T = \frac{2A_1}{A_1 + A_2}. \tag{3.13b}$$

Note that when $A_2 \to \infty$, $R \to -1$ and $T \to 0$ so that, on this theory, an open-ended duct reflects the whole signal back from the end and no sound escapes. As might be expected, when $A_2 = A_1$, $R = 0$ and $T = 1$ so the sound travels unaffected.

An application of changes in duct section is the exhaust muffler, such as those seen on the motorcycles of thoroughly respectable acoustics lecturers on the exhaust pipes of noisy brats. The simplest form of muffler, Figure 3.5 is simply a section of pipe with a greater cross-sectional area than the rest of the pipe.

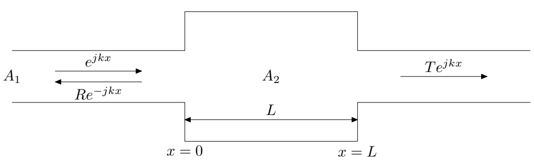


Figure 3.5: A simple exhaust muffler

A muffler has two functions: to reduce the noise radiated into the surroundings (which is why vehicles are obliged to have them) and to increase the engine power (which is why people fit new ones). The first function is fulfilled by modifying the pressure field which reaches the open end of the exhaust, the second by imposing a reflected wave which alters slightly the exhaust characteristics of the engine cylinder.

The muffler shown in figure 3.5 is the simplest device we can imagine but it will give us an idea of the behaviour of a realistic system. We need boundary conditions at $x = 0$ and at $x = L$. The pressure and continuity conditions at $x = 0$ are:

$$1 + R = T_2 + R_2, \tag{3.14a}$$

$$A_1(1 - R) = A_2(T_2 - R_2), \tag{3.14b}$$

and at $x = L$:

$$T_2 \mathrm{e}^{\mathrm{j}kL} + R_2 \mathrm{e}^{-\mathrm{j}kL} = T\mathrm{e}^{\mathrm{j}kL}, A_2(T_2 \mathrm{e}^{\mathrm{j}kL} - R_2 \mathrm{e}^{-\mathrm{j}kL}) = A_1 T \mathrm{e}^{\mathrm{j}kL}. \tag{3.15a}$$

Rearranging these equations, we can eliminate T_2 and R_2 (we are not very interested in what happens inside the muffler) to find T, the transmitted wave. Combining equations 3.14 yields:

$$(A_2 + A_1) - (A_1 - A_2)R = 2A_2T_2,$$
$$(A_2 - A_1) + (A_2 + A_1)R = 2A_2R_2,$$

and, writing $m = A_2/A_1$:

$$(m + 1) + (m - 1)R = 2mT_2,$$
$$(m - 1) + (m + 1)R = 2mR_2.$$

Similarly equations 3.15 can be combined:

$$2mT_2\mathrm{e}^{\mathrm{j}kL} = (m + 1)T\mathrm{e}^{\mathrm{j}kL},$$
$$2mR_2\mathrm{e}^{-\mathrm{j}kL} = (m - 1)T\mathrm{e}^{\mathrm{j}kL}.$$

We can eliminate R_2 and T_2 to find the transmitted wave:

$$T = \frac{\cos kL - \mathrm{j}\sin kL}{\cos kL - \mathrm{j}(m + m^{-1})/2 \sin kL} \tag{3.16}$$

The most interesting thing to know from an environmental point of view is the magnitude of the transmitted wave:

$$|T| = \left(1 + \frac{(m - m^{-1})^2}{4} \sin^2 kL\right)^{-1} \tag{3.17}$$

Looking at this equation, we can see that the transmitted wave amplitude is minimized for certain values of kL, if we take m fixed. The net effect is that the muffler acts as a low pass filter.

We can also calculate the reflected wave amplitude:

$$R = \frac{m+1}{m-1}(T-1), \tag{3.18}$$

showing that quite a strong wave is reflected back into the engine. With the correct timing, which depends on the length of the exhaust pipe leading up to the muffler, this can increase the engine power slightly.

3.4 The Helmholtz resonator

One of the most important resonant systems is the *Helmholtz resonator*, the classic example of which is the wine or beer bottle. It is modelled, figure 3.6, as a volume V connected to the outside world by a neck of length l and cross-sectional area S. We can estimate the resonant frequency of the system by considering the motion of a 'plug' of fluid in the neck of the bottle under the action of an external force and an internal restoring force due to the compressibility of the fluid in the bulb.

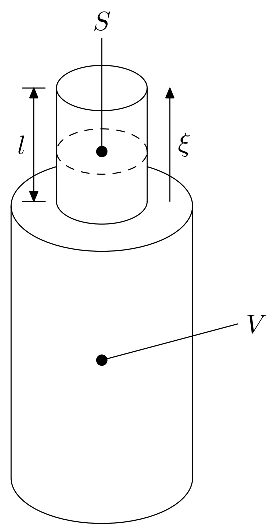


Figure 3.6: Helmholtz' bottle

Assuming that the process is adiabatic, the density and pressure in the bulb are related by:

$$p = k\rho^\gamma; \quad \frac{\mathrm{d}p}{\mathrm{d}\rho} = c^2,$$

as in §1.2. If the plug of fluid in the neck of the bottle is displaced by an amount ξ (assumed positive out of the neck), the volume of fluid inside the bulb changes by an amount $S\xi$. Using subscript 0 to indicate mean values, the resulting change in density is:

$$\begin{aligned}\frac{\rho}{\rho_0} &= \frac{V}{V - S\xi}, \\ &= \frac{1}{1-(S/V)\xi}, \\ &\approx 1 - \frac{S}{V}\xi,\end{aligned}$$

by the binomial theorem and the corresponding change in pressure is:

$$p - p_0 = -\rho_0 \frac{c^2 S}{V}\xi.$$

The equation of motion for the plug can then be written, noting that its mass $m = \rho_0 S l$:

$$\rho_0 S l \ddot{\xi} + \rho_0 \frac{c^2 S}{V}\xi = -p_a S,$$

where p_a is the externally applied pressure. This is the equation of motion for an oscillator with a resonant frequency:

$$\omega = \sqrt{\frac{c^2 S}{V l}}.$$

Helmholtz resonators can be used whenever you want to reduce noise at some known frequency. One of the main applications is in acoustic liners used in aircraft engines, which are made up of a large number of small Helmholtz resonators with dimensions chosen to absorb noise at a specified frequency.

Sound from a wine bottle

A wine bottle has internal volume $V \approx 7.5 \times 10^{-4}\mathrm{m}^3$ and a neck of length $l \approx 0.05\mathrm{m}$ and cross-sectional area $S \approx 7.854 \times 10^{-5}\mathrm{m}^2$. The resonant frequency is then about $492\mathrm{rad/s}$, or $78\mathrm{Hz}$.

3.5 Questions

1. A point source of wavenumber k is placed near a pressure release surface, on which the boundary condition is that the pressure be zero. Calculate the effect of the boundary on the radiated sound.

2. a) Calculate the wave reflected from the open end of a duct (i.e. a pressure release surface). This is a simple model for the behaviour of an engine exhaust or an organ pipe.

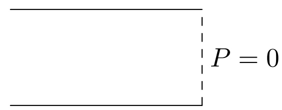


Figure 3.7: Open ended duct

   b) Calculate the resonant frequencies of a duct of length L which is open at both ends. This is a simple model of the resonant behaviour of an engine exhaust. Calculate the acoustic velocity at the end of the duct. Why might this be useful?

3. The density of Perspex is about $1200\mathrm{kg/m}^3$. Estimate the attenuation of a normal wave of frequency $100\mathrm{Hz}$, transmitted through an aircraft window of thickness $5\mathrm{mm}$. Perform the same calculation for an aluminium (density $2700\mathrm{kg/m}^3$) wall of thickness $2\mathrm{mm}$. Which path reduces the cabin noise most and what would be the first easy way to reduce the noise inside the aircraft? What happens to noise at $1\mathrm{kHz}$?

4. A turbofan engine has a main fan with 20 blades operating at 6000rpm. In order to reduce the radiated noise, it is required to line the inlet of the engine with a material composed of cells which act as Helmholtz resonators, figure 3.8. The maximum thickness w of the liner material is $3\mathrm{mm}$. For aerodynamic reasons, the cell opening diameter d is required to be $2\mathrm{mm}$ and the cell internal depth h is limited to $10\mathrm{mm}$. Estimate the cell diameter D required for the acoustic liner.

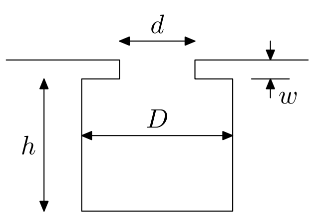


Figure 3.8: A cell of an acoustic liner

Chapter 4

Measuring sound

So far we have talked about sound without thinking about how we measure it. There are two important devices available to us for sound measurement: microphones and ears. They work in a similar manner, but with the important difference that ears are directly connected to a signal-processing system which extracts extra information about the sound field while microphones usually only give us a simple recording at one point.

4.1 Microphones

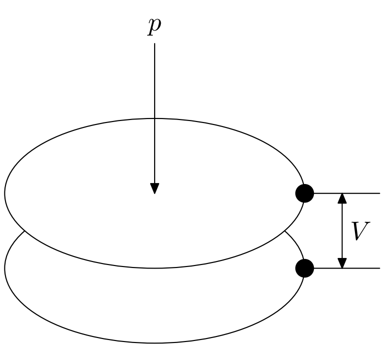


Figure 4.1: The principle of the condenser microphone: the deformation of the diaphragm changes the capacitance of the system which alters the output voltage V

The simplest device for the measurement of sound is a microphone. These are mechanical devices which convert the mechanical input of acoustic pressure fluctuations into an electrical signal. For high quality measurements, we usually use condenser microphones which are capacitors with one flexible plate which is exposed to the sound field. Movement of the plate changes the capacitance of the system and alters the voltage across the plates, generating an output signal, figure 4.1. The disadvantage of condenser microphones is that they need an external power supply, but they are still used where high quality measurements or recordings are needed. An alternative, which is more robust and simpler to use is the piezoelectric device which incorporates a solid which generates an electric charge in response to mechanical load.

In either case, the output from the system is a voltage which is proportional to the acoustic pressure which can then be processed using standard techniques. This can be done in real time (effects pedals) or using recorded data (ripping CDs). The main point to remember is that the Shannon sampling theorem tells us we have to record the data at a frequency (number of samples per second) at least twice as high as the highest frequency in our signal. The human ear can detect frequencies up to about $20\mathrm{kHz}$ so music is digitally recorded at $44.1\mathrm{kHz}$ to give reasonable reproduction.

4.2 Ears

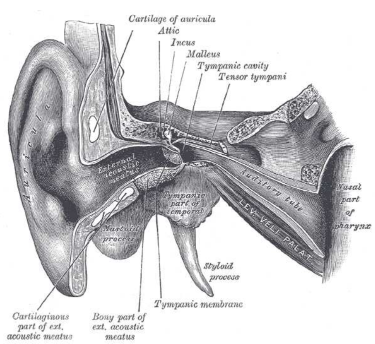


Figure 4.2: The human ear (from *Gray's anatomy*, via Wikipedia)

The human, or other animal, ear can be viewed as a type of microphone, although it has integrated signal processing and is mechanically a bit more complicated than the microphones we plug into our measurement systems. Figure 4.2 shows a section through the human ear. Sound coming from outside travels down the ear canal which terminates at the eardrum (tympanic membrane). The eardrum is connected to the inner ear by a mechanical linkage of three bones, the hammer, anvil and stirrup. This connects to the *cochlea*, a liquid filled organ which allows the ear to detect the amplitude and frequency of incoming sounds. A nerve takes the signal from the cochlea and transfers it to the brain where further signal processing allows us to extract more information about the sound we are hearing.

The cochlea is a tube but, because it tapers and has mechanical properties which vary along its length, different frequency components of the incoming sound propagate at different rates. This means that the components generate a maximum signal at different positions on the cochlea, decomposing the sound into elements which the brain can then process.

4.3 Multiple microphones

One thing we have noticed about our ears is that they tell us where sound is coming from. In part, this is because we can use head movement to tell us something about how the perceived sound changes with direction but it is mainly due to how our brains combine the signals from our two ears. We can do the same thing with microphones to characterize sound fields: the classic application is the detection of submarines by an oil-covered sweaty chap listening to headphones in a war movie.

Example: Dipole microphone

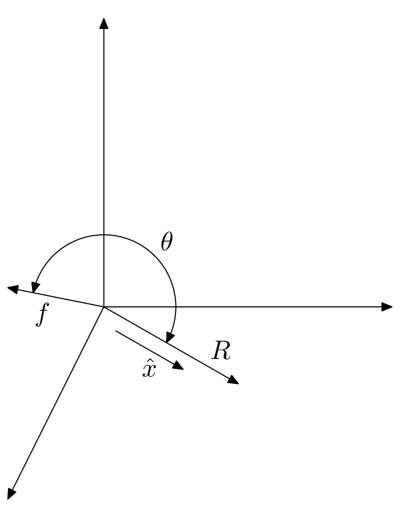


Figure 4.3: Dipole coordinate system

Very often we want to be able to measure sound from a particular direction, either to characterize a source or to reject noise from particular directions (in an aircraft microphone system, for example). The simplest method for doing this is to use two microphones joined together. We can work this out directly, or we can use the principle of *reciprocity*. This says that if we switch the source position and the microphone position, the microphone measures the same sound in both cases. You can see that this is so by switching $\mathbf{x}$ and $\mathbf{y}$ in Equation 2.8 and noting that the distance does not change. If we put two sources together and calculate the noise at some other point, this is equivalent to the noise measured by two microphones if noise is generated at the original microphone point. Because the sound field is made up of contributions from two sources, it is called a *dipole* system.

The form of the acoustic field for a dipole system can be derived from first principles. If we start with two sources of equal and opposite strength, separated by a small distance a, their positions are $(\pm a/2, 0, 0)$. Then the total sound at some point is:

$$p = \frac{q(t - R_+/c)}{4\pi R_+} - \frac{q(t - R_-/c)}{4\pi R_-}, \tag{4.1}$$
$$R_\pm = [(x \mp a/2)^2 + y^2 + z^2]^{1/2}.$$

We want to calculate the total radiated sound for (very) small values of a assuming that $f = aq$ the *dipole moment* remains constant. The easiest way to do this is to expand p in a Taylor series:

$$p \approx p|_{a=0} + \left.\frac{\mathrm{d}p}{\mathrm{d}a}\right|_{a=0} a + \ldots. \tag{4.2}$$

Differentiating (4.1):

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}a}\frac{q(t-R_\pm/c)}{4\pi R_\pm} &= -\frac{\partial R_\pm}{\partial a}\left(\frac{\dot q(t-R_\pm/c)}{4\pi R_\pm c}+\frac{q(t-R_\pm/c)}{4\pi R_\pm^2}\right),\\ \left.\frac{\partial R_\pm}{\partial a}\right|_{a=0} &= \mp\frac{1}{2}\frac{x}{R},\\ R &= (x^2+y^2+z^2)^{1/2}.\end{aligned}$$

Using these results in (4.1):

$$p \approx a\frac{x}{R}\left(\frac{\dot q(t-R/c)}{4\pi R c}+\frac{\dot q(t-R/c)}{4\pi R^2}\right). \tag{4.3}$$

We can rewrite this by noting that $f = aq$ and $x/R = \cos\theta$:

$$p = \left(\frac{\dot f(t-R/c)}{c}+\frac{f(t-R/c)}{R}\right)\frac{\cos\theta}{4\pi R}. \tag{4.4}$$

If we look at this as a sound generating system, it tells us that the maximum noise comes at $\theta = 0$ and the minimum (zero) at right angles to the line through the sources, because of cancellation effects. If we apply reciprocity, however, and assume that the *source* is at $\mathbf{x}$ and treat the dipole as a combination of two microphones, the output signal is one which amplifies sound from the $\theta = 0$ direction and cancels out noise from $\theta = \pi/2$. This allows you to use the system in noisy environments where you want to ensure that only sound from one direction is accepted in the system. A good example would be the headset microphones used by pilots: you want to ensure that the sound from the pilot is accepted in the system but the background noise is rejected.

Microphone arrays

If you want to be more particular about your measurements, you can add more microphones to set up a *microphone array*. This is a number of microphones whose signals are combined in such a way as to amplify the sound from a particular position or direction. One of the main applications of such arrays is a line of microphones towed behind a ship for submarine detection, although they are also used in acoustic experiments to characterize or locate noise sources.

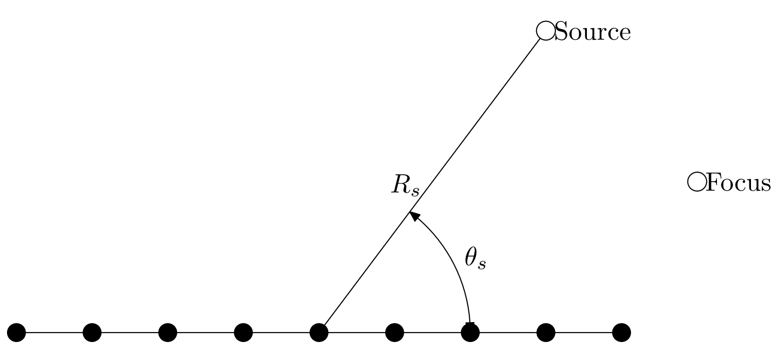


Figure 4.4: Schematic of a linear microphone array

Figure 4.4 shows the operation of an array. We are interested in sound from a 'focus' point. What we need to know is how much sound we will pick up from a source at some other position. The sound from the source will be $\exp(\mathrm{j}kR_s)/4\pi R_s$. The response of the array is approximately an integral along the line of microphones with each microphone's signal rephased to amplify sound coming from the 'focus' position:

$$S = \int_{-L/2}^{L/2} \frac{\mathrm{e}^{\mathrm{j}kR_s}}{4\pi R_s} \mathrm{e}^{-\mathrm{j}kR_f}\,\mathrm{d}x. \tag{4.5}$$

We can approximate the integral by noting that for distant sources $1/R_s$ is approximately constant for all microphones and, using a Taylor series, we can write:

$$R_s \approx R_s|_{x=0} - \left.\frac{x_s}{R_s}\right|_{x=0} x,$$
$$R_s - R_f \approx (R_s - R_f)|_{x=0} - (\cos\theta_s - \cos\theta_f)x.$$

Inserting this into equation 4.5:

$$S = \frac{L}{2}\frac{\mathrm{e}^{\mathrm{j}k(R_s-R_f)}}{2\pi R_s}\frac{\sin k(\cos\theta_s - \cos\theta_f)L/2}{k(\cos\theta_s - \cos\theta_f)L/2}.$$

The amplitude of the response has the form of a $\sin x/x$ curve. This has a maximum when $\cos\theta_s = \cos\theta_f$; this is no surprise, it simply means that we hear the most noise when we 'look' straight at the source. The shape of the curve is shown in Figure 4.5. We can see that as we move away from the focus position, the amplitude of the response is smaller: by looking in one direction, we reject noise from other directions. We can also see from the shape of the curve that increasing k (proportional to frequency), the amplitude of the response becomes smaller. So the array gives better discrimination at high frequency. We get the same effect by increasing L, the length of the array. The performance of the array is characterized by the parameter kL (so no change there then).

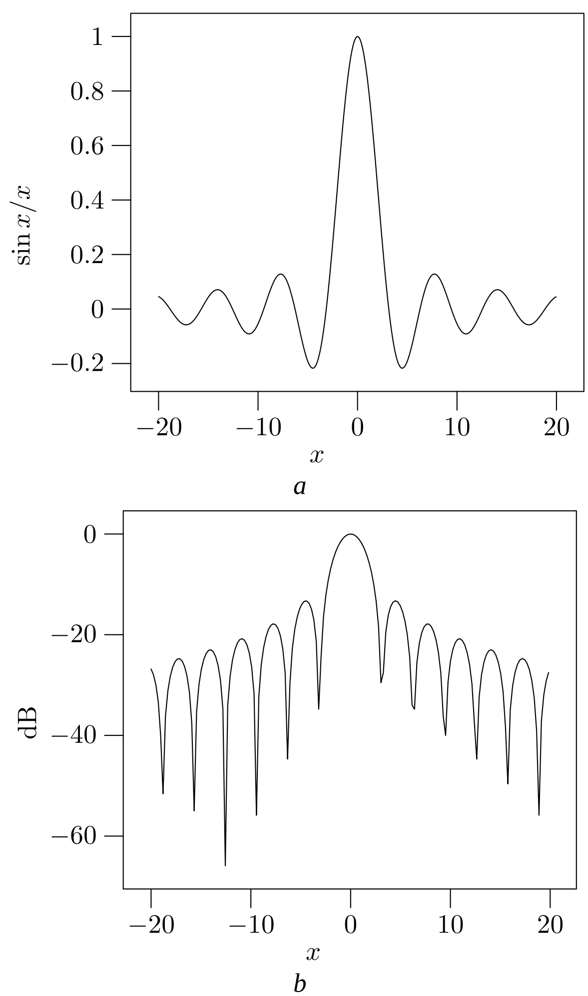


Figure 4.5: Array response $\sin x/x$: *a*: response; *b* amplitude in dB.

Chapter 5

Moving sources

Yeeeeeeeeehaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaaw.

Major T.J. “King” Kong (Slim Pickens) in *Doctor Strangelove or: How I learned to stop worrying and love the bomb*.

As you may be aware from the movies and the scream of Major Kong as he plummets to his doom astride a bomb, the sound heard from a source changes if the source is moving. As Major Kong falls Russia-wards, he accelerates (Isaac Newton says he has to). This acceleration changes the frequency of his shout as he falls.

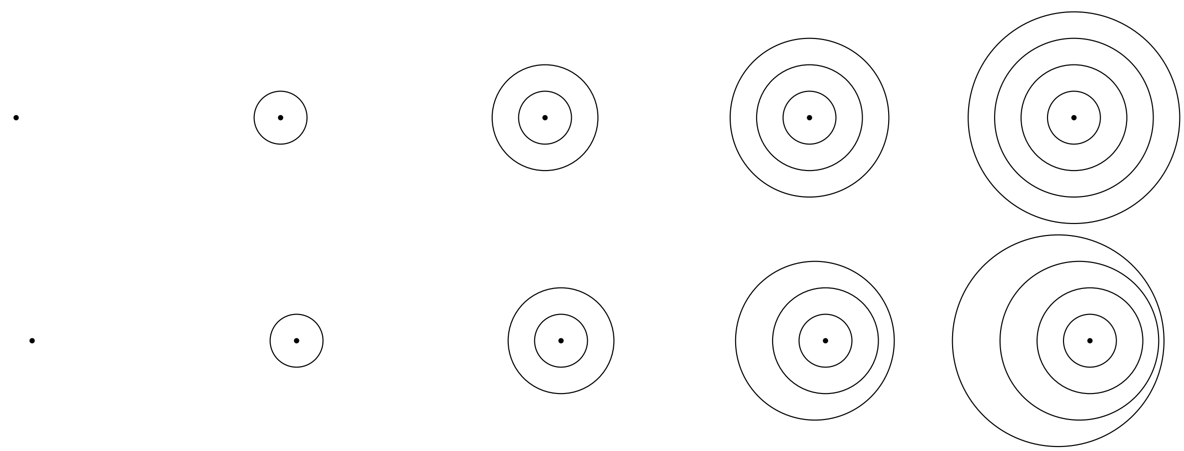


Figure 5.1: A simple model for the Doppler effect, *a*: stationary source; *b*: moving source.

Figure 5.1 shows what is happening. Figure 5.1*a* shows the wavefronts radiating from a stationary source. They propagate at the speed of sound and along any line from the source, they are equidistant. In figure 5.1*b*, the source moves to the right at some velocity V. The wavefronts still travel at the speed of sound, but each is generated a point successively further to the right. This causes the wavefronts to bunch up ahead of the source and stretch out behind it. This obviously leads to a change in the frequency of the sound at some observer position but also to a change in the amplitude, as more or fewer wavefronts arrive per unit time.

To quantify the effect of motion on the sound radiated by a source, we use the solution of the wave equation, equation 1.10, with a moving point source:

$$q(\mathbf{y}, t) = q(t)\delta(\mathbf{y} - \mathbf{y}_0(t)).$$

This represents a point source which is at $\mathbf{y} = \mathbf{y}_0$ at time t. Inserting this into equation 1.10:

$$p = \int_\tau \int_V q(t) \frac{\delta(\tau - t + R/c)}{4\pi R} \, \mathrm{d}V \, \mathrm{d}\tau. \tag{5.1}$$

This can be solved using the normal relationship for the delta function, but with the change of variables $\tau \to g$ where $g(\tau) = \tau - t + R/c$:

$$\int \delta(g(\tau)) f(\tau) \, \mathrm{d}\tau = \left. \frac{f(\tau)}{|dg/d\tau|} \right|_{g(\tau)=0}.$$

Integrating over τ in equation 5.1

$$\int_\tau \frac{\delta(\tau - t + R/c)}{4\pi R} \, \mathrm{d}\tau = \frac{1}{4\pi R |1 + \partial R/\partial\tau/c|}$$

where

$$\frac{\partial R}{\partial \tau} = -\frac{\partial \mathbf{y}_0}{\partial \tau} . \frac{\mathbf{x} - \mathbf{y}_0}{R},$$
$$\frac{1}{c} \frac{\partial \mathbf{y}_0}{\partial \tau} = \mathbf{M},$$

the source (vector) Mach number and

$$M_r = -\mathbf{M} . \frac{\mathbf{x} - \mathbf{y}_0}{R},$$

the relative Mach number of the source in the direction of the observer, so that

$$p = \int_V \frac{q(\tau)}{4\pi R |1 - M_r|} \, \mathrm{d}V.$$

Because q is a point source, we can integrate over V to find:

$$p = \frac{q(\tau)}{4\pi R |1 - M_r|}.$$

Finally, for a moving source with monopole strength q and dipole strength $\mathbf{f}$:

$$\boxed{p = \frac{\partial}{\partial t} \frac{q(\tau)}{4\pi R |1 - M_r|} + \nabla . \frac{\mathbf{f}(\tau)}{4\pi R |1 - M_r|}.} \tag{5.2}$$

The important thing to note here is that the sound is amplified by a factor $1/|1 - M_r|$, the *Doppler factor*. For a supersonic source, it can happen that $1 - M_r = 0$ and the pressure p is infinite. It is also important to realize that a source which is steady in its own reference frame (the loading on a propeller blade, for example) can still radiate noise if it is moving, due to variations in the Doppler factor.

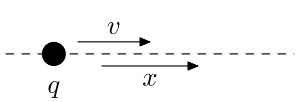


Figure 5.2: Source in rectilinear motion

We now look again at the problem of a monopole source moving in a straight line, figure 5.2. The position of the source is $x = vt$. The general problem is left as an exercise, and here we will only look at the sound radiated to an observer on the axis of motion. To work out the radiated noise for an observer ahead of the source, we need the following quantities:

$$R = c(t - \tau) = x - v\tau,$$
$$\tau = \frac{t - x/c}{1 - M},$$
$$R = \frac{x - vt}{1 - M},$$
$$M_r = M.$$

The source-observer Mach number M_r is equal to the source Mach number M for observer positions ahead of the source ($x > vt$) and $-M$ for observer positions behind the source ($x < vt$). Inserting the various quantities into equation 5.2:

$$p = \frac{\partial}{\partial t}\frac{1}{4\pi}\frac{q(\tau)}{x - vt}.$$

To look at the effect of motion on the frequency of the noise, consider a source with $q = \exp[-\mathrm{j}\omega t]$. The sound heard by an observer will be proportional to $\exp[-\mathrm{j}\omega\tau]$. Since $\tau = (t - x/c)/(1 - M)$, the sound at the observer will be proportional to

$$\exp[-\mathrm{j}\omega(t - x/c)/(1 - M)]$$

and the perceived frequency will be $\omega/(1-M)$. For points behind the source, $R = x+v\tau$ and the perceived frequency is $\omega/(1 + M)$.

5.1 Questions

1. Repeat the example on page 30 for an observer or microphone position which is not on the line of motion of the source. In writing the result in a compact form, you might find the definition $\beta^2 = 1 - M^2$ useful.

2. A turboprop aircraft has four-bladed propellers which rotate at 500rpm. A noise measurement is taken on the ground as the aircraft flies overhead at height 200m. If the measurement microphone is 400m ahead of the aircraft and the measured frequency of the first harmonic of the noise is 60Hz, how fast is the aircraft flying?

Chapter 6

Aircraft noise: propellers

The calculation of the noise generated by a general body in arbitrary motion is a hard problem. The sound radiated by a source undergoing motion as simple as pure rotation is qualitatively different from that of a source moving in a straight line. This is partly because the calculation of the retarded time and the Doppler factor is not as simple as in the linear motion case and partly because of the difficulty of calculating the source terms, the force and volume sources of equation 5.2.

6.1 Sound from rotating sources

To keep things as simple as possible without making them unrealistic, we will look at the problem of the sound radiated by a rotating point source. This is a very simple system but contains most of the behaviour of real rotors and will spare us the agonies of dealing with superfluous difficulties. The arrangement is shown in figure 6.1: a point source at radius a rotates at frequency Ω. We assume that there is no forward motion, so this system corresponds to a stationary propeller, or a helicopter rotor in hover.

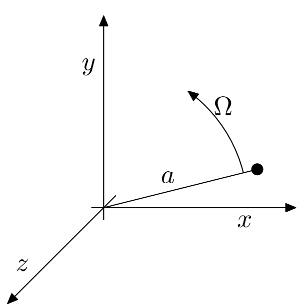


Figure 6.1: A rotating source

We will use cylindrical coordinates (r, θ, z) and assume that the observer is positioned at a point $(r, 0, z)$. Changing the angular position of the observer will only affect the phase of the sound and not its overall shape. To make things easier for ourselves, we will work in terms of the retarded time rather than the observer time.The position of the source at time τ is:

$$(a \cos \Omega\tau, a \sin \Omega\tau, 0).$$

Differentiating, its velocity is:

$$(-a\Omega \sin \Omega\tau, a\Omega \cos \Omega\tau, 0).$$

The source observer distance is (remember the observer does not move):

$$R^2 = R_0^2 + a^2 - 2ar \cos \Omega\tau,$$

where R_0 is the distance of the observer from the centre of rotation,

$$R_0 = [r^2 + z^2]^{1/2}.$$

We have the source-observer distance, but to calculate the Doppler factor we need to know the source–observer Mach number M_r:

$$\begin{aligned}
M_r &= -\frac{1}{c}\frac{\partial R}{\partial \tau}, \\
\frac{\partial R}{\partial \tau} &= a\frac{r}{R}\Omega \sin \Omega\tau, \\
M_r &= -\frac{r}{R} M_t \sin \Omega\tau.
\end{aligned}$$

Here $M_t = a\Omega/c$ is the rotational Mach number of the source. The Doppler factor is:

$$\frac{1}{|1-M_r|} = \frac{R}{|R + rM_t \sin\theta|},$$

where $\theta = \Omega\tau$ is the position of the source at time τ. The first obvious thing is to check if and when the Doppler factor becomes (nominally) infinite:

$$R = -rM_t \sin\theta.$$

This can be solved by squaring both sides and remembering that $\sin^2\theta = 1 - \cos^2\theta$:

$$M_t^2 r^2 \cos^2\theta - 2ar\cos\theta + R_0^2 + a^2 - M_t^2 r^2 = 0.$$

If we now scale all lengths on the source radius a, the equation becomes:

$$M_t^2 r^2 \cos^2\theta - 2r\cos\theta + R_0^2 + 1 - M_t^2 r^2 = 0, \tag{6.1}$$

which has two solutions:

$$\cos\theta = \frac{1}{M_t^2 r} \pm \frac{1}{M_t^2 r}[(1-M_t^2)(1-M_t^2 r^2) - M_t^2 z^2]^{1/2}. \tag{6.2}$$

If the source is to approach the observer at sonic velocity, the solution for $\cos\theta$ must be real. This means that the term inside the square root must not be negative:

$$(1-M_t^2)(1-M_t^2 r^2) - M_t^2 z^2 \geq 0.$$

Solving with this term set to zero:

$$z^2 = (M_t^2 - 1)\left(r^2 - \frac{1}{M_t^2}\right), \tag{6.3}$$

which defines a curve in the r–z plane dividing points where the source approaches at sonic velocity from points where it does not. For z^2 to be positive (i.e. a valid point in the plane) $M_t > 1$ and $r > 1/M_t$. This means that a source must be travelling supersonically if it is to approach an observer position at sonic velocity (hardly a surprise) and the observer position must lie outside the *sonic radius* $1/M_t$, which is the radius where the source has, or would have, sonic rotation velocity. Figure 6.2 shows the dividing curves for different values of M_t. The region inside the curve, labelled 'subsonic', never experiences the source approaching at sonic velocity, while the points in the outer region, labelled 'sonic', do.

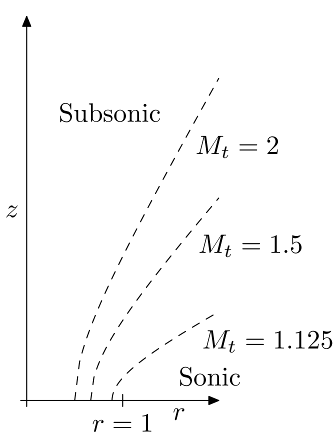


Figure 6.2: Points subject to Doppler radiation from a rotating source. The dashed lines indicate the curve $z^2 = (M_t^2 - 1)(r^2 - 1/M_t^2)$ for different tip Mach numbers.

We have managed to get this far without ever calculating the noise heard at some observation point. If we now calculate the quantities we need to work out the noise:

$$\begin{aligned} R &= [1 + r^2 + z^2 - 2r\cos\theta]^{1/2}, \\ 1 - M_r &= 1 + M_t \frac{r}{R}\sin\theta, \\ \Omega t &= \theta + M_t R, \\ \frac{1}{4\pi R|1-M_r|} &= \frac{1}{4\pi|R + rM_t\sin\theta|}, \end{aligned}$$

where lengths are still scaled on a and θ is still the source position at the retarded time τ.

To calculate the radiated noise, we simply take different values of θ, ranging from 0 to 2π and calculate the corresponding values of R and the

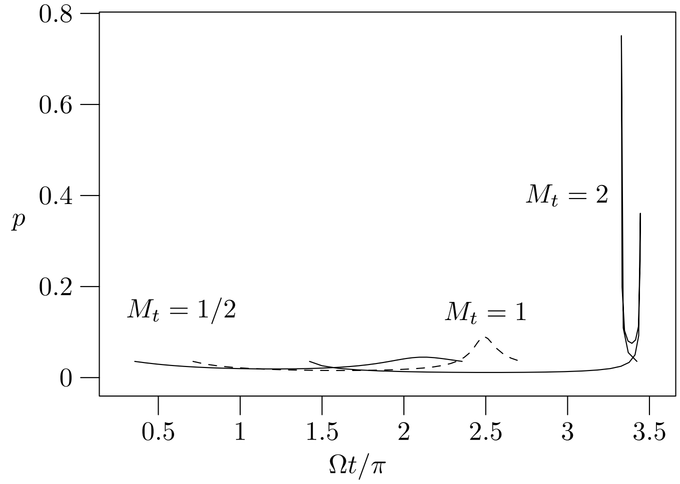


Figure 6.3: Time records for rotating source.

arrival times Ωt. If the values of θ are evenly spaced, we do not expect the values of Ωt to be evenly spaced, but they will cover a range of 2π.

Figure 6.3 shows $1/4\pi R|1 - M_r|$ plotted against $\Omega t/\pi$ for three different values of M_t. Note that in each case, Ωt covers a range of 2π. As you might expect, the noise for $M_t = 0.5$ is weaker (though not much weaker) than that for $M_t = 1$ which is very much weaker than that for $M_t = 2$. This is not unexpected but there is something strange about the noise record for $M_t = 2$: there are three values of pressure for some time points.

The reason for this is shown in figure 6.4 which shows the position θ as a function of Ωt. For $M_t = 2$, there is a range of Ωt for which there are three values of τ, meaning that the sound received at each time has a contribution from three different source positions. This is a feature unique to supersonically rotating sources and illustrates the manner in which noise from such sources is *qualitatively* different and is not just a louder version of subsonic source noise.For higher rotation speeds, there can be five, seven or more retarded times for a given arrival time.

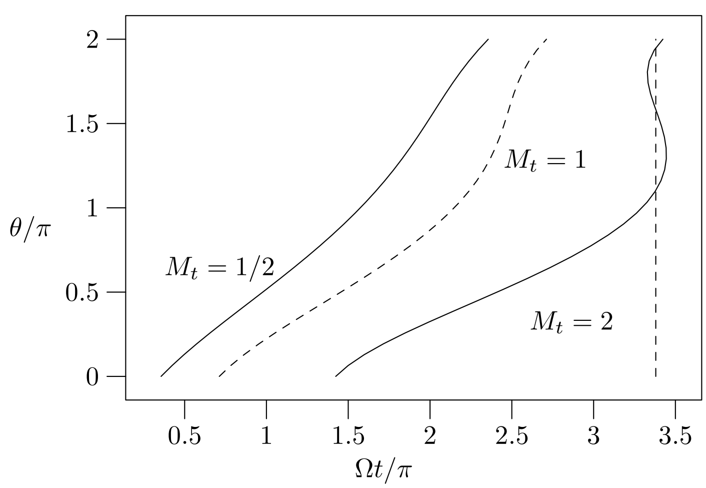
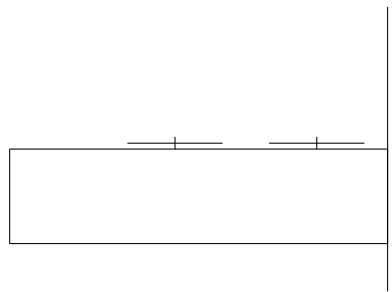


Figure 6.4: Retarded times for rotating source: the vertical dashed line indicates a value of t for which there are three values of τ.

6.2 Questions

1. Given a source rotating at Mach number M_t, at what azimuthal angle does it generate maximum acoustic pressure at an observer? Account for both supersonic and subsonic source Mach numbers.

2. The figure below shows schematically the layout of the propellers on one wing of a four-engined turboprop. The propellers are of diameter 3m and their hubs are 2.25m and 6.75m respectively from the fuselage (assumed to be of constant section). The propellers are advanced high speed designs rotating at 2300rpm. Calculate the blade tip Mach number and thus the extent of the region on the fuselage affected by supersonic source radiation.



3. A supersonic transport makes a turn of radius 300km at an altitude of 12000m. If the flight Mach number $M = 2$, calculate the radius of the 'quiet zone' below the aircraft. How would this change if M were reduced to 1.5?

4. Repeat question 3 of §2.5 but with the piston velocity distribution given by $v = V \exp[\mathrm{j}(n\theta - \omega t)]$ (you will probably need to consult a big maths book such as Gradshteyn & Ryzhik). The result tells you about how sound at a given frequency radiates from a rotating source.

Chapter 7

Aircraft noise: jets

The approach to sound generation by sources in a flow is that of Lighthill who developed the basis of modern aeroacoustics in the 1950s, as civil jet engines were being developed. The derivation given here follows Lighthill's original approach but is closer to that of Powell who developed a theory of sound generation by vorticity. The idea is to go through the motions of §1.2 but without linearizing the equations. The exact equations of inviscid fluid motion are:

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho \mathbf{v}) = 0, \tag{7.1a}$$

$$\rho \frac{\partial \mathbf{v}}{\partial t} + \rho \mathbf{v} \nabla \mathbf{v} + \nabla p = 0. \tag{7.1b}$$

As in §1.2, we differentiate equation 7.1a with respect to time, equation 7.1b with respect to space and subtract one from the other:

$$\nabla^2 p - \frac{\partial^2 \rho}{\partial t^2} = \nabla. \left(\nabla p + \frac{\partial}{\partial t}(\rho \mathbf{v}) \right). \tag{7.2}$$

To simplify this equation, we can rearrange equations 7.1. Multiplying equation 7.1b by $\mathbf{v}$ and adding it to equation 7.1a:

$$\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla.(\rho \mathbf{v}\mathbf{v}) + \nabla p = 0.$$

Inserting this into equation 7.2:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\nabla. \left(\frac{\partial}{\partial t}(\rho \mathbf{v}) + \nabla.(\rho \mathbf{v}\mathbf{v}) - \frac{\partial}{\partial t}(\rho \mathbf{v}) \right),$$

which includes the usual approximation for the relationship between ρ and p. The product $\rho\mathbf{v}\mathbf{v}$ is to be read as a tensor (like a matrix, or vector of vectors) which can be written:

$$\mathbf{T} = \begin{bmatrix} \rho v_x v_x & \rho v_y v_x & \rho v_z v_x \\ \rho v_x v_y & \rho v_y v_y & \rho v_z v_y \\ \rho v_x v_z & \rho v_y v_z & \rho v_z v_z \end{bmatrix},$$

or, more compactly, $T_{ij} = \rho v_i v_j$. The net result is then:

$$\nabla^2 p - \frac{1}{c_0^2}\frac{\partial^2 p}{\partial t^2} = -\nabla\nabla(\rho \mathbf{v}\mathbf{v}), \tag{7.3}$$

which is an approximation to Lighthill's theory of aerodynamically generated sound.

7.1 Lighthill's eighth power law for jet noise

Solving Lighthill's equation for different sources is more than we can manage in these notes, but we can derive a scaling law for jet noise which was one of the first great successes of the theory. The 'solution' of equation 7.3 is

$$p = -\nabla\nabla \int_V \frac{\mathbf{T}(\mathbf{y}, t - R/c_0)}{4\pi R}\,\mathrm{d}V,$$

where $\mathbf{T} = \rho\mathbf{v}\mathbf{v}$. In the far field, we can approximate this integral by differentiating it: when we do this, we will retain only terms which depend on $1/R$ (everything else decays much more rapidly). Setting coordinates so that the origin is inside the source region, $\mathbf{x} - \mathbf{y} \approx \mathbf{x}$ and

$$p \approx \frac{1}{4\pi}\frac{\mathbf{x}\mathbf{x}}{x^3}\int_V \frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\mathbf{T}(\mathbf{y}, t - R/c_0)\,\mathrm{d}V.$$

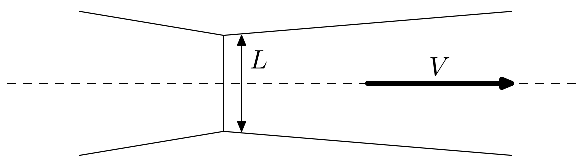


Figure 7.1: Parameters for jet noise.

There is no general solution for this equation, but we can derive a scaling law for the radiated acoustic power. Figure 7.1 shows a simple jet flow with the relevant parameters indicated. We take a characteristic length L, characteristic velocity V and a mean density ρ_0. Then:

$$\mathbf{T} \sim \rho_0 V^2, \quad \frac{\partial}{\partial t} \sim \frac{V}{L},$$
$$p \sim \frac{1}{4\pi}\frac{1}{x}\frac{1}{c_0^2}\left(\frac{V}{L}\right)^2 \rho_0 V^2 L^3,$$

and the pressure scales as:

$$p \sim \rho_0 \frac{V^4}{c_0^2}\frac{L}{x}.$$

From equation 1.16, the intensity scales as

$$\bar{I} \sim \rho_0 \frac{V^8}{c_0^5}\left(\frac{L}{x}\right)^2.$$

The total acoustic power W is the intensity integrated over a spherical surface of radius x and

$$W \sim \rho_0 \frac{V^8}{c_0^5} L^2. \tag{7.4}$$

The total acoustic power thus scales on the eighth power of jet velocity. This is Lighthill's eighth power law and was derived before experimental data were available to confirm it: it is one of the few scientific predictions to have been a genuine prediction. It is strictly only true for low speed flows, because we have implicitly assumed the source to be compact. At higher speeds, the characteristic frequency of the source increases and interference effects become important.

Figure 7.2: Trends in aircraft design: the Boeing 777 has two engines providing almost as much thrust as the four engines of the Boeing 747, a quieter, more fuel-efficient solution.

Example: Modern aircraft

Using Lighthill's scaling law, we can estimate the difference in noise from a twin-engine and four-engine aircraft. We know that the thrust from an engine is proportional to $\rho V^2 D^2$. The total thrust F is the same in both cases, and:

$$F = 4\rho V_4^2 D_4^2 = 2\rho V_2^2 D_2^2,$$

and the total noise W is:

$$\begin{aligned} W_4 &= 4V_4^8 D_4^2, \\ W_2 &= 2V_2^8 D_2^2. \end{aligned}$$

We can calculate the ratio of the total noise, by calculating the ratio of the jet velocities:

$$\begin{aligned} \frac{F/4}{F/2} &= \left(\frac{V_4}{V_2}\right)^2 \left(\frac{D_4}{D_2}\right)^2, \\ V_2 &= \sqrt{2}\frac{D_4}{D_2} V_4, \end{aligned}$$

and, if we assume that $D_2 = 2D_4$,

$$\begin{aligned} \frac{W_2}{W_4} &= \frac{1}{2}\left(\frac{\sqrt{2}}{2}\right)^8 (2)^2, \\ &= 1/8, \end{aligned}$$

which is a noise reduction of $9\mathrm{dB}$.

7.2 Questions

1. The thrust from a jet of diameter D scales as $\rho V^2 D^2$. For a fixed thrust, find a relationship between the noise from the jet and its diameter. What relevance do you think this relationship has for aircraft design?

2. Given that jet thrust scales as $\rho V^2 D^2$, estimate the noise reduction to be had by converting a four engined aircraft to use two engines of twice the exhaust diameter.

References

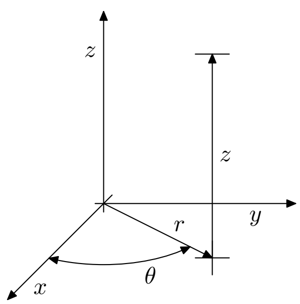
These notes only cover some of the basic elements of acoustics. Recommended texts if you want a different view or to deepen your knowledge:

- DOWLING, A. P. & FFOWCS WILLIAMS, J. E. 1983, *Sound and sources of sound*, Butterworth. This is quite a slim book compared to Pierce but it covers more of the things in these notes.
- CRIGHTON, D. G., DOWLING A. P., FFOWCS WILLIAMS, J. E., HECKL, M. & LEPPINGTON, F. G. 1992, *Modern methods in analytical acoustics*, Springer-Verlag. Very mathematical but covers a lot of material.
- HUBBARD, H. H. ed 1995, *Aeroacoustics of flight vehicles*, Acoustical Society of America. This is a two volume review of almost everything connected to noise from aircraft.
- LIGHTHILL, M. J. 1952, On sound generated aerodynamically: I General theory, *Proceedings of the Royal Society* A, **211**:564–587. This is the foundation of modern aeroacoustics and is surprizingly readable for a paper of such fundamental importance.
- PIERCE, A. 1994, *Acoustics: An introduction to its physical principles and applications*, American Institute of Physics, New York. This is the standard modern reference for acoustics. If you want to buy one comprehensive book on acoustics, this is the one. It doesn't really cover aerodynamically generated noise so you might want to look at Dowling & Ffowcs Williams as well.
- GRADSHTEYN, I. & RYZHIK, I. M. 1980, *Table of integrals, series and products*, Academic, London. A big book of all the mathematical formulae anyone could ever need.

Some useful mathematics

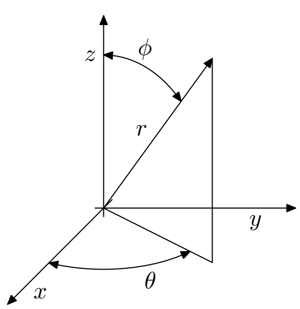
Coordinate systems

Cylindrical coordinates:



$$x = r\cos\theta, \quad y = r\sin\theta;$$
$$r = (x^2+y^2)^{1/2}, \quad \theta = \tan^{-1} y/x.$$

Spherical coordinates:



$$x = r\sin\phi\cos\theta, \quad y = r\sin\phi\sin\theta,$$
$$z = r\cos\phi;$$
$$r = (x^2+y^2+z^2)^{1/2}, \quad \theta = \tan^{-1} y/x,$$
$$\phi = \tan^{-1} z/(x^2+y^2)^{1/2}.$$

Differential operators

In Cartesian coordinates:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right),$$
$$\nabla.\mathbf{f} = \frac{\partial f_x}{\partial x} + \frac{\partial f_y}{\partial y} + \frac{\partial f_z}{\partial z},$$
$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

In cylindrical coordinates:

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right),$$
$$\nabla.\mathbf{f} = \frac{1}{r}\frac{\partial}{\partial r}(rf_r) + \frac{1}{r}\frac{\partial f_\theta}{\partial \theta} + \frac{\partial f_z}{\partial z},$$
$$\nabla^2 f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} + \frac{\partial^2 f}{\partial z^2}.$$

In spherical coordinates:

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \phi}, \frac{1}{r\sin\phi}\frac{\partial f}{\partial \theta}\right),$$
$$\nabla.\mathbf{f} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 f_r\right) + \frac{1}{r\sin\phi}\frac{\partial}{\partial \phi}(f_\phi \sin\phi) + \frac{1}{r\sin\phi}\frac{\partial f_\theta}{\partial \theta},$$
$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial \phi}\left(\sin\phi\frac{\partial f}{\partial \phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 f}{\partial \theta^2}.$$

Complex variables

We often use complex variable notation to make life easier. If we write a complex number $z = x + \mathrm{j}y$

where $\mathrm{j} = \sqrt{-1}$, then:

$$z = |z|\mathrm{e}^{\mathrm{j}\phi},$$
$$|z| = (x^2 + y^2)^{1/2},$$
$$\phi = \tan^{-1} y/x.$$

In dealing with constant frequency waves, we can use the relation:

$$\mathrm{e}^{-\mathrm{j}\omega t} = \cos\omega t - \mathrm{j}\sin\omega t$$

and if we wish to consider a general wave p of a fixed frequency, say, this can be written:

$$p(t) = P\mathrm{e}^{-\mathrm{j}\omega t},$$

where now P contains information about the amplitude and the phase.

The Dirac delta

The basic rule for integrating the delta function is:

$$\int_{-\infty}^{\infty} f(x)\delta(x - x_0)\,\mathrm{d}x = f(x_0),$$

and in the more complicated case where the argument of the delta function is itself a function:

$$\int_{-\infty}^{\infty} f(x)\delta(g(x))\,\mathrm{d}x = \frac{f(x_{g=0})}{|\mathrm{d}g/\mathrm{d}x_{g=0}|}.$$

He has never again encountered the most esteemed Arkady Apollonovich Sempleyarov in connection with acoustical problems. The latter was quickly transferred to Bryansk and appointed director of a mushroom-growing center. Nowadays, Moscow residents eat pickled saffron milk caps and marinated white mushrooms with endless relish and praise, and never stop rejoicing in the lucky transfer. Since it is all a matter of the past now, we feel free to say that Arkady Apollonovich never did make any headway with acoustics, and, for all his efforts to improve the sound, it remained as bad as it was.

The Master and Margarita, Mikhail Bulgakov

1.2 $\exp \mathrm{j}kR/4\pi R \exp[-\mathrm{j}\omega t]$.

1.4 set $20\log_{10} R_2/R_1 = 60$ and find $R_2 = 3000\mathrm{m}$.

2.3 treat the intake as a piston with radius $a = 1.5\mathrm{m}$, $k = 80 \times 2\pi/340 = 1.478/\mathrm{s}$ and $v = 0.02\mathrm{m/s}$. Then:

$$p = -\rho_0 cv(\mathrm{e}^{\mathrm{j}kR_a} - \mathrm{e}^{\mathrm{j}kz}),$$

with $z = 20\mathrm{m}$. Insert the numbers:

$$\begin{aligned}
R_a &= 20.056, \\
p &= -1.2 \times 340 \times 0.02(\cos 29.643 + \mathrm{j}\sin 29.643 - \cos 29.56 - \mathrm{j}\sin 29.56) \\
&= 8.16 \times (0.080299 - \mathrm{j}0.019970), \\
|p| &= 0.67519, \\
\mathrm{SPL} &= 20\log_{10}\frac{|p|}{2\times 10^{-5}}, \\
&= 90.5\mathrm{dB}
\end{aligned}$$

2.4

$$\begin{aligned}
p\mathrm{e}^{-\mathrm{j}\omega t} &= 2\frac{\partial}{\partial t}\iint_S \frac{q(\mathbf{y},\tau)}{4\pi R}\,\mathrm{d}S, \\
p &= -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\int_0^{2\pi}\int_0^a \frac{\mathrm{e}^{\mathrm{j}kR}}{R} r_1\,\mathrm{d}r_1\,\mathrm{d}\theta_1, \\
R &= (r^2 + r_1^2 - 2rr_1\cos\theta_1 + z^2)^{1/2},
\end{aligned}$$

In the far field $R \approx R_0 - r_1\sin\phi\cos\theta_1$ and $1/R \approx 1/R_0$ so that:

$$p \approx -\mathrm{j}\frac{\omega\rho_0 v}{2\pi}\frac{\mathrm{e}^{\mathrm{j}kR_0}}{R_0}\int_0^a\int_0^{2\pi} \mathrm{e}^{-\mathrm{j}kr_1\sin\phi\cos\theta_1}\,\mathrm{d}\theta_1, r_1\,\mathrm{d}r_1.$$

Using the integral definition of $J_0(\cdot)$:

$$p = -\mathrm{j}\omega\rho_0 v\frac{\mathrm{e}^{\mathrm{j}kR_0}}{R_0}\int_0^a J_0(kr_1\sin\phi)r_1\,\mathrm{d}r_1.$$

Changing variables $x = kr_1\sin\phi$:

$$p = -\mathrm{j}\omega\rho_0 v\frac{\mathrm{e}^{\mathrm{j}kR_0}}{(k\sin\phi)^2 R_0}\int_0^{ka\sin\phi} J_0(x)x\,\mathrm{d}x.$$

Integrating:

$$p = -\mathrm{j}\omega\rho_0 v\frac{\mathrm{e}^{\mathrm{j}kR_0}}{ka\sin\phi R_0}\frac{J_1(ka\sin\phi)}{ka\sin\phi}.$$

3.1 As in the notes, place at source at the image point in the boundary. The boundary condition is now that $p = 0$ so the sound field is:

$$p = \frac{\mathrm{e}^{\mathrm{j}kR_+}}{4\pi R_+} - \frac{\mathrm{e}^{\mathrm{j}kR_-}}{4\pi R_-}.$$

3.2 The incident and reflected waves are $A \exp \mathrm{j}kx$ and $B \exp[-\mathrm{j}kx]$ as before. Applying the boundary condition at $x = 0$:

$$
\begin{aligned}
A + B &= 0, \\
B &= -A, \\
p &= A(\mathrm{e}^{\mathrm{j}kx} - \mathrm{e}^{-\mathrm{j}kx}).
\end{aligned}
$$

3.3 As already proven:

$$
|T| = \left[1 + \left(\frac{\omega m}{2\rho c}\right)^2 \cos^2\theta\right]^{-1/2}.
$$

For Perspex, $m = 6\mathrm{kg/m}^2$. With $\omega = 2\pi \times 100$, $|T| = 1/4.727 = -13.5\mathrm{dB}$. For aluminium, $m = 5.4\mathrm{kg/m}^2$ and $|T| = 1/4.27 = -12.6\mathrm{dB}$. At 1kHz, $|T| = 1/46.2 = -33.3\mathrm{dB}$ for Perspex and $|T| = 1/41.59 = -32.4\mathrm{dB}$.

The noise reduction for each material is about the same: this means that increasing the thickness of one material will not help the noise reduction much. Both materials need to be made thicker (or insulated) to give good noise reduction. At 1kHz, the noise reduction is large so there is no need to increase it.

3.4 The basic frequency is $\omega = 2\pi \times 20 \times 6000/60 = 12.566 \times 10^3\mathrm{rad/s}$. Rearrange the formula for resonant frequency to find:

$$
\begin{aligned}
V &= \frac{c^2 S}{\omega^2 l} = 7.666 \times 10^{-7}\mathrm{m}^3, \\
V &= h\pi D^2/4, \\
D &= (4V/\pi h)^{1/2} = 10\mathrm{mm}.
\end{aligned}
$$

5.2 Given the source frequency and position, we can work out the relative Mach number. The source frequency is

$$
\begin{aligned}
f &= 4 \times 500/60, \\
&= 33.3\text{Hz}
\end{aligned}
$$

with the multiplication by 4 because there are four blades. The angle between the source velocity and the direction to the microphone is $\tan^{-1} 200/400 = 0.464$rad. If the measured frequency is f', then

$$
\begin{aligned}
f' &= \frac{f}{1 - M_r}, \\
M_r &= 1 - \frac{f}{f'} = M\cos\theta, \\
M &= \frac{1 - f/f'}{\cos\theta},
\end{aligned}
$$

where M is the flight Mach number. Inserting the numbers,

$$
\begin{aligned}
M &= \frac{1 - 33.3/60}{\cos 0.464} \\
&= 0.498
\end{aligned}
$$

and the flight velocity is $Mc = 0.498 \times 340 = 169$m/s.

6.1 In the subsonic case, the highest pressure occurs when M_r is a maximum, i.e. when the source approaches the observer at its highest velocity. In the supersonic case, the highest pressure occurs if the source approaches the observer at $M_r = 1$ and otherwise when M_r has a maximum.

6.2 First calculate the tip Mach number:

$$\begin{aligned} M_t &= \frac{\Omega a}{c}, \\ &= \frac{2\pi 2300}{60}\frac{1.5}{340}, \\ &= 1.063. \end{aligned}$$

From the notes the region affected by a supersonic source is:

$$z^2 = (M_t^2 - 1)\left(r^2 - \frac{1}{M_t^2}\right).$$

The affected region will be that due to the outboard propeller where, scaling on propeller radius, $r = 6.75/1.5 = 4.5$ and

$$\begin{aligned} z^2 &= (1.063^2 - 1)\left(4.5^2 - \frac{1}{1.063^2}\right), \\ &= 2.52. \end{aligned}$$

The affected region is upstream and downstream of the propeller so

$$z = \pm 1.59$$

and we have to rescale to get the full extent so the solution is

$$\begin{aligned} z &= \pm 1.59 \times 1.5, \\ &= \pm 2.39\mathrm{m} \end{aligned}$$

and the length of the affected region is 4.78m.

7.1 The thrust and noise power are given by:

$$\begin{aligned} T &= \rho V^2 D^2, \\ W &= \frac{\rho}{c^5} V^8 D^2. \end{aligned}$$

For fixed thrust:

$$V = \frac{1}{D}\left(\frac{T}{\rho}\right)^{1/2},$$

so that the noise is given by:

$$\begin{aligned} W &= \frac{\rho}{c^5}\frac{1}{D^8}\left(\frac{T}{\rho}\right)^4 D^2, \\ &= \frac{T^4}{\rho^3 c^5}\frac{1}{D^6}. \end{aligned}$$

The implication is that large jets are very much quieter than small ones which is why modern aircraft have such high bypass ratios: to reduce the jet exhaust velocity.

7.2 Assume a given total thrust T. On the two-engined aircraft:

$$
\begin{aligned}
\frac{T}{2} &= V_2^2 D_2^2, \\
W_2 &= 2V_2^8 D_2^2,
\end{aligned}
$$

where the scaling factors (ρ and c) have been ignored. On the four engined aircraft:

$$
\begin{aligned}
\frac{T}{4} &= V_4^2 D_4^2, \\
W_4 &= 2V_4^8 D_4^2,
\end{aligned}
$$

so the first thing we can write down is:

$$
\frac{W_2}{W_4} = \frac{2}{4}\left(\frac{V_2}{V_4}\right)^8 \left(\frac{D_2}{D_4}\right)^2.
$$

We know the diameter ratio so we now have to find the velocity ratio. We get this from the thrust:

$$
\begin{aligned}
\frac{T/4}{T/2} &= \left(\frac{V_4}{V_2}\right)^2 \left(\frac{D_4}{D_2}\right)^2, \\
V_2 &= \sqrt{2}\frac{D_4}{D_2}V_4.
\end{aligned}
$$

Inserting this into the expression for W_2/W_4

$$
\begin{aligned}
\frac{W_2}{W_4} &= \frac{1}{2}\left(\frac{1}{\sqrt{2}}\right)^8 (2)^2, \\
&= 0.125.
\end{aligned}
$$

In decibels, this is $10\log_{10}(W_2/W_4) = -9$dB.