FINANCIAL

MATHEMATICS

A Practical Guide for Actuaries

and other Business Professionals

Second Edition

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Contents

1 Interest Rates and Factors

Overview

- interest is the payment made by a borrower (i.e. the cost of doing business) for using a lender's capital assets (usually money); an example is a loan transaction
- interest rate is the percentage of interest to the capital asset in question
- interest takes into account the risk of default (risk that the borrower can't pay back the loan)
- the risk of default can be reduced if the borrower promises to release an asset of theirs in the event of their default (the asset is called collateral)

1.1 Interest

Interest on Savings Accounts

- a bank borrows a depositor's money and pays them interest for the use of their money
- the greater the need for money, the greater the interest rate offered

Interest Earned During the Period t to $t + s$: $AV_{t+s} - AV_t$

- the Accumulated Value at time n is denoted as AV_n
- interest earned during a period of time is the difference between the Accumulated Value at the end of the period and the Accumulated Value at the beginning of the period

The Effective Rate of Interest: i

- i is the amount of interest earned over a one-year period when 1 is invested
- i is also defined as the ratio of the amount of Interest Earned during the period to the Accumulated Value at the beginning of the period

$$i = \frac{AV_{t+1} - AV_t}{AV_t}$$

Interest on Loans

- compensation a borrower of capital pays to a lender of capital
- lender has to be compensated since they have temporarily lost use of their capital
- interest and capital are almost always expressed in terms of money

1.2 Simple Interest

- let the interest amount earned each year on an investment of X be constant where the annual rate of interest is i:

$$AV_t = X(1 + ti),$$

where $(1 + ti)$ is a linear function

- simple interest has the property that interest is **NOT** reinvested to earn additional interest

- amount of Interest Earned to time t is

$$I = AV_t - AV_0 = X(1 + it) - X = X \cdot it$$

1.3 Compound Interest

- let the interest amount earned each year on an investment of X also allow the interest earned to earn interest where the annual rate of interest is i:

$$AV_t = X(1 + i)^t,$$

where $(1 + i)^t$ is an exponential function

- compound interest produces larger accumulations than simple interest when $t > 1$

- note that a constant rate of compound interest implies a constant effective rate of interest

1.4 Accumulated Value

Accumulated Value Factor: AVF_t

- assume that AV_t is continuously increasing

- let X be the initial Principal invested $(X > 0)$ where $AV_0 = X$

- AV_t defines the Accumulated Value that amount X grows to in t years

- the Accumulated Value at time t is the product of the initial capital investment of X (Principal) made at time zero and the Accumulation Value Factor:

$$AV_t = X \cdot AVF_t,$$

where $AVF_t = (1 + it)$ if simple interest is being applied and $AVF_t = (1 + i)^t$ if compound interest is being applied

1.5 Present Value

Discounting

- Accumulated Value is a future value pertaining to payment(s) made in the past
- Discounted Value is a present value pertaining to payment(s) to be made in the future
- discounting determines how much must be invested initially (Z) so that X will be accumulated after t years

$$Z \cdot (1+i)^t = X \rightarrow \; Z = \frac{X}{(1+i)^t} = X(1+i)^{-t}$$

- Z represents the present value of X to be paid in t years
- let $v = \dfrac{1}{1+i}$, v is called a *discount factor* or *present value factor*

$$Z = X \cdot v^t$$

Discount Function (Present Value Factor): PVF_t

- simple interest: $PVF_t = \dfrac{1}{1+it}$
- compound interest: $PVF_t = \dfrac{1}{(1+i)^t} = v^t$
- compound interest produces smaller Discount Values than simple interest when $t > 1$

1.6 Rate of Discount: d

Definition

- an effective rate of interest is taken as a percentage of the balance at the beginning of the year, while an effective rate of discount is at the end of the year.
- eg. if 1 is invested and 6% interest is paid at the end of the year, then the Accumulated Value is 1.06
- eg. if 0.94 is invested after a 6% discount is paid at the beginning of the year, then the Accumulated Value at the end of the year is 1.00
- d is also defined as the ratio of the amount of interest (amount of discount) earned during the period to the amount invested at the end of the period

$$d = \frac{AV_{t+1} - AV_t}{AV_{t+1}}$$

- if interest is constant, then discount is constant
- the amount of discount earned from time t to $t+s$ is $AV_{t+s} - AV_t$

Relationships Between i and d

- if 1 is borrowed and interest is paid at the beginning of the year, then $1-d$ remains
- the accumulated value of $1-d$ at the end of the year is 1:

$$(1-d)(1+i) = 1$$

- interest rate is the ratio of the discount paid to the amount at the beginning of the period:

$$i = \frac{d}{1-d}$$

- discount rate is the ratio of the interest paid to the amount at the end of the period:

$$d = \frac{i}{1+i}$$

- the present value of end-of-year interest is the discount paid at the beginning of the year

$$iv = d$$

- the present value of 1 to be paid at the end of the year is the same as borrowing $1-d$ and repaying 1 at the end of the year (if both have the same value at the end of the year, then they have to have the same value at the beginning of the year)

$$1 \cdot v = 1 - d$$

- the difference between end-of-year, i, and beginning-of-year interest, d, depends on the difference that is borrowed at the beginning of the year and the interest earned on that difference

$$i - d = i[1 - (1-d)] = i \cdot d \geq 0$$

Discount Factors: PVF_t **and** AVF_t

– under the simple discount model the Discount Present Value Factor is:

$$PVF_t = 1 - dt \quad \text{for } 0 \leq t < 1/d$$

– under the simple discount model the Discount Accumulated Value Factor is:

$$AVF_t = (1 - dt)^{-1} \quad \text{for } 0 \leq t < 1/d$$

– under the compound discount model, the Discount Present Value Factor is:

$$PVF_t = (1 - d)^t = v^t \quad \text{for } t \geq 0$$

– under the compound discount model, the Discount Accumulated Value Factor is:

$$AVF_t = (1 - d)^{-t} \quad \text{for } t \geq 0$$

– a constant rate of simple discount implies an increasing effective rate of discount

– a constant rate of compound discount implies a constant effective rate of discount

1.7 Constant Force of Interest: δ

Definitions

– annual effective rate of interest is applied over a one-year period

– a constant annual force of interest can be applied over the smallest sub-period imaginable (at a moment in time) and is denoted as δ

– an instantaneous change at time t, due to interest rate δ, where the accumulated value at time t is X, can be defined as follows:

$$\begin{aligned}
\delta &= \frac{\frac{d}{dt}AV_t}{AV_t} \\
&= \frac{d}{dt}ln(AV_t) \\
&= \frac{\frac{d}{dt}X(1+i)^t}{X(1+i)^t} \\
&= \frac{(1+i)^t \cdot ln(1+i)}{(1+i)^t} \\
\delta &= ln(1+i)
\end{aligned}$$

– taking the exponential function of δ results in

$$e^{\delta} = 1+i$$

– taking the inverse of the above formula results in

$$e^{-\delta} = \frac{1}{1+i} = v$$

– Accumulated Value Factor (AVF_t) using constant force of interest is

$$AVF_t = e^{\delta t}$$

– Present Value Factor (PVF_t) using constant force of interest is

$$PVF_t = e^{-\delta t}$$

1.8 Varying Force of Interest

- let the constant force of interest δ now vary at each infitesimal point in time and be denoted as δ_t
- a change from time t_1 to t_2, due to interest rate δ_t, where the accumulated value at time t_1 is X, can be defined as follows:

$$\begin{aligned}
\delta_t &= \frac{\frac{d}{dt}AV_t}{AV_t} \\
&= \frac{d}{dt}ln(AV_t) \\
\int_{t_1}^{t_2} \delta_t \cdot dt &= \int_{t_1}^{t_2} \frac{d}{dt}ln(AV_t) \cdot dt \\
&= ln(AV_{t_2}) - ln(AV_{t_1}) \\
\int_{t_1}^{t_2} \delta_t \cdot dt &= ln\left(\frac{AV_{t_2}}{AV_{t_1}}\right) \\
e^{\int_{t_1}^{t_2} \delta_t \cdot dt} &= \frac{AV_{t_2}}{AV_{t_1}}
\end{aligned}$$

Varying Force of Interest Accumulation Factor - AVF_{t_1,t_2}

- let

$$AVF_{t_1,t_2} = e^{\int_{t_1}^{t_2} \delta_t \cdot dt}$$

represent an accumulation factor over the period t_1 to t_2, where the force of interest is varying

- if $t_1 = 0$, then the notation simplifies from AVF_{0,t_2} to AVF_t i.e. $AVF_t = e^{\int_0^t \delta_t \cdot dt}$
- if δ_t is readily integrable, then AVF_{t_1,t_2} can be derived easily
- if δ_t is not readily integrable, then approximate methods of integration are required

Varying Force of Interest Present Value Factor - PVF_{t_1,t_2}

- let

$$PVF_{t_1,t_2} = \frac{1}{AVF_{t_1,t_2}} = \frac{1}{e^{\int_{t_1}^{t_2} \delta_t \cdot dt}} = \frac{AV_{t_1}}{AV_{t_2}} = e^{-\int_{t_1}^{t_2} \delta_t \cdot dt}$$

represent a present value factor over the period t_1 to t_2, where the force of interest is varying

- if $t_1 = 0$, then the notation simplifies from PVF_{0,t_2} to PVF_t i.e. $PVF_t = e^{-\int_0^t \delta_t \cdot dt}$

1.9 Discrete Changes in Interest Rates

– the most common application of the accumulation and present value factors over a period of t years is

$$AVF_t = \prod_{k=1}^{t} (1 + i_k)$$

and

$$PFV_t = \prod_{k=1}^{t} \frac{1}{(1 + i_k)}$$

where i_k is the constant rate of interest between time $k - 1$ and time k

Exercises and Solutions

1.2 Simple Interest

Exercise (a)

At what rate of simple interest will 500 accumulate to 615 in 2.5 years?

Solution (a)

$$500[1 + i(2.5)] = 615$$

$$i = \frac{\frac{615}{500} - 1}{2.5} = 9.2\%$$

Exercise (b)

In how many years will 500 accumulate to 630 at 7.8% simple interest?

Solution (b)

$$500[1 + .078(n)] = 630$$

$$i = \frac{\frac{630}{500} - 1}{.078} = 3.33 \text{ years}$$

Exercise (c)

At a certain rate of simple interest 1,000 will accumulate to 1,100 after a certain period of time. Find the accumulated value of 500 at a rate of simple interest three fourths as great over twice as long a period of time.

Solution (c)

$$1,000[1 + i \cdot n] = 1,100 \rightarrow i \cdot n = .11$$

$$500[1 + \frac{3}{4} i \cdot 2n] = 500[1 + (1.5)(.11)] = 582.50$$

Exercise (d)

Simple interest of $i = 4\%$ is being credited to a fund. In which period is this equivalent to an effective rate of 2.5%?

Solution (d)

$$i_n = \frac{i}{1 + i(n-1)}$$

$$0.25 = \frac{.04}{1 + .04(n-1)} \rightarrow n = 16$$

1.3 Compound Interest

Exercise (a)

Fund A is invested at an effective annual interest rate of 3%. Fund B is invested at an effective annual interest rate of 2.5%. At the end of 20 years, the total in the two funds is 10,000. At the end of 31 years, the amount in Fund A is twice the amount in Fund B. Calculate the total in the two funds at the end of 10 years.

Solution (a)

Let the initial funds be A and B. Therefore, we have two equations and two unknowns:

$$A(1.03)^{20} + B(1.025)^{20} = 10,000$$

$$A(1.03)^{31} = 2B(1.025)^{31}$$

Solving for B in the second equation and plugging it into the first equation gives us $A = 3,624.73$ and $B = 2,107.46$. We seek $A(1.03)^{10} + B(1.025)^{10}$ which equals

$$3,624.73(1.03)^{10} + 2,107.46(1.025)^{10} = 7,569.07$$

Exercise (b)

Carl puts 10,000 into a bank account that pays an annual effective interest rate of 4% for ten years. If a withdrawal is made during the first five and one-half years, a penalty of 5% of the withdrawal amount is made. Carl withdrawals K at the end of each of years 4, 5, 6, 7. The balance in the account at the end of year 10 is 10,000. Calculate K.

Solution (b)

$$10,000(1.04)^{10} - 1.05K(1.04)^{6} - 1.05K(1.04)^{5} - K(1.04)^{4} - K(1.04)^{3} = 10,000$$

$$14,802 - K[(1.05)(1.04)^{6} + (1.05)(1.04)^{5} + (1.04)^{4} + (1.04)^{3}] = 10,000$$

$$4,802 = K \cdot 4.9 \rightarrow K = 980$$

1.4 Accumulated Value

Exercise (a)

100 is deposited into an account at the beginning of every 4-year period for 40 years.

The account credits interest at an annual effective rate of i.

The accumulated value in the account at the end of 40 years is X, which is 5 times the accumulated amount at the end of 20 years. Calculate X.

Solution (a)

$$100(1+i)^4+100(1+i)^8+...+100(1+i)^{40} = X = 5[100(1+i)^4+100(1+i)^8+...+100(1+i)^{20}]$$

$$100(1+i)^4[1+100(1+i)^4+...+100(1+i)^{36}] = 5\cdot 100(1+i)^4[1+100(1+i)^4+...+100(1+i)^{16}]$$

$$100(1+i)^4\left[\frac{1-[(1+i)^4]^{10}}{1-(1+i)^4}\right] = 5\cdot 100(1+i)^4\left[\frac{1-[(1+i)^4]^5}{1-(1+i)^4}\right]$$

$$1-(1+i)^{40} = 5[1-(1+i)^{20}]$$

$$(1+i)^{40}-5(1+i)^{20}+4=0 \rightarrow [(1+i)^{20}-1][(1+i)^{20}-4]=0 \rightarrow (1+i)^{20}=1 \text{ or } (1+i)^{20}=4$$

$$(1+i)^{20}=1 \rightarrow i=0\% \rightarrow AV_{40}=1000 \text{ and } AV_{20}=500, \text{ impossible since } AV_{40}=5AV_{20}$$

therefore, $(1+i)^{20}=4$

$$X = 100(1+i)^4\left[\frac{1-(1+i)^{40}}{1-(1+i)^4}\right] = 100(4^{\frac{1}{5}})\left[\frac{1-4^2}{1-4^{\frac{1}{5}}}\right] = 100(61.9472) = 6194.72$$

1.5 Present Value

Exercise (a)

Annual payments are made at the end of each year, forever. The payments at time n is defined as n^3 for the first n years. After year n, the payments remain constant at n^2. The present value of these payments at time 0 is $20n^2$ when the annual effective rate of interest is 0% for the first n years and 25% thereafter. Calculate n.

Solution (a)

$$[(1^3)v_{0\%}+(2^3)v_{0\%}^2+(3^3)v_{0\%}^3+...+(n^3)v_{0\%}^n]+v_{0\%}^n[(n^2)v_{25\%}^1+(n^2)v_{25\%}^2+(n^2)v_{25\%}^3+...]=20n^2$$

$$[(1^3)+(2^3)+(3^3)+...+(n^3)]+(n^2)v_{25\%}[1+v_{25\%}^1+v_{25\%}^2+...]=20n^2$$

$$\left[\frac{n^2(n+1)^2}{4}\right]+(n^2)v_{25\%}\left[\frac{1}{1-v_{25\%}}\right]=20n^2$$

$$\left[\frac{(n+1)^2}{4}\right]+(.8)\left[\frac{1}{1-.8}\right]=20$$

$$\left[\frac{(n+1)^2}{4}\right]+4=20\rightarrow\frac{(n+1)^2}{4}=16\rightarrow\frac{(n+1)}{2}=4\rightarrow n=7$$

Exercise (b)

At an effective annual interest rate of i, $i>0$, each of the following two sets of payments has present value K:

(i) A payment of 121 immediately and another payment of 121 at the end of one year.

(ii) A payment of 144 at the end of two years and another payment of 144 at the end of three years.

Calculate K.

Solution (b)

$$121+121v=144v^2+144v^3=K$$

$$121(1+v)=144v^2(1+v)\rightarrow v=\frac{11}{12}$$

$$K=121(1+\frac{11}{12})\rightarrow K=231.92$$

Exercise (c)

The present value of a series of payments of 2 at the end of every eight years, forever, is equal to 5. Calculate the effective rate of interest.

Solution (c)

$$2v^8 + 2v^{16} + 2v^{24} + ... = 5$$

$$2v^8[1 + v^8 + v^{16} + ... = 5$$

$$2v^8 \left[\frac{1}{1 - v^8}\right] = 5$$

$$2v^8 = 5 - 5v^8$$

$$7v^8 = 5$$

$$v^8 = \frac{5}{7}$$

$$(1 + i)^8 = \frac{7}{5}$$

$$1 + i = \left(\frac{7}{5}\right)^{\frac{1}{8}} \to i = .04296$$

1.6 Rate of Discount

Exercise (a)

A business permits its customers to pay with a credit card or to receive a percentage discount of r for paying cash.

For credit card purchases, the business receives 97% of the purchase price one-half month later.

At an annual effective rate of discount of 22%, the two payments are equivalent. Find r.

Solution (a)

$$\$1(1 - r) = \$0.97v^{\frac{1}{12} * \frac{1}{2}} = \$0.97v^{\frac{1}{24}}$$

$$(1 - r) = .97(1 - d)^{\frac{1}{24}} = .97(1 - .22)^{\frac{1}{24}} = .96$$

$$r = .04 = 4\%$$

Exercise (b)

You deposit 1,000 today and another 2,000 in five years into a fund that pays simple discounting at 5% per year.

Your friend makes the same deposits into another fund, but at time n and $2n$, respectively. This fund credits interest at an annual effective rate of 10%.

At the end of 10 years, the accumulated value of your deposits is exactly the same as the accumulated value of your friend's deposits.

Calculate n.

Solution (b)

$$1,000[1-5\%(10)]^{-1}+2,000[1-5\%(5)]^{-1}=1,000(1.10)^{10-n}+2,000(1.10)^{10-2n}$$

$$\frac{1,000}{.5}+\frac{2,000}{.75}=1,000(1.10)^{10}v^{n}+2,000(1.10)^{10}v^{2n}$$

$$4,666.67=2,593.74v^{n}+5,187.48v^{2n}\rightarrow \underbrace{5,187.48}_{a}v^{2n}+\underbrace{2,593.74}_{b}v^{n}\underbrace{-4,666.67}_{c}=0$$

$$v^{n}=\frac{-2,593.74+\sqrt{2,593.74^{2}-4(5,187.48)(-4,666.67)}}{2(5,187.48)}=.7306$$

$$(1.10)^{n}=\frac{1}{.7306}=1.36824\rightarrow n=\frac{ln(1.36824)}{ln(1.10)}=3.29$$

Exercise (c)

A deposit of X is made into a fund which pays an annual effective interest rate of 6% for 10 years.

At the same time , X/2 is deposited into another fund which pays an annual effective rate of discount of d for 10 years.

The amounts of interest earned over the 10 years are equal for both funds.

Calculate d.

Solution (c)

$$X(1.06)^{10}-X=\frac{X}{2}(1-d)^{-10}-\frac{X}{2}$$

$$d=0.0905$$

1.7 Constant Force of Interest

Exercise (a)

You are given that $AV_t = Kt^2 + Lt + M$, for $0 \le t \le 2$, and that $AV_0 = 100$, $AV_1 = 110$, and $AV_2 = 136$. Determine the force of interest at time $t = \frac{1}{2}$.

Solution (a)

$$AV_0 = M = 100$$

$$AV_1 = K + L + M = 110 \to K + L = 10$$

$$AV_2 = 4K + 2L + M = 136 \to 4K + 2L = 36$$

These equations solve for $K = 8$, $L = 2$, $M = 100$.

We know that

$$\delta_t = \frac{\frac{d}{dt}AV_t}{AV_t} = \frac{2Kt + L}{Kt^2 + Lt + M}$$

$$\delta_{\frac{1}{2}} = \frac{(2)(8)(\frac{1}{2}) + 2}{(8)(\frac{1}{2})^2 + (2)(\frac{1}{2}) + 100} = \frac{10}{103} = 0.09709$$

Exercise (b)

Fund A accumulates at a simple interest rate of 10%. Fund B accumulates at a simple discount rate of 5%. Find the point in time at which the forces of interest on the two funds are equal.

Solution (b)

$$AVF_t^A = 1 + .10t \text{ and } AVF_t^B = (1 + .05t)^{-1}$$

$$\delta_t^A = \frac{\frac{d}{dt}AVF_t^A}{AVF_t^A} = \frac{.10}{1 + .10t}$$

$$\delta_t^B = \frac{\frac{d}{dt}AVF_t^B}{AVF_t^B} = \frac{.05(1 - .05t)^{-2}}{1 - .05t)^{-1}} = .05(1 - .05t)^{-1}$$

Equating and solve for t

$$\frac{.10}{1 + .10t} = \frac{.05}{1 - .05t} \to .10 - .005t = .05 + .005t \to t = 5$$

1.8 Varying Force of Interest

Exercise (a)

On 15 March 2003, a student deposits X into a bank account. The account is credited with simple interest where i=7.5%

On the same date, the student's professor deposits X into a different bank account where interest is credited at a force of interest

$$\delta_t = \frac{2t}{t^2+k}, t \geq 0.$$

From the end of the fourth year until the end of the eighth year, both accounts earn the same dollar amount of interest.

Calculate k.

Solution (a)

Simple Interest Earned $= AV_8 - AV_4 = X[1+.075(8)] - X[1+.075(4)] = X(0.075)(4) = X(.3)$

$$\text{Compound Interest Earned} = AV_8 - AV_4 = Xe^{\int_0^8 \delta_t \cdot dt} - Xe^{\int_0^4 \delta_t \cdot dt}$$

$$Xe^{\int_0^8 \frac{2t}{t^2+k} \cdot dt} - Xe^{\int_0^4 \frac{2t}{t^2+k} \cdot dt} = Xe^{\int_0^8 \frac{f'(t)}{f(t)} \cdot dt} - Xe^{\int_0^4 \frac{f'(t)}{f(t)} \cdot dt} = X\frac{f(8)}{f(0)} - X\frac{f(4)}{f(0)}$$

$$X\frac{(8)^2+k}{(0)^2+k} - X\frac{(4)^2+k}{(0)^2+k} = X\frac{48}{k}, \text{compound interest}$$

$$X(.3) = X\frac{48}{k} \rightarrow .3 = \frac{48}{k} \rightarrow .30k = 48 \rightarrow k = 160$$

Exercise (b)

Payments are made to an account at a continuous rate of , $k^2+8tk^2+(tk)^2$, where $0 \leq t \leq 10$ and $k > 0$.

Interest is credited at a force of interest where

$$\delta_t = \frac{8+2t}{1+8t+t^2}$$

After 10 years, the account is worth $88,690$.

Calculate k.

Solution (b)

$$\int_0^{10} [k^2 + 8tk^2 + (tk)^2](1+i)^{10-t} dt = 88,690$$

$$(1+i)^{10-t} = e^{\int_t^{10} \delta_s \cdot ds} = e^{\int_t^{10} \frac{8+2s}{1+8s+s^2} \cdot ds} = e^{\int_t^{10} \frac{f'(s)}{f(s)} \cdot ds} = \frac{f(10)}{f(t)}$$

$$= \frac{1+8(10)+(10)^2}{1+8t+t^2} = \frac{181}{1+8t+t^2}$$

$$\int_0^{10} [k^2 + 8tk^2 + (tk)^2] \frac{181}{1+8t+t^2} dt = 88,690$$

$$k^2 \int_0^{10} [1+8t+t^2] \frac{181}{1+8t+t^2} dt = 88,690 \rightarrow k^2(181)(10) = 88,690 \rightarrow k^2 = 49 \rightarrow k = 7$$

Exercise (c)

Fund A accumulates at a constant force of interest of $\delta_t^A = \dfrac{.05}{1+.05t}$ at time t, for $t \geq 0$, and Fund B accumulates at a constant force of interest of $\delta^B = 5\%$. You are given:

(i) The amount in Fund A at time zero is 1,000.

(ii) The amount in Fund B at time zero is 500.

(iii) The amount in Fund C at any time t, $t \geq 0$, is equal to the sum of the amounts in Fund A and Fund B. Fund C accumulates at a force of interest of δ_t^C, for $t0$.

Calculate δ_2^C.

Solution (c)

$$AV_t^C = AV_t^A + AV_t^B = 1,000e^{\displaystyle\int_0^t \delta_s dt} + 500e^{.05t}$$

$$AV_t^C = 1,000e^{\displaystyle\int_0^t \frac{.05}{1+.05s} \cdot ds} + 500e^{.05t} = 1,000\left[\frac{f(t)}{f(0)}\right] + 500e^{.05t}$$

$$AV_t^C = 1,000\left[\frac{1+.05(t)}{1+.05(0)}\right] + 500(.05)e^{.05t} = 1000[1+.05t] + 500e^{.05t}$$

$$\frac{d}{dt}AV_t^C = 1,000(.05) + 500(.05)e^{.05t} = 50 + 25^{.05t}$$

$$\delta_t = \frac{\frac{d}{dt}AV_t^C}{AV_t^C} = \frac{50+25e^{.05t}}{1000[1+.05t]+500e^{.05t}}$$

$$\delta_2 = \frac{50+25e^{.05(2)}}{1,000[1+.05(2)]+500e^{.05(2)}} = 4.697\%$$

Exercise (d)

Fund F accumulates at the rate $\delta_t = \dfrac{1}{1+t}$. Fund G accumulates at the rate $\delta_t = \dfrac{4t}{1+2t^2}$. $F(t)$ is the amount in Fund F at time t, and $G(t)$ is the amount in fund G at time t, with $F(0) = G(0)$. Let $H(t) = F(t) - G(t)$. Calculate T, the value of time t when $H(t)$ is a maximum.

Solution (d)

Since $F(0) = G(0)$, we can assume an initial deposit of 1. Then we have,

$$F(t) = e^{\displaystyle\int_0^t \frac{1}{1+r} \cdot dr} = e^{ln(1+t)-ln(1)} = 1+t$$

$$G(t) = e^{\displaystyle\int_0^t \frac{4r}{1+2r^2} dr} = e^{ln(1+2t^2)-ln(1)} = 1+2t^2$$

$$H(t) = t - 2t^2 \text{ and } \frac{d}{dt}H(t) = 1 - 4t = 0 \rightarrow t = \frac{1}{4}$$

2 Level Annuities

Overview

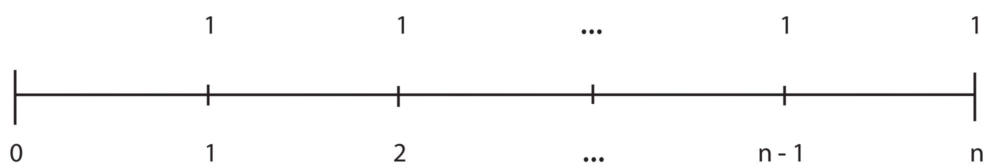
Definition of An Annuity

- a series of payments made at equal intervals of time (annually or otherwise)
- payments made for certain over a fixed period of time are called an annuity-certain
- payments made for an uncertain period of time are called a contingent annuity
- the payment frequency and the interest conversion period are equal
- the payments are level

2.1 Annuity-Immediate

Definition

- payments of 1 are made at the end of every year for n years



Annuity-Immediate Present Value Factor

- the present value (at $t = 0$) of an annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a_{\overline{n|}i}$ and is calculated as follows:

$$
\begin{aligned}
a_{\overline{n|}i} &= (1)v + (1)v^2 + \cdots + (1)v^{n-1} + (1)v^n \\
&= v(1 + v + v^2 + \cdots + v^{n-2} + v^{n-1}) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^n}{1-v}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^n}{d}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^n}{\frac{i}{1+i}}\right) \\
&= \frac{1-v^n}{i}
\end{aligned}
$$

Annuity-Immediate Accumulated Value Factor

– the accumulated value (at $t = n$) of an annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $s_{\overline{n}|_i}$ and is calculated as follows:

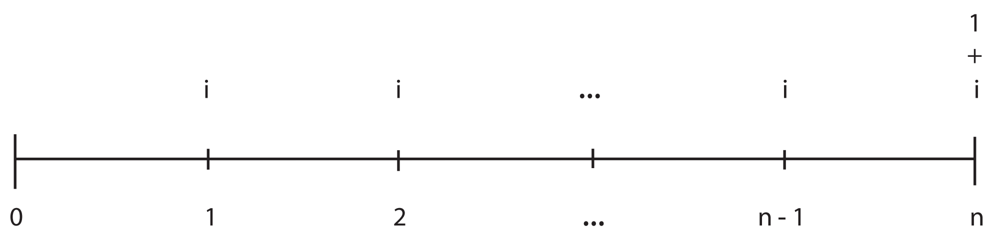
$$\begin{aligned} s_{\overline{n}|_i} &= 1 + (1)(1+i) + \cdots + (1)(1+i)^{n-2} + (1)(1+i)^{n-1} \\ &= \frac{1-(1+i)^n}{1-(1+i)} \\ &= \frac{1-(1+i)^n}{-i} \\ &= \frac{(1+i)^n - 1}{i} \end{aligned}$$

Basic Relationship $1 : 1 = i \cdot a_{\overline{n}|} + v^n$

Consider an n–year investment where 1 is invested at time 0.

The present value of this single payment income stream at $t = 0$ is 1.

Alternatively, consider a n–year investment where 1 is invested at time 0 and produces annual interest payments of $(1) \cdot i$ at the end of each year and then the 1 is refunded at $t = n$.



The present value of this multiple payment income stream at $t = 0$ is $i \cdot a_{\overline{n}|} + (1)v^n$.

Therefore, the present value of both investment opportunities are equal.

Also note that $a_{\overline{n}|} = \dfrac{1-v^n}{i} \rightarrow 1 = i \cdot a_{\overline{n}|} + v^n$.

Basic Relationship $2 : PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

– if the future value at time n, $s_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $a_{\overline{n}|}$

$$\begin{aligned} s_{\overline{n}|} \cdot v^n &= \left[\frac{(1+i)^n - 1}{i}\right] \cdot v^n \\ &= \frac{(1+i)^n \cdot v^n - v^n}{i} \\ &= \frac{1 - v^n}{i} \\ &= a_{\overline{n}|} \end{aligned}$$

– if the present value at time 0, $a_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $s_{\overline{n}|}$

$$\begin{aligned} a_{\overline{n}|} \cdot (1+i)^n &= \left[\frac{1 - v^n}{i}\right](1+i)^n \\ &= \frac{(1+i)^n - v^n(1+i)^n}{i} \\ &= \frac{(1+i)^n - 1}{i} \\ &= s_{\overline{n}|} \end{aligned}$$

Basic Relationship $3: \dfrac{1}{a_{\overline{n}|}} = \dfrac{1}{s_{\overline{n}|}} + i$

Consider a loan of 1, to be paid back over n years with equal annual payments of P made at the end of each year. An annual effective rate of interest, i, is used. The present value of this single payment loan must be equal to the present value of the multiple payment income stream.

$$P \cdot a_{\overline{n}|i} = 1$$

$$P = \frac{1}{a_{\overline{n}|i}}$$

Alternatively, consider a loan of 1, where the annual interest due on the loan, $(1)i$, is paid at the end of each year for n years and the loan amount is paid back at time n.

In order to produce the loan amount at time n, annual payments at the end of each year, for n years, will be made into an account that credits interest at an annual effective rate of interest i.

The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$D \cdot s_{\overline{n}|i} = 1$$

$$D = \frac{1}{s_{\overline{n}|i}}$$

The total annual payment will be the interest payment and account payment:

$$i + \frac{1}{s_{\overline{n}|i}}$$

Therefore, a level annual annuity payment on a loan is the same as making an annual interest payment each year plus making annual deposits in order to save for the loan repayment.

Also note that

$$\frac{1}{a_{\overline{n}|i}} = \frac{i}{1 - v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{i(1+i)^n}{(1+i)^n - 1}$$

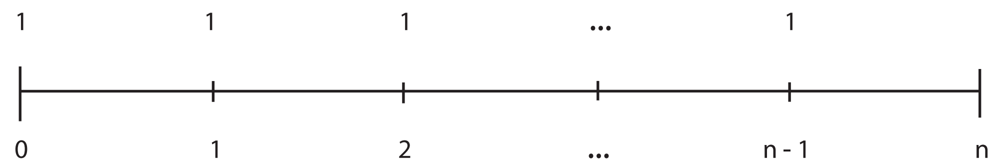
$$= \frac{i(1+i)^n + i - i}{(1+i)^n - 1} = \frac{i[(1+i)^n - 1] + i}{(1+i)^n - 1}$$

$$= i + \frac{i}{(1+i)^n - 1} = i + \frac{1}{s_{\overline{n}|}}$$

2.2 Annuity–Due

Definition

– payments of 1 are made at the beginning of every year for n years



Annuity-Due Present Value Factor

– the present value (at $t = 0$) of an annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
\ddot{a}_{\overline{n}|i} &= 1 + (1)v + (1)v^2 + \cdots + (1)v^{n-2} + (1)v^{n-1} \\
&= \frac{1 - v^n}{1 - v} \\
&= \frac{1 - v^n}{d}
\end{aligned}$$

Annuity-Due Accumulated Value Factor

– the accumulated value (at $t = n$) of an annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{s}_{\overline{n}|i}$ and is calculated as follows:

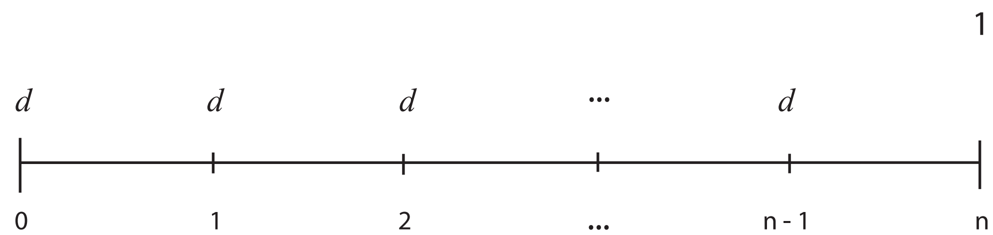
$$\begin{aligned}
\ddot{s}_{\overline{n}|i} &= (1)(1+i) + (1)(1+i)^2 + \cdots + (1)(1+i)^{n-1} + (1)(1+i)^n \\
&= (1+i)[1 + (1+i) + \cdots + (1+i)^{n-2} + (1+i)^{n-1}] \\
&= (1+i)\left[\frac{1 - (1+i)^n}{1 - (1+i)}\right] \\
&= (1+i)\left[\frac{1 - (1+i)^n}{-i}\right] \\
&= (1+i)\left[\frac{(1+i)^n - 1}{i}\right] \\
&= \frac{(1+i)^n - 1}{d}
\end{aligned}$$

Basic Relationship $1 : 1 = d \cdot \ddot{a}_{\overline{n}|} + v^n$

Consider an n–year investment where 1 is invested at time 0.

The present value of this single payment income stream at $t = 0$ is 1.

Alternatively, consider a n–year investment where 1 is invested at time 0 and produces annual interest payments of $(1)\cdot d$ at the beginning of each year and then have the 1 refunded at $t = n$.



The present value of this multiple payment income stream at $t = 0$ is $d \cdot \ddot{a}_{\overline{n}|} + (1)v^n$.

Therefore, the present value of both investment opportunities are equal.

Also note that $\ddot{a}_{\overline{n}|} = \dfrac{1 - v^n}{d} \rightarrow 1 = d \cdot \ddot{a}_{\overline{n}|} + v^n$.

Basic Relationship $2 : PV(1 + i)^n = FV$ **and** $PV = FV \cdot v^n$

– if the future value at time n, $\ddot{s}_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $\ddot{a}_{\overline{n}|}$

$$\ddot{s}_{\overline{n}|} \cdot v^n = \left[\frac{(1+i)^n - 1}{d}\right] \cdot v^n = \frac{(1+i)^n \cdot v^n - v^n}{d} = \frac{1 - v^n}{d} = \ddot{a}_{\overline{n}|}$$

– if the present value at time 0, $\ddot{a}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\ddot{s}_{\overline{n}|}$

$$\ddot{a}_{\overline{n}|} \cdot (1+i)^n = \left[\frac{1 - v^n}{d}\right](1+i)^n = \frac{(1+i)^n - v^n(1+i)^n}{d} = \frac{(1+i)^n - 1}{d} = \ddot{s}_{\overline{n}|}$$

Basic Relationship 3 : $\frac{1}{\ddot{a}_{\overline{n}|}} = \frac{1}{\ddot{s}_{\overline{n}|}} + d$

Consider a loan of 1, to be paid back over n years with equal annual payments of P made at the beginning of each year. An annual effective rate of interest, i, is used. The present value of the single payment loan must be equal to the present value of the multiple payment stream.

$$P \cdot \ddot{a}_{\overline{n}|i} = 1$$
$$P = \frac{1}{\ddot{a}_{\overline{n}|i}}$$

Alternatively, consider a loan of 1, where the annual interest due on the loan, $(1) \cdot d$, is paid at the beginning of each year for n years and the loan amount is paid back at time n.

In order to produce the loan amount at time n, annual payments at the beginning of each year, for n years, will be made into an account that credits interest at an annual effective rate of interest i.

The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$D \cdot \ddot{s}_{\overline{n}|i} = 1$$
$$D = \frac{1}{\ddot{s}_{\overline{n}|i}}$$

The total annual payment will be the interest payment and account payment:

$$d + \frac{1}{\ddot{s}_{\overline{n}|i}}$$

Therefore, a level annual annuity payment is the same as making an annual interest payment each year and making annual deposits in order to save for the loan repayment.

Also note that

$$\begin{aligned} \frac{1}{\ddot{a}_{\overline{n}|i}} &= \frac{d}{1-v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{d(1+i)^n}{(1+i)^n - 1} \\ &= \frac{d(1+i)^n + d - d}{(1+i)^n - 1} = \frac{d[(1+i)^n - 1] + d}{(1+i)^n - 1} \\ &= d + \frac{d}{(1+i)^n - 1} = d + \frac{1}{\ddot{s}_{\overline{n}|}} \end{aligned}$$

Basic Relationship 4**: Annuity Due= Annuity Immediate** $\times(1+i)$

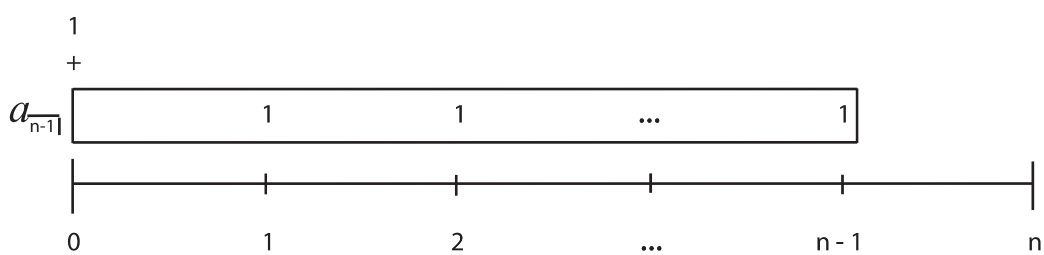
$$\ddot{a}_{\overline{n}|} = \frac{1-v^n}{d} = \frac{1-v^n}{i} \cdot (1+i) = a_{\overline{n}|} \cdot (1+i)$$
$$\ddot{s}_{\overline{n}|} = \frac{(1+i)^n - 1}{d} = \left[\frac{(1+i)^n - 1}{i}\right] \cdot (1+i) = s_{\overline{n}|} \cdot (1+i)$$

An annuity–due starts one period earlier than an annuity-immediate and as a result, earns one more period of interest, hence it will be larger.

Basic Relationship $5 : \ddot{a}_{\overline{n}|} = 1 + a_{\overline{n-1}|}$

$$
\begin{aligned}
\ddot{a}_{\overline{n}|} &= 1 + [v + v^2 + \cdots + v^{n-2} + v^{n-1}] \\
&= 1 + v[1 + v + \cdots + v^{n-3} + v^{n-2}] \\
&= 1 + v\left(\frac{1 - v^{n-1}}{1 - v}\right) \\
&= 1 + \left(\frac{1}{1+i}\right)\left(\frac{1 - v^{n-1}}{d}\right) \\
&= 1 + \left(\frac{1}{1+i}\right)\left(\frac{1 - v^{n-1}}{i/1+i}\right) \\
&= 1 + \frac{1 - v^{n-1}}{i} \\
&= 1 + a_{\overline{n-1}|}
\end{aligned}
$$

This relationship can be visualized with a time line diagram.

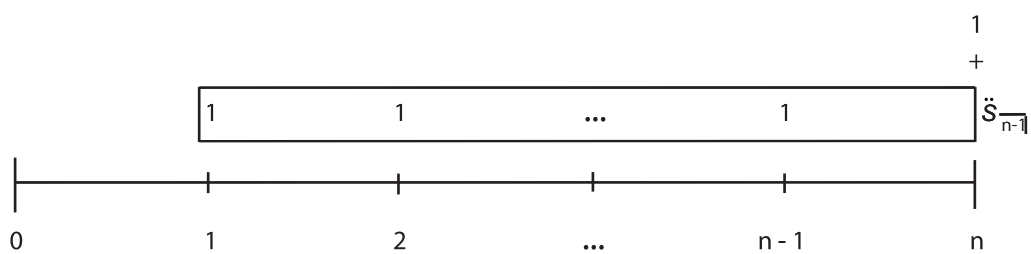


An additional payment of 1 at time 0 results in $a_{\overline{n-1}|}$ becoming n payments that now commence at the beginning of each year which is $\ddot{a}_{\overline{n}|}$.

Basic Relationship $6 : s_{\overline{n}|} = 1 + \ddot{s}_{\overline{n-1}|}$

$$
\begin{aligned}
s_{\overline{n}|} &= 1 + [(1+i) + (1+i)^2 + \cdots + (1+i)^{n-2} + (1+i)^{n-1}] \\
&= 1 + (1+i)[1 + (1+i) + \cdots + (1+i)^{n-3} + (1+i)^{n-2}] \\
&= 1 + (1+i)\left[\frac{1-(1+i)^{n-1}}{1-(1+i)}\right] \\
&= 1 + (1+i)\left[\frac{1-(1+i)^{n-1}}{-i}\right] \\
&= 1 + (1+i)\left[\frac{(1+i)^{n-1}-1}{i}\right] \\
&= 1 + \frac{(1+i)^{n-1}-1}{d} \\
&= 1 + \ddot{s}_{\overline{n-1}|}
\end{aligned}
$$

This relationship can also the visualized with a time line diagram.



An additional payment of 1 at time n results in $\ddot{s}_{\overline{n-1}|}$ becoming n payments that now commerce at the end of each year which is $s_{\overline{n}|}$

2.3 Deferred Annuities

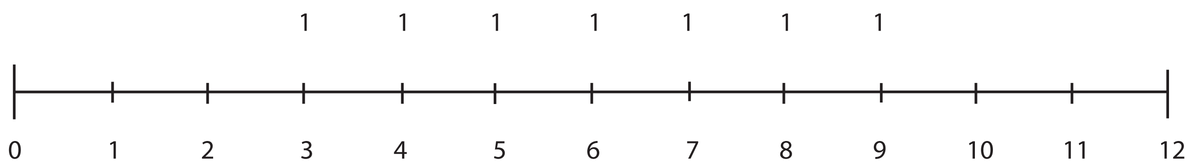
– There are three alternative dates to valuing annuities rather than at the beginning of the term ($t = 0$) or at the end of the term ($t = n$)

(i) present values more than one period before the first payment date

(ii) accumulated values more than one period after the last payment date

(iii) current value between the first and last payment dates

– The following example will be used to illustrate the above cases. Consider a series of payments of 1 that are made at time $t = 3$ to $t = 9$, inclusive.



Present Values More than One Period Before The First Payment Date
At $t = 2$, there exists 7 future end-of-year payments whose present value is represented by $a_{\overline{7}|}$. If this value is discounted back to time $t = 0$, then the value of this series of payments (2 periods before the first end-of-year payment) is

$$_{2|}a_{\overline{7}|} = v^2 \cdot a_{\overline{7}|}.$$

The general form is:

$$_{m|}a_{\overline{n}|i} = v^m \cdot a_{\overline{n}|i}.$$

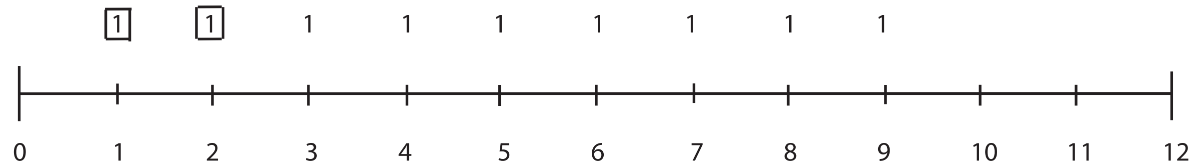
Alternatively, at $t = 3$, there exists 7 future beginning-of-year payments whose present value is represented by $\ddot{a}_{\overline{7}|}$. If this value is discounted back to time $t = 0$, then the value of this series of payments (3 periods before the first beginning-of-year payment) is

$$_{3|}\ddot{a}_{\overline{7}|} = v^3 \cdot \ddot{a}_{\overline{7}|}.$$

The general form is:

$$_{m|}\ddot{a}_{\overline{n}|} = v^m \cdot \ddot{a}_{\overline{n}|i}.$$

Another way to examine this situation is to pretend that there are 9 end-of-year payments. This can be done by adding 2 more payments to the existing 7. In this case, let the 2 additional payments be made at $t = 1$ and 2 and be denoted as $\boxed{1}$.



At t = 0, there now exists 9 end-of-year payments whose present value is $a_{\overline{9}|}$. This present value of 9 payments would then be reduced by the present value of the two imaginary payments, represented by $a_{\overline{2}|}$. Therefore, the present value at $t = 0$ is

$$a_{\overline{9}|} - a_{\overline{2}|},$$

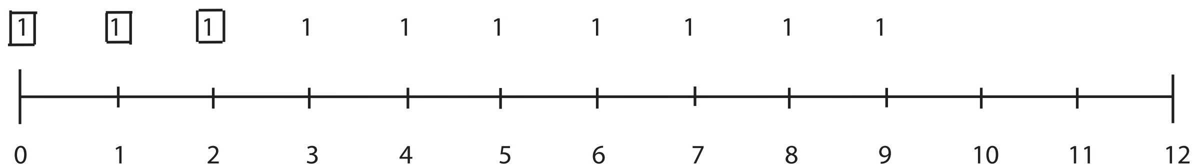
and this results in

$$v^2 \cdot a_{\overline{7}|} = a_{\overline{9}|} - a_{\overline{2}|}.$$

The general form is

$$v^m \cdot a_{\overline{n}|} = a_{\overline{m+n}|} - a_{\overline{m}|}.$$

With the annuity–due version, one can pretend that there are 10 payments being made. This can be done by adding 3 payments to the existing 7 payments. In this case, let the 3 additional payments be made at $t = 0$, 1 and 2 and be denoted as $\boxed{1}$.



At t = 0, there now exists 10 beginning-of-year payments whose present value is $\ddot{a}_{\overline{10}|}$. This present value of 10 payments would then be reduced by the present value of the three imaginary payments, represented by $\ddot{a}_{\overline{3}|}$. Therefore, the present value at $t = 0$ is

$$\ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|},$$

and this results in

$$v^3 \cdot \ddot{a}_{\overline{7}|} = \ddot{a}_{\overline{10}|} - \ddot{a}_{\overline{3}|}.$$

The general form is

$$v^m \cdot \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{m+n}|} - \ddot{a}_{\overline{m}|}.$$

Accumulated Values More Than One Period After The Last Payment Date

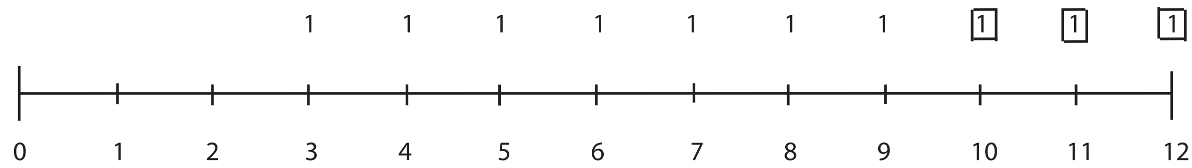
At $t = 9$, there exists 7 past end-of-year payments whose accumulated value is represented by $s_{\overline{7}|}$. If this value is accumulated forward to time $t = 12$, then the value of this series of payments (3 periods after the last end-of-year payment) is

$$s_{\overline{7}|} \cdot (1 + i)^3.$$

Alternatively, at $t = 10$, there exists 7 past beginning-of-year payments whose accumulated value is represented by $\ddot{s}_{\overline{7}|}$. If this value is accumulated forward to time $t = 12$, then the value of this series of payments (2 periods after the last beginning-of-year payment) is

$$\ddot{s}_{\overline{7}|} \cdot (1 + i)^2.$$

Another way to examine this situation is to pretend that there are 10 end-of-year payments. This can be done by adding 3 more payments to the existing 7. In this case, let the 3 additional payments be made at $t = 10$, 11 and 12 and be denoted as $\boxed{1}$.



At t = 12, there now exists 10 end-of-year payments whose present value is $s_{\overline{10}|}$. This future value of 10 payments would then be reduced by the future value of the three imaginary payments, represented by $s_{\overline{3}|}$. Therefore, the accumulated value at $t = 12$ is

$$s_{\overline{10}|} - s_{\overline{3}|},$$

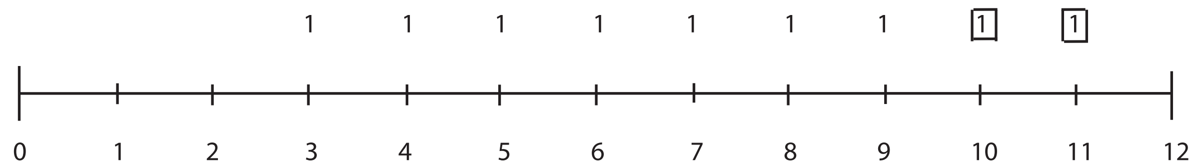
and this results in

$$s_{\overline{7}|} \cdot (1+i)^3 = s_{\overline{10}|} - s_{\overline{3}|}.$$

The general form is

$$s_{\overline{n}|} \cdot (1+i)^m = s_{\overline{m+n}|} - s_{\overline{m}|}.$$

With the annuity–due version, one can pretend that there are 9 payments being made. This can be done by adding 2 payments to the existing 7 payments. In this case, let the 2 additional payments be made at $t = 10$ and 11 and be denoted as $\boxed{1}$.



At t = 12, there now exists 9 beginning-of-year payments whose accumulated value is $\ddot{s}_{\overline{9|}}$. This future value of 9 payments would then be reduced by the future value of the two imaginary payments, represented by $\ddot{s}_{\overline{2|}}$. Therefore, the accumulated value at $t = 12$ is

$$\ddot{s}_{\overline{9|}} - \ddot{s}_{\overline{2|}},$$

and this results in

$$\ddot{s}_{\overline{7|}} \cdot (1+i)^2 = \ddot{s}_{\overline{9|}} - \ddot{s}_{\overline{2|}}.$$

The general form is

$$\ddot{s}_{\overline{n|}} \cdot (1+i)^m = \ddot{s}_{\overline{m+n|}} - \ddot{s}_{\overline{m|}}.$$

Current Values Between The First And Last Payment Dates

The 7 payments can be represented by an annuity-immediate or by an annuity-due depending on the time that they are evaluated at.

For example, at $t=2$, the present value of the 7 end-of-year payments is $a_{\overline{7}|}$. At $t=9$, the future value of those same payments is $s_{\overline{7}|}$. There is a point between time 2 and 9 where the present value and the future value can be accumulated to and discounted back, respectively. At $t=6$, for example, the present value would need to be accumulated forward 4 years, while the accumulated value would need to be discounted back 3 years.

$$a_{\overline{7}|} \cdot (1+i)^4 = v^3 \cdot s_{\overline{7}|}$$

The general form is

$$a_{\overline{n}|} \cdot (1+i)^m = v^{(n-m)} \cdot s_{\overline{n}|}$$

Alternatively, at $t=3$, one can view the 7 payments as being paid at the beginning of the year where the present value of the payments is $\ddot{a}_{\overline{7}|}$. The future value at $t=10$ would then be $\ddot{s}_{\overline{7}|}$. At $t=6$, for example, the present value would need to be accumulated forward 3 years, while the accumulated value would need to be discounted back 4 years.

$$\ddot{a}_{\overline{7}|} \cdot (1+i)^3 = v^4 \cdot \ddot{s}_{\overline{7}|}.$$

The general form is

$$\ddot{a}_{\overline{n}|} \cdot (1+i)^m = v^{(n-m)} \cdot \ddot{s}_{\overline{n}|}.$$

At any time during the payments, there will exists a series of past payments and a series of future payments.

For example, at $t=6$, one can define the past payments as 4 end-of-year payments whose accumulated value is $s_{\overline{4}|}$. The 3 end-of-year future payments at $t=6$ would then have a present value (at $t=6$) equal to $a_{\overline{3}|}$. Therefore, the current value as at $t=6$ of the 7 payments is

$$s_{\overline{4}|} + a_{\overline{3}|}.$$

Alternatively, if the payments are viewed as beginning-of-year payments at $t=6$, then there are 3 past payments and 4 future payments whose accumulated value and present value are respectively, $\ddot{s}_{\overline{3}|}$ and $\ddot{a}_{\overline{4}|}$. Therefore, the current value as at $t=6$ of the 7 payments can also be calculated as

$$\ddot{s}_{\overline{3}|} + \ddot{a}_{\overline{4}|}.$$

This results in

$$s_{\overline{4}|} + a_{\overline{3}|} = \ddot{s}_{\overline{3}|} + \ddot{a}_{\overline{4}|}.$$

The general form is

$$s_{\overline{m}|} + a_{\overline{n}|} = \ddot{s}_{\overline{n}|} + \ddot{a}_{\overline{m}|}.$$

2.4 Continuously Payable Annuities

– payments are made continuously every year for the next n years (i.e. $m = \infty$)

Continuously Payable Annuity Present Value Factor

– the present value (at $t = 0$) of a continuous annuity, where the annual effective rate of interest is i, shall be denoted as $\bar{a}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}\bar{a}_{\overline{n}|i} &= \int_0^n v^t dt \\ &= \int_0^n e^{-\delta t} dt \\ &= -\frac{1}{\delta} e^{-\delta t}\Big]_0^n \\ &= -\frac{1}{\delta}\left[e^{-\delta n} - e^{-\delta 0}\right] \\ &= \frac{1}{\delta}\left[1 - e^{-\delta n}\right] \\ &= \frac{1 - v_i^n}{\delta}\end{aligned}$$

Continuously Payable Annuity Accumulated Value Factor

– the accumulated value (at $t = n$) of a continuous annuity, where the annual effective rate of interest is i, shall be denoted as $\bar{s}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}\bar{s}_{\overline{n}|i} &= \int_0^n (1+i)^{n-t} dt \\ &= \int_0^n (1+i)^t dt \\ &= \int_0^n e^{\delta t} dt \\ &= \frac{1}{\delta} e^{\delta t} dt\Big]_0^n \\ &= \frac{1}{\delta}\left[e^{\delta n} - e^{\delta 0}\right] \\ &= \frac{(1+i)^n - 1}{\delta}\end{aligned}$$

Basic Relationship $1 : 1 = \delta \cdot \bar{a}_{\overline{n}|} + v^n$

Basic Relationship $2 : PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

- if the future value at time n, $\bar{s}_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $\bar{a}_{\overline{n}|}$

$$\begin{aligned}
\bar{s}_{\overline{n}|} \cdot v^n &= \left[\frac{(1+i)^n - 1}{\delta}\right] \cdot v^n \\
&= \frac{(1+i)^n \cdot v^n - v^n}{\delta} \\
&= \frac{1 - v^n}{\delta} \\
&= \bar{a}_{\overline{n}|}
\end{aligned}$$

- if the present value at time 0, $\bar{a}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\bar{s}_{\overline{n}|}$

$$\begin{aligned}
\bar{a}_{\overline{n}|} \cdot (1+i)^n &= \left[\frac{1 - v^n}{\delta}\right] (1+i)^n \\
&= \frac{(1+i)^n - v^n(1+i)^n}{\delta} \\
&= \frac{(1+i)^n - 1}{\delta} \\
&= \bar{s}_{\overline{n}|}
\end{aligned}$$

Basic Relationship $3: \dfrac{1}{\bar{a}_{\overline{n}|}} = \dfrac{1}{\bar{s}_{\overline{n}|}} + \delta$

- Consider a loan of 1, to be paid back over n years with annual payments of P that are paid continuously each year, for the next n years. An annual effective rate of interest, i, and annual force of interest, δ, is used. The present value of this single payment loan must be equal to the present value of the multiple payment income stream.

$$P \cdot \bar{a}_{\overline{n}|_i} = 1$$

$$P = \frac{1}{\bar{a}_{\overline{n}|_i}}$$

- Alternatively, consider a loan of 1, where the annual interest due on the loan, $(1) \times \delta$, is paid continuously during the year for n years and the loan amount is paid back at time n.
- In order to produce the loan amount at time n, annual payments of D are paid continuously each year, for the next n years, into an account that credits interest at an annual force of interest, δ.
- The future value of the multiple deposit income stream must equal the future value of the single payment, which is the loan of 1.

$$D \cdot \bar{s}_{\overline{n}|_i} = 1$$

$$D = \frac{1}{\bar{s}_{\overline{n}|_i}}$$

- The total annual payment will be the interest payment and account payment:

$$\delta + \frac{1}{\bar{s}_{\overline{n}|_i}}$$

- Note that

$$\frac{1}{\bar{a}_{\overline{n}|_i}} = \frac{\delta}{1 - v^n} \times \frac{(1+i)^n}{(1+i)^n} = \frac{\delta(1+i)^n}{(1+i)^n - 1}$$

$$= \frac{\delta(1+i)^n + \delta - \delta}{(1+i)^n - 1} = \frac{\delta[(1+i)^n - 1] + \delta}{(1+i)^n - 1}$$

$$= \delta + \frac{\delta}{(1+i)^n - 1} = \delta + \frac{1}{\bar{s}_{\overline{n}|}}$$

- Therefore, a level continuous annual annuity payment on a loan is the same as making an annual continuous interest payment each year plus making level annual continuous deposits in order to save for the loan repayment.

Basic Relationship $4 : \bar{a}_{\overline{n}|} = \dfrac{i}{\delta} \cdot a_{\overline{n}|}$, $\bar{s}_{\overline{n}|} = \dfrac{i}{\delta} \cdot s_{\overline{n}|}$

- Consider annual payments of 1 made continuously each year for the next n years. Over a one-year period, the continuous payments will accumulate at the end of the year to a lump sum of $\bar{s}_{\overline{1}|}$. If this end-of-year lump sum exists for each year of the n-year annuity-immediate, then the present value (at $t = 0$) of these end-of-year lump sums is the same as $\bar{a}_{\overline{n}|}$:

$$\begin{aligned} \bar{a}_{\overline{n}|} &= \bar{s}_{\overline{1}|} \cdot a_{\overline{n}|} \\ &= \frac{i}{\delta} \cdot a_{\overline{n}|} \end{aligned}$$

- Therefore, the accumulated value (at $t = n$) of these end-of-year lump sums is the same as $\bar{s}_{\overline{n}|}$:

$$\begin{aligned} \bar{s}_{\overline{n}|} &= \bar{s}_{\overline{1}|} \cdot s_{\overline{n}|} \\ &= \frac{i}{\delta} \cdot s_{\overline{n}|} \end{aligned}$$

Basic Relationship $5 : \bar{a}_{\overline{n}|} = \dfrac{d}{\delta} \cdot \ddot{a}_{\overline{n}|}$, $\bar{s}_{\overline{n}|} = \dfrac{d}{\delta} \cdot \ddot{s}_{\overline{n}|}$

- The mathematical development of this relationship is derived as follows:

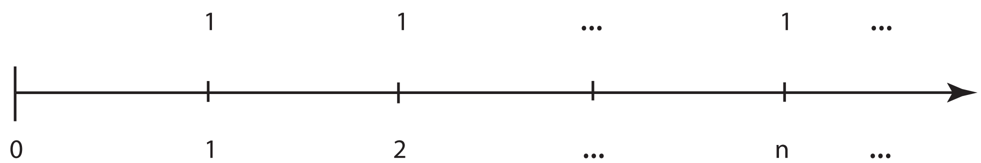
$$\ddot{a}_{\overline{n}|} \cdot \frac{d}{\delta} = \frac{1 - v^n}{d} \cdot \frac{d}{\delta} = \frac{1 - v^n}{\delta} = \bar{a}_{\overline{n}|}$$

$$\ddot{s}_{\overline{n}|} \cdot \frac{d}{\delta} = \frac{(1+i)^n - 1}{d} \cdot \frac{d}{\delta} = \frac{(1+i)^n - 1}{\delta} = \bar{s}_{\overline{n}|}$$

2.5 Perpetuities

Definition Of A Perpetuity-Immediate

– payments of 1 are made at the end of every year forever i.e. $n = \infty$



Perpetuity-Immediate Present Value Factor

– the present value (at $t = 0$) of a perpetuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a_{\overline{\infty|}i}$ and is calculated as follows:

$$
\begin{aligned}
a_{\overline{\infty|}i} &= (1)v + (1)v^2 + (1)v^3 + \cdots \\
&= v(1 + v + v^2 + \cdots) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-v^{\infty}}{1-v}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1-0}{d}\right) \\
&= \left(\frac{1}{1+i}\right)\left(\frac{1}{\frac{i}{1+i}}\right) \\
&= \frac{1}{i}
\end{aligned}
$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula:

$$a_{\overline{\infty|}i} = \frac{1-v^{\infty}}{i} = \frac{1-0}{i} = \frac{1}{i}$$

– Note that $\frac{1}{i}$ represents an initial amount that can be invested at $t = 0$. The annual interest payments, payable at the end of the year, produced by this investment is $\left(\frac{1}{i}\right) \cdot i = 1$.

– $s_{\overline{\infty|}}$ is not defined since it would equal ∞

Basic Relationship $1 : a_{\overline{n|}} = a_{\overline{\infty|}} - v_n \cdot a_{\overline{\infty|}}$

The present value formula for an annuity-immediate can be expressed as the difference between two perpetuity-immediates:

$$a_{\overline{n|}} = \frac{1 - v^n}{i} = \frac{1}{i} - \frac{v^n}{i} = \frac{1}{i} - v^n \cdot \frac{1}{i} = a_{\overline{\infty|}} - v^n \cdot a_{\overline{\infty|}}.$$

In this case, a perpetuity-immediate that is payable forever is reduced by perpetuity-immediate payments that start after n years. The present value of both of these income streams, at t = 0, results in end-of-year payments remaining only for the first n years.

Definition Of A Perpetuity-Due

– payments of 1 are made at the beginning of every year forever i.e. $n = \infty$

Perpetuity-Due Present Value Factor

– the present value (at $t = 0$) of a perpetuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}_{\overline{\infty}|i}$ and is calculated as follows:

$$\begin{aligned}\ddot{a}_{\overline{\infty}|i} &= (1) + (1)v^1 + (1)v^2 + \cdots \\ &= \left(\frac{1 - v^\infty}{1 - v}\right) \\ &= \left(\frac{1 - 0}{d}\right) \\ &= \frac{1}{d}\end{aligned}$$

– one could also derive the above formula by simply substituting $n = \infty$ into the original present value formula:

$$\ddot{a}_{\overline{\infty}|d} = \frac{1 - v^\infty}{d} = \frac{1 - 0}{d} = \frac{1}{d}$$

– Note that $\frac{1}{d}$ represents an initial amount that can be invested at $t = 0$. The annual interest payments, payable at the beginning of the year, produced by this investment is $\left(\frac{1}{d}\right) \cdot d = 1$.

– $\ddot{s}_{\overline{\infty}|}$ is not defined since it would equal ∞

Basic Relationship $1 : \ddot{a}_{\overline{n}|} = \ddot{a}_{\overline{\infty}|} - v_n \cdot \ddot{a}_{\overline{\infty}|}$

The present value formula for an annuity-due can be expressed as the difference between two perpetuity-dues:

$$\ddot{a}_{\overline{n}|} = \frac{1 - v^n}{d} = \frac{1}{d} - \frac{v^n}{d} = \frac{1}{d} - v^n \cdot \frac{1}{d} = \ddot{a}_{\overline{\infty}|} - v^n \cdot \ddot{a}_{\overline{\infty}|}.$$

In this case, a perpetuity-due that is payable forever is reduced by perpetuity-due payments that start after n years. The present value of both of these income streams, at $t = 0$, results in beginning-of-year payments remaining only for the first n years.

Definition Of A Continuously Payable Perpetuity Present Value Factor

– payments of 1 are made continuously every year forever

– the present value (at t=0) of a continuously payable perpetuity, where the annual effective rate of interest is i, shall be denoted as $\bar{a}_{\overline{\infty}|_i}$ and is calculated as follows:

$$\begin{aligned}
\bar{a}_{\overline{\infty}|_i} &= \int_0^\infty v^t dt \\
&= \int_0^\infty e^{-\delta t} dt \\
&= -\frac{1}{\delta} e^{-\delta t} \Big]_0^\infty \\
&= \frac{1}{\delta}
\end{aligned}$$

Basic Relationships: $\bar{a}_{\overline{\infty}|_i} = \frac{i}{\delta} a_{\overline{\infty}|}$ and $\bar{a}_{\overline{\infty}|_i} = \frac{d}{\delta} \ddot{a}_{\overline{\infty}|}$

– The mathematical development of these relationships are derived as follows:

$$a_{\overline{\infty}|} \cdot \frac{i}{\delta} = \frac{1}{i} \cdot \frac{i}{\delta} = \frac{1}{\delta} = \bar{a}_{\overline{\infty}|}$$

$$\ddot{a}_{\overline{\infty}|} \cdot \frac{d}{\delta} = \frac{1}{d} \cdot \frac{d}{\delta} = \frac{1}{\delta} = \bar{a}_{\overline{\infty}|}$$

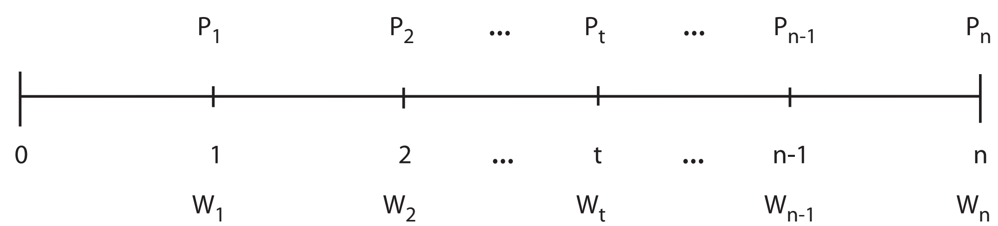
2.6 Equations of Value

– the value at any given point in time, t, will be either a present value or a future value (sometimes referred to as the time value of money)

– the time value of money depends on the calculation date from which payment(s) are either accumulated or discounted to

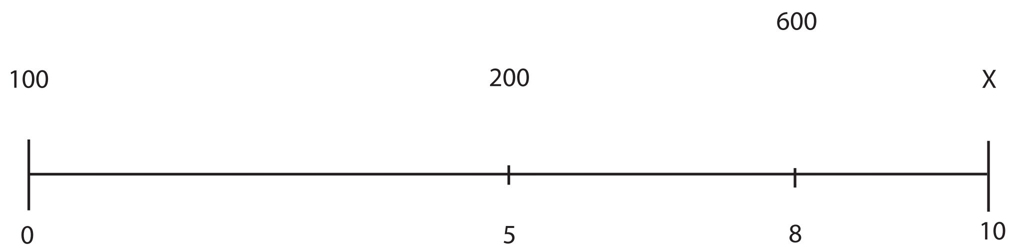
Time Line Diagrams

– it helps to draw out a time line and plot the payments and withdrawals accordingly

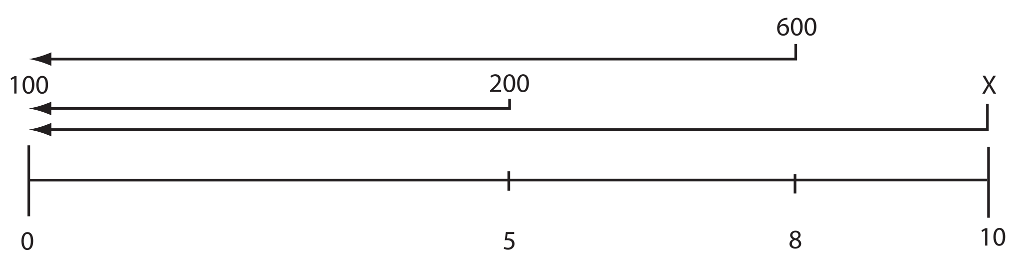


Example

– a payment of 600 is due in 8 years; the alternative is to receive 100 now,200 in 5 years and X in 10 years. If $i = 8\%$, find $\$X$, such that the value of both options is equal.

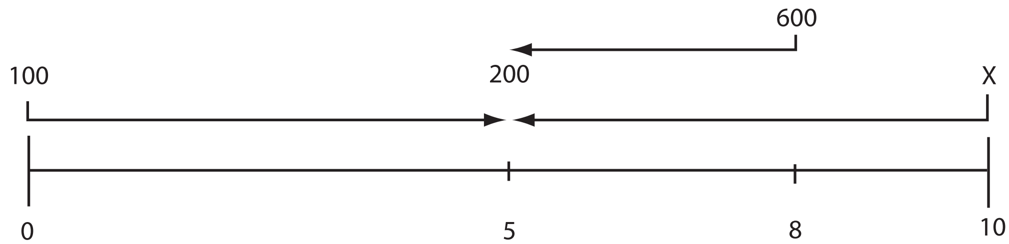


– compare the values at $t = 0$



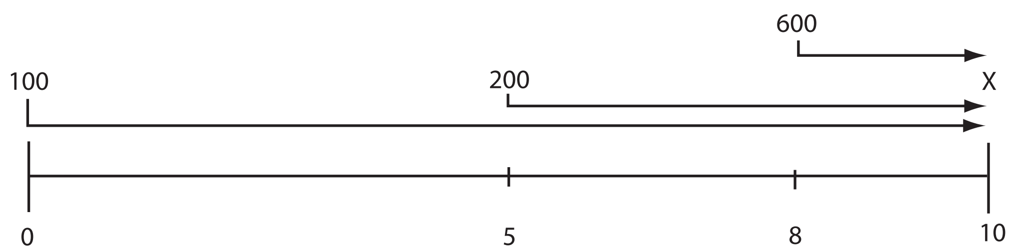
$$600v_{8\%}^{8} = 100 + 200v_{8\%}^{5} + Xv_{8\%}^{10}$$
$$X = \frac{600v_{8\%}^{8} - 100 - 200v_{8\%}^{5}}{v_{8\%}^{10}} = 190.08$$

– compare the values at $t = 5$



$$
\begin{aligned}
600v^3 &= 100(1+i)^5 + 200 + Xv^5 \\
X &= \frac{600v^3 - 100(1+i)^5 - 200}{v^5} = 190.08
\end{aligned}
$$

– compare the values at $t = 10$



$$
\begin{aligned}
600(1+i)^2 &= 100(1+i)^{10} + 200(1+i)^5 + X \\
X &= 600(1+i)^2 - 100(1+i)^{10} - 200(1+i)^5 = 190.08
\end{aligned}
$$

– all 3 equations gave the same answer because all 3 equations treated the value of the payments consistently at a given point of time.

Unknown Rate of Interest

– Assuming that you do not have a financial calculator

Linear Interpolation

- need to find the value of $a_{\overline{n}|}$ at two different interest rates where $a_{\overline{n}|i_1} = P_1 < P$ and $a_{\overline{n}|i_2} = P_2 > P$.

-

$$\left.\begin{aligned} a_{\overline{n}|i_1} &= P_1 \\ a_{\overline{n}|i} &= P \\ a_{\overline{n}|i_2} &= P_2 \end{aligned}\right\} \; i \approx i_1 + \frac{P_1 - P}{P_1 - P_2}(i_2 - i_1)$$

Exercises and Solutions

2.1 Annuity-Immediate

Exercise (a)

Fence posts set in soil last 9 years and cost Y each while fence posts set in concrete last 15 years and cost $Y+X$. The posts will be needed for 35 years. What is the value of X such that a fence builder would be indifferent between the two types of posts?

Solution (a)

The soil posts must be set 4 times (at t=0,9,18,27). The PV of the cost per post is $PV = Y \cdot \frac{a_{\overline{36}|}}{a_{\overline{9}|}}$.

The concrete posts must be set 3 times (at t=0,15,30). The PV of the cost per post is $PV = (Y+X) \cdot \frac{a_{\overline{45}|}}{a_{\overline{15}|}}$.

The breakeven value of X is the value for which $PV_s = PV_c$. Thus

$$Y \cdot \frac{a_{\overline{36}|}}{a_{\overline{9}|}} = (Y+X) \cdot \frac{a_{\overline{45}|}}{a_{\overline{15}|}} \text{ or}$$

$$X \cdot \frac{a_{\overline{45}|}}{a_{\overline{15}|}} = Y \left[\frac{a_{\overline{36}|}}{a_{\overline{9}|}} - \frac{a_{\overline{45}|}}{a_{\overline{15}|}} \right]$$

$$X = Y \left[\frac{a_{\overline{36}|} \cdot a_{\overline{15}|}}{a_{\overline{45}|} \cdot a_{\overline{9}|}} - 1 \right]$$

Exercise (b)

You are given $\delta_t = \dfrac{4+t}{1+8t+t^2}$ for $t \geq 0$.

Calculate $s_{\overline{4}|}$.

Solution (b)

$$\begin{aligned}
s_{\overline{4}|} &= 1 + (1+i)^{4-3} + (1+i)^{4-2} + (1+i)^{4-1} \\
&= 1 + e^{\int_3^4 \delta_s \cdot ds} + e^{\int_2^4 \delta_s \cdot ds} + e^{\int_1^4 \delta_s \cdot ds} \\
&= 1 + e^{\int_3^4 \frac{4+s}{1+8s+s^2} \cdot ds} + e^{\int_2^4 \frac{4+s}{1+8s+s^2} \cdot ds} + e^{\int_1^4 \frac{4+s}{1+8s+s^2}} \\
&= 1 + e^{\frac{1}{2} \int_3^4 \frac{f'(s)}{f(s)} \cdot ds} + e^{\frac{1}{2} \int_2^4 \frac{f'(s)}{f(s)} \cdot ds} + e^{\frac{1}{2} \int_1^4 \frac{f'(s)}{f(s)} \cdot ds} \\
&= 1 + \left[\frac{f(4)}{f(3)} \right]^{\frac{1}{2}} + \left[\frac{f(4)}{f(2)} \right]^{\frac{1}{2}} + \left[\frac{f(4)}{f(1)} \right]^{\frac{1}{2}} \\
&= 1 + \left[\frac{1+8(4)+(4)^2}{1+8(3)+(3)^2} \right]^{\frac{1}{2}} + \left[\frac{1+8(4)+(4)^2}{1+8(2)+(2)^2} \right]^{\frac{1}{2}} + \left[\frac{1+8(4)+(4)^2}{1+8(1)+(1)^2} \right]^{\frac{1}{2}} \\
&= 1 + 1.2005 + 1.5275 + 1.4878 = 5.2158
\end{aligned}$$

Exercise (c)

You are given the following information:

(i) The present value of a $6n$-year annuity-immediate of 1 at the end of every year is 9.7578.

(ii) The present value of a $6n$-year annuity-immediate of 1 at the end of every second year is 4.760.

(iii) The present value of a $6n$-year annuity-immediate of 1 at the end of every third year is K.

Determine K assuming an annual effective interest rate of i.

Solution (c)

$$9.758 = a_{\overline{6n}|i}$$

$$4.760 = a_{\overline{3n}|(1+i)^2-1} = \frac{a_{\overline{6n}|i}}{s_{\overline{2}|i}}$$

$$K = a_{\overline{2n}|(1+i)^3-1} = \frac{a_{\overline{6n}|i}}{s_{\overline{3}|i}}$$

$$\frac{9.758}{4.760} = s_{\overline{2}|i} = 2.05 = 1 + (1+i) \rightarrow i = 5\%$$

$$K = \frac{9.758}{s_{\overline{3}|5\%}} = \frac{9.758}{3.1525} = 3.095$$

2.2 Annuity-Due

Exercise (a)

Simplify $a_{\overline{15}|i}(1+i)^{45}\ddot{a}_{\overline{3}|j}$ to one actuarial symbol, given that $j = (1+i)^{15} - 1$.

Solution (a)

$$a_{\overline{15}|i}(1+i)^{45}\ddot{a}_{\overline{3}|j} = s_{\overline{15}|i}(1+i)^{30}(1 + v_j^1 + v_j^2)$$
$$= s_{\overline{15}|i}(1+i)^{30}(1 + v_i^{15} + v_i^{30}) = s_{\overline{15}|i}[(1+i)^{30} + (1+i)^{15} + 1]$$

There are 3 sets of 15 end-of-year payments (45 in total) that are being made and carried forward to $t = 45$. Therefore, at t=45 you will have 45 end-of-year payment that have been accumulated and whose value is $s_{\overline{45}|i}$.

Exercise (b)

A person deposits 100 at the beginning of each year for 20 years. Simple interest at an annual rate of i is credited to each deposit from the date of deposit to the end of the twenty year period. The total amount thus accumulated is 2,840. If instead, compound interest had been credited at an effective annual rate of i, what would the accumulated value of these deposits have been at the end of twenty years?

Solution (b)

$$100[(1+i)+(1+2i)+(1+3i)+...+(1+20i)] = 2,840$$

$$100[20+i(1+2+3+...+20)] = 2,840$$

$$20+i(\frac{20\cdot 21}{2}) = 28.40 \rightarrow i = .04 \text{ and } d = .03846$$

$$100\ddot{s}_{\overline{20}|4\%} = 3,097$$

Exercise (c)

You plan to accumulate 100,000 at the end of 42 years by making the following deposits:

X at the beginning of years 1-14

No deposits at the beginning of years 15-32; and

Y at the beginning of years 33-42.

The annual effective interest rate is 7%.

$X - Y = 100$. Calculate Y.

Solution (c)

$$\underbrace{X}_{100-Y}\ddot{s}_{\overline{14}|7\%}(1.07)^{28} + Y\ddot{s}_{\overline{10}|7\%} = 100,000$$

$$(100+Y)(160.42997) + Y(14.78368) = 100,000$$

$$Y = 479.17$$

2.3 Deferred Annuities

Exercise (a)

Using an annual effective interest rate $j \geq 0$, you are given:

(i) The present value of 2 at the end of each year for $2n$ years, plus an additional 1 at the end of each of the first n years, is 36.

(ii) The present value of an n-year deferred annuity-immediate paying 2 per year for n years is 6.

Calculate j.

Solution (a)

$$\text{(i) } 36 = 2a_{\overline{2n}|} + a_{\overline{n}|}$$

$$\text{(ii) } 6 = v^n \cdot 2a_{\overline{n}|} = 2a_{\overline{2n}|} - 2a_{\overline{n}|}$$

then subtracting (ii) from (i) gives us:

$$30 = 3a_{\overline{n}|} \rightarrow 10 = a_{\overline{n}|}$$

$$6 = v^n 2(10) \rightarrow .3 = v^n$$

$$10 = a_{\overline{n}|} = \frac{1 - v^n}{i} = \frac{1 - .3}{i} \rightarrow i = 7\%$$

Exercise (b)

A loan of 1,000 is to be repaid by annual payments of 100 to commence at the end of the fifth year and to continue thereafter for as long as necessary. The effective rate of discount is 5%. Find the amount of the final payment if it is to be larger than the regular payments.

Solution (b)

$$PV_0 = 1,000 = 100a_{\overline{n}|} - 100a_{\overline{4}|}$$

$$a_{\overline{n}|} = \frac{1,000 + 100a_{\overline{4}|}}{100} = 13.53438$$

$$\frac{1 - v^n}{.0526316} = 13.52438$$

$$.288190 = v^n$$

$$3.47 = (1.0526316)^n$$

$$n = \frac{ln(3.47)}{ln(1.0526316)} = 24.2553$$

$$n = 24$$

$$1,000 = 100a_{\overline{24}|} + Xv^{24} - 100a_{\overline{4}|}$$

$$X = \frac{1,000 - 100(a_{\overline{24}|} - a_{\overline{4}|})}{v^{24}}$$

$$X = 7.217526(1.0526316)^{24}$$

$$X = 24.72$$

Thus, the total final payment is 124.72.

2.4 Continuously Payable Annuities

Exercise (a)

There is 40,000 in a fund which is accumulating at 4% per annum convertible continuously. If money is withdrawn continuously at the rate of 2,400 per annum, how long will the fund last?

Solution (a)

If the fund is exhausted at $t = n$, then the accumulated value of the fund at that time must equal the accumulated value of the withdrawals. Thus we have:

$$40,000e^{.04n} = 2,400\bar{s}_{\overline{n}|} = 2400(\frac{e^{.04n} - 1}{.04})$$

$$\frac{40,000}{2,400}(.04) = 1 - e^{.04n}$$

$$n = \frac{ln[1 - (.04)\frac{40,000}{2,400}]}{-.04} = 27.47 \text{ years}$$

Exercise (b)

If $\bar{a}_{\overline{n}|} = 4$ and $\bar{s}_{\overline{n}|} = 12$ find δ.

Solution (b)

$$\bar{a}_{\overline{n}|} = \frac{1 - v^n}{\delta} = 4 \rightarrow v^n = 1 - 4\delta$$

$$\bar{s}_{\overline{n}|} = \frac{(1 + i)^n - 1}{\delta} = 12$$

$$(1 + i)^n = 1 + 12\delta$$

$$(1 + i)^n = v^{-n} \text{ then } 1 + 12\delta = \frac{1}{1 - 4\delta}$$

$$1 + 8\delta - 48\delta^2 = 1 \text{ or } \delta = \frac{4}{48} = \frac{1}{6}$$

2.5 Perpetuities

Exercise (a)

A perpetuity-immediate pays X per year. Kevin receives the first n payments, Jeffrey receives the next n payments and Hal receives the remaining payments. The present value of Kevin's payments is 20% of the present value of the original perpetuity. The present value of Hal's payments is K of the present value of the original perpetuity.

Calculate the present value of Jeffrey's payments as a percentage of the original perpetuity.

Solution (a)

The present value of the perpetuity is:

$$Xa_{\overline{\infty}|} = \frac{X}{i}$$

The present value of Kevin's payments is:

$$Xa_{\overline{n}|} = .2Xa_{\overline{\infty}|} = .2\frac{X}{i}$$

This leads to:

$$a_{\overline{n}|} = \frac{.2}{i} \rightarrow 1 - v^n = .2 \rightarrow v^n = .8$$

The present value of Hal's payments is:

$$Xv^{2n}a_{\overline{\infty}|} = K \cdot X \cdot a_{\overline{\infty}|} = K \cdot \frac{X}{i}.$$

Therefore,

$$X \cdot (.8)^2 \cdot a_{\overline{\infty}|} \rightarrow K = (.8)^2 = .64.$$

Therefore, Jeffrey owns $.16(1 - .2 - .64)$.

Exercise (b)

A perpetuity pays 1 at the beginning of every year plus an additional 1 at the beginning of every second year.

Determine the present value of this annuity.

Solution (b)

$$K = \ddot{a}_{\overline{\infty}|_i} + v_i \ddot{a}_{\overline{\infty}|_{(1+i)^2-1}} = \frac{1}{d_i} + \frac{1}{1+i} \cdot \frac{1}{d_{(1+i)^2-1}} = \frac{1+i}{i} + \frac{(1+i)}{(1+i)^2-1}$$

$$K = \frac{1+i}{i} + \frac{1}{(1+i)} \cdot \frac{1+(1+i)^2-1}{(1+i)^2-1} = \frac{1+i}{i} + \frac{1}{(1+i)} \cdot \frac{(1+i)^2}{(1+i)(1+i)^2-1}$$

$$K = \frac{1+i}{i} + \frac{(1+i)}{(1+i)^2-1} = (1+i)\left[\frac{1}{i} + \frac{1}{(1+i)^2-1}\right]$$

$$K = (1+i)\left[\frac{(1+i)^2-1+i}{i[(1+i)^2-1]}\right] = (1+i)\left[\frac{1+2i+i^2-1+i}{i[1+2i+i^2-1}\right]$$

$$K = (1+i)\left[\frac{2i+i^2+i}{i[2i+i^2]}\right] = (1+i)\left[\frac{2+i+1}{2i+i^2}\right] = (1+i)\left[\frac{3+i}{i(2+i)}\right] = \frac{3+i}{d(2+i)}$$

Exercise (c)

A perpetuity-immediate pays X per year. Nicole receives the first n payments, Mark receives the next n payments and Cheryl receives the remaining payments. The present value of Nicole's payments is 30% of the present value of the original perpetuity. The present value of Cheryl's payments is K% of the present value of the original perpetuity.

Calculate the present value of Cheryl's payments as a percentage of the original perpetuity.

Solution (c)

The present value of Nicole's payments is:

$$X \cdot a_{\overline{n}|} = .3 \cdot X \cdot a_{\overline{\infty}|} = .3 \cdot \frac{X}{i}$$

This leads to

$$a_{\overline{n}|} = \frac{.3}{i} \rightarrow 1 - v^n = .3 \rightarrow v^n = .7$$

The present value of Cheryl's payments is:

$$X \cdot v^{2n} \cdot a_{\overline{\infty}|} = K \cdot X \cdot a_{\overline{\infty}|} = K \cdot \frac{X}{i}$$

$$\text{Therefore } X \cdot (.7)^2 \cdot a_{\overline{\infty}|} = K \cdot X \cdot a_{\overline{\infty}|} \rightarrow K = (.7)^2 = .49$$

2.6 Equation of Value

Exercise (a)

An investment requires an initial payment of 10,000 and annual payments of 1,000 at the end of each of the first 10 years. Starting at the end of the eleventh year, the investment returns five equal annual payments of X.

Determine X to yield an annual effective rate of 10% over the 15-year period.

Solution (a)

PV of cash flow in=PV of cash flow out

$$10,000 + 1,000a_{\overline{10}|10\%} = v^{10} \cdot X \cdot a_{\overline{5}|10\%}$$

$$X = \frac{10,000 + 1,000a_{\overline{10}|10\%}}{v^{10}a_{\overline{5}|10\%}}$$

$$X = \frac{10,000 + 1,000(6.144571)}{(.3855433)(3.7907868)}$$

$$X = 11,046$$

Exercise(b)

At a certain interest rate the present value of the following two payment patterns are equal:

(i) 200 at the end of 5 years plus 500 at the end of 10 years.

(ii) 400.94 at the end of 5 years.

At the same interest rate, 100 invested now plus 120 invested at the end of 5 years will accumulate to P at the end of 10 years. Calculate P.

Solution (b)

$$200v^5 + 500v^{10} = 400.94v^5 \to 500v^{10} = 200.94v^5 \to v^5 = .40188$$

$$v^5 = .40188 \to (1+i)^5 = 2.48831$$

$$P = 100(1+i)^{10} + 120(1+i)^5 = 100(2.48831)^2 + 120(2.48831) = 917.76$$

Exercise (c)

Whereas the choice of a comparison date has no effect on the answer obtained with compound interest, the same cannot be said of simple interest. Find the amount to be paid at the end of 10 years which is equivalent to two payments of 100 each, the first to be paid immediately and the second to be paid at the end of 5 years. Assume 5% simple interest is earned from the date each payment is made and use a comparison date of the end of 10 years.

Solution (c)

Equating at $t = 10$

$$X = 100(1+10i) + 100(1+5i) = [100(1+10(.05)) + 100(1+5(.05)) = 275.00$$

3 Varying Annuities

Overview

– in this section, payments will now vary; but the interest conversion period will continue to coincide with the payment frequency

– annuities can vary in 3 different ways

(i) where the payments increase or decrease by a fixed amount (sections 3.1, 3.2, 3.3, 3.4 and 3.5)

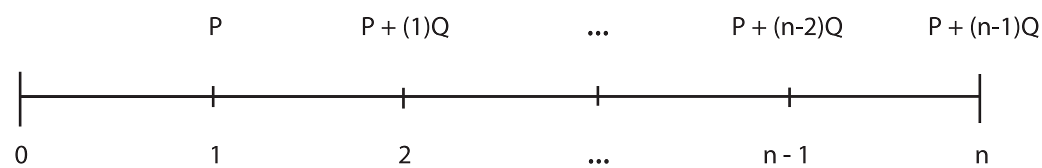
(ii) where the payments increase or decrease by a fixed rate (section 3.6)

(iii) where the payments increase or decrease by a variable amount or rate (section 3.7)

3.1 Increasing Annuity-Immediate

Annuity-Immediate

An annuity-immediate is payable over n years with the first payment equal to P and each subsequent payment increasing by Q. The time line diagram below illustrates the above scenario:



The present value (at $t = 0$) of this annual annuity–immediate, where the annual effective rate of interest is i, shall be calculated as follows:

$$\begin{aligned}
PV_0 &= [P]v + [P+Q]v^2 + \cdots + [P+(n-2)Q]v^{n-1} + [P+(n-1)Q]v^n \\
&= P[v + v^2 + \cdots + v^{n-1} + v^n] + Q[v^2 + 2v^3 \cdots + (n-2)v^{n-1} + (n-1)v^n] \\
&= P[v + v^2 + \cdots + v^{n-1} + v^n] + Qv^2[1 + 2v \cdots + (n-2)v^{n-3} + (n-1)v^{n-2}] \\
&= P[v + v^2 + \cdots + v^{n-1} + v^n] + Qv^2 \frac{d}{dv}[1 + v + v^2 + \cdots + v^{n-2} + v^{n-1}] \\
&= P \cdot a_{\overline{n}|i} + Qv^2 \frac{d}{dv}\left[\ddot{a}_{\overline{n}|i}\right] \\
&= P \cdot a_{\overline{n}|i} + Qv^2 \frac{d}{dv}\left[\frac{1-v^n}{1-v}\right] \\
&= P \cdot a_{\overline{n}|i} + Qv^2 \left[\frac{(1-v)\cdot(-nv^{n-1}) - (1-v^n)\cdot(-1)}{(1-v)^2}\right] \\
&= P \cdot a_{\overline{n}|i} + \frac{Q}{(1+i)^2}\left[\frac{-nv^n(v^{-1}-1) + (1-v^n)}{(i/1+i)^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^{n-1} - nv^n}{i^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^n(v^{-1}-1)}{i^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{(1-v^n) - nv^n(1+i-1)}{i^2}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{\frac{(1-v^n)}{i} - \frac{nv^n(i)}{i}}{i}\right] \\
&= P \cdot a_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right]
\end{aligned}$$

The accumulated value (at $t = n$) of an annuity–immediate, where the annual effective rate of interest is i, can be calculated using the same approach as above or calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_n &= PV_0 \cdot (1+i)^n \\
&= \left(P \cdot a_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i} - nv^n}{i}\right]\right)(1+i)^n \\
&= P \cdot a_{\overline{n}|i} \cdot (1+i)^n + Q\left[\frac{a_{\overline{n}|i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{i}\right] \\
&= P \cdot s_{\overline{n}|i} + Q\left[\frac{s_{\overline{n}|i} - n}{i}\right]
\end{aligned}$$

Let $P = 1$ and $Q = 1$. In this case, the payments start at 1 and increase by 1 every year until the final payment of n is made at time n.

Increasing Annuity-Immediate Present Value Factor

The present value (at $t = 0$) of this annual increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Ia)_{\overline{n|}i}$ and is calculated as follows:

$$\begin{aligned}(Ia)_{\overline{n|}i} &= (1) \cdot a_{\overline{n|}i} + (1) \cdot \left[\frac{a_{\overline{n|}i} - nv^n}{i}\right] \\ &= \frac{1 - v^n}{i} + \frac{a_{\overline{n|}i} - nv^n}{i} \\ &= \frac{1 - v^n + a_{\overline{n|}i} - nv^n}{i} \\ &= \frac{\ddot{a}_{\overline{n|}i} - nv^n}{i}\end{aligned}$$

Increasing Annuity-Immediate Accumulation Factor

The accumulated value (at $t = n$) of this annual increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Is)_{\overline{n|}i}$ and can be calculated using the same general approach as above, or alternatively, by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}(Is)_{\overline{n|}i} &= (Ia)_{\overline{n|}i} \cdot (1+i)^n \\ &= \left(\frac{\ddot{a}_{\overline{n|}i} - nv^n}{i}\right) \cdot (1+i)^n \\ &= \frac{\ddot{a}_{\overline{n|}i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{i} \\ &= \frac{\ddot{s}_{\overline{n|}i} - n}{i}\end{aligned}$$

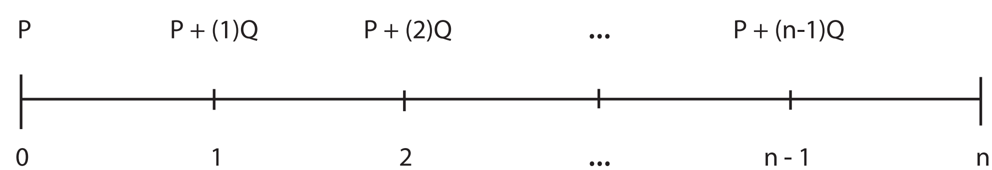
Increasing Perpetuity-Immediate Present Value Factor

The present value (at $t = 0$) of this annual increasing perpetuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Ia)_{\overline{\infty}|i}$ and is calculated as follows:

$$
\begin{aligned}
(Ia)_{\overline{\infty}|} &= v + 2v^2 + 3v^3 + \ldots \\
v(Ia)_{\overline{\infty}|} &= v^2 + 2v^3 + 3v^4 + \ldots \\
(Ia)_{\overline{\infty}|} - v(Ia)_{\overline{\infty}|} &= v + v^2 + v^3 + \ldots \\
(Ia)_{\overline{\infty}|}(1-v) &= a_{\overline{\infty}|} \\
(Ia)_{\overline{\infty}|} &= \frac{a_{\overline{\infty}|}}{1-v} = \frac{a_{\overline{\infty}|}}{d} = \frac{\frac{i}{d}}{\frac{i}{1+i}} = \frac{1+i}{i^2}
\end{aligned}
$$

3.2 Increasing Annuity-Due

An annuity-due is payable over n years with the first payment equal to P and each subsequent payment increasing by Q. The time line diagram below illustrates the above scenario:



The present value (at $t = 0$) of this annual annuity–due, where the annual effective rate of interest is i, shall be calculated as follows:

$$
\begin{aligned}
PV_0 &= [P] + [P+Q]v + \cdots + [P+(n-2)Q]v^{n-2} + [P+(n-1)Q]v^{n-1} \\
&= P[1+v+\cdots+v^{n-2}+v^{n-1}] + Q[v+2v^2\cdots+(n-2)v^{n-2}+(n-1)v^{n-1}] \\
&= P[1+v+\cdots+v^{n-2}+v^{n-1}] + Qv[1+2v\cdots+(n-2)v^{n-3}+(n-1)v^{n-2}] \\
&= P[1+v+\cdots+v^{n-2}+v^{n-1}] + Qv\frac{d}{dv}[1+v+v^2+\cdots+v^{n-2}+v^{n-1}] \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Qv\frac{d}{dv}\left[\ddot{a}_{\overline{n}|i}\right] \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Qv\frac{d}{dv}\left[\frac{1-v^n}{1-v}\right] \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Qv\left[\frac{(1-v)\cdot(-nv^{n-1})-(1-v^n)\cdot(-1)}{(1-v)^2}\right] \\
&= P\cdot \ddot{a}_{\overline{n}|i} + \frac{Q}{(1+i)}\left[\frac{-nv^n(v^{-1}-1)+(1-v^n)}{(i/1+i)^2}\right] \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{(1-v^n)-nv^{n-1}-nv^n}{i^2}\right]\cdot(1+i) \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{(1-v^n)-nv^n(v^{-1}-1)}{i^2}\right]\cdot(1+i) \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{(1-v^n)-nv^n(1+i-1)}{i^2}\right]\cdot(1+i) \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{\frac{(1-v^n)}{i}-\frac{nv^n(i)}{i}}{i}\right]\cdot(1+i) \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i}-nv^n}{i}\right]\cdot(1+i) \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i}-nv^n}{d}\right]
\end{aligned}
$$

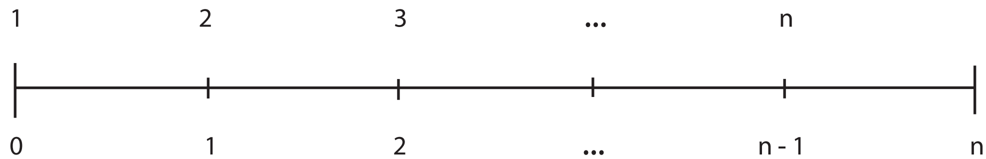
This present value of the annual annuity-due could also have been calculated using the basic principle that since payments under an annuity-due start one year earlier than under an annuity-immediate, the annuity–due will earn one more year of interest and thus, will be greater than an annuity-immediate by $(1 + i)$:

$$
\begin{aligned}
PV_0^{due} &= PV_0^{immediate} \times (1+i) \\
&= \left(P\cdot a_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i}-nv^n}{i}\right]\right)\times(1+i) \\
&= (P\cdot a_{\overline{n}|i})\times(1+i) + Q\left[\frac{a_{\overline{n}|i}-nv^n}{i}\right]\times(1+i) \\
&= P\cdot \ddot{a}_{\overline{n}|i} + Q\left[\frac{a_{\overline{n}|i}-nv^n}{d}\right]
\end{aligned}
$$

The accumulated value (at $t = n$) of an annuity–due, where the annual effective rate of interest is i, can be calculated using the same general approach as above, or alternatively, calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$
\begin{aligned}
FV_n &= PV_0 \cdot (1+i)^n \\
&= \left(P \cdot \ddot{a}_{\overline{n}|i} + Q \left[\frac{a_{\overline{n}|i} - nv^n}{d} \right] \right) (1+i)^n \\
&= P \cdot \ddot{a}_{\overline{n}|i} \cdot (1+i)^n + Q \left[\frac{a_{\overline{n}|i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{d} \right] \\
&= P \cdot \ddot{s}_{\overline{n}|i} + Q \left[\frac{s_{\overline{n}|i} - n}{d} \right]
\end{aligned}
$$

Let $P = 1$ and $Q = 1$. In this case, the payments start at 1 and increase by 1 every year until the final payment of n is made at time $n-1$.



Increasing Annuity-Due Present Value Factor

The present value (at $t = 0$) of this annual increasing annuity–due, where the annual effective rate of interest is i, shall be denoted as $(I\ddot{a})_{\overline{n}|i}$ and is calculated as follows:

$$
\begin{aligned}
(I\ddot{a})_{\overline{n}|i} &= (1) \cdot \ddot{a}_{\overline{n}|i} + (1) \cdot \left[\frac{a_{\overline{n}|i} - nv^n}{d}\right] \\
&= \frac{1 - v^n}{d} + \frac{a_{\overline{n}|i} - nv^n}{d} \\
&= \frac{1 - v^n + a_{\overline{n}|i} - nv^n}{d} \\
&= \frac{\ddot{a}_{\overline{n}|i} - nv^n}{d}
\end{aligned}
$$

Increasing Annuity-Due Accumulated Value Factor

The accumulated value (at $t = n$) of this annual increasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(I\ddot{s})_{\overline{n}|i}$ and can be calculated using the same approach as above or by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$
\begin{aligned}
(I\ddot{s})_{\overline{n}|i} &= (I\ddot{a})_{\overline{n}|i} \cdot (1+i)^n \\
&= \left(\frac{\ddot{a}_{\overline{n}|i} - nv^n}{d}\right) \cdot (1+i)^n \\
&= \frac{\ddot{a}_{\overline{n}|i} \cdot (1+i)^n - nv^n \cdot (1+i)^n}{d} \\
&= \frac{\ddot{s}_{\overline{n}|i} - n}{d}
\end{aligned}
$$

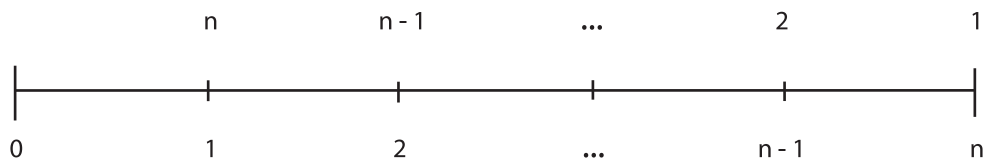
Increasing Perpetuity-Due Present Value Factor

The present value (at t=0) of this annual increasing perpetuity-due, where the annual effective rate of interest is i, shall be denoted as $(I\ddot{a})_{\overline{\infty|}i}$ and is calculated as follows:

$$
\begin{aligned}
(I\ddot{a})_{\overline{\infty|}} &= 1 + 2v + 3v^2 + 4v^3 + ... \\
v(I\ddot{a})_{\overline{\infty|}} &= v + 2v^2 + 3v^3 + ... \\
(I\ddot{a})_{\overline{\infty|}} - v(I\ddot{a})_{\overline{\infty|}} &= 1 + v + v^2 + ... \\
(I\ddot{a})_{\overline{\infty|}}(1-v) &= \frac{\ddot{a}_{\overline{\infty|}}}{1-v} = \frac{\frac{1}{d}}{d} = \frac{1}{d^2}
\end{aligned}
$$

3.3 Decreasing Annuity-Immediate

Let $P = n$ and $Q = -1$. In this case, the payments start at n and decrease by 1 every year until the final payment of 1 is made at time n.



Decreasing Annuity-Immediate Present Value Factor

The present value (at $t = 0$) of this annual decreasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Da)_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
(Da)_{\overline{n}|i} &= (n) \cdot a_{\overline{n}|i} + (-1) \cdot \left[\frac{a_{\overline{n}|i} - nv^n}{i} \right] \\
&= n \cdot \frac{1 - v^n}{i} - \frac{a_{\overline{n}|i} - nv^n}{i} \\
&= \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{i} \\
&= \frac{n - a_{\overline{n}|i}}{i}
\end{aligned}$$

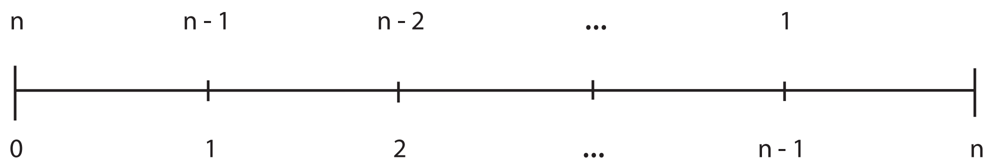
Decreasing Annuity-Immediate Accumulation Value Factor

The accumulated value (at $t = n$) of this annual decreasing annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $(Ds)_{\overline{n}|i}$ and can be calculated by using the same general approach as above, or alternatively, by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
(Ds)_{\overline{n}|i} &= (Da)_{\overline{n}|i} \cdot (1 + i)^n \\
&= \left(\frac{n - a_{\overline{n}|i}}{i} \right) \cdot (1 + i)^n \\
&= \frac{n \cdot (1 + i)^n - s_{\overline{n}|i}}{i}
\end{aligned}$$

3.4 Decreasing Annuity-Due

Let $P = n$ and $Q = -1$. In this case, the payments start at n and decrease by 1 every year until the final payment of 1 is made at time n.



Decreasing Annuity-Due Present Value Factor

The present value (at $t = 0$) of this annual decreasing annuity–due, where the annual effective rate of interest is i, shall be denoted as $(D\ddot{a})_{\overline{n}|i}$ and is calculated as follows:

$$
\begin{aligned}
(D\ddot{a})_{\overline{n}|i} &= (n) \cdot \ddot{a}_{\overline{n}|i} + (-1) \cdot \left[\frac{a_{\overline{n}|i} - nv^n}{d} \right] \\
&= n \cdot \frac{1 - v^n}{d} - \frac{a_{\overline{n}|i} - nv^n}{d} \\
&= \frac{n - nv^n - a_{\overline{n}|i} + nv^n}{d} \\
&= \frac{n - a_{\overline{n}|i}}{d}
\end{aligned}
$$

Decreasing Annuity-Due Accumulated Value Factor

The accumulated value (at $t = n$) of this annual decreasing annuity–due, where the annual effective rate of interest is i, shall be denoted as $(D\ddot{s})_{\overline{n}|i}$ and can be calculated using the same approach as above or by simply using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$
\begin{aligned}
(D\ddot{s})_{\overline{n}|i} &= (\ddot{D}a)_{\overline{n}|i} \cdot (1 + i)^n \\
&= \left(\frac{n - a_{\overline{n}|i}}{d} \right) \cdot (1 + i)^n \\
&= \frac{n \cdot (1 + i)^n - s_{\overline{n}|i}}{d}
\end{aligned}
$$

Basic Relationship $1 : \ddot{a}_{\overline{n|}} = i \cdot (Ia)_{\overline{n|}} + nv^n$

Consider an n–year investment where 1 is invested at the beginning of each year. The present value of this multiple payment income stream at $t = 0$ is $\ddot{a}_{\overline{n|}}$.

Alternatively, consider a n–year investment where 1 is invested at the beginning of each year and produces increasing annual interest payments progressing to $n \cdot i$ by the end of the last year with the total payments $(n \times 1)$ refunded at $t = n$.

The present value of this multiple payment income stream at $t = 0$ is $i \cdot (Ia)_{\overline{n|}} + nv^n$.

Therefore, the present value of both investment opportunities are equal.

Also note that $(Ia)_{\overline{n|}} = \dfrac{\ddot{a}_{\overline{n|}} - n \cdot v^n}{i} \rightarrow \ddot{a}_{\overline{n|}} = i \cdot (Ia)_{\overline{n|}} + nv^n$.

3.5 Continuously Payable Varying Annuities

– payments are made at a continuous rate although they increase at discrete times.

Continuously Payable Increasing Annuity Present Value Factors

The present value (at $t = 0$) of an increasing annuity, where payments are being made continuously at an annual rate t, increasing at a discrete time t and where the annual effective rate of interest is i, shall be denoted as $(I\bar{a})_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}(I\bar{a})_{\overline{n}|} &= \bar{a}_{\overline{1}|} + 2\bar{a}_{\overline{1}|}v + 3\bar{a}_{\overline{1}|}v^2 + ... + n\bar{a}_{\overline{1}|}v^{n-1} \\ &= \bar{a}_{\overline{1}|}(1 + 2v + 3v^2 + ... + nv^{n-1}) \\ &= \bar{a}_{\overline{1}|}(I\ddot{a})_{\overline{n}|} \\ &= \frac{1-v}{\delta} \cdot \left(\frac{\ddot{a}_{\overline{n}|} - nv^n}{d}\right) \\ &= \frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}\end{aligned}$$

Continuously Payable Increasing Annuity Accumulated Value Factor

The accumulated value (at $t = n$) of this annual increasing annuity where the annual effective rate of interest is i, shall be denoted as $(I\bar{s})_{\overline{n}|i}$ and can be calculated using the basic principle where an accumulated value is equal to its present value carried forward with interest.

$$\begin{aligned}(I\bar{s})_{\overline{n}|} &= (1+i)^n (I\bar{a})_{\overline{n}|} \\ &= (1+i)^n \left(\frac{\ddot{a}_{\overline{n}|} - nv^n}{\delta}\right) \\ &= \frac{\ddot{s}_{\overline{n}|} - n}{\delta}\end{aligned}$$

Continuously Payable Increasing Perpetuity Present Value Factor

The present value (at $t = 0$) of an increasing perpetuity, where payments are being made continuously at an annual rate of t, increasing at a discrete time t and where the annual effective rate of interest is i, shall be denoted as $(I\bar{a})_{\overline{\infty}|i}$ and is calculated as follows:

$$\begin{aligned}(I\bar{a})_{\overline{\infty}|} &= \bar{a}_{\overline{1}|} + 2\bar{a}_{\overline{1}|}v + 3\bar{a}_{\overline{1}|}v^2 + ... \\ &= \bar{a}_{\overline{1}|}(1 + 2v + 3v^2 + ...) \\ &= \bar{a}_{\overline{1}|}(I\ddot{a})_{\overline{\infty}|} = \frac{1-v}{\delta} \cdot \frac{1}{d^2} \\ &= \frac{1}{\delta \cdot d}\end{aligned}$$

Continuously Payable Decreasing Annuity Present Value Factor

The present value (at $t = 0$) of a decreasing annuity where payments are being made continuously at an annual rate t, decreasing at a discrete time t and where the annual effective rate of interest is i, shall be denoted as $(D\bar{a})_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}(D\bar{a})_{\overline{n}|} &= n\bar{a}_{\overline{1}|} + (n-1)\bar{a}_{\overline{1}|}v + (n-2)\bar{a}_{\overline{1}|}v^2 + ... + \bar{a}_{\overline{1}|}v^{n-1} \\ &= \bar{a}_{\overline{1}|}(n + (n-1)v + (n-2)v^2 + ... + v^{n-1}) \\ &= \bar{a}_{\overline{1}|}(D\ddot{a})_{\overline{n}|} = \frac{1-v}{\delta} \cdot \frac{n - a_{\overline{n}|}}{d} \\ &= \frac{n - a_{\overline{n}|}}{\delta}\end{aligned}$$

Continuously Payable Decreasing Annuity Accumulated Value Factor

The accumulated value (at $t = n$) of this annual decreasing annuity, where the effective rate of interest is i, shall be denoted as $(D\bar{s})_{\overline{n}|i}$ and can be calculated using the basic principle where an accumulated value is equal to its present value carried forward with interest.

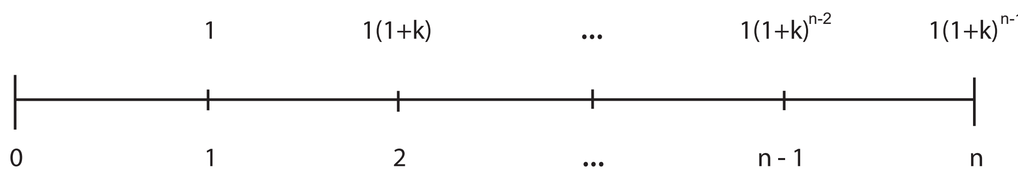
$$\begin{aligned}(D\bar{s})_{\overline{n}|} &= (1+i)^n (D\bar{a})_{\overline{n}|} \\ &= (1+i)^n \left(\frac{n - a_{\overline{n}|}}{\delta}\right) \\ &= \frac{n(1+i)^n - s_{\overline{n}|}}{\delta}\end{aligned}$$

3.6 Compound Increasing Annuities

Annuity-Immediate

An annuity-immediate is payable over n years with the first payment equal to 1 and each subsequent payment increasing by $(1+k)$.

The time line diagram below illustrates the above scenario:



Compound Increasing Annuity-Immediate Present Value Factor

The present value (at $t = 0$) of this annual geometrically increasing annuity–immediate, where the annual effective rate of interest is i, shall be calculated as follows:

$$\begin{aligned}
PV_0 &= (1)v_i + (1+k)v_i^2 + \cdots + (1+k)^{n-2}v_i^{n-1} + (1+k)^{n-1}v_i^n \\
&= v_i[1 + (1+k)v_i + \cdots + (1+k)^{n-2}v_i^{n-2} + (1+k)^{n-1}v_i^{n-1}] \\
&= \left(\frac{1}{1+i}\right)\left[1 + \left(\frac{1+k}{1+i}\right) + \cdots + \left(\frac{1+k}{1+i}\right)^{n-2} + \left(\frac{1+k}{1+i}\right)^{n-1}\right] \\
&= \left(\frac{1}{1+i}\right)\left[\frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)}\right] \\
&= \left(\frac{1}{1+i}\right)\left[\frac{1-v^n_{j=\frac{1+i}{1+k}-1}}{1-v_{j=\frac{1+i}{1+k}-1}}\right] \\
&= \left(\frac{1}{1+i}\right)\cdot \ddot{a}_{\overline{n}|j=\frac{1+i}{1+k}-1}
\end{aligned}$$

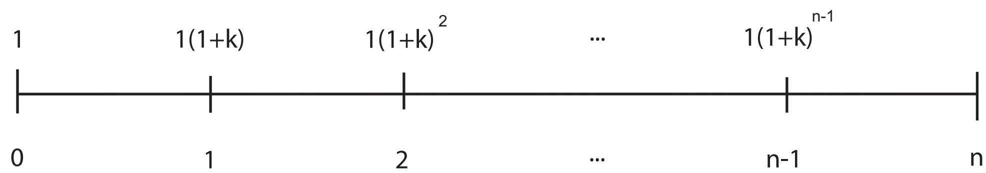
Compound Increasing Annuity-Immediate Accumulated Value Factor

The accumulated value (at $t = n$) of an annual geometric increasing annuity-immediate, where the annual effective rate of interest is i, can be calculated using the same approach as above or calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_n &= PV_0 \cdot (1+i)^n \\
&= \left(\frac{1}{1+i}\right) \cdot \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}} (1+i)^n \\
&= \left(\frac{1}{1+i}\right) \cdot \ddot{s}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}
\end{aligned}$$

Annuity-Due

An annuity-due is payable over n years with the first payment equal to 1 and each subsequent payment increasing by $(1+k)$. The time line diagram below illustrates the above scenario:



Compound Increasing Annuity-Due Present Value Factor

The present value (at $t=0$) of this annual geometrically increasing annuity–due, where the annual effective rate of interest is i, shall be calculated as follows:

$$\begin{aligned}
PV_0 &= (1) + (1+k)v_i + \cdots + (1+k)^{n-2}v_i^{n-2} + (1+k)^{n-1}v_i^{n-1} \\
&= 1 + \left(\frac{1+k}{1+i}\right) + \cdots + \left(\frac{1+k}{1+i}\right)^{n-2} + \left(\frac{1+k}{1+i}\right)^{n-1} \\
&= \frac{1-\left(\frac{1+k}{1+i}\right)^n}{1-\left(\frac{1+k}{1+i}\right)} \\
&= \frac{1-v^n_{j=\frac{1+i}{1+k}-1}}{1-v_{j=\frac{1+i}{1+k}-1}} \\
&= \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}
\end{aligned}$$

This present value could also have been achieved by simply multiplying the annuity-immediate version by $(1+i)$; $\left(\frac{1}{1+i}\right)\cdot \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}} \cdot (1+i)$.

Compound Increasing Annuity-Due Accumulated Value Factor

The accumulated value (at $t=n$) of an annual geometric increasing annuity-due, where the annual effective rate of interest is i, can be calculated using the same approach as above or calculated by using the basic principle where an accumulated value is equal to its present value carried forward with interest:

$$\begin{aligned}
FV_n &= PV_0 \cdot (1+i)^n \\
&= \ddot{a}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}(1+i)^n \\
&= \ddot{s}_{\overline{n}|_{j=\frac{1+i}{1+k}-1}}
\end{aligned}$$

3.7 Continuously Varying Payment Streams

Continuously Varying Payment Stream Present Value Factor

The present value (at t=a) of this continuously varying payment stream, where the payment stream ρ_t is from $t = a$ to $t = b$ and the annual force of interest is δ_t:

$$\int_a^b \rho_t \cdot e^{-\int_a^t \delta_s \cdot ds} dt$$

Continuously Varying Payment Stream Accumulated Value Factor

The accumulated value (at t=b) of this continuously varying payment stream, where the payment ρ is from $t = a$ to $t = b$ and the annual effective rate of interest is δ_t:

$$\int_a^b \rho_t \cdot e^{\int_a^t \delta_s \cdot ds} dt$$

3.8 Continuously Increasing Annuities

– payments are made continuously at a varying rate every year for the next n years

Continuously Increasing Annuity Present Value Factor

– The present value (at $t = 0$) of an increasing annuity, where payments are being made continuously at annual rate t at time t and where the annual effective rate of interest is i, shall be denoted as $\left(\bar{I}\bar{a}\right)_{\overline{n}|_i}$ and is calculated as follows (using integration by parts):

$$
\begin{aligned}
\left(\bar{I}\bar{a}\right)_{\overline{n}|_i} &= \int_0^n tv^t dt \\
&= \int_0^n \underbrace{t}_{u} \underbrace{e^{-\delta t} dt}_{dv} \\
&= \underbrace{t\frac{e^{-\delta t}}{-\delta}}_{u \cdot v}\Bigg]_0^n - \int_0^n \underbrace{\frac{e^{-\delta t}}{-\delta} dt}_{v \cdot du} \\
&= \frac{n \cdot e^{-\delta n}}{-\delta} + \frac{\bar{a}_{\overline{n}|_i}}{\delta} \\
&= \frac{\bar{a}_{\overline{n}|_i} - n \cdot v_i^n}{\delta}
\end{aligned}
$$

– The present value (at $t = 0$) of an annuity, where payment at time t is defined as $f(t)dt$ and where the annual effective rate of interest is i, shall be calculated as follows:

$$\int_0^n f(t) \cdot v_i^t dt$$

– if the force of interest becomes variable then the above formula becomes:

$$\int_0^n f(t) \cdot e^{\int_0^t \delta_s ds} dt$$

Continuously Increasing Annuity Accumulated Value Factor

– The accumulated value (at $t = n$) of an increasing annuity, where payments are being made continuously at annual rate t at time t and where the annual effective rate of interest is i, shall be denoted as $\left(\bar{I}\bar{s}\right)_{\overline{n}|_i}$ and is calculated as follows:

$$
\begin{aligned}
\left(\bar{I}\bar{s}\right)_{\overline{n}|_i} &= (1+i)^n \left(\bar{I}\bar{a}\right)_{\overline{n}|_i} \\
&= (1+i)^n \left(\frac{\bar{a}_{\overline{n}|} - nv_i^n}{\delta}\right) \\
&= \frac{\bar{s}_{\overline{n}|_i} - n}{\delta}
\end{aligned}
$$

Continuously Payable Continuously Increasing Perpetuity Present Value Factor

– The present value (at $t = 0$) of an increasing perpetuity, where payments are being made continuously at an annual rate of t, increasing at a continuous time t and where the annual effective rate of interest is i, shall be denoted as $(\bar{I}\bar{a})_{\overline{\infty}|i}$ and is calculated as follows:

$$
\begin{aligned}
(\bar{I}\bar{a})_{\overline{\infty}|i} &= \int_0^{\infty} tv^t dt \\
&= \frac{\bar{a}_{\overline{\infty}|i} - \infty \cdot v_i^{\infty}}{\delta} \\
&= \frac{\frac{1}{\delta} - 0}{\delta} \\
&= \frac{1}{\delta^2}
\end{aligned}
$$

3.9 Continuously Decreasing Annuities

– payments are made continuously at a varying rate every year for the next n years

Continuously Decreasing Annuity Present Value Factor

– The present value (at $t = 0$) of a decreasing annuity where payments are being made continuously at an annual rate t, decreasing at a continuous time t and where the annual effective rate of interest is i, shall be denoted as $(\bar{D}\bar{a})_{\overline{n}|i}$ and is calculated as follows:

$$
\begin{aligned}
(\bar{D}\bar{a})_{\overline{n}|i} &= \int_0^{n} (n-t) \cdot v^t dt \\
&= n \underbrace{\int_0^{n} v^t dt}_{\bar{a}_{\overline{n}|}} - \underbrace{\int_0^{n} t \cdot v^t dt}_{(\bar{I}\bar{a})_{\overline{n}|}} \\
&= n \cdot \bar{a}_{\overline{n}|} - \left(\frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} \right) \\
&= \frac{n \cdot (1 - v^n) - \bar{a}_{\overline{n}|} + nv^n}{\delta} \\
&= \frac{n - \bar{a}_{\overline{n}|}}{\delta}
\end{aligned}
$$

Continuously Decreasing Annuity Accumulated Value Factor

The accumulated value (at $t = n$) of this annual decreasing annuity, where the effective rate of interest is i, shall be denoted as $(\bar{D}\bar{s})_{\overline{n}|i}$ and can be calculated using the basic principle where an accumulated value is equal to its present value carried forward with interest.

$$
\begin{aligned}
(\bar{D}\bar{s})_{\overline{n}|i} &= (1+i)^n (\bar{D}\bar{a})_{\overline{n}|i} \\
&= (1+i)^n \left(\frac{n - \bar{a}_{\overline{n}|i}}{\delta} \right) \\
&= \frac{n(1+i)^n - \bar{s}_{\overline{n}|i}}{\delta}
\end{aligned}
$$

Exercises and Solutions

3.1 Increasing Annuity-Immediate

Exercise (a)

A deposit of 1 is made at the end of each year for 10 years into a bank account that pays interest at the end of each year at an annual effective rate of j.

Each interest payment is reinvested into another account where it earns an annual effective interest rate of $\frac{j}{2}$.

The accumulated value of these interest payments at the end of 10 years is 4.0122. Calculate j.

Solution (a)

$$j \cdot (Is)_{\overline{9}|\frac{j}{2}} = 4.0122$$

$$j \cdot \left[\frac{\ddot{a}_{\overline{9}|\frac{j}{2}} - 9}{\frac{j}{2}} \right] = 4.0122$$

$$\ddot{a}_{\overline{9}|\frac{j}{2}} = \frac{4.0122}{2} + 9 = 11.0061 \rightarrow \frac{j}{2} = 4\% \rightarrow j = 8\%$$

Exercise (b)

You are given two series of payments.

Series A is a perpetuity with payments of 1 at the beginning of each of the first three years, 2 at the beginning of each of the next three years, 3 at the beginning of each of the next three years, and so on.

Series B is a perpetuity with payments of K at the beginning of each of the first two years, $2K$ at the beginning of each of the next two years, $3K$ at the beginning of each of the next two years, and so on.

The present value of these two series is equal to each other.

Determine K.

Solution (b)

$$\ddot{a}_{\overline{3}|} \cdot (Ia)_{\overline{\infty}|_{i'=(1+i)^3-1}} = (K\ddot{a}_{\overline{2}|}) \cdot (Ia)_{\overline{\infty}|_{i'=(1+i)^2-1}}$$

$$\ddot{a}_{\overline{3}|} \cdot \frac{1}{i'd'} = (K\ddot{a}_{\overline{2}|}) \cdot \frac{1}{i'd'}$$

$$\ddot{a}_{\overline{3}|} \cdot \frac{1+i'}{i'i'} = (K\ddot{a}_{\overline{2}|}) \cdot \frac{1+i'}{i'i'}$$

$$\ddot{a}_{\overline{3}|} \cdot \frac{1+(1+i)^3-1}{[(1+i)^3-1]^2} = (Ka_{\overline{2}|}) \cdot \frac{1+(1+i)^2-1}{[(1+i)^2-1]^2}$$

$$\frac{(1+i)^3-1}{d} \cdot \frac{(1+i)^3}{[(1+i)^3-1]^2} = K\frac{(1+i)^2-1}{d} \cdot \frac{(1+i)^2}{[(1+i)^2-1]^2}$$

$$\frac{(1+i)}{(1+i)^3-1} = \frac{K}{(1+i)^2-1}$$

$$K = \frac{(1+i)^2-1}{(1+i)^3-1} \cdot (1+i) = \frac{s_{\overline{2}|}(1+i)}{s_{\overline{3}|}} = \frac{v^2 s_{\overline{2}|}}{v^3 s_{\overline{3}|}} = \frac{a_{\overline{2}|}}{a_{\overline{3}|}}$$

Exercise (c)

Danielle borrows 100,000 to be repaid over 30 years. You are given:

(i) Her first payment is X at the end of year 1.

(ii) Her payments increase at the rate of 100 per year for the next 19 years and remain level for the following 10 years.

(iii) The effective rate of interest is 5% per annum.

Calculate X.

Solution (c)

$$10,000 = Xa_{\overline{30}|} + v100(Ia)_{\overline{19}|} + v^{20}1,900a_{\overline{10}|}$$

$$10,000 = X\left[\frac{1-v^{30}}{i}\right] + 100v\left[\frac{\ddot{a}_{\overline{19}|} - 19v^{19}}{i}\right] + 1,900v^{20}\left[\frac{1-v^{10}}{i}\right]$$

$$X = 5,504.74$$

3.2 Increasing Annuity-Due

Exercise (a)

You buy an increasing perpetuity-due with annual payments starting at 5 and increasing by 5 each year until the payment reaches 100. The payments remain at 100 thereafter. The annual effective interest rate is 7.5%. Determine the present value of this perpetuity.

Solution (a)

$$(I\ddot{a})_{\overline{\infty}|} = \frac{1}{d^2} = 205.4444$$

$$5(I\ddot{a})_{\overline{\infty}|} - 5v^{20}(I\ddot{a})_{\overline{\infty}|} = 5(205.4444)(1 - .235413) = 785.40$$

3.3 Decreasing Annuity-Immediate

Exercise (a)

You can purchase one of two annuities:

Annuity 1: An 11-year annuity-immediate with payments starting at 1 and then increasing by 1 for 5 years, followed by payments starting at 5 and then decreasing by 1 for 4 years. The first payment is in two years.

Annuity 2: A 13-year annuity-immediate with payments starting at 5 and then increasing by 2 for 5 years, followed by payments starting at 16 and then decreasing by 4 for 3 years and then decreasing by 1 for 3 years. The first payment is in one year.

The present value of Annuity 1 is equal to 36.

Determine the present value of Annuity 2.

Solution (a)

$$PV_1 = v[\text{pyramid annuity}] = v\left[(Ia)_{\overline{6}|} + v^6 (Da)_{\overline{5}|}\right] = v\left[\frac{\ddot{a}_{\overline{6}|} - 6v^6}{i} + v^6 \frac{5 - a_{\overline{5}|}}{i}\right]$$

$$v\left[\ddot{a}_{\overline{6}|} \cdot a_{\overline{6}|}\right] = a_{\overline{6}|} \cdot a_{\overline{6}|} = \left[a_{\overline{6}|}\right]^2 = 36 \rightarrow a_{\overline{6}|} = 6 \rightarrow i = 0\%$$

$$PV_2 = 5 + 7 + 9 + 11 + 13 + 15 + 16 + 12 + 8 + 4 + 3 + 2 + 1 = 106$$

Exercise (b)

1,000 is deposited into Fund X, which earns an annual effective rate of interest of i. At the end of each year, the interest earned plus an additional 100 is withdrawn from the fund. At the end of the tenth year, the fund is depleted.

The annual withdrawals of interest and principal are deposited into Fund Y, which earns an annual effective rate of 9%.

The accumulated value of Fund Y at the end of year 10 is 2,085.

Determine i.

Solution (b)

$$\frac{1,000i}{10}(Ds)_{\overline{10}|9\%} + 100 \cdot s_{\overline{10}|9\%} = 100i\left[\frac{10(1.09)^{10} - s_{\overline{10}|9\%}}{.09}\right] + 100 \cdot (15.19293)$$

$$100i(94.23) + 1,519.29 = 2,085 \rightarrow i = 6\%$$

Exercise (c)

You are given an annuity-immediate paying 10 for 10 years, then decreasing by one per year for nine years and paying one per year thereafter, forever. The annual effective rate of interest is 4%. Calculate the present value of this annuity.

Solution (c)

$$PV = 10a_{\overline{10|}} + v^{10}(Da)_{\overline{9|}} + v^{19}a_{\overline{\infty|}}$$

$$= 10\left[\frac{1-v^{10}}{i}\right] + v^{10}\left[\frac{9-a_{\overline{9|}}}{i}\right] + v^{19}\cdot\frac{1}{i}$$

$$= 199.40$$

3.4 Decreasing Annuity-Due

Exercise (a)

Debbie receives her first annual payment of 5 today. Each subsequent payment decreases by 1 per year until time 4 years. After year 4, each payment increases by 1 until time 8 years. The annual interest rate is 6%. Determine the present value.

Solution (a)

The present value at time 0:

$$5 + 4v + 3v^2 + 2v^3 + v^4 + 2v^5 + 3v^6 + 4v^7 + 5v^8$$

Breaking up the payments from time 0 to time 4 is a decreasing annuity-due.

$$5 + 4v + 3v^2 + 2v^3 + v^4 = (D\ddot{a})_{\overline{5|}6\%}$$

$$(D\ddot{a})_{\overline{5|}6\%} = 13.914903$$

Then from time 5 to time 8 is an increasing annuity-due which is missing it's first payment.

$$2v + 3v^2 + 4v^3 + 5v^4 = (I\ddot{a})_{\overline{5|}6\%}$$

$$(I\ddot{a})_{\overline{5|}6\%} - 1 = 11.875734$$

$$PV_0 = [(I\ddot{a})_{\overline{5|}6\%} - 1]v^4_{6\%} = 23.32$$

3.5 Continuously Payable Varying Annuities

Exercise (a)

Calculate the accumulated value at time 10 years where payments are received continuously over each year. The payment is 100 during the first year and subsequent payments increase by 10 each year. The annual effective rate of interest is 4%.

Solution (b)

$$= 90\bar{s}_{\overline{10}|4\%} + 10(I\bar{s})_{\overline{10}|4\%}$$

$$= 90(12.244660) + 10(63.393834)$$

$$= 1165.41$$

3.6 Compounding Increasing Annuities

Exercise (a)

You have just purchased an increasing annuity immediate for 75,000 that makes twenty annual payments as follows:

(i) $5P, 10P, ..., 50P$ during years 1 through 10, and

(ii) $50P(1.05), 50P(1.05)^2, ..., 50P(1.05)^{10}$ during years 11 through 20.

The annual effective interest rate is 7% for the first 10 years and 5%, thereafter. Solve for P.

Solution (a)

$$75,000 = 5P \cdot (Ia)_{\overline{10}|7\%} + 50P(1.05) \cdot v_{7\%}^{10} \cdot v_{5\%} \cdot \ddot{a}_{\overline{10}|j=0}$$

$$75,000 = 5 \cdot P \left[\frac{\ddot{a}_{\overline{10}|7\%} - 10v_{7\%}^{10}}{i} \right] + 50P \cdot v_{7\%}^{10} \cdot (10)$$

$$P = \frac{75,000}{5\left[\frac{\ddot{a}_{\overline{10}|7\%} - 10v_{7\%}^{10}}{i}\right] + 500 \cdot v_{7\%}^{10}} = \frac{75,000}{427.8703} = 175.29$$

Exercise (b)

You have just purchased an annuity immediate for 75,000 that makes ten annual payments as follows:

(i) $12.45P, 4P, 6P, 8P, 10P$ for years 1 through 5 and

(ii) $10P(1.07), 10P(1.07)^2, 10P(1.07)^3, 10P(1.07)^4, 10P(1.07)^5$ for years 6 through 10.

The annual effective interest rate is 5% for the first 5 years and 7%, thereafter. Solve for P.

Solution (b)

$$75,000 = (10.45P)v_{5\%} + (2P) \cdot (Ia)_{\overline{5}|5\%} + v_{5\%}^5 \cdot v_{7\%} \cdot [10P(1.07)]\ddot{a}_{\overline{5}|i'=\frac{1.07}{1.07}-1=0\%}$$

$$75,000 = P \cdot \left[10.45v_{5\%} + 2\left(\frac{\ddot{a}_{\overline{5}|5\%} - 5v_{5\%}^5}{i}\right) + v_{5\%}^5 \cdot v_{7\%} \cdot [10(1.07)](5)\right]$$

$$75,000 = P \cdot (9.9524 + 25.1328 + 39.1763) = P \cdot (74.2615) \to P = 1,009.94.$$

Exercise (c)

You have just purchased an increasing annuity-immediate for 50,000 that makes twenty annual payments as follows:

(i) $P, 2P, ..., 10P$ in years 1 through 10; and

(ii) $10P(1.05), 10P(1.05)^2, ..., 10P(1.05)^{10}$ in years 11 through 20.

The annual effective interest rate is 7% for the first 10 years and 5% thereafter. Calculate P.

Solution (c)

$$50,000 = P(Ia)_{\overline{10}|7\%} + \left[10P\frac{(1.05)}{(1.05)} + 10P\frac{(1.05)^2}{(1.05)^2} + ... + 10P\frac{(1.05)^{10}}{(1.05)^{10}}\right]v_{7\%}^{10}$$

$$50,000 = P\left[\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{i}\right] + [10P \cdot 10]v_{7\%}^{10}$$

$$P = \frac{50,000}{\left[\frac{\ddot{a}_{\overline{10}|} - 10v^{10}}{i}\right] + 100v_{7\%}^{10}} = \frac{50,000}{85.57406}$$

$$P = 584.29$$

3.7 Continuously Varying Payment Streams

Exercise (a)

Find the present value of a continuous increasing annuity with a term of 10 years if the force of interest is $\delta = 0.04$ and if the rate of payment at time t is t^2 per annum.

Solution (a)

$$\int_0^n t^2 e^{-\delta \cdot t} dt = \int_0^{10} t^2 e^{-0.04t} dt$$

Using integration by parts:

$$= -\frac{t^2}{0.04} e^{-0.04t}\Big|_0^{10} + \frac{2}{0.04} \int_0^{10} t \cdot e^{-0.04t} dt$$

$$= -\frac{10^2}{0.04} e^{-0.04(10)} - \frac{2t}{(0.04)^2} e^{-0.04t}\Big|_0^{10} + \frac{2}{(0.04)^2} \int_0^{10} e^{-0.04t} dt$$

$$= -\frac{10^2}{0.04} e^{-0.4} - \frac{20}{(0.04)^2} e^{-0.04(10)} - \frac{2}{(0.04)^3} e^{-0.04t}\Big|_0^{10}$$

$$= -\frac{10^2}{0.04} e^{-0.4} - \frac{20}{(0.04)^2} e^{-.4} - \frac{2}{(0.04)^3} e^{-0.04(10)} + \frac{2}{(0.04)^3}$$

$$= \frac{2}{(0.04)^3} - e^{-0.4} \left[\frac{10^2}{0.04} + \frac{20}{(0.04)^2} + \frac{2}{(0.04)^3} \right]$$

$$= 247.6978$$

3.8 Continuously Increasing Annuities

Exercise (a)

Find the ratio of the total payments made under $(\bar{I}\bar{a})_{\overline{10|}}$ during the second half of the term of the annuity to those made during the first half.

Solution (a)

The payment is $t \cdot dt$ at time t.

During the second half the total payment is:

$$\int_5^{10} t \cdot dt = \frac{1}{2}t^2\Big]_5^{10} = 37.50$$

During the first half the total payment is:

$$\int_0^5 t \cdot dt = \frac{1}{2}t^2\Big]_0^5 = 12.50$$

The ratio is:

$$\frac{37.50}{12.50} = 3$$

Exercise (b)

Evaluate $(\bar{I}\bar{a})_{\overline{\infty|}}$ if $\delta = .08$.

Solution (b)

$$(\bar{I}\bar{a})_{\overline{\infty|}} = \int_0^{\infty} t \cdot dt$$

$$= \int_0^{\infty} t \cdot e^{-.08t} dt$$

$$= -\frac{t}{.08} \cdot e^{-.08t}\Big|_0^{\infty} + \frac{1}{.08}\int_0^{\infty} e^{-.08t} dt$$

$$= -\frac{t}{.08} \cdot e^{-.08t} - \left(\frac{1}{.08}\right)^2 \cdot e^{-.08t}\Big|_0^{\infty}$$

$$= \left(\frac{1}{.08}\right)^2 = 156.25$$

Exercise (c)

A perpetuity is payable continuously at the annual rate of $1+t^2$ at time t. If $\delta = .05$, find the present value of the perpetuity.

Solution (c)

$$PV = \int_0^\infty (1+t^2)e^{-.05t}dt$$

using integration by parts

$$= -\frac{1}{.05}e^{-.05t}\Big|_0^\infty - \frac{t^2}{.05}e^{-.05t}\Big|_0^\infty + \frac{2}{.05}\int_0^\infty t\cdot e^{-.05t}dt$$

$$= \frac{1}{.05} + \frac{2}{.05}\left[-\frac{t}{.05}e^{-.05t}\Big|_0^\infty + \frac{1}{.05}\int_0^\infty e^{-.05t}dt\right]$$

$$= \frac{1}{.05} - \frac{2}{(.05)^3}e^{-.05t}\Big|_0^\infty = \frac{1}{.05} + \frac{2}{(.05)^3} = 20 + 2(20)^3 = 16,020.$$

Exercise (d)

A continuously increasing annuity with a term of n years has payments payable at an annual rate t at time t. The force of interest is equal to $\frac{1}{n}$. Calculate the present value of this annuity.

Solution (d)

$$(\bar{I}\bar{a})_{\overline{n}|} = \frac{\bar{a}_{\overline{n}|} - nv^n}{\delta} = \frac{\frac{1-e^{-\delta n}}{\delta} - ne^{-\delta n}}{\delta}$$

$$= \frac{\frac{1-e^{-\frac{1}{n}\cdot n}}{\frac{1}{n}} - ne^{-\frac{1}{n}\cdot n}}{\frac{1}{n}}$$

$$= n\left(n(1-e^{-1}) - ne^{-1}\right) = n(n - ne^{-1} - ne^{-1}) = n(n-2ne^{-1}) = n^2(1-2e^{-1})$$

3.9 Continuously Decreasing Annuities

Exercise (a)

Find the ratio of the total payments made under $(\bar{D}\bar{a})_{\overline{10}|}$ during the second half of the term of the annuity to those made during the first half.

Solution (a)

The payment is $(10 - t) \cdot dt$ at time t.

During the second half the total payment is:

$$\int_5^{10} (10 - t)dt = -\frac{(10 - t)^2}{2}\bigg|_5^{10} = 12.5$$

During the first half the total payment is:

$$\int_0^{5} (10 - t)dt = -\frac{(10 - t)^2}{2}\bigg|_0^{5} = 37.5$$

The ratio is:

$$= \frac{12.5}{37.5} = \frac{1}{3}$$

4 Non-Annual Interest Rate and Annuities

Overview

- an effective rate of interest, i, is paid once per year at the end of the year
- a nominal rate of interest, $i^{(p)}$, is paid more frequently during the year (p times) and at the end of the sub-period (nominal rates are also quoted as annual rates)
- nominal rates are adjusted to reflect the rate to be paid during the sub–period; for example, $i^{(2)}$ is the nominal rate of interest convertible semi-annually

$$i^{(2)} = 10\% \rightarrow \frac{i^{(2)}}{2} = \frac{10\%}{2} = 5\% \text{ paid every 6 months}$$

4.1 Non-Annual Interest and Discount Rates

Non-Annual pthly Effective Interest Rates

- with effective interest, you have interest i, paid at p periods per year

$$(1+i)^{\frac{1}{p}} - 1$$

Non-Annual pthly Effective Discount Rates

- with effective discount, you have discount d, paid at p periods per year

$$1 - (1-d)^{\frac{1}{p}} = 1 - (1+i)^{-\frac{1}{p}}$$

4.2 Nominal p^{thly} Interest Rates: $i^{(p)}$

Equivalency to Effective Rates of Interest: $i, i^{(p)}$

- with effective interest, you have interest, i, paid at the end of the year
- with nominal interest, you have interest $\frac{i^{(p)}}{p}$, paid at the end of each sub-period and this is done p times over the year (p sub-periods per year)

$$(1+i) = \left(1 + \frac{i^{(p)}}{p}\right)^p$$

- if given an effective rate of interest, a nominal rate of interest can be determined:

$$i^{(p)} = p[(1+i)^{1/p} - 1]$$

- if given an effective rate of interest, the interest rate per sub-period can be determined:

$$\frac{i^{(p)}}{p} = (1+i)^{1/p} - 1$$

4.3 Nominal p^{thly} Discount Rates: $d^{(p)}$

Equivalency to Effective Rates of Discount: $d, d^{(p)}$

– with effective discount, you have discount, d, paid at the beginning of the year

– with nominal discount, you have discount $\frac{d^{(p)}}{p}$, paid at the beginning of each sub-period and this is done p times over the year (p sub-periods per year)

$$(1-d) = \left(1 - \frac{d^{(p)}}{p}\right)^{p}$$

– if given an effective rate of discount, a nominal rate of discount can be determined

$$d^{(p)} = p[1-(1-d)^{1/p}]$$

– the discount rate per sub-period can be determined, if given the effective discount rate

$$\frac{d^{(p)}}{p} = 1-(1-d)^{1/p}$$

Relationship Between $\frac{i^{(p)}}{p}$ **and** $\frac{d^{(p)}}{p}$

– when using effective rates, you must have $(1+i)$ or $(1-d)^{-1}$ by the end of the year

$$(1+i) = \frac{1}{v} = \frac{1}{(1-d)} = (1-d)^{-1}$$

– when replacing the effective rate formulas with their nominal rate counterparts, you have

$$\left(1 + \frac{i^{(m)}}{m}\right)^{m} = \left(1 - \frac{d^{(p)}}{p}\right)^{-p}$$

– when $p = m$

$$
\begin{aligned}
\left(1+\frac{i^{(m)}}{m}\right)^{m} &= \left(1-\frac{d^{(m)}}{m}\right)^{-m} \\
1+\frac{i^{(m)}}{m} &= \left(1-\frac{d^{(m)}}{m}\right)^{-1} \\
1+\frac{i^{(m)}}{m} &= \frac{m}{m\cdot d^{(m)}} \\
\frac{i^{(m)}}{m} &= \frac{m}{m\cdot d^{(m)}} - 1 = \frac{m-m+d^{(m)}}{m-d^{(m)}} \\
\frac{i^{(m)}}{m} &= \frac{d^{(m)}}{m-d^{(m)}} \\
\frac{i^{(m)}}{m} &= \frac{\frac{d^{(m)}}{m}}{1-\frac{d^{(m)}}{m}}
\end{aligned}
$$

– the interest rate over the sub-period is the ratio of the discount paid to the amount at the beginning of the sub-period (principle of the interest rate still holds)

$$
\frac{d^{(m)}}{m} = \frac{\frac{i^{(m)}}{m}}{1+\frac{i^{(m)}}{m}}
$$

– the discount rate over the sub–period is the ratio of interest paid to the amount at the end of the sub-period (principle of the discount rate still holds)

– the difference between interest paid at the end and at the beginning of the sub-period depends on the difference that is borrowed at the beginning of the sub-period and on the interest earned on that difference (principle of the interest and discount rates still holds)

$$
\begin{aligned}
\frac{i^{(m)}}{m} - \frac{d^{(m)}}{m} &= \frac{i^{(m)}}{m}\left[1-\left(1-\frac{d^{(m)}}{m}\right)\right] \\
&= \frac{i^{(m)}}{m}\cdot\frac{d^{(m)}}{m} \geq 0
\end{aligned}
$$

4.4 Annuities-Immediate Payable p^{thly}

Present Value Factor for a p^{thly} Annuity-Immediate

- payments of $\frac{1}{p}$ are made at the end of every $\frac{1}{p}$th of year for the next n years
- the present value (at $t = 0$) of an p^{th}ly annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $a^{(p)}_{\overline{n}|i}$ and is calculated as follows:

$$
\begin{aligned}
a^{(p)}_{\overline{n}|i} =& (\frac{1}{p})v_i^{\frac{1}{p}} + (\frac{1}{p})v_i^{\frac{2}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{p-1}{p}} + (\frac{1}{p})v_i^{\frac{p}{p}} && (1st\text{ year})\\
&+(\frac{1}{p})v_i^{\frac{p+1}{p}} + (\frac{1}{p})v_i^{\frac{p+2}{m}} + \cdots + (\frac{1}{p})v_i^{\frac{2p-1}{p}} + (\frac{1}{p})v_i^{\frac{2p}{p}} && (2nd\text{ year})\\
&\vdots\\
&+(\frac{1}{p})v_i^{\frac{(n-1)p+1}{p}} + (\frac{1}{p})v_i^{\frac{(n-1)p+2}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{np-1}{p}} + (\frac{1}{p})v_i^{\frac{np}{p}} && (\text{last year})\\
\\
=&(\frac{1}{p})v_i^{\frac{1}{p}}\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right] && (1st\text{ year})\\
&+(\frac{1}{p})v_i^{\frac{p+1}{p}}\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right] && (2nd\text{ year})\\
&\vdots\\
&+(\frac{1}{p})v_i^{\frac{(n-1)p+1}{p}}\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right] && (\text{last year})\\
\\
=&\left((\frac{1}{p})v_i^{\frac{1}{p}} + (\frac{1}{p})v_i^{\frac{p+1}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{(n-1)p+1}{m}})\right)\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right]\\
=&(\frac{1}{p})v_i^{\frac{1}{p}}\left(1 + v_i^{\frac{p}{p}} + \cdots + v_i^{(n-1)}\right)\left[\frac{1-(v_i^{\frac{1}{p}})^p}{1-v_i^{\frac{1}{p}}}\right]\\
=&(\frac{1}{p})\frac{1}{(1+i)^{\frac{1}{p}}}\left(\frac{1-v_i^n}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{p}}}\right]\\
=&(\frac{1}{p})\frac{1}{(1+i)^{\frac{1}{p}}}\left(\frac{1-v_i^n}{1-v_i^{\frac{1}{p}}}\right)\\
=&(\frac{1}{p})\left(\frac{1-v_i^n}{(1+i)^{\frac{1}{p}}-1}\right)\\
=&\frac{1-v_i^n}{p\left[(1+i)^{\frac{1}{p}}-1\right]}\\
=&\frac{1-v_i^n}{i^{(p)}} = \left(\frac{1}{p}\times p\right)\cdot\frac{1-v_i^n}{i^{(p)}}
\end{aligned}
$$

Accumulated Value Factor for a p^{thly} Annuity-Immediate

– the accumulated value (at $t = n$) of an p^{th}ly annuity–immediate, where the annual effective rate of interest is i, shall be denoted as $s^{(p)}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
s^{(p)}_{\overline{n}|i} =& (\frac{1}{p}) + (\frac{1}{p})(1+i)^{\frac{1}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{p-2}{p}} + (\frac{1}{p})(1+i)^{\frac{p-1}{p}} && \text{(last year)} \\
&+(\frac{1}{p})(1+i)^{\frac{p}{p}} + (\frac{1}{p})(1+i)^{\frac{p+1}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{2p-2}{p}} + (\frac{1}{p})(1+i)^{\frac{2p-1}{p}} && \text{(2}nd\text{ last year)} \\
&\vdots \\
&+(\frac{1}{p})(1+i)^{\frac{(n-1)p}{p}} + (\frac{1}{p})(1+i)^{\frac{(n-1)p+1}{m}} + \cdots + (\frac{1}{p})(1+i)^{\frac{np-2}{m}} + (\frac{1}{p})(1+i)^{\frac{np-1}{p}} && \text{(first year)} \\
\\
=& (\frac{1}{p}) \left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] && \text{(last year)} \\
&+(\frac{1}{p})(1+i)^{\frac{p}{p}} \left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] && \text{(2}nd\text{ last year)} \\
&\vdots \\
&+(\frac{1}{p})(1+i)^{\frac{(n-1)p}{p}} \left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] && \text{(first year)} \\
\\
=& \left((\frac{1}{p}) + (\frac{1}{p})(1+i)^{\frac{p}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{(n-1)p}{p})} \right) \left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] \\
=& (\frac{1}{p}) \left(1 + (1+i)^{\frac{p}{p}} + \cdots + (1+i)^{(n-1)}\right) \left[\frac{1 - ((1+i)^{\frac{1}{p}})^p}{1 - (1+i)^{\frac{1}{p}}}\right] \\
=& (\frac{1}{p}) \left(\frac{1 - (1+i)^n}{1 - (1+i)^1}\right) \cdot \left[\frac{1 - (1+i)^1}{1 - (1+i)^{\frac{1}{p}}}\right] \\
=& (\frac{1}{p}) \left(\frac{1 - (1+i)^n}{1 - (1+i)^{\frac{1}{p}}}\right) \\
=& \frac{(1+i)^n - 1}{p\left[(1+i)^{\frac{1}{p}} - 1\right]} \\
=& \frac{(1+i)^n - 1}{i^{(p)}} = \left(\frac{1}{p} \times p\right) \cdot \frac{(1+i)^n - 1}{i^{(p)}}
\end{aligned}$$

Basic Relationship $1: 1 = i^{(p)} \cdot a_{\overline{n|}}^{(p)} + v^n$

Basic Relationship $2: PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

– if the future value at time n, $s_{\overline{n|}}^{(p)}$, is discounted back to time 0, then you will have its present value, $a_{\overline{n|}}^{(p)}$

$$\begin{aligned} s_{\overline{n|}}^{(p)} \cdot v^n &= \left[\frac{(1+i)^n - 1}{i^{(p)}}\right] \cdot v^n \\ &= \frac{(1+i)^n \cdot v^n - v^n}{i^{(p)}} \\ &= \frac{1 - v^n}{i^{(p)}} \\ &= a_{\overline{n|}}^{(p)} \end{aligned}$$

– if the present value at time 0, $a_{\overline{n|}}^{(p)}$, is accumulated forward to time n, then you will have its future value, $s_{\overline{n|}}^{(p)}$

$$\begin{aligned} a_{\overline{n|}}^{(p)} \cdot (1+i)^n &= \left[\frac{1 - v^n}{i^{(p)}}\right](1+i)^n \\ &= \frac{(1+i)^n - v^n(1+i)^n}{i^{(p)}} \\ &= \frac{(1+i)^n - 1}{i^{(p)}} \\ &= s_{\overline{n|}}^{(p)} \end{aligned}$$

Basic Relationship $3: a_{\overline{n|}}^{(p)} = \frac{i}{i^{(p)}} \cdot a_{\overline{n|}}$**,** $s_{\overline{n|}}^{(p)} = \frac{i}{i^{(p)}} \cdot s_{\overline{n|}}$

– Consider payments of $\frac{1}{p}$ made at the end of every $\frac{1}{p}$th of year for the next n years. Over a one-year period, payments of $\frac{1}{p}$ made at the end of each p^{th} period will accumulate at the end of the year to a lump sum of $\left(\frac{1}{p} \times p\right) \cdot s_{\overline{1|}}^{(p)}$. If this end-of-year lump sum exists for each year of the n-year annuity-immediate, then the present value (at $t = 0$) of these end-of-year lump sums is the same as $\left(\frac{1}{p} \times p\right) \cdot a_{\overline{n|}}^{(p)}$:

$$\begin{aligned} \left(\frac{1}{p} \times p\right) \cdot a_{\overline{n|}}^{(p)} &= \left(\frac{1}{p} \times p\right) \cdot s_{\overline{1|}}^{(p)} \cdot a_{\overline{n|}} \\ a_{\overline{n|}}^{(p)} &= \frac{i}{i^{(p)}} \cdot a_{\overline{n|}} \end{aligned}$$

– Therefore, the accumulated value (at $t = n$) of these end-of-year lump sums is the same as $\left(\frac{1}{p} \times p\right) \cdot s_{\overline{n|}}^{(p)}$:

$$\begin{aligned} \left(\frac{1}{p} \times p\right) \cdot s_{\overline{n|}}^{(p)} &= \left(\frac{1}{m} \times p\right) \cdot s_{\overline{1|}}^{(p)} \cdot s_{\overline{n|}} \\ s_{\overline{n|}}^{(p)} &= \frac{i}{i^{(p)}} \cdot s_{\overline{n|}} \end{aligned}$$

4.5 Annuities-Due Payable p^{thly}

Present Value Factor of a p^{thly} Annuity-Due

- payments of $\frac{1}{p}$ are made at the beginning of every $\frac{1}{p}$th of year for the next n years
- the present value (at $t = 0$) of an p^{th}ly annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{a}^{(p)}_{\overline{n}|i}$ and is calculated as follows:

$$
\begin{aligned}
\ddot{a}^{(p)}_{\overline{n}|i} =&(\frac{1}{p}) + (\frac{1}{p})v_i^{\frac{1}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{p-2}{p}} + (\frac{1}{p})v_i^{\frac{p-1}{p}} && \text{(1}st\text{ year)}\\
&+(\frac{1}{p})v_i^{\frac{p}{p}} + (\frac{1}{p})v_i^{\frac{p+1}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{2p-2}{p}} + (\frac{1}{p})v_i^{\frac{2p-1}{p}} && \text{(2}nd\text{ year)}\\
&\vdots\\
&+(\frac{1}{p})v_i^{\frac{(n-1)p}{p}} + (\frac{1}{p})v_i^{\frac{(n-1)p+1}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{np-2}{p}} + (\frac{1}{p})v_i^{\frac{np-1}{p}} && \text{(last year)}\\
\\
&=(\frac{1}{p})\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right] && \text{(1}st\text{ year)}\\
&+(\frac{1}{p})v_i^{\frac{p}{p}}\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right] && \text{(2}nd\text{ year)}\\
&\vdots\\
&+(\frac{1}{p})v_i^{\frac{(n-1)p}{p}}\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right] && \text{(last year)}\\
\\
&=\left((\frac{1}{p}) + (\frac{1}{p})v_i^{\frac{p}{p}} + \cdots + (\frac{1}{p})v_i^{\frac{(n-1)p}{p}})\right)\left[1 + v_i^{\frac{1}{p}} + \cdots + v_i^{\frac{p-2}{p}} + v_i^{\frac{p-1}{p}}\right]\\
&=(\frac{1}{p})\left(1 + v_i^{\frac{p}{p}} + \cdots + v_i^{(n-1)}\right)\left[\frac{1-(v_i^{\frac{1}{p}})^p}{1-v_i^{\frac{1}{p}}}\right]\\
&=(\frac{1}{p})\left(\frac{1-v_i^n}{1-v_i^1}\right)\cdot\left[\frac{1-v_i^1}{1-v_i^{\frac{1}{p}}}\right]\\
&=(\frac{1}{p})\left(\frac{1-v_i^n}{1-v_i^{\frac{1}{p}}}\right)\\
&=(\frac{1}{p})\left(\frac{1-v_i^n}{1-(1-d)^{\frac{1}{p}}}\right)\\
&=\frac{1-v_i^n}{p\left[1-(1-d)^{\frac{1}{p}}\right]}\\
&=\frac{1-v_i^n}{d^{(p)}} = \left(\frac{1}{p}\times p\right)\cdot\frac{1-v_i^n}{d^{(p)}}
\end{aligned}
$$

– the accumulated value (at $t = n$) of an p^{th}ly annuity–due, where the annual effective rate of interest is i, shall be denoted as $\ddot{s}^{(p)}_{\overline{n}|i}$ and is calculated as follows:

$$\begin{aligned}
\ddot{s}^{(p)}_{\overline{n}|i} =&(\frac{1}{p})(1+i)^{\frac{1}{p}} + (\frac{1}{p})(1+i)^{\frac{2}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{p-1}{p}} + (\frac{1}{p})(1+i)^{\frac{p}{p}} && \text{(last year)}\\
&+(\frac{1}{p})(1+i)^{\frac{p+1}{p}} + (\frac{1}{p})(1+i)^{\frac{p+2}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{2p-1}{p}} + (\frac{1}{p})(1+i)^{\frac{2p}{p}} && \text{(2}nd\text{ last year)}\\
&\vdots\\
&+(\frac{1}{p})(1+i)^{\frac{(n-1)p+1}{p}} + (\frac{1}{p})(1+i)^{\frac{(n-1)p+2}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{np-1}{p}} + (\frac{1}{p})(1+i)^{\frac{np}{p}} && \text{(first year)}\\
\\
=&(\frac{1}{p})(1+i)^{\frac{1}{p}}\left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] && \text{(last year)}\\
&+(\frac{1}{p})(1+i)^{\frac{p+1}{p}}\left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] && \text{(2}nd\text{ last year)}\\
&\vdots\\
&+(\frac{1}{p})(1+i)^{\frac{(n-1)p+1}{p}}\left[1 + (1+i)^{\frac{1}{p}} + \cdots + (1+i)^{\frac{p-2}{p}} + (1+i)^{\frac{p-1}{p}}\right] && \text{(first year)}\\
\\
=&\left((\frac{1}{p})(1+i)^{\frac{1}{p}} + (\frac{1}{p})(1+i)^{\frac{p+1}{p}} + \cdots + (\frac{1}{p})(1+i)^{\frac{(n-1)p+1}{p}})\right)\left[\frac{1-((1+i)^{\frac{1}{p}})^p}{1-(1+i)^{\frac{1}{p}}}\right]\\
=&(\frac{1}{p})(1+i)^{\frac{1}{p}}\left(1 + (1+i)^{\frac{p}{p}} + \cdots + (1+i)^{(n-1)}\right)\left[\frac{1-((1+i)^{\frac{1}{p}})^p}{1-(1+i)^{\frac{1}{p}}}\right]\\
=&(\frac{1}{p})(1+i)^{\frac{1}{p}}\left(\frac{1-(1+i)^n}{1-(1+i)^1}\right)\cdot\left[\frac{1-(1+i)^1}{1-(1+i)^{\frac{1}{p}}}\right]\\
=&(\frac{1}{p})(1+i)^{\frac{1}{p}}\left(\frac{1-(1+i)^n}{1-(1+i)^{\frac{1}{p}}}\right)\\
=&\frac{(1+i)^n - 1}{p\cdot v_i^{\frac{1}{p}}\left[(1+i)^{\frac{1}{p}} - 1\right]}\\
=&\frac{(1+i)^n - 1}{p\left[1 - v_i^{\frac{1}{p}}\right]}\\
=&\frac{(1+i)^n - 1}{p\left[1 - (1-d)^{\frac{1}{p}}\right]}\\
=&\frac{(1+i)^n - 1}{d^{(p)}} = \left(\frac{1}{p}\times p\right)\cdot\frac{(1+i)^n - 1}{d^{(p)}}
\end{aligned}$$

Basic Relationship $1 : 1 = d^{(p)} \cdot \ddot{a}^{(p)}_{\overline{n}|} + v^n$

Basic Relationship $2 : PV(1+i)^n = FV$ **and** $PV = FV \cdot v^n$

- if the future value at time n, $\ddot{s}^{(p)}_{\overline{n}|}$, is discounted back to time 0, then you will have its present value, $\ddot{a}^{(p)}_{\overline{n}|}$

$$\begin{aligned}
\ddot{s}^{(p)}_{\overline{n}|} \cdot v^n &= \left[\frac{(1+i)^n - 1}{d^{(p)}}\right] \cdot v^n \\
&= \frac{(1+i)^n \cdot v^n - v^n}{d^{(p)}} \\
&= \frac{1 - v^n}{d^{(p)}} \\
&= \ddot{a}^{(p)}_{\overline{n}|}
\end{aligned}$$

- if the present value at time 0, $\ddot{a}^{(p)}_{\overline{n}|}$, is accumulated forward to time n, then you will have its future value, $\ddot{s}^{(p)}_{\overline{n}|}$

$$\begin{aligned}
\ddot{a}^{(p)}_{\overline{n}|} \cdot (1+i)^n &= \left[\frac{1 - v^n}{d^{(p)}}\right] (1+i)^n \\
&= \frac{(1+i)^n - v^n(1+i)^n}{d^{(p)}} \\
&= \frac{(1+i)^n - 1}{d^{(p)}} \\
&= \ddot{s}^{(p)}_{\overline{n}|}
\end{aligned}$$

Basic Relationship $3 : \ddot{a}^{(p)}_{\overline{n}|} = \dfrac{d}{d^{(p)}} \cdot \ddot{a}_{\overline{n}|}$**,** $\ddot{s}^{(p)}_{\overline{n}|} = \dfrac{d}{d^{(p)}} \cdot \ddot{s}_{\overline{n}|}$

- Consider payments of $\frac{1}{p}$ made at the beginning of every $\frac{1}{p}$th of year for the next n years. Over a one-year period, payments of $\frac{1}{p}$ made at the beginning of each p^{th} period will accumulate at the end of the year to a lump sum of $\left(\frac{1}{p} \times p\right) \cdot \ddot{s}^{(p)}_{\overline{1}|}$. If this end-of-year lump sum exists for each year of the n-year annuity-immediate, then the present value (at $t = 0$) of these end-of-year lump sums is the same as $\left(\frac{1}{p} \times p\right) \cdot \ddot{a}^{(p)}_{\overline{n}|}$:

$$\begin{aligned}
\left(\frac{1}{p} \times p\right) \cdot \ddot{a}^{(p)}_{\overline{n}|} &= \left(\frac{1}{p} \times p\right) \cdot \ddot{s}^{(p)}_{\overline{1}|} \cdot a_{\overline{n}|} \\
\ddot{a}^{(p)}_{\overline{n}|} &= \frac{i}{d^{(p)}} \cdot a_{\overline{n}|}
\end{aligned}$$

- Therefore, the accumulated value (at $t = n$) of these end-of-year lump sums is the same as $\left(\frac{1}{p} \times p\right) \cdot \ddot{s}^{(p)}_{\overline{n}|}$:

$$\begin{aligned}
\left(\frac{1}{p} \times p\right) \cdot \ddot{s}^{(p)}_{\overline{n}|} &= \left(\frac{1}{p} \times p\right) \cdot \ddot{s}^{(p)}_{\overline{1}|} \cdot s_{\overline{n}|} \\
\ddot{s}^{(p)}_{\overline{n}|} &= \frac{i}{i^{(p)}} \cdot s_{\overline{n}|}
\end{aligned}$$

Basic Relationship 5: Due= Immediate $\times(1+i)^{\frac{1}{p}}$

$$\ddot{a}^{(p)}_{\overline{n}|} = \frac{1-v^n}{d^{(p)}} = \frac{1-v^n}{\left(\frac{i^{(p)}}{1+\frac{i^{(p)}}{p}}\right)} = a^{(p)}_{\overline{n}|} \cdot \left(1+\frac{i^{(p)}}{p}\right) = a^{(p)}_{\overline{n}|} \cdot (1+i)^{\frac{1}{p}}$$

$$\ddot{s}^{(p)}_{\overline{n}|} = \frac{(1+i)^n-1}{d^{(p)}} = \frac{(1+i)^n-1}{\left(\frac{i^{(p)}}{1+\frac{i^{(p)}}{p}}\right)} = s^{(p)}_{\overline{n}|} \cdot \left(1+\frac{i^{(p)}}{p}\right) = s^{(p)}_{\overline{n}|} \cdot (1+i)^{\frac{1}{p}}$$

An p^{th}ly annuity–due starts one p^{th} of a year earlier than an p^{th}ly annuity-immediate and as a result, earns one p^{th} of a year more interest, hence it will be larger.

Exercises and Solutions

4.2 Nominal p^{th}ly Interest Rates

Exercise (a)

100 is deposited into a bank account. The account is credited at a nominal rate of interest convertible semiannually.

At the same time, 100 is deposited into another account where interest is credited at a force of interest, δ.

After 8.25 years, the value of each account is 300.

Calculate $(i - \delta)$.

Solution (a)

$$100\left(1+\frac{i^{(2)}}{2}\right)^{2\cdot(8.25)} = 300 \rightarrow 1+\frac{i^{(2)}}{2} = 3^{\frac{1}{16.5}} = 1.06885 \rightarrow i = \left(1+\frac{i^{(2)}}{2}\right)^{2} - 1 = .142439$$

OR

$$100(1+i)^{8.25} = 300 \rightarrow 1+i = 3^{\frac{1}{8.25}} \rightarrow i = 3^{\frac{1}{8.25}} - 1 = .142439$$

$$100e^{8.25\delta} = 300 \rightarrow e^{8.25\delta} = 3 \rightarrow 8.25\delta = ln[3] \rightarrow \delta = \frac{ln[3]}{8.25} = .133165$$

$$i - \delta = .142439 - .133165 = .009274 = .93\%$$

Exercise (b)

X is deposited into a savings account at time 0, which pays interest at a nominal rate of j convertible semiannually.

$2X$ is deposited into a savings account at time 0, which pays simple interest at an effective rate of j.

The same amount of interest is earned during the last 6 months of the 8th year under both savings accounts.

Calculate j.

Solution (b)

$$A(8) - A(7.5) = X\left[\left(1+\frac{j}{2}\right)^{2\cdot(8)} - \left(1+\frac{j}{2}\right)^{2\cdot(7.5)}\right]$$

$$X\left[\left(1+\frac{j}{2}\right)^{16} - \left(1+\frac{j}{2}\right)^{15}\right] = X\left(1+\frac{j}{2}\right)^{15}\left[1+\frac{j}{2}-1\right] = X\left(1+\frac{j}{2}\right)^{15}\left[\frac{j}{2}\right]$$

$$A(8) - A(7.5) = 2X[(1+j(8)) - (1+j(7.5))] = 2X(j(.5)) = X(j)$$

$$\left(1+\frac{j}{2}\right)^{15}\left[\frac{j}{2}\right] = j$$

$$\left(1+\frac{j}{2}\right)^{15} = 2 \rightarrow j = .094588 = 9.46\%$$

Exercise (c)

At time 0, K is deposited into Fund X, which accumulates at a force of interest of $\delta_t = \dfrac{t+3}{t^2+6t+9}$. At time m, $2K$ is deposited into Fund Y, which accumulates at an annual nominal rate of 10.25%, convertible quarterly.

At time n, where $n > m$, the accumulated value of each fund is $3K$. Calculate m.

Solution (c)

$$K \cdot e^{\int_0^n \delta_t \cdot dt} = 2K\left(1 + \frac{.1025}{4}\right)^{4(n-m)} = 3K \rightarrow e^{\int_0^n \delta_t \cdot dt} = 2\left(1 + \frac{.1025}{4}\right)^{4(n-m)} = 3$$

$$e^{\int_0^n \frac{t+3}{t^2+6t+9}dt} = 3 \rightarrow e^{\frac{1}{2}\int_0^n \frac{2t+6}{t^2+6t+9}} = 3 \rightarrow e^{\frac{1}{2}\int_0^n \frac{f'(t)}{f(t)}dt} = 3 \rightarrow \left(\frac{f(n)}{f(0)}\right)^{\frac{1}{2}} = 3$$

$$\left(\frac{n^2+6n+9}{9}\right)^{\frac{1}{2}} = 3 \rightarrow n^2+6n+9 = 81 \rightarrow (n-6)(n+12) = 0 \rightarrow n = 6$$

$$2\left(1 + \frac{.1025}{4}\right)^{4(6-m)} = 3 \rightarrow 4(6-m) = \frac{ln\left[\frac{3}{2}\right]}{ln\left[1 + \frac{.1025}{4}\right]} \rightarrow 6 - m = 4 \rightarrow m = 2$$

Exercise (d)

You make a deposit into a savings account, which pays interest at an annual nominal rate of $i^{(2)}$.

At the same time, your professor deposits 1,000 into a different savings account, which pays simple interest.

At the end of five years, the forces of interest, δ_5, on the two accounts are equal, and your professor's account has accumulated to 2,000. Calculate $i^{(2)}$.

Solution (d)

$$1,000(1+5i) = 2,000 \rightarrow i = 20\%$$

$$\delta_t = \frac{i}{1+it} \rightarrow \delta_5 = \frac{.2}{1+(.2)(5)} = 10\%$$

$$\left(\frac{1+i^{(2)}}{2}\right)^2 = e^{\delta(1)} \rightarrow \left(1 + \frac{i^{(2)}}{2}\right) = e^{\frac{\delta}{2}} = e^{.05}$$

$$i^{(2)} = 2(e^{.05} - 1) = 10.2542\%$$

Exercise (e)

If an investment will be tripled in 8 years at a force of interest δ, in how many years will an investment be doubled at a nominal rate of interest numerically equal to δ and convertible once every three years?

Solution (e)

$$e^{8\delta} = 3 \rightarrow \delta = \frac{ln[3]}{8}$$

$$(1 + 3\delta)^{\frac{n}{3}} = 2$$

$$\left(1 + \frac{3ln[3]}{8}\right)^{\frac{n}{3}} = 2$$

$$\frac{n}{3} = \frac{ln[2]}{ln\left[1 + \frac{3ln[3]}{8}\right]} \rightarrow \frac{n}{3} = 2 \rightarrow n = 6$$

Exercise (f)

Calculate the nominal rate of interest convertible once every four years that is equivalent to a nominal rate of discount convertible quarterly.

Solution (f)

$$i^{\frac{1}{4}} = \frac{1}{4}\left[(1+i)^4 - 1\right] = \frac{1}{4}\left[(1-d)^{-4} - 1\right]$$

$$= \frac{1}{4}\left[\left(1 - \frac{d^{(4)}}{4}\right)^{-4\cdot 4} - 1\right] = \frac{1}{4}\left[\left(1 - \frac{d^{(4)}}{4}\right)^{-16} - 1\right]$$

4.3 Nominal p^{th}ly Discount Rates

Exercise (a)

You are given a loan for which interest is to be charged over a four-year period, as follows:

(i) At an annual effective rate of interest of 6%, for the first year.

(ii) At an annual nominal rate of interest of 5%, convertible every two years, for the second year.

(iii) At an annual nominal rate of discount of 5%, convertible semiannually, for the third year.

(iv) At a force of interest of 5%, for the fourth year.

Calculate the annual effective rate of discount over thew four-year period.

Solution (a)

$$(1+i)\left(1+2i^{(\frac{1}{12})}\right)^{\frac{1}{2}}\left(1-\frac{d^{(2)}}{2}\right)^{-2}\cdot e^{\delta}=(1+i)^4=(1-d)^{-4}$$

$$(1.06)\left(1+2(.05)\right)^{\frac{1}{2}}\left(1-\frac{.05}{2}\right)^{-2}\cdot e^{.05}=(1-d)^{-4}$$

$$1.22944=(1-d)^{-4}\rightarrow d=1-1.22944^{-\frac{1}{4}}=5.03\%$$

Exercise (b)

Express $d^{(4)}$ as a function if $i^{(3)}$.

Solution (b)

$$\left(1+\frac{i^{(3)}}{3}\right)^3=\left(1-\frac{d^{(4)}}{4}\right)^{-4}$$

$$d^{(4)}=4\left[1-\left(1+\frac{i^{(3)}}{3}\right)^{\frac{-3}{4}}\right]$$

Exercise (c)

Express $i^{(6)}$ as a function of $d^{(2)}$.

Solution (c)

$$\left(1+\frac{i^{(6)}}{6}\right)^6=\left(1-\frac{d^{(2)}}{2}\right)^{-2}$$

$$i^{(6)}=6\left[\left(1-\frac{d^{(2)}}{2}\right)^{\frac{-1}{3}}-1\right]$$

Exercise (d)

Given a nominal interest rate of 7.5% convertible semiannually, determine the sum of the:

(i) force of interest; and

(ii) nominal discount rate compounded monthly.

Solution (d)

$$\left(1+\frac{i^{(2)}}{2}\right)^2=e^{\delta\cdot(1)}$$

$$\delta=2ln\left(1+\frac{.075}{2}\right)=.073627946$$

$$\left(1-\frac{d^{(12)}}{12}\right)^{-12}=\left(1+\frac{i^{(12)}}{2}\right)^2$$

$$d^{(12)}=12\left[1-\left(1+\frac{i^{(2)}}{2}\right)^{\frac{-2}{12}}\right]=.073402529$$

$$.073627946+.073402529=.147030475$$

4.4 Annuities-Immediate Payable p^{th}ly

Exercise (a)

On January 1, 1985, Michael has the following two options for repaying a loan:

(i) Sixty monthly payments of 100 beginning February 1, 1985.

(ii) A single payment of 6,000 at the end of K months.

Interest is at a nominal annual rate of 12% compounded monthly. The two options have the same present value. Determine K.

Solution (a)

$$\frac{i^{(12)}}{12} = .01/\text{month}$$

$$100a_{\overline{60}|.01} = 6000v^K$$

$$4495.5 = 6000v^K$$

$$\frac{6000}{4495.50} = (1.01)^K$$

$$K = \frac{ln\left(\frac{6000}{4495.50}\right)}{ln(1.01)} = 29 \text{ months}$$

Exercise (b)

Steve elects to receive his retirement benefit over 20 years at the rate of 2,000 per month beginning one month from now. The monthly benefit increases by 5% each year. At a nominal interest rate of 6% convertible monthly, calculate the present value of the retirement benefit.

Solution (b)

$$R = 2,000s_{\overline{12}|\frac{.06}{.12}} = 24,671.12$$

$$PV = Rv + R(1.05)v^2 + ... + R(1.05)^{19}v^{20}$$

$$PV = 24,671.12\left(\frac{1-\left(\frac{1.05}{1+i}\right)^{20}}{i-.05}\right), \text{where } i = (1.005)^{12} - 1 = .0616778$$

$$PV = 419,253.14$$

Exercise (c)

A pensioner elects to receive her retirement benefit over 20 years at a rate of 2,000 per month beginning one month from now. The monthly benefit increases by 5% each year.

At a nominal interest rate of 6% convertible monthly, calculate the present value of the retirement benefit.

Solution (c)

$$2,000a_{\overline{12}|\frac{1}{2}\%} = 23,237.86$$

$$PV = 23,237.86(1 + v(1.05) + v^2(1.05)^2 + ... + v^{19}(1.05)^{19})$$

$$= 23,237.86 \left(1 + \frac{1.05}{1+i} + \left(\frac{1.05}{1+i}\right)^2 + ... + \left(\frac{1.05}{1+i}\right)^{19}\right)$$

$$\text{where } i = \left(1 + \frac{.06}{12}\right)^{12} - 1 = .061678$$

$$= 23,237.86 \left(\frac{1 - \left(\frac{1.05}{1.061678}\right)^{20}}{.061678 - .05}\right)(1.061678)$$

$$PV = 419,252.46$$

Exercise (d)

If $3a^{(2)}_{\overline{n|}} = 2a^{(2)}_{\overline{2n|}} = 45s^{(2)}_{\overline{1|}}$, find i.

Solution (d)

$$3\left(\frac{1-v^n}{i^{(2)}}\right) = 2\left(\frac{1-v^{2n}}{i^{(2)}}\right) = 45\left(\frac{i}{i^{(2)}}\right)$$

$$3 - 3v^n = 2 - 2v^{2n}$$

$$2v^{2n} - 3v^n + 1 = 0$$

$$v^n = \frac{3-\sqrt{9-4(2)}}{4} = \frac{1}{2}$$

Then the equation:

$$3\left(\frac{1-v^n}{i^{(2)}}\right) = 45\left(\frac{i}{i^{(2)}}\right)$$

gives us

$$3\left(1-\frac{1}{2}\right) = 45i$$

$$i = \frac{1.5}{45} = \frac{1}{30}$$

4.5 Annuities-Due Payable p^{th}ly

Exercise (a)

At time $t = 0$ you deposit P into a fund which credits interest at an effective annual interest rate of 8%. At the end of each year in years 6 through 20, you withdraw an amount sufficient to purchase an annuity due of 100 per month for 10 years at a nominal interest rate of 12% compounded monthly. Immediately after the withdrawl at the end of year 20, the fund value is zero.

Calculate P.

Solution (a)

$$PMT = 100\ddot{a}_{\overline{120|}1\%} = 7,039.75$$

$$P(1.08)^6 = 7,039.75\ddot{a}_{\overline{15|}8\%}$$

$$P = 41,009.64$$

Exercise (b)

At time $t = 0$, Paul deposits P into a fund crediting interest at an effective annual interest rate of 8%. At the end of each year in year 6 through 20, Paul withdraws an amount sufficient to purchase an annuity-due of 100 per month for 10 years at a nominal interest rate of 12% compounded monthly. Immediately after the withdrawl at the end of year 20, the fund value is zero.

Calculate P.

Solution (b)

$$PMT = (100 \cdot 12)\ddot{a}^{(12)}_{\overline{10}|} = 100\ddot{a}_{\overline{120}|.01} = 7,039.75$$

$$0 = P(1.08)^{20} - v[\underbrace{(100 \cdot 12)\ddot{a}_{\overline{10}|^{(12)}}}_{100\ddot{a}_{\overline{120}|.01}}]\ddot{s}_{\overline{15}|.08} = 41,009$$

5 Project Appraisal and Loans

Overview

- this chapter extends the concepts and techniques covered to date and applies them to common financial situations

- simple financial transactions such as borrowing and lending are now replaced by a broader range of business/financial transactions

- taxes and investment expenses are to be ignored unless stated

5.1 Discounted Cash Flow Analysis

- by taking the present value of any pattern of future payments, you are performing a *discounted cash flow analysis*

- let CF_t^{in} represent the returns made back to the investor that are made at time t

- let CF_t^{out} represent the contributions by an investor that are made at time t

Net Present Value

- the net present value can be calculated by taking the present value of the cash-flows-in and reducing them by the present value of the cash-flows-out

$$
\begin{aligned}
NPV &= \sum_{t=0}^{n} \left(CF_t^{in} - CF_t^{out} \right) v_i^t \\
&= \sum_{t=0}^{n} CF_t^{in} \cdot v_i^t - \sum_{t=0}^{n} CF_t^{out} \cdot v_i^t
\end{aligned}
$$

Example

A 10–year investment project requires an initial investment of 1,000,000 and subsequent beginning-of-year payments of 100,000 for the following 9 years. The project is expected to produce 5 annual investment returns of 600,000 commencing 6 years after the initial investment.

From a cash flow perspective, we have

$$\begin{aligned} CF_0^{out} &= 1,000,000 \\ CF_1^{out} &= CF_2^{out} = \cdots = CF_9^{out} = 100,000 \\ CF_6^{in} &= CF_7^{in} = \cdots = CF_{10}^{in} = 600,000 \end{aligned}$$

The net present value can be calculated as:

$$\begin{aligned} NPV &= \sum_{t=0}^{n} \left(CF_t^{in} - CF_t^{out} \right) v_i^t \\ NPV &= \sum_{t=0}^{n} CF_t^{in} \cdot v_i^t - \sum_{t=0}^{n} CF_t^{out} \cdot v_i^t \\ &= 600,000 v_i^6 + 600,000 v_i^7 + \cdots + 600,000 v_i^{10} \\ &\quad - \left[1,000,000 + 100,000 v_i^1 + 100,000 v_i^2 + \cdots + 100,000 v_i^9 \right] \\ NPV &= 600,000 v_i^6 \ddot{a}_{\overline{5}|i} - 1,000,000 - 100,000 v_i^1 \ddot{a}_{\overline{9}|i} \\ &= -1,000,000 - 100,000 v_i^1 \ddot{a}_{\overline{5}|i} + 500,000 v_i^6 \ddot{a}_{\overline{4}|i} + 600,000 v_i^{10} \end{aligned}$$

The net present value depends on the annual effective rate of interest, i, that is adopted. Under either approach, if the net present value is negative, then the investment project would not be a desirable pursuit.

Internal Rate of Return

There exists a certain interest rate where the net present value is equal to 0.

$$NPV = \sum_{t=0}^{n} \frac{netCF_t}{(1+IRR)^t} = 0$$

Under the cash flow approach,

$$PV_0 = \sum_{t=0}^{n} CF_t^{in} \cdot v_i^t - \sum_{t=0}^{n} CF_t^{out} \cdot v_i^t = 0$$

$$\sum_{t=0}^{n} CF_t^{in} \cdot v_i^t = \sum_{t=0}^{n} CF_t^{out} \cdot v_i^t$$

In other words, there is an effective rate of interest that exists such that the present value of cash-flows-in will yield the same present value of cash-flows out. This interest rate is called a yield rate or an internal rate of return (IRR) as it indicates the rate of return that the investor can expect to earn on their investment (i.e. on their contributions or cash-flows-in).

This yield rate for the above investment project can be determined as follows:

$$\sum_{t=0}^{n} CF_t^{in} \cdot v_i^t = \sum_{t=0}^{n} CF_t^{out} \cdot v_i^t$$

$$600,000 v_i^6 \ddot{a}_{\overline{5}|i} = 1,000,000 + 100,000 v_i^1 \ddot{a}_{\overline{9}|i}$$

$$600,000 v_i^6 \ddot{a}_{\overline{5}|i} - 100,000 v_i^1 \ddot{a}_{\overline{9}|i} - 1,000,000 = 0$$

$$600,000 v_i^6 \cdot \frac{1 - v_i^5}{1 - v_i^1} - 100,000 v_i^1 \cdot \frac{1 - v_i^9}{1 - v_i^1} - 1,000,000 = 0$$

$$600,000 v_i^6 - 600,000 v_i^{11} - 100,000 v_i^1 + 100,000 v_i^{10} - 1,000,000(1 - v_i^1) = 0$$

$$600,000 v_i^6 - 600,000 v_i^{11} + 900,000 v_i^1 + 100,000 v_i^{10} - 1,000,000 = 0$$

The yield rate (solved by using a pocket calculator with advanced financial functions or by using Excel with its Goal Seek function) is 8.062%.

Using this yield rate, investment projects with a 10 year life can be compared to the above project. In general, those investment opportunities that have higher(lower) expected returns than the 8.062% may be more(less) desirable.

Reinvestment Rates

- the yield rate that is calculated assumes that the positive returns (or cash-flows-out) will be reinvested at the same yield rate
- the actual rate of return can be higher or lower than the calculated yield rate depending on the reinvestment rates

Example

- an investment of 1 is invested for n years and earns an annual effective rate of i. The interest payments are reinvested in an account that credits an annual effective rate of interest of j.
- if the interest is payable at the end of every year and earns a rate of j, then the accumulated value at time n of the interest payments and the original investment is:

$$FV_n = (i \times 1)s_{\overline{n}|j} + 1$$

- if $i = j$, then the accumulated value at time n is:

$$FV_n = i \times \frac{(1+i)^n - 1}{i} + 1 = (1+i)^n$$

Example

- an investment of 1 is made at the end of every year for n years and earns an annual effective rate of interest of i. The interest payments are reinvested in an account that credits an annual effective rate of interest of j.
- note that interest payments increase every year by $i \times 1$ as each extra dollar is deposited each year into the original account

- if the interest is payable at the end of every year and earns a rate of j, then the accumulated value at time n of the interest payments and the original investment is:

$$FV_n = i \times (Is)_{\overline{n-1}|j} + n \times 1$$

- if $i = j$, then the accumulated value at time n is:

$$\begin{aligned} FV_n &= i \times \frac{\ddot{s}_{\overline{n-1}|i} - (n-1)}{i} + n \\ &= \ddot{s}_{\overline{n-1}|i} - n + 1 + n \\ &= \ddot{s}_{\overline{n-1}|i} + 1 \\ &= s_{\overline{n}|i} \end{aligned}$$

5.2 Nominal vs. Real Interest Rates

Interest Rates and Inflation

– interest rates and inflation are assumed to move in the same direction over time since lenders will charge higher interest rates to make up for the loss of purchasing power due to higher inflation.

– the relationship is actually between the current rate of interest and the expected (not current) rate of inflation.

– a nominal interest rate is one that has not been adjusted for inflation

Nominal Interest Rates vs. Real Interest Rates

– let i_{real} represent the real rate of interest, i represent the nominal interest rate and π represent the rate of inflation where

$$i_{real} = \frac{1+i}{1+\pi} - 1$$

Present Value Formulas Using Nominal and Real Interest Rates

$$\begin{aligned} PV &= \sum_{t=0}^{n} nominalCF_t \cdot v_i^t \\ &= \sum_{t=0}^{n} \frac{nominalCF_t \cdot v_i^t}{(1+i_{real})^t} \\ &= \sum_{t=0}^{n} \frac{realCF_t}{(1+i_{real})^t} \end{aligned}$$

5.3 Investment Funds

Dollar-weighted Interest Rate

- investment funds typically experience multiple contributions and withdrawals during its life
- interest payments are often made at irregular periods rather than only at the end of the year
- here the yield rate is influenced by the dollar amount of the contribution, it is often referred to as a *dollar-weighted rate of interest*
- the dollar-weighted rate of interest is the actual return that the investor experiences over the year

$$F_0(1+i)^T + \sum_{s=1}^{n} c_s(1+i)^{T-t_s} = F_T$$

where F_0 is the value of the fund at time 0

where F_T is the value of the fund at time T

where c_s is the deposits or withdrawls of the fund

where t_s is the time and i is the interest rate

Simple Interest Approximation for Compound Interest

- to approximate the dollar-weighted rate of interest assume that $(1+i)^n = (1+ni)$
- this approach can produce results that are fairly close to the exact approach

Time-Weighted Interest Rate

– to measure annual fund performance without the influence of the contributions, one must look at the fund's performance over a variety of sub-periods. These sub-periods are triggered whenever a contribution takes place and end just before the next contribution (or when the end of the year is finally met).

– in general, the interest rate earned for the sub-period is determined by taking the ratio of the fund at the beginning of the sub-period versus the fund at the end of the sub-period.

– The yield rate derived from this method is called the *time-weighted rate of interest* and is determined using the following formula:

$$(1+i)^T = \frac{F_1}{F_0} \cdot \frac{F_2}{F_1 + c_1} \cdot \frac{F_3}{F_2 + c_2} \cdot ... \frac{F_T}{F_n + c_n}$$

where F_0 is the value of the fund at time 0

where F_T is the value of the fund at time T

where c_n is the external cash flows

where F_n is the value of the fund before each of the cash flows

5.4 Allocating Investment Income

- you are given a fund for which there are many investors. Each investor holds a share of the fund expressed as a percentage. For example, investor k might hold 5% of the fund at the beginning of the year
- over a one-year period, the fund will earn investment income, I
- there are two ways in which the fund's investment income can be distributed to the investors at the end of the year

  (i) Portfolio Method
  (ii) Investment Year Method

(i) Portfolio Method

- if investor k owns 5% of the fund at the beginning of the year, then investor k gets 5% of the investment income ($5\% \times I$).
- this is the same approach as if the fund's *dollar-weighted* rate of return was calculated and all the investors were credited with that same yield rate
- the disadvantage of the portfolio method is that it doesn't reward those investors who make good decisions
- For example, investor k may have contributed large amounts of money during the last six months of the year when the fund was earning, say 100%. Investor c may have withdrawn money during that same period, and yet both would be credited with the same rate of return.
- If the fund's overall yield rate was say, 0%, obviously, investor k would rather have their contributions credited with the actual 100% as opposed to $(1 + 0\%)^{\frac{1}{2}}$.

(ii) Investment Year Method

- an investors's contribution will be credited during the year with the interest rate that was in effect at the time of the contribution.
- this interest rate is often referred to as the *new-money* rate.
- Reinvestment rates can be handled in one of two ways:
  (a) Declining Index System - only principal is credited at new money rates
  (b) Fixed Index System - principal and interest is credited at new money rates
- this "earmarking" of money for new money rates will only go on for a specified period before the portfolio method commences

5.5 Loans: The Amortization Method

– there are two methods for paying off a loan

(i) Amortization Method - borrower makes installment payments at periodic intervals

(ii) Sinking Fund Method - borrower makes installment payments as the annual interest comes due and pays back the original loan as a lump-sum at the end. The lump-sum is built up with periodic payments going into a fund called a "sinking fund".

– this chapter also discusses how to calculate:

(a) the outstanding loan balance once the repayment schedule has begun, and

(b) what portion of an annual payment is made up of the interest payment and the principal repayment

Finding The Outstanding Loan

– There are two methods for determining the outstanding loan once the payment process commences

(i) Prospective Method

(ii) Retrospective Method

<u>(i) Prospective Method (see the future)</u>

– the original loan at time 0 represents the present value of future repayments. If the repayments, X, are to be level and payable at the end of each year, then the original loan can be represented as follows:

$$\text{Loan} = X \cdot a_{\overline{n}|i}$$

– the outstanding loan at time $t, O/S$ Loan$_t$, represents the present value of the remaining future repayments

$$O/S \text{ Loan}_t = P \cdot a_{\overline{n-t}|i}$$

– this also assumes that the repayment schedule determined at time 0 has been adhered to; otherwise, the prospective method will not work

<u>(ii) Retrospective Method (see the past)</u>

– If the repayments, X, are to be level and payable at the end of each year, then the outstanding loan at time t is equal to the accumulated value of the loan *less* the accumulated value of the payments made to date

$$O/S \text{ Loan}_t = \text{Loan} \cdot (1+i)^t - X \cdot s_{\overline{t}|i}$$

– this also assumes that the repayment schedule determined at time 0 has been adhered to; otherwise, the accumulated value of past payments will need to be adjusted to reflect what the actual payments were, with interest

Basic Relationship 1: Prospective Method = Retrospective Method

- let a loan be repaid with end-of-year payments of 1 over the next n years:

$$\text{Present Value of Payments} = \text{Present Value of Loan}$$
$$(1)a_{\overline{n}|i} = \text{Loan}$$

$$\text{Accumulated Value of Payments} = \text{Accumulated Value of Loan}$$
$$(1)a_{\overline{n}|i} \cdot (1+i)^t = \text{Loan} \cdot (1+i)^t$$

$$\begin{aligned} &\text{Accumulated Value of Past Payments} \\ &\quad + \text{Present Value Future Payments} = \text{Accumulated Value of Loan} \\ &(1)s_{\overline{t}|i} + (1)a_{\overline{n-t}|i} = \text{Loan} \cdot (1+i)^t = a_{\overline{n}|i} \cdot (1+i)^t \end{aligned}$$

$$\begin{aligned} \text{Present Value Future Payments} &= \text{Accumulated Value of Loan} \\ &\quad - \text{Accumulated Value of Past Payments} \\ (1)a_{\overline{n-t}|i} &= a_{\overline{n}|i} \cdot (1+i)^t - (1)s_{\overline{t}|i} \end{aligned}$$

$$\text{Prospective Method} = \text{Retrospective Method}$$

- the prospective method is preferable when the size of each level payment and the number of remaining payments is known
- the retrospective method is preferable when the number of remaining payments or a final irregular payment is unknown.

Amortization Schedules

- an original loan of $(1)a_{\overline{n}|i}$ is to be repaid with end-of-year payments of 1 over n years
- an annual end-of-year payment of 1 using the amortization method will contain an interest payment, I_t, and a principal repayment, P_t
- in other words, $1 = I_t + P_t$

Interest Payment

- I_t is intended to cover the interest obligation that is payable at the end of year t. The interest is based on the outstanding loan balance at the beginning of year t.
- B_{t-1} is the principal balance outstanding after a previous payment.
- using the prospective method for evaluating the outstanding loan balance, the interest payment is:

$$I_t = i \cdot B_{t-1}$$

Principal Repayment

- once the interest owed for the year is paid off, then the remaining portion of the amortization payment goes towards paying back the principal:

$$P_t = \frac{L}{a_{\overline{n}|i}} - I_t$$

5.6 Loans: The Sinking Fund Method

– let a loan of $(1)a_{\overline{n}|i}$ be repaid with single lump-sum payment at time n. If annual end-of-year interest payments of $i \cdot a_{\overline{n}|i}$ are being met each year, then the lump-sum required at $t = n$, is the original loan amount (i.e. the interest on the loan never gets to grow with interest).

– service payments are the interest payments that are paid to the lender

– let the lump-sum that is to be built up in a "sinking fund" be credited with interest rate j and the rate of interest on the loan be credited with interest rate i

Sinking Fund Loan: Equation of Value (when service payments equal interest due)

$$\begin{aligned} L &= \text{Accumulated Value of sinking fund payments (SFP) at time } n \\ &= SFP \cdot s_{\overline{n}|j} \end{aligned}$$

Sinking Fund Loan: Payments (when service payments equal interest due)

$$SFP = \frac{L}{s_{\overline{n}|j}}$$

Net Amount of Sinking Fund Loan (when service payments equal interest due)

– we define the loan amount that is not covered by the balance in the sinking fund as the *Net Amount of Loan* outstanding and is equal to:

$$\text{Net Amount of Loan}_t = L - SFP \cdot s_{\overline{t}|j}$$

– note that the *Net Amount of Loan* outstanding under the sinking fund method is the same value as the outstanding loan balance under the amortization method

Sinking Fund Loan: General Equation of Value

– usually, the interest rate on borrowing, i, is greater than the interest rate offered by investing in a fund, j

– the total payment under the sinking fund approach is then

$$L(1+i)^n = SP \cdot s_{\overline{n}|i} + SFP \cdot s_{\overline{n}|j}$$

– this is where SP is service payments and SPF is sinking fund payments

Net Amount of Sinking Fund Loan

$$\text{Net Amount of Loan}_t = L(1+i)^t - SP \cdot s_{\overline{t}|i} - SFP \cdot s_{\overline{t}|j}$$

Exercises and Solutions

5.1 Discounted Cash Flow Analysis

Exercise (a)

You invest 300 into a bank account at the beginning of each year for 20 years. The account pays out interest at the end of every year at an annual effective interest rate of i. The interest is reinvested at an annual effective rate of $\left(\frac{i}{2}\right)$. The annual effective yield rate on the entire investment over the 20-year period is 10%. Determine i.

Solution (a)

$$300\ddot{a}_{\overline{20}|10\%} = 20 \cdot 300 + (300i)(Is)_{\overline{20}|\frac{i}{2}}$$

$$18,900.75 = 6,000 + (300i)\left(\frac{a_{\overline{20}|\frac{i}{2}} - 20}{\frac{i}{2}}\right)$$

$$18,900.75 = 6,000 + (600)\left(s_{\overline{21}|\frac{i}{2}} - 21\right)$$

$$18,900.75 = 6,000 + 600s_{\overline{21}|\frac{i}{2}} - 12,600$$

$$s_{\overline{21}|\frac{i}{2}} = 42.50$$

use a calculator to find: $\frac{i}{2} = 6.531\% \rightarrow i = 2(6.531\%) = 13.062\%$

Exercise (b)

The internal rate of return for an investment in which $C_0 = 4,800$, $C_1 = X$, $R_1 = X + 4,000$ and $R_2 = 2,500$ can be expressed as $\frac{1}{n}$. Find n.

Solution (b)

$$\text{PV of } CF^{\text{in}} = \text{PV of } CF^{\text{out}}$$

$$C_0 + C_1 v = R_1 v + R_2 v^2$$

$$4,800 + Xv = (X + 4,000)v + 2,500v^2$$

$$2,500v^2 + 4,000v - 4,800 = 0$$

$$v = \frac{-4,000 + \sqrt{4,000^2 - 4(2,500)(-4,800)}}{2(2,500)} = \frac{4}{5} \rightarrow 1 + i = \frac{5}{4} \rightarrow i = \frac{1}{4}$$

Exercise (c)

An investor deposits 5,000 at the beginning of each year for five years in a fund earning an annual effective interest rate of 5%. The interest rate of 5%. The interest from this fund can be reinvested at an annual effective interest rate of 4%.

Prove that the future value of this investment at time $t = 10$ is equal to $6,250\left(s_{\overline{11}|4\%} - s_{\overline{6}|4\%} - 1\right)$.

Solution (c)

$$\begin{aligned}
\text{Interest in 1st year} &= 5,000(5\%) && = 250\\
\text{Interest in 2nd year} &= 10,000(5\%) && = 500\\
\text{Interest in 3rd year} &= 15,000(5\%) && = 750\\
\text{Interest in 4th year} &= 20,000(5\%) && = 1,000\\
\text{Interest in 5th year} &= 25,000(5\%) && = 1,250\\
\text{Interest in 6th year} &= 25,000(5\%) && = 1,250\\
\text{Interest in 7th year} &= 25,000(5\%) && = 1,250\\
\text{Interest in 8th year} &= 25,000(5\%) && = 1,250\\
\text{Interest in 9th year} &= 25,000(5\%) && = 1,250\\
\text{Interest in 10th year} &= 25,000(5\%) && = 1,250
\end{aligned}$$

$$FV_{10} = 5 \cdot 5,000 + 250(Is)_{\overline{5}|4\%}(1.04)^5 + 1,250 s_{\overline{5}|4\%}$$

$$FV_{10} = 25,000 + 250\left(\frac{\ddot{a}_{\overline{5}|4\%} - 5}{.04}\right)(1.04)^5 + 1,250\left(\frac{(1.04)^5 - 1}{.04}\right)$$

$$FV_{10} = 25,000 + 6,250\left(s_{\overline{6}|4\%} - 6\right)(1.04)^5 + 31,250\left((1.04)^5 - 1\right)$$

$$FV_{10} = 25,000 + 6,250\left(s_{\overline{6}|4\%} - 6\right)(1.04)^5 + 31,250(1.04)^5 - 31,250$$

$$FV_{10} = 6,250 s_{\overline{6}|4\%}(1.04)^5 - 6,250(1.04)^5 - 6,250$$

$$FV_{10} = 6,250\left(s_{\overline{6}|4\%}(1.04)^5 - (1.04)^5 - 1\right)$$

$$FV_{10} = 6,250\left(s_{\overline{11}|4\%} - s_{\overline{6}|4\%} - 1\right)$$

Exercise (d)

You have 10,000 and are looking for a solid investment. Your professor suggests that you lend him the 10,000 and agrees to repay you with 10 annual end-of-year payments that decrease arithmetically. The annual effective interest rate that you will charge him is 25% and you are able to reinvest the repayments at 10%. After 5 years, your professor flees the country and leaves you with nothing. What is your yield on this foolish investment?

Solution (d)

$$Payments = \frac{10,000}{(Da)_{\overline{10}|25\%}} = \frac{10,000}{\left(\frac{10 - a_{\overline{10}|25\%}}{25\%}\right)} = \frac{10,000}{25.719} = 388.83$$

$$\begin{aligned}
10,000(1+i)^5 &= 388.83\left[(Ds)_{\overline{5}|10\%} + 5s_{\overline{5}|10\%}\right] \\
10,000(1+i)^5 &= 388.83[19.475 + 30.526] \\
10,000(1+i)^5 &= 388.83[50.00] \\
10,000(1+i)^5 &= 19,441.65 \\
(1+i)^5 &= 1.944165 \\
i &= 14.22\%
\end{aligned}$$

5.2 Nominal vs. Real Interest Rates

Exercise (a)

The real interest rate is 6% and the inflation rate is 4%. Alan receives a payment of 1,000 at time 1, and subsequent payments increase by 50 for 5 more years. Determine the accumulated value of these payments at time 6 years.

Solution (b)

The nominal interest rate is:

$$i = (1.06)(1.04) - 1 = 10.24\%$$

The accumulated value of the cash flows is:

$$= 1,000(1.1024)^5 + 1,050(1.1024)^4 + 1,100(1.1024)^3 + 1,150(1.1024)^2 + 1,200(1.1024)^1 + 1,250$$

$$= 950s_{\overline{6}|10.24\%} + 50(Is)_{\overline{6}|10.24\%}$$

$$= 8623.08$$

5.3 Investment Funds

Exercise (a)

On January 1, an investment account is worth 100. On May 1, the value has increased to 120 and W is withdrawn. On November 1, the value is 100 and W is deposited. On January 1 of the following year, the investment account is worth 100. The time-weighted rate of interest is 0%. Calculate the dollar-weighted rate of interest.

Solution (a)

$$\left(\frac{120}{100}\right)\left(\frac{100}{120-W}\right)\left(\frac{100}{100+W}\right)-1=0\%$$

$$\left(\frac{100}{120-W}\right)\left(\frac{100}{100+W}\right)=\frac{1}{\frac{120}{100}}$$

$$\frac{10,000}{12,000+20W-W^2}=0.8333$$

$$0=16.67W-.8333W^2$$

$$0=W(16.67-.8333)$$

$$20=W$$

$$100(1+i)-20(1+i)^{\frac{8}{12}}+20(1+i)^{\frac{2}{12}}=100$$

Using simple interest approximation:

$$100(1+i)-20(1+\frac{8}{12}i)+20(1+\frac{2}{12}i)=100$$

$$100+90i=100$$

$$90i=0$$

$$i=0$$

Exercise (b)

On January 1, an investment is worth 100. On May 1, the value has increased to 120 and D is deposited. On November 1, the value is 100 and 40 is withdrawn. On January 1 of the following year, the investment account is worth 65. The time-weighted rate of return is 0%. Calculate the dollar-weighted rate of return.

Solution (b)

$$\left(\frac{120}{100}\right)\left(\frac{100}{120+D}\right)\left(\frac{65}{60}\right)-1=0\%$$

$$\left(\frac{120}{100}\right)\left(\frac{100}{120+D}\right)\left(\frac{65}{60}\right)=1$$

$$\frac{100}{120+D}=0.76923$$

$$100=92.3076+.76923D$$

$$10=D$$

$$100(1+i)+10(1+i)^{\frac{8}{12}}-40(1+i)^{\frac{2}{12}}=65$$

Using simple interest approximation:

$$100(1+i)+10(1+\frac{8}{12}i)-40(1+\frac{2}{12}i)=65$$

$$70+100i=65$$

$$100i=-5$$

$$i=-5\%$$

Exercise (c)

You are given the following information about two funds:

Date	Fund X Deposits	Fund Y Deposits	Fund X Withdrawls	Fund Y Withdrawls	Value of Fund X before Deposits and Withdrawls	Value of Fund Y before Deposits and Withdrawls
01-Jan-03					50,000	100,000
01-Mar-03					55,000	
01-May-03	24,000				50,000	
01-Jul-03				15,000		105,000
01-Nov-03			36,000		77,310	
01-Dec-03			10,000		F	
31-Dec-03					31,500	94,250

Fund Y's dollar-weighted rate of return in 2003 is equal to Fund X's time-weighted rate of return in 2003.

Calculate F.

Solution (c)

Calculate the dollar-weighted rate of return for Fund Y.

$$100,000(1+i) - 15,000(1+i)^{\frac{6}{12}} = 94,250$$

Under simple interest for compound interest:

$$100,000(1+i) - 15,000(1+\frac{6}{12}i) = 94,250$$

$$100,000 + 100,000i - 15,000 - 7,500i = 94,250$$

$$92,500i = 9,250$$

$$i = 10\%$$

Calculate the time-weighted rate of return for Fund X.

$$1.10 = (1+j_1)(1+j_2)(1+j_3)(1+j_4) = 1+i$$

$$1.10 = (1+j_1)(1+j_2)(1+j_3)(1+j_4)$$

$$1.10 = \frac{50,000}{50,000} \cdot \frac{77,310}{50,000+24,000} \cdot \frac{F}{77,310-36,000} \cdot \frac{31,500}{F-10,000} \rightarrow F = 36,260$$

5.4 Allocating Investment Income

Exercise (a)

The following table shows the annual effective interest rates being credited in an investment account, by the calendar year of the investment. The investment year method is applicable for the first three years, after which a portfolio rate is used:

Calendar Year	Investment Year Rates			Calendar Year	Portfolio
of Investment	i_1	i_2	i_3	of Portfolio Rate	Rate
1996	12%	12%	t%	1999	8%
1997	12%	5%	10%	2000	$(t-1)\%$
1998	8%	$(t-.02)\%$	12%	2001	6%
1999	9%	11%	6%	2002	9%
2000	7%	7%	10%	2003	10%

An investment of 100 is made at the beginning of years 1996, 1997 and 1998. The total amount of interest credited by the fund during 1999 is equal to 28.49. Calculate t.

Solution (a)

Interest earned in 1999 is equal to the 1999 interest rate multiplied by the balance at January 01, 1999.

1996 Money

$$100(1.12)(1.12)(1+t)(8\%)$$

1997 Money

$$100(1.12)(1.05)(10\%) = 11.76$$

1998 Money

$$100(1.08)(t-.02) = 108(t-.02)$$

$$\text{Total Interest} = 10.0353(1+t) + 11.76 + 108(t-.02) = 28.49$$

$$t = 7.5\%$$

5.5 Loans: The Amortization Method

Exercise (a)

You take out a loan for 2,000,000 that will be disbursed to you in three payments. The first payment of 1,000,000 is made immediately and is followed six months later by a payment of 500,000 and then six months after that by another payment of 500,000. The interest on the payments is calculated at a nominal rate of interest of 26.66%, convertible semi-annually.

After two years, you replace the outstanding loan with a 30-year loan at a nominal rate of interest of 12%, convertible monthly. The amount of the monthly payments for the first five years on this loan will be one-half of the monthly payment required after 5 years. Payments are to be made at the beginning of each month.

Calculate the amount of the 12th repayment.

Solution (a)

$$FV = 1,000,000(1.13333)^4 + 500,000(1.13333)^3 + 500,000(1.13333)^2$$

$$= 3,019,493.52$$

$$3,019,493.52 = P\ddot{a}_{\overline{60}|1\%} + v_{1\%}^{60} \cdot 2P\ddot{a}_{\overline{300}|1\%}$$

$$P = 20,000$$

Exercise (b)

A loan is amortized over five years with monthly payments at a nominal rate of interest of 6%, convertible monthly. The first payment is 1,000 and is to be paid one month after the date of the loan. Each succeeding monthly payment will be 5% lower than the prior payment. Calculate the outstanding loan balance immediately after the 40th payment is made.

Solution (b)

$$Payment_{41} = 1,000(.95)^{40} = 128.51$$

$$j = \frac{6\%}{12} = .005$$

$$O/SLoan_{40} = v_{.005}\left[128.51 \cdot a_{\overline{20}|i'=\frac{1+.005}{1-.95}-1=.05789}\right]$$

$$= v_{.005}[128.51 \cdot (12.344)]$$

$$= v_{.005}[1,586.33]$$

$$= 1,578.44$$

Exercise (c)

You borrow 1,000 at an annual effective interest rate of 4% and agree to repay it with 3 annual installments. The amount of each payment in the last 2 years is set at twice that in the first year. At the end of 1 year, you have the option to repay the entire loan with a final payment of X, in addition to the regular payment. This will yield the lender an annual effective rate of 4.5% over the 1-year period.

Solution (c)

$$1,000 = P \cdot v^1_{4\%} + 2P(v^2_{4\%} + v^3_{4\%}) \rightarrow P = \frac{1,000}{v^1_{4\%} + 2(v^2_{4\%} + v^3_{4\%})} = 217.93$$

$$1,000 = (217.93) \cdot v^1_{4.5\%} + Xv^1_{4.5\%} \rightarrow X = \frac{1,000 - (217.93) \cdot v^1_{4.5\%}}{v^1_{4.5\%}} = 827.07$$

Exercise (d)

A loan is to be repaid by annual installments of P at the end of each year for 10 years. You are given:

(i) The total principal repaid in the first 3 years is 290.35.

(ii) The total principal repaid in the last 3 years is 408.55.

Determine the total amount of interest paid during the life of the loan.

Solution (d)

$$290.35 = Pv^{10} + Pv^9 + Pv^8 = Pv^7(v^3 + v^4 + v^5)$$

$$408.55 = Pv^3 + Pv^4 + Pv^5 = P(v^3 + v^4 + v^5)$$

Dividing the 2 equations

$$\frac{290.35}{408.55} = v^7 \rightarrow i = 5\%$$

$$P = \frac{408.55}{v^3_{5\%} + v^4_{5\%} + v^5_{5\%}} = 150.03$$

$$L = P \cdot a_{\overline{10}|5\%} = 1,158.44$$

$$\text{Total Interest Paid=Total Payments-Loan} = 10(150.03) - 1,158.44 = 341.76$$

5.6 Loans: The Sinking Fund Method

Exercise (a)

An investor wishes to take out a loan at an annual effective interest rate of 9% and then buy a 10-year annuity whose present value is 1,000 calculated at an annual effective interest rate of 8%. The loan is to be repaid over the next 10 years with annual end-of-year interest payments. Annual end-of-year sinking fund deposits are also to be made that are credited at an annual effective interest rate of 7%.

Solution (a)

$$\text{Annual annuity payment is } \frac{1,000}{a_{\overline{10}|8\%}} = 149.03$$

The annuity payment must cover the interest payment and the sinking fund deposit.

$$149.03 = \frac{Price}{s_{\overline{10}|7\%}} + Price \cdot 9\%$$

$$Price = \frac{149.03}{\frac{1}{s_{\overline{10}|7\%}} + 9\%} = 917.80$$

Exercise (b)

A 5% 10-year loan of 10,000 is to be paid by the sinking fund method, with interest and sinking fund payments made at the end of each year. The effective rate of interest earned in the sinking fund is 3% per annum.

Immediately before the fifth payment is due, the lender requests that the loan be repaid immediately.

Calculate the net amount of the loan at that time.

Solution (b)

$$\text{Sinking fund deposit is } \frac{10,000}{s_{\overline{10}|3\%}} = 872.30$$

Sinking fund balance just before the fifth deposit is to be made is $892.30 s_{\overline{4}|3\%}(1.03) = 3,758.86$

$$\text{Value of the loan just before the interest payment is due is } 10,000(1.05) = 10,500$$

Net amount of the loan at time 5 is the value of the loan less the sinking fund balance is

$$10,500 - 3,786.86 = 6,741.14$$

Exercise (c)

You borrow 10,000 for 10 years at an annual effective interest rate of i and accumulate the amount necessary to repay the loan by using a sinking fund. Ten payments of X are made at the end of each year, which includes interest on the loan and the payment into the sinking fund, which earns an annual effective rate of 8%.

If the annual effective rate of the loan had been $2i$, your total annual payment would have been $1.5X$.

Calculate i.

Solution (c)

$$X = i \cdot 10,000 + \frac{10,000}{s_{\overline{10}|8\%}}$$

$$1.5X = 2i \cdot 10,000 + \frac{10,000}{s_{\overline{10}|8\%}}$$

$$0.5X = i \cdot 10,000 \rightarrow X = 2i \cdot 10,000$$

$$2i \cdot 10,000 = i \cdot 10,000 + \frac{10,000}{s_{\overline{10}|8\%}}$$

$$i = \frac{690.29}{10,000} = 6.9\%$$

6 Financial Instruments

Overview

– interest theory is used to evaluate the prices and values of:

1. bonds
2. equity (common stock, preferred stock)

– this chapter will show how to:

1. calculate the price of a security, given a yield rate
2. calculate the yield rate of a security, give the price

6.1 Types of Financial Instruments

There are seven types of common securities available in the financial markets that are discussed in this chapter:

6.1.1 Money Market Instruments

– provide high liquidity and attractive yields; some allow cheque writing

– contains a variety of short-term, fixed-income securities issued by governments and private firms

– credited rates fluctuate frequently with movements in short-term interest rates

– investors will "park" their money in an MMF while contemplating their investment options

Treasury Bills (T-bills)

– Government issued bonds are called:

- Treasury Bills (if less than one year)
- Treasury Notes (if in between one year and long-term)
- Treasury Bonds (if long-term)

– Treasury Bills are valued on a discount yield basis using actual/360

Question: Find the price of a 13-week T-bill that matures for 10,000 and is bought at discount to yield 7.5%.
Solution:

$$10,000\left[1-\frac{91}{360}(7.5\%)\right]=9,810.42$$

Certificate of Deposits (CD)

– rates are guaranteed for a fixed period of time ranging from 30 days to 6 months
– higher denominations will usually credit higher rates of interest
– yield rates are usually more stable than MMF's but less liquid
– withdrawal penalties tend to encourage a secondary trading market rather than cashing out

Commercial Paper

- is an unsecured debt note that usually lasts one to two months, up to 270 days
- usually issued by a large corporation
- the face amount is most likely in multiples of \$100,000

6.1.2 Bonds

- promise to pay interest over a specific term at stated future dates and then pay lump sum at the end of the term (similar qualities to an amortized loan approach).
- issued by corporations and governments as a way to raise money (i.e. borrowing).
- the end of the term is called a maturity date; some bonds can be repaid at the discretion of the bond issuer at any redemption date (callable bonds).
- interest payments from bonds are called coupon payments.
- bonds without coupons and that pay out a lump sum in the future are called accumulation bonds or zero-coupon bonds.
- a bond that has a fixed interest rate over the term of the bond is called fixed-rate bonds.
- a bond that has a fluctuated interest rate over the term of the bond is called a floating-rate bond.
- the risk that a bond issuer does not pay the coupon or principal payments is called default risk.
- a mortgage bond is a bond backed by collateral; in this case, by a mortgage on real property (more secure)
- an income bond pays coupons if company had sufficient funds; no threat of bankruptcy for missed coupon payment
- junk bonds have a high risk of defaulting on payments and therefore need to offer higher interest rates to encourage investment
- a convertible bond can be converted into the common stock of the company at the option of the bond owner
- borrowers in need of a large amount money may choose to issue serial bonds that have staggered maturity dates

6.1.3 Common Stock

- is an ownership security, like preferred stock, but does not have fixed dividends
- level of dividend is determined by company's directors
- common stock dividends are paid after interest payments for bonds and preferred stock are paid out
- variable dividend rates means prices are more volatile than bonds and preferred stock

6.1.4 Preferred Stock

– provides a fixed rate of return (similar to bonds); called a dividend

– ownership of stock means ownership of company (not borrowing)

– no maturity date

– for creditor purpose, preferred stock is second in line, behind bond owners (common stock is third)

– failure to pay dividend does not result in default

– cumulative preferred stock will make up for any missed dividends; regular preferred stock does not have to

– convertible preferred stock gives the owner the option of converting to common stock under certain conditions

6.1.5 Mutual Funds

– pooled investment accounts; an investor buys shares in the fund

– offers more diversification than what an individual can achieve on their own

– a money manager controls the investment; if the money manager trys to outbeat the market it is said that the investment strategy is active, if the money manager matches the index fund then the investment strategy is passive.

6.1.6 Guaranteed Investment Contracts (GIC)

– issued by insurance companies to large investors

– similar to CD's; market value does not change with interest rate movements

– GIC might allow for additional deposits and can offer insurance contract features i.e. annuity purchase options

– interest rates higher than CD's; closer to Treasury securities

– banks compete with their "bank investment contracts" (BIC)

6.1.7 Derivative Sercurities

- the value of a derivative security depends on the value of the underlying asset in the marketplace

Options

- a contract that allows the owner to buy or sell a security at a fixed price at a future date
- call option gives the owner the right to buy; put option gives the owner the right to sell
- European option can be used on a fixed date; American option can be used any time until its expiry date
- investors will buy(sell) call options or sell(buy) put options if they think a security's price is going to rise (fall)
- one motivation for buying or selling options is speculation; option prices depend on the value of the underlying asset (leverage)
- another motivation (and quite opposite to speculation) is developing hedging strategies to reduce investment risk (see Section 8.7 Short Sales)
- a warrant is similar to a call option, but has more distinct expiry dates; the issuing firm also has to own the underlying security
- a convertible bond may be considered the combination of a regular bond and a warrant

Futures

- this is a contract where the investor agrees, at issue, to buy or sell an asset at a fixed date (delivery date) at a fixed price (futures price)
- the current price of the asset is called the spot price
- an investor has two investment alternatives:
  - (i) buy the asset immediately and pay the spot price now, or
  - (ii) buy a futures contract and pay the futures price at the delivery date; earn interest on the money deferred, but lose the opportunity to receive dividends/interest payments

Forwards

- similar to futures except that forwards are tailored made between two parties (no active market to trade in)
- banks will buy and sell forwards with investors who want protection for currency rate fluctuations for one year or longer
- banks also sell futures to investors who wish to lock-in now a borrowing interest rate that will be applied to a future loan
- risk is that interest rates drop in the future and the investor is stuck with the higher interest rate

Swaps

– a swap is an exchange of two similar financial quantities

– for example, a change in loan repayments from Canadian dollars to American dollars is called a currency swap (risk depends on the exchange rate)

– an interest rate swap is where you agree to make interest payments based on a variable loan rate (floating rate) instead of on a predetermined loan rate (fixed rate)

Caps

– an interest rate cap places an upper limit on an interest rate to protect the investor against increasing interest rates.

$$Cap\ payment = Max\left[\frac{index\ rate\text{-}strike\ rate}{p}, 0\right] \cdot notational\ amount$$

where p is the amount of times per year interest is paid

Floors

– an interest rate floor a lower limit on an interest rate to protect the investor against decreasing interest rates.

$$Floor\ payment = Max\left[\frac{strike\ rate\text{-}index\ rate}{p}, 0\right] \cdot notational\ amount$$

where p is the amount of times per year interest is paid

6.2 Bond Valuation

– like any loan, the price(value) of a bond can be determined by taking the present value of its future payments

– prices will be calculated immediately after a coupon payment has been made, or alternatively, at issue date if the bond is brand new

– let P represent the price of a bond that offers coupon payments of (Fr) and a final lump sum payment of C. The present value is calculated as follows:

$$P = (Fr) \cdot a_{\overline{n}|i} + Cv_i^n$$

– F represents the *face amount(par value)* of a bond. It is used to define the coupon payments that are to be made by the bond.

– C represents the *redemption value* of a bond. This is the amount that is returned to the bond-holder(lender) at the end of the bond's term (i.e. at the maturity date).

– r represents the *coupon rate* of a bond. This is used with F to define the bond's coupon payments. This "interest rate" is usually quoted first on a nominal basis, convertible semi-annually since the coupon payments are often paid on a semi-annual basis. This interest rate will need to be converted before it can be applied.

– (Fr) represents the semi-annual *coupon payment* of a bond.

– n is the number of coupon payments remaining or the time until maturity.

– i is the bond's *yield rate* or *yield-to-maturity.* It is the IRR to the bond-holder for acquiring this investment. Recall that the yield rate for an investment is determined by setting the present value of cash-flows-in (the purchase price, P) equal to the present value of the cash-flows-out (the coupon payments, (Fr), and the redemption value, C).

– there are three formulas that can be used in order to determine the price of a bond:

(i) Basic Formula
(ii) Premium/Discount Formula
(iii) Base Amount Formula

<u>(i) Basic Formula</u>

$$P = (Fr) \cdot a_{\overline{n}|i} + Cv_i^n$$

As stated before, a bond's price is equal to the present value of its future payments.

<u>(ii) Premium/Discount Formula</u>

$$\begin{aligned} P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\ &= (Fr) \cdot a_{\overline{n}|i} + C\left(1 - i \cdot a_{\overline{n}|i}\right) \\ &= C + (Fr - Ci)\, a_{\overline{n}|i} \end{aligned}$$

If we let C loosely represent the loan amount that the bond-holder gets back, then $(Fr - Ci)$ represents how much better the actual payments, Fr, are relative to the "expected" interest payments, Ci.

When $(Fr - Ci) > 0$ the bond will pay out a "superior" interest payment than what the yield rate says to expect. As a result, the bond-buyer is willing to pay(lend) a bit more, $P - C$, than what will be returned at maturity. This extra amount is referred to as a "premium". On the other hand, if the coupon payments are less than what is expected according to the yield rate, $(Fr - Ci) < 0$, then the bond-buyer won't buy the bond unless it is offered at a price less than C or, in other words, at a "discount".

(iii) Base Amount Formula

Let G represent the *base amount* of the bond such that if multiplied by the yield rate, it would produce the same coupon payments that the bond is providing: $Gi = Fr \rightarrow G = \frac{Fr}{i}$.

The price of the bond is then calculated as follows:

$$\begin{aligned} P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\ &= (Gi) \cdot a_{\overline{n}|i} + C\left(1 - i \cdot a_{\overline{n}|i}\right) \\ &= G \cdot (1 - v_i^n) + Cv_i^n \\ &= G + (C - G)\, v_i^n \end{aligned}$$

If we now let G loosely represent the loan, then the amount that the bond-holder receives at maturity, in excess of the loan, would be a bonus. As a result, the bond becomes more valuable and a bond-buyer would be willing to pay a higher price than G. On the other hand, if the payout at maturity is perceived to be less than the loan G, then the bond-buyer will not purchase the bond unless the price is less than the loan amount.

Determination Of Yield Rates

– given a purchase price of a security, the yield rate can be determined under a number of methods.
Problem:
What is the yield rate convertible semi-annually for a 100 par value 10-year bond with 8% semi-annual coupons that is currently selling for 90?
Solution:

1. Use an Society of Actuaries recommended calculator with built in financial functions:

$$\boxed{\text{PV}} = 90,\ \boxed{\text{N}} = 2 \times 10 = 20,\ \boxed{\text{FV}} = 100,\ \boxed{\text{PMT}} = \frac{8\%}{2} \times 100 = 4,$$

$$\boxed{\text{CPT}}\,\boxed{\%\text{i}} \rightarrow 4.788\% \times 2 = 9.5676\%$$

2. Do a linear interpolation with bond tables (not a very popular method anymore).

Premium And Discount

- the redemption value, C, loosely represents a loan that is returned back to the lender after a certain period of time
- the coupon payments, (Fr), loosely represent the interest payments that the borrower makes so that the outstanding loan does not grow
- let the price of a bond, P, loosely represent the original value of the loan that the bond-buyer(lender) gives to the bond-issuer(borrower)
- the difference between what is lent and what is eventually returned, $P - C$, will represent the extra value (if $P - C > 0$) or the shortfall in value (if $P - C < 0$) that the bond offers
- the bond-buyer is willing to pay(lend) more than C if he/she perceives that the coupon payments, (Fr), are better than what the yield rate says to expect, which is (Ci).
- the bond-buyer will pay less than C if the coupon payments are perceived to be inferior to the expected interest returns, $Fr < Ci$.
- the bond is priced at a premium if $P > C$ (or $Fr > Ci$) or at a discount if $P < C$ (or $Fr < Ci$).
- recall the Premium/Discount Formula from the prior section:

$$\begin{aligned} P &= C + (Fr - Ci)\, a_{\overline{n}|i} \\ P - C &= (Fr - Ci)\, a_{\overline{n}|i} \\ &= (Cg - Ci)\, a_{\overline{n}|i} \\ &= C \cdot (g - i)\, a_{\overline{n}|i} \end{aligned}$$

- in other words, if the modified coupon rate, g, is better than the yield rate, i, the bond sells at a premium. Otherwise, if $g < i$, then the bond will have to be priced at a discount.
- $g = \frac{Fr}{C}$ represents the "true" interest rate that the bond-holder enjoys and is based on what the lump sum will be returned at maturity
- i represents the "expected" interest rate (the yield rate) and depends on the price of the bond
- the yield rate, i, is inversely related to the price of the bond.
- if the yield rate, i, is low, then the modified coupon rate, g, looks better and one is willing to pay a higher price.
- if the yield rate, i, is high, then the modified coupon rate, g, is not as attractive and one is not willing to pay a higher price (the price will have to come down).

Example

Let there exist a bond such that $C = 1$. The coupon payments are therefore equal to $g(Fr = Cg = g)$.

Let the price of the bond be denoted as $1 + p$ where p is the premium (if $p > 0$) or the discount (if $p < 0$).

$$\begin{aligned} P &= (Fr) \cdot a_{\overline{n}|i} + Cv_i^n \\ 1 + p &= g \cdot a_{\overline{n}|i} + (1)v_i^n \\ &= g \cdot a_{\overline{n}|i} + 1 - i \cdot a_{\overline{n}|i} \\ &= 1 + (g - i) \cdot a_{\overline{n}|i} \end{aligned}$$

Amortization of Premium and Accrual of Discount

Interest Earned: I_t

– using the prospective method for evaluating the value (current price) of the bond, the interest payment is derived as follows:

$$\begin{aligned} I_t &= i \cdot \ BV_{t-1} \\ &= i \left[g \cdot a_{\overline{n-(t-1)}|i} + (1)v_i^{n-(t-1)} \right] \\ &= g \cdot \left(1 - v_i^{n-(t-1)} \right) + v_i^{n-(t-1)} \\ &= g - (g - i) \cdot v_i^{n-(t-1)} \end{aligned}$$

– therefore, each coupon payment, g, is intended to represent a periodic interest payment plus(less) a return of portion of the premium(discount) that was made at purchase.

$$\begin{aligned} I_t &= g - (g - i) \cdot v_i^{n-(t-1)} \\ g &= I_t + (g - i) \cdot v_i^{n-(t-1)} \end{aligned}$$

Premium Amortization: PA_t

– the premium amortization in this case is equal to the coupon payment less the interest payment, $g - I_t$:

$$\begin{aligned} PA_t &= g - I_t \\ &= g - [g - (g - i) \cdot v_i^{n-(t-1)}] \\ PA_t &= (g - i) \cdot v_i^{n-(t-1)} \end{aligned}$$

Other Premium Amortization Formulas

$$PA_t = BV_{t-1} - BV_t$$

Amortization of Premium over the t-th period$=PA_t$

Accumulation of Discount over the t-th period$=-PA_t$

Book Value Of The Bond

– the bond's value starts at price P (or $1+p$, in our example) and eventually, it will become value C (or 1, in our example) at maturity.

– this value, or current price of the bond, at any time between issue date and maturity date can be determined using the prospective approach:

$$\begin{aligned} BV_t &= (Fr)\cdot a_{\overline{n-t}|_i} + Cv_i^{n-t} \\ 1+p &= g\cdot a_{\overline{n-t}|_i} + (1)v_i^{n-t} \\ &= g\cdot a_{\overline{n-t}|_i} + 1 - i\cdot a_{\overline{n-t}|_i} \\ &= 1 + (g-i)\cdot a_{\overline{n-t}|_i} \end{aligned}$$

– or it can be determined using the retrospective approach:

$$\begin{aligned} BV_t &= P(1+i)^t - Frs_{\overline{t}|_i} \\ &= P(1+i)^t - Cgs_{\overline{t}|_i} \end{aligned}$$

– when this asset is first acquired by the bond-buyer, its value is recorded into the accounting records, or the "books", at its purchase price. Since it is assumed that the bond will now be held until maturity, the bond's future value will continue to be calculated using the original expected rate of return (the yield rate that was used at purchase).

– the current value of the bond is often referred to as a "book value".

– the value of the bond will eventually drop or rise to value C depending if it was originally purchased at a premium or at a discount.
f

Book Value Formulas

$$\begin{aligned} BV_t &= Cga_{\overline{n-t}|_i} + Cv_t^{n-t} \\ &= P(1+t)^t + Cgs_{\overline{t}|_i} \\ &= BV_{t-1}(1+i) - Cg \\ &= BV_{t-1} + I_t - Cg \\ &= BV_{t-1} - PA_t \\ &= BV_{t-1} - \textit{Amortization of Premium} \\ &= BV_{t-1} + \textit{Accumulation of discount} \end{aligned}$$

Bond Amortization Schedule

– an amortization schedule for this type of loan can also be developed that will show how the bond is being *written down*, if it was purchased at a premium, or how it is being *written up*, if it was purchased at a discount.

– the following bond amortization schedule illustrates the progression of the coupon payments when $C = 1$ and when the original price of the bond was $1 + p = 1 + (g - i) \cdot a_{\overline{n}|i}$:

Period (t)	Coupon	Interest Earned I_t	Amorization of Premium PA_t	Book ValueBV_t
1	g	$g - (g-i) \cdot v_i^n$	$(g-i) \cdot v_i^n$	$1 + (g-i) \cdot a_{\overline{n-1}|i}$
2	g	$g - (g-i) \cdot v_i^{n-1}$	$(g-i) \cdot v_i^{n-1}$	$1 + (g-i) \cdot a_{\overline{n-2}|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	g	$g - (g-i) \cdot v_i^{n-(t-1)}$	$(g-i) \cdot v_i^{n-(t-1)}$	$1 + (g-i) \cdot a_{\overline{n-t}|i}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	g	$g - (g-i) \cdot v_i^2$	$(g-i) \cdot v_i^2$	$1 + (g-i) \cdot a_{\overline{1}|i}$
n	g	$g - (g-i) \cdot v_i^1$	$(g-i) \cdot v_i$	1
Total	$n \cdot g$	$n \cdot g - (g-i)a_{\overline{n}|i}$	$(g-i) \cdot a_{\overline{n}|i} = p$	

– note that the total of all the interest payments is represented by the total of all coupon payments *less* what is being returned as the premium (or *plus* what is being removed as discount since the loan is appreciating to C).

$$\begin{aligned}\sum_{k=1}^{n} I_k &= \sum_{k=1}^{n} g - (g-i)\cdot v_i^{n-(t-1)} \\ &= \sum_{k=1}^{n} g - \sum_{k=1}^{n}(g-i)\cdot v_i^{n-(k-1)} \\ &= ng - (g-i)a_{\overline{n}|i} \\ &= ng - p\end{aligned}$$

– note that the total of all the principal payments must equal the premium/discount

$$\sum_{k=1}^{n} P_k = \sum_{k=1}^{n}(g-i)v_i^{n-(k-1)} = (g-i)a_{\overline{n}|i} = p$$

– note that the book value at $t = n$ is equal to 1, the last and final payment back to the bond-holder.

– note that the principal (premium) repayments increase geometrically by $\left(1+\frac{i^{(2)}}{2}\right)$

i.e. $PA_{t+n} = PA_t\left(1+\frac{i^{(2)}}{2}\right)^n$.

– remember that the above example is based on a redemption value of $C = 1$. The above formulas need to be multiplied by the actual redemption value if C is not equal to 1

Straight Line Method

If an approximation for writing up or writing down the bond is acceptable, then the principal payments can be defined as follows:

$$PA_t = \frac{BV_0 - BV_n}{n}.$$

or $PA_t = \dfrac{1+p-1}{n} = \dfrac{p}{n}$, if $C = 1$.

I_t will then be the coupon payment less the premium amortization:

$$I_t = (Cg) - PA_t = (Cg) - \frac{BV_0 - BV_n}{n}.$$

or $I_t = g - \dfrac{p}{n}$, if $C = 1$.

The book value of the bond, BV_t, will then be equal the original price *less* the sum of the premium repayments made to date:

$$BV_t = BV_0 - \sum_{k=1}^{t} PA_k = BV_0 - t\left(\frac{BV_0 - BV_n}{n}\right) = BV_0 - tPA_t.$$

or $Price_t = 1 + p - t\dfrac{p}{n} = 1 + \left(\dfrac{n-t}{n}\right)p$, if $C = 1$.

The bond amortization table would then be developed as follows:

Period (t)	Payment	I_t	P_t	Book Value$_t$
1	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{n-1}{n}\right) p$
2	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{n-2}{n}\right) p$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
t	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{n-t}{n}\right) p$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n-1$	g	$g - \frac{p}{n}$	$\frac{p}{n}$	$1 + \left(\frac{1}{n}\right) p$
n	g	$g - \frac{p}{n}$	$\frac{p}{n}$	1
Total	ng	$ng - p$	$\frac{p}{n} \cdot n = p$	

Callable Bonds

– this is a bond where the issuer can redeem the bond prior to the maturity date if they so choose to; this is called a *call date*.

– the challenge in pricing callable bonds is trying to determine the most likely call date

– assuming that the redemption date is the same at any call date, then

(i) the call date will most likely be at the earliest date possible if the bond was sold at a premium (issuer would like to stop repaying the premium via the coupon payments as soon as possible).

(ii) the call date will most likely be at the latest date possible if the bond was sold at a discount (issuer is in no rush to pay out the redeption value).

– when the redemption date is not the same at every call date, then one needs to examine all possible call dates.

Example
A 100 par value 4% bond with semi-annual coupons is callable at the following times:

109.00, 5 to 9 years after issue

104.50, 10 to 14 years after issue

100.00, 15 years after issue.

Question: What price should an investor pay for the callable bond if they wish to realize a yield rate of (1) 5% payable semi-annually and (2) 3% payable semi-annually?

Solution:

(1) Since the market rate is better than the coupon rate, the bond would have to be sold at a discount and as a result, the issuer will wait until the last possible date to redeem the bond:

$$P = 2.00 \cdot a_{\overline{30}|2.5\%} + \$100.00 \cdot v^{30}_{2.5\%} = 89.53$$

(2) Since the coupon rate is better than the market rate, the bond would sell at a premium and as a result, the issuer will redeem at the earliest possible date for each of the three different redemption values:

$$\begin{aligned} P &= 2.00 \cdot a_{\overline{10}|1.5\%} + 109.00 \cdot v^{10}_{1.5\%} = 112.37 \\ P &= 2.00 \cdot a_{\overline{20}|1.5\%} + 104.50 \cdot v^{20}_{1.5\%} = 111.93 \\ P &= 2.00 \cdot a_{\overline{30}|1.5\%} + 100.00 \cdot v^{30}_{1.5\%} = 112.01 \end{aligned}$$

In this case, the investor would only be willing to pay 111.93.

Note that the excess of the redemption value over the par value is referred to as a *call premium* and starts at 9.00, before dropping to 4.50, before dropping to 0.00.

Bond Price Between Coupon Payment Dates

– up to now, bond prices and book values have been calculated assuming the coupon has just been paid.

– let P_t be the bond price (book value) just after a coupon payment has been made.

$$P_t = Fr.a_{\overline{n-t}|i} + C\,v_i^{n-t} = B_{t-1}(1+i) - Fr$$

– when buying an existing bond between its coupon dates, one must decide how to split up the coupon between the prior owner and the new owner.

– let Fr_k represent the amount of coupon (accrued coupon) that the prior owner is due where $0 < k < 1$

– the price of the bond (full price) to be paid to the prior owner at time $t+k$ should be based on the clean price of the bond and the accrued interest (accrued coupon)

$$P_{t+k} = \mathit{Clean\ Price} + \mathit{Accrued\ Interest}$$

Accrued Interest

– notice that the clean price of the bond, will only recognize future coupon payments; hence, the reason for a separate calculation to account for the accrued coupon, Fr_k.

– k may be calculated on an *actual/actual* or 30/360 basis if days are to be used.

$$\begin{aligned} AI &= \mathit{coupon} \cdot k \\ &= Fr \cdot k \end{aligned}$$

where $k = \dfrac{\textit{the number of days between the last coupon payment date and settlement date}}{\textit{total number days between coupon payment dates}}$

Full Price between Coupon Payments Dates

– full price is equal to the bond value as at the last coupon payment date, carried forward with compound interest.

– or the full price is equal to the sum of the next coupon Fr, and the price of the next coupon P_{t+1}, all discounted back to the settlement date.

$$\begin{aligned} P_{t+k} &= P_t(1+i)^k \\ &= P_{t+k} \\ &= (P_{t+1} + Fr)v_i^{(1-k)} \end{aligned}$$

Clean Price of Bond between Coupon Payment Dates

$$\begin{aligned} \mathit{Clean\ price} &= P_{t+k} - AI \\ &= P_t(1+i)^k - (\mathit{coupon} \cdot k) \\ &= P_t(1+i)^k - (Fr \cdot k) \end{aligned}$$

6.3 Stock Valuation

Common Stock

- issued by corporations i.e. borrowing, but not paying back the principal.
- pays out annually a dividend, Div, rather than interest with no requirement to guarantee payments. Also the dividend can be in any amount (not fixed income).
- an assumption is required with respect to the annual growth rate of dividends, k.
- Price is equal to the present value of future dividends at a given yield rate r and as given growth rate, g.

General DCF Stock Valuation Formula

$$PV stock = \sum_{t=1}^{\infty} \frac{div_t}{(1+r)^t}$$

Level Dividend Stock Valuation Formula

$$PV stock = \frac{div_1}{r}$$

Constant Dividend Growth

- the techniques to be used are exactly the same as those methods presented in Section 3.6, Compound Increasing Annuities.

$$PV = v_r \left(Div \cdot \ddot{a}_{\overline{\infty}|_{r'=\frac{1+r}{1+g}-1}} \right) = \frac{Div}{r-g}$$

Example

Assuming an annual effective yield rate of 10%, calculate the price of a common stock that pays a 2 annual dividend at the end of every year and grows at 5% for the first 5 years, 2.5% for the next 5 years and 0%, thereafter:

$$\begin{aligned} P &= v_{10\%} \left(2 \cdot \ddot{a}_{\overline{5}|_{r'=\frac{1+10\%}{1+5\%}-1}} \right) \\ &+ v_{10\%}^5 \left[v_{10\%} \left(2(1+5\%)^5 \cdot \ddot{a}_{\overline{5}|_{r'=\frac{1+10\%}{1+2.5\%}-1}} \right) \right] \\ &+ v_{10\%}^{10} \left[v_{10\%} \left(2(1+5\%)^5 (1+2.5\%)^5 \cdot \ddot{a}_{\overline{\infty}|_{r'=\frac{1+10\%}{1+0\%}-1}} \right) \right] \\ P &= 8.30 + 6.29 + 11.13 = 25.72 \end{aligned}$$

Price to Earning (P/E) ratio

$$P/E ratio = \frac{stock\ price\ per\ share}{earnings\ per\ share} = \frac{P_0}{EPS}$$

$$\text{where earnings per share} = \frac{net\ income}{number\ of\ outstanding\ shares}$$

Exercises and Solutions

6.2 Bond Valuation

Exercise (a)

You are given two $n-year$ 1000 par value bonds. Bond X has 14% semiannual coupons and a price of 1407.70, to yield i, compounded semiannually.

Bond Y has 12% semiannual coupons and a price of 1271.80, to yield the same rate i, compounded semiannually. Calculate the price of Bond X to yield $i - 1\%$.

Solution (a)

$$1,407.70 = 1,000 + 1,000\left(7\% - \frac{i}{2}\right) a_{\overline{2n}|\frac{i}{2}} \to .40770 = \left(7\% - \frac{i}{2}\right) a_{\overline{2n}|\frac{i}{2}}$$

$$1,271.80 = 1,000 + 1,000\left(6\% - \frac{i}{2}\right) a_{\overline{2n}|\frac{i}{2}} \to .27180 = \left(6\% - \frac{i}{2}\right) a_{\overline{2n}|\frac{i}{2}}$$

$$\frac{.40770}{.27180} = \frac{7\% - \frac{i}{2}}{6\% - \frac{i}{2}} \to \frac{i}{2} = .04 \to i = .08 \to .40770 = (7\% - 4\%)a_{\overline{2n}|4\%} \to 2n = 20$$

$$P = 1,000 + 1,000\left(7\% - \frac{8\% - 1\%}{2}\right) a_{\overline{20}|\frac{8\%-1\%}{2}} = 1,497.43$$

Exercise (b)

A bond with a par value of 1,000 and 6% semiannual coupons is redeemable for 1,100. You are given the following information.

(i) The bond is purchased a P to yield 8%, convertible semiannually.

(ii) The amount of the discount adjustment from the 16th coupon payment is 5.

Calculate P.

Solution (b)

$$Fr - iBV_{15} = -5 \to 30 - (4\%)BV_{15} = -5 \to BV_{15} = 875$$

$$BV_{15} = P(1+i)^{15} - Fr \cdot s_{\overline{15}|i}$$

$$875 = P(1.04)^{15} - 30s_{\overline{15}|4\%}$$

$$P = 819.41$$

Exercise (c)

A 700 par value five-year bond with 10% semiannual coupons is purchased for 670.60.

The present value of the redemption value is 372.05.

Calculate the redemption value.

Solution (c)

$$P = K + Fra_{\overline{n}|i}$$

$$P - K = Fr \cdot a_{\overline{n}|i}$$

$$670.60 - 372.05 = 35a_{\overline{10}|i} = 298.55 \to i = 3\%$$

$$372.05 = C \cdot v_{3\%}^{10} \to C = 500$$

Exercise (d)

You buy an $n - year$ 1,000 par value bond with 6.5% annual coupons at a price of 825.44, assuming an annual yield rate of i, $i > 0$.

After the first two years, the bond's book value has changed by 23.76. Calculate i.

Solution (d)

$$BV_2 - BV_0 = BV_0(1+i)^2 - Fr(1+i) - Fr - BV_0$$

$$23.76 = 825.44(1+i)^2 - 65(1+i) - 65 - 825.44$$

$$0 = \underbrace{825.44}_{a}(1+i)^2 - \underbrace{65}_{b}(1+i) - \underbrace{914.20}_{c}$$

$$1 + i = \frac{-(-65) + \sqrt{(65)^2 - 4(825.44)(-914.20)}}{2(825.44)} = 1.0925 \to i = 9.25\%$$

Exercise (e)

A 30-year 10,000 bond that pays 3% annual coupons matures at par. It is purchased to yield 5% for the first 15 years and 4% thereafter.

Calculate the premium or discount adjustment for year 8.

Solution (e)

$$Fr - iBV_7 = 300 - (5\%)\left(300a_{\overline{8}|5\%} + v^8_{5\%}300a_{\overline{15}|4\%} + 10,000v^8_{5\%}v^{15}_{4\%}\right)$$

$$= 300 - (5\%)(7954.82)$$

$$= 300 - 397.74$$

$$= -97.74$$

Exercise (f)

Among a company's asset and accounting records, an actuary finds an $n - year$ annual coupon-paying bond that was purchased at a discount. From the records, the actuary has determined the following.

(i) the amount for amortization of the discount in the $(n - 12th)$ coupon payment and $(n - 9th)$ coupon payment were 850 and 984, respectively

(ii) the discount is equal to 18,117.

What is the value of n.

Solution (f)

$$\frac{P_{n-9}}{P_{n-12}} = \frac{984}{850} = (1+i)^3 \rightarrow i = 5\%$$

$$P_1 = \frac{P_{n-12}}{(1+i)^{n-12-1}} = \frac{850}{(1+i)^{n-12-1}} = \frac{850}{(1.05)^{n-13}}$$

$$discount = P_1 + P_2 + P_3 + ... + P_n$$

$$18,117 = P_1 + P_1(1.05) + P_1(1.05)^2 + ... + P_1(1.05)^{n-1} = P_1[1 + (1.05) + (1.05)5^2 + ... + (1.05)^{n-1}]$$

$$18,117 = \frac{850}{(1.05^{n-13}}[s_{\overline{n}|5\%}] = \frac{850v^n_{5\%}}{(1.05)^{-13}}[s_{\overline{n}|5\%}] = 850(1.05)^{13}[a_{\overline{n}|5\%}]$$

$$a_{\overline{n}|5\%} = 11.3033 \rightarrow n = 17.07$$

Exercise (g)

A par value 10-year bond with 10% annual coupons is bought at a premium to yield an annual effective rate of 8% for the first 5 years and 6%, thereafter.

The interest portion of the 9th coupon is 12,880.

Calculate the par value.

Solution (g)

$$I_9 = 6\% \cdot BV_8$$

$$12,880 = 6\% \cdot \left[F(10\%)a_{\overline{2}|6\%} + Fv^2_{6\%}\right]$$

$$12,880 = 6\% \cdot F[1.07335707]$$

$$F = 200,000$$

Exercise (h)

A machine costs P. At the end of 10 years, it will have a salvage value of $0.28P$.

The book value at the end of 5 years using the sinking fund method at an annual effective interest rate of 6% is X.

The book value at the end of 5 years using the straight line method is Y.

You are given $X - Y$=650.92.

Calculate P.

Solution (h)

$$BV_5 = P - \frac{(P - .28P)}{s_{\overline{10}|6\%} \cdot s_{\overline{5}|6\%}} = X$$

$$BV_5 = P - \frac{5(P - .28P)}{10} = Y$$

$$P - \frac{(P - .28P)}{s_{\overline{10}|6\%} \cdot s_{\overline{5}|6\%} - \left(P - \frac{5(P-.28P)}{10}\right)} = 650.92$$

$$(P - .28P)\left(\frac{1}{2} - \frac{s_{\overline{5}|}}{s_{\overline{10}|}}\right) = 650.92$$

$$P = 12,500$$

6.3 Stock Valuation

Exercise (a)

You and a friend each sell a different stock short for a price of 1,000. The margin requirement is 60% of the selling price and the interest on the margin account is credited at an annual effective rate of 6%.

You buy back your stock one year later at a price of P. At the end of the year, the stock paid a dividend of X. Your friend also buys back their stock one year later but at a price of $(P-25)$. At the end of the year, their stock paid a dividend of $2X$.

Both you and your friend earn an annual effective yield rate of 20% on your sales.

Calculate P.

Solution (a)

$$\text{Margin (I)} = 1,000(60\%) = 600$$

$$\text{Interest on Margin} = 600(6\%) = 36$$

$$\text{Dividend} = -X$$

$$\text{Gain on Sale} = 1,000 - P$$

$$\text{Yield (I)} = \frac{1,000 - P + 36 - X}{600} = 20\% \rightarrow X = 916 - P$$

$$\text{Margin (II)} = 1,000(60\%) = 600$$

$$\text{Interest on Margin} = 600(6\%) = 36$$

$$\text{Dividend} = -2X$$

$$\text{Gain on Sale} = 1,000 - (P - 25)$$

$$\text{Yield (II)} = \frac{1,000 - P + 25 + 36 - 2X}{600} = \frac{1061 - P - 2(916 - P)}{600} = 20\% \rightarrow P = 891$$

Exercise (b)

You and a friend each sell a different stock short for a price of 1,000. The margin requirement is 50% of the selling price and the interest on the margin account is credited at an annual effective rate of 6%.

You buy back your stock one year later at a price of P. At the end of the year, the stock paid a dividend of X. Your friend also buys back their stock one year later but at a price of $(P - 25)$. At the end of the year, their stock paid a dividend of $2X$.

Both you and your friend earn an annual effective yield rate of 20% on your sales.

Calculate X.

Solution (b)

$$\text{Margin (I)} = 1,000(50\%) = 500$$

$$\text{Interest on Margin} = 500(6\%) = 30$$

$$\text{Dividend} = -X$$

$$\text{Gain on Sale} = 1,000 - P$$

$$\text{Yield (I)} = \frac{1,000 - P + 30 - X}{500} = 20\% \rightarrow X = 930 - P$$

$$\text{Margin (II)} = 1,000(50\%) = 500$$

$$\text{Interest on Margin} = 500(6\%) = 30$$

$$\text{Dividend} = -2X$$

$$\text{Gain on Sale} = 1,000 - (P - 25)$$

$$\text{Yield (II)} = \frac{1,000 - P + 25 + 30 - 2X}{500} = \frac{1055 - P - 2(930 - P)}{500} = 20\% \rightarrow X = 25$$

Exercise (c)

A common stock is purchased on January 1, 1992. It is expected to pay a dividend of 15 per share at the end of each year through December 31, 2001. Starting in 2002 dividends are expected to increase $K\%$ per year indefinitely, $K < 8\%$. The theoretical price to yield an annual effective rate of 8% is 200.90.

Calculate K.

Solution (c)

$$200.90 = 15a_{\overline{9}|8\%} + (1.08)^{-9}\left(\frac{15}{.08 - K}\right)$$

$$K = .01$$

Exercise (d)

Steven buys a stock for 200 which pays a dividend of 12 at the end of every 6 months. Steven deposits the dividend payments into a bank account earning a nominal interest rate of 10% convertible semiannually.

At the end of 10 years, immediately after receiving the 20th dividend payment of 12, Steven sells the stock. the sale price assumes a nominal yield of 8% convertible semiannually and that the semiannual dividend of 12 will continue forever.

Steven's annual effective yield over the 10-year period is i. Calculate i.

Solution (d)

$$AV\ of\ dividends = 12s_{\overline{20}|5\%} = 396.79$$

$$P = \frac{12}{.04} = 300$$

$$200(1+i)^{10} = 396.79 + 300$$

$$(1+i)^{10} = 3.4840$$

$$i = 13.29\%$$

7 Duration, Convexity and Immunization

Overview

- there are two types of bond sensitivity, they are duration and convexity.
- modified, Macaulay, and effective are three types of convexity and duration.
- the present value of a company's assets minus the present value of a companies liabilities is called a *surplus*.
- when a company's assets and liabilities are protected from interest rate fluctuations, this is called *immunization*.
- when matching asset cash flows to liability cash flows for the protection of the surplus, this is called *dedication*.

7.1 Price as a Function of Yield

- examining the price curve, price falls as the yield increases
- price is calculated as follows:

$$P = \sum_{t>0} CF_t \left(1 + \frac{y}{m}\right)^{-mt}$$

where P=asset price, y=yield (ie. $y = i^{(m)}$) and CF_t=cash flow at time t years

- the estimated percentage change in a bond's price is approximately:

$$\%\Delta P \approx (\Delta y)\frac{P'(y)}{P(y)}$$

7.2 Modified Duration

- Modified duration is calculated as follows

$$ModD = -\frac{P'(y)}{P(y)}$$

- where P is:

$$P = \sum CF_t \left(1 + \frac{y}{m}\right)^{-mt}$$

$$\frac{dP}{dy} = \sum -tCF_t \left(1 + \frac{y}{m}\right)^{-mt-1} = \frac{\sum -tCF_t \left(1 + \frac{y}{m}\right)^{-mt}}{\left(1 + \frac{y}{m}\right)}$$

$$ModD = \frac{\frac{dP}{dy}}{P} = \frac{\sum tCF_t \left(1 + \frac{y}{m}\right)^{-mt}}{P\left(1 + \frac{y}{m}\right)}$$

Modified Duration

$$ModD = \frac{\sum_{t=0} tCF_t \left(1+\frac{y}{m}\right)^{-mt}}{P\left(1+\frac{y}{m}\right)}$$

Relationship between price and modified duration

$$\%\Delta P \approx -(\Delta y)(ModD)$$

where one hundred basis points equals 1.0%

7.3 Macaulay Duration

– Macaulay Duration is calculated as follows:

$$MacD = -\frac{P'(\delta)}{P(\delta)} = -\left(\frac{dP}{d\delta}\right)\frac{1}{P}$$

– Solving for Macaulay Duration:

$$P = \sum CF_t e^{-\delta t}$$

$$\frac{dP}{d\delta} = \sum -t \cdot CF_t e^{-\delta t}$$

$$MacD = -\left(\frac{dP}{d\delta}\right)\frac{1}{P} = \frac{\sum t \cdot CF_t e^{-\delta t}}{\sum CF_t e^{-\delta t}}$$

$$MacD = \frac{\sum t \cdot CF_t e^{-\delta t}}{P} = \frac{\sum t \cdot CF_t \left(1+\frac{y}{m}\right)^{-mt}}{P} = \frac{\sum t \cdot CF_t \cdot v_i^t}{P}$$

Relationship between Macaulay Duration and Modified Duration

$$ModD = \frac{MacD}{\left(1+\frac{y}{m}\right)}$$

Example of Duration:
Assuming an interest rate of 8%, calculate the duration of:
(1) An n-year zero coupon bond:

$$MacD = \frac{\sum_{t=n}^{n} t \cdot v^t \cdot CF_t}{\sum_{t=n}^{n} v^t \cdot CF_t} = \frac{n \cdot v^n \cdot CF_n}{v^n \cdot CF_n} = n$$

(2) An n-year bond with 8% coupons:

$$MacD = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot CF_t}{\sum_{t=1}^{n} v^t \cdot CF_t} = \frac{8\% \cdot (Ia)_{\overline{n}|i} + 10v_i^n}{8\% \cdot a_{\overline{n}|i} + v_i^n}$$

(3) An n-year mortgage repaid with level payments of principal and interest:

$$MacD = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot CF_t}{\sum_{t=1}^{n} v^t \cdot CF_t} = \frac{(Ia)_{\overline{n}|i}}{a_{\overline{n}|i}}$$

(4) A preferred stock paying level dividends into perpetuity:

$$MacD = \frac{\sum_{t=1}^{\infty} t \cdot v^t \cdot CF_t}{\sum_{t=1}^{\infty} v^t \cdot CF_t} = \frac{(Ia)_{\overline{\infty}|i}}{a_{\overline{\infty}|i}}$$

Macaulay Duration for Bonds Priced at Par

$$MacD = \ddot{a}^{(m)}_{\overline{n}|i}$$

7.4 Effective Duration

– for bonds that do not have fixed cash flows, such as callable bonds, one can use effective duration

Effective Duration

$$EffD = \frac{P_{-} - P_{+}}{P_0(2\Delta y)}$$

where P_0 is the current price of a bond, P_+ is the bond price if interest rates shift up by Δy, and P_- is the bond price if interest rates shift down by Δy

Relationship between Price and Effective Duration

$$\%\Delta P \approx -(\Delta y)(EffD)$$

7.5 Convexity

– convexity is described as the rate of change in interest sensitivity. It is desirable to have positive (negative) changes in the asset values to be greater (less) than positive (negative) changes in liability values. If the changes were plotted on a curve against interest changes, you'd like the curve to be convex.

– convexity is calculated as follows:

$$Convexity = \frac{P''(y)}{P(y)}$$

– solving for convexity:

$$P = \sum CF_t \left(1 + \frac{y}{m}\right)^{-mt}$$

$$\frac{dP}{dy} = \sum -t \cdot CF_t \left(1 + \frac{y}{m}\right)^{-mt-1}$$

$$\frac{d^2P}{dy^2} = \sum t\left(t + \frac{1}{m}\right) CF_t \left(1 + \frac{y}{m}\right)^{-mt-2}$$

$$Convexity = \frac{d^2P}{dy^2} = \sum t\left(t + \frac{1}{m}\right) CF_t \left(1 + \frac{y}{m}\right)^{-mt-2}$$

Relationship between Price, Duration, and Convexity

$$\%\Delta P \approx -(\Delta y)(Duration) + \frac{(\Delta y)^2}{2}(Convexity)$$

7.5.1 Macaulay Convexity

solving for Macaulay convexity:

$$P = \sum CF_t e^{-\delta t}$$

$$\frac{dP}{d\delta} = \sum -t \cdot CF_t e^{-\delta t}$$

$$\frac{d^2P}{d\delta^2} = \sum t^2 \cdot CF_t e^{-\delta t}$$

$$MacC = \frac{\sum t^2 \cdot CF_t e^{-\delta t}}{P}$$

Dispersion

$$MacC = Dispersion + MacD^2$$

$$Dispersion = \frac{\sum (t - MacD)^2 \cdot CF_t \cdot e^{-\delta t}}{\sum CF_t \cdot e^{-\delta t}}$$

7.5.2 Effective Convexity

$$EffC = \frac{(P_+ + P_- - 2P_0)}{(\Delta y)^2 P_0}$$

7.6 Duration, Convexity and Prices: Putting it all Together

7.6.1 Revisiting the Percentage Change in Price

The general formula:

$$\%\Delta P \approx -(\Delta y)(Duration) + \frac{(\Delta y)^2}{2}(Convexity)$$

– one can calculate for $\%\Delta P$ using modified duration and convexity when they have the same frequency as the yield that is shifted

– modified duration and convexity can be used only when the bond's cash flows are fixed

– one can calculate for $\%\Delta P$ using Macaulay duration and Macaulay convexity when the yield is continuously compounding

– Macaulay duration and Macaulay convexity can be used only when the bond's cash flows are fixed

- one can calculate for $\%\Delta P$ using effective duration and effective convexity when they and Δy have the same frequency as the yield that is shifted and when the yield is continuously compounding
- effective duration and effective convexity can be used when the bond's cash flows are fixed and when they are not fixed

7.6.2 The Passage of Time and Duration

There are two opposite effects that occur as a bond ages:

- duration decreases with time since the timing for each payment decreases
- duration increases with time as more weight is given to the more distant cash flows (ie. it will "spike" as you geet closer to a cash flow)

The overall effect is that duration will *decrease*.

7.6.3 Portfolio Duration and Convexity

The duration of the portfolio is:

$$= \frac{P_1}{MV_{port}}D_1 + \frac{P_2}{MV_{port}}D_2 + ... + \frac{P_n}{MV_{port}}D_n$$

where n is how many bonds are in the portfolio, P_k is the price, D_k is the duration and MV_{port} is the market value of the portfolio

The convexity of the portfolio:

$$= \frac{P_1}{MV_{port}}C_1 + \frac{P_2}{MV_{port}}C_2 + ... + \frac{P_n}{MV_{port}}C_n$$

where C_k is the bond's convexity

7.7 Immunization

- it is very difficult for a financial enterprise to match the cash flows of their assets to the cash flows of their liabilities.
- especially when the cash flows can change due to changes in interest rates.

  Surplus

$$S(y) = PV_A - PV_L$$

  where y is the interest rate, PV_A is the present value of the assets and PV_L is the present value of the liabilities

A Problem with Interest Rates

- a bank issues a one-year deposit and guarantees a certain rate of return.
- if interest rates have gone up by the end of the year, then the deposit holder will not renew if the guaranteed rate is too low versus the new interest rate.
- the bank will need to pay out to the deposit holder and if the original proceeds were invested in long-duration assets ("going long"), then the bank needs to sell off its own assets (that have declined in value) in order to pay.
- if interest rates have gone down by the end of the year, then it is possible that the backing assets may not be able to meet the guaranteed rate; this becomes a greater possibility with short-duration assets ("going short").
- the bank may have to sell off some its assets to meet the guarantee.

A Solution to the Interest Rate Problem

- structure the assets so that their cash flows move at least the same amount as the liabilities' cash flows move when interest rates change.
- let A_t and L_t represent the cash flows at time t from an institution's assets and liabilities, respectively.
- let R_t represent the institution's net cash flows at time t such that $R_t = A_t - L_t$.
- if $P(i) = \sum_{t=1}^{n} v^t \cdot R_t = \sum_{t=1}^{n} v^t \cdot (A_t - L_t)$, then we would like the present value of asset cash flows to equal the present value of liability cash flows i.e. $P(i) = 0$:

$$\sum_{t=1}^{n} v^t \cdot A_t = \sum_{t=1}^{n} v^t \cdot L_t$$

- we'd also like the interest sensitivity (modified duration) of the asset cash flows to be equal to the interest sensitivity of the liabilities i.e. $\dfrac{P'(i)}{P(i)} = 0 \rightarrow P'(i) = 0$.

$$MacD_A = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot A_t}{\sum_{t=1}^{n} v^t \cdot A_t} = \frac{\sum_{t=1}^{n} t \cdot v^t \cdot L_t}{\sum_{t=1}^{n} v^t \cdot L_t} = MacD_L$$

– in addition, we'd also like the convexity of the asset cash flows to be equal to the convexity of the liabilities i.e. $\dfrac{P''(i)}{P(i)} = 0 \rightarrow P''(i) > 0$.

– an immunization strategy strives to meet these three conditions.

– recall that in Section 7.2, modified duration was determined by taking the 1^{st} derivative of the present value of the payments:

$$ModD = -\frac{\frac{d}{di}\left(\sum_{t=1}^{n} v^t \cdot R_t\right)}{\sum_{t=1}^{n} v^t \cdot R_t} = \frac{MacD}{1+i}$$

– we are now also interested in how sensitive the volatility itself is called convexity.

– We determine convexity by taking the 1^{st} derivative of $\bar{v}$:

$$Convexity = \frac{d}{di}\left(\frac{MacD}{1+i}\right) = \frac{\frac{d^2}{d^2 i}\left(\sum_{t=1}^{n} v^t \cdot R_t\right)}{\sum_{t=1}^{n} v^t \cdot R_t}$$

– note that the forces that control liability cash flows are often out of the control of the financial institution.

– as a result, immunization will tend to focus more on the structure of the assets and how to match its volatility and convexity to that of the liabilities.

– Immunization is a three-step process:

(1) the present value of cash inflows (assets) should be equal to the present value of cash outflows (liabilities).

(2) the interest rate sensitivity of the present value of cash inflows (assets) should be equal to the interest rate sensitivity of the present value of cash outflows (liabilities).

(3) the convexity of the present value of cash inflows (assets) should be greater than the convexity of the present value of cash outflows(liabilities). In other words, asset growth (decline) should be greater (less) than liability growth(decline).

Difficulties/Limitations of Immunization

(a) choice of i is not always clear.

(b) doesn't work well for large changes in i.

(c) yield curve is assumed to change with Δi; actually, short-term rates are more volatile than long-term rates.

(d) frequent rebalancing is required in order to keep the modified duration of the assets and liabilities equal.

(e) exact cash flows may not be known and may have to be estimated.

(f) convexity suggests that profit can be achieved or that arbitrage is possible.

(g) assets may not have long enough maturities of duration to match liabilities.

Example:

A bank is required to pay 1,100 in one year. There are two investment options available with respect to how monies can be invested now in order to provide for the 1,100 payback:

(i) a non-interest bearing cash fund, for which x will be invested, and

(ii) a two-year zero-coupon bond earning 10% per year, for which y will be invested.

Question: based on immunization theory, develop an asset portfolio that will minimize the risk that liability cash flows will exceed asset cash flows.

Solution:

- it is desirable to have the present value of the asset cash flows equal to that of the liability cash flows:

$$x + y(1.10)^2 \cdot v_i^2 = 1100 v_i^1$$

- it is desirable to have the modified duration of the asset cash flows equal to that of the liability cash flows so that they are equally sensitive to interest rate changes:

$$\frac{x}{x+y}\left(\frac{0}{1+i}\right) + \frac{y}{x+y}\left(\frac{2}{1+i}\right) = \frac{1}{1+i}$$

$$\frac{2y}{x+y} = 1$$

- it is desirable for the convexity of the asset cash flows to be greater than that of the liabilities:

$$\frac{\frac{d^2}{d^2i}\left(x + y(1.10)^2 \cdot v_i^2\right)}{x + y(1.10)^2 \cdot v_i^2} > \frac{\frac{d^2}{d^2i}\left(1100 v_i^1\right)}{1100 v_i^1}$$

$$\frac{y(1.10)^2(-2)(-3) \cdot v_i^4}{x + y(1.10)^2 \cdot v_i^2} > \frac{-1100(-2)v_i^3}{1100 v_i^1}$$

- if an effective rate of interest of 10% is assumed, then:

$$\left.\begin{array}{r} x + \frac{y(1.10)^2}{1.10^2} = \frac{1100}{1.1} \rightarrow x + y = 1000 \\ \frac{2y}{x+y} = 1 \end{array}\right\} \; x = 500, y = 500$$

- and the convexity of the assets is greater than convexity of the liabilities:

$$\frac{500(1.10)^2(-2)(-3) \cdot v_{10\%}^4}{500 + 500(1.10)^2 \cdot v_{10\%}^2} > \frac{-1100(-2)v_{10\%}^3}{1100 v_{10\%}^1}$$

$$2.479 > 1.653$$

- the interest volatility (modified duration) of the assets and liability are:

-

$$ModD_A = \left(\frac{x}{x+y}\right) \cdot \left(\frac{0}{1+i}\right) + \left(\frac{y}{x+y}\right) \cdot \left(\frac{2}{1+i}\right) = \left(\frac{500}{1000}\right) \cdot \left(\frac{2}{1.1}\right) = .90909$$

$$ModD_L = \left(\frac{1}{1+i}\right) = \left(\frac{1}{1.1}\right) = .90909$$

7.8 Full Immunization

- Redington immunization can be used when there are small shifts in a flat yield curve in order to protect the surplus
- full immunization can be used when there are small or large shifts in a flat yield curve in order to protect the surplus
- a fully immunized position fulfils all conditions of Redington immunization

Conditions For Full Immunization of a Single Liability Cash Flow

(1) Present value of assets=Present value of liability

(2) Duration of assets=Duration of liability

(3) The asset cash flows occur before and after the liability cash flow. That is: $(T-q) < T < (T+r)$

For a fully immunization position:

$$MacD_L = MacD_A = T$$

$$Dispersion_L = \frac{[T-T]^2 L \cdot e^{-\delta T}}{L \cdot e^{-\delta T}} = 0$$

$$Dispersion_A = \frac{[(T-q)-T]^2 Q \cdot e^{-\delta(T-q)} + [(T+r)-T]^2 R \cdot e^{-\delta(T+r)}}{Q^{-\delta(T-q)} + R \cdot e^{-\delta(T-r)}}$$

$$= \frac{q^2 Q e^{-\delta(T-q)} + r^2 R \cdot e^{-\delta(T+r)}}{Q \cdot e^{-\delta(T-q)} + R \cdot e^{-\delta(T+r)}}$$

$$Dispersion_A > 0$$

$$MacC_L = 0 + T^2 = T^2$$

$$MacC_A = Dispesion_A + T^2$$

$$MacC_A > MacC_L$$

Exercises and Solutions

7.2 Modified Duration

Exercise (a)

The annual effective yield on a bond is 6%. A 5 year bond pays annual coupons of 7%. Calculate the modified duration of the bond.

Solution (a)

$$CF_t = (0.07)(100) = 7$$

The price of the bond is:

$$P = \sum CF_t \left(1 + \frac{y}{m}\right)^{-mt}$$

$$= 7v + 7v^2 + 7v^3 + 7v^4 + 7v^5 + 100v^5$$

$$= 7(v + v^2 + v^3 + v^4 + v^5) + 100v^5$$

$$= 7a_{\overline{5}|6\%} + 100v^5$$

$$= 104.212$$

The price of modified duration:

$$ModD = \frac{\sum t \cdot CF_t \left(1 + \frac{y}{m}\right)^{-mt}}{P\left(1 + \frac{y}{m}\right)}$$

$$= \frac{1 \cdot 7v_{6\%} + 2 \cdot 7v_{6\%}^2 + 3 \cdot 7v_{6\%}^3 + 4 \cdot 7v_{6\%}^4 + 5 \cdot 7v_{6\%}^5 + 5 \cdot 100v_{6\%}^5}{104.212(1.06)}$$

$$= \frac{7(Ia)_{\overline{5}|6\%} + 5(100)v_{6\%}^5}{104.212(1.06)}$$

$$= 4.152$$

Exercise (b)

The current price of a bond is 116.73 and the current yield is 5%. The modified duration of the bond is 8.14. Use the modified duration to estimate the price of the bond if the yield increases to 6.30%.

Solution (b)

$$\%\Delta P \approx (-\Delta y)(ModD)$$

$$\approx (-0.013)(8.14)$$

$$\approx (-.10582)$$

The bond price is:

$$116.73(1 - 0.10582) = 104.378$$

Exercise (c)

A zero coupon bond matures in 10 years for 3,000. The bonds yield is 3% compounded semi-annually. Calculate the modified duration of the bond.

Solution (c)

$$ModD = \frac{\sum_t t \cdot CF_t \left(1 + \frac{y}{m}\right)^{-mt}}{P\left(1 + \frac{y}{m}\right)}$$

$$\frac{10(3,000)(1.015)^{-20}}{3,000(1.015)^{-20}(1.015)}$$

$$= \frac{10}{1.015} = 9.852$$

Exercise (d)

A two year bond has 6% annual coupons paid semi-annually. The bond's yield is 8% semi-annually. Calculate the modified duration of the bond.

Solution (d)

$$ModD = \frac{\sum t \cdot CF_t \left(1 + \frac{y}{m}\right)^{-mt}}{P\left(1 + \frac{y}{m}\right)}$$

The semi-annual coupons are $0.03(100) = 3$

The semi-annual yield is $\frac{0.08}{2} = 0.04$

$$Price = \sum CF_t \left(1 + \frac{y}{m}\right)^{-mt}$$

$$= 3v + 3v^2 + 3v^3 + 3v^4 + 100v^4$$

$$= 3a_{\overline{4}|4\%} + 100v^4$$

$$P = 96.3701$$

$$ModD = \frac{\frac{3(0.5)}{1.04} + \frac{3(1)}{(1.04)^2} + \frac{3(1.5)}{(1.04)^3}\frac{3(2)}{(1.04)^4} + \frac{100(2)}{(1.04)^4}}{96.3701 \cdot (1.04)}$$

$$= \frac{1.5(Ia)_{\overline{4}|4\%} + \frac{2(100)}{(1.04)^4}}{(96.3701)(1.04)}$$

$$= \frac{13.34528 + 170.96}{100.2249} = 1.8389$$

7.3 Macaulay Duration

Exercise (a)

The current price of a bond is 200. The derivative with respect to the yield to maturity is -800. The yield to maturity is 7%. Calculate the Macaulay duration.

Solution (a)

$$ModD = \frac{-P'(y)}{P(y)} = \frac{-(-800)}{200} = 4$$

$$ModD = \frac{MacD}{\left(1 + \frac{y}{m}\right)}$$

$$4 = \frac{MacD}{1.07}$$

$$MacD = 4.28$$

Exercise (b)

Calculate the Macaulay duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend is constant, and that the effective rate of interest is 8%.

Solution (b)

$$MacD = \frac{\sum_{t=1}^{\infty} t \cdot CF_t v^t}{\sum_{t=1}^{\infty} CF_t v^t}$$

$$= \frac{\sum_{t=1}^{\infty} t \cdot v^t}{\sum_{t=1}^{\infty} v^t}$$

Solving for the numerator:

$$\sum_{t=1}^{\infty} tv^t = v + 2v^2 + 3v^3 + 4v^4 + ... + \infty v^{\infty}$$

$$= \frac{1+i}{i^2}$$

Solving for the denominator:

$$\sum_{t=1}^{\infty} v^t = v + v^2 + v^3 + v^4 + ...v^{\infty}$$

$$= \frac{1}{i}$$

$$= \frac{\sum_{t=1}^{\infty} t \cdot v^t}{\sum_{t=1}^{\infty} v^t} = \frac{\frac{1+i}{i^2}}{\frac{1}{i}} = \frac{1+i}{i} = \frac{1.08}{0.08} = 13.5$$

Exercise (c)

Calculate the Macaulay Duration of a common stock that pays dividends at the end of each year into perpetuity. Assume that the dividend increases by 3% each year and the effective rate is 5%.

Solution (c)

$$MacD = \frac{\sum_{t=1}^{\infty} t \cdot CF_t v^t}{\sum_{t=1}^{\infty} CF_t v^t}$$

$$MacD = \frac{\sum_{t=1}^{\infty} t \cdot v^t (1.03)^{t-1}}{\sum_{t=1}^{\infty} v^t (1.03)^{(t-1)}}$$

Solving for the numerator:

$$N = v(1.03) + 2v^2(1.03) + 3v^3(1.03)^2 + ... + nv^n(1.03)^{n-1}$$

$$(1.03v)N = (1.03)v^2 + 2v^3(1.03)^2 + 3v^4(1.03)^3 + ... + (n-1)v^n(1.03)^{n-1} + ...$$

$$N - N(1.03)v = v + v^2(1.03) + v^3(1.03)^2 + ... + (1.03)^{n-1}v^n + ...$$

$$N(1-(1.03)v) = v(1 + v(1.03) + v^2(1.03)^2 + ... + (1.03)^n v^n$$

$$N(1-(1.03)v) = v\left[\frac{1-\left(\frac{1.03}{1+i}\right)^{\infty}}{1-\left(\frac{1.03}{1+i}\right)}\right]$$

$$N(1-(1.03)v) = v\left[\frac{1}{1-(1.03)v}\right]$$

$$N(1-(1.03)v) = \frac{1}{1+i}\left[\frac{1}{1-\frac{1.03}{1+i}}\right]$$

$$N(1-(1.03)v) = \frac{\frac{1}{1+i}}{\frac{1+i-1.03}{1+i}} = \frac{1}{i-0.03}$$

$$N = \frac{1+i}{(i-0.03)^2}$$

Now solving for the denominator:

$$= v + v^2(1.03) + v^3(1.03)^2 + ... + v^n(1.03)^{n-1}$$

$$= v(1 + v(1.03) + v^2(1.03)^2 + ... + v^{n-1}(1.03)^{n-1}$$

$$= v\left[\frac{1-\left(\frac{1.03}{1+i}\right)^{\infty}}{1-\frac{1.03}{1+i}}\right]$$

$$= \frac{v}{1+i}\left[\frac{1}{1-\frac{1.03}{1+i}}\right]$$

$$= \frac{\frac{1}{1+i}}{\frac{i-.03}{1+i}} = \frac{1}{i-.03}$$

$$MacD = \frac{\frac{1+i}{(i-.03)^2}}{\frac{1}{i-.03}} = \frac{1+i}{i-.03} = \frac{1.05}{0.05-0.03} = \frac{1.05}{0.02} = 52.5$$

<u>Exercise (d)</u>

The duration of a bond at interest rate i is defined as:

$$\frac{\sum_{t=1} t \cdot CF_t v^t}{\sum_{t=1} CF_{t=1} v^t}$$

where CF_t represents the net cash flow from the coupons and the maturity value of the bond at time t. You are given a 1,000 par value 20-year bond with 4% annual coupons and a maturity value of 1,000. Calculate the duration of this bond at 5% interest.

<u>Solution (d)</u>

Duration:

$$= \frac{40(Ia)_{\overline{20}|5\%} + 20(1,000)v^{20}_{5\%}}{40a_{\overline{20}|5\%} + 1,000v^{20}_{5\%}} = \frac{40\left(\frac{\ddot{a}_{\overline{20}|5\%} - 20v^{20}_{5\%}}{i}\right) + 20(1,000)v^{20}_{5\%}}{40a_{\overline{20}|5\%} + 1,000v^{20}_{5\%}}$$

$$\frac{11,975.81}{875.38} = 13.678$$

7.4 Effective Duration

Exercise (a)

a 10-year bond yielding 8% has a price of 112.76. If the bond yield falls to 7.75% then the price of the bond will increase to 115.32. If the bond's yield increases to 8.25% then the price of the bond will fall to 110.25. Calculate the effective duration of the bond.

Solution (a)

$$EffD = \frac{P_{-} - P_{+}}{P_0(2\Delta y)}$$

$$= \frac{115.32 - 110.25}{112.76(2)(0.0025)} = \frac{5.07}{0.5638} = 8.9926$$

Exercise (b)

A six year bond with coupons of 5% pays coupons semi-annually. The effective yield is 4%. It's current price is 98.71. If the bond increases by 10 basis points then it's price will fall to 97.84. If the bond's yield falls 10 basis points, then it's price rises to 99.13. Caculate the bond's effective duration.

Solution (b)

$$EffD = \frac{P_{-} - P_{+}}{P_0(2\Delta y)}$$

$$= \frac{99.13 - 97.84}{(98.71)(2)(0.001)} = \frac{1.29}{.19742} = 6.5346$$

7.5 Convexity

Exercise (a)

The effective duration is 5.4. The effective convexity is calculated by a 20 year bond that has an effective yield of 7% and a price of 103.10. If the bond's yield falls to 6.5% then the price increases to 103.51. If the bond's yield falls to 7.5% then the price decreases to 102.94. Calculate the estimated new price of the bond if it increases by 75 basis points.

Solution (a)

$$EffC = \frac{P_+ + P_- - 2P_0}{P_0(\Delta y)^2}$$

$$= \frac{103.51 + 102.94 - (2)(1.03.10)}{(103.10)(.005)^2}$$

$$= \frac{0.25}{.0025775} = 96.9932$$

$$\%\Delta P \approx -(\Delta y)(Duration) + \frac{(\Delta y)^2}{2}(Convexity)$$

$$= -(.0075)(5.4) + \frac{(.0075)^2}{2}(96.9932)$$

$$= -0.0378$$

7.6 Duration, Convexity, and Prices: Putting it all Together

Exercise (a)

Leah purchases three bonds to form a portfolio as follows:

Bond A has semi-annual coupons at 5%, a duration of 20.57 years and was purchased for 880.

Bond B is an 11-year bond with a duration of 11.75 years and was purchased for 1,124.

Bond C has a duration of 17.31 years and was purchased for 1,000.

Calculate the duration of the portfolio at the time of purchase.

Solution (a)

The duration of the portfolio is:

$$= \frac{P_1}{MV_{port}} D_1 + \frac{P_2}{MV_{port}} D_2 + ... + \frac{P_n}{MV_{port}} D_n$$

$$MV_{port} = P_1 + P_2 + ... P_n$$

$$MV_{port} = 880 + 1,124 + 1,000 = 3,004$$

$$Duration_{port} = \frac{(880)(20.57)}{3,004} + \frac{(1,124)(11.75)}{3,004} + \frac{(1,000)(17.31)}{3,004} = \frac{4,8618.6}{3,004} = 16.18$$

8 The Term Structure of Interest Rates

Overview

– this chapter will show how to:

1. calculate a spot rate
2. calculate a forward rate

8.1 Yield-to-Maturity

– an interest rate is called a yield rate or an internal rate of return (IRR) as it indicates the rate of return that the investor can expect to earn on their investment.

finding price using a yield rate:

$$P = \sum_{t>0} \frac{CF_t}{(1+y)^t}$$

where P is price, t is time, CF_t is the cash flow and y is the yield rate

Yield Curves

– usually long-term market interest rates are higher than short-term market interest rates

– yield curves are usually upsloping, although they can be flat, downsloping, a peak or a valley

– when the yield is constructed by government securities, it is called an *on-the-run yield curve*

– when each bond's coupon rate is assumed to equal a bond's yield rate, it is called *a par yield curve*

8.2 Spot Rates

– are the annual interest rates that make up the yield curve.

– let s_t represent the spot rate for period t.

– let P represent the net present value of a series of future payments (positive or negative) discounted using spot rates:

$$P = \sum_{t=0}^{n} (1+s_t)^{-t} \cdot CF_t$$

– The present value of annuities can also be found using spot rates:

$$a_{\overline{n}|} = \frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + ... + \frac{1}{(1+s_n)^n}$$

$$\ddot{a}_{\overline{n}|} = 1 + \frac{1}{1+s_1} + \frac{1}{(1+s_2)^2} + ... + \frac{1}{(1+s_{n-1})^{n-1}}$$

Bootstrapping

– determining future spot rates using the price of a coupon bond

– is used assuming arbitrage is impossible

– arbitrage is risk-free profit

Price using yield=Price using spot rates

$$\sum \frac{CF_t}{(1+y)^t} = \sum \frac{CF_t}{(1+s_t)^t}$$

Forward Rates

– are considered to be future reinvestment rates.

the price of a security is calculated as follows:

$$P = \sum_{t>0} \frac{CF_t}{(1+f_0)(1+f_1)\cdots(1+f_{t-1})}$$

where f_t is the annual effective forward rate from time t to $t+1$

– similar to the bootstrapping concept, forward rates can be calculated using the yield curve

Price of yield curve=Price using forward rates

Relationships between Forward Rates and Spot Rates

$$(1+s_t)^t = (1+f_0)(1+f_1)\cdots(1+f_{t-1})$$

$$s_t = \sqrt[t]{(1+f_0)(1+f_1)\cdots(1+f_{t-1})} - 1$$

$$f_{t-1} = \frac{(1+s_t)^t}{(1+s_{t-1})^{t-1}} - 1$$

Example:

A firm wishes to borrow money repayable in two years, where the one-year and two-year spot rates are 8% and 7%, respectively.

The estimated one-year deferred one-year spot rate is called the forward rate, f, and is calculated by equating the two interest rates such that

$$(1.08)^2 = (1.07)(1+f) \rightarrow f = 9.01\%$$

If the borrower thinks that the spot rate for the 2^{nd} year will be greater(less) than the forward rate, then they will select (reject) the 2-year borrowing rate.