

Integration Rules and Techniques

Antiderivatives of Basic Functions

Power Rule (Complete)

$$\int x^n dx = \begin{cases} \frac{x^{n+1}}{n+1} + C, & \text{if } n \neq -1 \\ \ln|x| + C, & \text{if } n = -1 \end{cases}$$

Exponential Functions

With base a :

$$\int a^x dx = \frac{a^x}{\ln(a)} + C$$

With base e , this becomes:

$$\int e^x dx = e^x + C$$

If we have base e and a linear function in the exponent, then

$$\int e^{ax+b} dx = \frac{1}{a}e^{ax+b} + C$$

Trigonometric Functions

$$\begin{array}{ll} \int \sin(x) dx = -\cos(x) + C & \int \cos(x) dx = \sin(x) + C \\ \int \sec^2(x) dx = \tan(x) + C & \int \csc^2(x) dx = -\cot(x) + C \\ \int \sec(x) \tan(x) dx = \sec(x) + C & \int \csc(x) \cot(x) dx = -\csc(x) + C \end{array}$$

Inverse Trigonometric Functions

$$\begin{aligned}\int \frac{1}{\sqrt{1-x^2}} dx &= \arcsin(x) + C \\ \int \frac{1}{x\sqrt{x^2-1}} dx &= \operatorname{arcsec}(x) + C \\ \int \frac{1}{1+x^2} dx &= \arctan(x) + C\end{aligned}$$

More generally,

$$\int \frac{1}{a^2+x^2} dx = \frac{1}{a} \arctan\left(\frac{x}{a}\right) + C$$

Hyperbolic Functions

$$\begin{aligned}\int \sinh(x) dx &= \cosh(x) + C & \int -\operatorname{csch}(x) \coth(x) dx &= \operatorname{csch}(x) + C \\ \int \cosh(x) dx &= \sinh(x) + C & \int -\operatorname{sech}(x) \tanh(x) dx &= \operatorname{sech}(x) + C \\ \int \operatorname{sech}^2(x) dx &= \tanh(x) + C & \int -\operatorname{csch}^2(x) dx &= \operatorname{coth}(x) + C\end{aligned}$$

Integration Theorems and Techniques

u -Substitution

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I , then

$$\int f(g(x)) g'(x) dx = \int f(u) du$$

If we have a definite integral, then we can either change back to xs at the end and evaluate as usual; alternatively, we can leave the anti-derivative in terms of u , convert the limits of integration to us , and evaluate everything in terms of u without changing back to xs :

$$\int_a^b f(g(x)) g'(x) dx = \int_{g(a)}^{g(b)} f(u) du$$

Integration by Parts

Recall the Product Rule:

$$\frac{d}{dx} [u(x)v(x)] = v(x) \frac{du}{dx} + u(x) \frac{dv}{dx}$$

Integrating both sides and solving for one of the integrals leads to our Integration by Parts formula:

$$\int u dv = uv - \int v du$$

Integration by Parts (which I may abbreviate as IbP or IBP) “undoes” the Product Rule.

When choosing u and dv , we want a u that will become simpler (or at least no more complicated) when we differentiate it to find du , and a dv what will also become simpler (or at least no more complicated) when we integrate it to find v . If you’re having trouble deciding what u and dv should be to accomplish this, you can use “LIATE” to choose u (choose as high on the list as possible):

1. **L**ogarithmic
2. **I**nverse Trigonometric
3. **A**lgebraic, such as polynomials (including powers of x) and rational functions.
4. **T**rigonometric
5. **E**xponential

and then whatever is left is dv . This doesn’t always work, but it’s a good place to start.

With definite integrals, the formula becomes

$$\int_a^b u dv = u(x)v(x)]_a^b - \int_a^b v du.$$

(This just means we find the antiderivative using IBP and then plug in the limits of integration the way we do with other definite integrals.)

Trigonometric Integrals

For integrals involving only powers of sine and cosine (both with the same argument):

- If at least one of them is raised to an odd power, pull off one to save for a u -sub, use a Pythagorean identity ($\cos^2(x) = 1 - \sin^2(x)$ or $\sin^2(x) = 1 - \cos^2(x)$) to convert the remaining (now even) power to the other trig function, then make a u -sub with u =(whichever trig function you didn’t save) and the trig function you set aside earlier will be part of du .
- If they are both raised to an even power, use a half-angle formula ($\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ or $\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$) to convert to cosines, expand the result and apply half-angle formulas again if needed (keep doing this until you no longer have any powers of cosine), then integrate (may need a simple u -sub).

For integrals involving only powers of secant and tangent (both with the same argument):

- If the secant is raised an even power, pull off two of them to save for a u -sub, use the Pythagorean identity $\sec^2(x) = 1 + \tan^2(x)$ to convert the remaining powers to tangents, then make a u -sub with $u = \tan(x)$ and the $\sec^2(x)$ you set aside earlier will be part of du .
- If the tangent is raised to an odd power, pull off one of each to save for a u -sub, use the Pythagorean identity $\tan^2(x) = \sec^2(x) - 1$ to convert the remaining powers to tangents, then make a u -sub with $u = \sec(x)$ and the $\sec(x)\tan(x)$ you set aside earlier will be part of du .

Trigonometric Substitutions

With certain integrals we can use right triangles to help us determine a helpful substitution:

If the integral contains an expression of the form...	...then make the substitution...	...and...
$\sqrt{a^2 - x^2}$	$x = a \sin \theta$	$dx = a \cos(\theta) d\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta$	$dx = a \sec^2(\theta) d\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta$	$dx = a \sec(\theta) \tan(\theta) d\theta$

You can memorize these rules if you wish, but we can also figure them out using a right triangle.

Partial Fraction Decomposition

Given a rational function to integrate, follow these steps:

1. If the degree of the numerator is greater than or equal to that of the denominator perform long division.
2. Factor the denominator into unique linear factors or irreducible quadratics.
3. Split the rational function into a sum of partial fractions with unknown constants on top as follows:

$$\underbrace{\frac{A}{ax + b}}_{\text{for a linear factor}} + \underbrace{\frac{B}{cx + d} + \frac{C}{(cx + d)^2} + \dots}_{\text{for a repeated linear factor}} + \underbrace{\frac{Dx + E}{ex^2 + fx + g}}_{\text{for an irreducible quadratic}}$$

For example:

$$\frac{x^2 + 7}{(2x + 1)(x - 3)^3(x^2 + 3x + 1)^2} = \frac{A}{2x + 1} + \frac{B}{x - 3} + \frac{C}{(x - 3)^2} + \frac{D}{(x - 3)^3} + \frac{Ex + F}{x^2 + 3x + 1} + \frac{Gx + H}{(x^2 + 3x + 1)^2}$$

4. Multiply both sides by the entire denominator and simplify.
5. Solve for the unknown constants by using a system of equations or picking appropriate numbers to substitute in for x .
6. Integrate each partial fraction. You may need to use u -substitution and/or $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$.

Integration Using Tables

While computer algebra systems such as *Mathematica* have reduced the need for integration tables, sometimes the tables give a nicer or more useful form of the answer than the one that the CAS will yield. Oftentimes we will need to do some algebra or use u -substitution to get our integral to match an entry in the tables.

What follows is a selection of entries from the integration tables in Stewart's Calculus, 7e:

$$21. \int \sqrt{a^2 + u^2} du = \frac{u}{2} \sqrt{a^2 + u^2} + \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$$

$$22. \int u^2 \sqrt{a^2 + u^2} du = \frac{u}{8} (a^2 + 2u^2) \sqrt{a^2 + u^2} - \frac{a^4}{8} \ln(u + \sqrt{a^2 + u^2}) + C$$

$$23. \int \frac{\sqrt{a^2 + u^2}}{u} du = \sqrt{a^2 + u^2} - a \ln \left| \frac{a + \sqrt{a^2 + u^2}}{u} \right| + C$$

$$24. \int \frac{\sqrt{a^2 + u^2}}{u^2} du = -\frac{\sqrt{a^2 + u^2}}{u} + \ln(u + \sqrt{a^2 + u^2}) + C$$

$$25. \int \frac{du}{\sqrt{a^2 + u^2}} = \ln(u + \sqrt{a^2 + u^2}) + C$$

$$26. \int \frac{u^2 du}{\sqrt{a^2 + u^2}} = \frac{u}{2} \sqrt{a^2 + u^2} - \frac{a^2}{2} \ln(u + \sqrt{a^2 + u^2}) + C$$

$$27. \int \frac{du}{u\sqrt{a^2 + u^2}} = -\frac{1}{a} \ln \left| \frac{\sqrt{a^2 + u^2} + a}{u} \right| + C$$

Approximating Definite Integrals

We can approximate the **net** area under the graph of $f(x)$ over the interval $[a, b]$ using n rectangles and the indicated end/mid-point, where

$$\Delta x = \frac{b-a}{n}; \quad x_0 = a, \quad x_i = a + i \cdot \Delta x, \quad x_n = b; \quad \bar{x}_i = \frac{x_{i-1} + x_i}{2} \quad (\text{the midpoint of the interval } [x_{i-1}, x_i])$$

Left Endpoint Approximation: Uses the left endpoint of the subinterval to find the height of the corresponding rectangle.

$$\begin{aligned} \int_a^b f(x) dx &\approx L_n = \Delta x f(x_0) + \Delta x f(x_1) + \dots + \Delta x f(x_{n-1}) \\ &= \Delta x [f(x_0) + f(x_1) + \dots + f(x_{n-1})] \\ &= \sum_{i=0}^{n-1} f(x_i) \Delta x \end{aligned}$$

Right Endpoint Approximation: Uses the right endpoint of the subinterval to find the height of the corresponding rectangle.

$$\begin{aligned} \int_a^b f(x) dx &\approx R_n = \Delta x f(x_1) + \Delta x f(x_2) + \dots + \Delta x f(x_n) \\ &= \Delta x [f(x_1) + f(x_2) + \dots + f(x_n)] \\ &= \sum_{i=1}^n f(x_i) \Delta x \end{aligned}$$

Midpoint Approximation: Uses the midpoint of the subinterval to find the height of the corresponding rectangle.

$$\begin{aligned} \int_a^b f(x) dx &\approx M_n = \Delta x f(\bar{x}_1) + \Delta x f(\bar{x}_2) + \dots + \Delta x f(\bar{x}_n) \\ &= \Delta x [f(\bar{x}_1) + f(\bar{x}_2) + \dots + f(\bar{x}_n)] \\ &= \sum_{i=1}^n f(\bar{x}_i) \Delta x \end{aligned}$$

Trapezoid Approximation: Uses the trapezoids whose heights come from the function values at the left and right endpoints of the corresponding subinterval.

$$\begin{aligned} \int_a^b f(x) dx &\approx T_n = \frac{\Delta x}{2} [f(x_0) + 2f(x_1) + 2f(x_2) + \dots + 2f(x_{n-1}) + f(x_n)] \\ &= \frac{\Delta x}{2} \left(f(x_0) + \sum_{i=1}^{n-1} 2f(x_i) + f(x_n) \right) \end{aligned}$$

Simpson's Rule (Quadratic Approximation): Uses a quadratic to approximate the function at the top of the "rectangle" over the corresponding subinterval.

$$\int_a^b f(x) dx \approx S_n = \frac{\Delta x}{3} [f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \dots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Note: In order for the alternating pattern of 2's and 4's to work out correctly, n must be even! If n is not even, then Simpson's Rule cannot be used (in practice this probably won't be an issue since we could just choose to use one more or one less rectangle).

Simpson's Rule and the Midpoint and Trapezoid approximations are related by this formula:

$$S_{2n} = \frac{1}{3}T_n + \frac{2}{3}M_n.$$

Error Bound Formulas

Let E_M and E_T be the errors in the Midpoint and Trapezoidal Approximations, respectively. If

$$|f''(x)| \leq K, \quad a \leq x \leq b$$

then

$$|E_M| \leq \frac{K(b-a)^3}{24n^2} \quad \text{and} \quad |E_T| \leq \frac{K(b-a)^3}{12n^2}.$$

Let E_S be the error in the Simpson's Rule approximation. If

$$|f^{(4)}(x)| \leq K, \quad a \leq x \leq b$$

then

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}.$$