GRAPH THEORY: BASIC DEFINITIONS AND THEOREMS

1. Definitions

Definition 1. A **graph** $G = (V, E)$ consists of a set V of **vertices** (also called **nodes**) and a set E of **edges**.

Definition 2. If an edge connects to a vertex we say the edge is **incident** to the vertex and say the vertex is an **endpoint** of the edge.

Definition 3. If an edge has only one endpoint then it is called a **loop edge**.

Definition 4. If two or more edges have the same endpoints then they are called **multiple** or **parallel** edges.

Definition 5. Two vertices that are joined by an edge are called **adjacent** vertices.

Definition 6. A **pendant** vertex is a vertex that is connected to exactly one other vertex by a single edge.

Definition 7. A **walk** in a graph is a sequence of alternating vertices and edges $v_1e_1v_2e_2 \ldots v_ne_nv_{n+1}$ with $n \geq 0$. If $v_1 = v_{n+1}$ then the walk is **closed**. The **length** of the walk is the number of edges in the walk. A walk of length zero is a **trivial walk**.

Definition 8. A **trail** is a walk with no repeated edges. A **path** is a walk with no repeated vertices. A **circuit** is a closed trail and a **trivial circuit** has a single vertex and no edges. A trail or circuit is **Eulerian** if it uses every edge in the graph.

Definition 9. A **cycle** is a nontrivial circuit in which the only repeated vertex is the first/last one.

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Definition 10. A **simple graph** is a graph with no loop edges or multiple edges. Edges in a simple graph may be specified by a set $\{v_i, v_j\}$ of the two vertices that the edge makes adjacent. A graph with more than one edge between a pair of vertices is called a **multigraph** while a graph with loop edges is called a **pseudograph**.

Definition 11. A **directed graph** is a graph in which the edges may only be traversed in one direction. Edges in a simple directed graph may be specified by an ordered pair (v_i, v_j) of the two vertices that the edge connects. We say that v_i is **adjacent to** v_j and v_j is **adjacent from** v_i.

Definition 12. The **degree** of a vertex is the number of edges incident to the vertex and is denoted $\deg(v)$.

Definition 13. In a directed graph, the **in-degree** of a vertex is the number of edges **incident to** the vertex and the **out-degree** of a vertex is the number of edges **incident from** the vertex.

Definition 14. A graph is **connected** if there is a walk between every pair of distinct vertices in the graph.

Definition 15. A graph H is a **subgraph** of a graph G if all vertices and edges in H are also in G.

Definition 16. A **connected component** of G is a connected subgraph H of G such that no other connected subgraph of G contains H.

Definition 17. A graph is called **Eulerian** if it contains an Eulerian circuit.

Definition 18. A **tree** is a connected, simple graph that has no cycles. Vertices of degree 1 in a tree are called the **leaves** of the tree.

Definition 19. Let G be a simple, connected graph. The subgraph T is a **spanning tree of** G if T is a tree and every node in G is a node in T.

Definition 20. A **weighted graph** is a graph $G = (V, E)$ along with a function $w : E \to \mathbb{R}$ that associates a numerical weight to each edge. If G is a weighted graph, then T is a **minimal spanning tree of** G if it is a spanning tree and no other spanning tree of G has smaller total weight.

Definition 21. The **complete graph** on n nodes, denoted K_n, is the simple graph with nodes $\{1, \ldots, n\}$ and an edge between every pair of distinct nodes.

Definition 22. A graph is called **bipartite** if its set of nodes can be partitioned into two disjoint sets S_1 and S_2 so that every edge in the graph has one endpoint in S_1 and one endpoint in S_2.

Definition 23. The **complete bipartite graph** on n, m nodes, denoted $K_{n,m}$, is the simple bipartite graph with nodes $S_1 = \{a_1, \ldots, a_n\}$ and $S_2 = \{b_1, \ldots, b_m\}$ and with edges connecting each node in S_1 to every node in S_2.

Definition 24. Simple graphs G and H are called **isomorphic** if there is a bijection f from the nodes of G to the nodes of H such that $\{v, w\}$ is an edge in G if and only if $\{f(v), f(w)\}$ is an edge of H. The function f is called an **isomorphism**.

Definition 25. A simple, connected graph is called **planar** if there is a way to draw it on a plane so that no edges cross. Such a drawing is called an **embedding** of the graph in the plane.

Definition 26. For a planar graph G embedded in the plane, a **face** of the graph is a region of the plane created by the drawing. The area of the plane outside the graph is also a face, called the unbounded face.

2. Theorems

Theorem 1. *Let G be a connected graph. Then G is Eulerian if and only if every vertex in G has even degree.*

Theorem 2 (Handshaking Lemma)**.** *In any graph with n vertices v_i and m edges*

$$\sum_{i=1}^{n} \deg(v_i) = 2m$$

Corollary 1. *A connected non-Eulerian graph has an Eulerian trail if and only if it has exactly two vertices of odd degree. The trail begins and ends these two vertices.*

Theorem 3. *If T is a tree with n edges, then T has $n+1$ vertices.*

Theorem 4. *Two graphs that are isomorphic to one another must have*

1. *The same number of nodes.*
2. *The same number of edges.*
3. *The same number of nodes of any given degree.*
4. *The same number of cycles.*
5. *The same number of cycles of any given size.*

Theorem 5 (Kuratowski's Theorem)**.** *A graph G is nonplanar if and only if it contains a "copy" of $K_{3,3}$ or K_5 as a subgraph.*

Theorem 6 (Euler's Formula for Planar Graphs)**.** *For any connected planar graph G embedded in the plane with V vertices, E edges, and F faces, it must be the case that*

$$V + F = E + 2.$$