CHAPTER ONE

VECTOR GEOMETRY

1.1 INTRODUCTION

In this chapter vectors are first introduced as geometric objects, namely as directed line segments, or arrows. The operations of addition, subtraction, and multiplication by a scalar (real number) are defined for these directed line segments. Two and three dimensional Rectangular Cartesian coordinate systems are then introduced and used to give an algebraic representation for the directed line segments (or vectors). Two new operations on vectors called the dot product and the cross product are introduced. Some familiar theorems from Euclidean geometry are proved using vector methods.

1.2 SCALARS AND VECTORS

Some physical quantities such as length, area, volume and mass can be completely described by a single real number. Because these quantities are describable by giving only a magnitude, they are called **scalars**. [The word scalar means representable by position on a line; having only magnitude.] On the other hand physical quantities such as displacement, velocity, force and acceleration require both a magnitude and a direction to completely describe them. Such quantities are called **vectors**.

If you say that a car is traveling at 90 km/hr, you are using a scalar quantity, namely the number 90 with no direction attached, to describe the speed of the car. On the other hand, if you say that the car is traveling due north at 90 km/hr, your description of the car's velocity is a vector quantity since it includes both magnitude and direction.

To distinguish between scalars and vectors we will denote scalars by lower case italic type such as *a, b, c* etc. and denote vectors by lower case boldface type such as **u**, **v**, **w** etc. In handwritten script, this way of distinguishing between vectors and scalars must be modified. It is customary to leave scalars as regular hand written script and modify the symbols used to represent vectors by either underlining, such as $\underline{u}$ or $\underline{v}$, or by placing an arrow above the symbol, such as $\vec{u}$ or $\vec{v}$.

1.2 Problems

1. Determine whether a scalar quantity, a vector quantity or neither would be appropriate to describe each of the following situations.

   a. The outside temperature is 15º C.

   b. A truck is traveling at 60 km/hr.

   c. The water is flowing due north at 5 km/hr.

   d. The wind is blowing from the south.

   e. A vertically upwards force of 10 Newtons is applied to a rock.

   f. The rock has a mass of 5 kilograms.

   g. The box has a volume of .25 m^3.

   h. A car is speeding eastward.

   i. The rock has a density of 5 gm/cm^3.

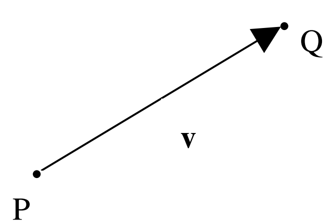
   j. A bulldozer moves the rock eastward 15m.

   k. The wind is blowing at 20 km/hr from the south.

   l. A stone dropped into a pond is sinking at the rate of 30 cm/sec.

1.3 GEOMETRICAL REPRESENTATION OF VECTORS

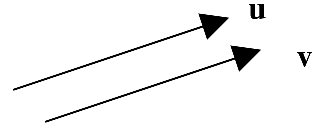
Because vectors are determined by both a magnitude and a direction, they are represented geometrically in 2 or 3 dimensional space as **directed line segments** or **arrows**. The length of the arrow corresponds to the magnitude of the vector while the direction of the arrow corresponds to the direction of the



vector. The tail of the arrow is called the **initial point** of the vector while the tip of the arrow is called the **terminal point** of the vector. If the vector **v** has the point P as its initial point and the point Q as its terminal point we will write $\mathbf{v} = \overrightarrow{PQ}$.

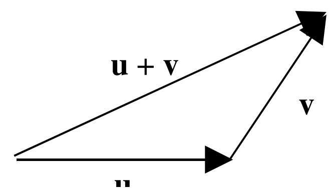
Equal vectors

Two vectors **u** and **v**, which have the same length and same direction, are said to be **equal vectors** even though they have different initial points and different terminal points. If **u** and **v** are equal vectors we write **u** = **v**.



Sum of two vectors

The **sum** of two vectors **u** and **v**, written **u** + **v** is the vector determined as follows. Place the vector **v** so that its initial point coincides with the terminal point of the vector **u.** The vector **u** + **v** is the vector whose initial point is the initial point of **u** and whose terminal point is the terminal point of **v**.



Zero vector

The **zero vector**, denoted **0**, is the vector whose length is 0. Since a vector of length 0 does not have any direction associated with it we shall agree that its direction is arbitrary; that is to say it can be assigned any direction we choose. The zero vector satisfies the property: $\mathbf{v} + \mathbf{0} = \mathbf{0} + \mathbf{v} = \mathbf{v}$ for every vector **v**.

Negative of a vector

If **u** is a nonzero vector, we define the **negative of u**, denoted **–u**, to be the vector whose magnitude (or length) is the same as the magnitude (or length) of the vector **u,** but whose direction is opposite to that of **u**.

If $\overrightarrow{AB}$ is used to denote the vector from point A to point B, then the vector from point B to point A is denoted by $\overrightarrow{BA}$, and $\overrightarrow{BA} = -\overrightarrow{AB}$.

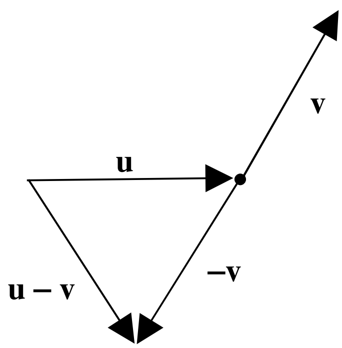
Difference of two vectors

If **u** and **v** are any two vectors, we define the **difference of u and v**, denoted **u – v**, to be the vector **u + (–v)**. To construct the vector **u – v** we can either

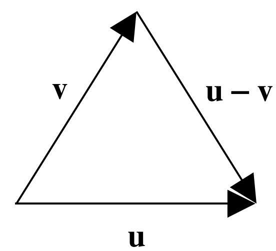
(i) construct the sum of the vector **u** and the vector **–v;** or

(ii) position **u** and **v** so that their initial points coincide; then the vector from the terminal point of **v** to the terminal point of **u** is the vector **u – v**.

(i)

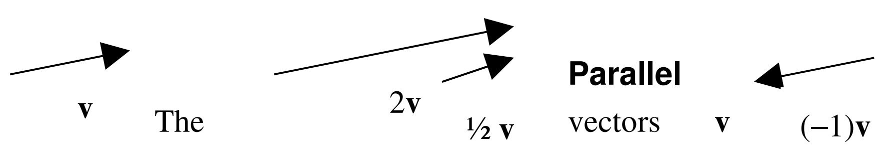


(ii)



Multiplying a vector by a scalar

If **v** is a nonzero vector and c is a nonzero scalar, we define the product of c and **v**, denoted c**v**, to be the vector whose length is $|c|$ times the length of **v** and whose direction is the same as that of **v** if $c > 0$ and opposite to that of **v** of $c < 0$. We define c**v** = **0** if $c = 0$ or if **v** = **0.**

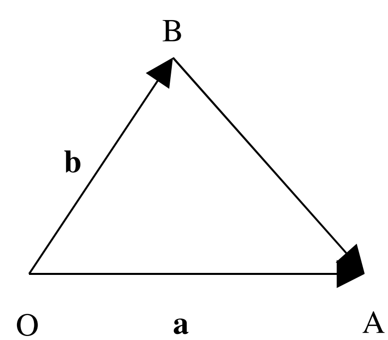


Parallel vectors

The vectors **v** and c**v** are **parallel** to each other. Their directions coincide if $c > 0$ and the directions are opposite to each other if $c < 0$. If **u** and **v** are parallel vectors, then there exists a scalar c such that **u =** c**v.** Conversely, if **u =** c**v** and $c \neq 0$, then **u** and **v** are parallel vectors.

Example

Let O, A and B be 3 points in the plane. Let $\overrightarrow{OA} = \mathbf{a}$ and let $\overrightarrow{OB} = \mathbf{b}$. Find an expression for the vector $\overrightarrow{BA}$ in terms of the vectors **a** and **b**.

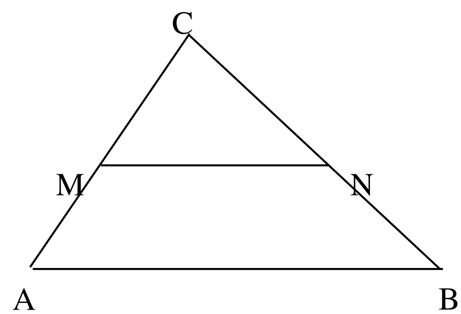


Solution

$$\begin{aligned}\overrightarrow{BA} &= \overrightarrow{BO} + \overrightarrow{OA}\\ &= -\overrightarrow{OB} + \overrightarrow{OA}\\ &= \overrightarrow{OA} - \overrightarrow{OB}\\ &= \mathbf{a} - \mathbf{b}\,.\end{aligned}$$

Example Prove that the line joining the mid points of two sides of a triangle is parallel to and one-half the length of the third side of the triangle.

Solution



Let $\Delta\,ABC$ be given. Let M be the mid point of side AC and let N be the mid point of side BC. Then

$$\overrightarrow{MN} = \overrightarrow{MC} + \overrightarrow{CN} = \tfrac{1}{2}\overrightarrow{AC} + \tfrac{1}{2}\overrightarrow{CB} = \tfrac{1}{2}(\overrightarrow{AC} + \overrightarrow{CB}) = \tfrac{1}{2}\overrightarrow{AB}\,.$$

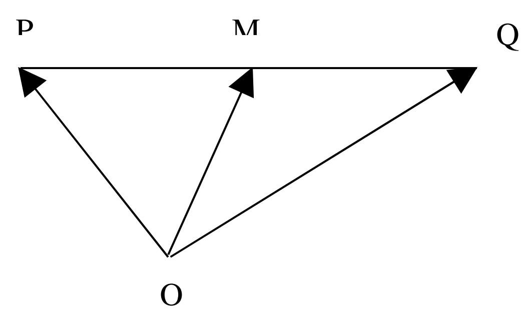
This shows that MN is one-half the length of AB and also that MN is parallel to AB [since the two vectors $\overrightarrow{MN}$ and $\tfrac{1}{2}\overrightarrow{AB}$ are equal, they have the same direction and hence are parallel, so $\overrightarrow{MN}$ and $\overrightarrow{AB}$ will also be parallel].

Example

Let M be the mid point of the line segment PQ. Let O be a point not on the line PQ. Prove that $\overrightarrow{OM} = \tfrac{1}{2}\overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{OQ}$.

Solution

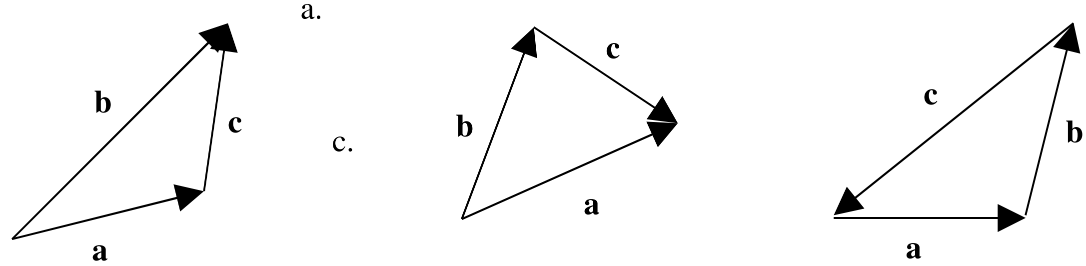
$$\begin{aligned}\overrightarrow{OM} = \overrightarrow{OP} + \overrightarrow{PM} &= \overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{PQ}\\ &= \overrightarrow{OP} + \tfrac{1}{2}(\overrightarrow{PO} + \overrightarrow{OQ})\\ &= \overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{PO} + \tfrac{1}{2}\overrightarrow{OQ}\\ &= \overrightarrow{OP} - \tfrac{1}{2}\overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{OQ}\\ &= \tfrac{1}{2}\overrightarrow{OP} + \tfrac{1}{2}\overrightarrow{OQ}\end{aligned}$$

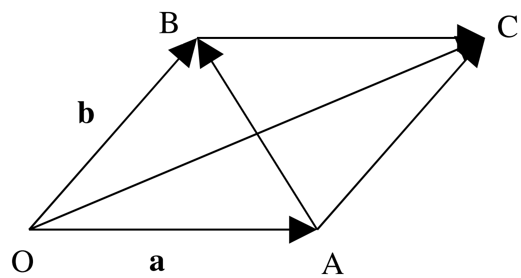


1.3 Problems

1. For each of the following diagrams, find an expression for the vector **c** in terms of the vectors **a** and **b**.

a. b. c.





2. Let OACB be the parallelogram shown. Let $\mathbf{a} = \overrightarrow{OA}$ and let $\mathbf{b} = \overrightarrow{OB}$. Find expressions for the diagonals $\overrightarrow{OC}$ and $\overrightarrow{AB}$ in terms of the vectors **a** and **b**.

3. Let ABC be a triangle. Let M be a point on AC such that the length of AM = ½ length of MC. Let N be a point on BC such that the length of BN = ½ length of NC. Show that MN is parallel to AB and that the length of MN is ⅔ the length of AB.

4. Let the point M divide the line segment AB in the ratio $t{:}s$ with $t + s = 1$. Let O be a point not on the line AB. Prove $\overrightarrow{OM} = s\overrightarrow{OA} + t\overrightarrow{OB}$.

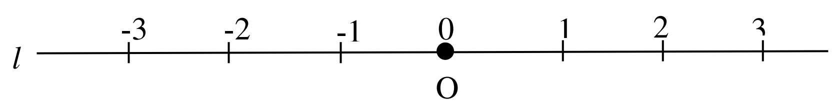
5. Prove that the diagonals of a parallelogram bisect each other.

6. Prove that the medians of a triangle are concurrent.

1.4 COORDINATE SYSTEMS

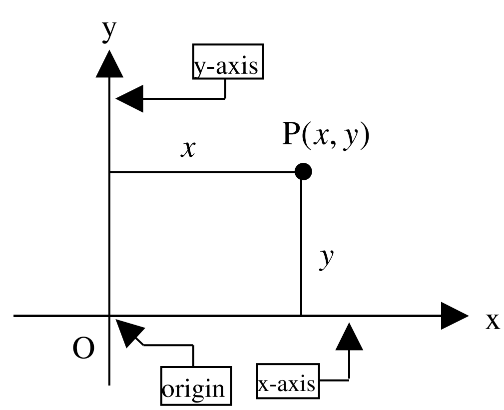
In order to further our study of vectors it will be necessary to consider vectors as algebraic entities by introducing a coordinate system for the vectors. A coordinate system is a frame of reference that is used as a standard for measuring distance and direction. If we are working with vectors in two-dimensional space we will use a two-dimensional rectangular Cartesian coordinate system. If we are working with vectors in three-dimensional space, the coordinate system that we use is a three-dimensional rectangular Cartesian coordinate system. To understand these two and three-dimensional rectangular coordinate systems we first introduce a one-dimensional coordinate system also known as a real number line.

Let **R** denote the set of all real numbers. Let *l* be a given line. We can set up a one-to-one relationship between the real numbers **R** and the points on *l* as follows. Select a point O, which will be called the **origin**, on the line *l*. To this point we associate the number 0. Select a unit of length and use it to mark off equidistantly placed points on either side of O. The points on one side of O, called the positive side, are assigned the numbers 1, 2, 3 etc. while the points on the other side of O, called the negative side are assigned the numbers –1, –2, –3 etc. A one-to-one correspondence now exists between all the real numbers **R** and the points on *l*. The resulting line is called a **real number line** or more simply a **number line** and the number associated with any given point on the line is called its **coordinate.** We have just constructed a one-dimensional coordinate system.



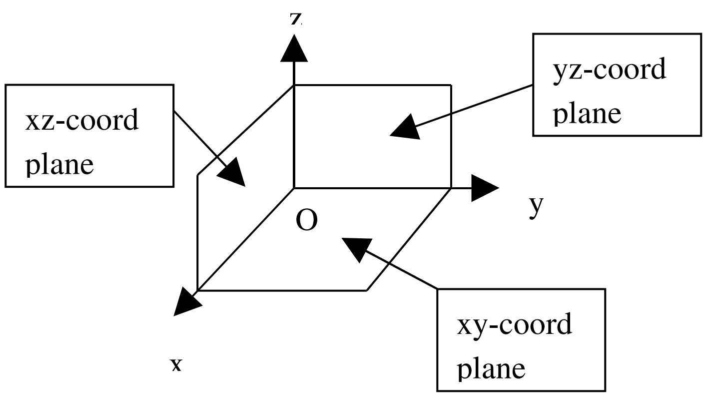
Two-dimensional rectangular Cartesian coordinate system

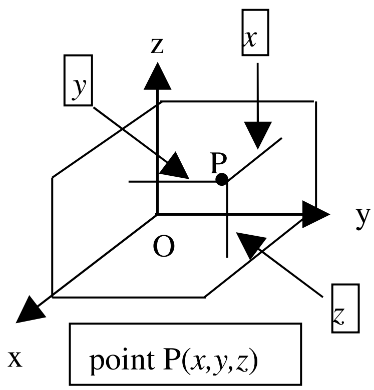
The two-dimensional Cartesian coordinate system has as its frame of reference two number lines that intersect at right angles. The horizontal number line is called the **x-axis** and the vertical number line is the **y-axis**. The point of intersection of the two axes is called the **origin** and is denoted by O. To each point P in two-dimensional space we associate an **ordered pair** of real numbers (*x, y*) called the coordinates of the point. The number *x* is called the **x-coordinate** of the point and the number *y* is the **y-coordinate** of the point. The x-coordinate *x* is the horizontal distance of the point P from the y-axis while the y-coordinate *y* is the vertical distance of the point P from the x-axis. The set of all ordered pairs of real numbers is denoted $\mathbf{R}^2$.



Three-dimensional rectangular Cartesian coordinate system

The three-dimensional Cartesian coordinate system has as its frame of reference three number lines that intersect at right angles at a point O called the origin. The number lines are called the **x-axis**, the **y-axis** and the **z-axis**. To each point P in three-dimensional space we associate an **ordered triple** of real numbers (x, y, z) called the **coordinates** of the point. The number x is the distance of the point P from the **yz-coordinate plane**. The number y is the distance of the point P from the **xz-coordinate plane**. The number z is the distance of the point P from the **xy-coordinate plane**. The set of all ordered triples of real numbers is denoted by $\mathbf{R}^3$. When the coordinate axes are labeled as shown in the

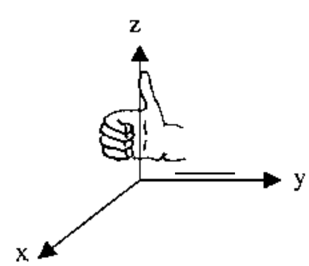




following diagrams, the coordinate system is said to be a right-handed Cartesian coordinate system.

Right-handed Cartesian coordinate system

A **right-handed Cartesian coordinate system** is one in which the coordinate axes are so labeled that if we curl the fingers on our right hand so as to point from the positive x-axis towards the positive y-axis, the thumb will point in the direction of the positive z-axis. [If the thumb is pointing in the direction opposite to the direction of the positive z-axis, the coordinate system is a left-handed coordinate system.]



1.4 Problems

1. Draw a right-handed three-dimensional Cartesian coordinate system, and plot the following points with the given coordinates.

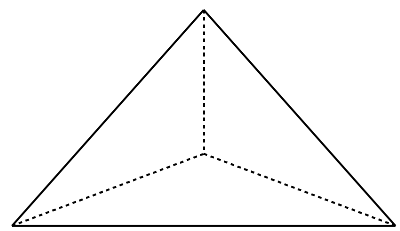
   a. P (2, 1, 3) b. Q (3, 4, 5) c. R (2, 1, -2) d. S (0, -2, -1)

2. A cube has one vertex at the origin, and the diagonally opposite vertex is the point with coordinates (1, 1, 1). Find the coordinates of the other vertices of the cube.

3. A rectangular parallelepiped (box) has one vertex at the origin and the diagonally opposite vertex at the point (2, 3, 1). Find the coordinates of the other vertices.

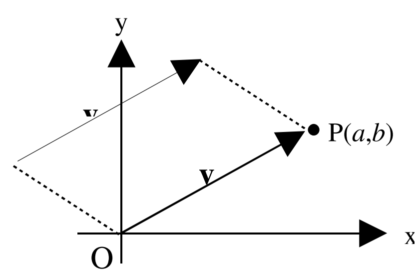
4. A pyramid has a square base located on the xy-coordinate plane. Diagonally opposite vertices of the square base are located at the points with coordinates (0, 0, 0) and (2, 2, 0). The height of the pyramid is 2 units. Find the coordinates of the other vertices of the pyramid. [Assume that the top of the pyramid lies directly above the centre of the square base.]

5. A regular tetrahedron is a solid figure with 4 faces, each of which is an equilateral triangle. If a regular tetrahedron has one face lying on the xy-coordinate plane with vertices at (0, 0, 0) and (0, 1, 0), find the coordinates of the other two vertices if all coordinates are nonnegative



tetetrahedron

1.5 DEFINING VECTORS ALGEBRAICALLY



Since a vector is determined solely by its magnitude and direction, any given vector may be relocated with respect to a given coordinate system so that its initial point is at the origin O. Such a vector is said to be in **standard position**. When a given vector **v** is in standard position there exists a unique terminal point P such that $\mathbf{v} = \overrightarrow{OP}$. This one-to-one relationship between the vector **v** and the terminal point P enables us to give an algebraic definition for the vector **v**. If **v** is a vector in two-dimensional space and P(*a, b*) is the unique point P such that $\mathbf{v} = \overrightarrow{OP}$, then we will identify the vector **v** with the ordered pair of real numbers (a,b) and write $\mathbf{v} = (a, b)$. Similarly if **v** is a vector in three-dimensional space and P(*a, b, c*) is the unique point P such that $\mathbf{v} = \overrightarrow{OP}$, then we will identify **v** with the ordered triple of real numbers (a,b,c) and write **v** = (*a, b, c*). The two-dimensional vector **v** = (*a, b*) is said to have **components** *a* and *b* and the three-dimensional vector $\mathbf{v} = (a,b,c)$ is said to have **components** *a*, *b* and *c*.

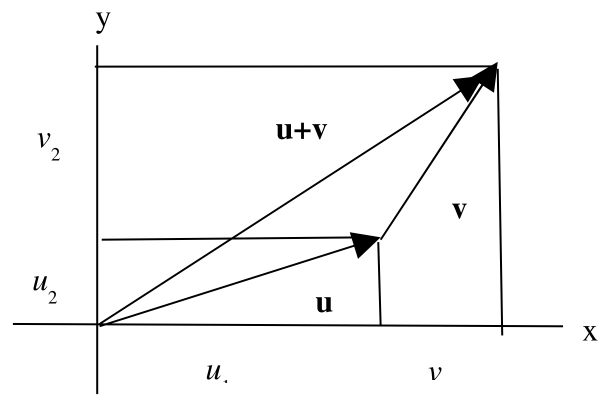
To avoid confusion, when dealing with the components of several vectors at the same time it is customary to denote the components of a given vector by subscripted letters that agree with the letter used to designate the vector. Thus we will write **v** = (v_1, v_2) if **v** is a vector in $\mathbf{R}^2$ and **v** = (v_1, v_2, v_3) if **v** is a vector in $\mathbf{R}^3$.

Equal vectors

If equal vectors **u** and **v** are located so that their initial points are at the origin, then their terminal points will coincide, and hence the corresponding components of **u** and **v** must be equal to each other. Thus **u** = **v** in $\mathbf{R}^2$ if and only if $u_1 = v_1$ and $u_2 = v_2$ while for vectors in $\mathbf{R}^3$, **u** = **v** if and only if $u_1 = v_1$, $u_2 = v_2$ and $u_3 = v_3$.

Sum of two vectors

Let **u** = (u_1, u_2) and **v** = (v_1, v_2) be two vectors in $\mathbf{R}^2$. If the vectors are located so that their initial points are at the origin, then their terminal points are the points with coordinates (u_1, u_2) and (v_1, v_2). If **v** is now placed so that its initial point is at (u_1, u_2), which is the terminal point of **u**, then the terminal point of **v** is the point with coordinates $(u_1 + v_1, u_2 + v_2)$.
Hence **u** + **v** = $(u_1 + v_1, u_2 + v_2)$.



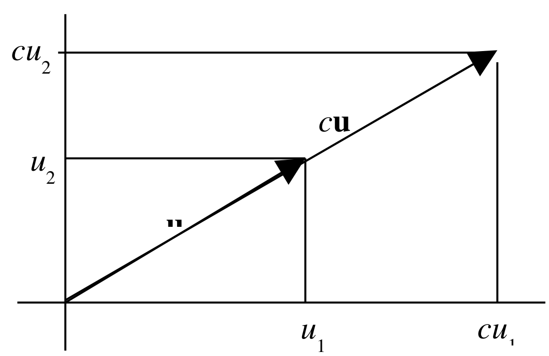
A similar argument for the vectors **u** = (u_1, u_2, u_3) and **v** = (v_1, v_2, v_3) in $\mathbf{R}^3$ gives
u + v = $(u_1 + v_1, u_2 + v_2, u_3 + v_3)$.

Example

Let **u** = (1, 2, 3) and **v** = (4, 1, 5). Then **u + v** = (1 + 4, 2 + 1, 3 + 5) = (5, 3, 8).

Multiplying a vector by a scalar

If **u** = (u_1, u_2) is a vector in $\mathbf{R}^2$ that has its initial point at the origin, then the terminal point of **u** is the point with coordinates (u_1, u_2). If $c>0$, then the vector c**u** has the same direction as **u** and is c times as long as **u** so its terminal point is the point with coordinates (cu_1, cu_2). A similar argument applies if $c<0$, except in this case the direction is reversed. In either case we have c**u** = (cu_1, cu_2).



If instead **u** is a vector in $\mathbf{R}^3$, then a similar argument will show that c**u** = (cu_1, cu_2, cu_3).

Example

If $\mathbf{u} = (3, 1, 2)$, then $5\mathbf{u} = (5\times3, 5\times1, 5\times2) = (15, 5, 10)$.

Difference of two vectors

The vector $\mathbf{u} - \mathbf{v}$ is defined to be equal to the vector sum $\mathbf{u} + (-1)\mathbf{v}$.

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are two vectors in $\mathbf{R}^2$, then

$\mathbf{u} - \mathbf{v} = (u_1, u_2) + (-1)(v_1, v_2) = (u_1, u_2) + (-v_1, -v_2) = (u_1 - v_1, u_2 - v_2)$.

Similarly, in $\mathbf{R}^3$ we have $\mathbf{u} - \mathbf{v} = (u_1 - v_1,\ u_2 - v_2,\ u_3 - v_3)$.

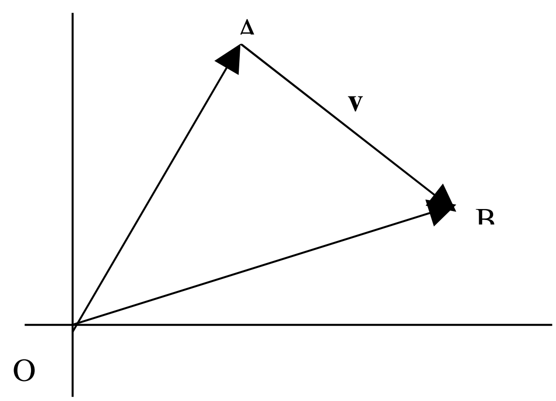
Example

If $\mathbf{u} = (4,5,2)$ and $\mathbf{v} = (2,-1,3)$ then $\mathbf{u} - \mathbf{v} = \left(4-2,\ 5-(-1),\ 2-3\right) = (2,6,-1)$.

Vector representation of a directed line segment

Let $\mathbf{v} = \overrightarrow{AB}$ where A is the point with coordinates (a_1, a_2) and B is the point with coordinates (b_1, b_2). Then

$$\mathbf{v} = \overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$$
$$= -\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OB} - \overrightarrow{OA}$$
$$= (b_1, b_2) - (a_1, a_2) = (b_1 - a_1, b_2 - a_2).$$



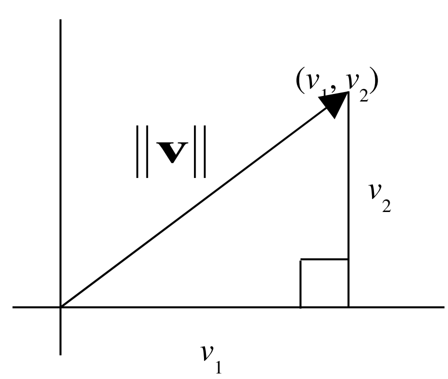
In $\mathbf{R}^3$, if $A = (a_1, a_2, a_3)$ and $B = (b_1, b_2, b_3)$ then $\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, b_3 - a_3)$.

Example

If $A = (1, 2, 3)$ and $B = (4, 6, 9)$, then $\overrightarrow{AB} = (4-1, 6-2, 9-3) = (3, 4, 6)$

Length of a vector

If $\mathbf{v} = (v_1, v_2)$ then the length of $\mathbf{v}$ is equal to the length of the directed line segment from the origin $(0, 0)$ to the point (v_1, v_2). We will use the symbol



$\|\mathbf{v}\|$ to represent the length of the vector **v.** Using Pythagoras' theorem for right triangles we can calculate that length to be $\sqrt{v_1^{\,2} + v_2^{\,2}}$ and so we have the formula $\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2}}$. A similar argument for a vector $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbf{R}^3$, using Pythagoras' theorem twice, gives $\|\mathbf{v}\| = \sqrt{v_1^{\,2} + v_2^{\,2} + v_3^{\,2}}$.

Theorem If c is a scalar and **v** is a vector in $\mathbf{R}^2$ or $\mathbf{R}^3$, then $\|c\,\mathbf{v}\| = |c|\;\|\mathbf{v}\|$.

Proof The following proof is for **v** in $\mathbf{R}^2$. The proof for **v** in $\mathbf{R}^3$ is similar.

$$\|c\,\mathbf{v}\| = \|(cv_1, cv_2)\| = \sqrt{(cv_1)^2 + (cv_2)^2} = \sqrt{c^2(v_1 \;+ v_2^{\,2})} = \sqrt{c^2}\sqrt{v_1^{\,2} + v_2^{\,2}} = |c|\;\|\mathbf{v}\|.$$

Unit vector

If $\|\mathbf{v}\| = 1$ we say **v** is a **unit vector**. Because the length of a vector is a positive quantity, the length of the vector c**v** is $|c|\;\|\mathbf{v}\|$. To find a unit vector in the direction of a given vector **v,** multiply the vector **v** by the scalar $\frac{1}{\|\mathbf{v}\|}$. The resulting vector $\frac{1}{\|\mathbf{v}\|}\mathbf{v}$ is a unit vector in the direction of **v.** A unit vector in the direction opposite to **v** is $-\frac{1}{\|\mathbf{v}\|}\mathbf{v}$.

Example

If **v** = (2, 2, 1), then the length of **v** is $\|\mathbf{v}\| = \sqrt{2^2 + 2^2 + 1^2} = \sqrt{4+4+1} = \sqrt{9} = 3$ and a unit vector in the direction of **v** is $\frac{1}{\|\mathbf{v}\|}\mathbf{v} = \frac{1}{3}(2,2,1) = \left(\frac{2}{3},\frac{2}{3},\frac{1}{3}\right)$. A unit vector in the direction opposite to that of **v** is $\left(-\frac{2}{3},-\frac{2}{3},-\frac{1}{3}\right)$.

1.5 Problems

Let $\mathbf{u} = (2, 1, 3)$, $\mathbf{v} = (3, 1, -2)$ and $\mathbf{w} = (4, -1, 1)$.

1. Find the following vectors.
   a. $\mathbf{u} + \mathbf{v}$ b. $\mathbf{u} - \mathbf{v}$ c. $2\mathbf{w}$ d. $2\mathbf{u} - 3\mathbf{v}$ e. $\mathbf{u} + 2\mathbf{v} - 3\mathbf{w}$ f. $2\mathbf{u} + 3\mathbf{v} - \mathbf{w}$

2. Find the following lengths.
   a. $\|\mathbf{u}\|$ b. $\|\mathbf{v}\|$ c. $\|2\mathbf{w}\|$ d. $\|\mathbf{u}+\mathbf{v}\|$ e. $\|\mathbf{u}-\mathbf{v}\|$ f. $\|\mathbf{v}-\mathbf{w}\|$

3. Find components of the vector equal to the directed line segment $\overrightarrow{PQ}$.
   a. P = (1, 2, 3) Q = (2, 4, 7) b. P = (3, 1, 4) Q = (5, 7, 1)
   c. P = (–2, 5, 1) Q = (4, –3, 2) d. P = (0, 3, 2) Q = (2, 0, 5)

4. Let $\mathbf{v} = \overrightarrow{AB}$. If $\mathbf{v}$ and A are as given below, find the coordinates of B.
   a. $\mathbf{v} = (3, 5, 4)$ A = (1, 3, 2) b. $\mathbf{v} = (2, 5, 4)$ A = (1, –2, 2)

5. Let $\mathbf{v} = \overrightarrow{AB}$. If $\mathbf{v}$ and B are as given below, find the coordinates of A.
   a. $\mathbf{v} = (3, 5, 4)$ B = (2, 5, 6) b. $\mathbf{v} = (2, 5, 4)$ B = (4, 1, 7).

6. Let $\mathbf{v}$ be the given vector. Find a unit vector in the direction of $\mathbf{v}$ and find a unit vector in the direction opposite to that of $\mathbf{v}$.
   a. $\mathbf{v} = (2, 2, 1)$ b. $\mathbf{v} = (3, 0, 4)$ c. $\mathbf{v} = (1, 2, 3)$ d. $\mathbf{v} = (-2, 3, -4)$.

7. If $\mathbf{v} = (3a, 4a, 5a)$ and $\|\mathbf{v}\| = 10$, find the value of a.

1.6 THE DOT PRODUCT (SCALAR PRODUCT)

The dot product is a method for multiplying two vectors. Because the product of the multiplication is a scalar, the dot product is sometimes referred to as the scalar product. The dot product will be used to find an angle between two vectors and will have applications in finding distances between points and lines, points and planes, etc.

If $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ are two vectors in $\mathbf{R}^2$, we define their **dot product**, denoted $\mathbf{u} \bullet \mathbf{v}$, as follows: $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2$.

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors in $\mathbf{R}^3$, we define their dot product to be $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3$.

Example

Let $\mathbf{u} = (1, 2, 3)$ and $\mathbf{v} = (4, 5, 6)$.
Then $\mathbf{u} \bullet \mathbf{v} = (1)(4) + (2)(5) + (3)(6) = 4 + 10 + 18 = 32$.
The following theorem relates the length of a vector to the dot product of the vector with itself.

Theorem For any vector $\mathbf{u}$ in $\mathbf{R}^2$ or in $\mathbf{R}^3$, $\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}}$.

Proof The following proof is for $\mathbf{R}^2$. The proof for $\mathbf{R}^3$ is similar.
Let $\mathbf{u} = (u_1, u_2)$. Then $\mathbf{u} \bullet \mathbf{u} = (u_1, u_2) \bullet (u_1, u_2) = u_1^2 + u_2^2 = \|\mathbf{u}\|^2$.
Taking square roots gives $\|\mathbf{u}\| = \sqrt{\mathbf{u} \bullet \mathbf{u}}$.

The next theorem lists some algebraic properties of the dot product.

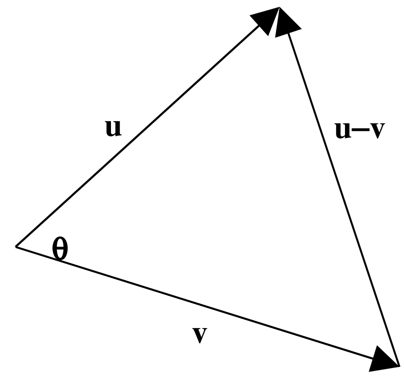
Theorem Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$, and let c be a scalar. Then
(a) $\mathbf{u} \bullet \mathbf{v} = \mathbf{v} \bullet \mathbf{u}$
(b) $c(\mathbf{u} \bullet \mathbf{v}) = (c\mathbf{u}) \bullet \mathbf{v} = \mathbf{u} \bullet (c\mathbf{v})$
(c) $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w}) = \mathbf{u} \bullet \mathbf{v} + \mathbf{u} \bullet \mathbf{w}$
(d) $\mathbf{u} \bullet \mathbf{0} = 0$.

Proof (a) Let $\mathbf{u} = (u_1, u_2)$ and $\mathbf{v} = (v_1, v_2)$ be any two vectors in $\mathbf{R}^2$.
Then $\mathbf{u} \bullet \mathbf{v} = u_1v_1 + u_2v_2 = v_1u_1 + v_2u_2 = \mathbf{v} \bullet \mathbf{u}$. The proof for $\mathbf{R}^3$ is similar
The proofs for parts (b), (c) and (d) are similar straightforward computations.

The following theorem shows how the dot product of two vectors $\mathbf{u}$ and $\mathbf{v}$ is related to the angle between the vectors.

Theorem Let **u** and **v** be two vectors in $\mathbf{R}^2$ or $\mathbf{R}^3$. Let θ be the angle between **u** and **v**. Then $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\| \, \|\mathbf{v}\| \cos\theta$.

Proof Let **u** and **v** to be a pair of adjacent sides of a triangle whose third side is **u – v.** Using the cosine law for triangles we get



$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - \mathbf{2}\,\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$
$$(\mathbf{u} - \mathbf{v}) \bullet (\mathbf{u} - \mathbf{v}) = \mathbf{u} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} - \mathbf{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$
$$\mathbf{u} \bullet \mathbf{u} - \mathbf{u} \bullet \mathbf{v} - \mathbf{v} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} = \mathbf{u} \bullet \mathbf{u} + \mathbf{v} \bullet \mathbf{v} - \mathbf{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$
$$-\mathbf{2}\,\mathbf{u} \bullet \mathbf{v} = -\mathbf{2}\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$
$$\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Angle between two vectors

The preceding theorem provides a method for finding the cosine of the angle between two vectors and hence finding the angle between the two vectors. Solving $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$ for $\cos\theta$ gives the formula $\cos\theta = \dfrac{\mathbf{u} \bullet \mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}$.

Example

Find the cosine of the angle between the vectors **u** = (3, 1, 2) and **v** = (1, 4, 3).

Solution

$$\cos\theta = \frac{(3,1,2) \bullet (1,4,3)}{\|(3,1,2)\|\,\|(1,4,3)\|} = \frac{3+4+6}{\sqrt{9+1+4}\sqrt{1+16+9}} = \frac{13}{\sqrt{14}\sqrt{26}} = \frac{13}{\sqrt{2}\sqrt{7}\sqrt{2}\sqrt{13}} = \frac{\sqrt{13}}{2\sqrt{7}}$$

Having found the cosine of the angle θ, we can find the angle $\theta = \cos^{-1}\left(\dfrac{\sqrt{13}}{2\sqrt{7}}\right) = 47°$.

Orthogonal vectors

Vectors **u** and **v** are said to be **orthogonal** or perpendicular to each other if they meet at right angles. If **u** and **v** are orthogonal, then $\mathbf{u} \bullet \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos(\pi/2) = 0$. [Since $\cos(\pi/2)$ = 0.] Conversely, if **u • v** = 0 we must have either **u** = **0** or **v** = **0** or $\mathbf{u} \perp \mathbf{v}$. Since the zero vector **0** can have any direction, we will agree that **0** is orthogonal to any vector. Hence we say that **u** and **v** are orthogonal if and only if **u • v** = 0.

Example

Show that the vectors **u** = (1, 2, 2) and **v** = (2, 1, –2) are orthogonal vectors.

Solution

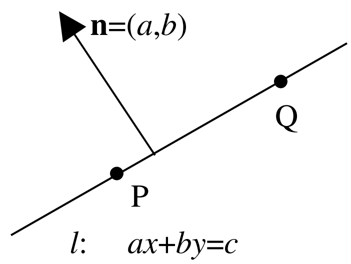
$\mathbf{u} \bullet \mathbf{v} = (1, 2, 2) \bullet (2, 1, -2) = 2 + 2 - 4 = 0$. Hence $\mathbf{u} \perp \mathbf{v}$.

Normal vector

If l is a line in $\mathbf{R}^2$ or in $\mathbf{R}^3$ and **n** is a vector that is orthogonal to the line l, we call **n** a **normal vector** to the line l.

Theorem Let $ax + by = c$ be the equation of a line l in $\mathbf{R}^2$. Then the vector $\mathbf{n} = (a, b)$ is a normal vector to the line l.

Proof First select two points P and Q on l. Select $P = (c/a, 0)$ and $Q = (0, c/b)$, then the vector $\overrightarrow{PQ}$ lies on l.
But $\overrightarrow{PQ} = (0, c/b) - (c/a, 0) = (-c/a, c/b)$. To show that $\mathbf{n} \perp \overrightarrow{PQ}$ we take the dot product.
$\mathbf{n} \bullet \overrightarrow{PQ} = (a, b) \bullet (-c/a, c/b) = -c + c = 0$. This proves that the vector **n** is a normal vector to the line l.



Example

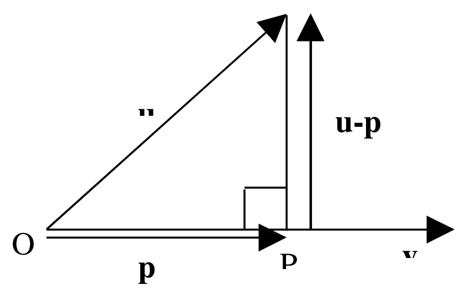
Find a vector that is normal to the line $2x + 3y = 5$.

Solution

From the previous theorem the vector $\mathbf{n} = (2, 3)$ is normal to the given line $2x + 3y = 5$ since the coefficients of x and y are 2 and 3.

Projections

Let **u** and **v** be two given vectors with $\mathbf{v} \neq \mathbf{0}$. The **projection of u along v**, denoted $\text{proj}_{\mathbf{v}}\mathbf{u}$ is the vector **p** found as follows. Drop a perpendicular from the terminal point of **u** that intersects the line through **v** at the point P. Then $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{p} = \overrightarrow{OP}$.



We find **p** as follows. Since **p** lies along **v**, there is a scalar k such that $\mathbf{p} = k\mathbf{v}$. Now $\mathbf{u}-\mathbf{p}$ is orthogonal to **v** so $(\mathbf{u}-\mathbf{p}) \bullet \mathbf{v} = 0$. But

$(\mathbf{u}-\mathbf{p})\bullet\mathbf{v} = 0 \Leftrightarrow \mathbf{u}\bullet\mathbf{v} - \mathbf{p}\bullet\mathbf{v} = 0 \Leftrightarrow \mathbf{u}\bullet\mathbf{v} - k\mathbf{v}\bullet\mathbf{v} = 0 \Leftrightarrow k = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}} = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}$.

Hence $\text{proj}_{\mathbf{v}}\mathbf{u} = \mathbf{p} = k\mathbf{v} = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v} = \dfrac{\mathbf{u}\bullet\mathbf{v}}{\|\mathbf{v}\|^2}\mathbf{v}$.

Example

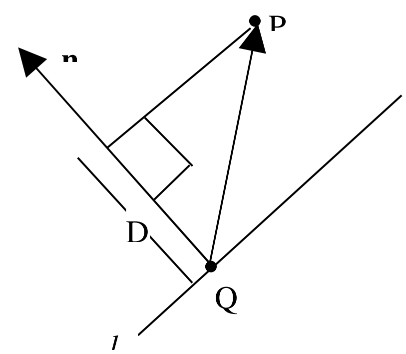
Let $\mathbf{u} = (8, 1, 4)$ and let $\mathbf{v} = (1, 2, 2)$. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$.

Solution

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \frac{\mathbf{u}\bullet\mathbf{v}}{\mathbf{v}\bullet\mathbf{v}}\mathbf{v} = \frac{(8, 1, 4)\bullet(1, 2, 2)}{(1, 2, 2)\bullet(1, 2, 2)}(1, 2, 2) = \frac{8+2+8}{1+4+4}(1, 2, 2) = 2(1, 2, 2) = (2, 4, 4)$$

Distance between a point and a line in $\mathbf{R}^2$

To find the distance D between a point P and a line l in $\mathbf{R}^2$, we select a point Q on the line l, then the distance D is the length of the projection of $\overrightarrow{QP}$ on $\mathbf{n}$, a normal vector to the line l.



$$D = \left\|\text{proj}_{\mathbf{n}}\overrightarrow{QP}\right\| = \left\|\frac{\overrightarrow{QP}\bullet\mathbf{n}}{\mathbf{n}\bullet\mathbf{n}}\mathbf{n}\right\|$$

$$= \frac{\left|\overrightarrow{QP}\bullet\mathbf{n}\right|}{\|\mathbf{n}\|\,\|\mathbf{n}\|}\|\mathbf{n}\| = \frac{\left|\overrightarrow{QP}\bullet\mathbf{n}\right|}{\|\mathbf{n}\|} = \frac{\left|\overrightarrow{PQ}\bullet\mathbf{n}\right|}{\|\mathbf{n}\|}$$

Note that $\left|\overrightarrow{QP}\bullet\mathbf{n}\right| = \left|\overrightarrow{PQ}\bullet\mathbf{n}\right|$ and so either of the last two forms for the distance D can be used interchangeably.

Example

Find the distance between the point P = (9, 1) and the line 3x + 4y = 6.

Solution

The point Q = (2, 0) lies on the line 3x + 4y = 6 so $\overrightarrow{QP} = (9, 1) - (2, 0) = (7, 1)$.

Since $\mathbf{n} = (3, 4)$, the distance is $D = \dfrac{\left|\overrightarrow{QP}\bullet\mathbf{n}\right|}{\|\mathbf{n}\|} = \dfrac{|(7, 1)\bullet(3, 4)|}{\|(3, 4)\|} = \dfrac{|21+4|}{\sqrt{9+16}} = \dfrac{25}{5} = 5$

1.6 Problems

In problems 1 to 3 below, let $\mathbf{u} = (1, 2, 1)$, $\mathbf{v} = (3, 2, 4)$ and $\mathbf{w} = (1, -1, 3)$.

1. Calculate the following dot products.
   a. $\mathbf{u} \bullet \mathbf{v}$ b. $\mathbf{u} \bullet \mathbf{w}$ c. $\mathbf{v} \bullet \mathbf{w}$ d. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w})$ e. $\mathbf{u} \bullet (2\mathbf{v} + 3\mathbf{w})$

2. Find the length of each of each of the following vectors.
   a. $\mathbf{u}$ b. $\mathbf{v}$ c. $\mathbf{w}$ d. $\mathbf{u} + \mathbf{v}$ e. $2\mathbf{u} - 3\mathbf{v}$

3. Find the cosine of the angle between the following pairs of vectors.
   a. $\mathbf{u}$ and $\mathbf{v}$ b. $\mathbf{u}$ and $\mathbf{w}$ c. $\mathbf{v}$ and $\mathbf{w}$ d. $\mathbf{u} + \mathbf{v}$ and $\mathbf{u} - \mathbf{v}$

4. Show that the following pairs of vectors are orthogonal.
   a. $(2, 1, 3)$ and $(1, 1, -1)$ b. $(1, 3, 5)$ and $(2, 1, -1)$
   c. $(4, 5, 1)$ and $(2, -1, -3)$ d. $(1, 0, 1)$ and $(0, 1, 0)$

5. Find a vector $\mathbf{n}$ which is normal to the given line in $\mathbf{R}^2$.
   a. $2x + 3y = 5$ b. $x - 2y = 3$ c. $3x + y = 4$ d. $x + 3y = 1$

6. Find $\text{proj}_{\mathbf{v}}\mathbf{u}$ for each of the following pairs of vectors $\mathbf{u}$ and $\mathbf{v}$.
   a. $\mathbf{u} = (1, 2, 1)$ and $\mathbf{v} = (3, 1, 0)$ b. $\mathbf{u} = (3, 1, 4)$ and $\mathbf{v} = (1, 2, 2)$
   c. $\mathbf{u} = (5, 4, 3)$ and $\mathbf{v} = (3, 1, 1)$ d. $\mathbf{u} = (1, 1, 2)$ and $\mathbf{v} = (3, 4, 1)$

7. Find the distance between the point P and the line l in $\mathbf{R}^2$.
   a. $P = (2, 3)$ l: $3x + 4y = 1$ b. $P = (5, 1)$ l: $3x - 4y = 2$
   c. $P = (5, 3)$ l: $5x + 12y = 1$ d. $P = (3, 4)$ l: $x + 2y = 3$

8. Prove Pythagoras' theorem: The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.

9. Prove that the angle inscribed in a semi circle is a right angle.

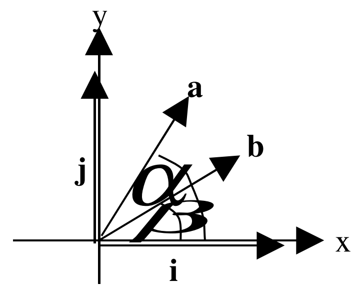
10. Prove that the sum of the squares of the diagonals of a parallelogram equals the sum of the squares of its sides.

11. Prove that the diagonals of a rhombus (parallelogram with equal sides) are perpendicular.

12. Prove that the mid point of the hypotenuse of a right triangle is equidistant from the three vertices of the triangle

13. Prove that the altitudes of a triangle are concurrent.

14. Let **a** and **b** be unit vectors in the xy-plane making angles α and β respectively with the x-axis. Let **i** and **j** be the vectors (1, 0) and (0, 1) respectively.
    a. Show that $\mathbf{i} \bullet \mathbf{i} = 1$, $\mathbf{i} \bullet \mathbf{j} = 0$ and $\mathbf{j} \bullet \mathbf{j} = 1$.
    b. Show that $\mathbf{a} = \cos\alpha\,\mathbf{i} + \sin\alpha\,\mathbf{j}$ and $\mathrm{b} = \cos\beta\,\mathbf{i} + \sin\beta\,\mathbf{j}$
    c. Prove that $\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$.



1.7 THE CROSS PRODUCT (VECTOR PRODUCT)

In the previous section we were introduced to the dot product of two vectors. The result of taking the dot product of two vectors is a scalar quantity. We now introduce a second method of multiplying two vectors from $\mathbf{R}^3$ that results in a vector quantity. The symbol used to denote this product is a cross $\times$, hence the name "cross product". Because the result is a vector, the term "vector product" is sometimes used for this product.

The cross product has a number of applications. We will use the cross product to find the areas of triangles and parallelograms. It will also be used to calculate the volume of a parallelepiped and later to find the distance between a point and a line in $\mathbf{R}^3$.

Cross product (vector product)

If $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ are two vectors in $\mathbf{R}^3$, the **cross product $\mathbf{u}\times\mathbf{v}$** is the vector in $\mathbf{R}^3$ defined as follows.

$$\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1).$$

Example

Let $\mathbf{u} = (3, 1, 2)$ and let $\mathbf{v} = (4, 6. 5)$.

Then $\mathbf{u}\times\mathbf{v} = (1\times5 - 2\times6,\ 2\times4 - 3\times5,\ 3\times6 - 1\times4) = (-7,\ -7,\ 14)$.

Although the definition of the cross product as given above may be difficult to remember, the concept of a 2×2 determinant can be used to simplify the process.

Consider the 2×2 array of numbers $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. The **determinant** of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, written, det $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ or $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$, is defined to be the number $ad - bc$. Then the cross product of $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$, using determinants, can be written as the vector

$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, \ -\begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \ \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right).$$

We remember the components of $\mathbf{u}\times\mathbf{v}$ as follows.

1) Form the 2×3 rectangular array $\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix}$ where the first row consists of the components of the vector **u** and the second row consists of the components of vector **v**.

2) To find the first component of $\mathbf{u}\times\mathbf{v}$, delete the first column and take the determinant of the remaining 2×2 array; to find the second component of $\mathbf{u}\times\mathbf{v}$, delete the second column and take the negative of the determinant of the remaining 2×2 array; to find the third component of $\mathbf{u}\times\mathbf{v}$, delete the third column and take the determinant of the remaining 2×2 array.

Example

Find $\mathbf{u}\times\mathbf{v}$ if $\mathbf{u} = (2, 3, 4)$ and $\mathbf{v} = (5, 6, 7)$.

Solution

Construct the rectangular array $\begin{bmatrix} 2 & 3 & 4 \\ 5 & 6 & 7 \end{bmatrix}$. Then

$$\begin{aligned} \mathbf{u}\times\mathbf{v} &= \left(\begin{vmatrix} 3 & 4 \\ 6 & 7 \end{vmatrix}, -\begin{vmatrix} 2 & 4 \\ 5 & 7 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix} \right) \\ &= (3\times7-4\times6, \ -(2\times7-4\times5), \ 2\times6-3\times5) \\ &= (21-24, \ -(14-20), \ 12-15) \\ &= (-3, \ 6, \ -3) \end{aligned}$$

Theorem $\mathbf{u}\times\mathbf{v} = -\,\mathbf{v}\times\mathbf{u}$

Proof $\mathbf{u}\times\mathbf{v} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)$

$= -\,(u_3v_2 - u_2v_3,\ u_1v_3 - u_3v_1,\ u_2v_1 - u_1v_2)$

$= -\,(v_2u_3 - v_3u_2,\ v_3u_1 - v_1u_3,\ v_1u_2 - v_2u_1)$

$= -\,\mathbf{v}\times\mathbf{u}$

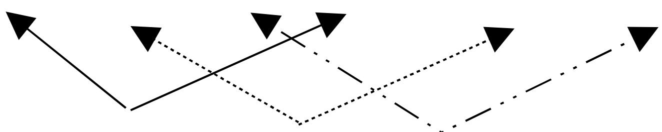
Theorem $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**.

Proof We show that $\mathbf{u}\times\mathbf{v}$ is orthogonal to **u** by showing that the dot product of $\mathbf{u}\times\mathbf{v}$ and **u** is equal to zero. The proof that $\mathbf{u}\times\mathbf{v}$ is orthogonal to **v** is similar.

$(\mathbf{u}\times\mathbf{v})\bullet\ \mathbf{u} = (u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1)\bullet\ (u_1,\ u_2,\ u_3)$

$= (u_2v_3 - u_3v_2)\,u_1 + (u_3v_1 - u_1v_3)\,u_2 + (u_1v_2 - u_2v_1)\,u_3$

$= u_2v_3\,u_1 - u_3v_2\,u_1 + u_3v_1\,u_2 - u_1v_3\,u_2 + u_1v_2\,u_3 - u_2v_1\,u_3$



arrows indicate canceling pairs

$= 0$

Since $(\mathbf{u}\times\mathbf{v})\bullet\ \mathbf{u} = 0$, $\mathbf{u}\times\mathbf{v}$ and **u** are orthogonal.

Example

Find a vector orthogonal to both $\mathbf{u} = (1, 3, 2)$ and $\mathbf{v} = (4, 0, 1)$.

Solution

The vector $\mathbf{u}\times\mathbf{v}$ is orthogonal to both **u** and **v**, so we calculate $\mathbf{u}\times\mathbf{v}$.

$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix},\ -\begin{vmatrix} 1 & 2 \\ 4 & 1 \end{vmatrix},\ \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} \right) = (3-0,\ -(1-8),\ 0-12) = (3,\ 7,\ -12)$$

The next theorem is a useful result that can be applied to calculate the area of a triangle and the area of a parallelogram. It is also used to calculate the volume of a parallelepiped in $\mathbf{R}^3$ and to find the distance between a point and a line in $\mathbf{R}^3$.

Theorem $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\|\sin\theta$ where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$.

Proof The proof consists of 2 steps.

(1) We first show $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\ \|\mathbf{v}\|^2 - (\mathbf{u}\bullet\mathbf{v})^2$ by computing the left and right hand sides separately and showing that they are equal to each other.

$\|\mathbf{u}\times\mathbf{v}\|^2 = (u_2v_3 - u_3v_2)^2 + (u_3v_1 - u_1v_3)^2 + (u_1v_2 - u_2v_1)^2$ ……………………………...(i)

$\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u}\bullet\mathbf{v})^2 = (u_1{}^2 + u_2{}^2 + u_3{}^2)(v_1{}^2 + v_2{}^2 + v_3{}^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2$ ………..(ii)

A lengthy computation shows right hand sides of (i) and (ii) are equal and so we conclude $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\ \|\mathbf{v}\|^2 - (\mathbf{u}\bullet\mathbf{v})^2$.

(2) Starting with $\|\mathbf{u}\times\mathbf{v}\|^2 = \|\mathbf{u}\|^2\ \|\mathbf{v}\|^2 - (\mathbf{u}\bullet\mathbf{v})^2$ we expand the dot product on the right

$$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\|\mathbf{u}\|\ \|\mathbf{v}\|\cos\theta)^2$$
$$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2(1-\cos^2\theta)$$
$$= \|\mathbf{u}\|^2\|\mathbf{v}\|^2\sin^2\theta$$

Taking square roots gives the required result: $\|\mathbf{u}\times\mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\|\sin\theta$.

The next theorem lists several properties of the cross product. The properties are established by straightforward computations and so the proofs are omitted.

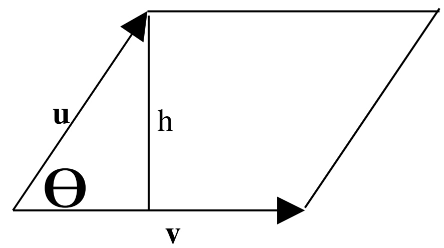
Theorem Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors in $\mathbf{R}^3$. Then $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ satisfy the following properties.

(a) $\mathbf{u}\times(\mathbf{v}+\mathbf{w}) = \mathbf{u}\times\mathbf{v} + \mathbf{u}\times\mathbf{w}$
(b) $(\mathbf{u}+\mathbf{v})\times\mathbf{w} = \mathbf{u}\times\mathbf{w} + \mathbf{v}\times\mathbf{w}$
(c) $\mathbf{u}\times\mathbf{0} = \mathbf{0}\times\mathbf{u} = \mathbf{0}$
(d) $\mathbf{u}\times\mathbf{u} = \mathbf{0}$

The area of a parallelogram

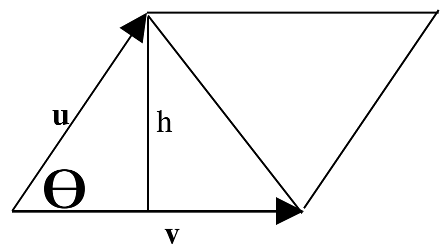
Let **u** and **v** be the adjacent sides of a parallelogram. The area of a parallelogram is length of base × height. From the adjoining diagram we have that the length of the base is $\|\mathbf{v}\|$ and the height is h. From trigonometry we get $\frac{h}{\|\mathbf{u}\|} = \sin\theta$ so $h = \|\mathbf{u}\|\sin\theta$. Therefore the area A is given by

$A = \text{base} \times \text{height} = \|\mathbf{v}\|\ \|\mathbf{u}\|\sin\theta = \|\mathbf{u}\times\mathbf{v}\|$



The area of a triangle

Let **u** and **v** be the adjacent sides of a triangle. Since the area of the triangle is one-half the area of the parallelogram with **u** and **v** as its adjacent sides, the area of the triangle is $A = \frac{1}{2}\|\mathbf{u}\times\mathbf{v}\|$.



Example

Find the area of the parallelogram having adjacent sides **u** = (2, 3, 1) and **v** = (4, 0 2).

Solution

$$\mathbf{u}\times\mathbf{v} = \left(\begin{vmatrix}3 & 1\\ 0 & 2\end{vmatrix},\ -\begin{vmatrix}2 & 1\\ 4 & 2\end{vmatrix},\ \begin{vmatrix}2 & 3\\ 4 & 0\end{vmatrix}\right) = (6,\ 0,\ -12)$$

$$\text{Area} = \|\mathbf{u}\times\mathbf{v}\| = \|(6, 0, -12)\| = \sqrt{36+0+144} = \sqrt{180} = \sqrt{36\times 5} = \sqrt{36}\sqrt{5} = 6\sqrt{5}$$

Example

Find the area of the triangle whose vertices are A = (1, 2, 2), B = (3, 4, 5) and C = (5, 6, 4)

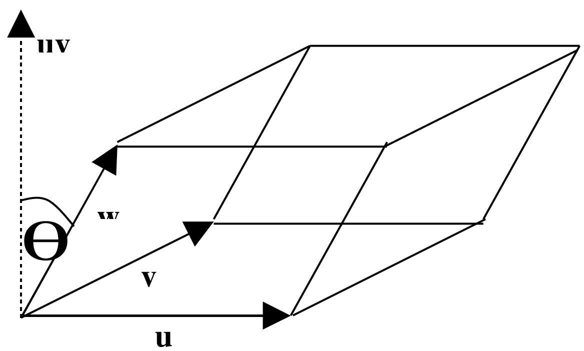
Solution

Let $\mathbf{u} = \overrightarrow{AB} = (3, 4, 5) - (1, 2, 2) = (2, 2, 3)$ and
let $\mathbf{v} = \overrightarrow{AC} = (5, 6, 4) - (1, 2, 2) = (4, 4, 2)$.

Then $\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}, -\begin{vmatrix} 2 & 3 \\ 4 & 2 \end{vmatrix}, \begin{vmatrix} 2 & 2 \\ 4 & 4 \end{vmatrix} \right) = (-8, 8, 0)$

Area of triangle ABC $= \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| = \frac{1}{2}\|(-8, 8, 0)\| = \frac{1}{2}\sqrt{64 + 64 + 0} = 4\sqrt{2}$

The volume of a parallelepiped



A **parallelepiped** is a solid (3-dimensional) figure having six faces with opposite pairs of faces being congruent parallelograms. A parallelepiped can be specified by giving 3 vectors **u**, **v** and **w** that form the 3 edges emanating from a common vertex. The volume of the parallelepiped is the area of the base × height. The area of the base is the area of the parallelogram with **u** and **v** as adjacent sides and is equal to $\|\mathbf{u} \times \mathbf{v}\|$. The height is the length of the projection of **w** onto **u**×**v** = $\|\text{proj}_{\mathbf{u}\times\mathbf{v}}\mathbf{w}\|$.

But $\|\text{proj}_{\mathbf{u}\times\mathbf{v}}\mathbf{w}\| = \left\| \frac{\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})}{(\mathbf{u} \times \mathbf{v}) \bullet (\mathbf{u} \times \mathbf{v})} (\mathbf{u} \times \mathbf{v}) \right\| = \frac{|\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|^2} \|\mathbf{u} \times \mathbf{v}\| = \frac{|\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|}$.

Thus the volume of the parallelepiped is $V = \|\mathbf{u} \times \mathbf{v}\| \frac{|\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})|}{\|\mathbf{u} \times \mathbf{v}\|} = |\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})|$.

Example

Find the volume of the parallelepiped having the following three vectors as edges.
u = (2, 3, 1), **v** = (3, 4, 3) and **w** = (4, 5, 6)

Solution

$\mathbf{u} \times \mathbf{v} = \left(\begin{vmatrix} 3 & 1 \\ 4 & 3 \end{vmatrix}, -\begin{vmatrix} 2 & 1 \\ 3 & 3 \end{vmatrix}, \begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} \right) = (5, -3, -1)$

$\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v}) = (4, 5, 6) \bullet (5, -3, -1) = 20 - 15 - 6 = -1$

Volume $= |\mathbf{w} \bullet (\mathbf{u} \times \mathbf{v})| = |-1| = 1$

1.7 Problems

For problems 1 to 5 let **u** = (4, 3, 2), **v** = (5, 1, 3) and **w** = (2, 1, 4).

1. Find a. $\mathbf{u}\times\mathbf{v}$ b. $\mathbf{u}\times\mathbf{w}$ c. $\mathbf{v}\times\mathbf{w}$ d. $\mathbf{u}\times(\mathbf{v}+\mathbf{w})$

2. Find a. $\mathbf{u}\times(\mathbf{v}\times\mathbf{w})$ b. $(\mathbf{u}\times\mathbf{v})\times\mathbf{w}$ c. $(\mathbf{u}\times\mathbf{v})\bullet\mathbf{v}$

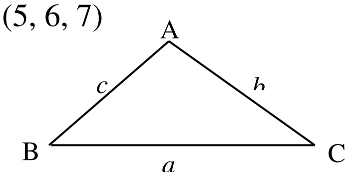
3. Find a vector orthogonal to a. **u** and **v** b. **u** and **w**

4. Find the area of the parallelogram whose adjacent sides are
   a. **u** and **v** b. **u** and **w** c. **v** and **w**

5. Find the area of the triangle whose adjacent sides are
   a. **u** and **v** b. **u** and **w** c. **v** and **w**

6. Find the area of the triangle whose vertices are given.
   a. (1, 2, 3), (2, 4, 5), (4, 5, 8) b. (2, 2, 1), (4, 3, 5), (5, 6, 7)

7. Prove the law of sines for triangles. $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.



8. Let **u** and **v** be two nonzero vectors in $\mathbf{R}^3$.
   a. Prove that if **u** and **v** are parallel vectors, then $\mathbf{u}\times\mathbf{v}=\mathbf{0}$.
   b. Prove that if $\mathbf{u}\times\mathbf{v}=\mathbf{0}$, then **u** and **v** are parallel vectors.

1.8 STANDARD BASIS VECTORS FOR $\mathbf{R}^3$

The following three unit vectors $\mathbf{i}=(1,0,0)$, $\mathbf{j}=(0,1,0)$ and $\mathbf{k}=(0,0,1)$ play a special role in $\mathbf{R}^3$ They are called the **standard basis vectors for $\mathbf{R}^3$.** Every vector in $\mathbf{R}^3$ can be written as a unique combination of these three vectors as follows. Let **v** = (*a*, *b*, *c*) be an arbitrary vector in $\mathbf{R}^3$**.** Then we can write

$$\mathbf{v}=(a,b,c)=(a,0,0)+(0,b,0)+(0,0,c)=a(1,0,0)+b(0,1,0)+c(0,0,1)=a\mathbf{i}+b\mathbf{j}+c\mathbf{k}.$$

Example

If $\mathbf{v}=(2,3,5)$, then $\mathbf{v}=2\mathbf{i}+3\mathbf{j}+5\mathbf{k}$.

For the dot product of the standard basis vectors with each other, we have the following results, which can be verified by a direct computation.

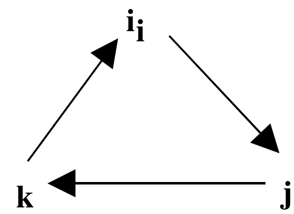
$\mathbf{i} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{k} = 1$ and $\mathbf{i} \bullet \mathbf{j} = \mathbf{j} \bullet \mathbf{i} = \mathbf{j} \bullet \mathbf{k} = \mathbf{k} \bullet \mathbf{j} = \mathbf{k} \bullet \mathbf{i} = \mathbf{i} \bullet \mathbf{k} = 0$.

For the cross product of the standard basis vectors with each other, we have the following results which can also be verified by a direct computation.

$\mathbf{i} \times \mathbf{i} = \mathbf{j} \times \mathbf{j} = \mathbf{k} \times \mathbf{k} = \mathbf{0}$ and

$\mathbf{i} = \mathbf{j} \quad \mathbf{k} \qquad \mathbf{j} = \mathbf{k} \quad \mathbf{i} \qquad \mathbf{k} = \mathbf{i} \quad \mathbf{j} \qquad \mathbf{j} = \mathbf{i} - \quad \mathbf{k} \qquad \mathbf{k} = \mathbf{j} - \quad \mathbf{i} \qquad \mathbf{i} = \mathbf{k} - \quad \mathbf{j}$.

The results for the cross products of any two of the three standard basis vectors can be remembered by using the adjoining diagram. The product of any two successive vectors in the diagram, when moving clockwise, is the third vector in the diagram. The product of any two successive vectors in the diagram, when moving counterclockwise, is the negative of the third vector in the diagram.



1.8 Problems

1. Write each of the following vectors as a combination of the three standard basis vectors **i**, **j,** and **k**.
   a. **u** = (4, 3, 7) b. **v** = (3, –1, 2) c. **w** = (–2, 5, 6) d. **r** = (1, 0, 2)

2. Verify the following results for the standard basis vectors
   i = (1, 0, 0), **j =** (0, 1, 0), and **k** = (0, 0, 1).
   a. $\mathbf{i} \bullet \mathbf{i} = 1$ b. $\mathbf{j} \bullet \mathbf{j} = 1$ c. $\mathbf{i} \bullet \mathbf{j} = 0$ d. $\mathbf{j} \bullet \mathbf{k} = 0$
   e. $\mathbf{i} \times \mathbf{i} = \mathbf{0}$ f. $\mathbf{i} \times \mathbf{j} = \mathbf{k}$ g. $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$ h. $\mathbf{j} \times \mathbf{k} = \mathbf{i}$

3. Compute the following dot products.
   a. (2**i** + 3**j** + **k**)• (3**i** – 2**j** + 5**k**) b. (3**i** + **j** + 4**k**)• (2**i** – 5**j** + 6**k**)
   c. (2**i** – 5**j** + 3**k**)• (4**i** + 2**j** – 3**k**) d. (**i** – 3**j** + 2**k**)• (6**i** + **j** – 3**k**)

4. Compute the following cross products.
   a. (2**i** + 3**j** + **k**) × (3**i** – 2**j** + 5**k**) b. (3**i** + **j** + 4**k**) × (2**i** – 5**j** + 6**k**)
   c. (2**i** – 5**j** + 3**k**) × (4**i** + 2**j** – 3**k**) d. (**i** – 3**j** + 2**k**) × (6**i** + **j** – 3**k**)

5. Find *a* if the following pairs of vectors are orthogonal.

a. $\mathbf{u} = a\mathbf{i} + 2a\mathbf{j} + \mathbf{k},\ \mathbf{v} = \mathbf{i} - 3\mathbf{j} - 5\mathbf{k}$ b. $\mathbf{u} = 3a\mathbf{i} + 2\mathbf{j} - 3\mathbf{k},\ \mathbf{v} = 2\mathbf{i} - 6\mathbf{j} + a\mathbf{k}$

1.9 VECTORS IN $\mathbf{R}^m$

We have already seen that the set of all real numbers **R** can be identified with a one-dimensional number line; the set of all ordered pairs of real numbers $\mathbf{R}^2$ can be identified with a two-dimensional plane and that the set of all ordered triples of real numbers $\mathbf{R}^3$ can be identified with three-dimensional space. Continuing in this manner would suggest that the set of all ordered four-tuples could be identified with a four-dimensional space and more generally the set of all ordered *m*-tuples could be identified with an *m*-dimensional space.

We use the symbol $\mathbf{R}^m$ to denote the set of all ordered *m*-tuples **u** = $(u_1, u_2, u_3, \cdots\cdots, u_m)$. We will refer to the *m*-tuples as **vectors** in the space $\mathbf{R}^m$ and the entries u_1, u_2, etc. as the **components** of the vector **u**.

Two vectors from $\mathbf{R}^m$ are said to be **equal** if their corresponding components are equal to each other. That is **u = v** if and only if $u_1 = v_1$, $u_2 = v_2$, etc.

We define the **sum** of **u** and **v** by **u + v =** $(u_1 + v_1, u_2 + v_2, \cdots\cdots, u_m + v_m)$**.**

We define **multiplication of a vector v by a scalar** *c* as *c***u** = $(cu_1, cu_2, \cdots\cdots cu_m)$.

The **length** of the vector **u** is denoted $\|\mathbf{u}\|$ and is defined by $\|\mathbf{u}\| = \sqrt{u_1^{\ 2} + u_2^{\ 2} + \cdots + u_m^{\ 2}}$.

The **dot product** is defined to be $\mathbf{u} \bullet \mathbf{v} = u_1 v_1 + u_2 v_2 + \cdots\cdots + u_m v_m$.

If $\mathbf{u} \bullet \mathbf{v} = 0$ we say that the vectors **u** and **v** are orthogonal to each other.

Note that there is no cross product defined for $\mathbf{R}^m$ when $\mathrm{m} \neq 3$.

Example

Let **u** = (1, 3, 2, 4) and **v** = (2, –1, 4, 3) be two vectors in $\mathbf{R}^4$. Then

u + **v** = (1, 3, 2, 4) + (2, –1, 4, 3) = (1+2, 3–1, 2+4, 4+3) = (3, 2, 6, 7)

u – **v** = (1, 3, 2, 4) – (2, –1, 4, 3) = (1–2, 3+1, 2–4, 4–3) = (–1, 4, –2, 1)

$\mathbf{u} \bullet \mathbf{v} = (1,3,2,4) \bullet (2,-1,4,3) = (1)(2) + (3)(-1) + (2)(4) + (4)(3) = 2 - 3 + 8 + 12 = 19$

3**u** = 3(1, 3, 2, 4) = $(3\times 1, 3\times 3, 3\times 2, 3\times 4)$ = (3, 9, 6, 12)

$\|\mathbf{u}\| = \sqrt{1^2 + 3^2 + 2^2 + 4^2} = \sqrt{1+9+4+16} = \sqrt{30}$

1.9 Problems

For questions 1 to 6, let **u** = (1, 3, 2, 4), **v** = (5, 3 ,0, 1), and **w** = (3, 2, –1, 4).

1. Find a. **u + v** b. 2**u** – 3**v** c. **u + v – w**

2. Find a. $\mathbf{u} \bullet \mathbf{v}$ b. $\mathbf{v} \bullet \mathbf{w}$ c. $\mathbf{u} \bullet (\mathbf{v} + \mathbf{w})$

3. Find a. $\|\mathbf{u}\|$ b. $\|\mathbf{v} + \mathbf{w}\|$ c. $\|\mathbf{u} - \mathbf{v}\|$

4. Find a unit vector in the direction of a. **u** b. **v** c. **w**

5. Show that the following pairs of vectors are orthogonal by showing that their dot product is 0.
   a. (1, 2, 3, 1) (3, 1, 1, –8) b. (2, 0, 3, –1) (5, 6, –2, 4)
   c. (1, 2, 3, 4, 5) (4, 4, –3, –2, 1) d. (1, 3, 5, 2, 4) (3, –4, –1, 3, 2)

6. Show that the following sets of vectors are mutually orthogonal by showing that each vector in the set is orthogonal to all the other vectors in the set.

   a. (1, 1, 0, 0) (1, –1, 2, 3) (2, –2, 1, –2)
   b. (2, 1, –11, 4) (3, 2, 0, –2) (2, –1, 1, 2)
   c. (1, –1, 1, –1) (2, 2, 3, 3) (3, 3, –2, –2)
   d. (1, 0, 2, 1) (2, 3, –1, 0) (6, –5, –3, 0)

7. Consider the four unit vectors
   $\mathbf{e}_1 = (1,0,0,0)$, $\mathbf{e}_2 = (0,1,0,0)$, $\mathbf{e}_3 = (0,0,1,0)$ and $\mathbf{e}_4 = (0,0,0,1)$ in $\mathbf{R}^4$. Write each of the following vectors from $\mathbf{R}^4$ as a combination of the vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$.
   a. (2, 3, 5, 4) b. (3, 1, 0, –2) c. (5, 7, 2, 3)

8. Prove the following results for $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and $\mathbf{e}_4$.

a. $\|\mathbf{e}_1\| = 1$ b. $\mathbf{e}_1 \bullet \mathbf{e}_2 = 0$ c. $(\mathbf{e}_1 + \mathbf{e}_2) \bullet (\mathbf{e}_1 - \mathbf{e}_2) = 0$