Introduction to Advanced Structural Mechanics

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Aim of the Introductory Course

The aim of these slides is to help students review some basic concepts of structural mechanics that will be exploited during the course of Advanced Structural Mechanics.

Outline of the Introductory Course

- PART I: Cross-sectional Properties.
- PART II: Solid Mechanics: Displacements and Strains, Stress and Equilibrium, Constitutive Equations.
- PART III: Internal Forces in Beams: Axial Force, Bending Moment, Shear Force, and Torque. The Euler-Bernoulli beam model.
- PART IV   Analysis of Statically determinate and indeterminate Structures.

Links and resources

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Suggested reading:

- Beer, Johnston, DeWolf, *Mechanics of Materials.*
- Gere and Timoshenko, *Mechanics of Materials.*

PART I

Cross-sectional Properties

Outline of PART I

- Beam: Geometric Model.

- Cross-Sectional Properties:
  - Area
  - First Moments of Area
  - Centroid
  - Second Moments of Area
  - Transfer of Axis Theorem and Rotation of Axes
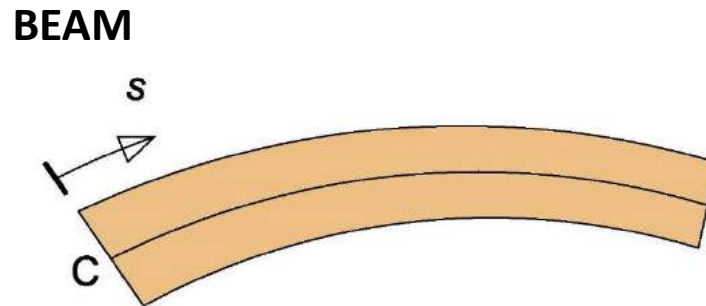 - Principal Axes and Central Ellipse of Inertia.

Beam: Geometric Model

A **beam** is a structural element generated by a planar figure Ω (i.e. **cross section**) that moves in the space remaining normal to the trajectory described by its centroid.
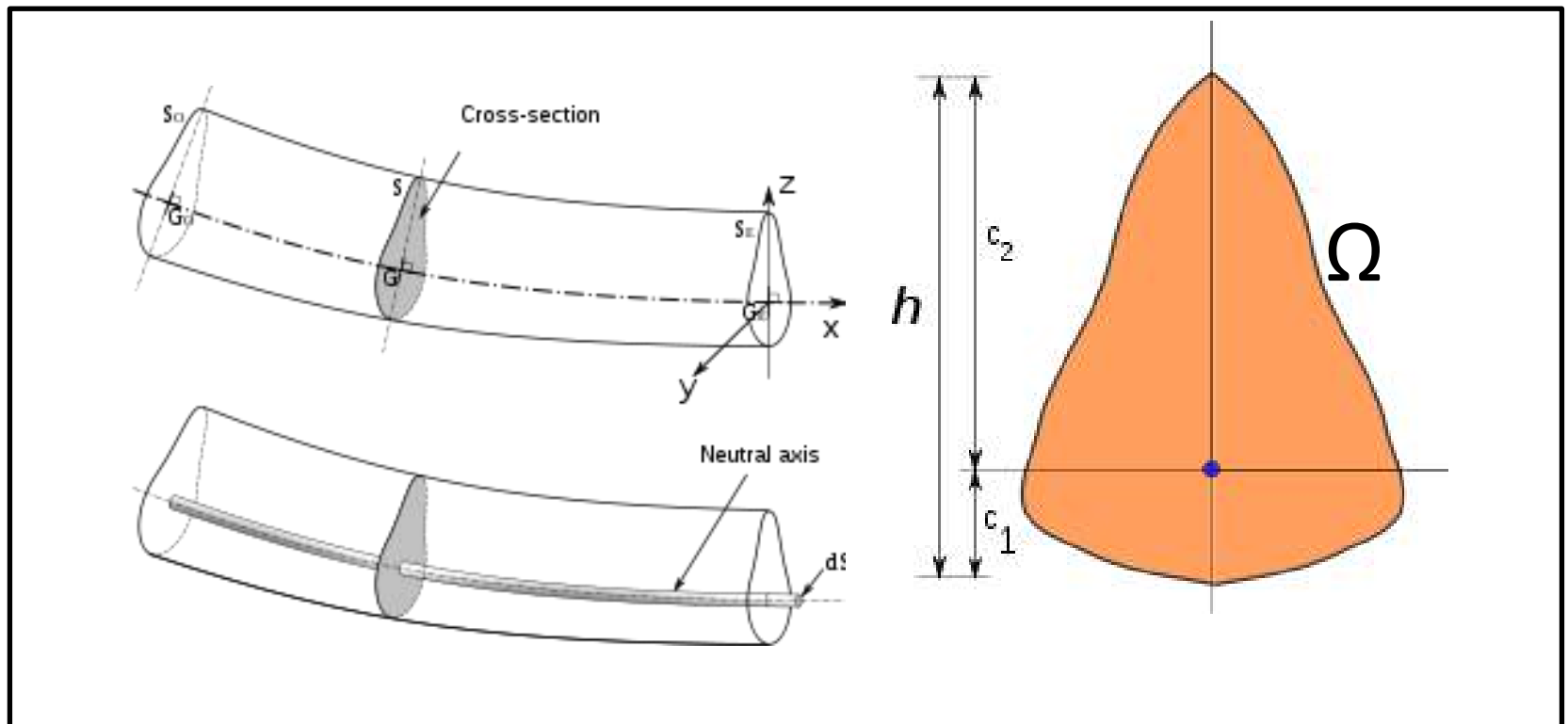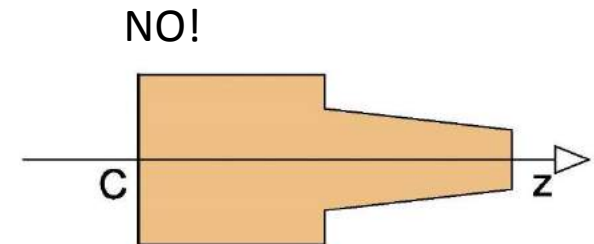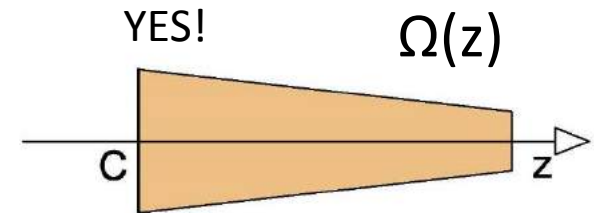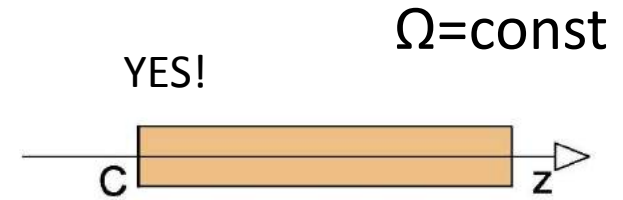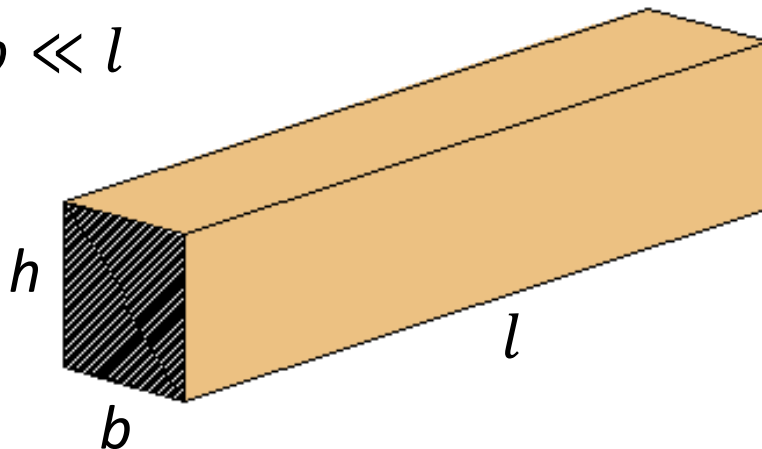
Beam: Geometric Model

Geometric Requirements:

- $\Omega(s)$ constant or can vary continuously:
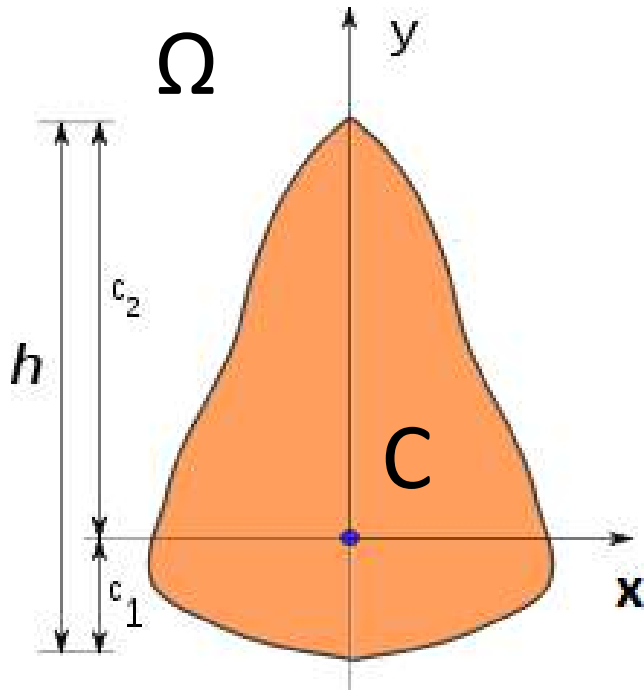
Ω=const

YES!

BEAM

YES! $\Omega(z)$

NO!
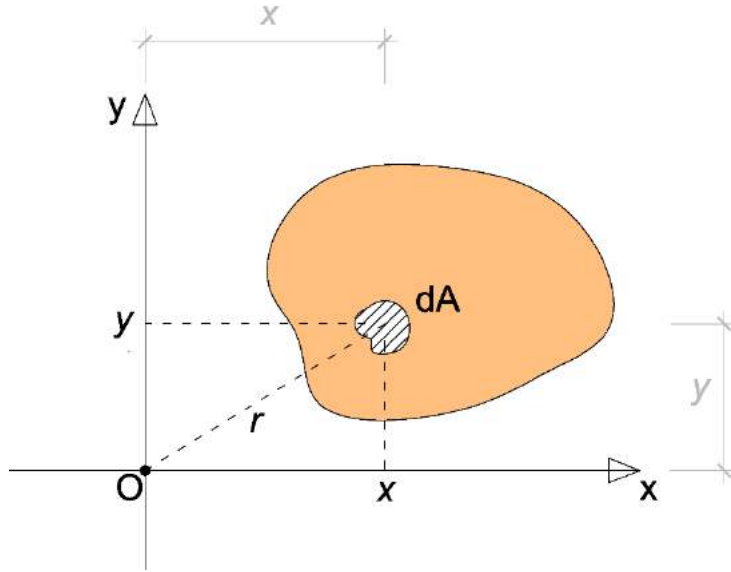
- $h \ll l$
  $b \ll l$

Cross-sectional Properties

For any cross-section Ω, it is possible to define some quantities that are related only to the cross-section geometry.

- Area A

- Static Moment of Area

- Centroid C

- Moments and Product of Inertia

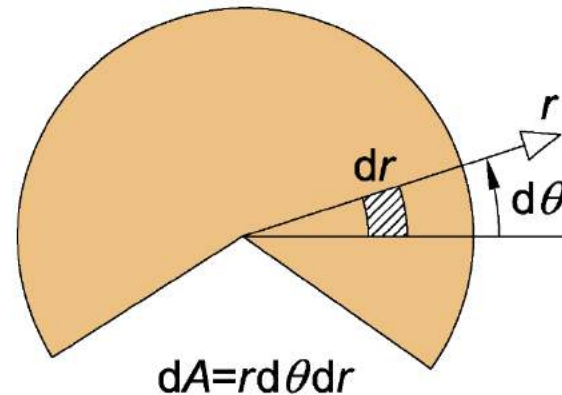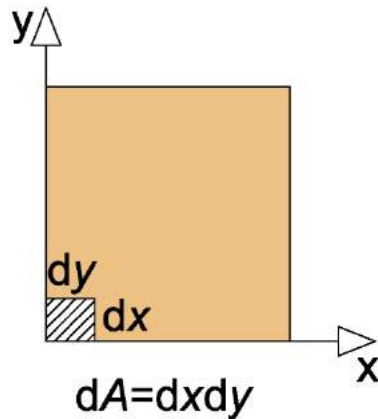# Cross-sectional Properties

Area:

$$A = \int_{\Omega} dA$$

$$A = [m^2]$$

dA:

dA=dxdy

dA=rdθdr

Cross-sectional Properties

First Moment of Area: Static Moments
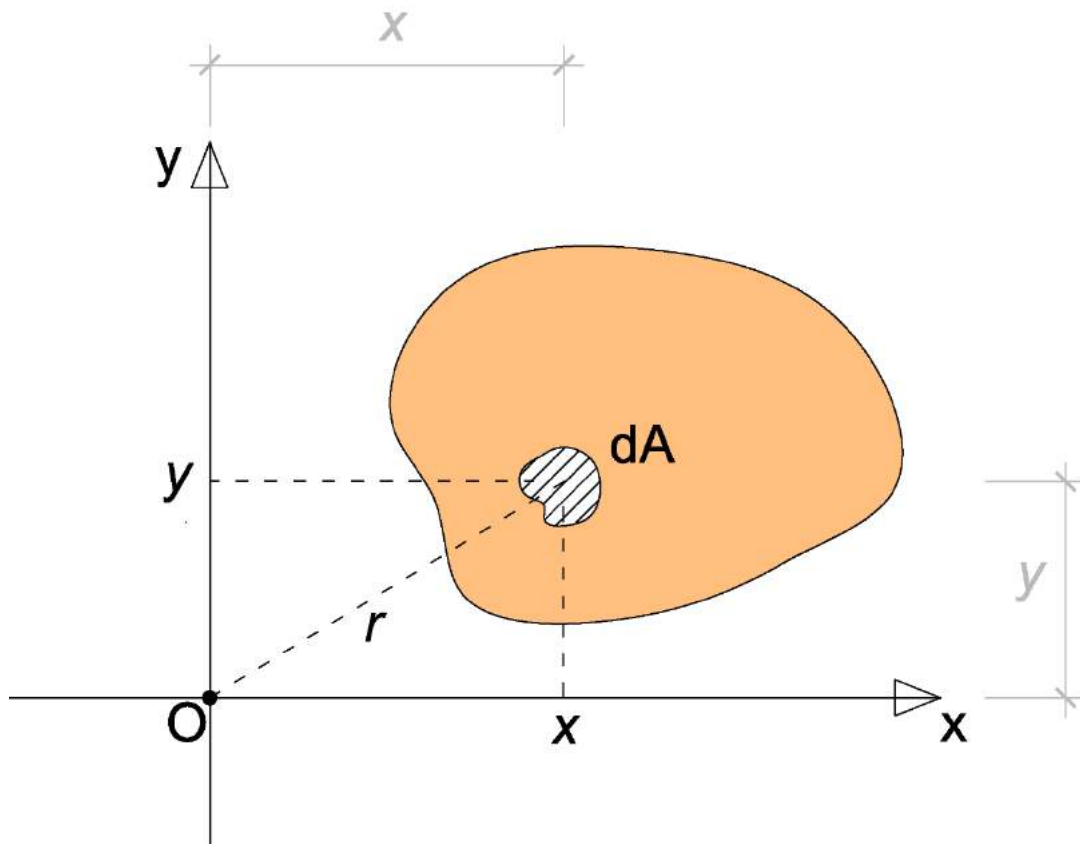
$$S_x = \int_\Omega y\,dA$$

$$S_y = \int_\Omega x\,dA$$

$$S = [L^3]$$

Cross-sectional Properties
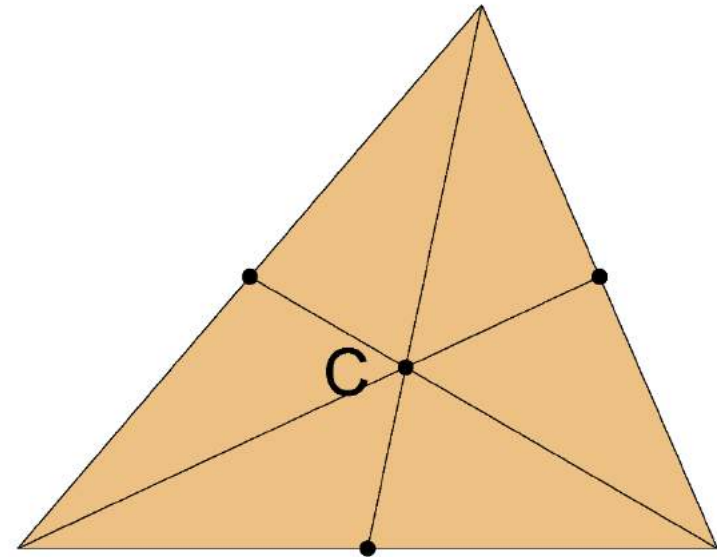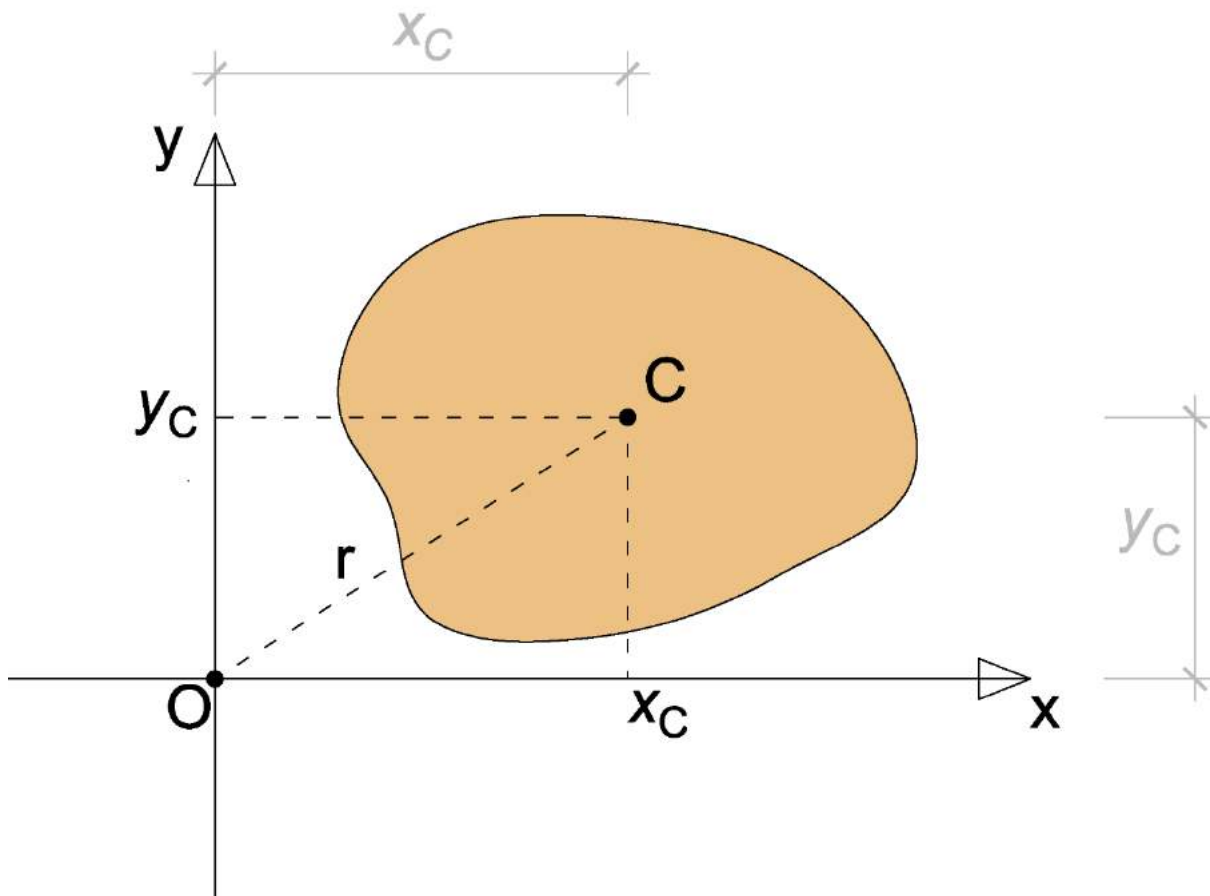
Centroid:

The centroid C of a plane figure or two-dimensional shape is the arithmetic mean position of all the points in the shape.

The centroid C of an area is the point of intersection of all the straight lines that subdivide the plane figure in equal parts

Cross-sectional Properties

Centroid:

$$x_C = \frac{S_y}{A}$$

$$y_C = \frac{S_x}{A}$$

$$x_C, y_C = [L]$$

Cross-sectional Properties

Static Moment & Centroid: Properties

- S_x and S_y can be $\gtreqless 0$
- The static moment S is zero if calculated with respect to a centroidal axis (i.e. the centroid lies on the axis)
- The Static Moment calculated with respect to an axis of symmetry = 0.
- If an area has an axis of symmetry, the centroid C lies on the axis.
- If an area has two axes of symmetry, the centroid C is located at the intersection of the axes .
- $S = \sum_i S_i$ (domain of integration can be added: geometric decomposition)

Cross-sectional Properties

Why is it called first moment?
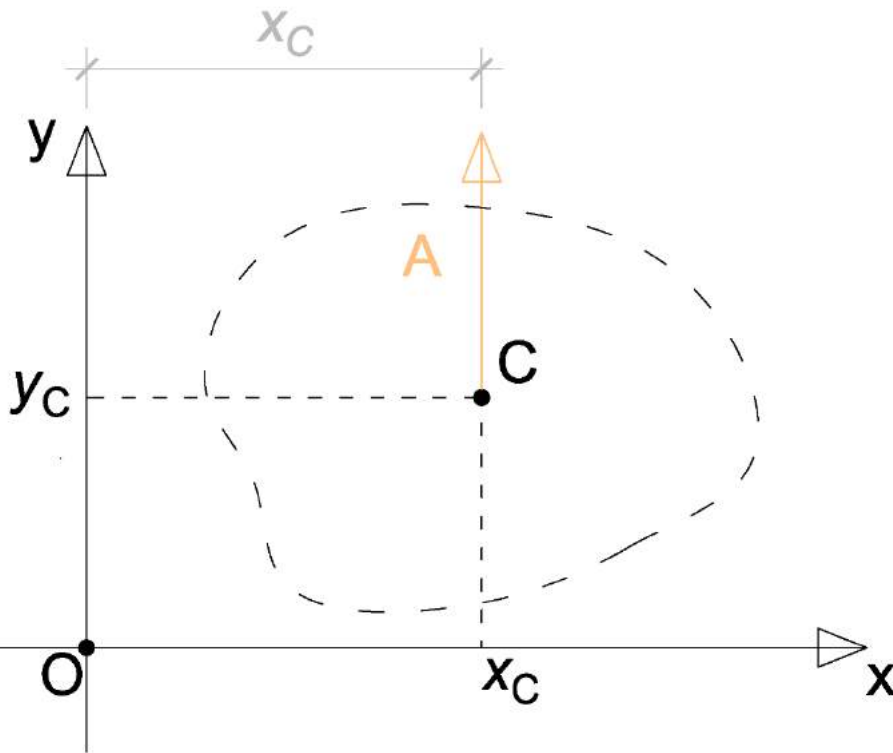
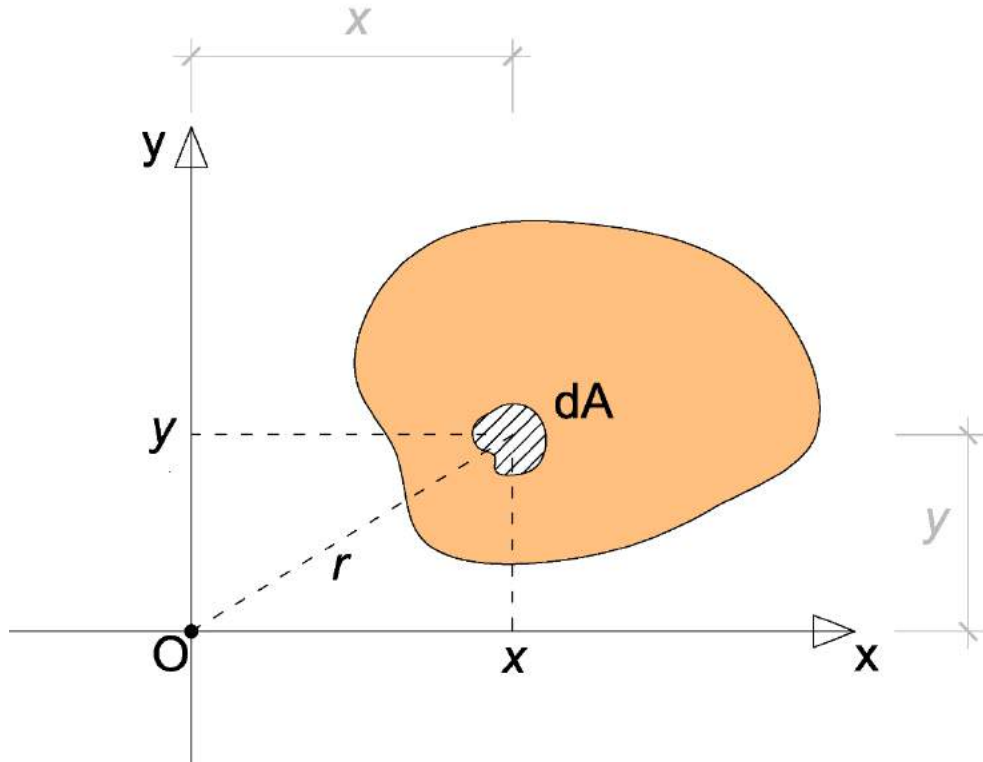Let's assume to apply a vector at C whose magnitude is the area A of the region Ω (the dimensions of this vector are [L^2]).
The moment of this vector with respect to y is Ax_C , which is the static moment S_y

Cross-sectional Properties

Second Moment of Area: Moment of Inertia:

$$I_x = \int_\Omega y^2 dA$$

$$I_y = \int_\Omega x^2 dA$$

$$I = [L^4]$$

Cross-sectional Properties

Second Moment of Area: Product of Inertia and Polar Moment of Area

$$I_{xy} = \int_{\Omega} xy\,dA \quad \text{Product o Inertia}$$

$$I_O = \int_{\Omega} r^2\,dA \quad \text{Polar Moment of Area}$$

$$I_{xy}, I_O = [L^4]$$
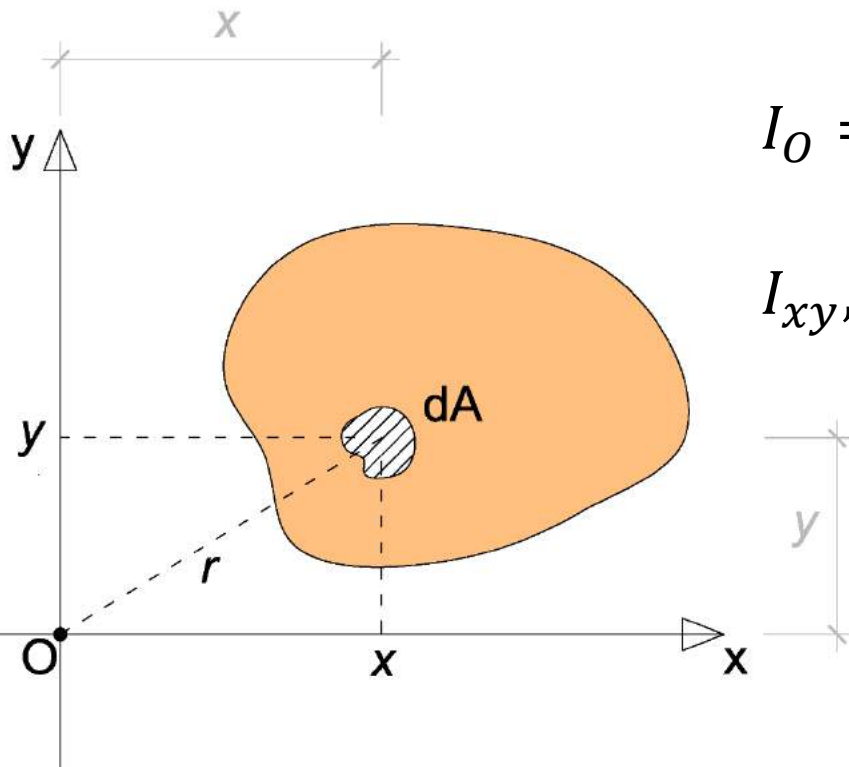
Cross-sectional Properties

Second Moments of Area: Properties

- I_x and $I_y > 0$
- $I_O > 0$
- $I_O = I_x + I_y$ when O is the origin of the x and y axes.
- $I_{xy} \gtreqless 0$
- $I_{xy} = 0$ if either x or y is an axis of symmetry
- $I = \sum_i I_i$ (valid for all the Second Moments of Area)

Cross-sectional Properties

Change of coordinates
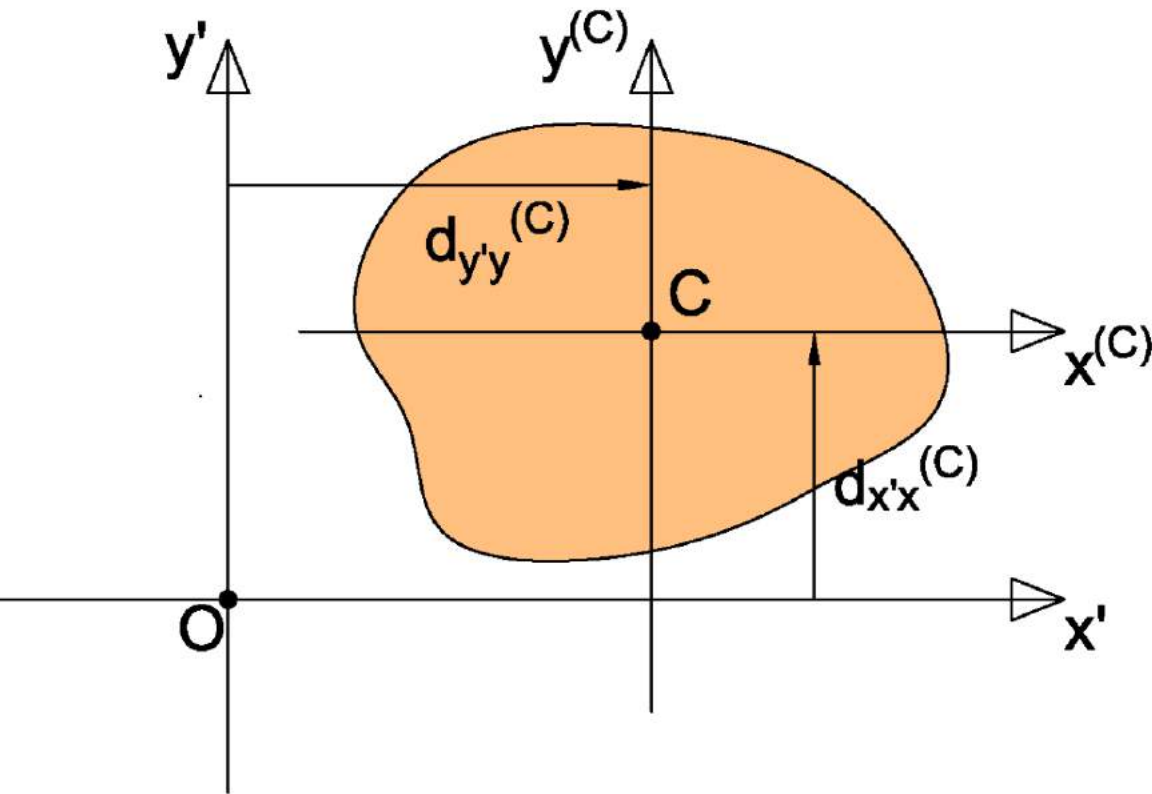
$$x' = x^{(C)} + d_{y'y^{(C)}}$$
$$y' = y^{(C)} + d_{x'x^{(C)}}$$

Note that $d_{y'y^{(C)}}$ and $d_{x'x^{(C)}}$ are the coordinates of the centroid C with respect to the Cartesian system Ox'y'. Thus $d_{y'y^{(C)}} = x'_C$ and $d_{x'x^{(C)}} = y'_C$

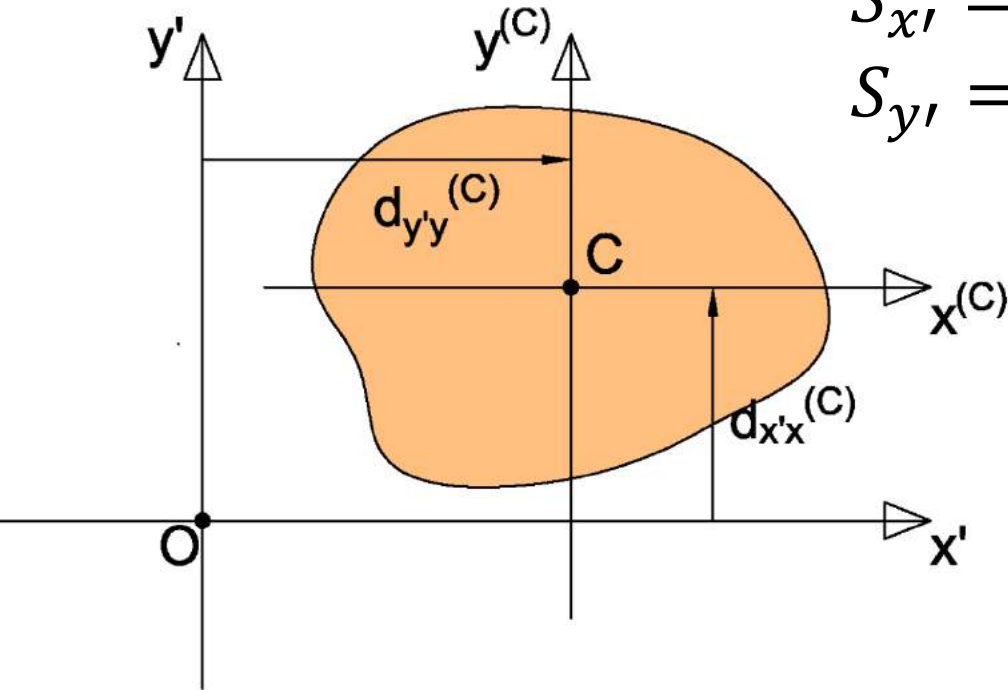# Cross-sectional Properties

Parallel Axes: Static Moment

$$S_{x\prime} = S_{x(C)} + Ad_{x\prime x(C)} = Ad_{x\prime x(C)}$$
$$S_{y\prime} = S_{y(C)} + Ad_{y\prime y(C)} = Ad_{y\prime y(C)}$$

Note that $S_{x(C)} = S_{y(C)} = 0$

$$S_{x\prime} = \int_{\Omega} y\prime dA = \int_{\Omega} (y^{(C)} + d_{x\prime x(C)})dA = S_{x(C)} + Ad_{x\prime x(C)} = Ad_{x\prime x(C)}$$

Cross-sectional Properties

Transfer-of-axis Theorem: Second Moments of Area
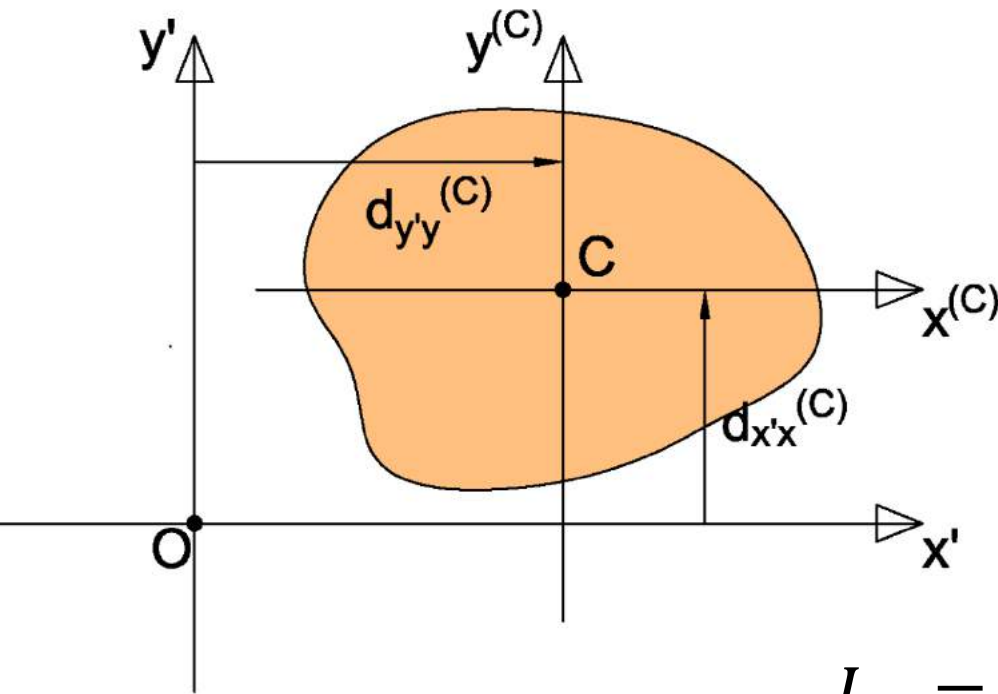
Transfer-of-axis Theorem

$$I_{x'} = I_{x^{(C)}} + A\left(d_{x'x^{(C)}}\right)^2$$

$$I_{y'} = I_{y^{(C)}} + A\left(d_{y'y^{(C)}}\right)^2$$

$$I_{x'y'} = I_{x^{(C)}y^{(C)}} + A d_{x'x^{(C)}} d_{y'y^{(C)}}$$

Cross-sectional Properties

Rotation of the Axes:

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = y\cos\alpha - x\sin\alpha$$
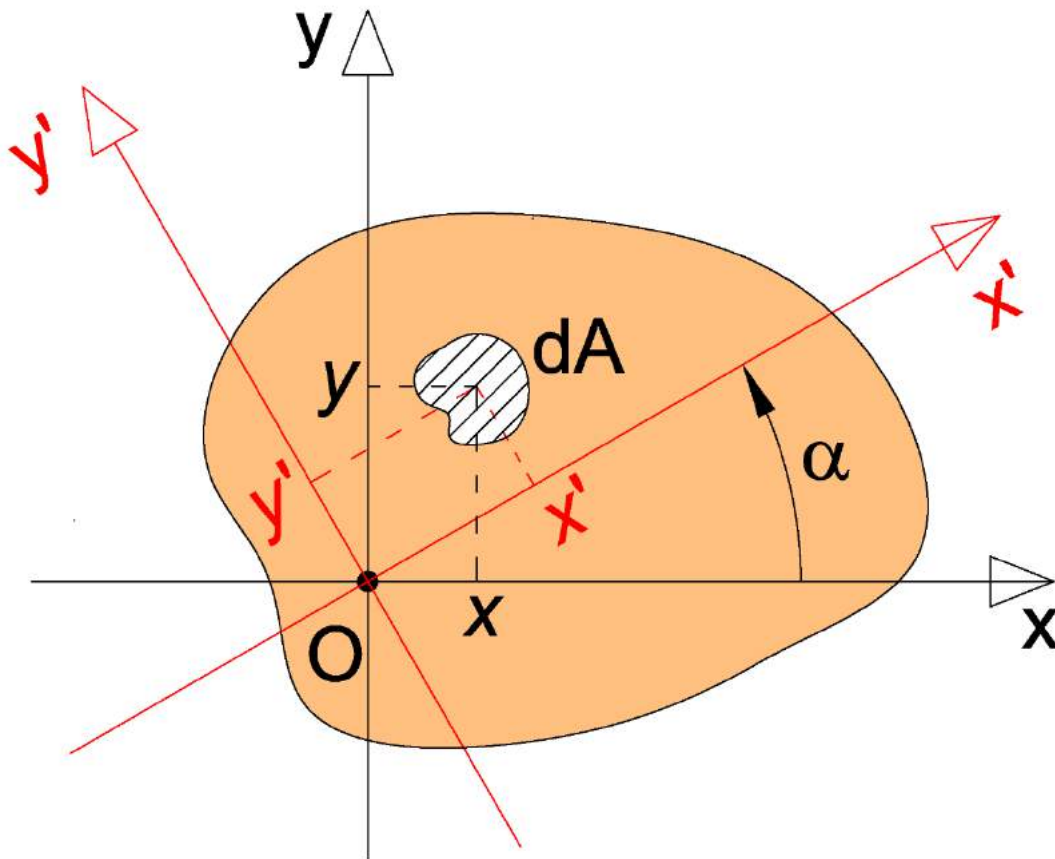
Cross-sectional Properties

Rotation of the axes: Second Moment of Area

$$I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2} \cos 2\alpha - I_{xy} sin2\alpha$$

$$I_{y'} = \frac{I_x + I_y}{2} - \frac{I_x - I_y}{2} \cos 2\alpha + I_{xy} sin2\alpha$$

$$I_{x'y'} = \frac{I_x - I_y}{2} sin2\alpha + I_{xy} cos2\alpha$$

Cross-sectional Properties

Principal axes (1/3)

Goal: determine the value of α_0 for which $I_{x'}$ and $I_{y'}$ are the maximum and minimum moments of inertia for the cross section (or viceversa)

$$\frac{dI_{y'}}{d\alpha} = 0 \qquad \frac{dI_{x'}}{d\alpha} = 0$$

$$tg2\alpha_0 = -\frac{2I_{xy}}{I_x - I_y}$$

Note that if we enforce $I_{x'y'} = 0$ we obtain the expression above. Thus, when $\alpha = \alpha_0$ $I_{x'}$ and $I_{y'}$ attain the maximum and minimum values (or vice versa) and simultaneously $I_{x'y'} = 0$

Cross-sectional Properties

Principal axes (2/3)

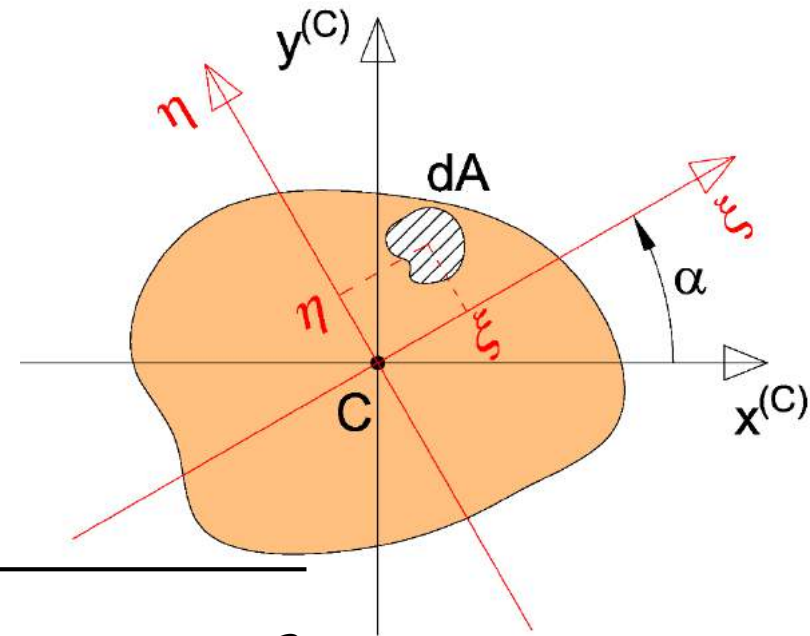With $C \equiv O$ we define ξ, η as the _centroidal principal axes_:

- I_ξ, I_η _principal moments of inertia (minimum/maximum moment of inertia or viceversa)_

with:

- $I_{\xi\eta} = 0$

and:

- $I_\xi , I_\eta = \dfrac{I_{x^{(C)}} + I_{y^{(C)}}}{2} \pm \sqrt{\left(\dfrac{I_{x^{(C)}} - I_{y^{(C)}}}{2}\right)^2 + \left(I_{x^{(C)} y^{(C)}}\right)^2}$

Cross-sectional Properties

Principal axes (3/3): Properties

- If a figure has an axis of symmetry, one of the principal axis is the axis of symmetry.

- Any other axis perpendicular to the first one (of symmetry) is the second principal axis.

Cross-sectional Properties

Mohr circle:

Given the centroidal principal axes ξ, η, with $I_\xi > I_\eta$

$$I_x = \frac{I_\xi + I_\eta}{2} + \frac{I_\xi - I_\eta}{2} cos2\alpha$$

$$I_y = \frac{I_\xi + I_\eta}{2} - \frac{I_\xi - I_\eta}{2} cos2\alpha$$

$$I_{xy} = \frac{I_\xi - I_\eta}{2} sin2\alpha$$

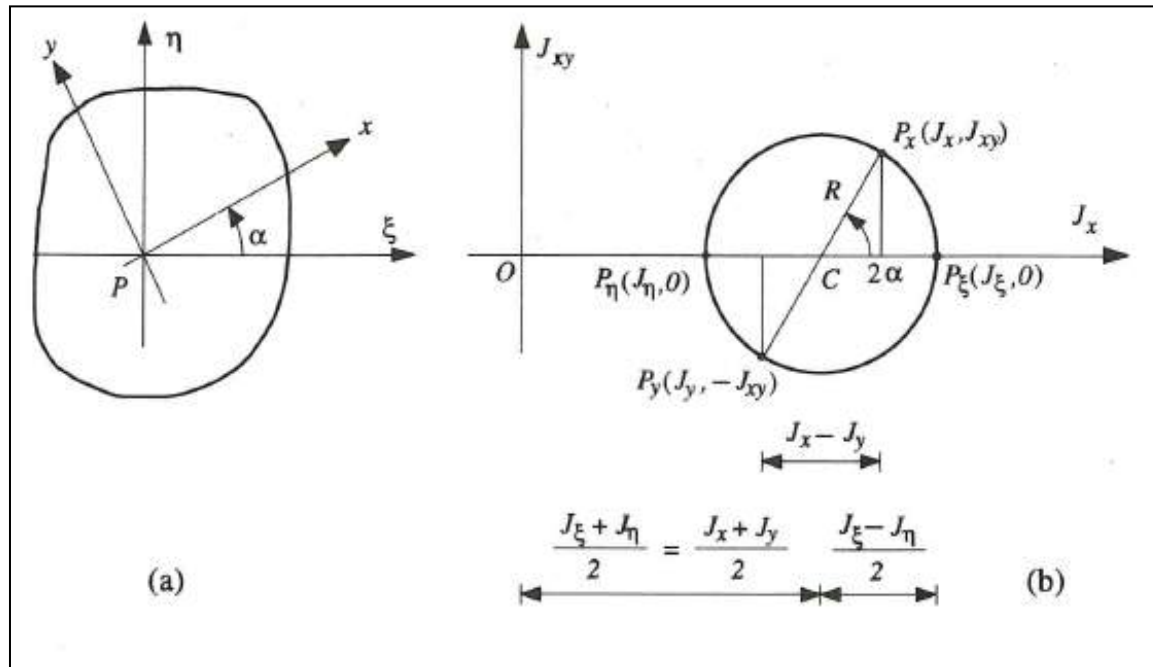Parametric equations of a circle on the plane I_x, I_{xy}.

Note that the expressions above are written as though x and y are
two axes that rotate of an angle α with respect to the princial axes
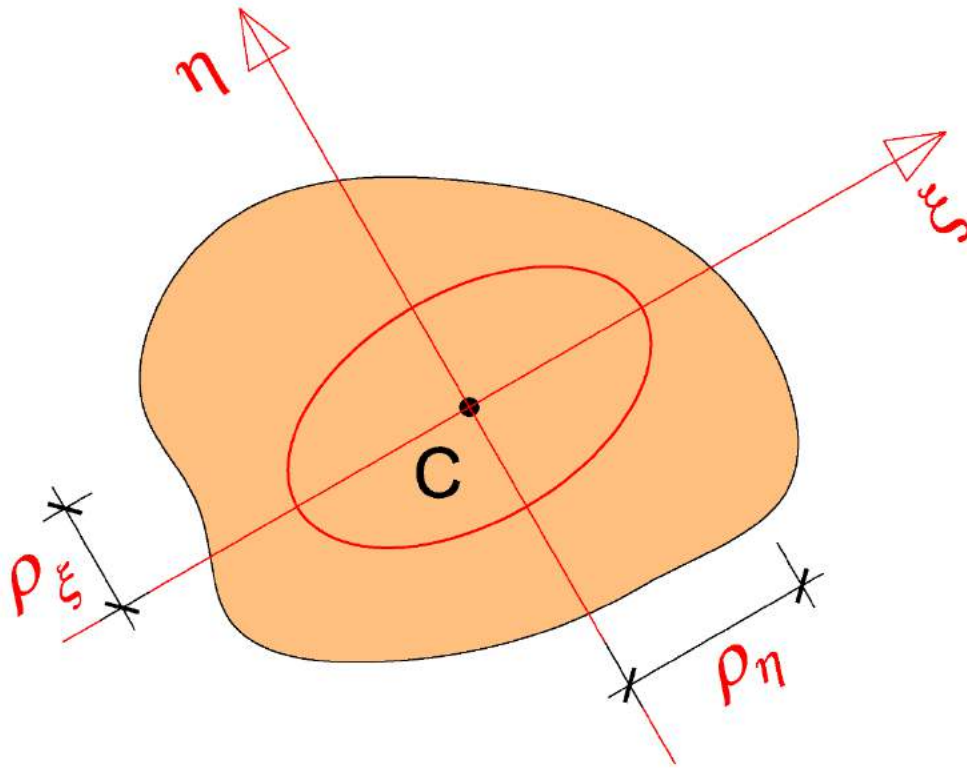
Cross-sectional Properties

Mohr circle:

Parametric equations of a circle in the plane I_x, I_y.

$$R = \frac{I_\xi - I_\eta}{2}; \; C = \left(\frac{I_\xi + I_\eta}{2}, 0 \right)$$

(a)

(b)

$$\frac{J_\xi + J_\eta}{2} = \frac{J_x + J_y}{2} \quad \frac{J_\xi - J_\eta}{2}$$

*A. Di Tommaso. Geometria delle Masse

Cross-sectional Properties

Radii of Gyration & Ellipse of Inertia

$$\rho_\xi = \sqrt{\frac{I_\xi}{A}} \qquad I_\xi = A\rho_\xi^2$$

$$\rho_\eta = \sqrt{\frac{I_\eta}{A}} \qquad I_\eta = A\rho_\eta^2$$

Analytical expression:

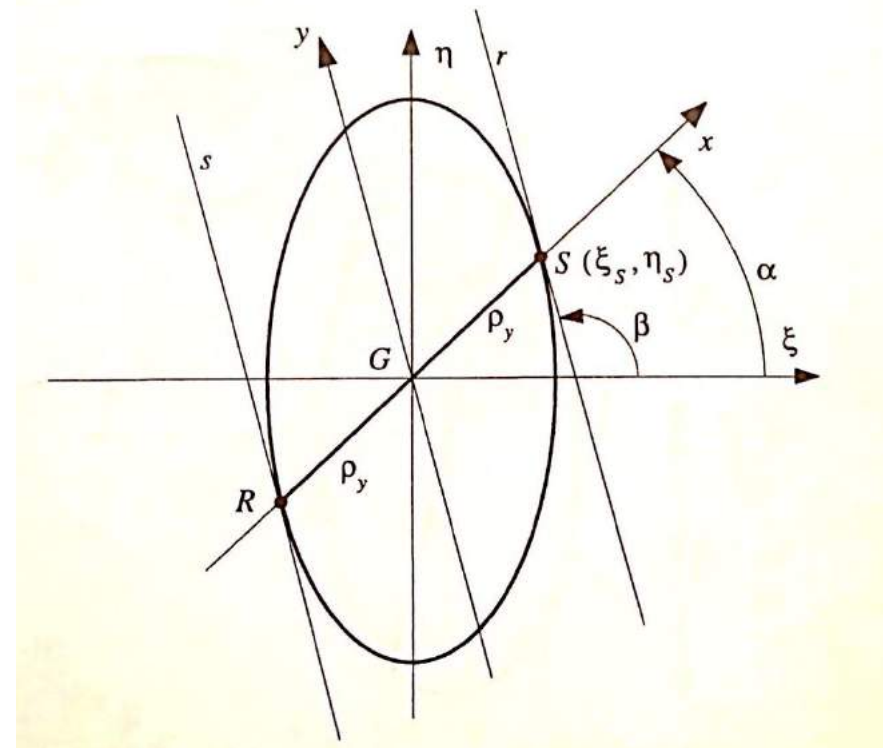$$\frac{\xi^2}{\rho_\eta^2} + \frac{\eta^2}{\rho_\eta^2} = 1$$

Cross-sectional Properties

Radii of Gyration & Ellipse of Inertia

The Ellipse of Inertia provides a graphical representation of the inertia properties of the cross-section.

$$\rho_x = \sqrt{\frac{I_x}{A}} \qquad I_x = A\rho_x^2$$

$$\rho_y = \sqrt{\frac{I_y}{A}} \qquad I_y = A\rho_y^2$$
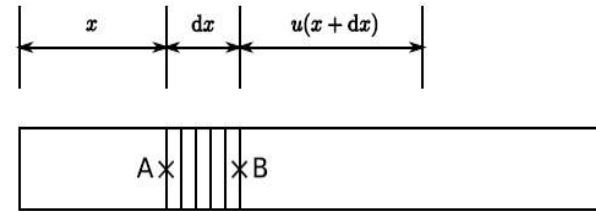
PART II

Solid Mechanics: Displacements and Strains, Strains and Stresses, Stresses and Forces
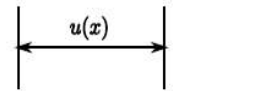
Outline of PART II

- Displacements and Strains: *compatibility equations*

- Strains and Stresses: *constitutive equations*

- Stresses and Forces: *equilibrium equations*
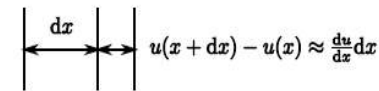
Displacements and Strains (1D)

- displacement:

$u(x)$

- axial strain:

Phillips, Wadee. Pre course Reading Solid Mechanics [1]

$$\epsilon(x) = \frac{A'B' - AB}{AB} = \frac{du(x)}{dx}$$

different notation

$$\epsilon_{11} = \frac{du_1}{dx_1}$$

Note: the strain is assumed to be positive if the material/solid elongates and negative viceversa

Displacements and Strains (2D)

- displacement:

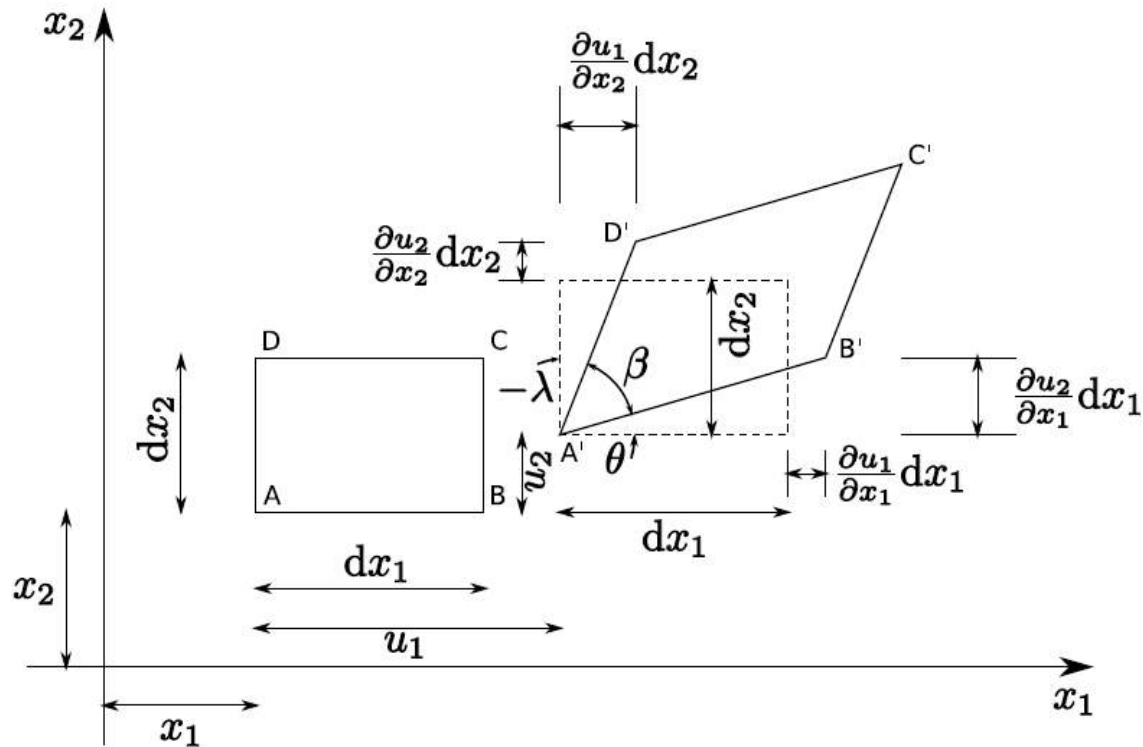$u_1(x_1, x_2)$ and $u_2(x_1, x_2)$

- axial strain:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}$$

$$\epsilon_{22} = \frac{\partial u_2}{\partial x_2}$$

- shear strain:

$$\gamma_{12} = \frac{\pi}{2} - \beta = \theta - \lambda = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \qquad \text{with: } \gamma_{12} = 2\epsilon_{12}$$
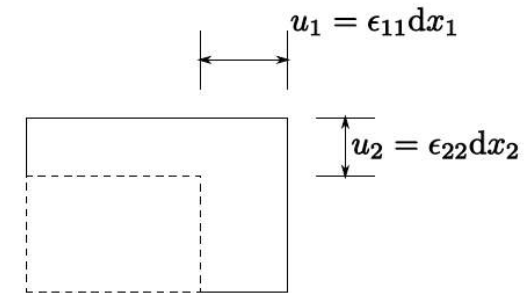
[1]

Displacements and Strains (3D)

- displacement:

$u_1(x_1, x_2, x_3), u_2(x_1, x_2, x_3)$ and $u_3(x_1, x_2, x_3)$

- axial strain:

$$\epsilon_{11} = \frac{\partial u_1}{\partial x_1}, \epsilon_{22} = \frac{\partial u_2}{\partial x_2}, \epsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

$u_1 = \epsilon_{11} dx_1$

$u_2 = \epsilon_{22} dx_2$

(a) Dilation

- shear strain:

$$\gamma_{12} = \frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}$$
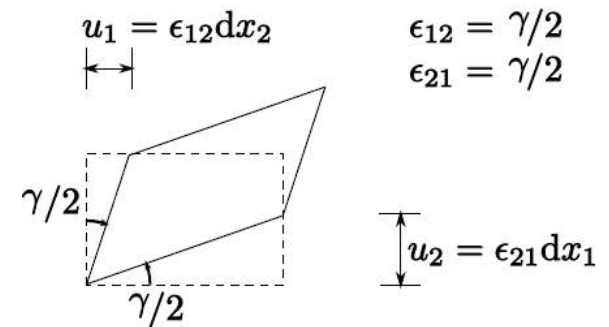with: $\gamma_{12} = 2\epsilon_{12}$

$$\gamma_{13} = \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}$$
with: $\gamma_{13} = 2\epsilon_{13}$

$$\gamma_{23} = \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}$$
with: $\gamma_{23} = 2\epsilon_{23}$

$u_1 = \epsilon_{12} dx_2$

$\epsilon_{12} = \gamma/2$
$\epsilon_{21} = \gamma/2$

$\gamma/2$

$u_2 = \epsilon_{21} dx_1$

$\gamma/2$

(b) Distortion

Displacements and Strains (3D)

The **compatibility equations** link displacements and strains

in 1D:

$$\epsilon_{11} = \frac{du_1}{dx_1}$$

in 2D:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ 2\epsilon_{12} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 \\ 0 & \dfrac{\partial}{\partial x_2} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

in 3D:

$$\begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial}{\partial x_1} & 0 & 0 \\ 0 & \dfrac{\partial}{\partial x_2} & 0 \\ 0 & 0 & \dfrac{\partial}{\partial x_3} \\ \dfrac{\partial}{\partial x_2} & \dfrac{\partial}{\partial x_1} & 0 \\ \dfrac{\partial}{\partial x_3} & 0 & \dfrac{\partial}{\partial x_1} \\ 0 & \dfrac{\partial}{\partial x_3} & \dfrac{\partial}{\partial x_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

In compact notation:

$$\epsilon = \mathbf{D}u$$

Displacements and Strains

In 3D for a given strain tensor

$$\boldsymbol{\epsilon} = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & \epsilon_{13} \\ \epsilon_{12} & \epsilon_{22} & \epsilon_{23} \\ \epsilon_{13} & \epsilon_{23} & \epsilon_{33} \end{bmatrix}$$

the principal strains ϵ_a and principal directions of strain $\boldsymbol{a} \equiv [a_1, a_2, a_3]$, i.e. those directions in which there exist only axial strain and no distorsions, can be found solving the eigenvalue problem:

$$(\boldsymbol{\epsilon} - \epsilon_a \boldsymbol{I})\boldsymbol{a} = 0$$

$\boldsymbol{\epsilon}$ 2nd order strain tensor

ϵ_a principal strain

$\boldsymbol{a}$ principal directions

$\boldsymbol{I}$ identity matrix

quantities to be detrmined

Displacements and Strains

In 2D the principal strains and the principal directions, in the plane $x_1 - x_2$, can be determined also as

* principal strains:

$$\epsilon_1, \epsilon_2 = \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \sqrt{\left(\frac{\epsilon_{11} - \epsilon_{22}}{2}\right)^2 + \epsilon_{12}{}^2}$$

* the angle α of the principal directions a_1, a_2 w.r.t. the x_1, x_2 axes:

$$tg2\alpha = -\frac{2\epsilon_{12}}{\epsilon_{11} - \epsilon_{22}}$$

Note: Mohr circle of radius $\mathrm{R} = \frac{\epsilon_1 - \epsilon_2}{2}$ and centre $C = \frac{\epsilon_1 + \epsilon_2}{2}$

Strains and Stresses

The *constitutive equations* are the relations between kinetics (stress, stress-rate) quantities and kinematics (strain, strain-rate) quantities for a material.

They describe mathematically the actual behavior of a material.
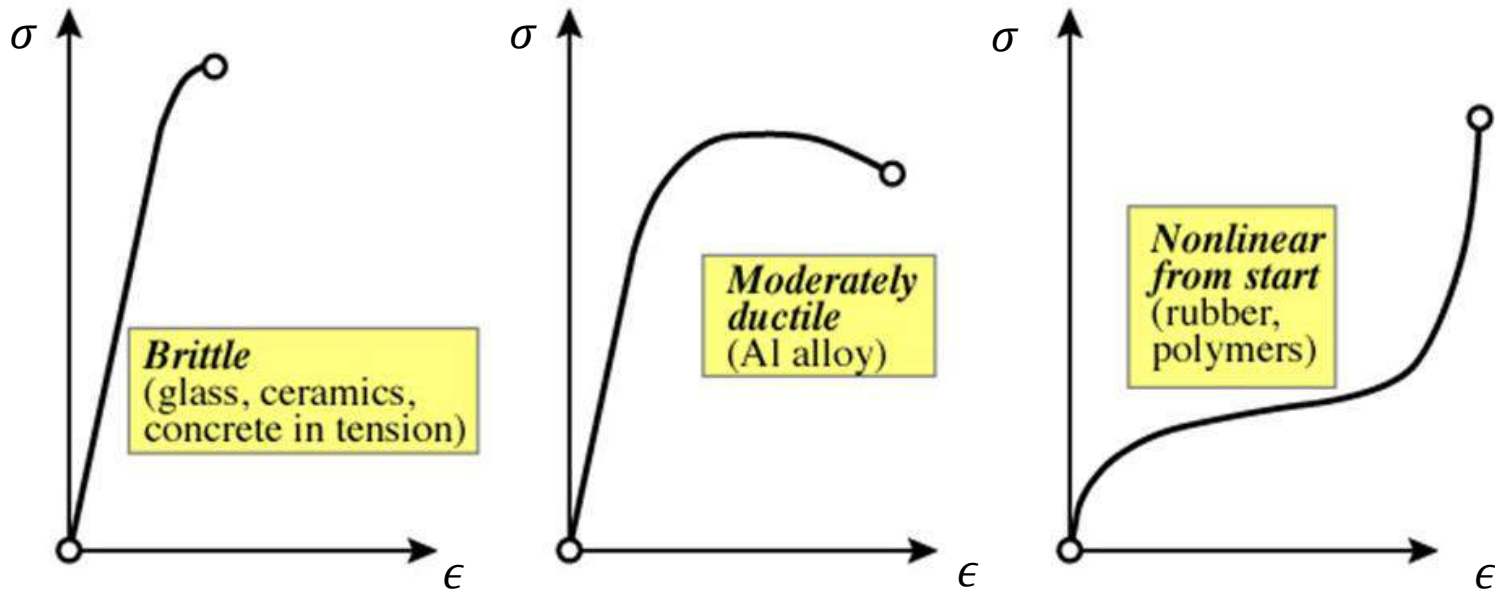
Uniaxial (1D) stress-strain curves

Strains and Stresses

In 3D the constitutive equations in linear elasticity read

$$\boldsymbol{\sigma} = \boldsymbol{C}\boldsymbol{\epsilon}$$

where $\boldsymbol{C}$ is the Elasticity matrix, $\boldsymbol{\sigma}$ is the vector collecting the stress components and $\boldsymbol{\epsilon}$ is the vector of the strain components

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66} \end{bmatrix} \begin{bmatrix} \epsilon_{11} \\ \epsilon_{22} \\ \epsilon_{33} \\ 2\epsilon_{12} \\ 2\epsilon_{13} \\ 2\epsilon_{23} \end{bmatrix}$$

Alike $\boldsymbol{\epsilon} = \boldsymbol{S}\boldsymbol{\sigma}$, where $\boldsymbol{S} = \boldsymbol{C}^{-1}$ is the Compliance matrix. In the general case the material has a $\boldsymbol{C}$ matrix characterized by 36 indipendent coefficients.

Strains and Stresses

MATERIAL NAME	INDEPENDENT COEFFICIENTS	NOTE
TRICLINIC (GENERAL ANISOTROPIC)	21	The material has NO planes of symmetry, i.e. the material properties differ in all directions. It is possible to prove that the elasticity and compliance tensor are symmetric. The number of independent coefficients reduces to 21.
MONOCLINIC	13	The material has 1 plane of symmetry . Number of coefficients reduces to 13.
ORTHOTROPIC	9	The material has 3 mutually perpendicular planes of symmetry. This implies no interaction between normal and shear stresses and strains. Number of coefficients reduces to 9.
TRANSVERSELY ISOTROPIC	5	The material has one plane in which material properties are independent of the orientation. If $x_1 - x_2$ is the plane, subscripts 1 and 2 are interchangeable. Number of coefficients reduces to 5.
ISOTROPIC	2	The material has infinite planes of symmetry, i.e. the material properties are independent of orientation. All subscripts are interchangeable.

Strains and Stresses

For an **Isotropic material**:

$$C = \frac{E}{(1-2v)(1+v)}\begin{bmatrix} 1-v & v & v & 0 & 0 & 0 \\ v & 1-v & v & 0 & 0 & 0 \\ v & v & 1-v & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2v) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2v) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2v) \end{bmatrix}.$$

$$S = C^{-1} = \frac{1}{E}\begin{bmatrix} 1 & -v & -v & 0 & 0 & 0 \\ -v & 1 & -v & 0 & 0 & 0 \\ -v & -v & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+v) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+v) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+v) \end{bmatrix}$$

where E is the Young's modulus, v is the Poisson's ratio

Stresses and Forces

Cauchy Principle states that *upon any surface that divides the body, the action of one part of the body on the other is equivalent (equipollent) to the system of distributed forces and couples on the surface dividing the body.*

Stress Vector:

$$T(n, x)$$

Body force:

$$f$$

Surface force:

$$P$$
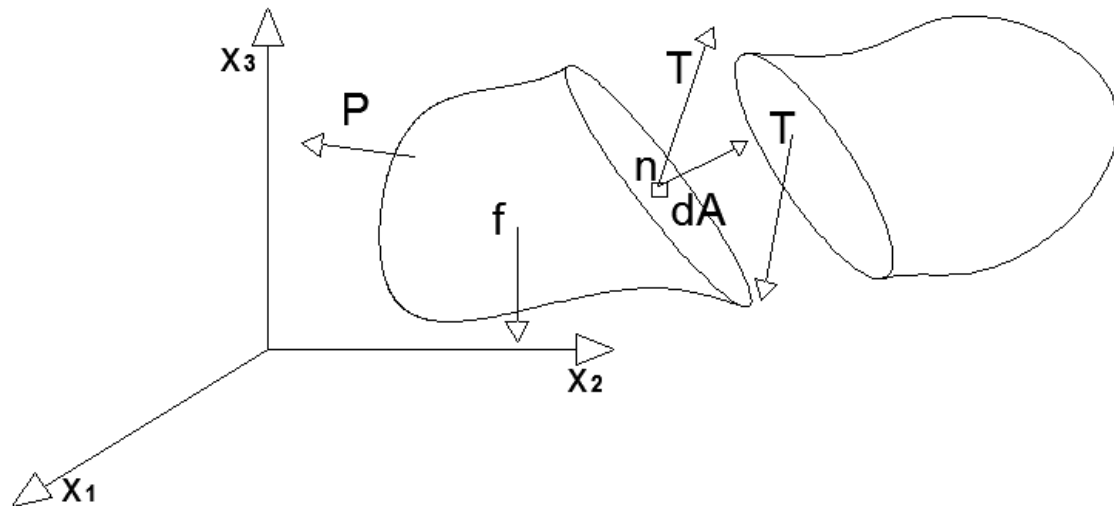
Stresses and Forces

The stress vector T, in general, has a component σ_n along the normal n to the surface, and two tangential components, τ_c and τ_b, on the surface.

$$\sigma_n = T(n, x)\, n$$

$$\tau_c = T(n, x)\, c$$

$$\tau_b = T(n, x)\, b$$

$$\tau = \sqrt{\tau_c^2 + \tau_b^2}$$
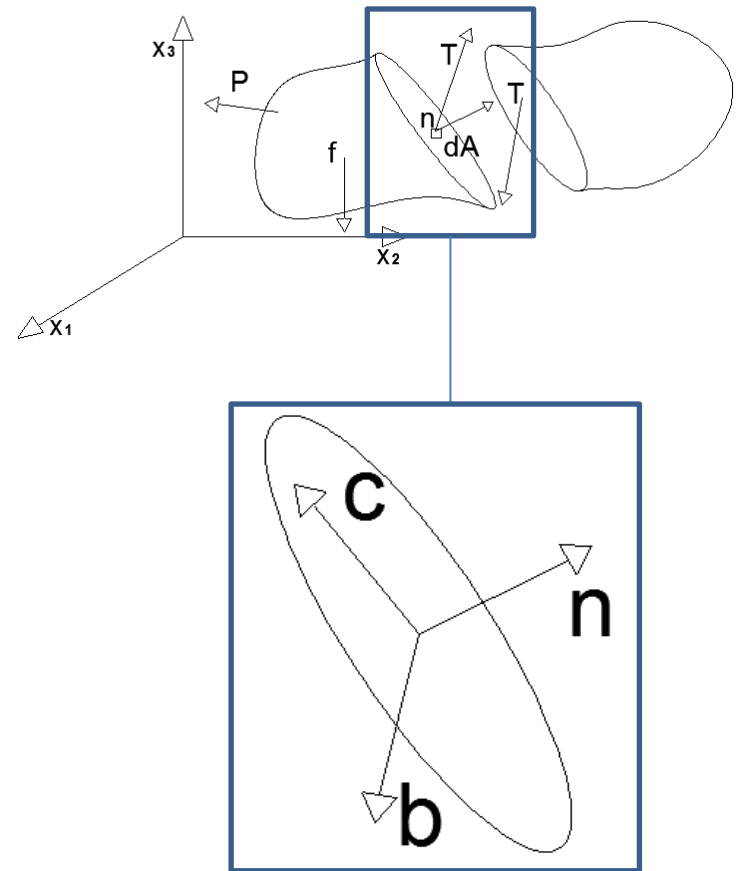
Stresses and Forces

The stress vector T can be related to the the Cauchy stress tensor

$$\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

as

$$T(n, x) = \sigma n = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$
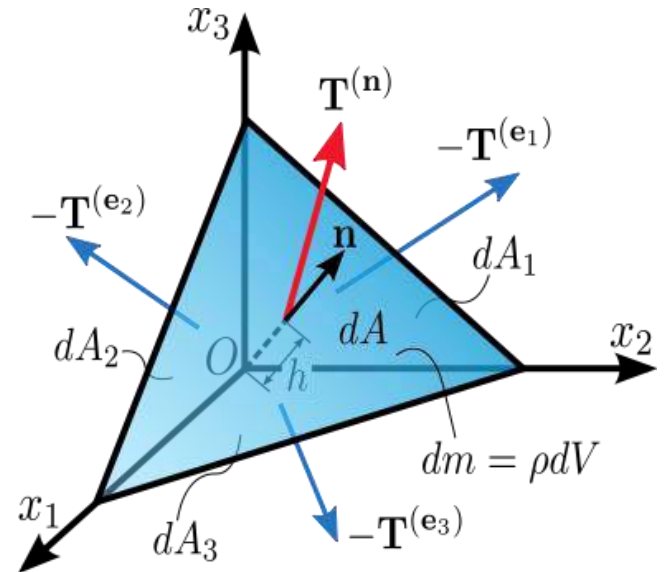
Note: from the balance of angular moment it can be proved that $\sigma_{12}=\sigma_{21}$, $\sigma_{13}=\sigma_{31}$ and $\sigma_{23}=\sigma_{32}$, i.e. the stress tensor has 6 indipendent components.
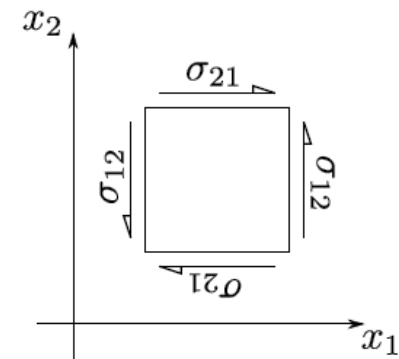
Stresses and Forces

The **equilibrium equations** link forces and stresses:

$$div \; \boldsymbol{\sigma} + \boldsymbol{f} = \boldsymbol{0}$$

scalar equations:

$$\frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} + f_1 = 0$$

$$\frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} + f_2 = 0$$

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} + f_3 = 0$$

on the boundary:

$$\boldsymbol{P} = \boldsymbol{\sigma n}$$

Stresses and Forces

In 3D for a given stress tensor

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

the principal stresses σ_a and principal directions of stress $\boldsymbol{a} \equiv [a_1, a_2, a_3]$, i.e. those directions in which there exist only axial stress and no shear stress, can be found solving the eigenvalue problem:

$$(\boldsymbol{\sigma} - \sigma_a \boldsymbol{I})\boldsymbol{a} = 0$$

$\boldsymbol{\sigma}$ 2nd order strain tensor

σ_a principal strain

$\boldsymbol{a}$ principal directions

$\boldsymbol{I}$ identity matrix

quantities to be detrmined

PART III

Internal Forces

Outline of PART III

- Internal forces:
  - Area
  - First Moments of Area
  - Centroid
  - Second Moments of Area
  - Transfer of Axis Theorem and Rotation of Axes
  - Principal Axes and Central Ellipse of Inertia.

Introduction 1/2

- A beam is in equilibrium under the action of external forces.

- Each portion of the beam must be in equilibrium under the action of external forces.

- If we cut the beam, the equilibrium of each portion is ensured by a distribution of stresses equivalent to a force $\underline{S}$ applied at the centroid of the cross-section and a couple whose moment is $\underline{M}$.
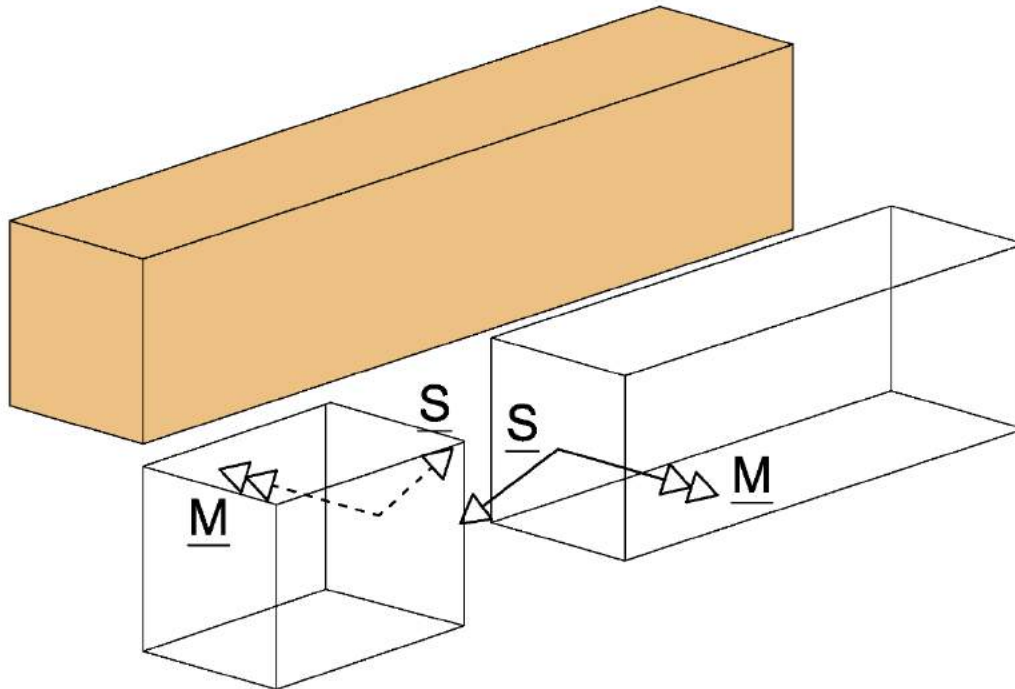
Note:
The external forces are not shown!

Introduction 2/2

The components of force <u>S</u> with respect to the Cartesian axes:

- P axial force;

- V_x shear force in the x direction;

- V_y shear force in the y direction;
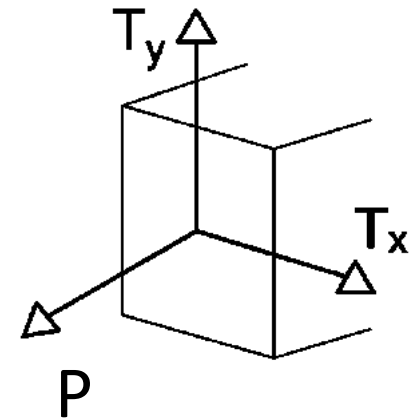
The components of the moment <u>M</u> of the couple with respect to the Cartesian axes:

- M_y bending moment;

- M_z bending moment;

- M_x torsion;
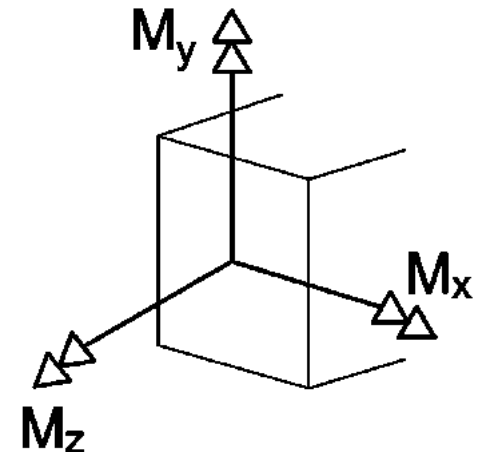
Axial Loading

Equilibrium:

- The resultant of the internal forces S for an axially loaded member is normal to a section cut perpendicular to the member axis. S=N.

- $M = 0 \rightarrow M_x, M_y, M_z = 0$;

- The only non-zero internal force is N.

- $N > 0$ tension; $N < 0$ compression.

  Axial Force:
  $$N = P$$

  Normal stress:
  $$\sigma = \frac{P}{A}$$

(a)   (b)
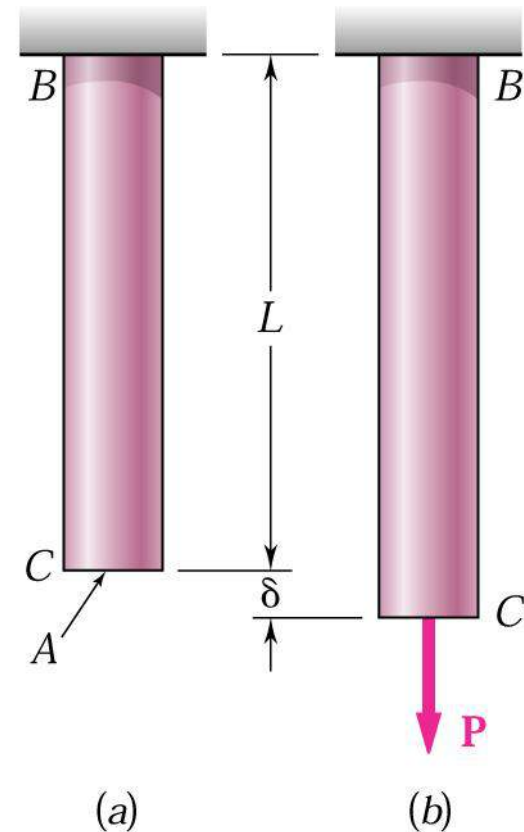
Axial Loading

Displacement and strains:

- The Cross-section translates along the beam axis and remains normal to the axis.

- The total elongation is δ

Axial Strain:

$$\varepsilon = \frac{\delta}{L}$$

(a)   (b)

Axial Loading

Stress-strain Curve: Ductile Material
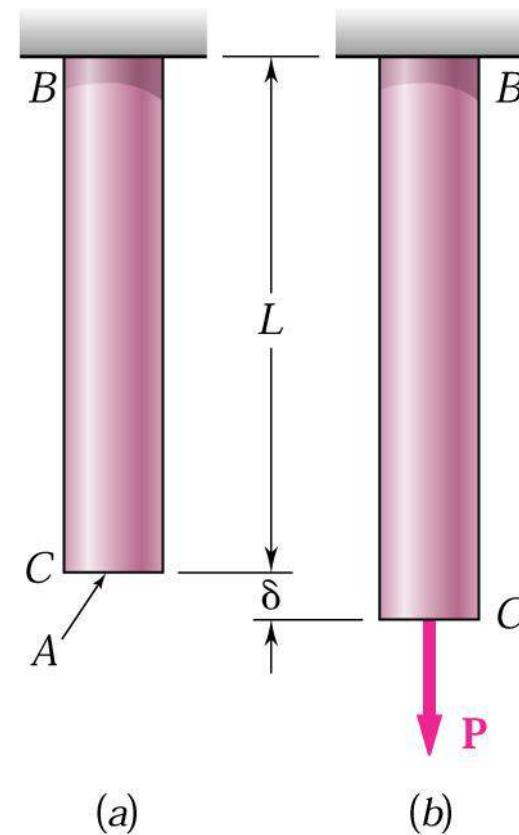
Constitutive Equation:
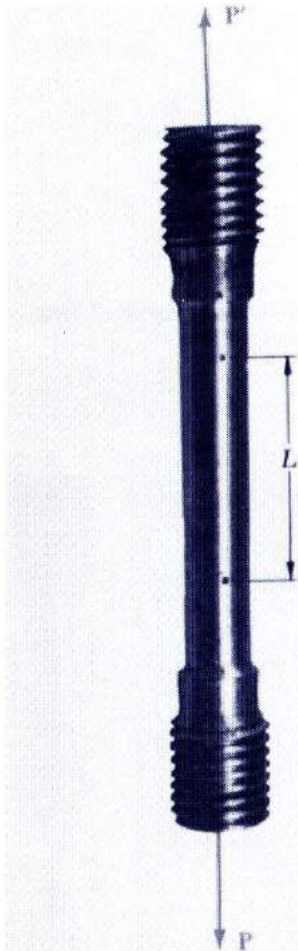
Fig. 2.8 Test specimen with tensile load.

σ_U 60

σ_Y 40

20

0.0012

Yield E Strain-hardening Necking

0.02 0.2 0.25

Rupture

σ_B

ϵ

σ
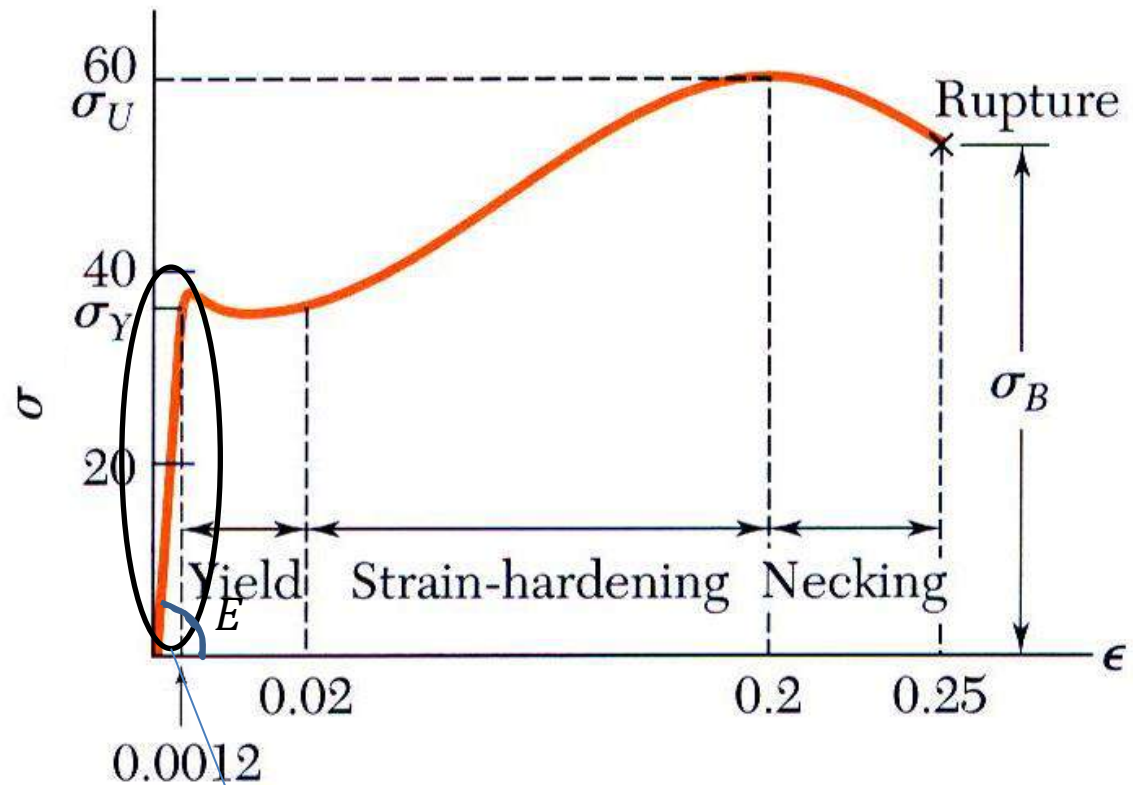
(a) Low-carbon steel

Linear Elasticity

Hooke's law:
$$\sigma = E\varepsilon$$

Axial Loading

Deformation under axial loading
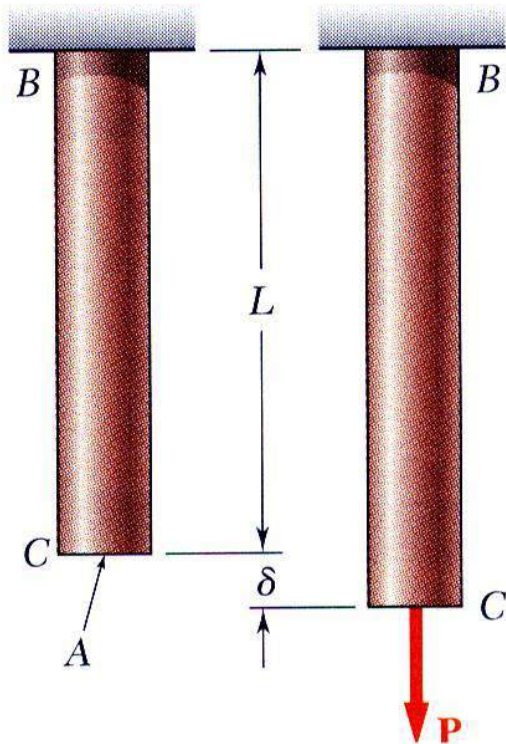
- From Hooke's Law:

$$\sigma = \mathrm{E}\varepsilon \qquad \varepsilon = \frac{\sigma}{\mathrm{E}}$$

- From Equilibrium:

$$\sigma = \frac{P}{A} \qquad \varepsilon = \frac{P}{\mathrm{E}A}$$

- From the definition of strain:

$$\varepsilon = \frac{\delta}{L}$$

- Equating and solving for the deformation,

$$\delta = \frac{PL}{AE}$$

Pure Bending Loading

Equilibrium:

- The beam is subjected to equal and opposite couples, whose moment is M, acting in the same longitudinal plane.

- The internal forces in the generic cross-section must satisfy the condition.

$$F_x = \int \sigma_x \, dA = 0$$
$$M_y = \int z\sigma_x \, dA = 0$$
$$M_z = \int -y\sigma_x \, dA = M$$

Pure Bending Loading

Displacement and Strains 1/2
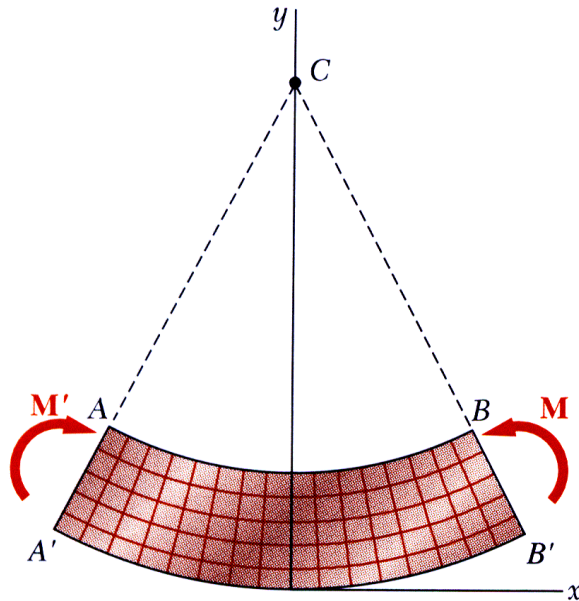
(a) Longitudinal, vertical section
(plane of symmetry)
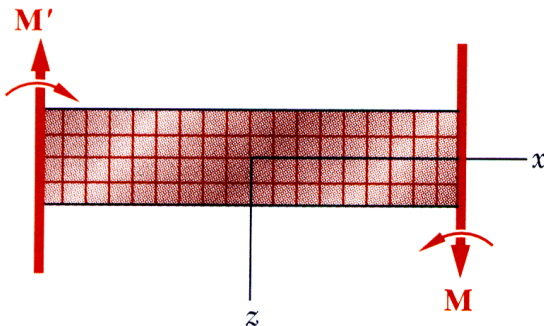
(b) Longitudinal, horizontal section

2D Beam in pure bending:

- member remains symmetric

- bends uniformly to form a circular arc

- Cross-sections remain plane and perpendicular to the axis of the beam

- The top outermost fibers will shorten and the bottom outermost fibers will elongate

- a *neutral surface* must exist that is parallel to the upper and lower surfaces and for which the length does not change

Pure Bending Loading

Consider a beam segment of length *L*.

After deformation, the length of the neutral surface remains *L*.

At other sections:

$$L' = (\rho - y)\theta$$

$$\delta = L' - L = (\rho - y)\theta - \rho\theta = -y\theta$$

$$\varepsilon_x = \frac{\delta}{L} = -\frac{y\theta}{\rho\theta} = -\frac{y}{\rho} \quad \text{(strain varies linearly)}$$

$$\varepsilon_m = \frac{c}{\rho} \quad \text{or} \quad \rho = \frac{c}{\varepsilon_m}$$

$$\varepsilon_x = -\frac{y}{c}\varepsilon_m$$

Pure Bending Loading

Navier's Formula:
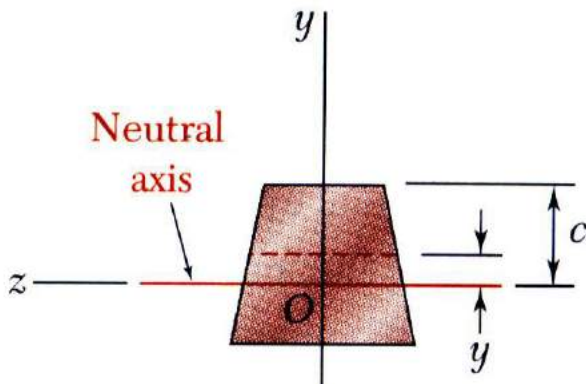
Constitutive Equation:

- For a linearly elastic material:

$$\sigma_x = E\varepsilon_x = -\frac{y}{c}E\varepsilon_m$$

$$= -\frac{y}{c}\sigma_m \quad \text{(stress varies linearly)}$$

σ_m

c

Neutral surface

σ_x

- 1) Static equilibrium:

$$F_x = 0 = \int \sigma_x \, dA = \int -\frac{y}{c}\sigma_m \, dA$$

$$0 = -\frac{\sigma_m}{c}\int y \, dA$$

First moment with respect to neutral plane is zero

$$\int y \, dA = S_z = 0$$

The neutral surface must pass through the centroid of the cross-section.
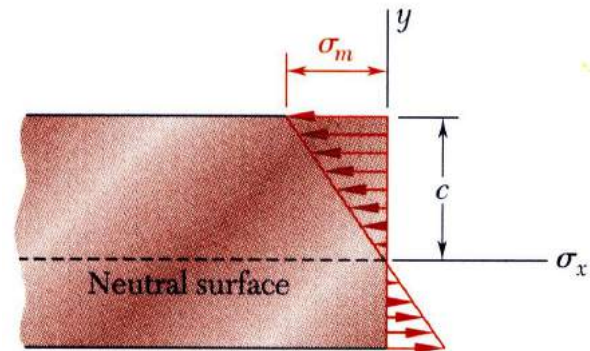
- 2) Static equilibrium:

$$M = \int -y\sigma_x \, dA = \int -y\left(-\frac{y}{c}\sigma_m\right)dA$$

$$M = \frac{\sigma_m}{c}\int y^2 \, dA = \frac{\sigma_m I}{c}$$

$$\sigma_m = \frac{Mc}{I}$$

Substituting $\quad \sigma_x = -\frac{y}{c}\sigma_m$

$$\sigma_x = -\frac{M}{I}y$$

Navier's Formula

Pure Bending Loading

A

Maximum normal stress & Section Modulus:

h

b
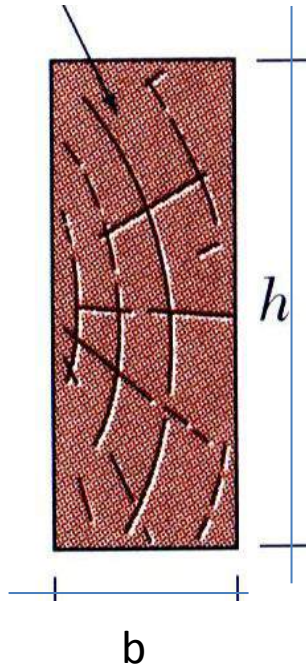
- The maximum normal stress due to bending,

$$\sigma_m = \frac{M}{I}c = \frac{M}{S}$$

I = section moment of inertia

$$W = \frac{I}{c} = \text{section modulus}$$

- For a rectangular beam cross section,

$$W = \frac{I}{c} = \frac{\frac{1}{12}bh^3}{h/2} = \tfrac{1}{6}bh^2 = \tfrac{1}{6}Ah$$

N. A.
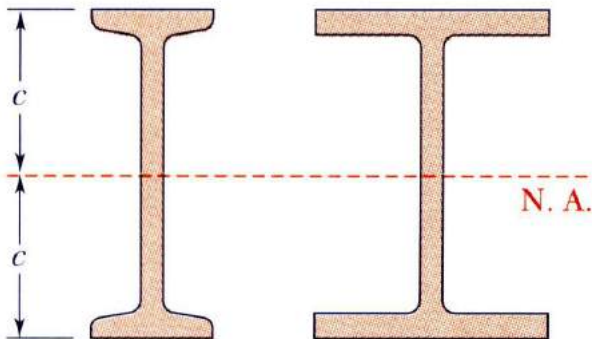
- Structural steel beams (I beams and H beams) are designed to have a large section modulus.

Pure Bending Loading

Deformation under pure bending

- Deformation due to bending moment M is quantified by the curvature of the neutral surface
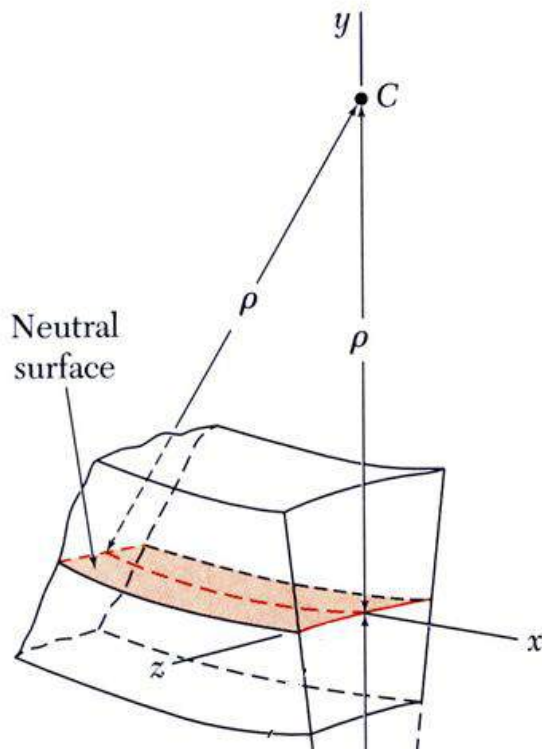
$$\chi = \frac{1}{\rho} = \frac{\varepsilon_m}{c} = \frac{\sigma_m}{Ec} = \frac{1}{Ec}\frac{Mc}{I}$$

$$\chi = \frac{M}{EI}$$

Eccentric Axial Loading in a Plane of Symmetry

(a)

(b)
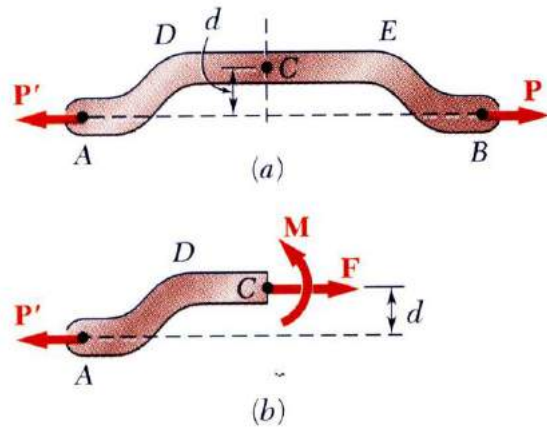
- The eccentric loading determines an axial force F and a couple Pd at cross section C.

$$F = P$$

$$M = Pd$$

- Principle of superposition: stress distribution due to eccentric loading is determined by superposing the uniform stress due to a centric load and the linear stress distribution due to pure bending

$$\sigma_x = (\sigma_x)_{\text{centric}} + (\sigma_x)_{\text{bending}}$$

$$= \frac{P}{A} - \frac{My}{I}$$
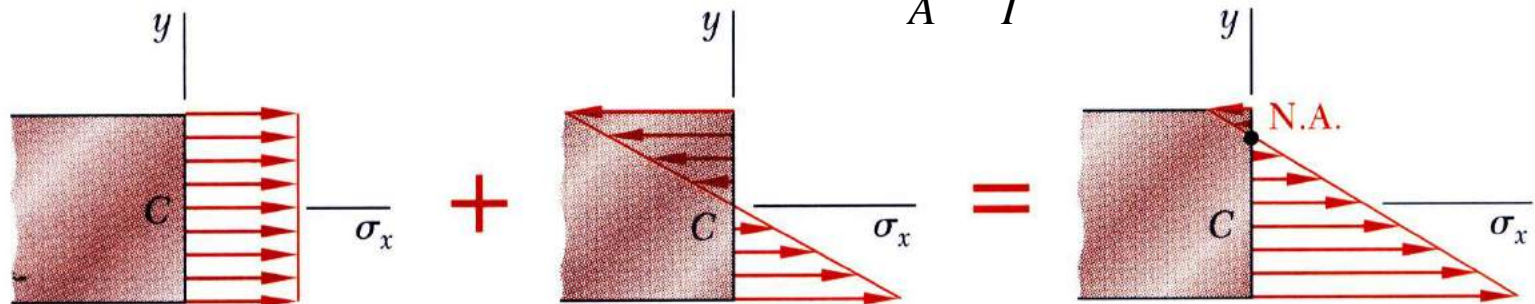
Unsymmetric bending
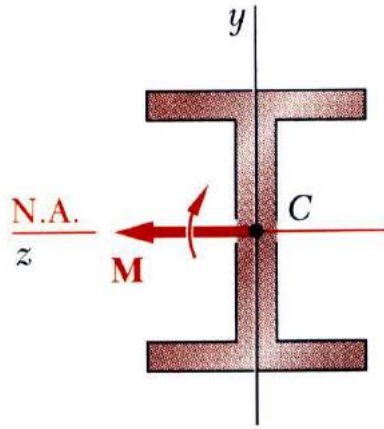
- Analysis of pure bending is limited to members subjected to bending couples acting in a plane of symmetry.

- Members remain symmetric and bend in the plane of symmetry and the neutral axis of the cross-section coincides with the axis of the couple.
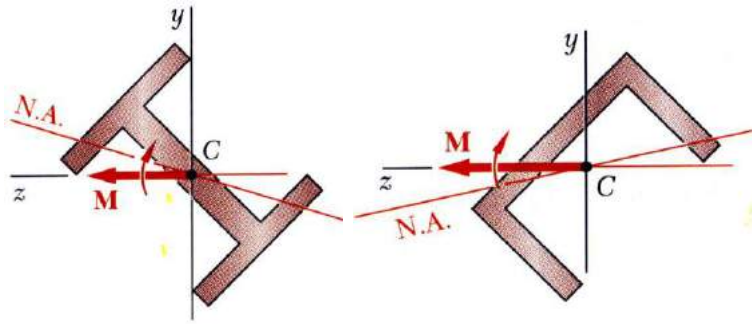
- For situations in which the bending couples do not act in a plane of symmetry, the neutral axis of the cross-section will not coincide with the axis of the couple (x) and the beam will not bend in the plane of the couple (y-z).
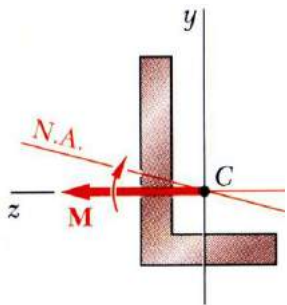
Unsymmetric bending
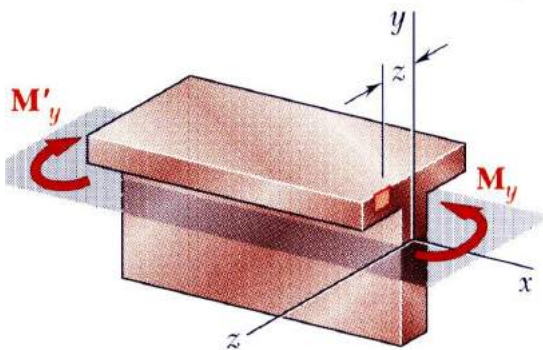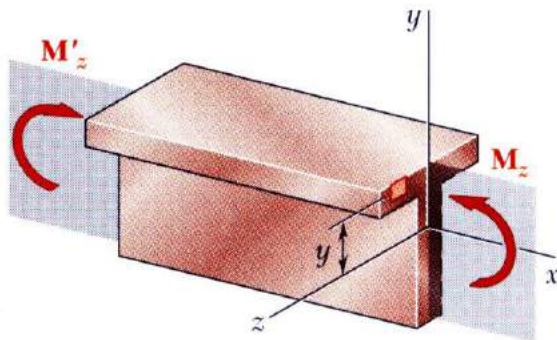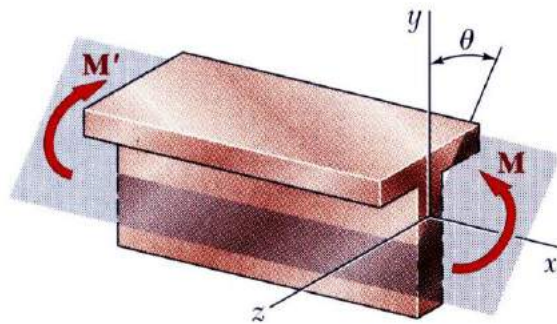
Principle of superposition :

- Resolve the couple vector into components along the centroidal principal axes.

$$M_z = M\cos\theta \qquad M_y = M\sin\theta$$

- Superpose the stress distributions

$$\sigma_x = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- The neutral axis is found by enforcing:

$$\sigma_x = 0 = -\frac{M_z y}{I_z} + \frac{M_y z}{I_y} = -\frac{(M\cos\theta)y}{I_z} + \frac{(M\sin\theta)z}{I_y}$$
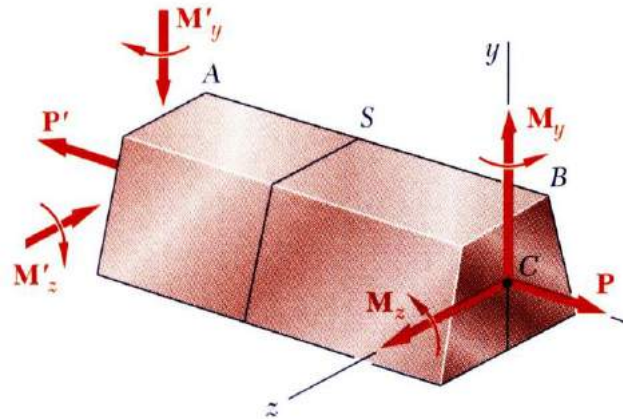
$$\tan\phi = \frac{y}{z} = \frac{I_z}{I_y}\tan\theta$$
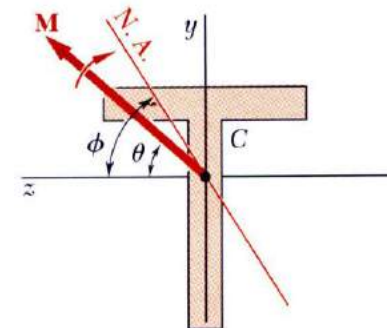
General Case of Eccentric Axial Loading
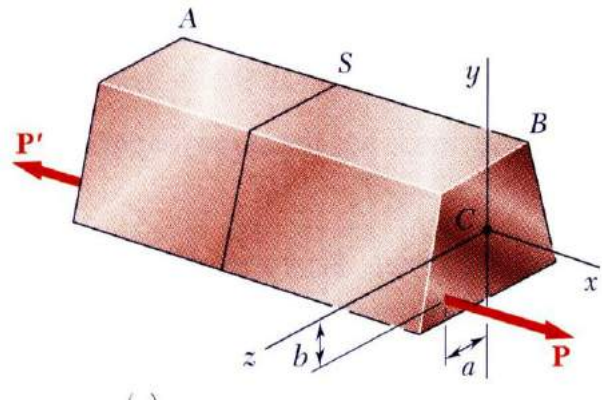
- Consider a straight member subject to equal and opposite eccentric forces.

- The eccentric force is equivalent to the system of a centric force and two couples.

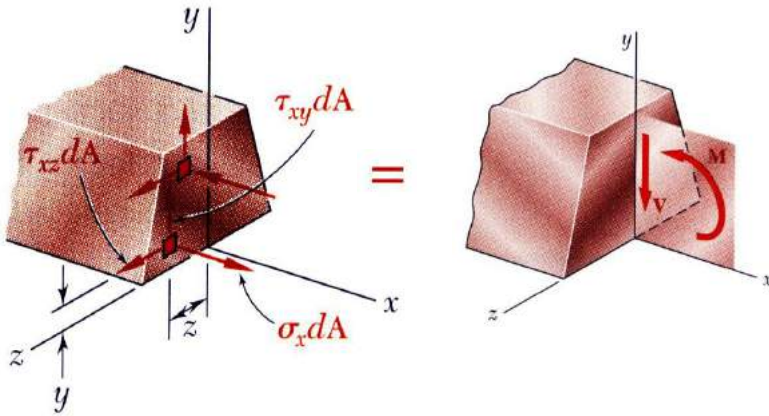$$P = \text{centric force}$$

$$M_y = Pa \qquad M_z = Pb$$

- By the principle of superposition, the combined stress distribution is

$$\sigma_x = \frac{P}{A} - \frac{M_z y}{I_z} + \frac{M_y z}{I_y}$$

- If the neutral axis lies on the cross-section, it may be found from

$$\frac{M_z}{I_z} y - \frac{M_y}{I_y} z = \frac{P}{A}$$

Shear Flow on the Horizontal Face

- Transverse loading applied to a beam results in normal and shearing stresses in transverse sections.

- Distribution of normal and shearing stresses satisfies

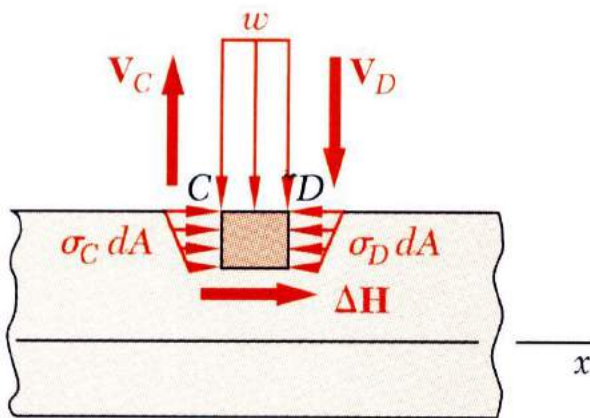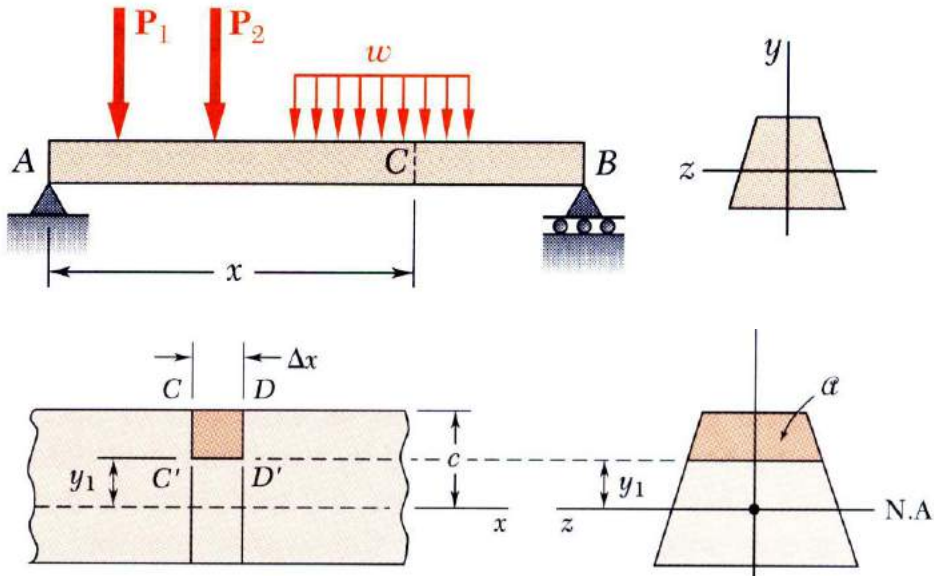$$F_x = \int \sigma_x dA = 0 \qquad M_x = \int (y\tau_{xz} - z\tau_{xy})dA = 0$$

$$F_y = \int \tau_{xy} dA = -V \qquad M_y = \int z\sigma_x dA = 0$$

$$F_z = \int \tau_{xz} dA = 0 \qquad M_z = \int (-y\sigma_x) = 0$$
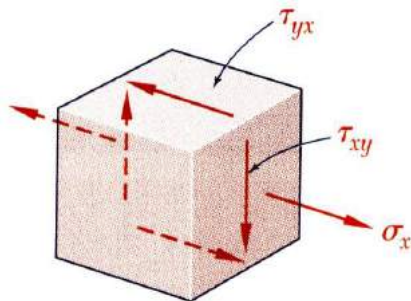
- By reciprocity of shear stress, when shearing stresses are exerted on the vertical faces of an element, equal stresses must be exerted on the horizontal faces

- Longitudinal shearing stresses must exist in any member subjected to transverse loading.

Shear flow 1/2

- For equilibrium of beam element

$$\sum F_x = 0 = \Delta H + \int_A (\sigma_C - \sigma_D) dA$$

$$\Delta H = \frac{M_D - M_C}{I} \int_A y \, dA$$
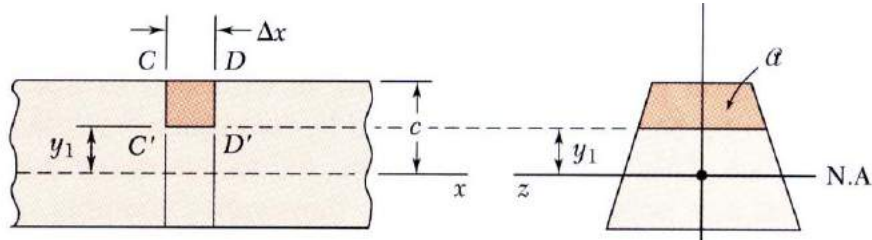
- Note:

$$S_z = \int_A y \, dA$$

$$M_D - M_C = \frac{dM}{dx} \Delta x = V \Delta x$$

- Substituting:

$$\Delta H = \frac{VQ}{I} \Delta x$$

$$q = \frac{\Delta H}{\Delta x} = \frac{VS_z}{I} = shear \ flow$$

Shear flow 2/2

- Shear flow,

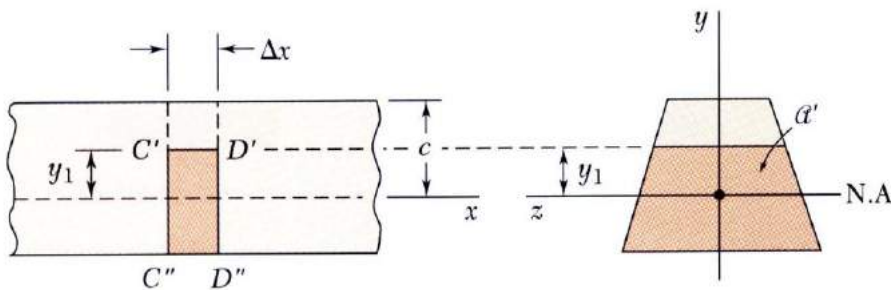$$q = \frac{\Delta H}{\Delta x} = \frac{V S_z}{I} = shear\ flow$$

- where

$$S_z = \int_A y\, dA$$

$= $ first moment of area above y_1

$$I = \int_{A+A'} y^2\, dA$$

$= $ second moment of full cross section
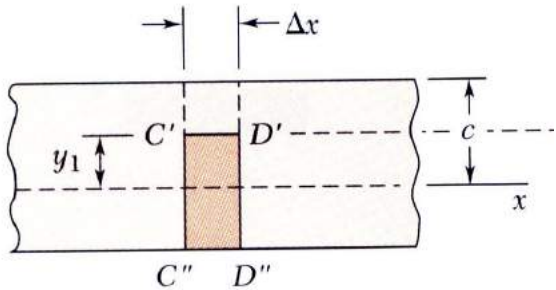
- Same result found for lower area

$$q' = \frac{\Delta H'}{\Delta x} = \frac{V S'}{I} = -q'$$

$$S + S' = 0$$

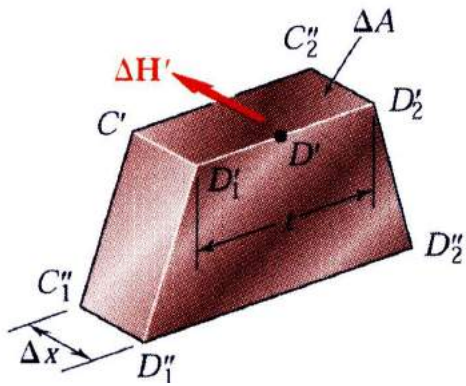$= $ first moment with respect
to neutral axis

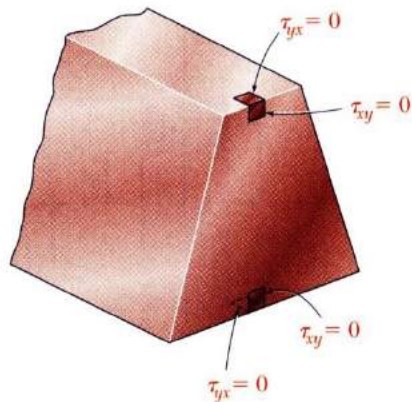$$\Delta H' = -\Delta H$$

Shear stress

- The *average* shearing stress on the horizontal face of the element is obtained by dividing the shearing force on the element by the area of the face.
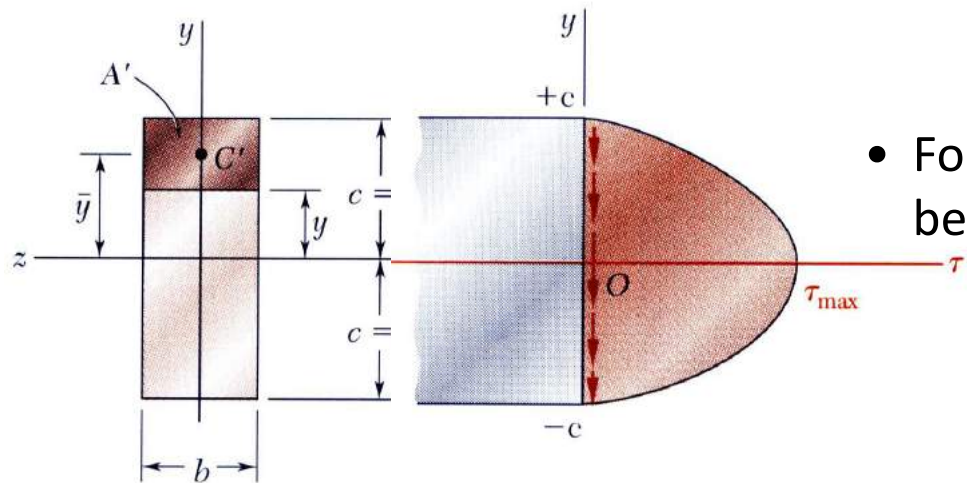
$$\tau_{ave} = \frac{\Delta H}{\Delta A} = \frac{q\,\Delta x}{\Delta A} = \frac{VS_z}{I_z}\frac{\Delta x}{b\,\Delta x}$$

$$= \frac{VS_z}{I_z b}$$

- On the upper and lower surfaces of the beam, $t_{yx} = 0$. It follows that $t_{xy} = 0$ on the upper and lower edges of the transverse sections.
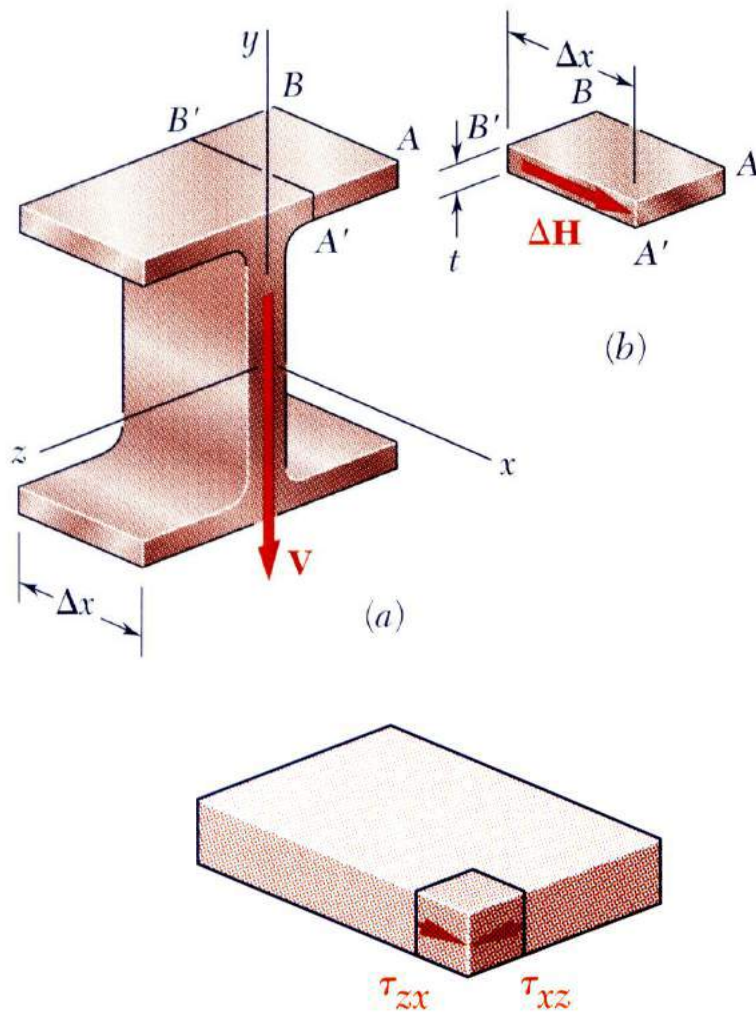
Shear stress: Example

- For a narrow rectangular beam:

$$\tau_{xy} = \frac{VS_z}{I_z b} = \frac{3}{2}\frac{V}{A}\left(1 - \frac{y^2}{c^2}\right)$$

$$\tau_{max} = \frac{3}{2}\frac{V}{A}$$

Shearing Stresses in Thin-Walled Members

(a)

(b)

- Consider a segment of a wide-flange beam subjected to the vertical shear V.

- The longitudinal shear force on the element is

$$\Delta H = \frac{VS}{I} \Delta x$$

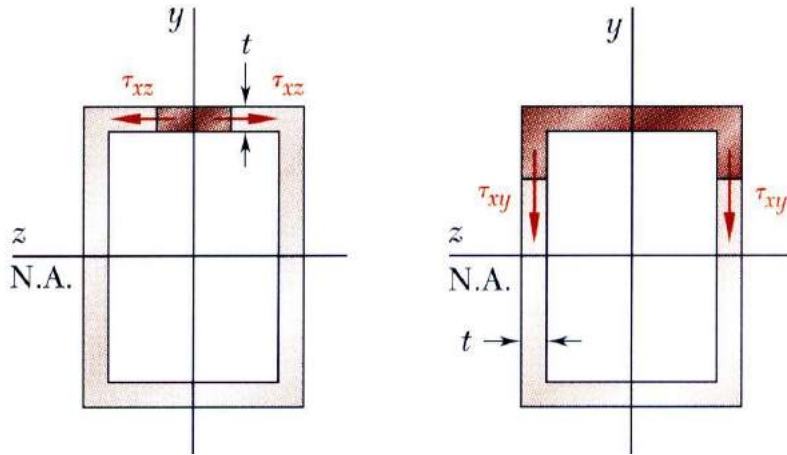- The corresponding shear stress is

$$\tau_{zx} = \tau_{xz} \approx \frac{\Delta H}{t\,\Delta x} = \frac{VS}{It}$$

- Previously found a similar expression for the shearing stress in the web
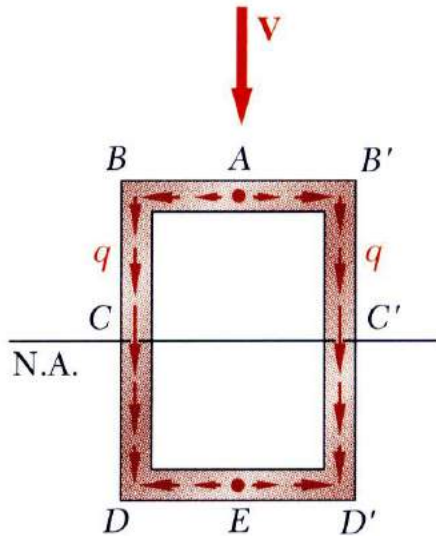
$$\tau_{xy} = \frac{VS}{It}$$

- NOTE: $\tau_{xy} \approx 0$ in the flanges

  $\tau_{xz} \approx 0$ in the web

Shearing Stresses in Thin-Walled Members

- The variation of shear flow across the section depends only on the variation of the first moment.
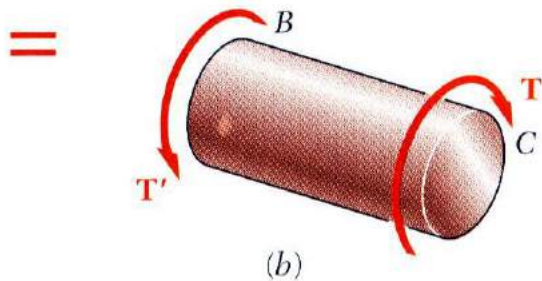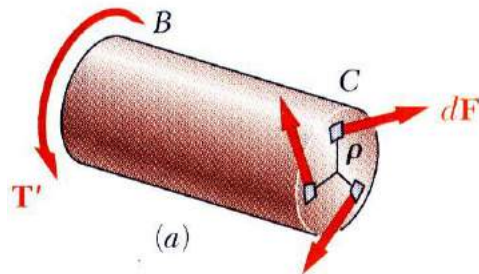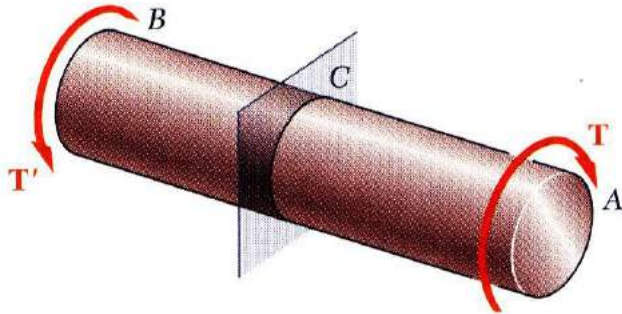
$$q = \tau t = \frac{VS}{I}$$

- For a box beam, q grows smoothly from zero at A to a maximum at C and C' and then decreases back to zero at E.
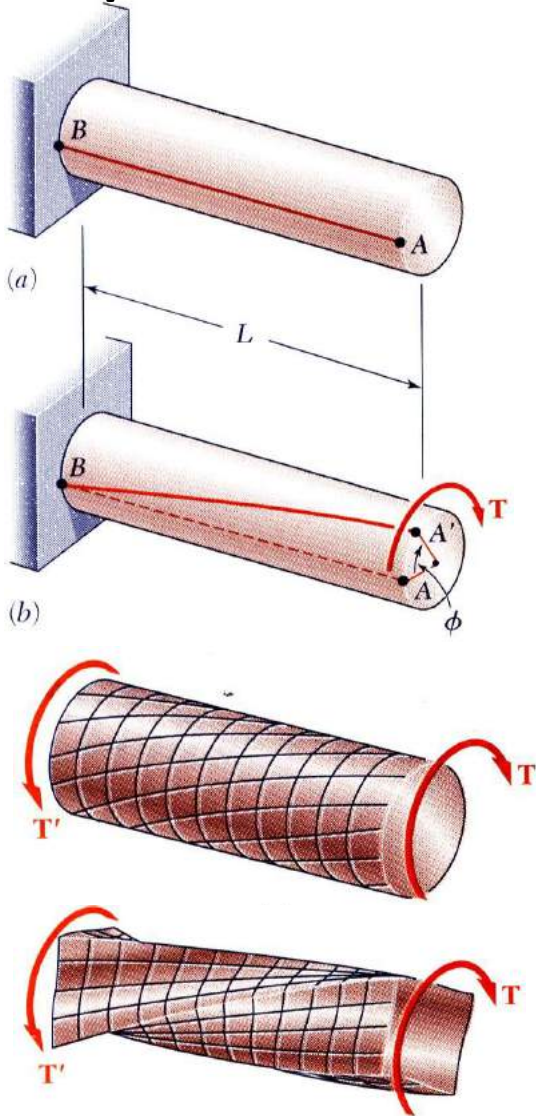
Torque

Equilibrium:

- Net of the internal shearing stresses is an internal torque, equal and opposite to the applied torque

$$T = \int \rho \, dF = \int \rho(\tau \, dA)$$

- Although the net torque due to the shearing stresses is known, the distribution of the stresses is not

- Distribution of shearing stresses is statically indeterminate – must consider beam deformations

- Unlike the normal stress due to axial loads, the distribution of shearing stresses due to torsional loads can not be assumed uniform.

Torque

Displacement

- The angle of twist is proportional to the applied torque and to the element length.

$$\phi \propto T$$

$$\phi \propto L$$

- When subjected to torsion, every cross-section of a **circular beam** remains plane and undistorted.
- Cross-sections for hollow and solid circular beam remain plain and undistorted because a circular beam is axisymmetric.

- Cross-sections of noncircular (non-axisymmetric) beam are distorted when subjected to torsion (warping displacement).

Torque
Shearing Strain

(a)

(b)

(c)
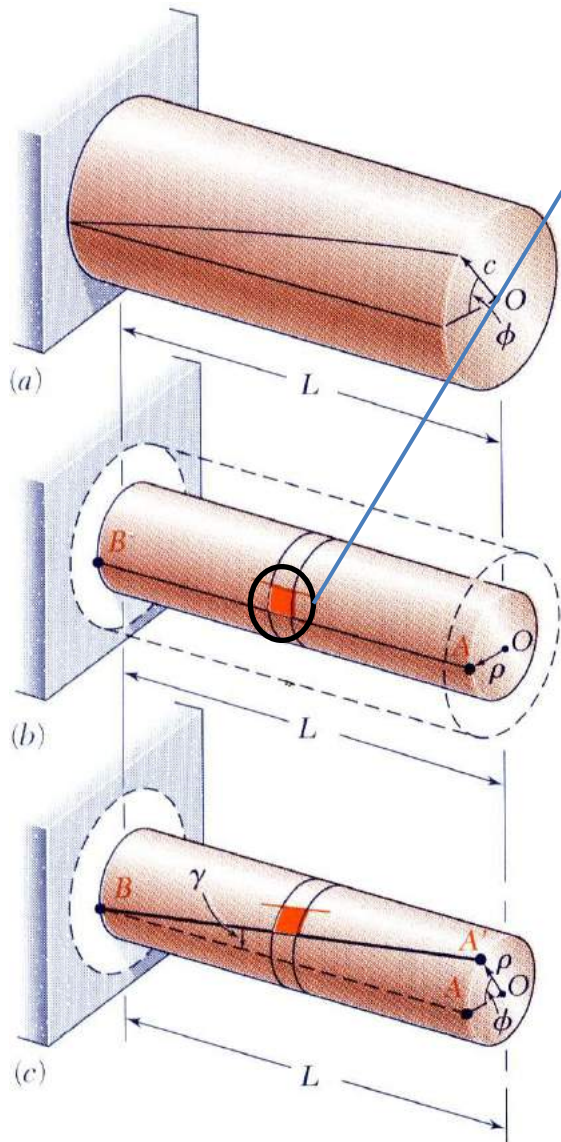
- Consider an interior section of the beam. As a torsional load is applied, an element on the interior cylinder deforms into a rhombus.

- Since the ends of the element remain planar, the shear strain is equal to angle of twist.

- It follows that

$$L\gamma = \rho\phi \quad \text{or} \quad \gamma = \frac{\rho\phi}{L}$$

- Shear strain is proportional to twist and radius

$$\gamma_{max} = \frac{c\phi}{L} \quad \text{and} \quad \gamma = \frac{\rho}{c}\gamma_{max}$$

Torque

$$J = \frac{1}{2}\pi c^4$$

$$J = \frac{1}{2}\pi\left(c_2^4 - c_1^4\right)$$

Constitutive Equation:

- Multiplying the previous equation by the shear modulus

$$G\gamma = \frac{\rho}{c}G\gamma_{max}$$

From Hooke's Law, $\tau = G\gamma$, so

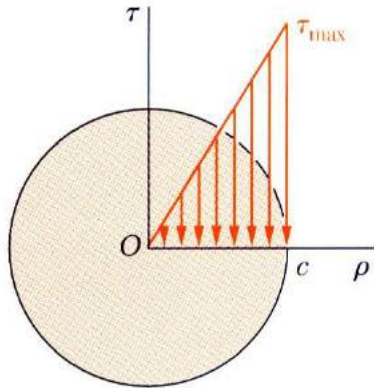$$\tau = \frac{\rho}{c}\tau_{max}$$

The shearing stress varies linearly with the radial position in the section.

- Recall that the sum of the moments from the internal stress distribution is equal to the torque on the beam at the section,
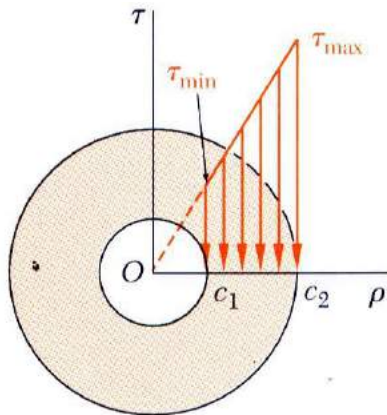
$$T = \int \rho\tau \, dA = \frac{\tau_{max}}{c}\int \rho^2 \, dA = \frac{\tau_{max}}{c}J$$

- The results are known as the elastic torsion formulas,

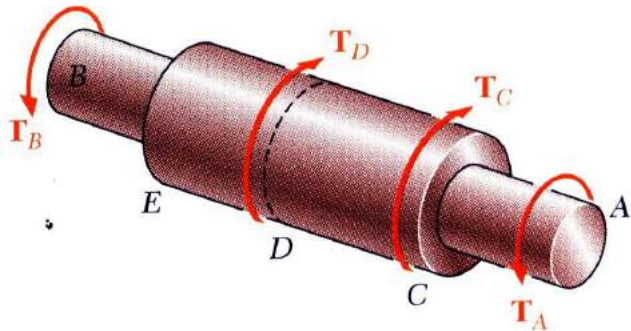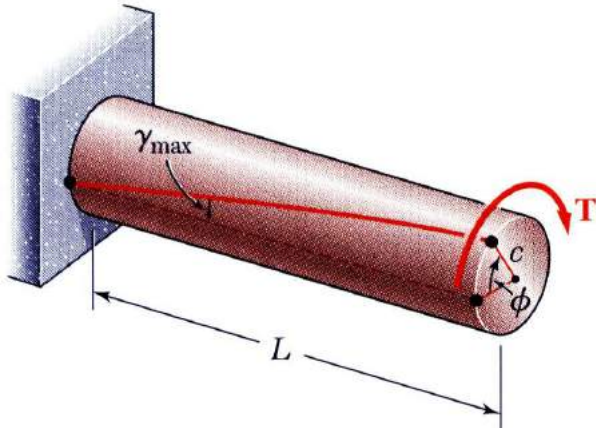$$\tau_{max} = \frac{Tc}{J} \quad \text{and} \quad \tau = \frac{T\rho}{J}$$

Deformation under Torque

- Recall that the angle of twist and maximum shearing strain are related,

$$\gamma_{\max} = \frac{c\phi}{L}$$

- In the elastic range, the shearing strain and shear are related by Hooke's Law,

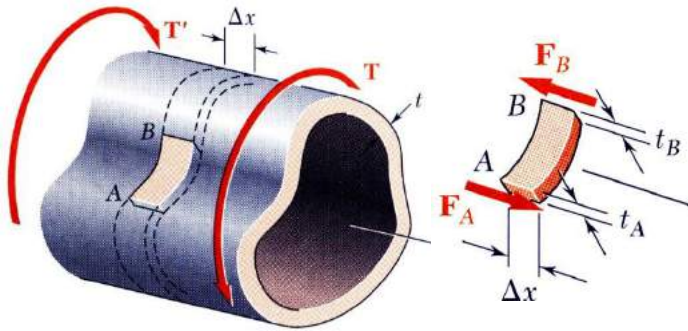$$\gamma_{\max} = \frac{\tau_{\max}}{G} = \frac{Tc}{JG}$$

- Equating the expressions for shearing strain and solving for the angle of twist,

$$\phi = \frac{TL}{JG}$$

- If the torsional loading or beam cross-section changes along the length, the angle of rotation is found as the sum of segment rotations

$$\phi = \sum_i \frac{T_i L_i}{J_i G_i}$$
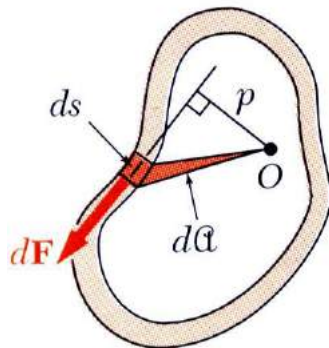
Thin-Walled Hollow section

- Summing forces in the x-direction on *AB*,

$$\sum F_x = 0 = \tau_A(t_A\Delta x) - \tau_B(t_B\Delta x)$$

$$\tau_A t_A = \tau_B t_B = \tau t = q = \text{shear flow}$$
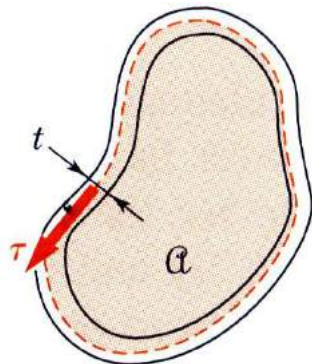
  shear stress varies inversely with thickness

- Compute the beam torque from the integral of the moments due to shear stress

$$dM_0 = p\,dF = p\,\tau(t\,ds) = q(p\,ds) = 2q\,dA$$

$$T = \oint dM_0 = \oint 2q\,dA = 2qA$$

$$\tau = \frac{T}{2tA}$$

- Angle of twist:

$$\phi = \frac{TL}{4A^2G}\oint\frac{ds}{t}$$
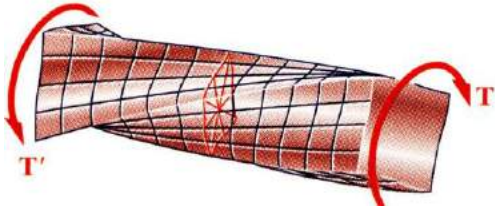
Torsion of Noncircular Members

TABLE 3.1. Coefficients for Rectangular Bars in Torsion

a/b	c_1	c_2
1.0	0.208	0.1406
1.2	0.219	0.1661
1.5	0.231	0.1958
2.0	0.246	0.229
2.5	0.258	0.249
3.0	0.267	0.263
4.0	0.282	0.281
5.0	0.291	0.291
10.0	0.312	0.312
∞	0.333	0.333

- Previous torsion formulas are valid for circular section

- Planar cross-sections of noncircular element do not remain planar and stress and strain distribution do not vary linearly

- For uniform rectangular cross-sections,

$$\tau_{\max} = \frac{T}{c_1 ab^2} \qquad \phi = \frac{TL}{c_2 ab^3 G}$$

- At large values of a/b, the maximum shear stress and angle of twist for other open sections are the same as a rectangular bar.