14

Homogeneous Linear Equations — The Big Theorems

Let us continue the discussion we were having at the end of section 12.3 regarding the general solution to any given homogeneous linear differential equation. By then we had seen that any linear combination of particular solutions,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_M y_M(x) \quad ,$$

is another solution to that homogeneous differential equation. In fact, we were even beginning to suspect that this expression could be used as a general solution to the differential equation provided the y_k's were suitably chosen. In particular, we suspected that the general solution to any second-order, homogeneous linear differential equation can be written

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

were c_1 and c_2 are arbitrary constants, and y_1 and y_2 are any two solutions that are not constant multiples of each other.

These suspicions should have been reinforced in the last chapter in which general solutions were obtained via reduction of order. In the examples and exercises, you should have noticed that the solutions obtained to the given homogeneous differential equations could all be written as just described.

It is time to confirm these suspicions, and to formally state the corresponding results. These results will not be of merely academic interest. We will use them for much of the rest of this text.

For practical reasons, we will split our discussion between this and the next chapter. This chapter will contain the statements of the most important theorems regarding the solutions to homogeneous linear differential equations, along with a little discussion to convince you that these theorems have a reasonable chance of being true. More convincing (and lengthier) analysis will be carried out in the next chapter.

14.1 Preliminaries and a Little Review

We are discussing general homogeneous linear differential equations. If the equation is of second order, it will be written as

$$ay'' + by' + cy = 0 \quad .$$

More generally, it will be written as

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

where N, the order, is some positive integer. The coefficients — a, b and c in the second order case, and the a_k's in the more general case — will be assumed to be continuous functions over some open interval $\mathcal{I}$, and the first coefficient — a or a_0 — will be assumed to be nonzero at every point in that interval.

Recall the "principle of superposition": If $\{y_1, y_2, \ldots, y_K\}$ is a set of particular solutions over $\mathcal{I}$ to a given homogeneous linear equation, then any linear combination of these solutions,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_K y_K(x) \qquad \text{for all } x \text{ in } \mathcal{I} \quad ,$$

is also a solution over $\mathcal{I}$ to the the given differential equation. Also recall that this set of y's is called a *fundamental set of solutions* (over $\mathcal{I}$) for the given homogeneous differential equation if and only if both of the following hold:

1. The set is linearly independent over $\mathcal{I}$ (i.e., none of the y_k's is a linear combination of the others over $\mathcal{I}$).

2. Every solution over $\mathcal{I}$ to the given differential equation can be expressed as a linear combination of the y_k's.

14.2 Second-Order Homogeneous Equations

Let us limit our attention to the possible solutions to a second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0 \quad . \tag{14.1}$$

We will first look at what we can derive just from the reduction of order method (with a few assumptions), and then see how that can be extended by some basic linear algebra. Because of some of the assumptions we will make, our discussion here will not be completely rigorous, but it will lead to some of the more important ideas regarding general solutions to second-order homogeneous linear differential equations. After that, I will tell you what can be rigorously proven regarding these general solutions. If you are impatient, you can skip ahead and read that part (theorem 14.1 on page 302).

The Form of the Reduction of Order Solution

As I hope you observed, the reduction of order method applied to an equation of the form (14.1) always led (in the previous chapter, at least) to a general solution of the form

$$y(x) = c_1 y_1(x) + c_R y_R(x)$$

where $\{y_1, y_R\}$ is a linearly independent set of solutions on the interval of interest (we are using the "subscript R" just to emphasize that this part came from reduction of order).

!►Example 14.1: *In section 13.2, we illustrated the reduction of order method by solving*

$$x^2y'' - 3xy' + 4y = 0$$

on the interval $\mathcal{I} = (0, \infty)$ *. After first observing that*

$$y_1(x) = x^2$$

was one solution to this differential equation, we applied the method of reduction of order to obtain the general solution

$$y(x) = x^2[A\ln|x| + B] = Ax^2\ln|x| + Bx^2$$

(where A *and* B *denote arbitrary constants). Observe that this is in the form*

$$y(x) = c_1y_1(x) + c_Ry_R(x) \quad .$$

In this case,

$$y_1(x) = x^2 \qquad \text{and} \qquad y_R(x) = x^2\ln|x| \quad ,$$

and c_1 *and* c_R *are simply the arbitrary constants* A *and* B *, renamed. Observe also that, here,* y_1 *and* y_R *are clearly not constant multiples of each other. So*

$$\{\, y_1,\ y_R \,\} = \left\{\, x^2,\ x^2\ln|x| \,\right\}$$

is a linearly independent pair of functions on the interval of interest. And since every other solution to our differential equation can be written as a linear combination of this pair, this set is a fundamental set of solutions for our differential equation.

Let's look a little more closely at the solution to equation (14.1),

$$ay'' + by' + cy = 0 \quad ,$$

generally obtained via reduction of order. Assuming we have one known nontrivial particular solution y_1 , we set

$$y = y_1u \quad ,$$

plug this into the differential equation, and obtain (after simplification) an equation of the form

$$Au'' + Bu' = 0 \quad , \tag{14.2}$$

which can be treated as the first-order differential equation

$$Av' + Bv = 0$$

using the substitution $v = u'$. Assuming A and B are reasonable functions on our interval of interest, you can easily verify that the general solution to this first-order equation is of the form

$$v(x) = c_Rv_0(x)$$

where c_R is an arbitrary constant, and v_0 is any particular (nontrivial) solution to this first-order equation. (More about this differential equation, along with some important properties of its solutions, is derived in the next chapter.)

Since $u' = v$, we can then recover the general formula for u from the general formula for v by integration:

$$\begin{aligned} u(x) &= \int v(x)\,dx = \int c_R v_0(x)\,dx = c_R \int v_0(x)\,dx \\ &= c_R\,[u_0(x) + c_0] = c_1 + c_R u_0(x) \end{aligned}$$

where u_0 is any single antiderivative of v_0, c_0 is the (arbitrary) constant of integration and $c_1 = c_R c_0$. This, with our initial substitution, yields the general solution

$$y(x) = y_1(x)u(x) = y_1(x)\,[c_1 + c_R u_0(x)]$$

which, after letting

$$y_R(x) = y_1(x)u_0(x) \quad ,$$

simplifies to

$$y(x) = c_1 y_1(x) + c_R y_R(x) \quad .$$

Thus, we have written a general solution to our second-order homogeneous differential equation as a linear combination of just two particular solutions. The question now is whether the set $\{y_1, y_R\}$ is linearly independent or not. But if not, then $y_R = u_0 y_1$ is a constant multiple of y_1, which means u_0 is a constant and, consequently,

$$v_0 = {u_0}' = 0 \quad ,$$

contrary to the known fact that v_0 is a *non*trivial solution to equation (14.2). So u_0 is not a constant, $y_R = u_0 y_1$ is not a constant multiple of y_1, and the pair $\{y_1, y_R\}$ is linearly independent. And since all other solutions can be written as linear combinations of these two solutions, $\{y_1, y_R\}$ is a fundamental set of solutions for out differential equation.

What we have just shown is that, assuming

1. a nontrivial solution y_1 to the second-order differential equation exists, and

2. the functions A and B are ‘reasonable’ over the interval of interest

then the reduction of order method yields a general solution to differential equation (14.1) of the form

$$y(x) = c_1 y_1(x) + c_R y_R(x)$$

where $\{y_1, y_R\}$ is a linearly independent set of solutions.[1]

Applying a Little Linear Algebra

But what if we start out with any linearly independent pair of solutions $\{y_1, y_2\}$ to differential equation (14.1)? Using y_1, we can still derive the general solution

$$y(x) = c_1 y_1(x) + c_R y_R(x)$$

[1] In fact, theorem 11.2 on page 253 can be used to show that a y_1 exists. The real difficulty is in verifying that A and B are ‘reasonable’, especially if y_1 is zero at some point in the interval.

where y_R is that second solution obtained through the reduction of order method. And since this is a general solution and y_2 is a particular solution, there must be constants κ_1 and κ_R such that

$$y_2(x) = \kappa_1 y_1(x) + \kappa_R y_R(x) \quad .$$

Moreover, because $\{y_1, y_2\}$ is (by assumption) linearly independent, y_2 cannot be a constant multiple of y_1. Thus $\kappa_R \neq 0$ in the above equation, and that equation can be solved for y_R, obtaining

$$y_R(x) = -\frac{\kappa_1}{\kappa_R} y_1(x) + \frac{1}{\kappa_R} y_2(x) \quad .$$

Consequently,

$$\begin{aligned} y(x) &= c_1 y_1(x) + c_R y_R(x) \\ &= c_1 y_1(x) + c_R\left[-\frac{\kappa_1}{\kappa_R} y_1(x) + \frac{1}{\kappa_R} y_2(x)\right] = C_1 y_1(x) + C_2 y_2(x) \end{aligned}$$

where

$$C_1 = c_1 - \frac{c_R \kappa_1}{\kappa_R} \qquad \text{and} \qquad C_2 = \frac{c_R}{\kappa_R} \quad .$$

So any linear combination of y_1 and y_R can also be expressed as a linear combination of y_1 and y_2. This means

$$y(x) = C_1 y_1(x) + C_2 y_2(x)$$

can also be used as a general solution, and, hence, $\{y_1, y_2\}$ is also a fundamental set of solutions for our differential equation.

So what? Well, if you are lucky enough to easily find a linearly independent pair of solutions to a given second-order homogeneous equation, then you can use that pair as your fundamental set of solutions — there is no need to grind through the reduction of order computations.[2]

The Big Theorem on Second-Order Homogeneous Linear Differential Equations

Let me repeat what we've just derived:

> *The general solution of a second-order homogeneous linear differential equation is given by*
>
> $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$
>
> *where $\{c_1, c_2\}$ is a pair of arbitrary constants and $\{y_1, y_2\}$ is any linearly independent pair of particular solutions to that differential equation.*

In deriving this statement, we made some assumptions about the existence of solutions, and the 'reasonableness' of the first-order differential equation arising in the reduction of order method. In the next chapter, we will rigorously rederive this statement without making these assumptions. We will also examine a few related issues regarding the linear independence of solution sets and the solvability of initial-value problems. What we will discover is that the following theorem can be proven. This can be considered the "Big Theorem on Second-Order Homogeneous Linear

[2] If you've had a course in linear algebra, you may recognize that a "fundamental set of solutions" is a "basis set" for the "vector space of all solutions to the given homogeneous differential equation". This is worth noting, if you understand what is being noted.

Differential Equations". It will be used repeatedly, often without comment, in the chapters that follow.

Theorem 14.1 (general solutions to second-order homogenous linear differential equations)
Let $\mathcal{I}$ be some open interval, and suppose we have a second-order homogeneous linear differential equation

$$ay'' + by' + cy = 0$$

where, on $\mathcal{I}$, the functions a, b and c are continuous, and a is never zero. Then the following statements all hold:

1. *Fundamental sets of solutions for this differential equation (over $\mathcal{I}$) exist.*
2. *Every fundamental solution set consists of a pair of solutions.*
3. *If $\{y_1, y_2\}$ is any linearly independent pair of particular solutions over $\mathcal{I}$, then:*
   (a) *$\{y_1, y_2\}$ is a fundamental set of solutions.*
   (b) *A general solution to the differential equation is given by*

   $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

   where c_1 and c_2 are arbitrary constants.
   (c) *Given any point x_0 in $\mathcal{I}$ and any two fixed values A and B, there is exactly one ordered pair of constants $\{c_1, c_2\}$ such that*

   $$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

   also satisfies the initial conditions

   $$y(x_0) = A \quad \text{and} \quad y'(x_0) = B \quad .$$

The statement about "initial conditions" in the above theorem assures us that second-order sets of initial conditions are appropriate for second-order linear differential equations. It also assures us that a fundamental solution set for a second-order linear homogeneous differential equation can not become "degenerate" at any point in the interval $\mathcal{I}$. In other words, there is no need to worry about whether an initial-value problem (with x_0 in $\mathcal{I}$) can be solved. It has a solution, and only one solution. (To see why we might be worried about "degeneracy", see exercise 14.2 on page 308.)

To illustrate how this theorem is used, let us solve a differential equation that you may recall solving in chapter 11 (see page 247). Comparing the approach used there with that used here should lead you to greatly appreciate the theory we've just developed.

!►*Example 14.2:* *Consider (again) the homogeneous second-order linear differential equation*

$$y'' + y = 0 \quad .$$

In example 12.2 on page 266 we discovered ("by inspection") that

$$y_1(x) = \cos(x) \quad \text{and} \quad y_2(x) = \sin(x)$$

are two solutions to this differential equation, and in example 12.4 on page 268 we observed that the set of these two solutions is linearly independent pair. The above theorem now assures us that, indeed, this pair,

$$\{ \cos(x) \, , \, \sin(x) \} \quad ,$$

is a fundamental set of solutions for the above second-order homogeneous linear differential equation, and that

$$y(x) \; = \; c_1 \cos(x) \; + \; c_2 \sin(x)$$

is a general solution.

!▶Example 14.3: *Now, consider the initial-value problem*

$$y'' \; + \; y \; = \; 0 \qquad \text{with} \quad y(0) = 3 \quad \text{and} \quad y'(0) = 5 \quad .$$

We just found that

$$y(x) \; = \; c_1 \sin(x) \; + \; c_2 \cos(x)$$

is a general solution to the differential equation. Taking derivatives, we have

$$y'(x) \; = \; [c_1 \sin(x) \; + \; c_2 \cos(x)]' \; = \; c_1 \cos(x) \; - \; c_2 \sin(x) \quad .$$

Using this in our set of initial conditions, we get

$$3 \; = \; y(0) \; = \; c_1 \sin(0) \; + \; c_2 \cos(0) \; = \; c_1 \cdot 0 \; + \; c_2 \cdot 1$$

and

$$5 \; = \; y'(0) \; = \; c_1 \cos(0) \; - \; c_2 \sin(0) \; = \; c_1 \cdot 1 \; - \; c_2 \cdot 0 \quad .$$

Hence,

$$c_1 \; = \; 5 \qquad \text{and} \qquad c_2 \; = \; 3 \quad ,$$

and the solution to our initial-value problem is

$$\begin{aligned} y(x) \; &= \; c_1 \sin(x) \; + \; c_2 \cos(x) \\ &= \; 5 \sin(x) \; + \; 3 \cos(x) \quad . \end{aligned}$$

Finding fundamental sets of solutions for most homogeneous linear differential equations will not be as easy as it was for the differential equation in the last two examples. Fortunately, fairly straightforward methods are available for finding fundamental sets for some important classes of differential equations. Some of these methods are partially described in the exercises, and will be more completely developed in later chapters.

14.3 Homogeneous Linear Equations of Arbitrary Order

The big theorem on second-order homogeneous equations, theorem 14.1, can be extended to an analogous theorem covering homogeneous linear equations of all orders. That theorem is:

Theorem 14.2 (general solutions to homogenous linear differential equations)
Let $\mathcal{I}$ be some open interval, and suppose we have an N^{th}-order homogeneous linear differential equation

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

where, on $\mathcal{I}$, the a_k's are all continuous functions with a_0 never being zero. Then the following statements all hold:

1. *Fundamental sets of solutions for this differential equation (over $\mathcal{I}$) exist.*
2. *Every fundamental solution set consists of exactly N solutions.*
3. *If $\{y_1, y_2, \ldots, y_N\}$ is any linearly independent set of N particular solutions over $\mathcal{I}$, then:*
   (a) *$\{y_1, y_2, \ldots, y_N\}$ is a fundamental set of solutions.*
   (b) *A general solution to the differential equation is given by*

   $$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$

   where c_1, c_2, ... and c_N are arbitrary constants.
   (c) *Given any point x_0 in $\mathcal{I}$ and any N fixed values A_1, A_2, ... and A_N, there is exactly one ordered set of constants $\{c_1, c_2, \ldots, c_N\}$ such that*

   $$y(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_N y_N(x)$$

   also satisfies the initial conditions

   $$y(x_0) = A_1 \quad , \quad y'(x_0) = A_2 \quad ,$$
   $$y''(x_0) = A_2 \quad , \quad \cdots \quad \text{and} \quad y^{(N-1)}(x_0) = A_N \quad .$$

A proof of this theorem is given in the next chapter.

14.4 Linear Independence and Wronskians

Let $\{y_1, y_2, \ldots, y_N\}$ be a set of N (sufficiently differentiable) functions on an interval $\mathcal{I}$. The corresponding *Wronskian*, denoted by either W or $W[y_1, y_2, \ldots, , y_N]$, is the function on $\mathcal{I}$ generated by the following determinant of a matrix of derivatives of the y_k's:

$$W = W[y_1, y_2, \ldots, , y_N] = \begin{vmatrix} y_1 & y_2 & y_3 & \cdots & y_N \\ {y_1}' & {y_2}' & {y_3}' & \cdots & {y_N}' \\ {y_1}'' & {y_2}'' & {y_3}'' & \cdots & {y_N}'' \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ {y_1}^{(N-1)} & {y_2}^{(N-1)} & {y_3}^{(N-1)} & \cdots & {y_N}^{(N-1)} \end{vmatrix} \quad .$$

In particular, if $N = 2$,

$$W = W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ {y_1}' & {y_2}' \end{vmatrix} = y_1 {y_2}' - {y_1}' y_2 \quad .$$

!►Example 14.4: *Let's find* $W[y_1, y_2]$ *on the real line when*

$$y_1(x) = x^2 \qquad \text{and} \qquad y_2(x) = x^3 \quad .$$

In this case,

$$y_1{}'(x) = 2x \qquad \text{and} \qquad y_2{}'(x) = 3x^2 \quad ,$$

and

$$W[y_1, y_2] = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1{}'(x) & y_2{}'(x) \end{vmatrix} = \begin{vmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{vmatrix} = x^2 3x^2 - 2xx^3 = x^4$$

Wronskians naturally arise when dealing with initial-value problems. For example, suppose we have a pair of functions y_1 and y_2, and we want to find constants c_1 and c_2 such that

$$y(x) = c_1 y_1(x) + c_2 y_2(x)$$

satisfies

$$y(x_0) = 2 \qquad \text{and} \qquad y'(x_0) = 5$$

for some given point x_0 in our interval of interest. In solving for c_1 and c_2, you can easily show that

$$c_1 W(x_0) = 2y_2{}'(x_0) - 5y_2(x_0) \qquad \text{and} \qquad c_2 W(x_0) = 5y_1(x_0) - 2y_1{}'(x_0) \quad .$$

Thus, if $W(x_0) \neq 0$, then there is exactly one possible value for c_1 and one possible value for c_2, namely,

$$c_1 = \frac{2y_2{}'(x_0) - 5y_2(x_0)}{W(x_0)} \qquad \text{and} \qquad c_2 = \frac{5y_1(x_0) - 2y_1{}'(x_0)}{W(x_0)} \quad .$$

However, if $W(x_0) = 0$, then the system reduces to

$$0 = 2y_2{}'(x_0) - 5y_2(x_0) \qquad \text{and} \qquad 0 = 5y_1(x_0)) - 2y_1{}'(x_0)$$

which cannot be solved for c_1 and c_2.[3]

More generally, the vanishing of a Wronskian of a set of functions signals that the given set is not a good choice in constructing solutions to initial-value problems. The value of this fact is enhanced by the following remarkable theorem:

Theorem 14.3 (Wronskians and fundamental solution sets)
Let W *be the Wronskian of any set* $\{y_1, y_2, \ldots, y_N\}$ *of* N *particular solutions to an* N^{th}*-order homogeneous linear differential equation*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2} y'' + a_{N-1} y' + a_N y = 0$$

on some interval open $\mathcal{I}$. *Assume further that the* a_k*'s are continuous functions with* a_0 *never being zero on* $\mathcal{I}$. *Then:*

[3] Of course, the choice of 2 and 5 as the initial values was not important; any other values could have been used (we were just trying to reduce the number of symbols to keep track off). What is important is whether $W(x_0)$ is zero or not.

1. *If* $W(x_0) = 0$ *for any single point* x_0 *in* $\mathcal{I}$ *, then* $W(x) = 0$ *for every point* x *in* $\mathcal{I}$ *, and the set* $\{y_1, y_2, \ldots, y_N\}$ *is not linearly independent (and, hence, is not a fundamental solution set) on* $\mathcal{I}$ *.*

2. *If* $W(x_0) \neq 0$ *for any single point* x_0 *in* $\mathcal{I}$ *, then* $W(x) \neq 0$ *for every point* x *in* $\mathcal{I}$ *, and* $\{y_1, y_2, \ldots, y_N\}$ *is a fundamental solution set solutions for the given differential equation on* $\mathcal{I}$ *.*

This theorem (proven in the next chapter) gives a relatively easy to use test for determining when a set of solutions to a linear homogeneous differential equation is a fundamental set of solutions. This test is especially useful when the order of the differential equation 3 or higher.

!►*Example 14.5:* *Consider the functions*

$$y_1(x) = 1 \quad , \quad y_2(x) = \cos(2x) \quad \text{and} \quad y_3(x) = \sin^2(x) \quad .$$

You can easily verify that all are solutions (over the entire real line) to the homogeneous third-order linear differential equation

$$y''' + 4y' = 0 \quad .$$

So, is

$$\left\{ 1,\ \cos(2x)\,,\ \sin^2(x) \right\}$$

a fundamental set of solutions for this differential equation? To check we compute the first-order derivatives

$$y_1{}'(x) = 0 \quad , \quad y_2{}'(x) = -2\sin(2x) \quad , \quad y_3{}'(x) = 2\sin(x)\cos(x) \quad ,$$

the second-order derivatives

$$y_1{}''(x) = 0 \quad , \quad y_2{}''(x) = -4\cos(2x) \quad \text{and} \quad y_3{}''(x) = 2\cos^2(x) - 2\sin^2(x) \quad ,$$

and form the corresponding Wronskian,

$$W(x) = W[1, \cos(2x)\,, \sin^2(x)] = \begin{vmatrix} 1 & \cos(2x) & \sin^2(x) \\ 0 & -2\sin(2x) & 2\sin(x)\cos(x) \\ 0 & -4\cos(2x) & 2\cos^2(x) - 2\sin^2(x) \end{vmatrix} \quad .$$

Rather than compute out this determinant for 'all' values of x *(which could be very tedious), let us simply pick a convenient value for* x *, say* $x = 0$ *, and compute the Wronskian at that point:*

$$W(0) = \begin{vmatrix} 1 & \cos(2\cdot 0) & \sin^2(0) \\ 0 & -2\sin(2\cdot 0) & 2\sin(0)\cos(0) \\ 0 & -4\cos(2\cdot 0) & 2\cos^2(0) - 2\sin^2(0) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & -4 & 2 \end{vmatrix} = 0 \quad .$$

Theorem 14.3 assures us that, since this Wronskian vanishes at that one point, it must vanish everywhere. More importantly for us, this theorem also tells us that $\{1,\ \cos(2x)\,,\ \sin^2(x)\}$ *is not a fundamental set of solutions for our differential equation.*

!► *Example 14.6:* *Now consider the functions*

$$y_1(x) = 1 \quad , \quad y_2(x) = \cos(2x) \quad \text{and} \quad y_3(x) = \sin(2x) \quad .$$

Again, you can easily verify that all are solutions (over the entire real line) to the homogeneous third-order linear differential equation

$$y''' + 4y' = 0 \quad .$$

So, is

$$\{1,\ \cos(2x)\,,\ \sin(2x)\}$$

a fundamental set of solutions for our differential equation, above? To check we compute the appropriate derivatives and form the corresponding Wronskian,

$$\begin{aligned} W(x) &= W[1, \cos(2x)\,, \sin(2x)] \\ &= \begin{vmatrix} y_1 & y_2 & y_3 \\ y_1{}' & y_2{}' & y_3{}' \\ y_1{}'' & y_2{}'' & y_3{}'' \end{vmatrix} = \begin{vmatrix} 1 & \cos(2x) & \sin(2x) \\ 0 & -2\sin(2x) & 2\cos(2x) \\ 0 & -4\cos(2x) & -2\sin(2x) \end{vmatrix} \quad . \end{aligned}$$

Letting $x = 0$ *, we get*

$$W(0) = \begin{vmatrix} 1 & \cos(2\cdot 0) & \sin(2\cdot 0) \\ 0 & -2\sin(2\cdot 0) & 2\cos(2\cdot 0) \\ 0 & -4\cos(2\cdot 0) & -2\sin(2\cdot 0) \end{vmatrix} = \begin{vmatrix} 1 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & -4 & 0 \end{vmatrix} = 8 \neq 0 \quad .$$

Theorem 14.3 assures us that, since this Wronskian is nonzero at one point, it is nonzero everywhere, and that $\{1,\ \cos(2x)\,,\ \sin(2x)\}$ *is a fundamental set of solutions for our differential equation. Hence,*

$$y(x) = c_1 \cdot 1 + c_2\cos(2x) + c_3\sin(2x)$$

is a general solution to our third-order differential equation.

Additional Exercises

14.1 a. *Assume* y *is a solution to*

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + \sin(x)\,y = 0$$

over the interval $(0,\infty)$ *. Keep in mind that this automatically requires* y *,* y' *and* y'' *to be defined at each point in* $(0,\infty)$ *. Thus, both* y *and* y' *are differentiable on this interval and, as you learned in calculus, this means that* y *and* y' *must be continuous on* $(0,\infty)$ *. Now, rewrite the above equation to obtain a formula for* y'' *in terms of* y *and* y' *, and, using this formula, show that* y'' *must also be continuous on* $(0,\infty)$ *. Why can we not be sure that* y'' *is continuous at* 0 *?*

b. *Let $\mathcal{I}$ be some interval, and assume y satisfies*

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0$$

over $\mathcal{I}$. Assume, further, that a, b and c, as well as both y and y' are continuous functions over $\mathcal{I}$, and that a is never zero on $\mathcal{I}$. Show that y'' also must be continuous on $\mathcal{I}$. Why do we require that a never vanishes on $\mathcal{I}$?

c. *Let $\mathcal{I}$ be some interval, and assume y satisfies*

$$a_0 y^{(N)} + a_1 y^{(N-1)} + \cdots + a_{N-2}y'' + a_{N-1}y' + a_N y = 0$$

over $\mathcal{I}$. Assume, further, that the a_k's, as well as y, y', $\ldots$ and $y^{(n-1)}$ are continuous functions over $\mathcal{I}$, and that a_0 is never zero on $\mathcal{I}$. Show that $y^{(N)}$ also must be continuous on $\mathcal{I}$. Why do we require that a_0 never vanishes on $\mathcal{I}$?

14.2. *The following exercises all refer to theorem 14.1 on page 302 and the following pair of functions:*

$$\{y_1, y_2\} = \left\{x^2, x^3\right\} \quad .$$

a. *Using the theorem, verify that*

$$\left\{x^2, x^3\right\}$$

is a fundamental solution set for

$$x^2y'' - 4xy' + 6y = 0$$

over the interval $(0, \infty)$.

b. *Find the constants c_1 and c_2 so that*

$$y(x) = c_1x^2 + c_2x^3$$

satisfies the initial conditions

$$y(1) = 0 \quad \text{and} \quad y'(1) = -4 \quad .$$

c. *Attempt to find the constants c_1 and c_2 so that*

$$y(x) = c_1x^2 + c_2x^3$$

satisfies the initial conditions

$$y(0) = 0 \quad \text{and} \quad y'(0) = -4 \quad .$$

What 'goes wrong'. Why does this not violate the claim in theorem 14.1 about initial-value problems being solvable?

14.3. *Particular solutions to the differential equation in each of the following initial-value problems can found by assuming*

$$y(x) = e^{rx}$$

where r is a constant to be determined. To determine these constants, plug this formula for y into the differential equation, observe that the resulting equation miraculously simplifies to a simple algebraic equation for r, and solve for the possible values of r.

Do that with each equation; then use those solutions and the big theorem on general solutions to second order, homogeneous linear equations (theorem 14.1 on page 302) to construct a general solution, and, finally, solve the given initial-value problem:

a. $y'' + y' - 2y = 0$ *with* $y(0) = 1$ *and* $y'(0) = 1$

b. $y'' + 4y' + 3y = 0$ *with* $y(0) = 2$ *and* $y'(0) = -1$

c. $6y'' - 5y' + y = 0$ *with* $y(0) = 4$ *and* $y'(0) = 0$

d. $y'' + 3y' = 0$ *with* $y(0) = -2$ *and* $y'(0) = 3$

14.4. *Find solutions of the form*

$$y(x) = e^{rx}$$

where r is a constant (as in the previous exercise) and use the solutions found (along with the results given in theorem 14.2 on page 304) to construct general solutions to the following differential equations:

a. $y''' - 9y' = 0$

b. $y^{(4)} - 10y'' + 9y = 0$