Paper-1: FOUNDATIONS OF BIOPHYSICS
Module-10: Trigonometry

Introduction
History of trigonometry:

- Hipparchus is considered the founder of trigonometry. He was born in Nicaea, Bithynia (now Iznik, Turkey).
- In Indian astronomy, the study of trigonometric functions flowered in the Gupta period, especially due to Aryabhata (6th century CE).
- Trigonometry deals with right-angled triangles, the ratios and relationships between the triangle's sides and angles.
- Trigonometry can be decomposed into trigono + metry. The first part of the word is from Greek trigon "triangle". The second part of trigonometry is from Greek metron "a measure."

Objective:

- To understand the importance and relationship between angles and lengths in a triangle
- To extend the same concept to calculate angles or lengths in different geometrical entities by constructing an imaginary triangles
- To understand the various applications of trigonometric functions

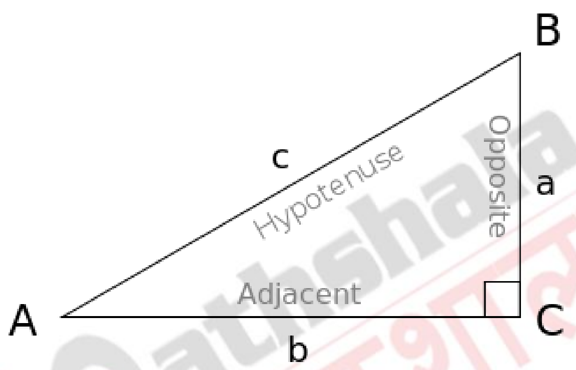
1. Linear and Angular measurement:
- Linear measurements are measured along a line or curve i.e. length. S.I. unit of measurement is meter.
- Angular measurement is done by measuring angle. Plane angle is measured in radians and solid angle in steradian.
- Solid angle is the ratio between the area subtended and the square of its distance from the vertex. Whereas planer angle is the ratio of arc to the distance from the vertex.
- In case of sphere, surface area is $4\pi r^2$ and distance from the vertex is r. Hence, solid angle is = $4\pi r^2 / r^2 = 4\pi$.
- Angular measurement in 2D is measured in degree and radians. The relation between them is given by, $\theta_d = \theta_r X 180/2\pi$. Where, θ_d=angle in degree and θ_r =angle in radian.

2. Trigonometric functions:
- For a triangle, there are 3 sides and 3 angles which describes it. In right angled triangle, one angle is 90. Hence 3 sides can completely describe the triangle. The ratios of sides of triangle are called trigonometric functions. There can be 6 permutations of 3 sides in the ratio of two sides at a time.

- In the figure A, B, C are the angles whereas, AB, BC and AC are the sides of triangles. AB is hypotenuse, as it is opposite to right angle. BC is opposite to angle A and AC is adjacent to the angle A. Now describing the different ratio as:

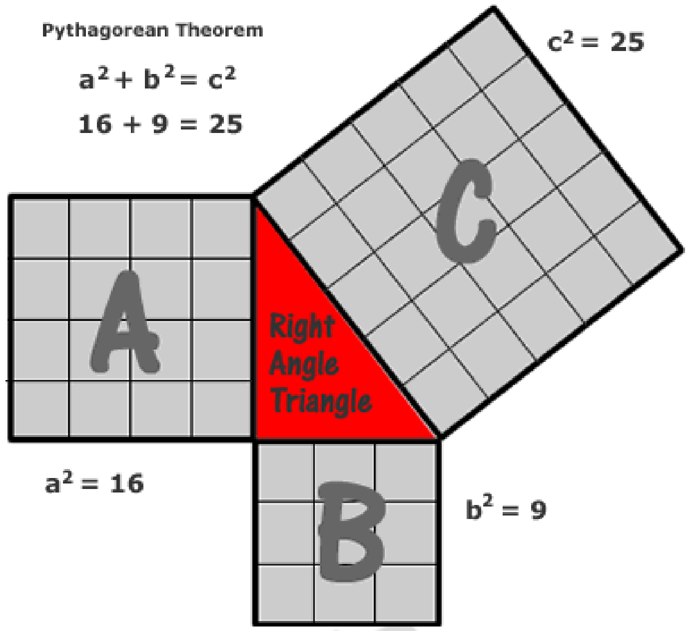
$$\sin A = \frac{\text{Opposite}}{\text{Hypotenuse}} = \frac{BC}{AB} \qquad \cos A = \frac{\text{Adjacent}}{\text{Hypotenuse}} = \frac{AC}{AB} \qquad \tan A = \frac{\text{Opposite}}{\text{Adjacent}} = \frac{BC}{AC}$$

$$\text{cosec} A = \frac{\text{Hypotenuse}}{\text{Opposite}} = \frac{AB}{BC} \qquad \sec A = \frac{\text{Hypotenuse}}{\text{Adjacent}} = \frac{AB}{AC} \qquad \cot A = \frac{\text{Adjacent}}{\text{Opposite}} = \frac{AC}{BC}$$



3. Pythagorean Theorem:

- It states that the square of the hypotenuse (the side opposite the right angle) is equal to the sum of the squares of the other two sides. i.e. $c^2=a^2+b^2$ (as shown in figure)
- An example is right angled triangle with its sides as 3, 4 and 5 respectively. 3^2, 4^2 and 5^2 can represented as square of 3X3, 4X4 and 5X5 respectively. It can be verified that $3^2+ 4^2 =5^2$. Hence, Pythagorean Theorem is verified.



4. Trigonometric Identities

- $\sin^2\theta + \cos^2\theta = 1$
  Proof:
  As we know that,
  $c^2 = a^2 + b^2$ (from Pythagorean theorem)
  Dividing both side by c^2, we get
  $1 = a^2/c^2 + b^2/c^2$
  But, we know that $a/c = \sin\theta$ and $b/c = \cos\theta$.
  Hence, $1 = \sin^2\theta + \cos^2\theta$

- $\sec^2\theta - \tan^2\theta = 1$
  Proof:
  As we know that,
  $1 = \sin^2\theta + \cos^2\theta$
  Dividing both side by $\cos^2\theta$, we get
  $1/\cos^2\theta = \sin^2\theta/\cos^2\theta + 1$
  But, we know that $1/\cos\theta = \sec\theta$ and $\sin\theta/\cos\theta = \tan\theta$.
  Hence, $\sec^2\theta = 1 + \tan^2\theta$ or $\sec^2\theta - \tan^2\theta = 1$

- $\mathrm{Cosec}2\theta - \mathrm{Cot}2\theta = 1$
  Proof:
  As we know that,
  $1 = \sin^2\theta + \cos^2\theta$
  Dividing both side by $\sin^2\theta$, we get
  $1/\sin^2\theta = 1 + \cos^2\theta/\sin^2\theta$
  But, we know that $1/\sin\theta = \mathrm{cosec}\,\theta$ and $\cos\theta/\sin\theta = \cot\theta$.
  Hence, $\mathrm{cosec}^2\theta = 1 + \cot^2\theta$ or $\mathrm{cosec}^2\theta - \cot^2\theta = 1$

5. Series Solutions

- There exist a series for every trigonometric function, which approximates for small values of angle i.e. *x (using Taylor series)*

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!},$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}.$$

6. Sine Formula:

- It relates angle and sides of a triangle trough sine function.

$$\frac{\mathrm{a}}{\sin A} = \frac{\mathrm{b}}{\sin B} = \frac{\mathrm{c}}{\sin C}$$

- Proof:

The area of any triangle can be written as one half of its base times its height. Depending on which side one chooses to be the base, the area can be written as:

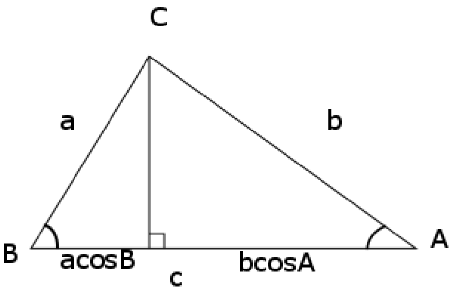
$$\text{Area} = \frac{1}{2} b(c \sin A) = \frac{1}{2} c(a \sin B) = \frac{1}{2} a(b \sin C).$$

Now dividing by abc/2, we have

$$\frac{2\text{Area}}{abc} = \frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

7. Cosine formula:

- It relates angle and length of sides of a triangle trough cosine function.
  $\mathrm{a}^2 = \mathrm{b}^2 + \mathrm{c}^2 - 2\mathrm{bc} \cos A$
- It generalize the Pythagorean theorem i.e. for A=π/2, we have $\mathrm{a}^2 = \mathrm{b}^2 + \mathrm{c}^2$.



Proof:
In the figure we can say that c = a. cos$B$ + b. cos$A$.
Now multiply through out by c we have c^2 = c. a. cosB+c. b. cosA.
Similarly, a^2 = a. b. cos$C$ + a. c. cos$B$ and b^2 = b. a. cosC+b. c. cosA
Adding b^2, c^2 and subtracting a^2, we have
$b^2+c^2-a^2$ = 2bc. cos$A$
Or, $a^2 = b^2 + c^2$ - 2bc. cos$A$

8. Euler formula:

- Euler's formula provides a powerful connection between Complex analysis and trigonometry, and provides an interpretation of the sine and cosine functions as weighted sums of the exponential function. It gives equal to exponential decay and exponential blow.

$$\sin A = \frac{e^{iA} - e^{-iA}}{2i}, \qquad \cos A = \frac{e^{iA} + e^{-iA}}{2}.$$

- Adding sin A and cos A we get,
$$e^{ix} = \cos x + i \sin x$$

9. Angle transformation formulae:

Identity 1:
$$\cos(A \pm B) = (\cos A \cos B \mp \sin A \sin B)$$
$$\sin(A \pm B) = (\sin A \cos B \pm \sin B \cos A)$$

Proof:
As we know that, from Euler formula we have
$$e^{i\theta} = \cos\theta + i\sin\theta$$
Now, putting $\theta = A + B$ we have
$$e^{i(A+B)} = \cos(A+B) + i\sin(A+B)$$
Also using the properties of exponential functions, we have
$$e^{i(A+B)} = e^{iA}e^{iB} = (\cos A + i\sin A)(\cos B + i\sin B)$$
$$e^{i(A+B)} = (\cos A\cos B - \sin A\sin B) + i(\sin A\cos B + \sin B\cos A)$$

Hence, we have

$$\cos(A+B)+i\sin(A+B)=(\cos A\cos B-\sin A\sin B)+i(\sin A\cos B+\sin B\cos A)$$

Comparing real and imaginary parts both sides, we get

$$\cos(A+B)=\cos A\cos B-\sin A\sin B$$

$$\sin(A+B)=\sin A\cos B+\cos A\sin B$$

Identity 2:

$$\tan(A\pm B)=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}\qquad\qquad \cot(A\pm B)=\frac{\cot A\cot B\mp 1}{\cot B\pm\cot A}$$

As you know

$$\tan(A\pm B)=\frac{\sin(A\pm B)}{\cos(A\pm B)}=\frac{\sin A\cos B\pm\cos A\sin B}{\cos A\cos B\mp\sin A\sin B}$$

Dividing by cos A. cos B, we have

$$\tan(A\pm B)=\frac{\dfrac{\sin A}{\cos A}\pm\dfrac{\sin B}{\cos B}}{1\mp\dfrac{\sin A}{\cos A}.\dfrac{\sin B}{\cos B}}=\frac{\tan A\pm\tan B}{1\mp\tan A\tan B}$$

Similarly, we get

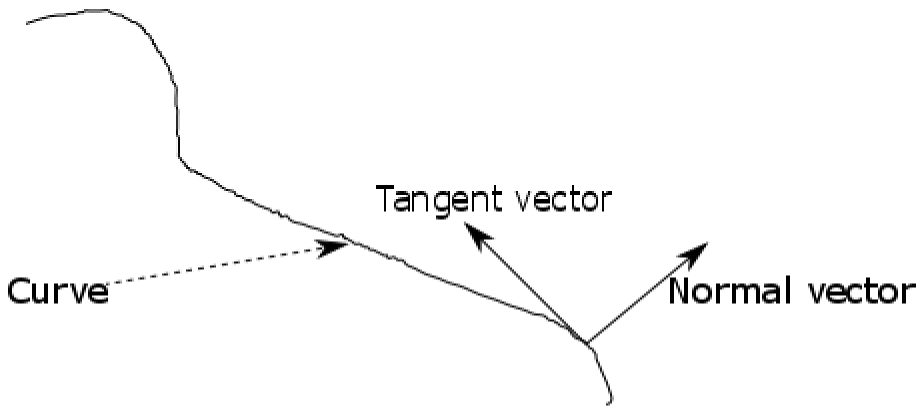
$$\cot(A\pm B)=\frac{\cos(A\pm B)}{\sin(A\pm B)}=\frac{\cos A\cos B\mp\sin A\sin B}{\sin A\cos B\pm\cos A\sin B}$$

Dividing by sin A. sin B

$$\cot(A\pm B)=\frac{\cot A\cot B\mp 1}{\cot B\pm\cot A}$$

<u>10. Calculus of trigonometric functions:</u>

- A curve has a normal and tangent vector at a point. Derivative at a point to the curve is given by its tangent.
- Any vector can be decomposed along the normal and tangent vector to the point.



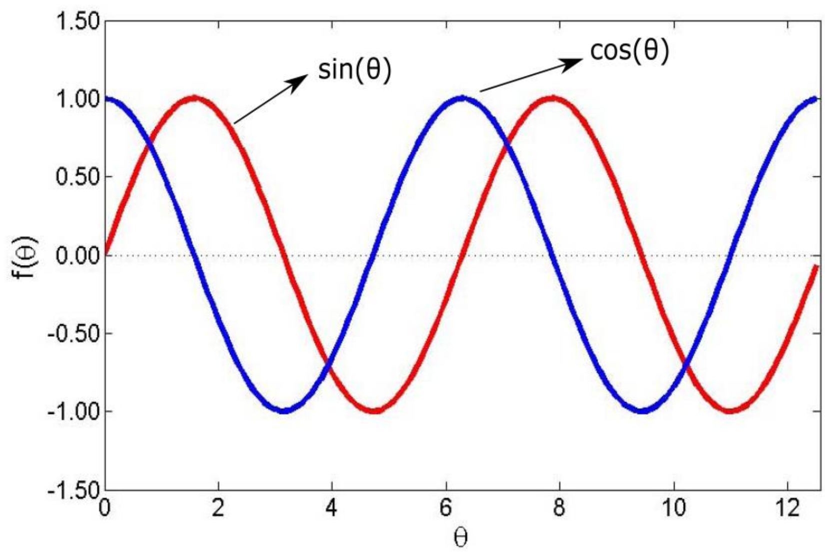
11. Calculus of trigonometric functions

Derivative of a sin and cos is given by, $\frac{d\sin x}{dx} = \cos x, \frac{d\cos x}{dx} = -\sin x$.

Let us consider $\sin x$, then its derivative is $\cos x$.

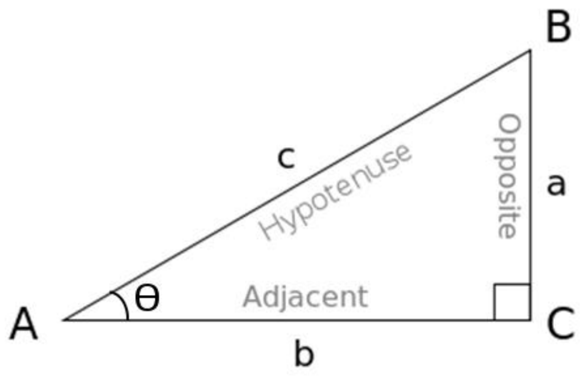
At $x = 0$, $\sin x$ is 0 and increasing but its slope is max and decreasing to 0 at $x = \pi/2$.

Hence $\cos x$ is maximum and decreasing at $x = 0$. Similarly, for $\cos x$, as cosx is decreasing at $x = 0$, slope is increasing and negative. At $x = \pi$, slope becomes horizontal i.e. 0.



12. Inverse Trigonometric functions:

Inverse trigonometric functions are the inverse functions of the trigonometric functions. Specifically, they are the inverses of the sine, cosine, tangent, cotangent, secant, and cosecant functions. They are used to obtain an angle from any of the angle's trigonometric ratios.



If $a/c = \sin\theta$, then $\theta = \sin^{-1}(a/c)$ i.e. θ is inverse of sin of a/c.

Where, θ is in radians.

Similarly, $\theta = \cos^{-1}(b/c)$, $\theta = \tan^{-1}(a/b)$,
$\theta = \sec^{-1}(c/b)$, $\theta = \cot^{-1}(b/a)$, $\theta = \mathrm{cosec}^{-1}(c/a)$

Summary - Trigonometry:

- What it is?
    - It is study of triangles to measure its angles and sides. It has great utility in geometric measurement in day today life.
- How?
    - It represents different ratio of sides as trigonometric functions and uses various relation base don geometry to measure the angles or vice-versa.
- Why it is useful?
    - It gives the complete information of tringle (all angles and side), based on very few information.
    - It gives meaning to calculus as dot and cross product of vectors and tensors are related through these trigonometric functions only.
    - It helps in getting normal and tangential component of vector or tensor.