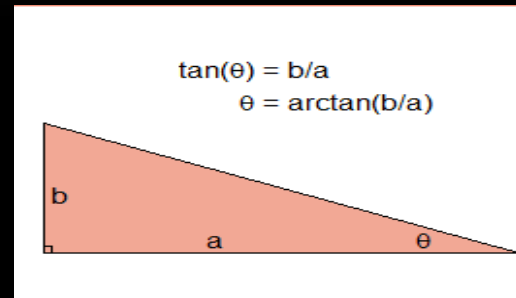


INVERSE TRIGONOMETRIC FUNCTIONS

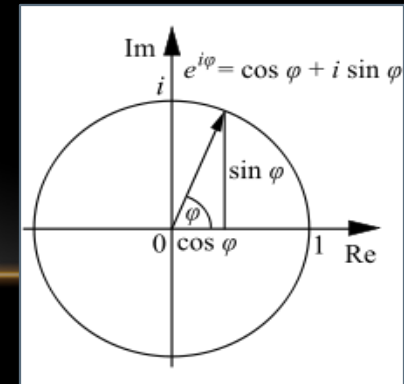
Colin Cox

WHAT IS AN INVERSE TRIG FUNCTION?

- Used to solve for the angle when you know two sides of a right triangle. For example if a ramp is resting against a trailer, then you could find the *angle* of its incline if you know the distance and the height to the trailer.



- Solving for angles on the unit circle if given either the sine or cosine value at a point. For example, if the $\cos x = \frac{1}{2}$, the $\arccos(\frac{1}{2}) = \frac{\pi}{3}$. Likewise, if the $\sin x = \frac{\sqrt{3}}{2}$, then the $\arcsin(\frac{\sqrt{3}}{2}) = \pi/3$ as well. *There is only one solution in the domain of the arccos and arcsin function for this value but we will get to that later.



HISTORY

- The Greeks were the original pioneers of trigonometry.
- Euclid, a Greek mathematician (300BC), defined triangles as a ratio of lines rather than the lengths of lines in his book, *Elements*.
- Isaac Newton took a bold step toward modern trigonometry by founding differential calculus in the 17th century.
- Euler established a deep relationship between the trigonometric function with Euler's Formula, $e^{i\theta} = \cos(\theta) + i \sin(\theta)$ This equation led us to the unit circle.



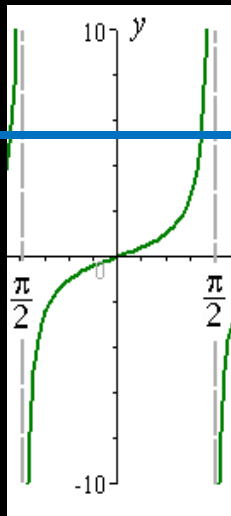
INTRODUCTION

- To begin, one must understand that an Inverse function is simply an exchange between the x and y in a ONE-TO-ONE function.
 - One-to-one simply means that every element of the range of the function corresponds with exactly one element of the domain (This will be crucial in the explanation of Inverse Trig Functions).
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Horizontal Line Test

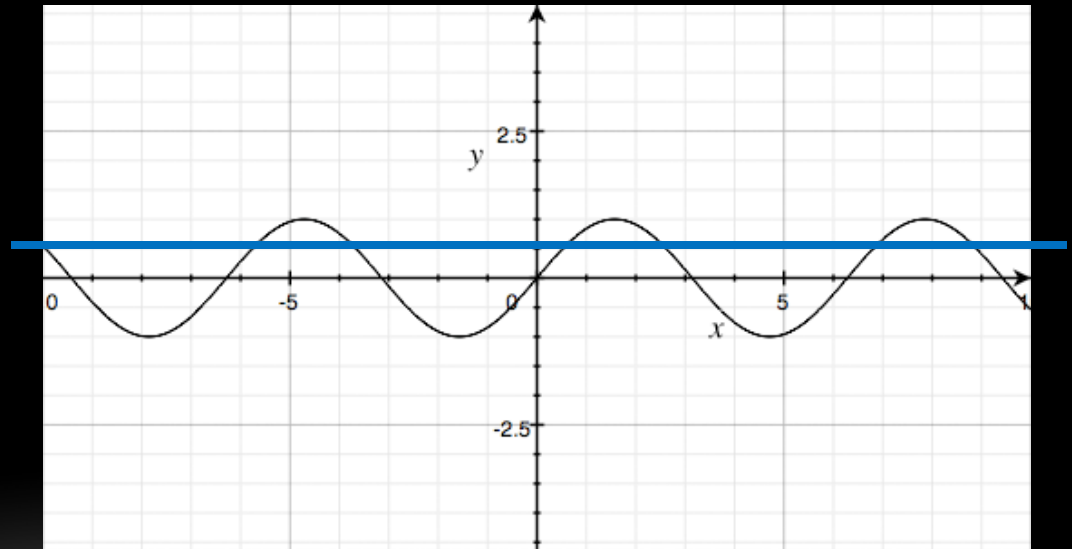
- A function must pass the Horizontal Line Test to be considered one-to-one. No horizontal line can intersect the function more than once.

Passes!



$$f(x) = \tan(x), -\pi/2 < x < \pi/2$$

Does not pass!



$$f(x) = \sin(x)$$

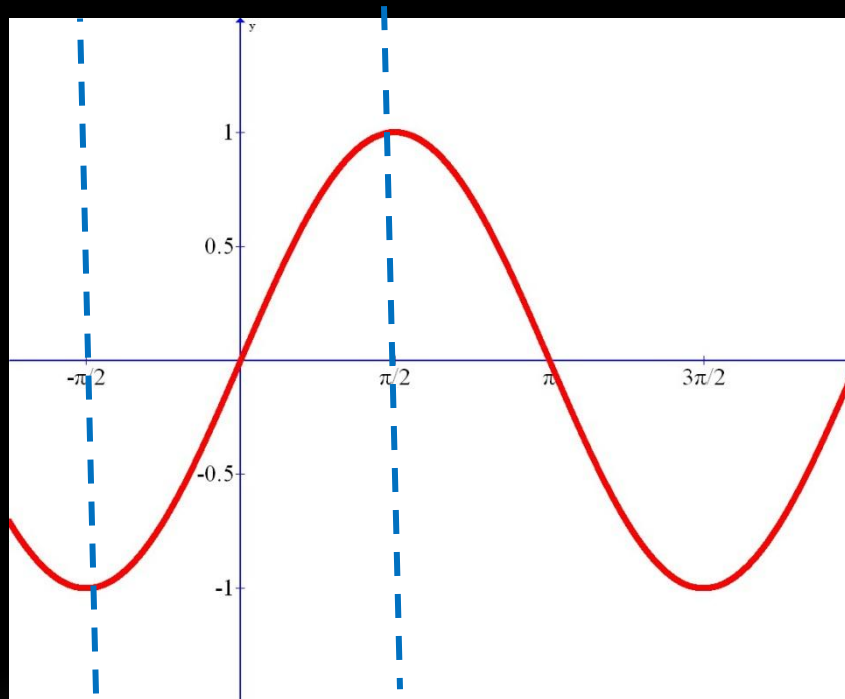
DEFINITION

The Definition of Inverse trig functions can be seen as the following formulas. Each is the inverse of their respective trigonometric function. Also, each inverse trig function also has a unique domain and range that make them one-to-one functions.

- Inverse Sine function= $\arcsin x$
- Inverse Cosine Function= $\arccos x$
- Inverse Tangent Function= $\arctan x$
- Inverse Secant Function= $\operatorname{arcsec} x$
- Inverse Cotangent Function= $\operatorname{arccot} x$
- Inverse Cosecant Function= $\operatorname{arccsc} x$

Example

We must limit the Domain of the $f(x)=\sin(x)$ function to obtain a one-to-one function.



$$D(\sin x) = [-\pi/2, \pi/2]$$

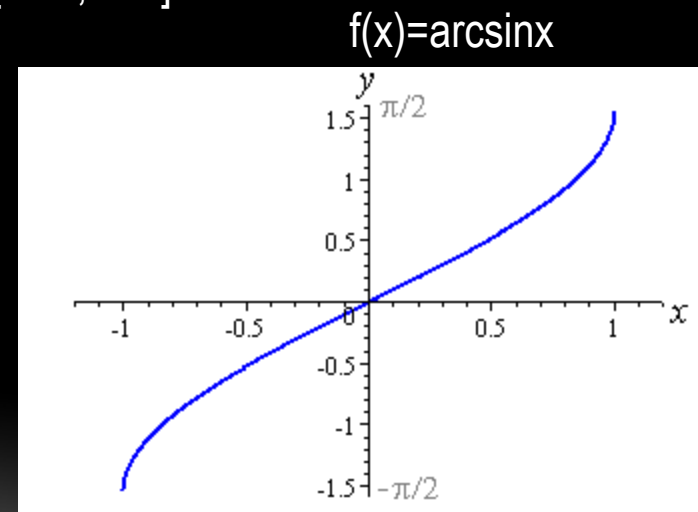
Inverse Sin Function

	Domain	Range
$f(x)=\sin x$	$[-\pi/2, \pi/2]$	$[-1, 1]$
$f(x)=\arcsin x$	$[-1, 1]$	$[-\pi/2, \pi/2]$

A few cancelation equations

$\arcsin(\sin x) = x$ (As long as the domain is between $[-\pi/2, \pi/2]$)

$\sin(\arcsin x) = x$ (If domain is between $[-1, 1]$)



***Remember arcsin x does NOT equal $1/\sin x$**

Inverse Cos Function

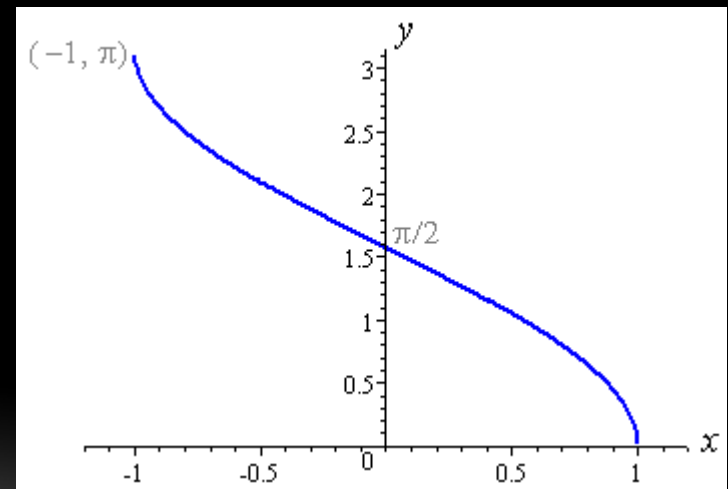
	Domain	Range
$f(x)=\cos x$	$[0, \pi]$	$[-1, 1]$
$f(x)=\arccos x$	$[-1, 1]$	$[0, \pi]$

A few cancelation equations

$\arccos(\cos x) = x$ (As long as the domain is between $[0, \pi]$)

$\cos(\arccos x) = x$ (If domain is between $[-1, 1]$)

$f(x)=\arccos x$



***Remember arccosx does NOT equal $1/\cos x$**

Inverse Tan Function

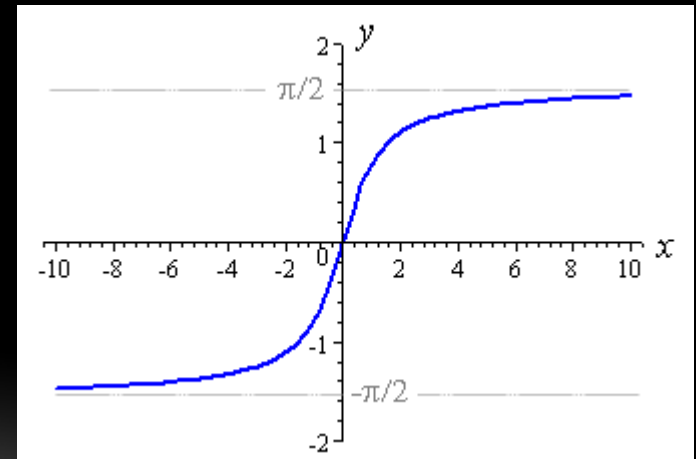
	Domain	Range
$f(x)=\tan x$	$[-\pi/2, \pi/2]$	$[-\infty, \infty]$
$f(x)=\arctan x$	$[-\infty, \infty]$	$[-\pi/2, \pi/2]$

A few cancelation equations

$\arctan(\tan x) = x$ (As long as the domain is between $[-\pi/2, \pi/2]$)

$\tan(\arctan x) = x$ (domain is all real numbers)

$f(x)=\arctan x$

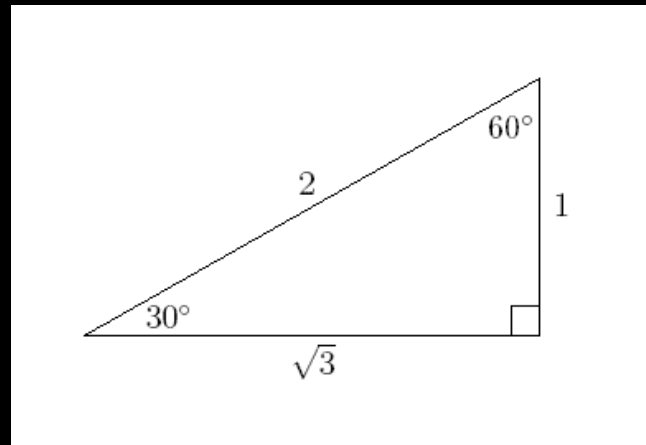


***Remember $\arctan x$ does NOT equal $1/\tan x$**

METHODS OF SOLVING

Lets look at a point on the unit circle, $(\frac{\sqrt{3}}{2}, 1/2)$

- Start by visualizing the object
(such as drawing a triangle)

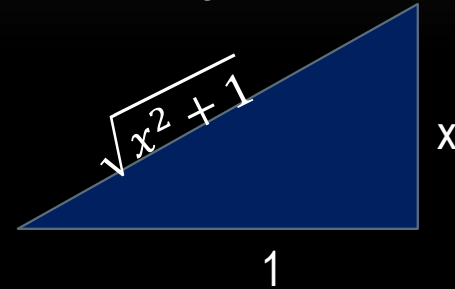


- Essentially, you are looking for an angle that y is equal to when $\cos(x) = \frac{\sqrt{3}}{2}$ or $\sin(x) = \frac{1}{2}$
- Solve $\arccos\frac{\sqrt{3}}{2} = \frac{\pi}{6}$ and likewise $\arcsin\frac{1}{2} = \frac{\pi}{6}$

METHODS OF SOLVING

Simplify the expression $\sin(\arctan x)$

- Essentially, you are looking for the sin of a number whose tangent is x
- Start by visualizing the object (such as drawing a triangle)



- We know that the legs of the triangle are x and 1 because of $\arctan x = \frac{x}{1}$
- Since the range of $\arctan x$ is $[-\pi/2, \pi/2]$, we only consider positive values of x
- The next step is finding the hypotenuse, which is $\sqrt{x^2 + 1}$ in this case
- Finally, we read off the sine ratio to get $\sin(\arctan x) = \frac{x}{\sqrt{x^2 + 1}}$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}(\sin^{-1} x) = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\csc^{-1} x) = \frac{-1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\cos^{-1} x) = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}(\sec^{-1} x) = \frac{1}{|x|\sqrt{x^2-1}}$$

$$\frac{d}{dx}(\tan^{-1} x) = \frac{1}{1+x^2}$$

$$\frac{d}{dx}(\cot^{-1} x) = \frac{-1}{1+x^2}$$

*It is necessary to memorize these formulas

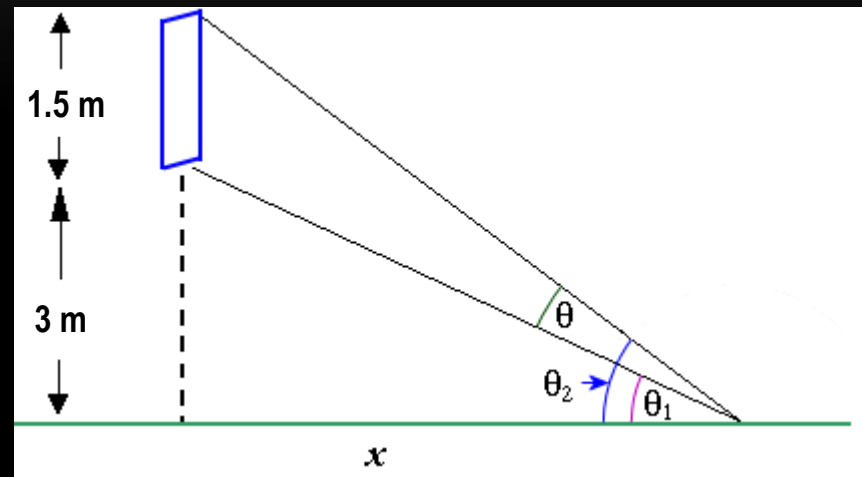
EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

Suppose that you are going at Buffalo Wild Wings to watch the Aggie game. The television is 1.5 meters tall and 3 meters above eye level. If the best view of the screen is obtained when the viewing angle is subtended at a maximum from eye level, find the distance horizontal distance from the screen?



EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

FIRST, DRAW A VISUAL

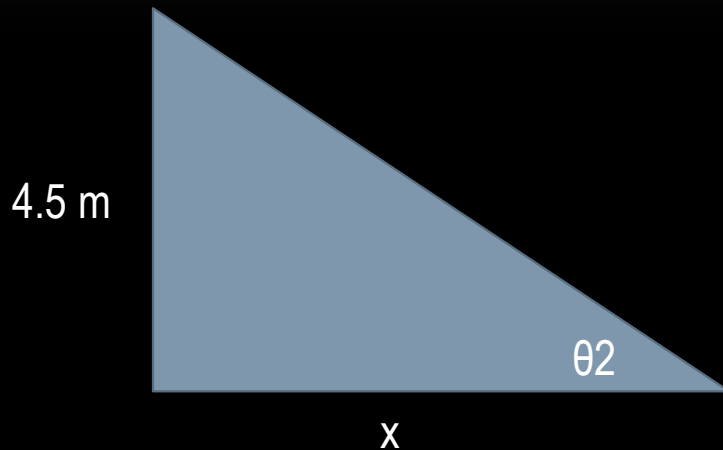


Some things to recognize before you begin:

- In order to maximize theta, we need to find $\frac{d(\theta)}{dx}$
- $\theta = \theta_2 - \theta_1$
- The derivative for the inverse tangent is $\frac{1}{1+x^2}$

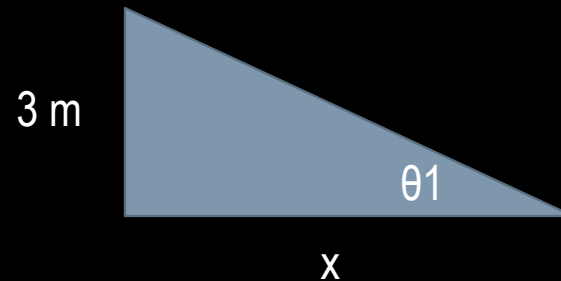
EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

Start by setting up triangles for θ_1 and θ_2 .



$$\tan(\theta_2) = 4.5/x$$

$$\text{Therefore, } \arctan(4.5/x) = \theta_2$$



$$\tan(\theta_1) = 3/x$$

$$\text{Therefore, } \arctan(3/x) = \theta_1$$

EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

Since $\theta = \theta_2 - \theta_1$, we can combine the previous equations into this one.

$$\theta(x) = \arctan(4.5/x) - \arctan(3/x)$$

The next step would be to take the derivative of this function with respect to theta. For simplicity, I am going to bring the x up to numerator to be x^{-1} . Keep in mind that there are functions inside of functions.

$$\theta'(x) = \frac{d}{dx}(\arctan(4.5x^{-1})) - \frac{d}{dx}(\arctan(3x^{-1}))$$

EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

*Remember that the inverse tangent function derivative is $1/(1+x^2)$.
The next step is writing out the derivative of the function.*

$$\theta'(x) = \frac{1}{(1+(4.5/x)^2)}(-4.5/x^2) - \frac{1}{(1+(3/x)^2)}(-3/x^2)$$

Now we can simplify the function by pulling out the $1/x^2$.

$$\theta'(x) = (1/x^2)((-4.5/1+(4.5/x)^2) + (3/1+(3/x)^2))$$

EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

Now we can multiply the x^2 into the denominators of the 2 fractions to cancel some of the terms out.

$$\theta'(x) = -10.9/(x^2+4.5^2) + 8.5/(x^2+3^2)$$

Adding the two fractions together gives us the following

$$\theta'(x) = (-1.5x^2 + 20.25)/((x^2+4.5^2)(x^2+8.5^2))$$

EXAMPLE: FINDING MAXIMUM VIEWING ANGLE

To find when this fraction equals 0, we only need to set the numerator equal to 0.

$$0 = -1.5x + 20.25$$

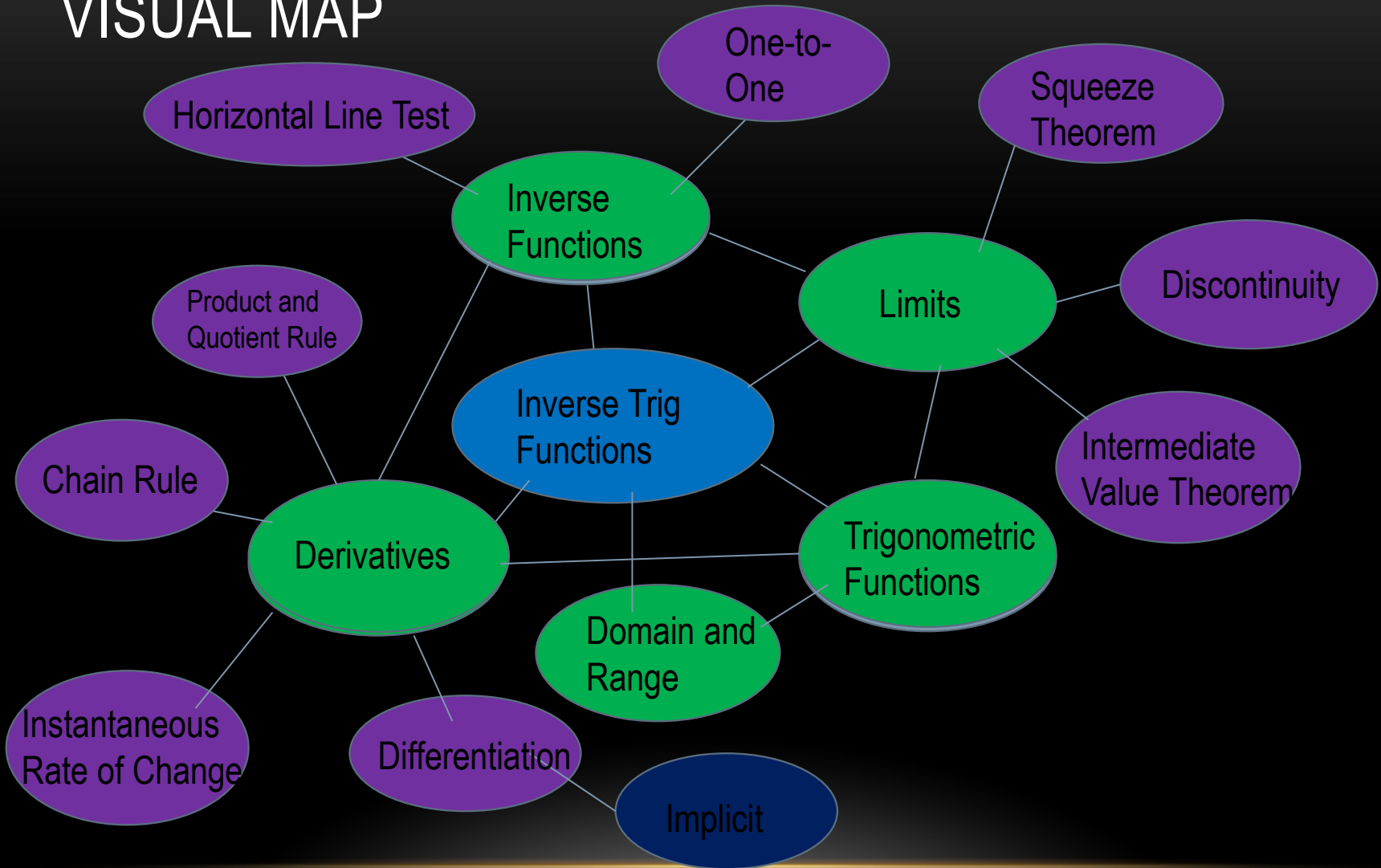
Solving for gives us the following:

$$x = 13.5$$

The distance from the screen that will give you the best viewing angle is 13.5 meters.



VISUAL MAP



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