Basic Mathematics

Definite Integration

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The aim of this package is to provide a short self assessment programme for students who want to be able to calculate basic definite integrals.

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The full range of these packages and some instructions, should they be required, can be obtained from our web page Mathematics Support Materials.

1. Introduction

It is possible to determine a function $F(x)$ from its derivative $f(x)$ by calculating the anti-derivative or integral of $f(x)$, i.e.,

$$\text{if} \quad \frac{dF}{dx} = f(x)\,, \quad \text{then} \quad F(x) = \int f(x)dx + C$$

where C is an integration constant (see the package on **indefinite integration**). In this package we will see how to use integration to calculate the area under a curve.

As a revision exercise, try this quiz on indefinite integration.

Quiz Select the indefinite integral of $\int (3x^2 - \frac{1}{2}x)dx$ with respect to x

(a) $6x - \dfrac{1}{2} + C\,,$

(b) $\dfrac{3}{2}x^3 - x^2 + C\,,$

(c) $x^2 + \dfrac{1}{4}x^2 + C\,,$

(d) $x^3 - \dfrac{1}{4}x^2 + C\,.$

Hint: If $n \neq -1$, the integral of x^n is $x^{n+1}/(n+1)$.

2. Definite Integration

We **define** the definite integral of the function $f(x)$ with respect to x from a to b to be

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)\,,$$

where $F(x)$ is the anti-derivative of $f(x)$. We call a and b the lower and upper limits of integration respectively. The function being integrated, $f(x)$, is called the integrand. Note the minus sign!

Note integration constants are not written in definite integrals since they always cancel in them:

$$\begin{aligned}\int_a^b f(x)dx &= F(x)\Big|_a^b \\ &= (F(b) + C) - (F(a) + C) \\ &= F(b) + C - F(a) - C \\ &= F(b) - F(a)\,.\end{aligned}$$

Example 1 Calculate the definite integral $\int_1^2 x^3 dx$.

From the rule $\int ax^n dx = \dfrac{a}{n+1}x^{n+1}$ we have

$$\begin{aligned}\int_1^2 x^3 dx &= \frac{1}{3+1}x^{3+1}\Big|_1^2 \\ &= \frac{1}{4}x^4\Big|_1^2 = \frac{1}{4}\times 2^4 - \frac{1}{4}\times 1^1 \\ &= \frac{1}{4}\times 16 - \frac{1}{4} = 4 - \frac{1}{4} = \frac{15}{4}\,.\end{aligned}$$

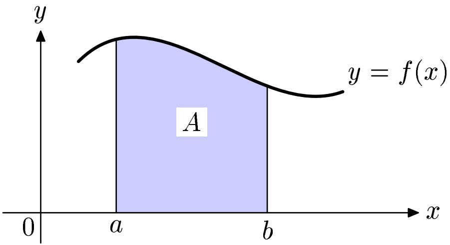
EXERCISE 1. Calculate the following definite integrals: (click on the **green** letters for the solutions)

(a) $\int_0^3 x dx$, (b) $\int_{-1}^2 x dx$,

(c) $\int_1^2 (x^2 - x) dx$, (d) $\int_{-1}^2 (x^2 - x) dx$.

3. The Area Under a Curve

The definite integral of a function $f(x)$ which lies above the x axis can be interpreted as the area under the curve of $f(x)$.
Thus the area shaded blue below



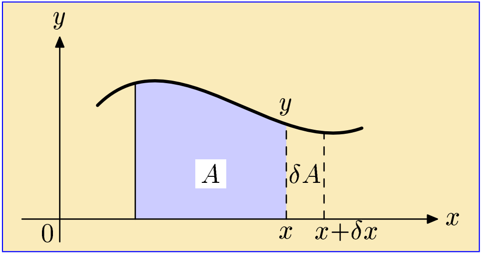
is given by the definite integral

$$\int_a^b f(x)dx = F(x)\Big|_a^b = F(b) - F(a)\,.$$

This is demonstrated on the next page.

Consider the area, A, under the curve, $y = f(x)$. If we increase the value of x by δx, then the increase in area, δA, is approximately

$$\delta A = y\,\delta x \quad \Rightarrow \frac{\delta A}{\delta x} = y\,.$$



Here we approximate the area of the thin strip by a rectangle of width δx and height y. In the limit as the strips become thin, $\delta x \to 0$, this means:

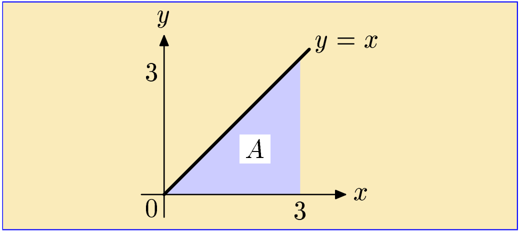
$$\frac{dA}{dx} = \lim_{\delta x \to 0} \frac{\delta A}{\delta x} = y\,.$$

The function (height of the curve) is the derivative of the area and **the area below the curve is an anti-derivative or integral of the function.**

N.B. so far we have assumed that $y = f(x)$ lies above the x axis.

Example 2 Consider the integral $\int_0^3 x dx$. The integrand $y = x$ (a straight line) is sketched below. The area underneath the line is the blue shaded triangle. The area of any triangle is half its base times the height. For the blue shaded triangle, this is

$$A = \frac{1}{2} \times 3 \times 3 = \frac{9}{2}.$$



As expected, the integral yields the same result:

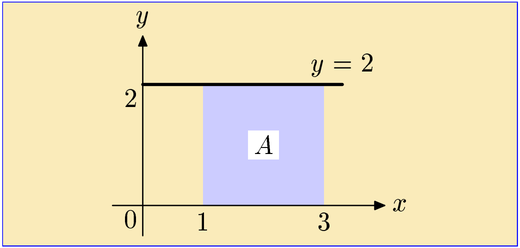
$$\int_0^3 x dx = \left.\frac{x^2}{2}\right|_0^3 = \frac{3^2}{2} - \frac{0^2}{2} = \frac{9}{2} - 0 = \frac{9}{2}.$$

Here is a quiz on this relation between definite integrals and the area under a curve.

Quiz Select the value of the definite integral

$$\int_1^3 2dx\,,$$

which is sketched in the following diagram:

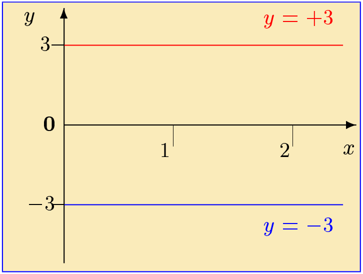


(a) $6\,$, (b) $2\,$, (c) $4\,$, (d) $8\,$.

Hint: 2 may be written as $2x^0$, since $x^0 = 1$.

Example 3 Consider the two lines: $y = 3$ and $y = -3$.

Let us integrate these functions in turn from $x = 0$ to $x = 2$.



a) For $y = 3$:

$$\int_0^2 (+3)dx = 3x\Big|_0^2 = 3\times2-3\times0 = 6\,.$$

and 6 is indeed the area of the rectangle of height 3 and length 2.

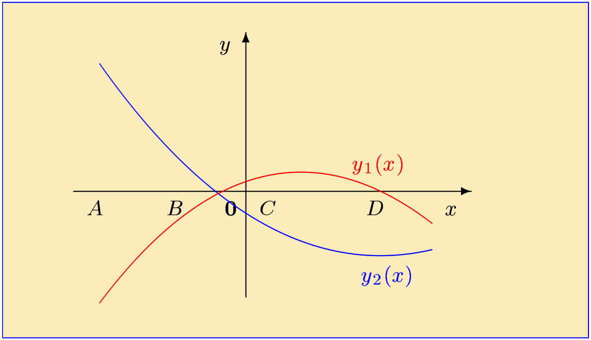
b) However, for $y = -3$:

$$\int_0^2 (-3)dx = -3x\Big|_0^2 = -3\times2-(-3\times0) = -6\,.$$

Although both rectangles have the same area, the *sign* of this result is negative because the curve, $y = -3$, lies below the x axis. This indicates the **sign convention:**

If a function lies below the x axis, its integral is negative.
If a function lies above the x axis, its integral is positive.

Exercise 2.



From the diagram above, what can you say about the signs of the following definite integrals? (Click on the **green** letters for the solutions)

(a) $\int_A^B y_1(x)dx$, (b) $\int_B^D y_1(x)dx$,

(c) $\int_A^0 y_2(x)dx$, (d) $\int_C^D y_2(x)dx$.

Example 4 To calculate $\int_{-4}^{-2} 6x^2 dx$, use $\int ax^n dx = \dfrac{a}{n+1}x^{n+1}$. Thus

$$\begin{aligned}\int_{-4}^{-2} 6x^2 dx &= \frac{6}{2+1}x^{2+1}\Big|_{-4}^{-2} \\ &= \frac{6}{3}x^3\Big|_{-4}^{-2} = 2x^3\Big|_{-4}^{-2} \\ &= 2\times(-2)^3 - 2\times(-4)^3 = -16 + 128 = 112\,.\end{aligned}$$

Note that even though the integration range is for negative x (from -4 to -2), the integrand, $f(x) = 6x^2$, is a positive function. The definite integral of a positive function is positive. (Similarly it is negative for a negative function.)

Quiz Select the definite integral of $y = 5x^4$ with respect to x if the lower limit of the integral is $x = -2$ and the upper limit is $x = -1$

(a) -31, (b) 31, (c) 29, (d) -27.

Exercise 3. Use the integrals listed below to calculate the following definite integrals. (Click on the **green** letters for the solutions)

$f(x)$	x^n for $n \neq -1$	$\sin(ax)$	$\cos(ax)$	e^{ax}	$\dfrac{1}{x}$
$\int f(x)dx$	$\dfrac{1}{n+1}x^{n+1}$	$-\dfrac{1}{a}\cos(ax)$	$\dfrac{1}{a}\sin(ax)$	$\dfrac{1}{a}\mathrm{e}^{ax}$	$\ln(x)$

(a) $\int_4^9 3\sqrt{t}\,dt$,

(b) $\int_{-1}^1 (x^2 - 2x + 4)dx$,

(c) $\int_0^\pi \sin(x)dx$,

(d) $\int_0^3 4\mathrm{e}^{2x}dx$,

(e) $\int_1^2 \dfrac{3}{t}dt$,

(f) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos(4w)dw$.

Quiz Find the correct result for the definite integral

$$\int_a^{2b} x^2 dx\,.$$

(a) $\frac{8}{3}b^3 - \frac{1}{3}a^3\,,$ (b) $4b - 2a\,,$

(c) $\frac{8}{3}b^3 + \frac{1}{3}a^3\,,$ (d) $\frac{1}{3}b^3 - \frac{1}{3}a^3\,.$

Quiz Select the correct result for the definite integral

$$\int_2^{3} \frac{1}{x^2} dx\,,$$

from the answers offered below

(a) $-1\,,$ (b) $\frac{1}{5}\,,$ (c) $\frac{1}{36}\,,$ (d) $\frac{1}{6}\,.$

4. Final Quiz

Begin Quiz Choose the solutions from the options given.

1. What is the area under the curve of the following positive function $y = 10x^4 + 3x^2$ between $x = -1$ and $x = 2$?

(a) 75 , (b) 53 , (c) 69 , (d) 57 .

2. What is the definite integral of $3\sin(2x)$ from $x = 0$ to $x = \pi/2$?

(a) -3 , (b) 0 , (c) 3 , (d) $\frac{5}{2}$.

3. Find the (non-zero) value of b for which the definite integral $\int_0^b (2s - 3)ds$ vanishes

(a) 1 , (b) 5 , (c) 3 , (d) 2 .

4. Select below the definite integral $\int_{-2}^{2} e^{2x} dx$ with respect to x.

(a) $2\left(e^4 - e^{-4}\right)$, (b) $\frac{1}{2}\left(e^4 - \sqrt[4]{e}\right)$, (c) 0 , (d) $\frac{1}{2}\left(e^4 - e^{-4}\right)$.

End Quiz Score: Correct

Solutions to Exercises

Exercise 1(a) To calculate $\int_0^3 x dx$, use the formula

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

with $n = 1$. This yields

$$\begin{aligned} \int_0^3 x dx &= \frac{1}{1+1} x^{1+1} \bigg|_0^3 = \frac{1}{2} x^2 \bigg|_0^3 \\ &= \frac{1}{2} \times (3)^2 - \frac{1}{2} \times (0)^2 \\ &= \frac{1}{2} \times 9 - 0 = \frac{9}{2} \,. \end{aligned}$$

Click on the **green** square to return

□

Exercise 1(b) To calculate $\int_{-1}^{2} x dx$, use the formula for the indefinite integral

$$\int x^n dx = \frac{1}{n+1} x^{n+1}$$

with $n = 1$. This yields

$$\begin{aligned}
\int_{-1}^{2} x dx &= \left. \frac{1}{1+1} x^{1+1} \right|_{-1}^{2} = \left. \frac{1}{2} x^2 \right|_{-1}^{2} \\
&= \frac{1}{2} \times (2)^2 - \frac{1}{2} \times (-1)^2 \\
&= \frac{1}{2} \times 4 - \frac{1}{2} \times (+1) \\
&= 2 - \frac{1}{2} = \frac{3}{2} .
\end{aligned}$$

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Exercise 1(c) To evaluate the definite integral $\int_1^2 (x^2 - x)dx$ we rewrite it as the sum of two integrals and use $\int x^n dx = \frac{1}{n+1}x^{n+1}$ with $n = 2$ in the first integral and with $n = 1$ in the second one

$$\begin{aligned}
\int_1^2 x^2\,dx - \int_1^2 x\,dx &= \frac{1}{2+1}x^{2+1}\Big|_1^2 - \frac{1}{1+1}x^{1+1}\Big|_1^2 \\
&= \frac{1}{3}x^3\Big|_1^2 - \frac{1}{2}x^2\Big|_1^2 \\
&= \frac{1}{3}\times 2^3 - \frac{1}{3}\times 1^3 - \left(\frac{1}{2}\times 2^2 - \frac{1}{2}\times 1^2\right) \\
&= \frac{1}{3}\times 8 - \frac{1}{3}\times 1 - \left(\frac{1}{2}\times 4 - \frac{1}{2}\times 1\right) \\
&= \frac{7}{3} - \frac{3}{2} = \frac{14}{6} - \frac{9}{6} = \frac{5}{6}\,.
\end{aligned}$$

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Exercise 1(d) To find the integral $\int_{-1}^{2}(x^2 - x)dx$ we rewrite it as the sum of two integrals and use the result of the previous part to write it as

$$
\begin{aligned}
\int_{-1}^{2} x^2\,dx - \int_{-1}^{2} x\,dx &= \frac{1}{3}x^3\Big|_{-1}^{2} - \frac{1}{2}x^2\Big|_{-1}^{2} \\
&= \frac{1}{3}\times 2^3 - \frac{1}{3}\times(-1)^3 - \left(\frac{1}{2}\times 2^2 - \frac{1}{2}\times(-1)^2\right) \\
&= \frac{1}{3}\times 8 + \frac{1}{3}\times 1 - \left(\frac{1}{2}\times 4 - \frac{1}{2}\times 1\right) = \frac{9}{3} - \frac{3}{2} \\
&= 3 - \frac{3}{2} = \frac{3}{2}\,.
\end{aligned}
$$

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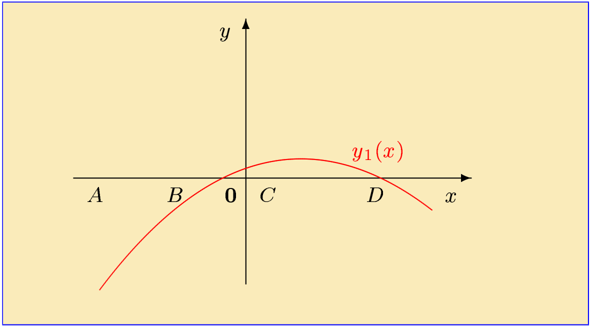
Exercise 2(a)



The sign of the definite integral, $\int_A^B y_1(x)dx$, must be negative. This is because the function $y_1(x)$ is negative for all values of x between A and B. The area is all below the x axis.

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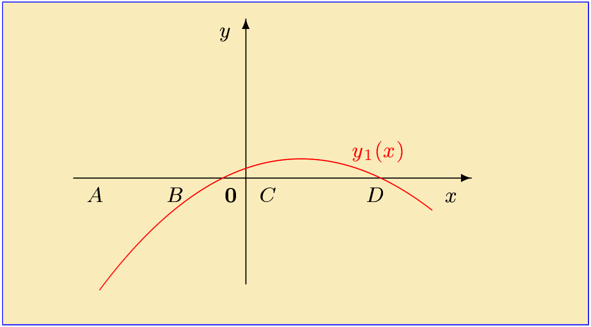
Exercise 2(b)



The sign of the definite integral, $\int_B^D y_1(x)dx$, must be positive. This is because, between the integration limits B and D, there is more area above the x axis than below the x axis.

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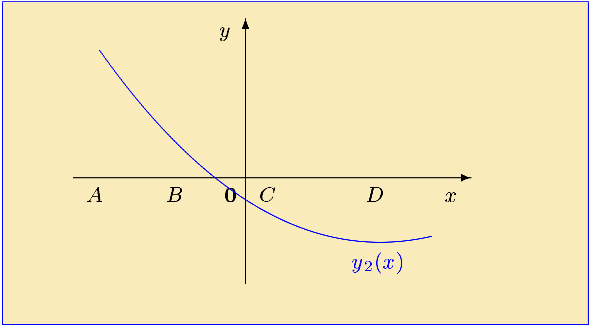
Exercise 2(c)



The sign of the definite integral, $\int_A^0 y_2(x)dx$, must be positive. This is because, between the integration limits A and 0, there is more area above the x axis than below it.

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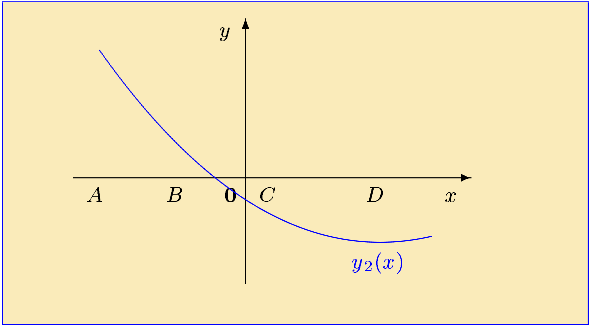
Exercise 2(d)



The sign of the definite integral, $\int_C^D y_2(x)dx$, must be negative. This is because, between the integration limits C and D, the integrand $y_2(x)$ is always negative.

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Exercise 3(a) To calculate the definite integral $\int_4^9 3\sqrt{t}dt$ we rewrite it as

$$\int_4^9 3\sqrt{t}dt = 3 \times \int_4^9 t^{1/2}dt$$

and use $\int x^n dx = \frac{1}{n+1}x^{n+1}$ for $n = \frac{1}{2}$

$$\begin{aligned} 3 \times \int_4^9 t^{\frac{1}{2}}dt &= 3 \times \frac{1}{\frac{1}{2}+1}t^{\frac{1}{2}+1}\Big|_4^9 = 3 \times \frac{1}{\frac{3}{2}} \times t^{\frac{3}{2}}\Big|_4^9 = 3 \times \frac{2}{3}t^{\frac{3}{2}}\Big|_4^9 \\ &= 2t^{\frac{3}{2}}\Big|_4^9 = 2 \times (9)^{\frac{3}{2}} - 2 \times (4)^{\frac{3}{2}} = 2 \times (9^{\frac{1}{2}})^3 - 2 \times (4^{\frac{1}{2}})^3 \\ &= 2 \times (3)^3 - 2 \times (2)^3 = 2 \times 27 - 2 \times 8 = 54 - 16 = 38\,. \end{aligned}$$

N.B. dividing by a fraction is equivalent to multiplying by its inverse (see the package on **fractions**).

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Exercise 3(b) To calculate the definite integral $\int_{-1}^{1}(x^2-2x+4)dx$ we rewrite it as a sum of integrals $\int_{-1}^{1} x^2dx - 2\times\int_{-1}^{1} xdx + 4\times\int_{-1}^{1} 1dx$ and use $\int x^n dx = \frac{1}{n+1}x^{n+1}$ with $n=2$ in the first integral,

$$\int_{-1}^{1} x^2 dx \quad = \quad \frac{1}{2+1}x^{2+1}\Big|_{-1}^{1} = \frac{1}{3}x^3\Big|_{-1}^{1} = \frac{1}{3}\left(1^3-(-1)^3\right) = \frac{2}{3}\,,$$

with $n=1$ in the second integral

$$2\times\int_{-1}^{1} xdx \quad = \quad 2\times\frac{1}{1+1}x^{1+1}\Big|_{-1}^{1} = x^2\Big|_{-1}^{1} = -\left(1^2-(-1)^2\right) = 0\,,$$

and with $n=0$ in the last integral

$$4\times\int_{-1}^{1} 1dx \quad = \quad 4\times\frac{1}{0+1}x^{0+1}\Big|_{-1}^{1} = 4x\Big|_{-1}^{1} = 4\left(1-(-1)\right) = 8\,.$$

Summing up these numbers we obtain $2/3+0+8=26/3$.

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Exercise 3(c) To calculate the definite integral $\int_0^\pi \sin(x)dx$ we note from the table that

$$\int \sin(ax)dx = -\frac{1}{a}\cos(x).$$

This yields (with $a = 1$)

$$\begin{aligned} \int_0^\pi \sin(x)dx &= -\cos(x)\Big|_0^\pi \\ &= -(\cos(\pi) - \cos(0)) = -((-1) - 1) = 2\,. \end{aligned}$$

N.B. It is worth emphasizing that the angles in calculus formulae for trigonometric functions are measured in radians.

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Exercise 3(d) To calculate the definite integral $\int_0^3 4e^{2x}dx$, write

$$\int_0^3 4e^{2x}dx = 4 \times \int_0^3 e^{2x}dx$$

and use from the table

$$\int e^{ax}dx = \frac{1}{a}e^{ax}.$$

This gives for $a = 2$

$$\begin{aligned} 4 \times \int_0^3 e^{2x}dx &= 4 \times \frac{1}{2}e^{2x}\Big|_0^3 = 2e^{2x}\Big|_0^3 \\ &= 2e^{(2\times 3)} - 2e^{(2\times 0)} \\ &= 2e^6 - 2e^0 = 2e^6 - 2 \times 1 = 2e^6 - 2\,. \end{aligned}$$

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Exercise 3(e) To evaluate the definite integral $\int_1^2 \frac{3}{t} dt$ we write

$$\int_1^2 \frac{3}{t} dt = 3 \times \int_1^2 \frac{1}{t} dt$$

and use

$$\int dt \frac{1}{t} = \ln(t) .$$

This yields

$$\begin{aligned} 3 \times \int_1^2 \frac{1}{t} dt &= 3 \times \ln(t) \Big|_1^2 \\ &= 3 \times \ln(2) - 3 \times \ln(1) \\ &= 3 \times \ln(2) - 3 \times 0 \\ &= 3 \ln(2) . \end{aligned}$$

N.B. $\ln(0) = 1$, since $\mathrm{e}^0 = 1$.

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Exercise 3(f) To find the definite integral $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos(4w)dw$ use

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 2\cos(4w)dw = 2 \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(4w)dw\,.$$

and $\int \cos(ax)dx = \frac{1}{a}\sin(x)$. This gives for $a = 4$

$$\begin{aligned} 2 \times \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos(4w)dw &= 2 \times \frac{1}{4}\sin(4w)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \frac{1}{2}\sin(4w)\Big|_{\frac{\pi}{4}}^{\frac{\pi}{2}} \\ &= \frac{1}{2}\sin(4 \times \frac{\pi}{2}) - \frac{1}{2}\sin(4 \times \frac{\pi}{4}) \\ &= \frac{1}{2}\sin(2\pi) - \frac{1}{2}\sin(\pi) \\ &= \frac{1}{2} \times 0 - \frac{1}{2} \times 0 = 0\,. \end{aligned}$$

Click on the **green** square to return □

Solutions to Quizzes

Solution to Quiz: To find the indefinite integral $\int(3x^2 - \frac{1}{2}x)dx$ we use the sum rule for integrals, rewriting it as the sum of two integrals

$$\begin{aligned}\int(3x^2 - \frac{1}{2}x)\,dx &= \int 3x^2\,dx + \int(-\frac{1}{2}x)\,dx \\ &= 3\int x^2\,dx - \frac{1}{2}\int x\,dx\,.\end{aligned}$$

Using $\int x^n dx = \frac{1}{n+1}x^{n+1}\,, n \neq -1$ with $n = 2$ in the first integral and with $n = 1$ in the second one gives

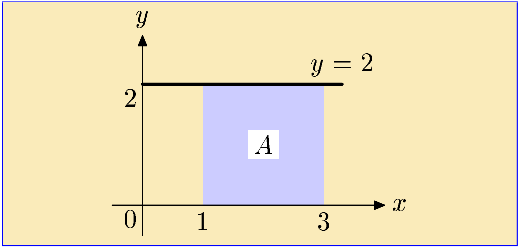
$$\begin{aligned}3\int x^2\,dx - \frac{1}{2}\int x\,dx &= 3 \times \frac{1}{1+2}x^{1+2} - \frac{1}{2} \times \frac{1}{1+1}x^{1+1} + C \\ &= \frac{3}{3}x^3 - \frac{1}{2(1+1)}x^2 + C = x^3 - \frac{1}{4}x^2 + C\,.\end{aligned}$$

Check that differentiation of this result gives $3x^2 - \frac{1}{2}x$. End Quiz

Solution to Quiz: Using $2 = 2x^0$, the integral

$$\begin{aligned}\int_1^3 2dx &= 2\int_1^3 x^0 dx = 2x\Big|_1^3 \\ &= 2\times 3 - 2\times 1 = 6-2 \\ &= 4\,.\end{aligned}$$

Indeed from the diagram



the area under the curve between the integration limits is the area of a square of side 2. This has area $2 \times 2 = 4$. End Quiz

Solution to Quiz: The definite integral of $y = 5x^4$ with respect to x if the lower limit of the integral is $x = -2$ and the upper limit $x = -1$ can be written as

$$\int_{-2}^{-1} 5x^4 dx\,.$$

From the basic result $\int_A^B ax^n dx = \frac{a}{n+1} x^{n+1}\Big|_A^B$ we obtain

$$\begin{aligned}\int_{-2}^{-1} 5x^4 dx &= \frac{5}{5}x^5\Big|_{-2}^{-1} = x^5\Big|_{-2}^{-1} \\ &= (-1)^5 - (-2)^5 \\ &= -1 - (-32) \\ &= -1 + 32 = 31\,.\end{aligned}$$

Note that since the integrand $5x^4$ is positive for all x, the negative suggested solutions could not be correct. End Quiz

Solution to Quiz: To calculate the definite integral $\int_a^{2b} x^2 dx$ use the basic indefinite integral

$$\int x^n dx = \frac{1}{n+1} x^{n+1} .$$

with $n = 2$. This gives

$$\begin{aligned}
\int_a^{2b} x^2 dx &= \frac{1}{2+1} x^{(2+1)} \Big|_a^{2b} = \frac{1}{3} x^3 \Big|_a^{2b} \\
&= \frac{1}{3} \times (2b)^3 - \frac{1}{3} \times (a)^3 \\
&= \frac{1}{3} \times (2)^3 \times b^3 - \frac{1}{3} \times a^3 \\
&= \frac{1}{3} \times 8 \times b^3 - \frac{1}{3} \times a^3 \\
&= \frac{8}{3} b^3 - \frac{1}{3} a^3 .
\end{aligned}$$

End Quiz

Solution to Quiz: To evaluate the definite integral

$$\int_2^3 \frac{1}{x^2} dx = \int_2^3 x^{-2} dx$$

use $\int_A^B x^n dx = \frac{1}{n+1} x^{n+1} \Big|_A^B$ with $n = -2$

$$\begin{aligned}
\int_2^3 x^{-2} dx &= \frac{1}{(-2+1)} x^{-2+1} \Big|_2^3 = \frac{1}{(-1)} x^{-1} \Big|_2^3 \\
&= (-1) \times \frac{1}{x} \Big|_2^3 = -\frac{1}{x} \Big|_2^3 = -\frac{1}{3} - (-\frac{1}{2}) \\
&= -\frac{1}{3} + \frac{1}{2} = -\frac{2}{6} + \frac{3}{6} \\
&= \frac{-2+3}{6} = \frac{1}{6}\,.
\end{aligned}$$

End Quiz