AST1100
Introduction to astrophysics

Lecture Notes

Institute of
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AST1100 Lecture Notes

1–2 Celestial Mechanics

1 Kepler's Laws

Kepler used Tycho Brahe's detailed observations of the planets to deduce three laws concerning their motion:

1. The orbit of a planet is an ellipse with the Sun in one of the foci.
2. A line connecting the Sun and the planet sweeps out equal areas in equal time intervals.
3. The orbital period around the Sun and the semimajor axis (see figure 4 on page 8 for the definition) of the ellipse are related through:

$$P^2 = a^3, \tag{1}$$

where P is the period in years and a is the semimajor axis in AU (astronomical units, 1 AU = the distance between the Earth and the Sun).

Whereas the first law describes the shape of the orbit, the second law is basically a statement about the orbital velocity: When the planet is closer to the Sun it needs to have a higher velocity than when far away in order to sweep out the same area in equal intervals. The third law is a mathematical relation between the size of the orbit and the orbital period. As an example we see that when the semimajor axis doubles, the orbital period increases by a factor $2\sqrt{2}$ (do you agree?).

The first information that we can extract from Kepler's laws is a relation between the velocity of a planet and the distance from the Sun. When the distance from the Sun increases, does the orbital velocity increase or decrease? If we consider a nearly circular orbit, the distance traveled by the planet in one orbit is $2\pi a$, proportional to the semimajor axis. The mean velocity can thus be expressed as $v_m = 2\pi a/P$ which using Kepler's third law simply gives $v_m \propto a/(a^{3/2}) \propto 1/\sqrt{a}$ (check that you understood this!). Thus, the mean orbital velocity of a planet decreases the further away it is from the Sun.

When Newton discovered his law of gravitation,

$$\vec{F} = \frac{Gm_1 m_2}{r^2}\vec{e}_r,$$

he was able to deduce Kepler's laws from basic principles. Here $\vec{F}$ is the gravitational force between two bodies of mass m_1 and m_2 at a distance r and G is the gravitational constant. The unit vector in the direction of the force is denoted by $\vec{e}_r$.

2 General solution to the two-body problem

Kepler's laws is a solution to the *two-body problem*: Given two bodies with mass m_1 and m_2 at a positions $\vec{r}_1$ and $\vec{r}_2$ moving with speeds $\vec{v}_1$ and $\vec{v}_2$ (see figure 1). The only force acting on these two masses is their mutual gravitational attraction. How can we describe their future motion as a function of time? The rest of this lecture will be devoted to this problem.

In order to solve the problem we will now describe the motion from the rest frame of mass 1: We will sit on m_1 and describe the observed motion of m_2, i.e. the motion of m_2 with respect to m_1. (As an example this could be the Sun-Earth system, from the Earth you view the motion of the Sun). The only force acting on m_2 (denoted $\vec{F}_2$) is the gravitational pull from m_1. Using Newton's second law for m_2 we get

$$\vec{F}_2 = -G\frac{m_1 m_2}{|\vec{r}|^3}\vec{r} = m_2\ddot{\vec{r}}_2, \tag{2}$$

where $\vec{r} = \vec{r}_2 - \vec{r}_1$ the vector pointing from m_1 to m_2 (or from the Earth to the Sun in our example). Overdots describe derivatives with respect to time,

$$\dot{\vec{r}} = \frac{d\vec{r}}{dt}$$
$$\ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2}$$

Sitting on m_1, we need to find the vector $\vec{r}(t)$ as a function of time (in our example this would be the position vector of the Sun as seen from the Earth). This function would completely describe the motion of m_2 and be a solution to the two-body problem (do you see this?).

Using Newton's third law, we have a similar equation for the force acting on m_1

$$\vec{F}_1 = -\vec{F}_2 = G\frac{m_1 m_2}{|\vec{r}|^3}\vec{r} = m_1\ddot{\vec{r}}_1. \tag{3}$$

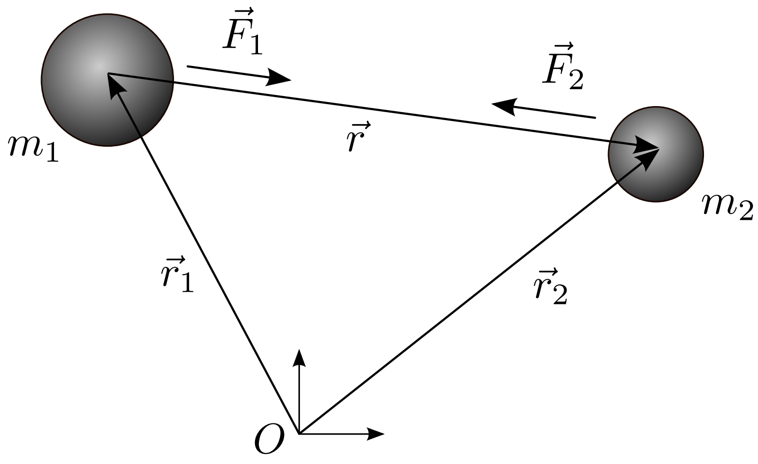

Figure 1: The two-body problem.

Subtracting equation (3) from (2), we can eliminate $\vec{r}_1$ and $\vec{r}_2$ and obtain an equation only in $\vec{r}$ which is the variable we want to solve for,

$$\ddot{\vec{r}} = \ddot{\vec{r}}_2 - \ddot{\vec{r}}_1 = -G\frac{m_1 + m_2}{|\vec{r}\,|^3}\vec{r} \equiv -m\frac{\vec{r}}{r^3}, \tag{4}$$

where $r = |\vec{r}\,|$ and $m = G(m_1 + m_2)$. This is the equation of motion of the two-body problem,

$$\ddot{\vec{r}} + m\frac{\vec{r}}{r^3} = 0. \tag{5}$$

We are looking for a solution of this equation with respect to $\vec{r}(t)$, this would be the solution to the two-body problem predicting the movement of m_2 with respect to m_1.

To get further, we need to look at the geometry of the problem. We introduce a coordinate system with m_1 at the origin and with $\vec{e}_r$ and $\vec{e}_\theta$ as unit vectors. The unit vector $\vec{e}_r$ points in the direction of m_2 such that $\vec{r} = r\vec{e}_r$ and $\vec{e}_\theta$ is perpendicular to $\vec{e}_r$ (see figure 2). At a given moment, the unit vector $\vec{e}_r$ (which is time dependent) makes an angle θ with a given fixed (in time) coordinate system defined by unit vectors $\vec{e}_x$ and $\vec{e}_y$. From figure 2 we see that (do you really see this? Draw some figures to convince yourself!)

$$\begin{aligned}
\vec{e}_r &= \cos\theta\vec{e}_x + \sin\theta\vec{e}_y \\
\vec{e}_\theta &= -\sin\theta\vec{e}_x + \cos\theta\vec{e}_y
\end{aligned}$$

The next step is to substitute $\vec{r} = r\vec{e}_r$ into the equation of motion (equation 5). In this process we will need the time derivatives of the unit vectors,

$$\begin{aligned}
\dot{\vec{e}}_r &= -\dot{\theta}\sin\theta\vec{e}_x + \dot{\theta}\cos\theta\vec{e}_y \\
&= \dot{\theta}\vec{e}_\theta \\
\dot{\vec{e}}_\theta &= -\dot{\theta}\cos\theta\vec{e}_x - \dot{\theta}\sin\theta\vec{e}_y \\
&= -\dot{\theta}\vec{e}_r
\end{aligned}$$

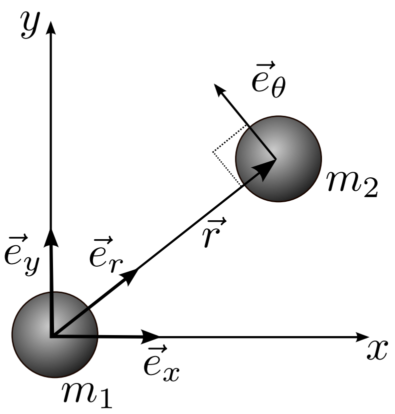

Figure 2: Geometry of the two-body problem.

Using this, we can now take the derivative of $\vec{r} = r\vec{e}_r$ twice,

$$\begin{aligned}\dot{\vec{r}} &= \dot{r}\vec{e}_r + r\dot{\vec{e}}_r \\ &= \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta \\ \ddot{\vec{r}} &= \ddot{r}\vec{e}_r + \dot{r}\dot{\vec{e}}_r + (\dot{r}\dot{\theta} + r\ddot{\theta})\vec{e}_\theta + r\dot{\theta}\dot{\vec{e}}_\theta \\ &= (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\vec{e}_\theta.\end{aligned}$$

Substituting $\vec{r} = r\vec{e}_r$ into the equation of motion (equation 5), we thus obtain

$$(\ddot{r} - r\dot{\theta}^2)\vec{e}_r + \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})\vec{e}_\theta = -\frac{m}{r^2}\vec{e}_r.$$

Equating left and right hand sides, we have

$$\ddot{r} - r\dot{\theta}^2 = -\frac{m}{r^2} \tag{6}$$

$$\frac{d}{dt}(r^2\dot{\theta}) = 0 \tag{7}$$

The vector equation (equation 5) has thus been reduced to these two scalar equations. Go back and check that you understood the transition.

The last of these equations indicates a constant of motion, something which does not change with time (why?). What constant of motion enters in this situation? Certainly the angular momentum of the system should be a constant of motion so let's check the expression for the angular momentum vector $\vec{h}$ (note that h is defined as angular momentum per mass, $(\vec{r}\times\vec{p})/m_2$ (remember that m_1 is at rest in our current coordinate frame)):

$$|\vec{h}| = |\vec{r} \times \dot{\vec{r}}| = |(r\vec{e}_r) \times (\dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta)| = r^2\dot{\theta}.$$

So equation (7) just tells us that the magnitude of the angular momentum $h = r^2\dot{\theta}$ is conserved, just as expected.

To solve the equation of motion, we are left with solving equation (6). In order to find a solution we will

1. solve for r as a function of angle θ instead of time t. This will give us the distance of the planet as a function of angle and thus the orbit.
2. Make the substitution $u(\theta) = 1/r(\theta)$ and solve for $u(\theta)$ instead of $r(\theta)$. This will transform the equation into a form which can be easily solved.

In order to substitute u in equation (6), we need its derivatives. We start by finding the derivatives of u with respect to θ,

$$\frac{du(\theta)}{d\theta} = \dot{u}\frac{dt}{d\theta} = -\frac{\dot{r}}{r^2}\frac{1}{\dot{\theta}} = -\frac{\dot{r}}{h}$$
$$\frac{d^2u(\theta)}{d\theta^2} = -\frac{1}{h}\frac{d}{d\theta}\dot{r} = -\frac{1}{h}\ddot{r}\frac{1}{\dot{\theta}}.$$

In the last equation, we substitute $\ddot{r}$ from the equation of motion (6),

$$\frac{d^2u(\theta)}{d\theta^2} = \frac{1}{h\dot{\theta}}(\frac{m}{r^2} - r\dot{\theta}^2) = \frac{m}{h^2} - \frac{1}{r} = \frac{m}{h^2} - u,$$

where the relation $h = r^2\dot{\theta}$ was used twice. We thus need to solve the following equation

$$\frac{d^2u(\theta)}{d\theta^2} + u = \frac{m}{h^2}$$

This is just the equation for a harmonic oscillator (if you have not encountered the harmonic oscillator in other courses yet, it will soon come, it is simply the equation of motion for an object which is attached to a spring in motion) with known solution:

$$u(\theta) = \frac{m}{h^2} + A\cos(\theta - \omega),$$

where A and ω are constants depending on the initial conditions of the problem. Try now to insert this solution into the previous equation to see that this is indeed the solution. Substituting back we now find the following expression for r:

The general solution to the two-body problem

$$r = \frac{p}{1 + e\cos f} \tag{8}$$

where $p = h^2/m$, $e = (Ah^2/m)$ and $f = \theta - \omega$.

We recognize this expression as the general expression for a conic section.

3 Conic sections

Conic sections are curves defined by the intersection of a cone with a plane as shown in figure 3. Depending on the inclination of the plane, conic sections can be divided into three categories with different values of p and e in the general solution to the two-body problem (equation 8),

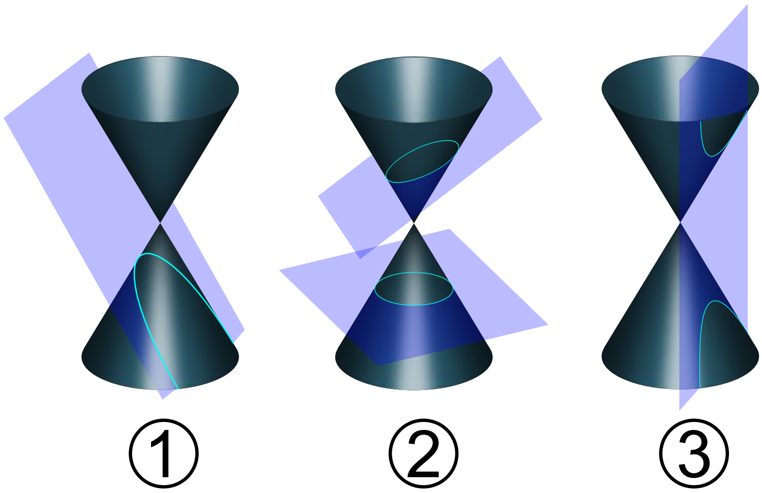

Figure 3: Conic sections: **Circle:** e=0,p=a, **Ellipse:** $0 \leq e < 1$, $p = a(1-e^2)$, **Parabola:** $e = 1$, $p = 2a$, **Hyperbola:** $e > 1$ and $p = a(e^2 - 1)$

1. the ellipse, $0 \leq e < 1$ and $p = a(1 - e^2)$ (of which the circle, $e = 0$, is a subgroup),
2. the parabola, $e = 1$ and $p = 2a$,
3. the hyperbola, $e > 1$ and $p = a(e^2 - 1)$.

In all these cases, a is defined as a positive constant $a \geq 0$. Of these curves, only the ellipse represents a bound orbit, in all other cases the planet just passes the star and leaves. We will discuss the details of an elliptical orbit later. First, we will check which conditions decides which trajectory an object will follow, an ellipse, parabola or hyperbola. Our question is thus: If we observe a planet or other object close to a star, is it in orbit around the star or just passing by? For two masses to be gravitationally bound, we expect that their total energy, kinetic plus potential, would be less than zero, $E < 0$. Clearly the total energy of the system is an important initial condition deciding the shape of the trajectory.

We will now investigate how the trajectory $r(\theta)$ depends on the total energy. In the exercises you will show that the total energy of the system can be written:

Total energy of a two-body system

$$E = \frac{1}{2}\hat{\mu}v^2 - \frac{\hat{\mu}m}{r}, \tag{9}$$

where $v = |\dot{\vec{r}}|$, the velocity of m_2 observed from m_1 (or vice versa) and $\hat{\mu} = m_1 m_2/(m_1 + m_2)$.

We will now try to rewrite the expression for the energy E in a way which will help us to decide the relation between the energy of the system and the shape of the orbit. We will start by rewriting the velocity in terms of its radial and tangential components using the fact that $\vec{v} = \dot{\vec{r}} = \dot{r}\vec{e}_r + r\dot{\vec{e}}_r$

$$v^2 = v_r^2 + v_\theta^2 = \dot{r}^2 + (r\dot{\theta})^2, \tag{10}$$

decomposed into velocity along $\vec{e}_r$ and $\vec{e}_\theta$ (check that you got this!). We need the time derivative of r. Taking the derivative of equation (8),

$$\dot{r} = \frac{pe\sin f}{(1+e\cos f)^2}\dot{\theta},$$

we get from equation (10) for the velocity

$$v^2 = \dot{\theta}^2\frac{p^2e^2\sin^2 f}{(1+e\cos f)^4} + r^2\dot{\theta}^2.$$

Next step is in both terms to substitute $\dot{\theta} = h/r^2$ (where did this come from?) and then using equation (8) for r giving

$$v^2 = \frac{h^2e^2\sin^2 f}{p^2} + \frac{h^2(1+e\cos f)^2}{p^2}.$$

Collecting terms and remembering that $\cos^2 f + \sin^2 f = 1$ we obtain

$$v^2 = \frac{h^2}{p^2}(1 + e^2 + 2e\cos f).$$

We will now get back to the expression for E. Substituting this expression for v as well as r from equation (8) into the energy expression (equation 9), we obtain

$$E = \frac{1}{2}\hat{\mu}\frac{h^2}{p^2}(1 + e^2 + 2e\cos f) - \hat{\mu}m\frac{1+e\cos f}{p} \tag{11}$$

Total energy is conserved and should therefore be equal at any point in the orbit, i.e. for any angle f. We may therefore choose an angle f which is such that this expression for the energy will be easy to evaluate. We will consider the energy at the point for which $\cos f = 0$,

$$E = \frac{1}{2}\hat{\mu}\frac{h^2}{p^2}(1+e^2) - \frac{\hat{\mu}m}{p}$$

We learned above (below equation 8) that $p = h^2/m$ and thus that $h = \sqrt{mp}$. Using this to eliminate h from the expression for the total energy we get

$$E = \frac{\hat{\mu}m}{2p}(e^2 - 1).$$

If the total energy $E = 0$ then we immediately get $e = 1$. Looking back at the properties of conic sections we see that this gives a parabolic trajectory. Thus, masses which have just too much kinetic energy to be bound will follow a parabolic trajectory. If the total energy is different from zero, we may rewrite this as

$$p = \frac{\hat{\mu}m}{2E}(e^2 - 1).$$

We now see that a negative energy E (i.e. a bound system) gives an expression for p following the expression for an ellipse in the above list of properties for conic sections (by defining $a = \hat{\mu}m/(2|E|)$. Similarly a positive energy gives the expression for a hyperbola. We have shown that the total energy of a system determines whether the trajectory will be an ellipse (bound systems $E < 0$), hyperbola (unbound system $E > 0$) or parabola ($E = 0$). We have just shown Kepler's first law of motion, stating that a bound planet follows an elliptical orbit. In the exercises you will also show Kepler's second and third law using Newton's law of gravitation.

4 The elliptical orbit

We have seen that the elliptical orbit may be written in terms of the distance r as

$$r = \frac{a(1-e^2)}{1+e\cos f}.$$

In figure (4) we show the meaning of the different variables involved in this equation:

- a is the *semimajor axis*
- b is the *semiminor axis*
- e is the *eccentricity* defined as $e = \sqrt{1-(b/a)^2}$
- m_1 is located in the *principal focus*
- the point on the ellipse closest to the principal focus is called *perihelion*
- the point on the ellipse farthest from the principal focus is called *aphelion*
- the angle f is called the *true anomaly*

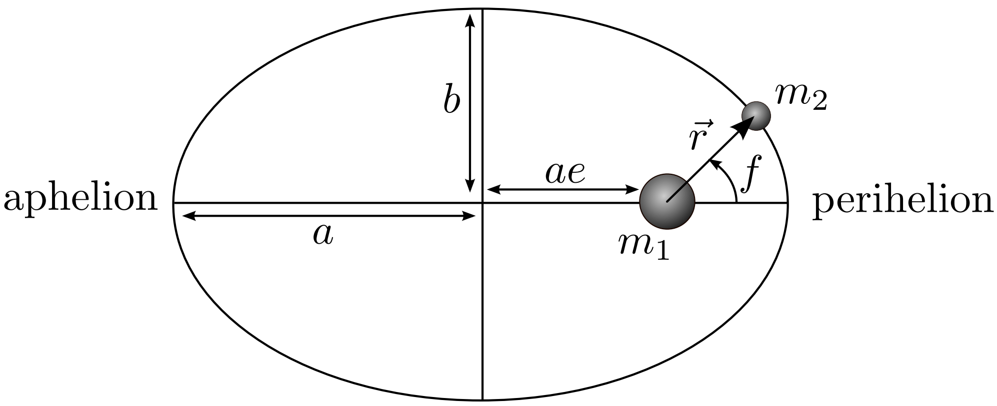

Figure 4: The ellipse.

The eccentricity is defined using the ratio b/a. When the semimajor and semiminor axis are equal, $e = 0$ and the orbit is a circle. When the semimajor axis is much larger than the semiminor axis, $e \to 1$.

5 Center of mass system

In the previous section we showed that seen from the rest frame of one of the masses in a two-body system, the other mass follows an elliptical / parabolic / hyperbolic trajectory. How does this look from a frame of reference which is not at rest with respect to one of the masses? We know that both masses m_1 and m_2 are moving due to the gravitational attraction from the other. If we observe a distant star-planet system, how does the planet *and* the star move with respect to each other? We have only shown that sitting on either the planet or the star, the other body will follow an elliptical orbit.

An elegant way to describe the full motion of the two-body system (or in fact an N-body system) is to introduce *center of mass coordinates*. The center of mass position $\vec{R}$ is located at a point on the line between the two masses m_1 and m_2. If the two masses are equal, the center of mass position is located exactly halfway between the two masses. If one mass is larger than the other, the center of mass is located closer to the more massive body. The center of mass is a weighted mean of the position of the two masses:

$$\vec{R} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2, \tag{12}$$

where $M = m_1 + m_2$. We can similarly define the center of mass for an N-body system as

$$\vec{R} = \sum_{i=1}^{N} \frac{m_i}{M}\vec{r}_i, \tag{13}$$

where $M = \sum_i m_i$ and the sum is over all N masses in the system. Newton's second law for one object in the system is

$$\vec{f}_i = m_i \ddot{\vec{r}}_i$$

where $\vec{f}_i$ is the total force on object i. Summing over all bodies in the system, we obtain Newton's second law for the full N-body system

$$\vec{F} = \sum_{i=1}^{N} m_i \ddot{\vec{r}}_i, \tag{14}$$

where $\vec{F}$ is the total force on all masses in the system. We may divide the total force on all masses into one contribution from internal forces between masses and one contribution from external forces,

$$\vec{F} = \sum_i \sum_{j \neq i} \vec{f}_{ij} + \vec{F}_{\text{ext}},$$

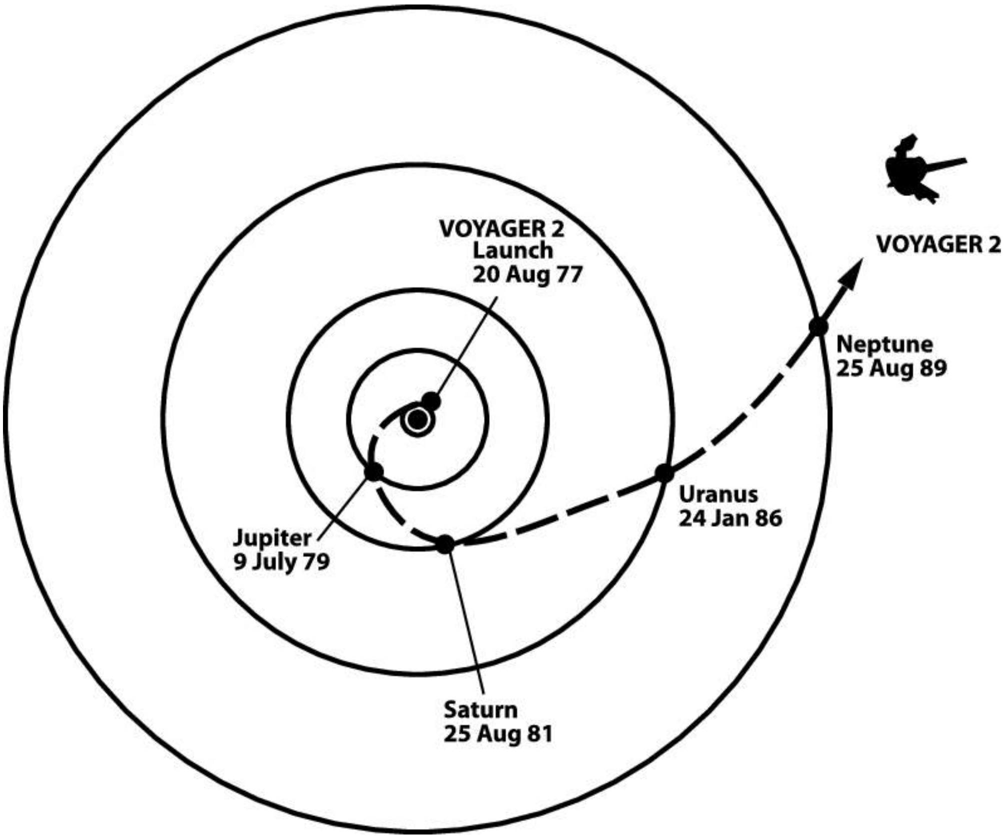

Figure 5: Info-figure: A diagram of the trajectory that enabled NASA's Voyager 2 spacecraft to tour all the four gas giants and achieve a large enough velocity to escape our solar system. Celestial mechanics obviously played an integral part in the extremely careful planning that was needed in order to carry out the probe's ambitious tour of the outer solar system. The planetary flybys not only allowed for close-up observations of the planets and their moons, but also accelerated the probe so that it could reach the next object. In 2012 Voyager 2 was at a distance of roughly 100 AU from the Sun, traveling outward at around 3.3 AU per year. It is expected to keep transmitting weak radio messages until at least 2025.

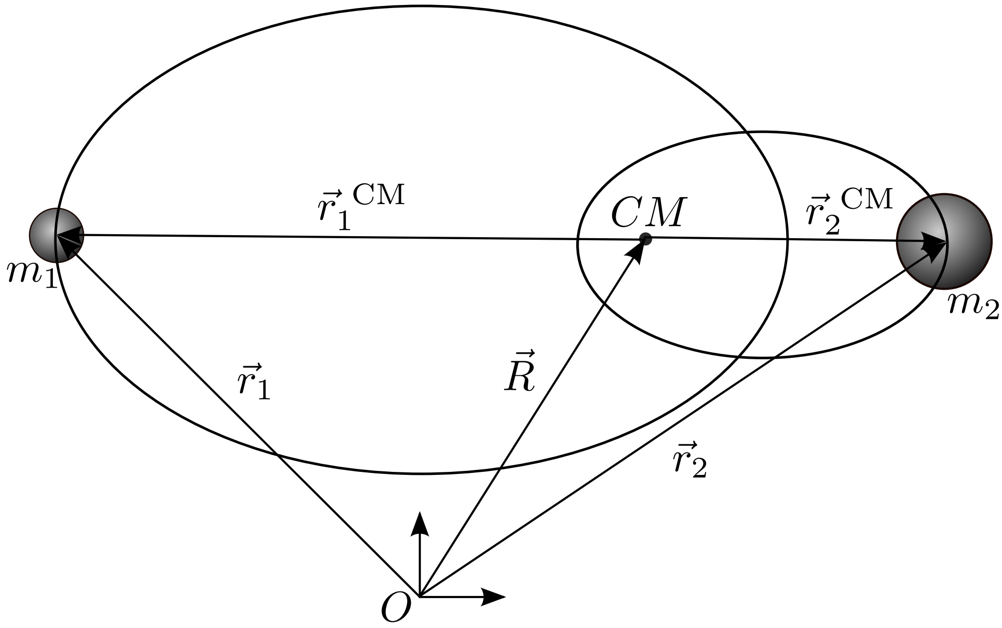

Figure 6: The center of mass system: The center of mass (CM) is indicated by a small point. The two masses m_1 and m_2 orbit the center of mass in elliptical orbits with the center of mass in one focus of both ellipses. The center of mass vectors $\vec{r}_1^{\mathrm{CM}}$ and $\vec{r}_2^{\mathrm{CM}}$ start at the center of mass and point to the masses.

where $\vec{f}_{ij}$ is the gravitational force on mass i from mass j. Newton's third law implies that the sum over all internal forces vanish ($\vec{f}_{ij} = -\vec{f}_{ji}$). The right side of equation (14) can be written in terms of the center of mass coordinate using equation (13) as

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i = M\ddot{\vec{R}},$$

giving

$$M\ddot{\vec{R}} = \vec{F}_{\mathrm{ext}}.$$

(Check that you followed this deduction!). If there are no external forces on the system of masses ($\vec{F}_{\mathrm{ext}} = 0$), this equation tells us that the center of mass position does not accelerate, i.e. if the center of mass position is at rest it will remain at rest, if the center of mass position moves with a given velocity it will keep moving with this velocity. We may thus divide the motion of a system of masses into the motion of the center of mass and the motion of the individual masses with respect to the center of mass.

We now return to the two-body system assuming that no external forces act on the system. The center of mass moves with constant velocity and we decide to deduce the motion of the masses with respect to the center of mass system, i.e. the rest frame of the center of mass. We will thus be sitting at the center of mass which we define as the origin of our coordinate system, looking at the motion of the two masses. When we know the motion of the two masses with respect to the center of mass, we know the full motion of the system since we already know the motion of the center of mass position.

Since we take the origin at the center of mass location, we have $\vec{R} = 0$. Using equation (12) we get

$$0 = \frac{m_1}{M}\vec{r}_1^{\,\mathrm{CM}} + \frac{m_2}{M}\vec{r}_2^{\,\mathrm{CM}},$$

where CM denotes position in the center of mass frame (see figure 6). Combining this equation with the fact that $\vec{r} = \vec{r}_2 - \vec{r}_1 = \vec{r}_2^{\,\mathrm{CM}} - \vec{r}_1^{\,\mathrm{CM}}$ we obtain

$$\vec{r}_1^{\,\mathrm{CM}} = -\frac{\hat{\mu}}{m_1}\vec{r}, \tag{15}$$

$$\vec{r}_2^{\,\mathrm{CM}} = \frac{\hat{\mu}}{m_2}\vec{r}, \tag{16}$$

The **reduced mass** $\hat{\mu}$ is defined as

$$\hat{\mu} = \frac{m_1 m_2}{m_1 + m_2}.$$

The relative motion of the masses with respect to the center of mass can be expressed in terms of $\vec{r}_1^{\,\mathrm{CM}}$ and $\vec{r}_2^{\,\mathrm{CM}}$ as a function of time, or as we have seen before, as a function of angle f. We already know the motion of one mass with respect to the other,

$$|\vec{r}| = \frac{p}{1 + e\cos f}.$$

Inserting this into equations (15) and (16) we obtain

$$|\vec{r}_1^{\,\mathrm{CM}}| = \frac{\hat{\mu}}{m_1}|\vec{r}| = \frac{\hat{\mu}p}{m_1(1 + e\cos f)}$$

$$|\vec{r}_2^{\,\mathrm{CM}}| = \frac{\hat{\mu}}{m_2}|\vec{r}| = \frac{\hat{\mu}p}{m_2(1 + e\cos f)}$$

For a bound system we thus have

$$|\vec{r}_1^{\,\mathrm{CM}}| = \frac{\frac{\hat{\mu}}{m_1}a(1 - e^2)}{1 + e\cos f} \equiv \frac{a_1(1 - e^2)}{1 + e\cos f}$$

$$|\vec{r}_2^{\,\mathrm{CM}}| = \frac{\frac{\hat{\mu}}{m_2}a(1 - e^2)}{1 + e\cos f} \equiv \frac{a_2(1 - e^2)}{1 + e\cos f}$$

We see from these equations that for a gravitationally bound system, *both* masses move in elliptical orbits with the center of mass in one of the foci (how do you see this?). The semimajor axis of these two masses are given by

$$a_1 = \frac{\hat{\mu} a}{m_1},$$
$$a_2 = \frac{\hat{\mu} a}{m_2},$$
$$a = a_1 + a_2$$

(check that you understand how these equations come about) where a_1 and a_2 are the semimajor axis of m_1 and m_2 respectively and a is the semimajor axis of the elliptical orbit of one of the masses seen from the rest frame of the other. Note that the larger the mass of a given body with respect to the other, the smaller the ellipse. This is consistent with our intuition: The more massive body is less affected by the same force than is the less massive body. The Sun moves in an ellipse around the center of mass which is much smaller than the elliptical orbit of the Earth. Figure (6) shows the situation: the planet and the star orbit the common center of mass situated in one common focus of both ellipses.

6 Problems

Problem 1 (20–45 min.)

The scope of this problem is to deduce Kepler's second law. Kepler's second law can be written mathematically as

$$\frac{dA}{dt} = \text{constant},$$

i.e. that the area A swept out by the vector $\vec{r}$ per time interval is constant. We will now show this step by step:

1. Show that the infinitesimal area dA swept out by the radius vector $\vec{r}$ for an infinitesimal movement dr and $d\theta$ is $dA = \frac{1}{2} r^2 d\theta$.
2. Divide this expression by dt and you obtain an expression for dA/dt in terms of the radius r and the tangential velocity v_θ.
3. By looking back at the above derivations, you will see that the tangential velocity can be expressed as $v_\theta = h/r$.
4. Show Kepler's second law.

Problem 2 (20–45 min.)

The scope of this problem is to deduce Kepler's third law. Again we will solve this problem step by step:

1. In the previous problem we found an expression for dA/dt in terms of a constant. Integrate this equation over a full period P and show

that

$$P = \frac{2\pi ab}{h}$$

(Hint: the area of an ellipse is given by πab).

2. Use expressions for h and b found in the text to show that

$$P^2 = \frac{4\pi^2}{G(m_1 + m_2)} a^3 \qquad (17)$$

3. This expression obtained from Newtonian dynamics differs in an important way from the original expression obtain empirically by Kepler (equation 1). How? Why didn't Kepler discover it?

Problem 3 (15–30 min.)

1. How can you measure the mass of a planet in the solar system by observing the motion of one of its satellites? Assume that we know only the semimajor axis and orbital period for the elliptical orbit of the satellite around the planet. **Hint 1:** Kepler's third law (the exact version). **Hint 2:** You are allowed to make reasonable approximations.

2. Look up (using Internet or other sources) the semimajor axis and orbital period of Jupiter's moon Ganymede.

   (a) Use these numbers to estimate the mass of Jupiter.

   (b) Then look up the mass of Jupiter. How well did your estimate fit? Is this an accurate method for computing planetary masses?

   (c) Which effects could cause discrepancies from the real value and your estimated value?

Problem 4 (70–90 min.)

1. Show that the total energy of the two-body system in the center of mass frame can be written as

$$E = \frac{1}{2}\hat{\mu}v^2 - \frac{GM\hat{\mu}}{r},$$

where $v = |d\vec{r}/dt|$ is the relative velocity between the two objects, $r = |\vec{r}|$ is their relative distance, $\hat{\mu}$ is the reduced mass and $M \equiv m_1 + m_2$ is the total mass. **Hint:** make the calculation in the center of mass frame and use equation (15) and (16).

2. Show that the total angular momentum of the system in the center of mass frame can be written

$$\vec{P} = \hat{\mu}\vec{r} \times \vec{v},$$

3. Looking at the two expressions you have found for energy and angular momentum of the system seen from the center of mass frame: Can you find an equivalent two-body problem with two masses m_1' and m_2' where the equations for energy and momentum will be of the same form as the two equations which you have just derived? What are m_1' and m_2'? If you didn't understand the question, here is a rephrasing: If you were given these two equations without knowing anything else, which physical system would you say that the describe?

Problem 5 (optional 30–45 min.)

1. At which points in the elliptical orbit (for which angels f) is the velocity of a planet at maximum or minimum?
2. Using only the mass of the Sun, the semimajor axis and eccentricity of Earth's orbit (which you look up in Internet or elsewhere), can you find an estimate of Earth's velocity at aphelion and perihelion?
3. Look up the real maximum and minimum velocities of the Earth's velocity. How well do they compare to your estimate? What could cause discrepancies between your estimated values and the real values?
4. Use Python (or Matlab or any other programming language) to plot the variation in Earth's velocity during one year.

Hint 1: Use one or some of the expressions for velocity found in section (3) as well as expressions for p and h found in later sections (including the above problems). **Hint 2:** You are allowed to make reasonable approximations.

Problem 6 (optional 10–30 min.)

1. Find our maximum and minimum distance to the center of mass of the Earth-Sun system.
2. Find Sun's maximum and minimum distance to the center of mass of the Earth-Sun system.
3. How large are the latter distances compared to the radius of the Sun?

Problem 7: Numerical solution to the 2-body/3-body problem

In this problem you are first going to solve the 2-body problem numerically by a well-known numerical method. We will start by considering the ESA satellite Mars Express which entered an orbit around Mars in December 2003 (`http://www.esa.int/esaMI/Mars\_Express/index.html`). The goal of Mars Express is to map the surface of Mars with high resolution images. When Mars Express is at an altitude of 10107 km above the surface of Mars with a velocity 1166 m/s (with respect to Mars), the engines are turned off and the satellite has entered the orbit.

In this exercise we will use that the radius of Mars is 3400 km and the mass of Mars is 6.4×10^{23} kg. Assume the weight of the Mars Express spacecraft to be 1 ton.

1. (**4–5 hours**) A distance of 10107 km is far too large in order to obtain high resolution images of the surface. Thus, the orbit of Mars Express need to be very eccentric such that it is very close to the surface of Mars each time it reaches perihelion. We will now check this by calculating the orbit of Mars Express numerically. We will introduce a fixed Cartesian coordinate system to describe the motion of Mars and the satellite. Assume that at time $t = 0$ Mars has position $[x_1 = 0,\ y_1 = 0,\ z_1 = 0]$ and Mars express has position $[x_2 = 10107 + 3400$ km, $y_2 = 0,\ z_2 = 0]$ in this fixed coordinate system (see Figure 8).

   The velocity of Mars express is only in the positive y-direction at this moment. In our fixed coordinate system, the initial velocity vectors are therefore $\vec{v}_1 = 0$ (for Mars) and $\vec{v}_2 = 1166 \frac{\text{m}}{\text{s}} \vec{j}$ (for Mars Express), where $\vec{j}$ is the unitvector along the y-axis. There is no velocity component in the z-direction so we can consider the system as a 2-dimensional system with movement in the (x, y)-plane.

   Use Newton's second law,

   $$m\frac{d^2\vec{r}}{dt^2} = \vec{F},$$

   to solve the 2-body problem numerically. Use the Euler-Cromer method for differential equations. Plot the trajectory of Mars Express. Do 10^5 calculations with timestep $dt = 1$ second. Is the result what you would expect?
   Hints - Write Newton's second law in terms of the velocity vector.

   $$m\frac{d\vec{v}}{dt} = \vec{F} \Rightarrow m\left(\frac{d\vec{v_x}}{dt}\vec{i} + \frac{d\vec{v_y}}{dt}\vec{j}\right) = F_x\vec{i} + F_y\vec{j}$$

   Then we have the following relation between the change in the components of the velocity vector and the components of the force vector;

   $$\frac{dv_x}{dt} = \frac{F_x}{m}$$

   $$\frac{dv_y}{dt} = \frac{F_y}{m}$$

   These equations can be solved directly by the Euler-Cromer method and the given initial conditions. For each timestep (use a for- or while-loop), calculate the velocity $v_{x/y}(t + dt)$ (Euler's method) and the position $x(t + dt)$, $y(t + dt)$ (standard kinematics) for Mars and

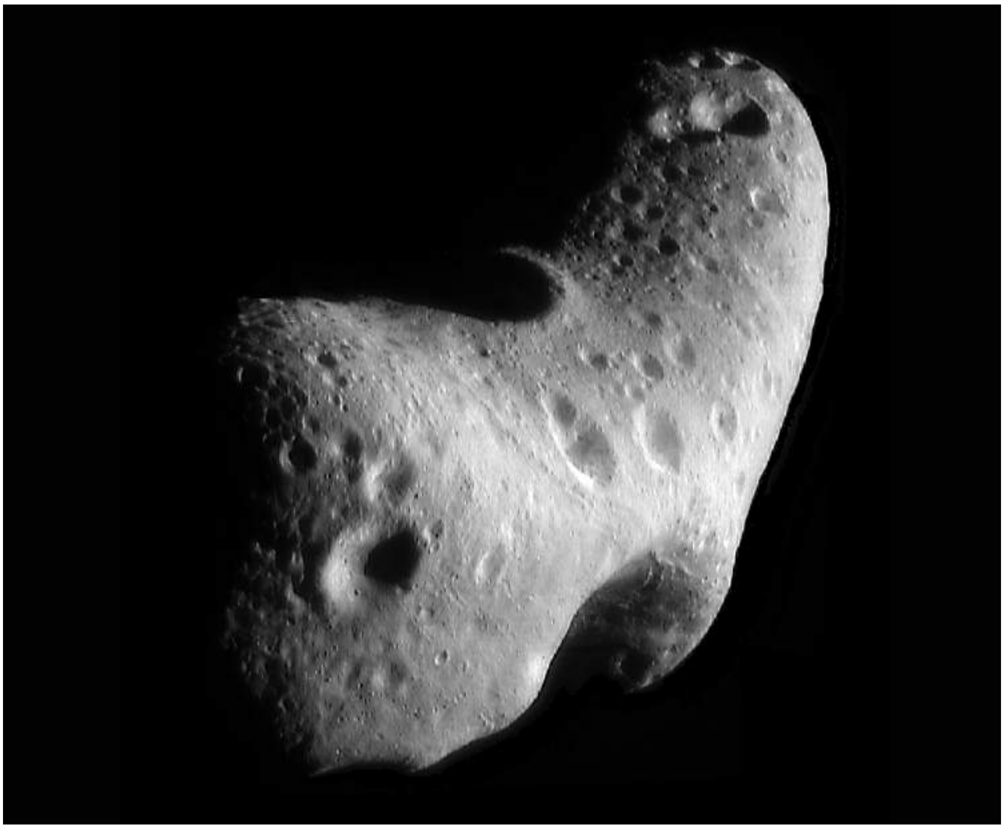

Figure 7: Info-figure: 433 Eros was the target of the first long-term, close-up study of an asteroid. After a four year journey the NEAR-Shoemaker space probe was inserted into orbit around the 33 km long, potato-shaped asteroid in February 2000 and encircled it 230 times from various distances before touching down on its surface. The primary scientific objective was to return data on the composition, shape, internal mass distribution, and magnetic field of Eros. Asteroids are a class of rocky small solar system bodies that orbit the Sun, mostly in the asteroid belt between Mars and Jupiter. They are of great interest to astronomers as they are leftover material from when the solar system formed some 4.6 billion years ago..

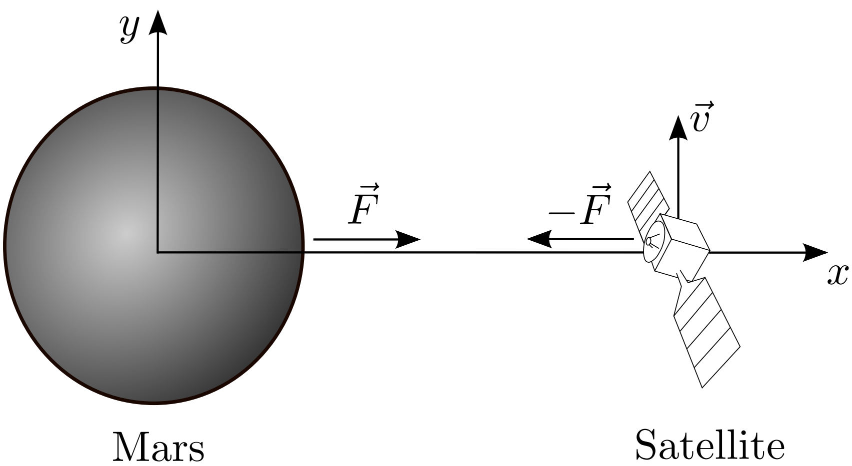

Figure 8: Mars and Mars Express at time t = 0.

the space craft (or even easier: use vectors). A good approach is to make a function that calculates the gravitational force. From the loop you send the previous timestep positiondata (4 values) to the function which calculates and returns the components of the force (4 values, two for Mars and two for the space craft) with correct negative/positive sign. Collect the values in arrays and use the standard python/scitools plot-command.

2. (**60–90 min**) Mars Express contained a small lander unit called Beagle 2. Unfortunately contact was lost with Beagle 2 just after it should have reached the surface. Here we will calculate the path that Beagle 2 takes down to the surface (this is not the real path that was taken). We will assume that the lander does not have any engines and is thus moving under the influence of only two forces: the force of gravity from Mars and the force of friction from the Martian atmosphere. The friction will continously lower the altitude of the orbit until the lander hits the surface of the planet. We will assume the weight of the lander to be 100 kg.

   We will now assume that Mars Express launches Beagle 2 when Mars Express is at perihelion. We will assume that it adjusts the velocity of Beagle 2 such that it has a velocity of 4000 m/s (with respect to Mars) at this point. Thus we have 2-body problem as in the previous exercise. At t = 0, the position of Mars and the lander is $[x_1 = 0,\ y_1 = 0,\ z_1 = 0]$ and $[x_2 = -298 - 3400$ km, $y_2 = 0,\ z_2 = 0]$ respectively (Figure 9). The initial velocity vector of the lander is $\vec{v_1} = -4000\ \frac{\mathrm{m}}{\mathrm{s}}\vec{j}$ with respect to Mars. Due to Mars'atmosphere a force of friction acts on the lander which is always in the direction opposite to the velocity vector. A simple model of this force is given by

   $$\vec{f} = -k\vec{v},$$

   where $k = 0.00016$ kg/s is the friction constant due to Mars' atmosphere. We will assume this to have the same value for the full orbit.

   Plot the trajectory that the lander takes down to the surface of Mars. Set $dt = 1$ second.

   Hints - You can use most of the code from the previous exercise. First, we write Newton's second law in terms of the cartesian components;

   $$\frac{dv_x}{e} = \frac{F_x + f_x}{m}$$
   $$\frac{dv_y}{dt} = \frac{F_y + f_y}{m}$$

   (or use the vector from direcly if you prefer). The best approach is to make one more function that calculates the force of friction with the lander's velocity components as arguments. In this problem you

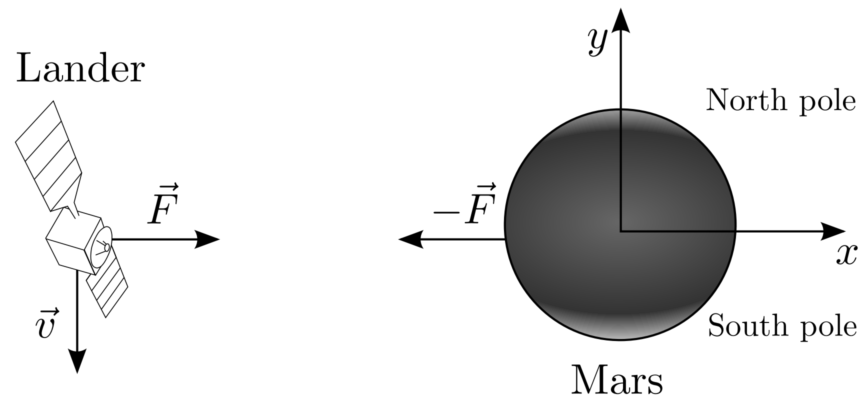

Figure 9: Mars and Beagle 2 at time t = 0.

should use a while-loop. For each evaluation, first call the gravitational function (as before) and then the friction function. Remember to send the space craft's velocity components from the previous timestep. In the friction function, first calculate the total force F, and then (if you do not use vectors) the components F_x and F_y (use simple trigonometry) with correct positive/negative-sign by checking the sign of the velocity components. Then return the force components to the loop. For each evaluation (in the while-loop) check whether the spaceship has landed or not.

3. Use the trajectory of the previous exercise to check the landing site: Was the lander supposed to study the ice of the Martian poles or the rocks at the Martian equator? Use figure 9 to identify the position of the poles with respect to the geometry of the problem (the result does not have any relation with the objectives or landing site of the real Beagle 2 space craft)

4. (**90–120 min.**) Finally, we will use our code to study the 3-body problem. There is no analytical solution to the 3-body problem, so in this case we are forced to use numerical calculations. The fact that most problems in astrophysics consider systems with a huge number of objects strongly underlines the fact that numerical solutions are of great importance.

   About half of all the stars are binary stars, two stars orbiting a common center of mass. Binary star systems may also have planets orbiting the two stars. Here we will look at one of many possible shapes of orbits of such planets. We will consider a planet with the mass identical to the mass of Mars. One of the stars has a mass identical to the mass of the Sun (2×10^{30} kg), the other has a mass 4 times that of the Sun.

   The initial positions are [$x_1 = -1.5$ AU, $y_1 = 0$, $z_1 = 0$] (for the planet), [$x_2 = 0$, $y_2 = 0$, $z_2 = 0$] (for the small star) and [$x_3 = 3$ AU, $y_3 = 0$, $z_3 = 0$] (for the large star) (Figure 3). The initial velocity vectors are $\vec{v_1} = -1\ \frac{\text{km}}{\text{s}}\vec{j}$ (for the planet), $\vec{v_2} = 30\ \frac{\text{km}}{\text{s}}\vec{j}$ (for the small

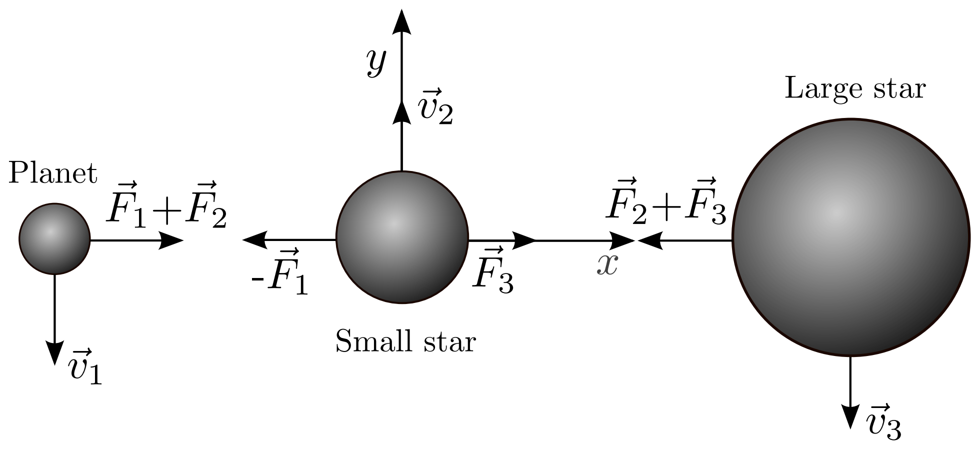

Figure 10: The binary star system with the planet at time t = 0.

star) and $\vec{v_3} = -7.5\ \frac{\text{km}}{\text{s}}\vec{j}$ (for the large star).

Plot the orbit of the planet and the two stars in the same figure. Use timestep $dt = 400$ seconds and make 10^6 calculations. It should now be clear why it is impossible to find an analytical solution to the 3-body problem. Note that the solution is an approximation. If you try to change the size and number of time steps you will get slighly different orbits, small time steps cause numerical problems and large time steps is too inaccurate. The given time step is a good trade-off between the two problems but does not give a very accurate solution. Accurate methods to solve this problem is outside the scope of this course. Play around and try some other starting positions and/or velocities.

Hints: There is really not much more code you need to add to the previous code to solve this problem. Declare arrays and constants for the three objects. In your for-/while-loop, calculate the total force components for each object. Since we have a 3-body problem we get two contributions to the total force for each object. In other words, you will have to call the function of gravitation three times for each time-evaluation. For each time step, first calculate the force components between the planet and the small star, then the force components between the planet and the large star, and finally the force components between the small and the large star. Then you sum up the contributions that belong to each object.

5. Look at the trajectory and try to imagine how the sky will look like at different epochs. If we assume that the planet has chemical conditions for life equal to those on earth, do you think it is probable that life will evolve on this planet? Use your tracetory to give arguments.

AST1100 Lecture Notes

3 Extrasolar planets

1 Detecting extrasolar planets

Most models of star formation tell us that the formation of planets is a common process. We expect most stars to have planets orbiting them. Why then, has only a very few planets (about ten by fall 2010) around other stars been seen directly? There are two main reasons for this:

1. The planet's orbit is often close to the star. If the star is far away from us, the angular distance between the star and the planet is so small that the telescope cannot separate the two objects.

2. The light from the star is much brighter than the starlight reflected from the planet. It is very difficult to detect a faint signal close to a very bright source.

How large is the angular distance on the sky between Earth and Sun seen from our closest star, Proxima Centauri 4.22 light years away? Look at the geometry in figure 1. The distance r is 4.22 light years, the distance Sun-Earth d is 150×10^6 km. Using the small angle formula from geometry (and this is indeed a very small angle),

$$d = r\theta$$

we find $\theta = 0.00021^\circ$ (check!). In astrophysics we usually specify small angles in terms of *arcminutes* and *arcseconds*, denoted $'$ and $''$. There are 60 arcminutes in one degree and 60 arcseconds in one arcminute. Thus the angular distance between Sun and Earth as seen from Proxima Centauri is $0.77''$. From the ground, the best resolution a normal telescope can reach is $0.4''$ under very good atmospheric conditions (actually using so-called adaptive optics better resolutions may be attained). This means that two objects with a smaller angular distance on the sky cannot be separated by the telescope. So the green men on a planet orbiting our nearest star would just be able to see the Earth with the best telescopes under very good atmospheric conditions (provided the atmosphere on this planet is similar to the Earth's)! The Hubble Space Telescope which is not limited by the atmosphere can reach a resolution of $0.1''$. For the people on a

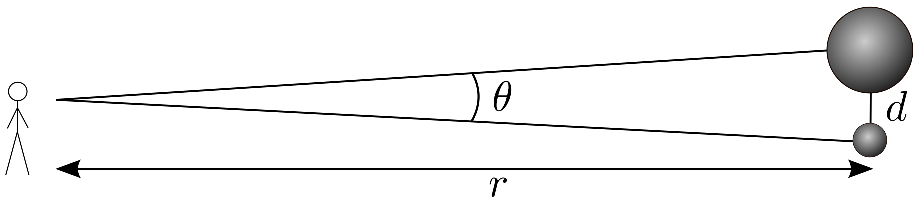

Figure 1: The angular extension of a distant planet's orbit around its star.

planet orbiting a star located 100 light years away from Earth, the angular distance between the Earth and the Sun would be 0.03″. From this planet, our green friends would be unable to see the Earth using the Hubble Space Telescope! A huge advance in optics and telescope technology is needed in the future in order to resolve planets which are orbiting close to their mother star.

Still, about 500 planets orbiting other stars have been detected (by fall 2010, but the number is now rapidly increasing after the launch of the Kepler satellite in 2009 (`http://kepler.nasa.gov/`)). The reason for this can be found in the previous lecture: In a star-planet system, the planet *and* the star are orbiting their common center of mass. Thus, the star is moving in an elliptical orbit. If the velocity of the star can be measured, then a regular variation of the star's velocity as it orbits the center of mass should be detected. This is the way most of the extrasolar planets have been discovered so far (this is also about to change with Kepler which discovers extrasolar planets by eclipses which we will come back to later)

One way to measure the velocity of a star is by the Doppler effect, that electromagnetic waves (light) from the star change their wavelength depending on whether the star is moving towards us or away from us. When the star is approaching, we observe light with shorter wavelength, the light is *blueshifted*. On the contrary, when the star is receding, the light is *redshifted*. By measuring the displacement of spectral lines in the stellar spectra (more details about this in a later lecture), we can measure velocities of stars by the impressive precision of 1 m/s, the walking speed of a human being. In this way, even small variations in the star's velocity can be measured. Recall the formula:

Change in wavelength due to the Doppler effect

$$\frac{\lambda - \lambda_0}{\lambda_0} = \frac{v_r}{c},$$

where λ is the observed wavelength and λ_0 is the wavelength seen from the rest frame of the object emitting the wave.

There is one drawback of this method: only radial velocity can be measured. Tangential velocity, movements perpendicular to the line of sight, does not produce any Doppler effect. The orbital plane of a planet (which

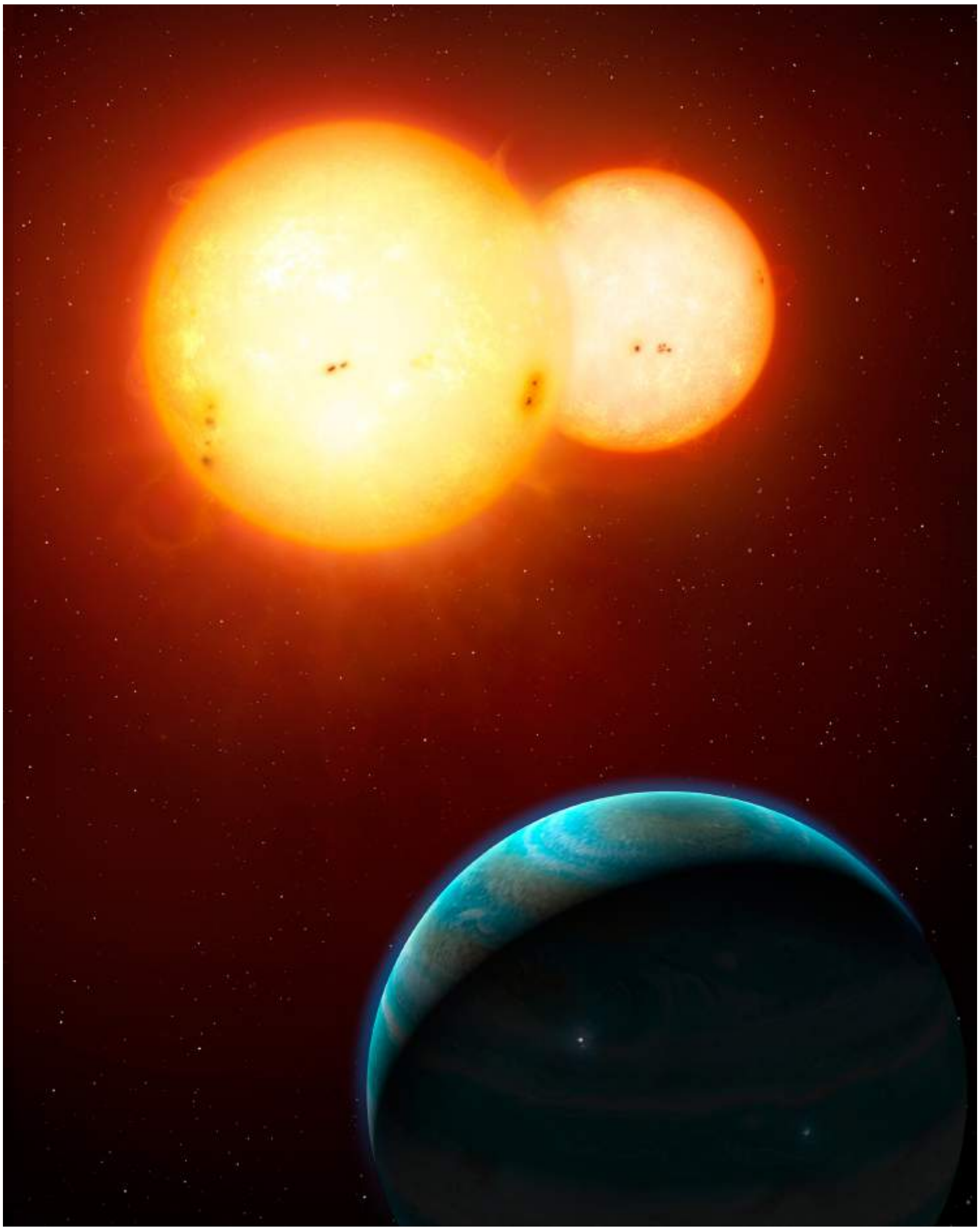

Figure 2: Info-figure: Artistic rendition of Kepler-35b, a Saturn-sized planet orbiting a pair of Sun-like stars. The first confirmed detection of a planet orbiting a main-sequence star other than the Sun was made in 1995. Since then hundreds of exoplanets have been discovered; see http://exoplanet.eu for a complete and up-to-date list. Astronomers employ several methods for finding exoplanets, e.g. the radial velocity or Doppler method, the transit method, gravitational microlensing, astrometry, pulsar timing, and even direct imaging. Recent surveys have shown that planets around stars in the Milky Way are the rule rather than the exception! Exoplanet research is one the hottest fields in astronomy today.(M. Garlick)

is the same as the orbital plane of the star) will have a random orientation. We will therefore only be able to measure one component of the star's velocity, the radial velocity.

In figure 3 we have plotted the situation. The angle i is called the *inclination* of the orbit. It is simply the angle between the line of sight and a line perpendicular to the orbital plane (see figure 3). When the inclination $i = 90^\circ$, the plane of the orbit is aligned with the line of sight and the velocity measured from the Doppler effect is the full velocity. For an inclination $i = 0^\circ$, there is no radial component of the velocity and no Doppler effect is seen. A regular variation in a star's radial velocity could be the sign of a planet orbiting it.

We will in the following assume circular orbits (i.e. the eccentricity $e = 0$). This will make calculations easier, the distance from the center of mass a is always the same and more importantly, the velocity v is the same for all points in the orbit. In figure 4 we show how the radial velocity changes during the orbit of the star around the center of mass. If the inclination is $i = 90^\circ$, then the radial velocity v_r equals the real velocity v in the points B and D in the figure. For other inclinations, the radial velocity v_r in points B and D is given by

$$|v_r| = v \sin i. \tag{1}$$

This is found by simple geometry, it is the component of the velocity vector taken along the line of sight (do you see this?). Note: The velocity v discussed here is the orbital velocity of the star, i.e. the velocity of the star with respect to the center of mass. Normally the star/planet system, i.e. the center of mass, has a (approximately) constant velocity with respect to the observer. This velocity v_{pec} is called the peculiar velocity and must be subtracted in order to obtain the velocity with respect to the center of mass. Recall from the previous lecture that the velocity of the star can be decomposed into the velocity of the center of mass (peculiar velocity) and the velocity of the star with respect to the center of mass (which is the one we need).

2 Determining the mass of extrasolar planets

We know that Kepler's third law (Newton's version as you deduced it in the exercises of the previous lectures) connects the orbital period P, the semimajor axis a (radius in the case of a circular orbit) and mass m of the planet/star (do you remember how?). From observations of the radial velocity of a star we can determine the orbital period of the star/planet system. Is there a way to combine this with Kepler's laws in order to obtain the mass of the planet? The goal of this section is to solve this

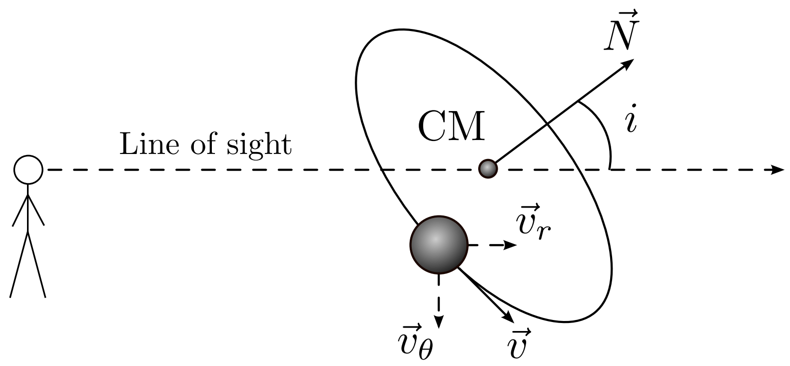

Figure 3: Inclination: The angle between the line of sight and the normal $\vec{N}$ to the orbital plane is called the inclination i. The maximum radial velocity of the star equals $v \sin i$.

problem. We will deduce a way to determine the mass of an extrasolar planet with as little information as possible.

In the following we will use m_*, a_*, v_* for mass, radius of the orbit and velocity of the star in its orbit around the center of mass. Similarly we will use m_p, a_p and v_p for the corresponding quantities regarding the planet. The constant velocities may be expressed as,

$$v_* = \frac{2\pi a_*}{P} \qquad v_\mathrm{p} = \frac{2\pi a_\mathrm{p}}{P}. \tag{2}$$

Note again that this is velocity with respect to center of mass, any peculiar velocity has been subtracted. In the lecture notes for lecture 1-2, section 5, we found expressions for the position of the two bodies m_1 and m_2 taken in the center of mass frame, $\vec{r}_1^{\,\mathrm{CM}}$ and $\vec{r}_2^{\,\mathrm{CM}}$. Before reading on, look back at these lecture notes now and make sure you remember how these expressions were obtained!

Did you check those lecture notes? Ok, then we can continue. Take these masses to be the star and the planet. Using these expressions, we obtain (check!)

$$\frac{|\vec{r}_*^{\,\mathrm{CM}}|}{|\vec{r}_\mathrm{p}^{\,\mathrm{CM}}|} = \frac{m_\mathrm{p}}{m_*} = \frac{a_*}{a_\mathrm{p}},$$

where the expressions for the semimajor axes a_1 and a_2 from lecture 1–2

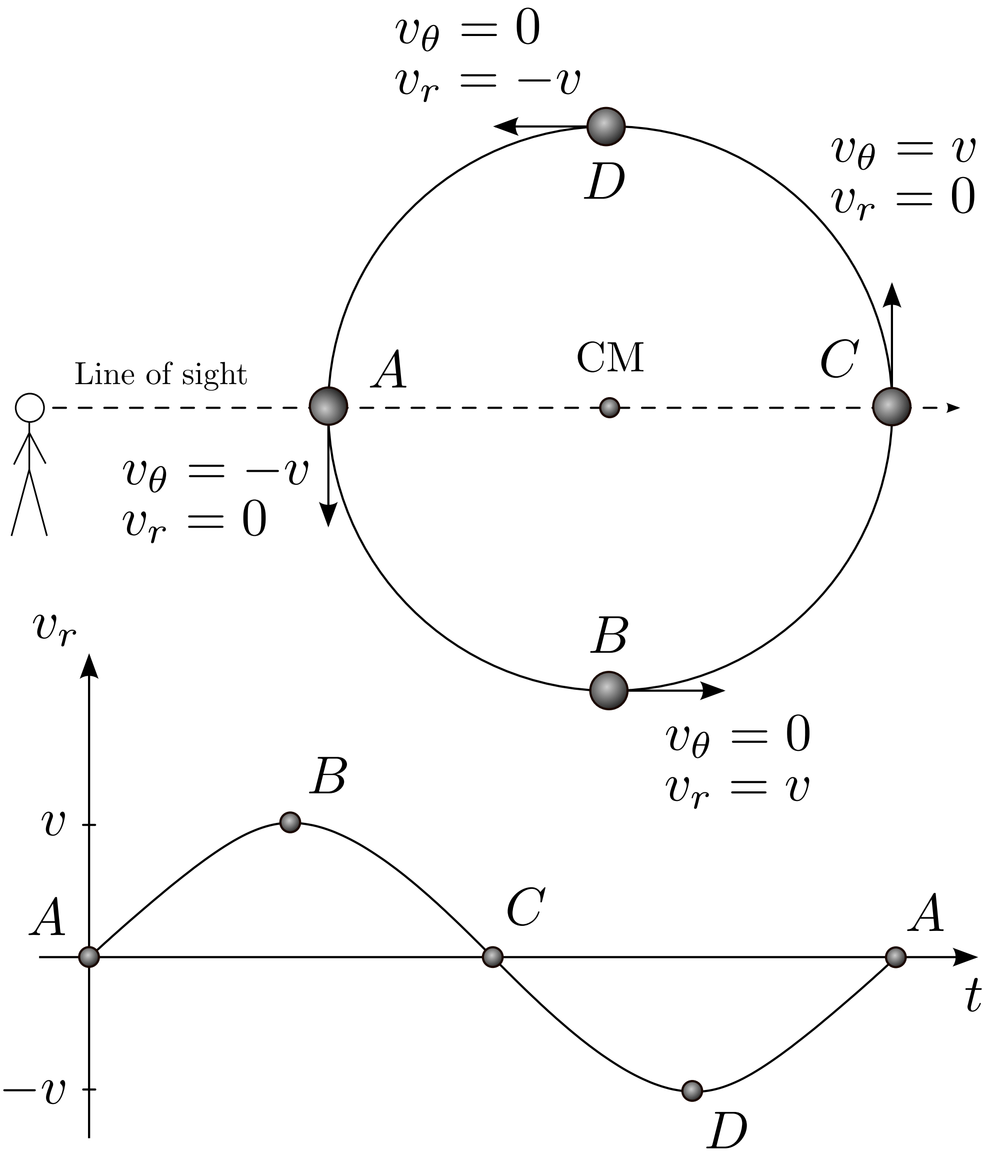

Figure 4: The velocity curve of a star orbiting the common center of mass with a planet. The points where the component of the velocity vector along the line of sight is zero (A and C) as well as the points where the radial component equals the full velocity (B and D) are indicated. In the figure, we have assumed an inclination of 90°.

were used. Using equation (2), we also have that

$$\frac{a_*}{a_{\rm p}} = \frac{v_*}{v_{\rm p}} = \frac{v_{*r}/\sin i}{v_{{\rm p}r}/\sin i} = \frac{v_{*r}}{v_{{\rm p}r}},$$

where equation (1) was used. Note: Here, the radial velocities v_{*r} and $v_{{\rm p}r}$ refer to the velocity at the point B in figure 4, the point for which the radial velocity is maximal. We may use these two equation to eliminate the unknown velocity of the planet

$$v_{{\rm p}r} = v_{*r}\frac{m_*}{m_{\rm p}}. \tag{3}$$

We will now return to Kepler's third law,

$$m_* + m_{\rm p} = \frac{4\pi^2 a^3}{P^2 G},$$

where we have used the exact expression for Kepler's third law, derived in problem 2 in lecture notes 1–2. From section 5 in those notes, we also had that

$$a = a_* + a_{\rm p},$$

the semimajor axis a (of the orbit of the planet seen from the star or vice versa) equals the sum of the semimajor axes of the orbits of the planet and star about the center of mass. We can now express these in terms of velocities (equation 2)

$$a = \frac{P}{2\pi}(v_* + v_{\rm p}).$$

Inserting this into Kepler's third law, we have

$$m_* + m_{\rm p} = \frac{P}{2\pi G}(v_* + v_{\rm p})^3.$$

Normally we are only able to measure radial velocities, not the absolute velocity. We thus use equation (1) as well as equation (3) to obtain

$$m_* + m_{\rm p} = \frac{P}{2\pi G}\frac{(v_{*r} + v_{{\rm p}r})^3}{\sin^3 i} = \frac{P v_{*r}^3}{2\pi G \sin^3 i}\left(1 + \frac{m_*}{m_{\rm p}}\right)^3.$$

Assuming that the star is much more massive than the planet (which is normally the case, for instance $m_{\rm Jupiter}/m_{\rm Sun} \sim 10^{-3}$) we get

$$m_* = \frac{P v_{*r}^3}{2\pi G \sin^3 i}\frac{m_*^3}{m_{\rm p}^3},$$

which solved for the mass of the planet (which is the quantity we are looking for) gives

$$m_{\rm p} \sin i = \frac{m_*^{2/3} v_{*r} P^{1/3}}{(2\pi G)^{1/3}}.$$

Normally, the mass of the star is known from spectroscopic measurements. The radial velocity of the star and the orbital period can both be inferred from measurements of the Doppler effect. Thus, the expression $m_\mathrm{p} \sin i$ can be calculated. Unfortunately, we normally do not know the inclination angle i. Therefore, this approach for measuring the planet's mass can only put a lower limit on the mass. By setting $i = 90°$ we find $m_\mathrm{p}^\mathrm{min}$. If the inclination angle is smaller, then the mass is always greater than this lower limit by a factor of $1/\sin i$. In the next section however, we will discuss a case in which we can actually know the inclination angle.

3 Measuring the radius and the density of extrasolar planets

If the inclination is close to $i \sim 90°$, the planet passes in front of the stellar disc and an eclipse occurs: The disc of the planet obscures a part of the the light from the star. When looking at the light curve of the star, a dip will occur with regular intervals corresponding to the orbital period. In figure 5 we show a typical light curve. When the disc of the planet enters the disc of the star, the light curve starts falling. When the entire disc of the planet is inside the disc of the star, the light received from the star is now constant but lower than before the eclipse. When the disc of the planet starts to leave the disc of the star, the light curve starts rising again. When such a light curve is observed for a star where a planet has been detected with the radial velocity method described above, we know that the inclination of the orbit is close to $i = 90°$ and the mass estimate above is now a reliable estimate of the planet's mass rather than a lower limit.

In these cases, where the effect of the eclipse can be seen, the radius of the planet may also be measured. If we know the time of first contact (time t_0 in figure 5), the time when the disc of the planet has fully entered the disc of the star (time t_1) as well as the velocity of the planet with respect to the star, we can measure the radius of the planet. If the radius of the planet is R_p, then it took the disc of the planet with diameter $2R_\mathrm{p}$ a time $t_1 - t_0$ to fully enter the disc of the star. The planet moves with a velocity $v_* + v_\mathrm{p}$ with respect to the star (the velocity v_p is only the velocity with respect to the center of mass). Using simply that distance equals velocity times interval, we have

$$2R_\mathrm{p} = (v_* + v_\mathrm{p})(t_1 - t_0)$$

As we have seen, we can obtain t_1 and t_0 from the light curve. We can also obtain the velocity of the planet (the velocity of the star is measured directly by the Doppler effect) by using equation (3),

$$v_\mathrm{p} = v_* \frac{m_*}{m_\mathrm{p}}.$$

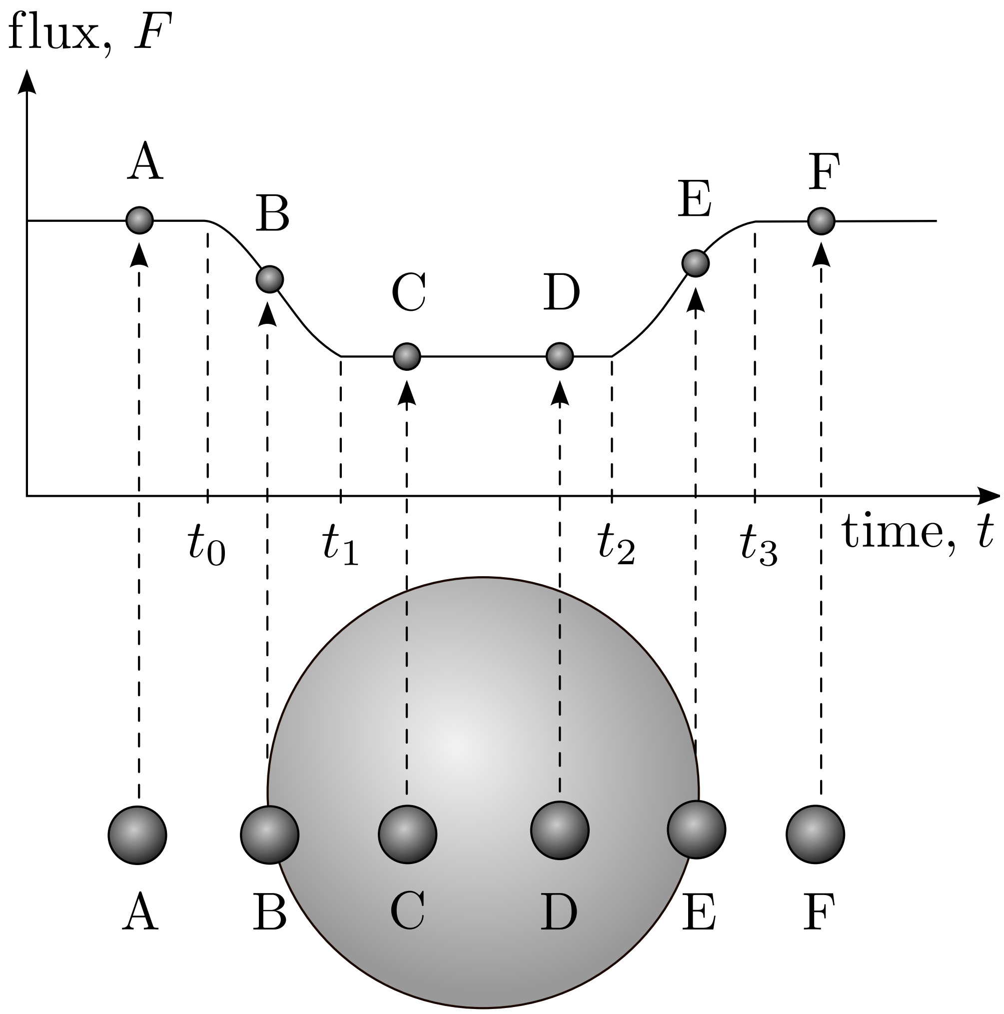

Figure 5: The lower part of the figure shows a planet eclipsing a star. The upper part shows a plot of the flux variation with time at the different points during the eclipse. The moments at which the eclipse starts $t = t_0$ and ends $t = t_3$ as well as the moments when the full disc of the planet enters $t = t_1$ and leaves $t = t_2$ the star are indicated.

Here the mass of the planet m_p has been calculated since we know that the inclination is $i \sim 90°$. Thus, the radius R_p of the planet is easily obtained. Combining the measured mass and radius of the planet we get an estimate of the mean density

$$\rho_\mathrm{p} = \frac{m_\mathrm{p}}{4/3\pi R_\mathrm{p}^3}.$$

We can use this to determine whether the detected planet is *terrestrial planet* with a solid surface like the inner planets in the solar system, or a *gas planet* consisting mainly of gas and liquids like the outer planets in our solar system. The terrestrial planets in our solar system have densities of order 4−5 times the density of water whereas the gas planets have densities of order $0.7 - 1.7$ times the density of water. If the detected planet is a terrestrial planet, it could also have life.

Finally, note that also the radius R_* of the observed star can be obtain by the same method using the time it takes for the planet to cross the disc of the star,

$$2R_* = (v_* + v_\mathrm{p})(t_2 - t_0).$$

We have discussed two ways of discovering extrasolar planets,

- by measuring radial velocity
- by measuring the light curve

In the following problems you will also encounter a third way,

- by measuring tangential velocity

For very close stars, the tangential movement of the star due to its motion in the orbit about the center of mass may be seen directly on the sky. The velocity we measure in this manner is the projection of the total velocity onto the plane perpendicular to the line of sight. There are two more methods which will briefly be discussed in later lectures,

- by gravitational lensing
- by pulsar timing

4 Exercise to be presented on the blackboard: The atmosphere of extrasolar planets

In figure 7 we show observations of the radial velocity of a star over a large period of time. We assume that these data is a collection of data from several telescopes around the world. Real data contain several additional complicated systematic effects which are not included in this figure. For

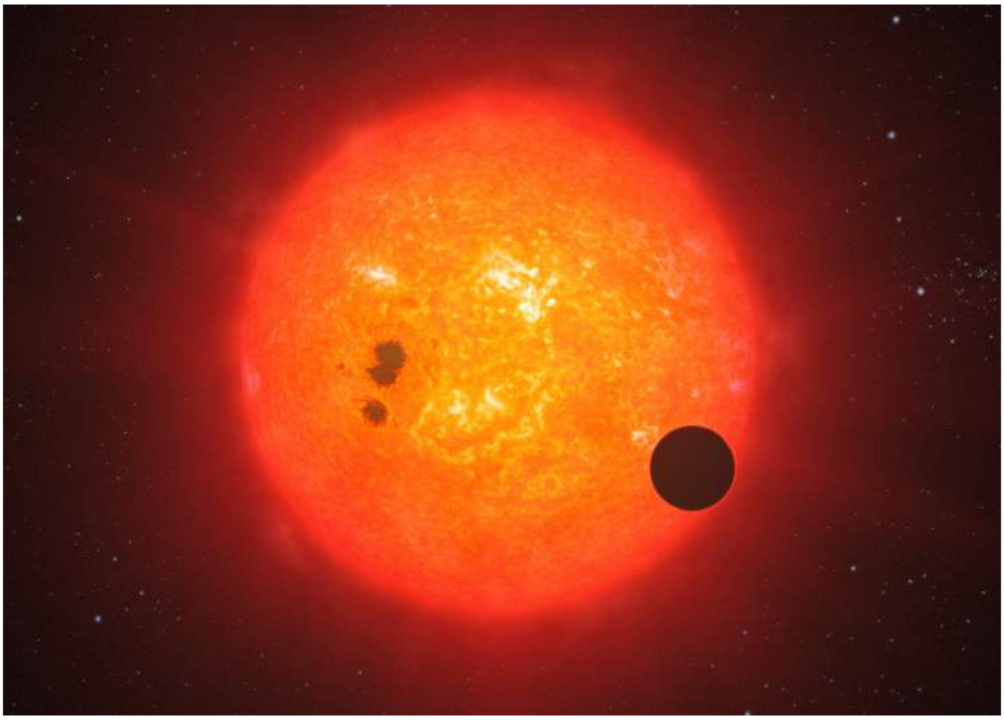

Figure 6: Info-figure: This artist's impression shows how the super-Earth surrounding the star GJ1214 may look. The planet was discovered by the transit method: the brightness of its host star decreased by a tiny amount as the (unseen) planet crossed in front of it. Spectroscopic follow-up observations, i.e. radial velocity measurements, were needed in order to confirm the planetary nature of the object and to obtain its mass. The planet is the second super-Earth (defined as a planet between one and ten times the mass of the Earth) for which astronomers have determined the mass and radius, giving vital clues about its structure. It is also the first super-Earth around which an atmosphere has been found. The planet is too hot to support life as we know it.(ESO/L. Calcada)

instance, changes in the velocity of the Earth need to be corrected for in velocity measurements. Here we assume that these corrections have already been made. Even if this plot does not show you all the complications of real life, it does give an impression of how data from observations may look like and how to use them to say something about extrasolar planets. You see that this is not a smooth curve, several systematic effects as for instance atmospheric instabilities give rise to what we call 'noise'.

1. Does this star move towards us or away from us? Use the figure to give an estimate of the peculiar velocity.

2. Use the curve to find the maximum radial velocity v_{r*} of the star (with respect to the center of mass) and the orbital period of the planet.

3. Spectroscopic measurements have shown the mass of the star to be 1.1 solar masses. Give an estimate of the lower bound for the mass of the planet. The result should be given in Jupiter masses.

4. In figure 8 we show a part of the light curve (taken at the wavelength 600 nm) of the star for the same period of time. Explain how this curve helps you to obtain the real mass of the planet, not only the lower bound, and give an estimate of this mass.

5. Use the light curve to find the radius of the planet. Note: there are 5 minutes between each cross in the plot.

6. In figure 9 we show a part of the light curve taken at the same time as the previous light curve but at a wavelength 1450 nm which is an absorption line of water vapor. Use the figure to determine if this planet may have an atmosphere containing water vapor and estimate the thickness of the atmosphere.

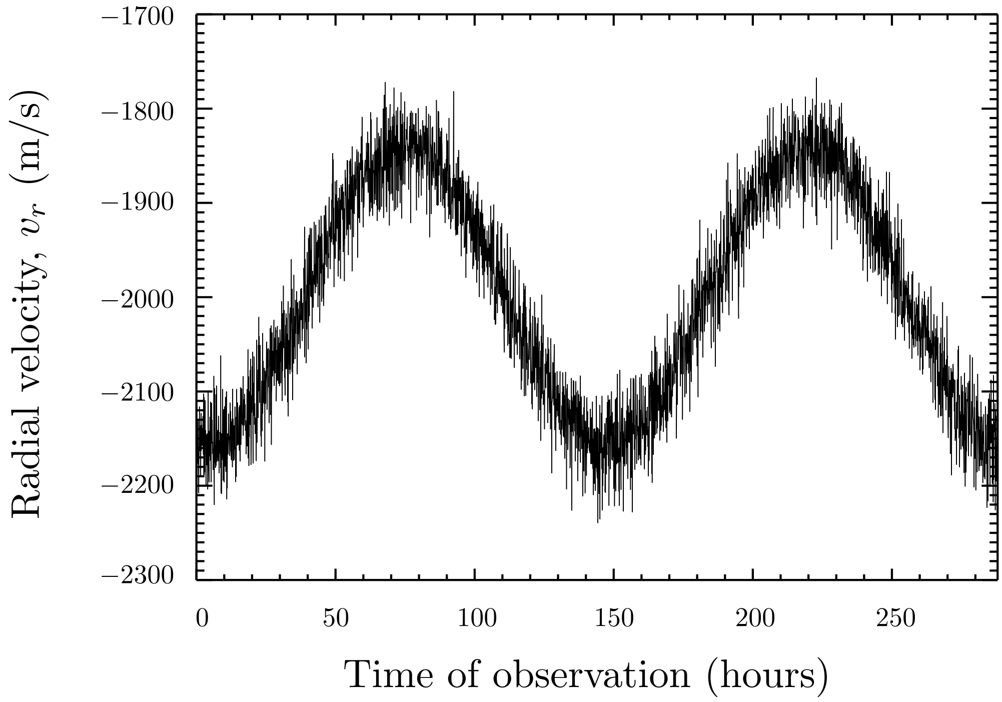

Figure 7: Velocity measurements of a star

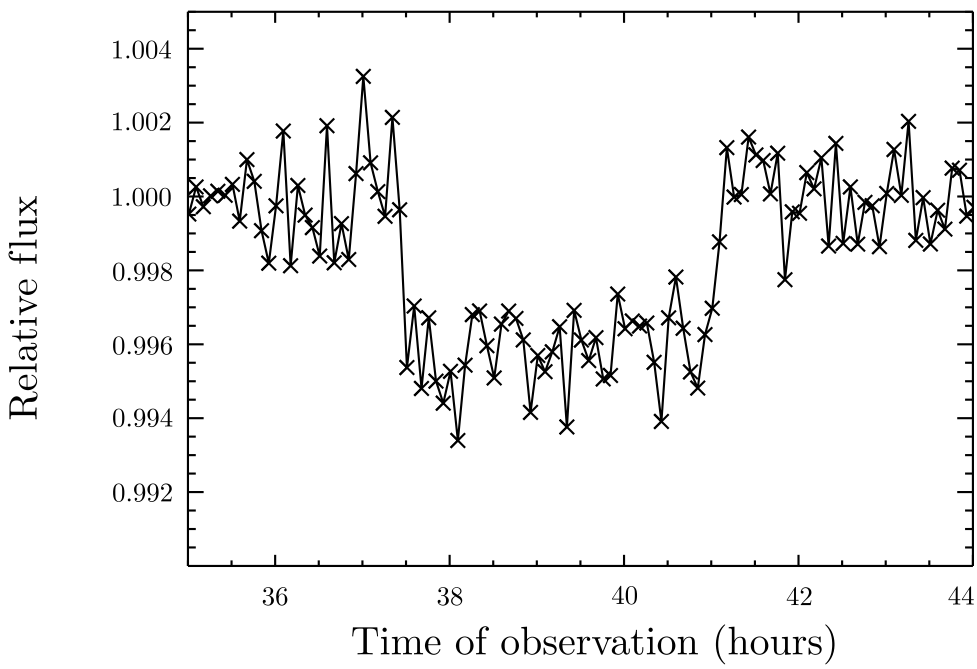

Figure 8: The light curve of a star at 600 nm. There are 5 minutes between each cross.

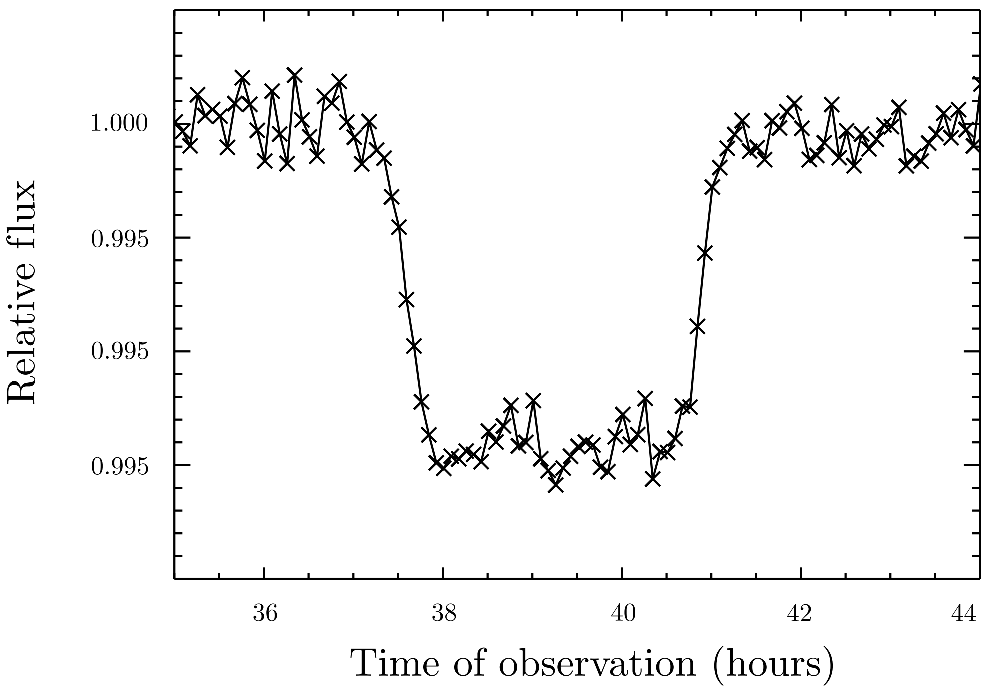

Figure 9: The light curve of a star at 1450 nm. There are 5 minutes between each cross.

5 Problems

Problem 1 (10–20 min.)

1. The precision in measurements of radial velocities by the Doppler effect is currently 1 m/s. Can a Jupiter like planet orbiting a star similar to the Sun at a distance from the mother star equal to the Sun-Jupiter distance be detected? (Use `www` or other sources to find the mass of Jupiter, the Sun and the distance between the two which are the only data you are allowed to use).

2. What about an Earth like planet in orbit at a distance 1 AU from the same star?

3. Using the radial velocity method, is it easier to detect planets orbiting closer or further away from the star?

4. In what distance range (from the mother star) does an Earth like planet need to be in order to be detected with the radial velocity method? (Again use a star similar to the Sun). Compare with the distance Sun-Mercury, the planet in our solar system which is closest to the Sun.

Problem 2 (20–30 min.)

For stars which are sufficiently close to us, their motion in the orbit about a common center of mass with a planet may be detected by observing the motion of the star directly on the sky. A star will typically move with a constant velocity in some given direction with respect to the Sun. If the star has a planet it will also be wobbling up and down (see figure 10). We will now study the necessary conditions which might enable the observation of this effect.

1. The Hubble Space Telescope (HST) has a resolution of about $0.1''$. How close to us does a star similar to the Sun with a Jupiter like planet (at the distance from the mother start equal to the Sun-Jupiter distance) need to be in order for the HST to observe the tangential wobbling of the star?

2. What about an Earth like planet at the distance of one AU from the same star?

3. The closest star to the Sun is Proxima Centauri at a distance of 4.22 l.y.. How massive does a planet orbiting Proxima Centauri at the distance of 1 AU need to be in order for the tangential wobbling of the star to be observed?

4. What about a planet at the distance from Proxima Centauri equal to the Sun-Jupiter distance?

5. If we can measure the tangential velocity (perpendicular to the line of sight) component of a star, we can get an estimate of the mass of

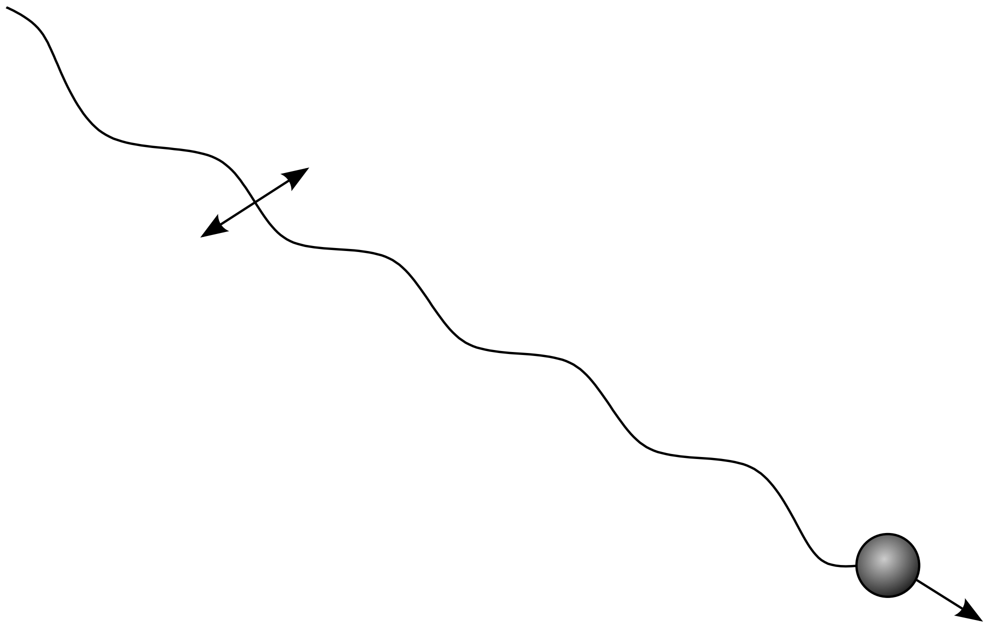

Figure 10: The transversal wobbling of a nearby star due to its orbital motion about the common center of mass with a planet. The angular extension of the orbit is indicated by two small arrows.

the planet not only a lower limit. Show that the exact mass of the planet can be expressed as (for any inclination i)

$$m_\mathrm{p} = \left(\frac{m_*^2 P}{2\pi G}\right)^{1/3} v_{t*}$$

(tangential velocity v_{t*} here is measured when the radial velocity is zero).

Problem 3 (45 min.–1 hour)

In figure 11 we show observations of the radial velocity of a star over a large period of time. We assume that these data is a collection of data from several telescopes around the world. Real data contain several additional complicated systematic effects which are not included in this figure. For instance, changes in the velocity of the Earth need to be corrected for in velocity measurements. Here we assume that these corrections have already been made. Even if this plot does not show you all the complications of real life, it does give an impression of how data from observations may look like and how to use them to say something about extrasolar planets. You see that this is not a smooth curve, several systematic effects as for instance atmospheric instabilities give rise to what we call 'noise'.

1. This plot shows a curve with a wave like shape, can you explain the shape of the curve?
2. Use this plot to give an estimate for the the 'perculiar velocity' of the

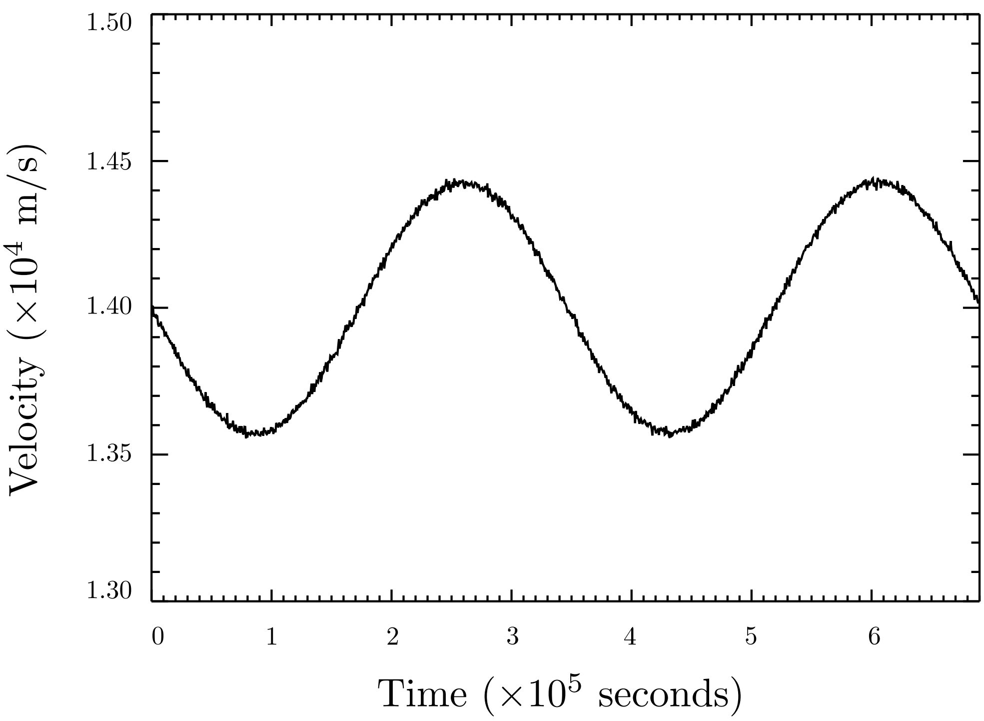

Figure 11: Velocity measurements of a star

star. 'Peculiar velocity' is a term used to describe the average motion of the star with respect to us, not taking into account oscillations from planets.

3. Use the curve to find the maximum radial velocity v_{r*} of the star (with respect to the center of mass) and the orbital period of the planet.

4. Spectroscopic measurements have shown the mass of the star to be 1.3 solar masses. Give an estimate of the lower bound for the mass of the planet. The result should be given in Jupiter masses.

5. In figure 12 we show the light curve of the star for the same period of time. Explain how this curve helps you to obtain the real mass of the planet, not only the lower bound, and give an estimate of this mass.

6. In figure 13 we have zoomed in on a part of the light curve. Use the figure to give a rough estimate of the density of the planet.

7. Is this a gas planet or a terrestrial planet?

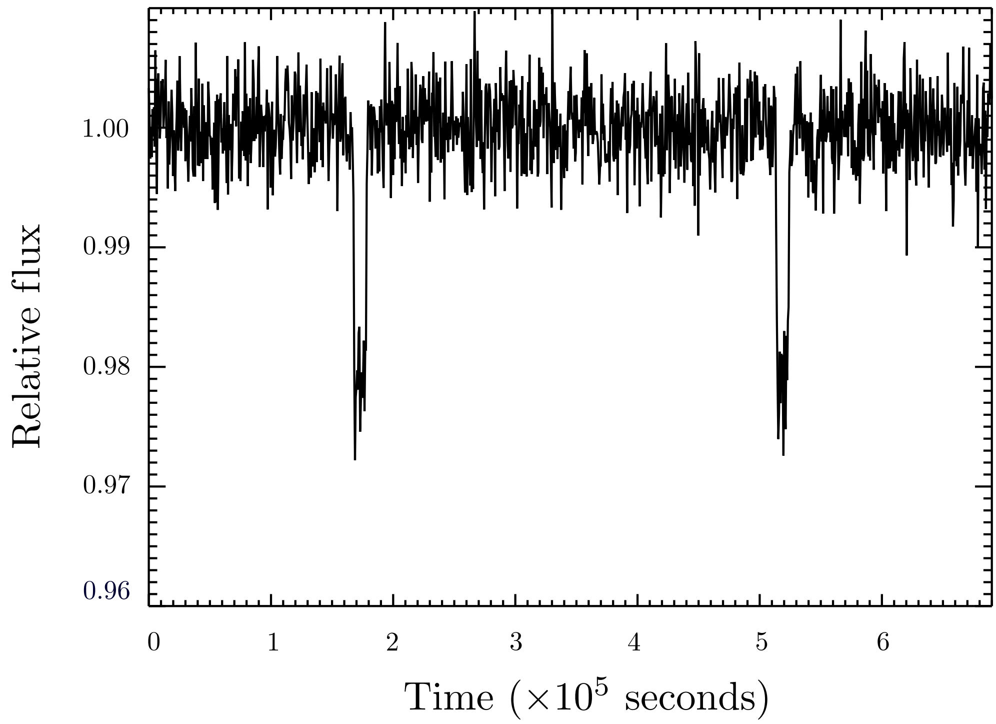

Figure 12: The light curve of a star.

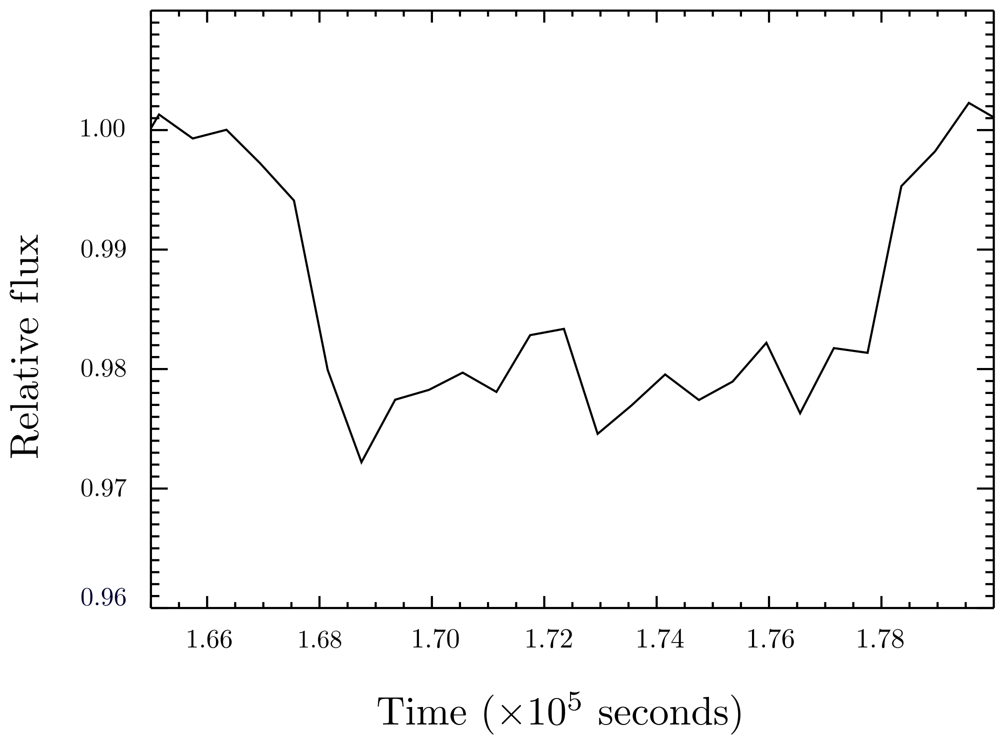

Figure 13: The light curve of a star.

Problem 4 (4–5 hours)

At the following link you will find some files containing simulated velocity and light curves of 10 stars:

`http://folk.uio.no/frodekh/AST1100/lecture3/`

Real data contain several additional complicated systematic effects which are not included in these files. For instance, changes in the velocity of the Earth need to be corrected for in velocity measurements. Here we assume that these corrections have already been made. Even if these data do not show you all the complications of real life, they will still give an impression of how data from observations may look like and how to use them to say something about extrasolar planets. Each file contains three rows, the first row is the time of observation, counted in seconds from the first observation which we define to be $t = 0$. We assume that these data is a collection of data from several telescopes around the world, studying these stars intensively for a given period of time (the length of this observing period is different for each star). The second row gives the observed wavelength λ of a spectral line (The $H\alpha$ line) at $\lambda_0 = 656.3$ nm in nm $= 10^{-9}$ m. You need to use the Doppler formula to obtain radial velocities yourself. You will see that this is not a smooth curve, several systematic effects, i.e. atmospheric instabilities give rise to what we call 'noise'. As you will see, this noise makes exact observations difficult. The third row shows the measured flux relative to the maximum flux for the given star. Again, also these data contain noise.

Use Python, Matlab or other software/programming languages to solve the following problems:

1. Estimate the peculiar velocity (the mean velocity of the star with respect to Earth) for each of the 10 stars, taking the mean of the velocity over all observations. Plot the velocity curves (subtract the mean velocity from the velocity for each observation) and light curves for the ten stars. Which of the stars appear to have a planet orbiting? Which of these planets are eclipsing their mother star?

2. The mass of the stars have been measured by other means, these are 0.8, 2.8, 0.5, 0.5, 1.8, 0.7, 1.6, 2.1, 7 and 8 solar masses for star 0-9 respectively. Can you, by looking at the velocity curves (velocity as a function of time), find the lower limit for the mass of the planet for the stars where you detected a planet. Find the numbers for the periods and max radial velocities by eye.

3. If you, by looking at the light curve, discovered that some of the planets are actually eclipsing the star, can you also estimate the radius and density of these planets. Again, you will need to estimate the time of eclipse by eye. Does any of the planets you have detected have the possibilities for life (at least in the form that we know life)?

4. You have made estimates of mass and radius using 'by-eye' mea-

surements. This is not the way that astrophysicists are working. Often, advanced signal processing methods are employed in order to get the best possible estimates. Also, scientific measurements always have uncertainties. The detailed methods for analyzing these data are outside the scope of this course, but you will encounter this in more advanced courses in astrophysics. Here we will show you a simple way to obtain estimates which are more exact than the 'by-eye' observations above. A similar method will be used in other problems in this course. The key to this method is the method of 'least squares'. We will use this to obtain more accurate periods and max radial velocities from the velocity measurements. We will model the velocity curves as cosine curves in the following way,

$$v_r^{\text{model}}(t) = v_r \cos(\frac{2\pi}{P}(t - t_0)), \tag{4}$$

where $v_r^{\text{model}}(t)$ is the theoretical model of the radial velocity as a function of time, v_r is the maximal radial velocity, P is the period of revolution and t_0 is some point for which the radial velocity is maximal (you see that if $t = t_0$ then the cosine term equals one). The unknown parameters in this model are v_r, P and t_0. Only the two first parameters, v_r and P, are necessary in order to estimate the mass of the planet, but we need to estimate all three in order to have consistent estimates of the first two. We will now try to find a combination of these three parameters, such that equation (4) gives a good description of the data. To do this, you need to write a computer code which calculates the difference, or actually the square of the difference, between the data and your model for a large number of values for the three parameters t_0, P and v_r. You need to define a function (an array in you computer) $\Delta(t_0, P, v_r)$ given as

$$\Delta(t_0, P, v_r) = \sum_{t=t_0}^{t=t_0+P} (v_r^{\text{data}}(t) - v_r^{\text{model}}(t, t_0, P, v_r))^2$$

This function gives you the difference between the data and your model for different values of t_0, P and v_r. What you want to find is the function which best fits your data, that is, the model which gives the minimum difference between the data and your model. You simply want to find for which parameters t_0, P and v_r that the function $\Delta(t_0, P, v_r)$ is minimal. How do you find the parameters t_0, P and v_r which minimize Δ? In this case it is quite easy, try to follow these steps:

(a) Choose one of your stars which clearly has a planet orbiting.

(b) Look at your data: You know that for $t = t_0$, the velocity is maximal. Look for the first peak in the curve and define a range in time around this curve for which you think that the exact peak must be. Define a minimum possible t_0 and a maximum possible t_0 (being sure that exact peak is somewhere between these two values). Then define a set of, say 20 (you choose what is more convenient in each case) values of t_0 which are equally spaced between the minimum and maximum value.

(c) Do the same for v_r, try to find a minimum and a maximum v_r which are such that you see by eye that the real exact v_r is between these two values. Then divide this range into about 20 equally spaced values (maybe less depending on the case).

(d) Do the same thing for the period. Look at the time difference between two peaks, and find a set of possible periods.

(e) Now, calculate the function Δ for all these values of t_0, P and v_r which you have found to be possible values. Find which of these about 20^3 combination of values which gave the smallest Δ, thus the smallest difference between data and model. These values are now your best estimates of P and v_r.

(f) Calculate the mass of the planet again with these values for P and v_r and compare with your previous 'by-eye' estimates. How well did you do in estimating 'by-eye'?

(g) Now repeat the procedure to estimate the exact mass for two other stars with planets and compare again with your 'by-eye' estimates.

AST1100 Lecture Notes

4 Stellar orbits and dark matter

1 Using Kepler's laws for stars orbiting the center of a galaxy

We will now use Kepler's laws of gravitation on much larger scales. We will study stars orbiting the center of galaxies. Our own galaxy, the Milky Way, contains more than 2×10^{11} stars. The diameter of the galaxy is about 100 000 light years and the Sun is located at a distance of about 25 000 light years from the center. It takes about 226 million years for the Sun to make one full revolution in its orbit.

The Milky way is a spiral galaxy where most of the stars are located in the *galactic disc* surrounding the center of the galaxy and in the galactic bulge, a spherical region about 10 000 light years in diameter located at the center (see figure 1). We will apply Newton/Kepler's laws to stars in the outer parts of a galaxy, at a large distance r from the center. For these stars, we may approximate the gravitational forces acting on the star to be the force of a mass $M(r)$ (which equals the total mass inside the orbit of the star) located at the center of the galaxy. Kepler's third law (Newton's modified version of it, see lecture notes 1–2, problem 2) for this star reads

$$P^2 = \frac{4\pi^2}{G(M(r)+m_*)}r^3,$$

where we assume a circular orbit with radius r. The orbital velocity of the star at distance r is (check!)

$$v(r) = \frac{2\pi r}{P} = \frac{2\pi r}{\sqrt{4\pi^2 r^3/(G(M(r)+m_*))}} \approx \sqrt{\frac{GM(r)}{r}}. \qquad (1)$$

where we used Kepler's third law and assumed that the total mass inside the star's orbit is much larger than the mass of the star, $M(r) \gg m_*$.

The density of stars is seen to fall off rapidly away from the center of the galaxy. Observations indicate that the stellar density decreases as $1/r^{3.5}$. Therefore, for stars in the outer parts of the galactic disc, we may consider

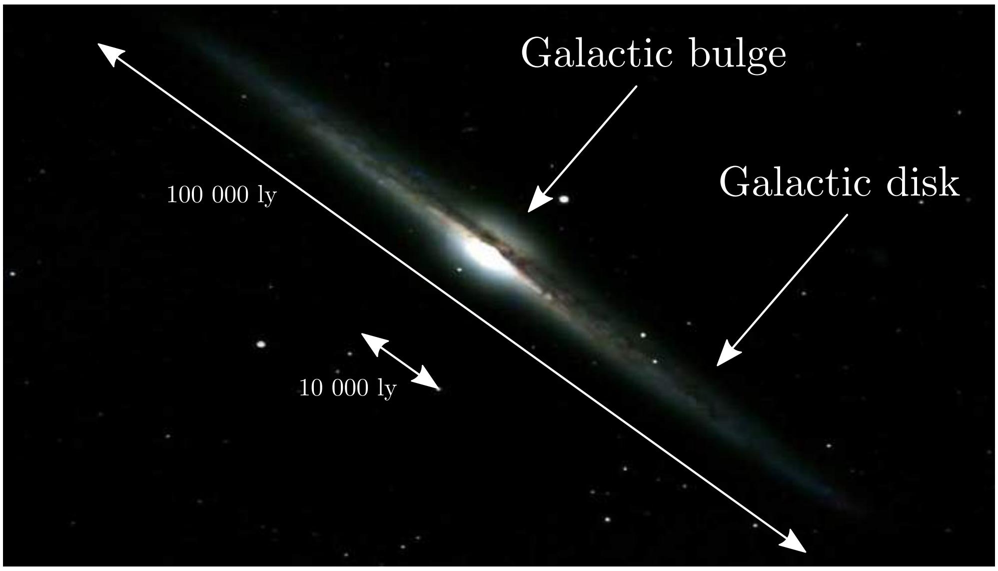

Figure 1: Dimensions of a typical galaxy.

the amount of mass inside the orbit to be the total mass M of the galaxy (since there is not much more material outside the star's orbit which can contribute to the total mass), that is to say $M(r) \to M$ asymptotically for large values of r. In this case expression (1) above can be written as

$$v(r) = \sqrt{\frac{GM}{r}}.$$

Thus, we expect the orbital velocity of stars in the outer parts of the galaxy to fall off as $1/\sqrt{r}$ with the distance r from the galactic center.

By measuring the Doppler effect, we can estimate the velocity of stars orbiting a galaxy at different distances r from the center. A huge number of observations show that the *galactic rotation curve*, the curve showing the orbital velocity as a function of distance r, is almost flat for large r for a large number of galaxies. Instead of falling off as $v \propto 1/\sqrt{r}$, the orbital velocity turns out not to decrease with distance (see figure 3). This came as a big surprise when it was first discovered. There must be something wrong about the assumptions made above. The main assumption made in our derivation was that the density of stars traces the mass density in the galaxy. Using the fact that the density of stars falls of rapidly for large r, we also assumed the total mass density to fall off similarly. This is true if the only constituents of the galaxy were stars. However, if there are other objects in the galaxy which do not emit light, which we cannot see, and which has a different distribution of mass than the stars, the assumptions leading to the $\propto 1/\sqrt{r}$ relation does not hold. One could explain this discrepancy between theory and data if there was an additional invisible matter component, *dark matter*.

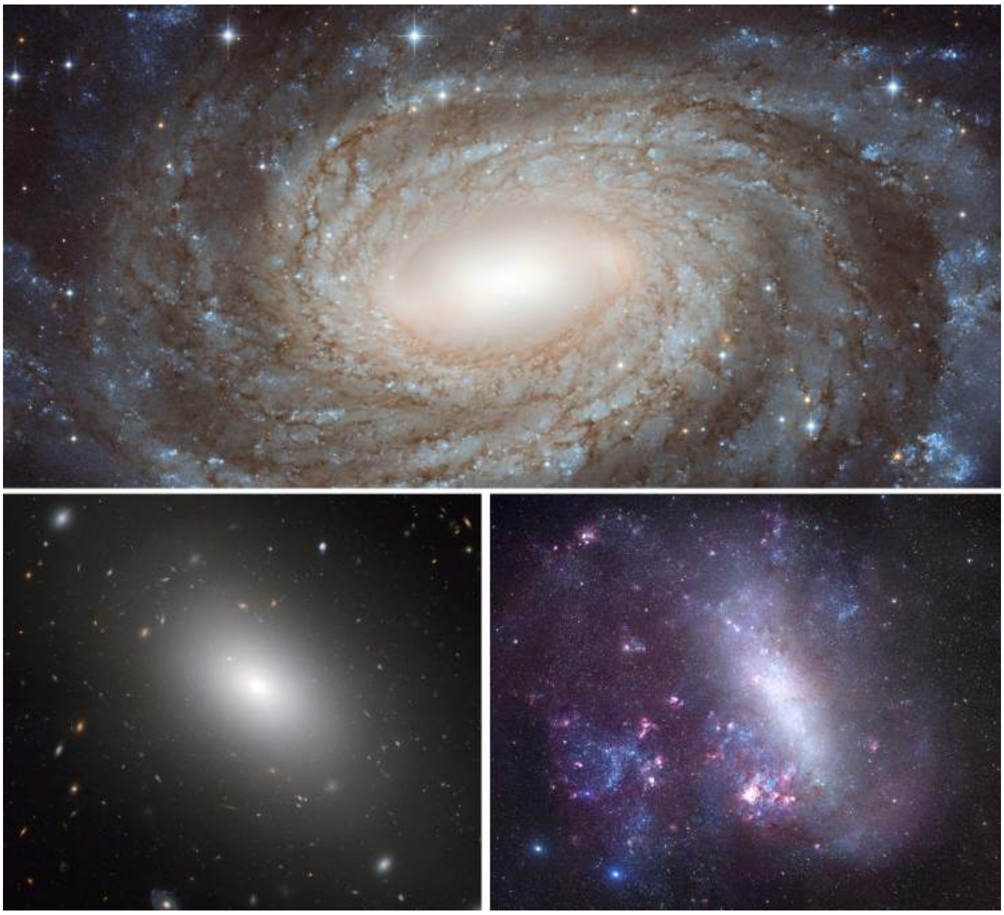

Figure 2: Info-figure: A galaxy is a massive, gravitationally bound system that consists of stars, stellar remnants, an interstellar medium of gas and dust, and a poorly understood component called dark matter which accounts for around 90% of the mass of most galaxies. Examples of galaxies range from dwarfs with as few as ten million (107) stars to giants with a hundred trillion (1014) stars. There are numerous ways to classify these objects, but as far as apparent shape is concerned, there are three main types: spiral galaxies, elliptical galaxies, and irregular galaxies. Pictured above are NGC 6384 (spiral), NGC 1132 (elliptical), and the Large Magellanic Cloud (irregular).(top: NASA, ESA, and the Hubble Heritage Team; lower left: ESA/Hubble & NASA; lower right: R. Gendler)

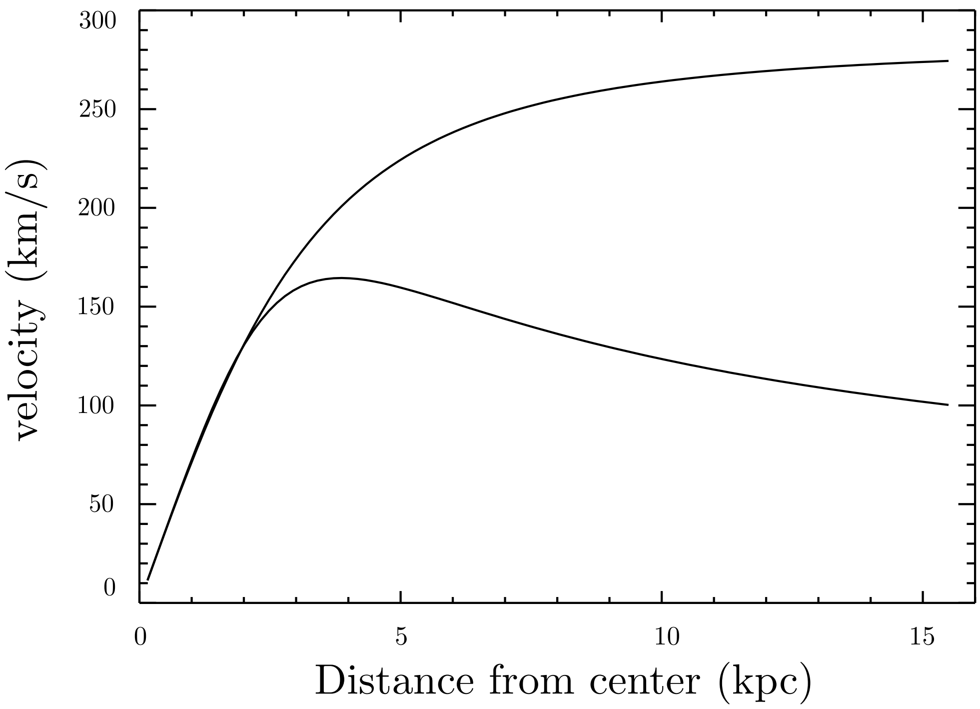

Figure 3: Models of galactic rotation curves. The lower curve is the curve expected from Kepler's laws (taking into account that M(r) is a function of r for lower radii), the upper curve is a model for the observed velocity curve.

2 Modeling the mass density field of a galaxy

Assuming that there is indeed an unknown matter component which has a different density profile $\rho(r)$ than the stars, we could make an attempt to find out how this dark matter is distributed in the galaxy. How can we map the matter distribution of invisible matter? We can simply look at its gravitational effect on visible matter. We have already seen traces of such an effect: the invisible matter changes the rotation curve of stars in the galaxies. Is there a way to use the rotation curve $v(r)$ to estimate the density profile $\rho(r)$ of the dark matter?

In the lack of better models, we will assume the distribution of dark matter to be spherically symmetric about the center of the galaxy. Thus, we assume that the density can be written as a function of distance r to the center only. We know that the total mass dM of a spherical shell of infinitesimal thickness dr at a distance r from the center of the galaxy can be written as

$$dM = 4\pi r^2 \rho(r) dr.$$

The surface of a spherical shell at distance r is $4\pi r^2$, the volume of the same shell of thickness dr is $4\pi r^2 dr$. Multiplying with the density $\rho(r)$ we obtain the total mass of the shell given in the previous expression. We now look back at equation (1), write it in terms of $M(r)$ and take the derivative of $M(r)$ with respect to r

$$\frac{dM}{dr} = \frac{v(r)^2}{G}.$$

Here we used the fact that $v(r)$ (taken from observations) seems independent of r such that $dv/dr \approx 0$ for large distances from the center. This is strictly not a necessary assumption, for any power law in the velocity $v(r) \propto r^n$ (where n is an arbitrary index) this expression holds up to a constant factor (check by taking the derivative of $M(r)$ setting $v(r) \propto r^n$). Thus, the following expressions will be valid for more general forms of the velocity $v(r)$ and is therefore also valid for more central regions.

We now have two equations for dM/dr. Setting these two expressions equal, we obtain

$$\rho(r) = \frac{v(r)^2}{4\pi G r^2}. \tag{2}$$

This is a simple expression for the matter density in the galaxy at distance r from the center, expressed only in terms of the rotational velocity $v(r)$. Note that for spherical symmetry, this expression holds also for small values of r. One could think that for stars close to the center, the matter outside the star's orbit would also contribute to the gravitational forces. However, it can be shown that the gravitational forces from a spherical shell add to zero everywhere inside this shell. Thus, simply by a set of Doppler measurements of orbital velocities at different distances r in the galaxy we are able to obtain a map of the matter distribution in terms of the density profile $\rho(r)$.

Recall that observations have shown the rotation curve $v(r)$ to be almost flat, i.e. independent of r, at large distances from the center. Looking at equation (2) this means that the total density in the galaxy falls of like $1/r^2$. Recall also that observations have shown the density of stars to fall off as $1/r^{3.5}$. Thus, the dark matter density falls of much more slowly than the density of visible matter. The dark matter is not concentrated in the center to the same degree as visible matter, it is distributed more evenly throughout the galaxy. Moreover, the density $\rho(r)$ which we obtain by this method is the total density, i.e.

$$\rho(r) = \rho(r)^{\mathrm{LM}} + \rho(r)^{DM},$$

the sum of the density due to luminous matter (LM) and the density due to dark matter (DM). Since the density of luminous matter falls off much more rapidly $\rho(r)^{\mathrm{LM}} \propto r^{-3.5}$ than the dark matter, the outer parts of the galaxy must be dominated by dark matter.

What happens to the mass density as we approach the center? Doesn't it diverge using $\rho(r) \propto r^{-2}$? Actually, it turns out that the rotation curve $v(r) \propto r$ close to the center. Looking at equation (2) we see that this implies a constant density in the central regions of the galaxy. A density profile which fits the observed density well over most distances r is given by

$$\rho(r) = \frac{\rho_0}{1 + (r/R)^2}, \quad (3)$$

where ρ_0 and R are constants which are estimated from data and which vary from galaxy to galaxy. For small radii, $r \ll R$ we obtain $\rho = \rho_0 =$ constant. For large radii $r \gg R$ we get back $\rho(r) \propto r^{-2}$.

Before you proceed, check that you now understand well why we think that dark matter must exist! Can you imagine other possible explanations of the strange galactic rotation curves without including dark matter?

3 What is dark matter?

Possible candidates to dark matter:

- planets and asteroids?
- brown dwarf stars?
- something else?

From our own solar system, it seems that the total matter is dominated by the Sun, not the planets. The total mass of the planets only make up about one part in 1000 of the total mass of the solar system. If this is the normal ratio, and we have no reason to believe otherwise, then the planets can only explain a tiny part of the invisible matter. Brown dwarf stars (more about these in later lectures) are stars which had too little mass to start nuclear reactions. They emit thermal radiation, but their temperature is low and they are therefore almost invisible. Observations of brown dwarfs in our neighborhood indicates that the number density is not large enough to fully explain the galactic rotation curves.

We are left with the last option, 'something else'. Actually, different kinds of observations in other areas of astrophysics (we will come back to this in the lectures on cosmology) indicate that the dark matter must be *non-baryonic matter*. Non-baryonic matter is matter which does not (or only very weakly) interact with normal visible matter in any other way than through gravitational interactions. From particle physics, we learn that the particle of light, the photon, is always created as a result of electromagnetic interactions. Non-baryonic matter does not take part in electromagnetic interactions (or only very weakly), only gravitational interactions, and can therefore not emit or absorb photons. Theoretical particle physics has predicted the existence of such non-baryonic matter for decades but it has been impossible to make any direct detections in the laboratory since these particles hardly interact with normal matter. We can only see them through their gravitational interaction on huge

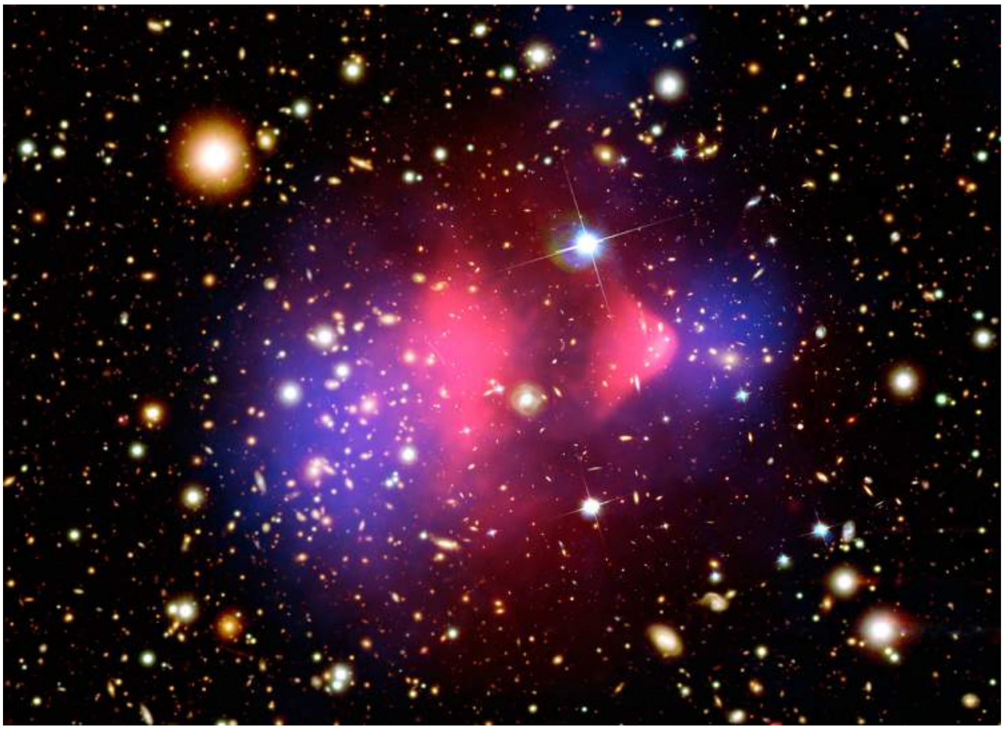

Figure 4: Info-figure: Composite image of the galaxy cluster 1E 0657-56, better known as the "Bullet cluster", which was formed after the collision of two large clusters of galaxies. Hot gas detected in X-rays is seen as two pink clumps in the image and contains most of the "normal" (i.e. baryonic) matter in the two clusters. The bullet-shaped clump on the right is the hot gas from one cluster, which passed through the hot gas from the other cluster during the collision. An optical image shows the galaxies in orange and white. The blue areas are where astronomers find most of the mass in the clusters, determined using the gravitational lensing effect where light from distant objects is distorted by intervening matter. Most of the matter in the clusters (blue) is clearly separate from the normal matter (pink), giving direct evidence that nearly all of the matter in the clusters is dark. (X-ray: NASA/CXC/CfA/M.Markevitch et al.; optical: NASA/STScI; Magellan/U.Arizona/D. Clowe et al.; lensing map: NASA/STScI; ESO WFI; Magellan/U.Arizona/D. Clowe et al.)

structures in the universe, such as galaxies. This is one example of how one can use astrophysics, the science of the largest structures in the universe, to study particle physics, the science of the smallest particles in the universe.

Dark matter is usually divided into two groups,

1. warm dark matter (WDM): light particles with high velocities ($v \approx c$)
2. cold dark matter (CDM): massive particles with low velocities ($v \ll c$)

One candidate to WDM are the *neutrinos* although these actually belong to baryonic matter. Neutrinos are very light particles which are associated with the electron and other elementary particles. When an electron is created in a particle collision, a neutrino is normally created in the same collision. Until a few years ago, neutrinos were thought not to have mass. Only some recent experiments have detected that they have a small but non-zero mass. Neutrinos, even if they are baryons, react only weakly with other particles and are therefore difficult to detect. One has been able to show that neutrinos do not make an important contribution to the total mass of galaxies. Nowadays, the most popular theories for dark matter are mostly theories based on CDM. Many different candidates for CDM exist in theoretical particle physics, but so far one has not been able to identify which particle might be responsible for the dark matter in galaxies.

Dark matter has been seen in many other types of observations as well. For instance by observing the orbits of galaxies about a common center of mass in clusters of galaxies, a similar effect has been seen: the orbits cannot be explained by including only the visible matter. Traces of dark matter has also been seen through observations of gravitational lenses (which we will come back to later) as well as other observations in cosmology.

4 Problems

Problem 1 (45 min.–1 hour)

Two galaxies with similar sizes orbit a common center of mass. Their distance from us has been estimated to 220 Mpc (one parsec=3.26 light years, 1 Mpc=10^6 parsecs). Their angular separation on the sky has been measured to $3.1'$. Their velocity with respect to the center of mass has been estimated to $v = 100$ km/s for both galaxies, one approaching us the other receding. Assume circular orbits. Assume that the velocities of the galaxies only have a radial component such that the given velocity is the full velocity of the galaxies.

1. What is the mass of the galaxies? (**Hint:** here you need to go back to the two-body problem. First calculate the radius of the orbit and

then use equations from the lectures on celestial mechanics. You will need to play a little with the equations.)

2. The size of the galaxies indicate that they contain roughly the same number of stars as the Milky Way, about 2×10^{11}. The average mass of a star in these two galaxies equals the mass of the Sun. What is the total mass of one of the galaxies counting only the mass of the stars?

3. What is the ratio of dark matter to luminous matter in these galaxies? This is an idealize example, but the result gives you the real average ratio of dark to luminous matter observed in the universe.

Problem 2 (90 min.–2 hours)

In the following link I have put three files with simulated (idealized) data taken from three galaxies:

`http://folk.uio.no/frodekh/AST1100/lecture4/`

Each file contains two columns, the first column is the position where the observation is made given as the angular distance (in arcseconds) from the center of the galaxy. These data are observations of the so-called 21 cm line. Neutral hydrogen emits radiation with wavelength 21.2 cm from a so-called forbidden transition in the atom. Radiation at this wavelength indicates the presence of neutral hydrogen. Galaxies usually contain huge clouds of neutral hydrogen. Measurements of the rotation curves of galaxies are usually made measuring the Doppler effect on this line at different distances from the center. The second column in these files is just that, the received wavelength of the 21.2 cm radiation. Again you need to use the Doppler formula to translate these wavelengths into radial velocities.

The three galaxies are estimated to be at distances 32, 4 and 12 Mpc. The total velocity of the galaxies has been measured to be 120, -75 and 442 km/s (positive velocity for galaxy moving away from us).

1. Make a plot of the rotation curves of these galaxies, plot distance in kpc and velocity in km/s.

2. Make a plot of the density profile of the galaxies (assuming that equation (2) is valid for all distances), again plot the distance in kpc and the density in solar masses per parsec^3.

3. Finally, assume that the density profile of these galaxies roughly follow equation (3). Find ρ_0 and R for these three galaxies (in the units you used for plotting). **Hints:** Looking at the expression for the density, it is easy to read ρ_0 off directly from the plot of the density profile. Having ρ_0 you can obtain R by trial and error, overplotting the density profile equation (3) for different R on top of your profile obtained from the data.

AST1100 Lecture Notes

5 The virial theorem

1 The virial theorem

We have seen that we can solve the equation of motion for the two-body problem analytically and thus obtain expressions describing the future motion of these two bodies. Adding just one body to this problem, the situation is considerably more difficult. There is no general analytic solution to the three-body problem. In astrophysics we are often interested in systems of millions or billions of bodies. For instance, a galaxy may have more than 2×10^{11} stars. To describe exactly the motion of stars in galaxies we would need to solve the 2×10^{11}-body problem. This is of course impossible, but we can still make some simple considerations about the general properties of such a system. We have already encountered one such general property, the fact that the center of mass maintains a constant velocity in the absence of external forces. A second law governing a large system is the *virial theorem* which we will deduce here. The virial theorem has a wide range of applications in astrophysics, from the formation of stars (in which case the bodies of the system are the atoms of the gas) to the formation of the largest structures in the universe, the clusters of galaxies. We will then apply the virial theorem to some of these problems in the coming lectures. Here we will show how to prove the theorem.

The virial theorem is a relation between the total kinetic energy and the total potential energy of a system in equilibrium. We will come back to the exact definition of the equilibrium state at the end of the proof.

We will consider a system of N particles (or bodies) with mass m_i, position vector $\vec{r}_i$, velocity vector $\vec{v}_i$ and momentum $\vec{p}_i = m_i\vec{v}_i$ (see figure 1). We will take the origin of our system to be the center of mass for reasons which we will see at the end. For this system, the total moment of inertia is given by (remember from your mechanics classes?)

$$I = \sum_{i=1}^{N} m_i|\vec{r}_i|^2 = \sum_{i=1}^{N} m_i\vec{r}_i \cdot \vec{r}_i.$$

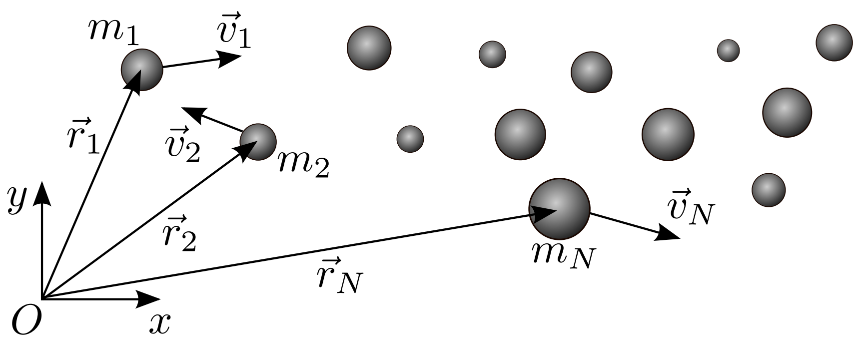

Figure 1: The N-body system.

In mechanics one usually takes the moment of intertia with respect to a given axis, here we take the moment of intertia with respect to the origin. The time derivative of the moment of inertia is called the *virial*,

$$Q = \frac{1}{2}\frac{dI}{dt} = \sum_{i=1}^{N} \vec{p}_i \cdot \vec{r}_i.$$

To deduce the virial theorem we need to take the time derivative of the virial

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \frac{d\vec{p}_i}{dt} \cdot \vec{r}_i + \sum_{i=1}^{N} \vec{p}_i \cdot \vec{v}_i,$$

where Newton's second law gives

$$d\vec{p}_i/dt = \vec{F}_i$$

$\vec{F}_i$ being the sum of all forces acting on particle i. We may write this as

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i + \sum_{i=1}^{N} m_i v_i^2,$$

where the last term may be expressed in terms of the total kinetic energy of the system $K = \sum_i 1/2 m_i v_i^2$

$$\frac{dQ}{dt} = \sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i + 2K. \qquad (1)$$

We will now try to simplify the first term on the right hand side. If no external forces work on the system and the only force which acts on a given particle is the gravitational force from all the other particles, we can write

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = \sum_{i=1}^{N} \sum_{j \neq i} \vec{f}_{ij} \cdot \vec{r}_i,$$

where $\vec{f}_{ij}$ is the gravitational force on particle i from particle j. The last sum is a sum over all particles j except particle $j = i$. The double sum thus expresses a sum over all possible combinations of two particles i and j, except the combination where $i = j$. We may view this as an $N \times N$ matrix where we sum over all elements ij in the matrix, except the diagonal elements ii. We divide this sum into two parts separated by the diagonal (see figure 2),

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = \underbrace{\sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_i}_{\equiv A} + \underbrace{\sum_{i=1}^{N} \sum_{j>i} \vec{f}_{ij} \cdot \vec{r}_i}_{\equiv B}$$

We now rewrite the sum B as

$$B = \sum_{i=1}^{N} \sum_{j>i} \vec{f}_{ij} \cdot \vec{r}_i = \sum_{j=1}^{N} \sum_{i<j} \vec{f}_{ij} \cdot \vec{r}_i,$$

where the sums have been interchanged (you can easily convince yourself that this is the same sum by looking at the matrix in figure 2). We can also interchange the name of the indices i and j (this is just renaming the indices, nothing else)

$$B = \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ji} \cdot \vec{r}_j.$$

From Newton's third law, we have $\vec{f}_{ij} = -\vec{f}_{ji}$,

$$B = -\sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_j.$$

Totally, we have,

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = A + B = \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_i - \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot \vec{r}_j = \sum_{i=1}^{N} \sum_{j<i} \vec{f}_{ij} \cdot (\vec{r}_i - \vec{r}_j). \tag{2}$$

Did you follow all steps so far? Here, the force $\vec{f}_{ij}$ is nothing else than the well known gravitational force,

$$\vec{f}_{ij} = G \frac{m_i m_j}{r_{ij}^3} (\vec{r}_j - \vec{r}_i),$$

where $r_{ij} = |\vec{r}_j - \vec{r}_i|$. Note that the force points in the direction of particle j. Inserting this into equation (2) gives

$$\sum_{i=1}^{N} \vec{F}_i \cdot \vec{r}_i = -\sum_{i=1}^{N} \sum_{j<i} G \frac{m_i m_j}{r_{ij}^3} r_{ij}^2 = \sum_{i=1}^{N} \sum_{j<i} U_{ij},$$

where U_{ij} is the gravitational potential energy between particle i and j. This sum is the total potential energy of the system (do you see this?), the sum of the potential between all possible pairs of particles (note that

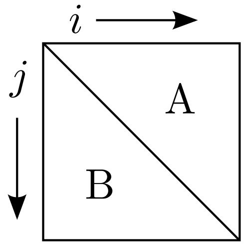

Figure 2: The matrix visualizing the summation

one pair of particle should be counted only once, this is why there is a $j < i$ in the latter sum). Thus, we have obtained an expressions for the two terms in equation (1) expressing the time derivative of the virial

$$\frac{dQ}{dt} = U + 2K.$$

Finally we will use the equilibrium condition. We will take the mean value of this expression over a long period of time,

$$\langle \frac{dQ}{dt} \rangle = \langle U \rangle + 2\langle K \rangle,$$

where

$$\langle \rangle = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau dt.$$

For the term on the left hand side, we find

$$\lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \frac{dQ}{dt} dt = \lim_{\tau \to \infty} \frac{Q(\tau) - Q(0)}{\tau} \equiv 0,$$

for a system in equilibrium. The last equality here is the definition of the equilibrium state in which the system needs to be for the virial theorem to hold: the mean value of the time derivative of the virial must go to zero. In order for this to be fulfilled, the quantities $Q(\tau)$ and $Q(0)$ need to have finite values. If, for instance, the system is bound and the particles go in regular orbits, the virial Q will oscillate regularly between two finite values. In this case, the last expression above will go to zero as $\tau \to \infty$. If Q had not been limited, which could happen for a system which is not bound, then Q could attain large values with time and it would not be clear that this expression would approach zero as $\tau \to \infty$. Using the above equation and the equilibrium condition we see that a bound system in equilibrium obeys

The Virial Theorem

$$\langle K \rangle = -\frac{1}{2}\langle U \rangle.$$

In order to obtain $\langle K \rangle$ and $\langle U \rangle$ we need to take the average of the kinetic and potential energy over a long time period. In the case of the solar system, this is easy: The orbits are periodic so it suffices to take the average over the longest orbital period. Please note that we have done the calculations in the center of mass frame. If we did it from a different frame of reference, our system of particles would move at a constant speed with respect to us and the distance to the system would increase indefinitely. All the distances would grow to infinity and the time derivative of the virial would not go to zero.

Averaging a system over a long time period may be equal to averaging the system over the ensemble. This is the *ergodic hypothesis*. Mathematically it can be written as

$$\lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau dt \to \lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^{N}.$$

If a bound system has a huge number of particles ($N \to \infty$), it is equivalent to seeing the system over a long period of time ($\tau \to \infty$). Thus, we can apply the virial theorem to a galaxy by taking the mean of the kinetic and potential energy of all stars in the galaxy in a given instant. According to the ergodic hypothesis, it is not necessary in this case to take the mean of the kinetic and potential energy over a very long period of time. Since the time scales for changes for such huge systems is very long, it is much easier to simply take the average over all stars. The ergodic theorem thus says that we can replace the mean value from being a time average to be an average over all bodies in the system.

2 Applying the virial theorem to a collapsing cloud of gas

To show the power of the virial theorem we will apply it to a system with very many particles and show how properties of this complex system may be calculated. In the exercises you will find two more examples of applications of the virial theorem to problems of a very different nature. The example presented in this section is also an appetiser for the lectures on stellar evolution coming later.

Before the advent of the theory of relativity, the source of the energy that powers stars was sought. One suggestion was that the stellar energy was gravitational energy that is being radiated away as the cloud of gas retracts. A star starts out as a huge cloud of gas which starts collapsing due to its own force of gravity. Gas falls towards the center of the cloud and releases gravitational energy in the form of electromagnetic radiation as it falls. As long as the cloud keeps collapsing, energy is radiated away and could possibly explain the energy production in stars. To check if this is a plausible explanation, we will need to calculate the total energy, kinetic plus potential, that the star could possibly radiate away during its collapse and compare this with the energy output from the Sun. To calculate the total energy of such a cloud, we need to invoke the virial theorem. A collapsing cloud of gas is a bound many-body system and the virial theorem should apply.

We will assume that the cloud is spherically symmetric with radius R and mass M. We need to calculate the total energy, kinetic plus potential, of such a cloud. Thanks to the virial theorem, it suffices to calculate only the potential energy. The total energy is given by

$$E = K + U = -\frac{1}{2}U + U = \frac{1}{2}U,$$

where K is kinetic energy and U is potential energy. Using the virial theorem $K = -U/2$, we replace K by U and obtain an expression for the total energy given only in terms of the total potential energy. I have skipped time average here since this is a system with very many particles and we can use the ergodic hypothesis and simply sum over all particles.

We see that if we are able to calculate the total potential energy of the cloud, we would also obtain the total mechanical energy (kinetic+potential). To obtain the total potential energy, we will start by considering the potential du of a tiny particle of mass dm inside the cloud at a distance r from the center. We have learned (see the lectures on dark matter) that the gravitational forces from a spherical shell of matter add to zero inside this shell. Thus we need only to consider the gravitational attraction on the mass dm from the sphere of matter inside the position of the mass. This is a sphere of radius r with mass $M(r)$. Being a sphere, Newton's law of gravitation applies as if it were a point mass located at the center with mass $M(r)$. Thus the potential energy between the particle dm and the rest of the cloud (the part inside the particle) is

$$du = -G\frac{M(r)dm}{r}.$$

We integrate this equation over all masses dm in the shell of thickness dr at distance r from the center. We assume that the mass density in the shell is given by $\rho(r)$. We then obtain the potential energy dU between the shell and the spherical mass $M(r)$ inside the shell.

$$dU = -G\frac{M(r)4\pi r^2\rho(r)dr}{r}.$$

To obtain the total potential energy U, we need to integrate this expression over all radii r out to the edge of the cloud at $r = R$,

$$U = -4\pi G\int_0^R M(r)\rho(r)r\,dr.$$

We would generally need to know the density $\rho(r)$ in order to obtain $M(r)$ and to integrate this equation. The scope here is to obtain an approximate expression giving us an idea about the mass and radius dependence of the energy and to obtain an order of magnitude estimate. For this purpose, we assume that the density is constant with a value equal to the mean density of the cloud,

$$\rho = \frac{M}{(4/3)\pi R^3}.$$

This gives $M(r) = (4/3)\pi r^3 \rho$ and we can integrate the equation

$$U = -4\pi G \left(\frac{M}{(4/3)\pi R^3} \right)^2 (4/3)\pi \int_0^R r^4 \, dr,$$

$$U = -\frac{3GM^2}{5R}.$$

From the virial theorem, the total energy is then (check!)

$$E = \frac{1}{2}U = -\frac{3GM^2}{10R}.$$

This is the total energy of a cloud of gas with mass M and radius R. The energy that the Sun has radiated away during its lifetime can be written as

$$E_{\text{radiated}} = E(\text{big}R) - E(R_\odot),$$

where 'big R' refers to the radius of the cloud when it started collapsing and $R_\odot$ is the current radius of the Sun. The total energy of the cloud goes as $\propto 1/R$, so for the initial cloud this quantity can be approximated to zero. Thus we are left with

$$E_{\text{radiated}} = \frac{3GM_\odot^2}{10R_\odot},$$

where $M_\odot$ is the mass of the Sun. Inserting numbers for the mass and radius of the Sun we obtain $E_{\text{radiated}} \approx 1.1 \times 10^{41} J$. Assuming that the Sun has been radiating with the same luminosity $L_\odot$ (dE/dt) during its full lifetime, we can calculate the age of the Sun,

$$\Delta t = \frac{E_{\text{radiated}}}{L_\odot} \approx 10^7 \text{years}.$$

If gravitational collapse was indeed the source of solar energy, the Sun couldn't have lived longer than about 10 millions years. Several geological findings have shown that the Earth and therefore also the Sun has existed for about 500 times as long. Thus using the virial theorem we have shown (using some assumptions) that gravitational collapse cannot satisfactory explain the generation of energy in the Sun.

3 Problems

Problem 1 (10–20 min.)

In a way we can look at the virial theorem as a generalization of Kepler's third law to a many-body system. Show that for the two-body problem, the virial theorem is identical to Kepler's third law in the Newtonian form (as deduced in the exercises in lecture notes 1–2). Assume circular orbits. Start with the virial theorem, insert expressions for the energies and show Kepler's third law. (You won't get more help here...).

Problem 2 (2–2.5 hours)

Fritz Zwicky was the first to note that there is some missing matter in the universe. In 1933, several years before the discovery of the flat rotation curves in the galaxies, he used the virial theorem to calculate the mass of galaxies in the Coma Cluster. A cluster of galaxies is a cluster of a few hundred galaxies orbiting a common center of mass. The Coma Cluster is one of our neighbouring clusters of galaxies. He found that the mass of the Coma Cluster calculated using the virial theorem was much larger than the mass expected from the visible luminous matter. In this problem we will try to follow his example and estimate the mass of galaxies in a cluster of galaxies. We will consider a simulated cluster of about 100 galaxies. We will assume that the cluster consists of these 100 brightest galaxies and assume that the remaining galaxies are too small to affect our calculations significantly.

1. Looking in the telescope we see that the cluster is spherical, the galaxies are evenly distributed inside a spherical volume. The distance to the cluster is 85 Mpc. You observe the radius of the cluster to be $32'$. What is the radius of the cluster in Mpc?

2. All galaxies in the cluster appear to be very similar to the Milky Way, both in the number of stars and the type of stars. The galaxies look so similar to each other that we can assume that all the galaxies have the same mass m. We know that the Milky Way has about 2×10^{11} stars. Assuming that the mean mass of a star equals the mass of the Sun, what is the estimated total luminous mass m of these galaxies?

3. Use the virial theorem to show that the mass m of a galaxy in the cluster can be written as

$$m = \frac{\sum_{i=1}^{N} v_i^2}{G \sum_{i=1}^{N} \sum_{j>i} 1/r_{ij}},$$

   where r_{ij} is the distance between galaxy i and galaxy j and v_i is the velocity of galaxy i with respect to the center of mass.

Figure 3: Info-figure: Fritz Zwicky was the first to use the virial theorem to infer the existence of unseen matter, which he referred to as "dunkle Materie" dark matter. He used the theorem in 1933 to calculate the mass of the Coma cluster of galaxies (aka. Abell 1656) and found that it was much larger than the mass expected from the luminous matter. The cluster contains more than one thousand galaxies, most of them ellipticals. It lies in the constellation of Coma Berenices, at a mean distance of roughly 100 Mpc. The central region is dominated by two giant elliptical galaxies, which are easily spotted in the above image. The bright blue-white source above the center is a foreground star in our own galaxy. (Figure: J. Misti)

4. You will find a file with data for each of the galaxies here:

   `http://folk.uio.no/frodekh/AST1100/lecture5/galaxies.txt`

   The first column in the file is the observed angular distance (in arcminutes) from the center of the cluster along an x-axis. The second column in the file is the observed angular distance (in arcminutes) from the center of the cluster along an y-axis. (the x-y coordinate system is chosen with an arbitrary orientation on the plane of observation (which is perpendicular to the line of sight)). The third column is the measured distance to the galaxy (from Earth) in Mpc. The fourth column is the position of the spectral line at 21.2 cm for the given galaxy in units of m.

   (a) Using these data, what is the radial velocity of the cluster with respect to us? Remember that the velocity of a galaxy can be written as

   $$v(\text{gal}) = v(\text{cluster}) + v(\text{rel}),$$

   where v(gal) is the total velocity of the galaxy with respect to us, v(cluster) is the velocity of the cluster (of the center of mass of the cluster) with respect to us and v(rel) is the relative velocity of a galaxy with respect to the center of mass of the cluster. The relative velocities with respect to the center of mass are random, so for a large number of galaxies the mean

   $$\frac{1}{N}\sum_{\text{i=1}}^{N} v_i(\text{rel}) \rightarrow 0$$

   goes to zero.

   (b) Make a plot showing how this cluster appears in the telescope: draw the x-y axes (using arcminutes as units on the axes) and make a dot at the position for each galaxy. Remember that in Python you can plot for instance a circle at each data point by using `plot(x,y,'o')`.

   (c) Use these data and the expression above for the mass of a galaxy from the virial theorem to obtain a minimum estimate of the total mass of a galaxy in the cluster. How does it compare to the estimate you obtained for luminous matter above? **Hint 1:** To make the double sum in Python you can construct two FOR-loops, one over the index i and one over the index j. Inside the two FOR-loops, you add the expression inside the sum for indices i and j to the final result. **Hint 2:** To find the distance between two galaxies i and j, it is convenient to find the x, y and z coordinates of each galaxy in meters.

   (d) Your measured velocities are based on the Doppler effect and are therefore radial velocities. Because the inclinations of the velocities with the line of sight is not 90°, your estimate is a minimum estimate of the mass. We will now use the fact that you have many galaxies and that you know that the orientation is random to get a more exact estimate. As a first step you will need to find the mean of $\sin^2 i$ (where i is inclination) taken over many galaxies with random orientations: What is the expected mean value taken over many galaxies of the expression $\sin^2 i$? We assume that the inclination is random (with a uniform distribution). Remember that the mean value of a function $f(x)$ is defined statistically by

$$\langle f(x) \rangle = \frac{\int dx f(x) P(x)}{\int dx P(x)},$$

where $P(x)$ is the statistical distribution, i.e. the probability of having a value x. In this case, the distribution is uniform, meaning that there is an equal probability for getting any value of the inclination i. We may thus set $P(x) = 1$. The integration in this general expression is done over all possible values of x.

(e) Can you use this to obtain a more accurate estimate of the mass?

AST1100 Lecture Notes

6 Electromagnetic radiation

1 The electromagnetic spectrum

To obtain information about the distant universe we have the following sources available:

1. **electromagnetic waves** at many different wavelengths.
2. **cosmic rays:** high energy elementary particles arriving from supernovae or black holes in our galaxy as well as from distant galaxies. The galactic magnetic field changes the direction of these particles making it impossible to determine the incoming direction and therefore the exact sources of the rays.
3. **neutrinos:** these extremely light elementary particles interact very rarely with other particles and can therefore arrive from huge distances without being scattered on the way. This property also makes neutrinos very difficult to detect and therefore a source of information with limited usefulness until better detection methods are discovered.
4. **gravitational waves:** spacetime distortions traveling through space as a wave. These are predicted by Einstein's general theory of relativity. Gravitational waves have still not been directly detected, but experiments are on their way.

Of these sources, electromagnetic waves is by far the most important. Practical problems limit the amount of information we can obtain from other sources with current technology. Since electromagnetic radiation is almost the only source which we use to get information about the distant universe, it is of high importance in astrophysics to know the processes which produce this kind of radiation. Here we will discuss some of the most important processes along with some discussion on how the radiation from these different processes is used to obtain information about the universe. Some important types of radiation are

- **thermal radiation:** the thermal motion of atoms produces electromagnetic radiation at all wavelengths. For a *black body* (see later),

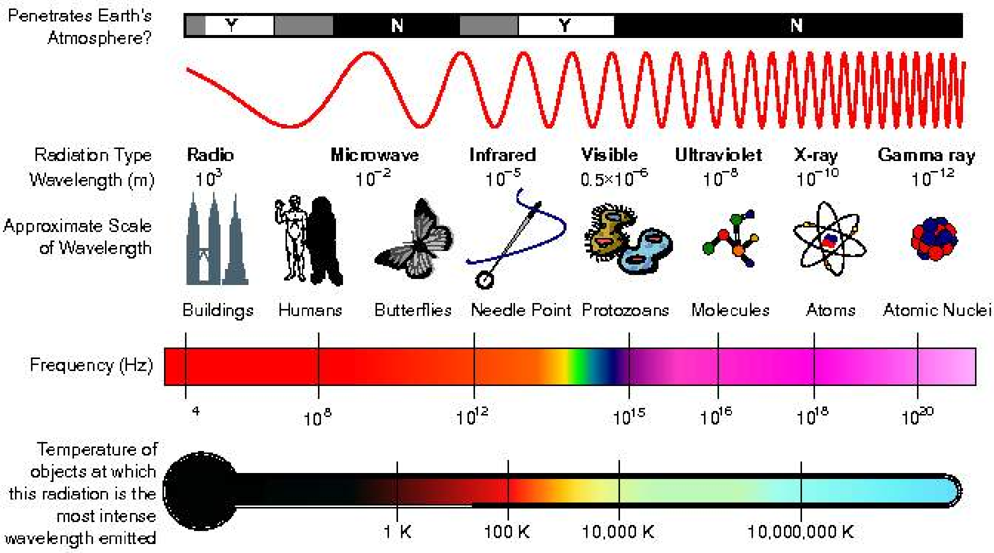

Figure 1: A diagram of the electromagnetic spectrum, showing various properties across the range of frequencies and wavelengths. The spectrum is a continuum, but is often divided into the following main regions of decreasing wavelength and increasing energy: radio, microwave, infrared, visible, ultraviolet, X-ray, and gamma-ray. Note that the Earth's atmosphere is transparent only to visible light, a part of the radio spectrum and a few narrow wavelength intervals in the infrared, thus limiting the types of celestial objects and astrophysical processes that can be studied using ground-based telescopes. (Figure:Wikipedia)

the radiation emitted at a given frequency is distributed according to Planck's law of radiation.

- **synchrotron radiation:** radiation produced by energetic charged particles accelerated in a magnetic field. This process emits electromagnetic radiation at different wavelengths depending on the energies involved in the process. Our own galaxy emits synchrotron radiation as radio waves due to the acceleration of cosmic ray electrons in the magnetic field of the galaxy.
- **Bremsstrahlung:** radiation produced by the 'braking' of a charged particle, usually an electron, by another charged particle, typically a proton or atomic nucleus. Due to electromagnetic forces from ions, electrons are deflected, and hence accelerated, producing electromagnetic radiation at all wavelengths. The space between galaxies in the clusters of galaxies is called the *intergalactic medium (IGM)*. It contains a very hot plasma of electrons and ions emitting brehmsstralung mainly as X-rays. These X-rays constitute an important source of information about distant clusters of galaxies.
- **21 cm radiation:** Neutral hydrogen emits radiation with wavelength 21 cm due to a so-called spin-flip: The quantum spin of the electron and proton may change direction such that the spin vectors go from having their orientation in the same direction to having their orientation in opposite directions. In this process, the total energy of the atom decreases and the energy difference between the two states is emitted as 21 cm radiation. This is a so-called forbidden transition, meaning that it occurs very rarely. For a single atom one would on average need to wait about 10 millions years for the process to occur. However, in huge clouds of gas the number of hydrogen atoms is so large that the intensity of 21 cm radiation can be quiet large even for such a rare process.

2 Solid angles

Before embarking on the properties of radiation, we will first introduce a new concept which will be widely used: *the solid angle*. The solid angle is a generalization of the concept of an *angle* from one to two dimensions. Looking at figure 3, we see that an angle measured in radians is simply a distance Δs taken along the rim of the unity circle

$$\theta = \Delta s.$$

To convince you about this, remember that the circumference of the unity circle, the full distance taken around the circle, is 2π. Now, the solid angle is measured in units of *steradians*, for short *sr*, and is a part of the *area* of the surface of the unit sphere as seen in figure 4. Thus,

$$\Omega = \Delta A.$$

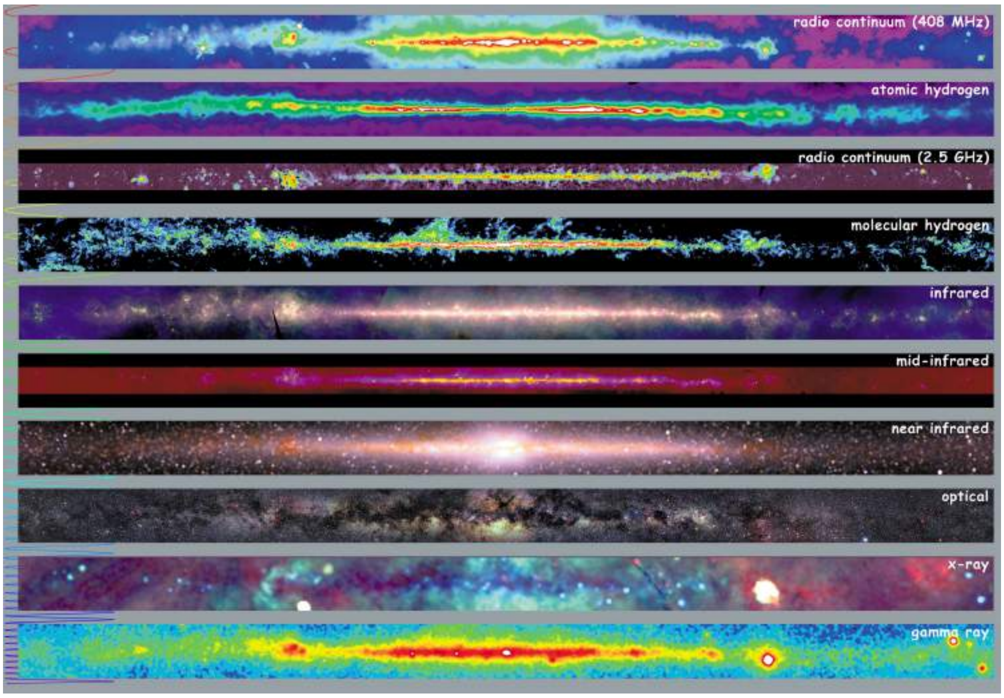

Figure 2: Info-figure: The Milky Way band observed in several wavelength regions (ultraviolet light is missing, though). The development of new detectors and, in particular, space telescopes has enabled us to study the universe at all wavelengths. We can now learn about celestial objects and physical processes that were completely unknown to astronomers only a few decades ago.(Figure: NASA)

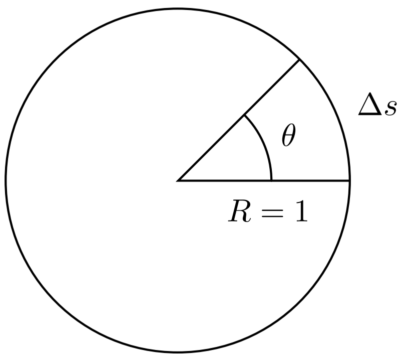

Figure 3: The angle measured in radians is defined as the length taken along the rim of the unit circle.

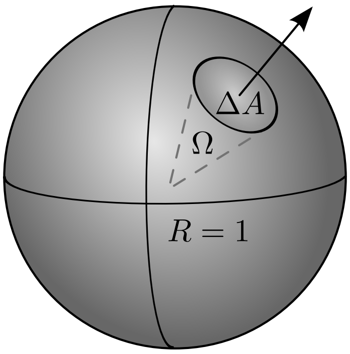

Figure 4: The solid angle measured in steradians is defined as the area taken on the surface of the unit sphere.

The solid angle corresponding to the full unit sphere is then 4π sr which is the full area of the surface of the unit sphere. If we imagine a source of radiation in the center of the unit sphere, the solid angle can be used to describe the amount of radiation going in a certain direction as the energy transported per steradian. This is widely used in the study of radiative processes in stars.

3 Black body radiation

Thermal radiation is emitted from an object of temperature T because of the thermal motion of atoms at this temperature. Black body radiation is thermal radiation from a black body. A black body is defined as a body which absorbs all radiation it receives, no radiation is reflected or can pass through. Many objects in astrophysics are close to being a black body, a star is a typical example. For a black body, an expression for the intensity of the thermal radiation as a function of wavelength/frequency can be obtained analytically. A black body emits thermal radiation at all frequencies, but which frequency has the largest intensity depends on the temperature of the black body. To calculate the distribution of radiation per frequency quantum physics is needed. We will therefore not make the calculation here (you will come to this in physics courses later), but rather state the result:

Planck's law of radiation

$$B(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}.$$

where ν is the frequency, T is the temperature of the black body, h is Planck's constant and k is the Boltzmann constant.

The quantity $B(\nu)$ is *intensity* defined such that

$$\Delta E = B(\nu) \cos\theta \Delta\nu \Delta A \Delta\Omega \Delta t \quad (1)$$

is the small energy passing through a small area ΔA into a small solid angle $\Delta\Omega$ (see figure 5) per small time interval Δt in the small frequency range $[\nu, \nu + \Delta\nu]$. Intensity is measured in units of W/m^2/sr/Hz. Here the factor $\cos\theta$ comes from the fact that energy per solid angle per area is lower by a factor $\cos\theta$ for an observer making an angle θ with the normal to the area emitting radiation. Example: Imagine you have a light bulb which emits black body radiation at a certain temperature. You set up a wall between you and the light bulb and let light pass only through a small hole in the wall of area $\Delta A = 0.1$ mm^2. Just around the hole you construct a unit sphere and put a detector at an angle $\theta = 30°$ with a line orthogonal to the wall. The detector occupies about 1/1000 of the unit sphere and thus absorbes light from $\Delta\Omega = 4\pi/1000$ sr. Finally, the detector contains a material which only absorbes and measures radiation in the wavelenght range 600–600.1 nm , such that $\Delta\nu = 0.1$ nm. The energy that the detector measures from the light during a period of $10^{-3}s$ is then:

$$\Delta E = B(600 \text{ nm}) \times \cos(30°) \times 0.1 \text{ mm}^2 \times (4\pi/1000) \text{ sr} \times 0.1 \text{ nm} \times 10^{-3} \text{ s}$$

In reality, the definition is made when we let all Δ be infinitesimally small, such that the definition reads

$$dE = B(\nu) \cos\theta \, d\nu \, dA \, d\Omega \, dt \qquad (2)$$

When we use differentials instead of finite differences Δ, we can use integrals to obtain the energy over large intervals in area, frequency, solid angle or time.

Note that in order to write Planck's law in terms of wavelength λ instead of frequency ν one can *not* simply replace $\nu = c/\lambda$. $B(\nu)$ is defined in terms of differentials, so we need to take these into account. When changing from frequency to wavelength, the energy must be the same, we are only changing variables, not the physics. Using that the energy ΔE is the same, we get from equation 2 that $B(\nu)d\nu = -B(\lambda)d\lambda$ (the minus sign comes from the fact that λ and ν increase in opposite directions,

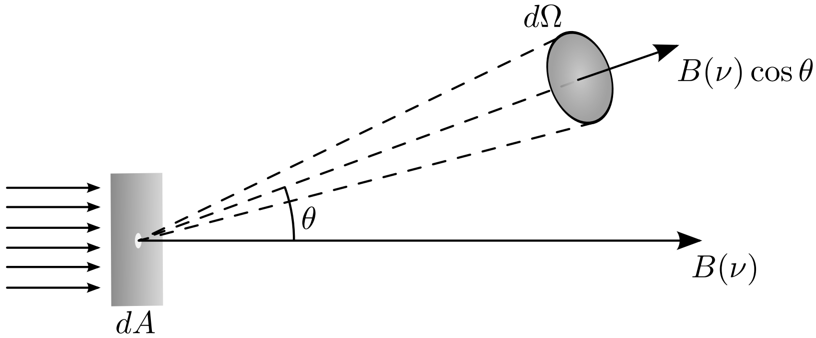

Figure 5: Intensity is the energy of radiation passing through area dA into a solid angle $d\Omega$ per time, per wavelength.

$\lambda + |\delta\lambda| \to \nu - |\delta\nu|$). We can write

$$B(\nu)d\nu = -B(\nu)\frac{d\nu}{d\lambda}d\lambda \equiv B(\lambda)d\lambda,$$

We therefore obtain

$$B(\lambda) = -B(\nu)\frac{d\nu}{d\lambda} = -B(\nu)\left(-\frac{c}{\lambda^2}\right) = \frac{2hc^2}{\lambda^5}\frac{1}{e^{hc/(kT\lambda)}-1}.$$

Figure (6) shows the intensity as a function of wavelength for black bodies with different temperature T. We see that the wavelength of maximum intensity is different for different temperatures. We can use the position of this peak to determine the temperature of a black body. We can find an analytical expression for the position of the peak by setting the derivative of Planck's law equal to zero,

$$\frac{dB(\lambda)}{d\lambda} = 0$$

In the exercises you will show that the result gives:

Wien's displacement law

$$T\lambda_{\mathrm{max}} = 2.9 \times 10^{-3}\ \mathrm{Km}.$$

Another way to obtain the temperature of a black body is by taking the area under the Planck curve, i.e. by integrating Planck's law over all wavelengths. This area is also different for different temperatures T. Integrating this over all solid angles $d\Omega$ and frequencies $d\nu$, we obtain an expression for the *flux*, energy per time per area,

$$F = \frac{dE}{dA\,dt}.$$

The integral can be written as (here we are just integrating equation (2) over $d\nu$ and $d\Omega$)

$$F = \int_0^\infty d\nu \int d\Omega B(\nu)\cos\theta.$$

Using that $d\Omega = d\phi\sin\theta d\theta = -d\phi(d\cos\theta)$ and substituting $u = h\nu/kT$, we get

$$\begin{aligned} F &= \int_0^{2\pi} d\phi \int_0^1 d\cos\theta\cos\theta \int d\nu \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)}-1} \\ &= \frac{2k^4T^4\pi}{h^3c^2}\int\frac{u^3du}{e^u-1} \\ &= \frac{2\pi k^4T^4}{h^3c^2}\underbrace{\zeta(4)}_{\pi^4/90}\underbrace{\Gamma(4)}_{3!} \\ &= \underbrace{\frac{2\pi^5k^4}{15h^3c^2}}_{\equiv\sigma}T^4. \end{aligned}$$

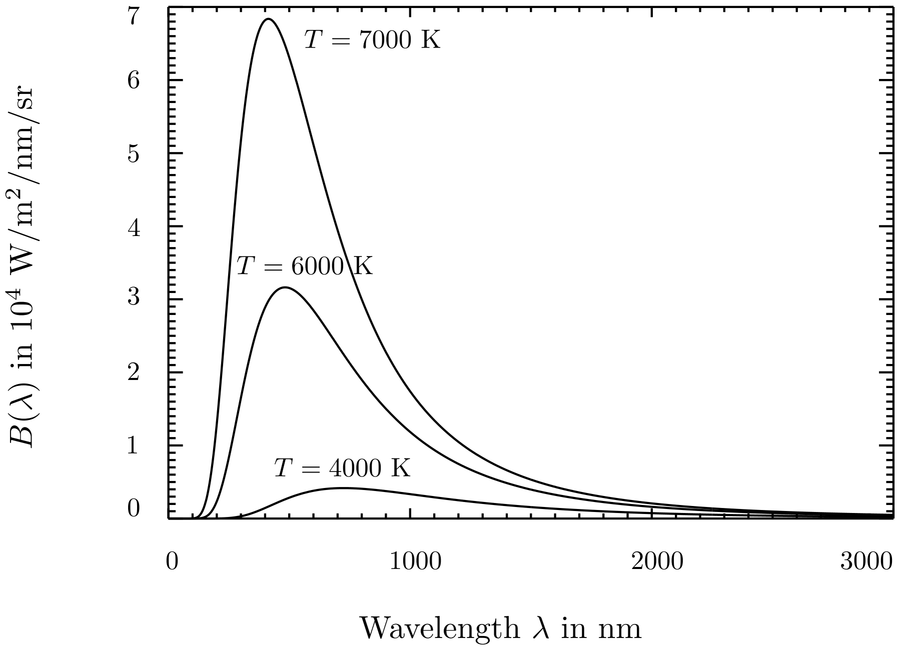

Figure 6: Planck's law for different black body temperatures.

Here the solution of the u-integral can be found in tables of integrals expressed in terms of ζ, the Riemann zeta-function and Γ, the gamma-function, both of which can be found in tables of mathematical functions. The final result is thus:

Stefan-Boltzmann law

The flux emitted from a black body is proportional to the temperature to the fourth power.

$$F = \sigma T^4,$$

where σ is a constant.

We see that we have two ways of measuring the temperature of a star, by looking for the wavelength were the intensity is maximal, or by measuring the energy per area integrated over all wavelengths. If a star had been a black body, these two temperatures would have agreed. However, a star is not a perfect black body. A star has different temperatures at different depths in the star's atmosphere. At different wavelengths we receive radiation from different depths and the final radiation is a combination of Planck radiation at several temperatures. Since the intensity as a function of wavelength is not a perfect Planck curve at a fixed temperature T, the two ways of measuring the temperature will also disagree,

- From Wien's displacement law, we get the *color temperature*, $T = \text{constant}/\lambda_{\text{max}}$.
- From Stefan-Boltzmann's law we get the *effective temperature*, $T = (F/\sigma)^{1/4}$.

The first temperature is called the color temperature since it shows for which wavelength the radiation has it's maximal intensity and hence which color the star appears to have. The second temperature is based on the total energy emitted.

We have so far introduced two measures for the energy of electromagnetic radiation:

Intensity

$$I(\nu) = \frac{dE}{cos\theta\, d\nu\, dA\, d\Omega\, dt}$$

energy received per frequency, per area, per solid angle and per time.

Flux (or total flux)

$$F = \frac{dE}{dA\, dt}$$

total energy received per area and per time.

You will now soon meet the following expressions:

Flux per frequency

$$F(\nu) = \frac{dE}{dA\, dt\, d\nu}$$

total energy received per area, per time and per frequency.

Luminosity

$$L = \frac{dE}{dt}$$

total energy received per unit of time.

Luminosity per frequency

$$L(\nu) = \frac{dE}{dt\, d\nu}$$

total energy received per frequency per time.

You will soon see more uses of all these expressions in practise, but it is already now a good idea to memorize the meaning of intensity, flux and luminosity.

4 Spectral lines

When looking at the spectra of stars you will discover that they have thin dark lines at some specific wavelengths. Something has obscured the radiation at these wavelengths. When the radiation leaves the stellar surface it passes through the stellar atmosphere which contains several atoms/ions absorbing the radiation at specific wavelengths corresponding to energy gaps in the atoms. According to Bohr's model of the atom, the electrons in the atom may only take certain energy levels $E_0, E_1, E_2, ...$ The electron cannot have an energy between these levels. This means that when a photon with energy $E = h\nu$ hits an atom, the electron can only absorb the energy of the photon if the energy $h\nu$ corresponds exactly to the difference between two energy levels $\Delta E = E_i - E_j$. Only in this case is the photon absorbed and the electron is excited to a higher energy level in the atom. Photons which do not have the correct energy will pass the atom without being absorbed. For this reason, only radiation at frequency ν with photon energy $E = h\nu$ corresponding to the difference in the energy level of the atoms in the stellar atmosphere will be absorbed. We will thus have dark lines in the spectra at the wavelengths corresponding to the energy gaps in the atoms in the stellar atmosphere (see figure 7). By studying the position of these dark lines, the *absorption lines*, in the spectra we get information about which elements are present in the stellar atmosphere.

The opposite effect also takes place. In the hotter parts of the stellar atmospheres, electrons are excited to higher energy levels due to collisions with other atoms. An electron can only stay in an excited energy level for a limited amount of time after which it spontaneously returns to the lowest energy level, emitting the energy difference as a photon. In these cases we will see bright lines, *emission lines*, in the stellar spectra at the wavelength corresponding to the energy difference, $h\nu = \Delta E$ (see figure 8).

The exact energy levels in the atoms and thus the wavelengths of the absorption and emission lines can be calculated using quantum physics, or they can be measured in the laboratory. However, the actual wavelength where the spectral line is found in a stellar spectra may differ from the predicted value. One reason for this could be the Doppler effect. If the star has a non-zero radial velocity with respect to the Earth, all wavelengths and hence also the position of the spectral lines will move according to

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v_r}{c},$$

where v_r is the radial component of the velocity. By taking the difference $\Delta\lambda$ between the observed wavelength (λ) and predicted wavelength (λ_0) of the spectral line, one can measure the velocity of a star or any other astrophysical object as we discussed in the lecture on extrasolar planets.

Note that even if the star has zero-velocity with respect to Earth, we will still measure a Doppler effect: The atoms in a gas are always moving in random directions with different velocities. This thermal motion of the atoms will induce a Doppler effect and hence a shift of the spectral line. Since the atoms have a large number of different speeds and directions, they will also induce a large number of different Doppler shifts $\Delta\lambda$ with the result that a given spectral line is not seen as a narrow line exactly at $\lambda = \lambda_0$, but as a sum of several spectral lines with different Doppler shifts $\Delta\lambda$. The total effect of all these spectral lines is one single broad

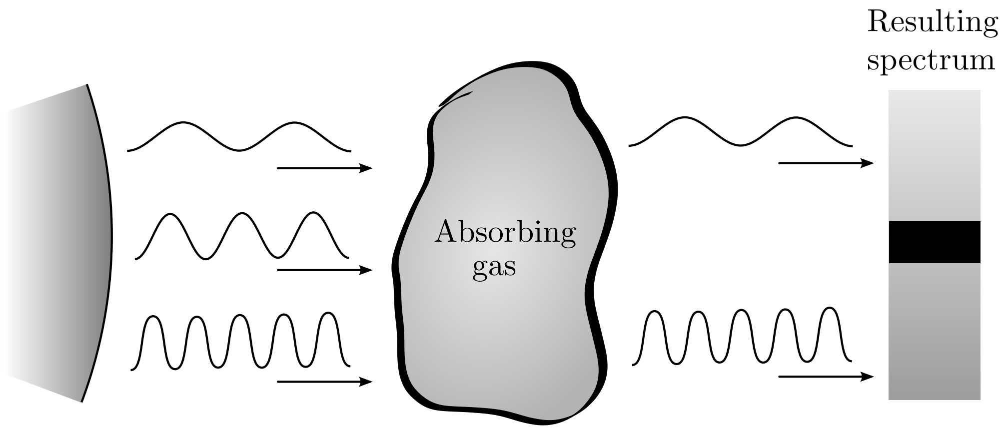

Figure 7: Formation of absorption lines.

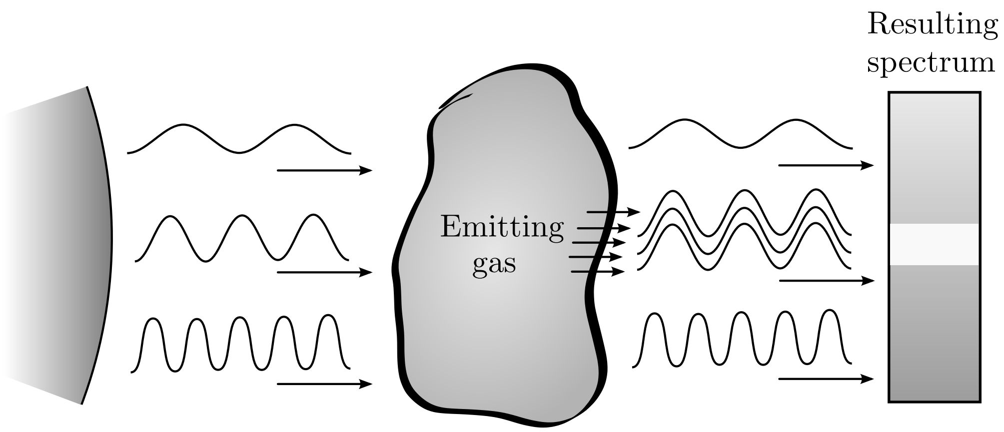

Figure 8: Formation of emission lines.

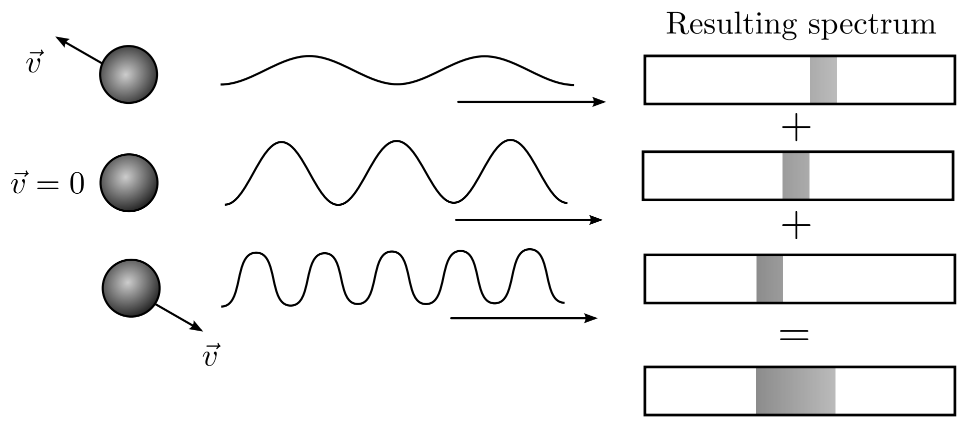

Figure 9: Broadening of spectral lines due to thermal motion.

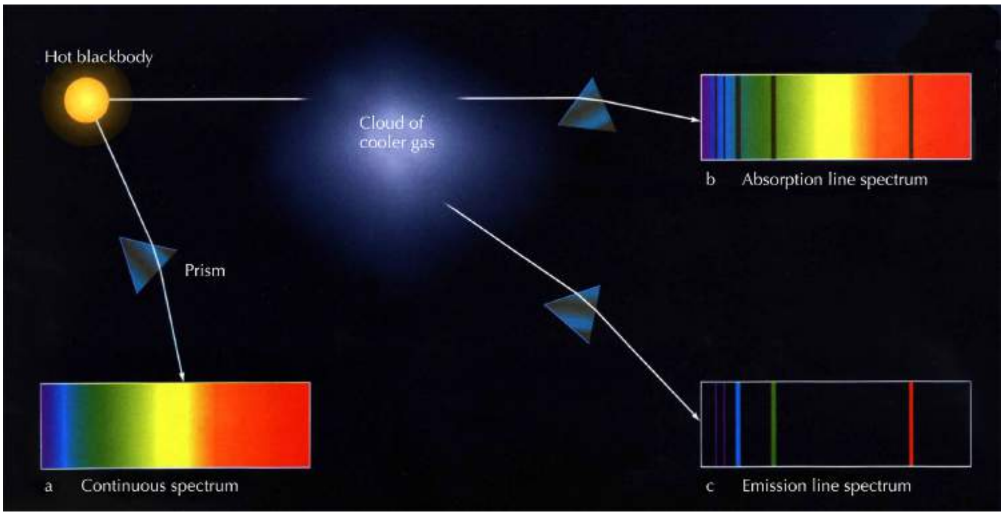

Figure 10: Info-figure: By studying the spectra of objects in the universe, you can do "remote learning" that is, from millions and even billions of light-years away you can figure out the object's chemical composition and velocity. a) If you look directly at a blackbody through a prism or a modern spectrograph, you will see a continuous spectrum. b) Clouds of gas absorb certain wavelengths of light. A continuous spectrum that hits a cloud of cool gas will be partially absorbed. The transmitted spectrum is called an absorption line spectrum, and is continuous except for the wavelengths that were absorbed by the gas. c) Anything that absorbs also emits. A cloud of cool gas that absorbs certain wavelengths from a blackbody will emit exactly those wavelengths as the gas atoms de-excite. If we look at the cloud without the blackbody in our line of sight, we will see an emission line spectrum. (Figure: www.nthu.edu.tw)

line centered at $\lambda = \lambda_0$ (see figure 9). The width of the spectral line will depend on the temperature of the gas, the higher the temperature, the higher the dispersion in velocities and thus in shifts $\Delta\lambda$ of wavelengths.

We can estimate the width of a line by using some elementary thermodynamics. From the above discussion, we see that we will need information about the velocity of the atoms in the gas. For an ideal gas at temperature T (measured in Kelvin K), the number density of atoms (number of atoms per volume) in a given velocity range $[v, v + dv]$ is given by the

Maxwell-Boltzmann distribution function

$$n(v)dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv.$$

where m is the mass of the atoms in the gas and n is the total number density of atoms per unit volume.

The meaning of the function is the following: if you insert the mass of the atoms in the gas m, the temperature T of the gas and the number density of the gas, n, then you can find out how many atoms in this gas which have a velocity in the range $[v, v + \Delta v]$. Say you need to find out how many atoms have the velocity in the range between 2 and 2.01 km/s. Then you insert $v = 2$ km/s to obtain $n(v)$, and use $\Delta v = 0.01$ km/s and use $n(v)\Delta v$ to find the total number of atoms with the given velocity per volume.

In figure 11 we see two such distributions (what is plotted is $n(v)/n$), both for hydrogen gas (the mass m has been set equal to the mass of the hydrogen atom), solid line for temperature $T = 6000$ K which is the temperature of the solar surface and dashed line for $T = 373$ K (which equals 100 °C). We can thus use this distribution to find the percentage of molecules in a gas which has a certain velocity. Now, before you read on, go back and make sure that you understand well the meaning of the function $n(v)$.

We see that the peak of this distribution, i.e. the velocity that the largest number of atoms have, depends on the temperature of the gas,

$$\frac{dn(v)}{dv} = 0 \rightarrow \frac{d}{dv}(e^{-mv^2/(2kT)}v^2) = 0.$$

Taking the derivative and setting it to zero gives the following relation

$$v_{\max}^2 = \frac{2kT}{m},$$

i.e. the most probable velocity for an atom in the gas is given by $v_{\max}$ (Note: 'max' does *not* mean highest velocity, but highest *probability*). Most of the atoms will have a velocity close to this velocity (see again figure 11).

The Maxwell-Boltzmann distribution only tells you the absolute value v of the velocity. When measuring the Doppler effect, only the radial (along

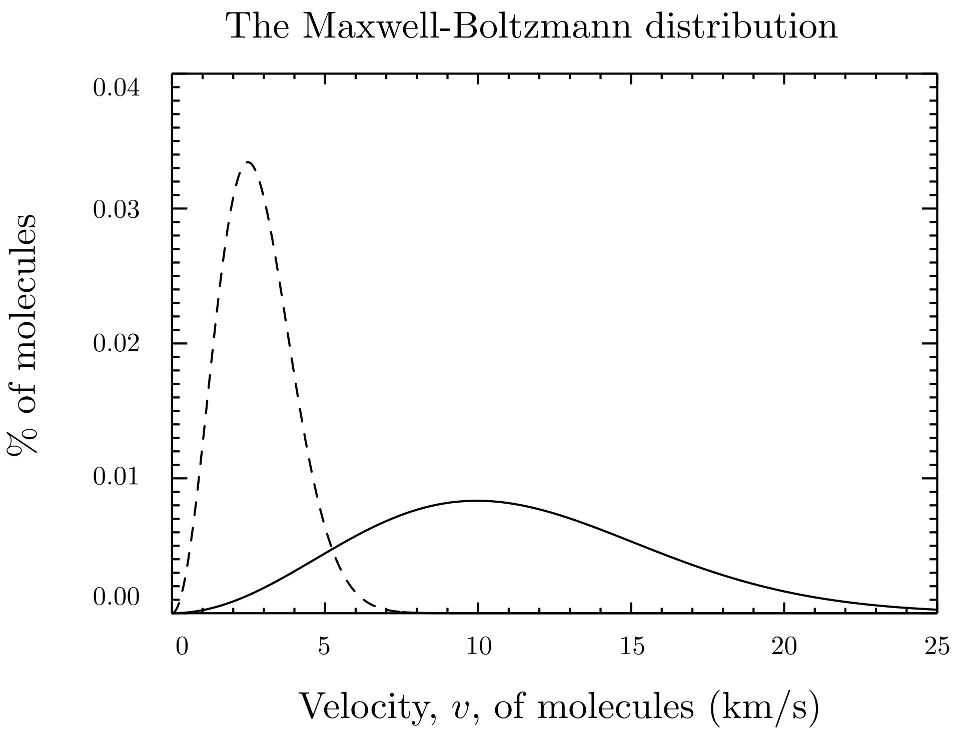

Figure 11: The Maxwell-Boltzmann distribution for hydrogen gas, showing the percentage of molecules in the gas having a certain thermal velocity at temperature $T = 6000$ K (solid line) and $T = 373$ K (dashed line).

the line of sight) component v_r has any effect. The atoms in a gas have random directions and therefore atoms with absolute velocity v will have radial velocities scattered uniformly in the interval $v_r = [-v, v]$ (why this interval? do you see it?). Since the most probable absolute velocity is $v_{\max}$ the most probable radial velocity will be all velocities in the interval $v_r = [-v_{\max}, v_{\max}]$ (you see that for instance $v_r = 0$ is in this interval, do you understand why $v_r = 0$ is at as common as $v_r = v_{\max}$?). The atoms with absolute velocity $v_{\max}$ will thus give Doppler shifts uniformly distributed between $\Delta\lambda/\lambda_0 = -v_{\max}/c$ and $\Delta\lambda/\lambda_0 = v_{\max}/c$. Few atoms have a much higher velocity than $v_{\max}$ and therefore the spectral line starts to weaken (less absorption/emission) after $|\Delta\lambda|/\lambda_0 = v_{\max}/c$. We will thus see a spectral line with the width given roughly by

$$2\Delta\lambda = \frac{2\lambda_0}{c} v_{\max} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT}{m}},$$

using the expression for $v_{\max}$ above. Do you see how this comes about? Try to imagine how the spectral line will look like, thinking how atoms at different velocities (above and below the most probable velocity) will contribute to v_r and thereby to the the spectral line. Try to make a rough plot of how $F(\lambda)$ for a spectral line should look like. Do not proceed until you have made a suggestion for a plot for $F(\lambda)$.

Of course, there are atoms at speeds other than $v_{\max}$ contributing to the spectral line as well. The resulting spectral line is thus not seen as a sudden drop/rise in the flux at $\lambda_0 - \Delta\lambda$ and a sudden rise/drop again at $\lambda_0 + \Delta\lambda$. Contributions from atoms at all different speeds make the spectral line appear like a Gaussian function with strongest absorption/emission at $\lambda = \lambda_0$. We say that the *line profile* is Gaussian. More accurate thermodynamic calculations show that we can approximate an absorption line with the Gaussian function

$$F(\lambda) = F_{\text{cont}}(\lambda) + (F_{\text{min}} - F_{\text{cont}}(\lambda))e^{-(\lambda-\lambda_0)^2/(2\sigma^2)}, \tag{3}$$

where $F_{\text{cont}}(\lambda)$ is the *continuum flux*, the flux $F(\lambda)$ which we would have if the absorption line had been absent. The width of the line is defined by σ. For a Gaussian curve one can write σ in terms of the Full Width at Half the Maximum (FWHM, see figure 12) as $\sigma = \Delta\lambda_{\text{FWHM}}/\sqrt{8\ln 2}$ where

$$\Delta\lambda_{\text{FWHM}} = \frac{2\lambda_0}{c}\sqrt{\frac{2kT\ln 2}{m}},$$

We see that this exact line width differs from our approximate calculations above only by $\sqrt{\ln 2}$. With this expression we also have a tool for measuring the temperature of the elements in the stellar atmosphere.

5 Stellar magnitudes

The Greek astronomer Hipparchus (about 150 BC) made a catalogue of about 850 stars and divided them into 6 *magnitude* classes, depending on their brightness: the brightest stars were classified as magnitude 1 stars, and the stars which could barely be seen were classified as magnitude 6. Little did Hipparchus know about the fact that more than 2000 years later his system would still be used, and not only that, it would be used by all astronomers in the (now much bigger) world. Whereas Hipparchus classified the stars by eye, a more scientific method is used today. The eye reacts to differences in the logarithm of the brightness. For this reason, the magnitude classification is logarithmic in the flux that we receive (energy received per area per time $F = \frac{dE}{dt\,dA}$). For a difference in magnitude of 5 between two stars, the ratio of the fluxes of these stars is defined to be exactly 100.

The flux we receive from a star depends on the distance to the star. We define the *luminosity* L of a star to be the total energy emitted by the whole star per unit time (dE/dt). This energy is radiated equally in all directions. If we put a spherical shell around the star at distance r, the energy received per unit area on this shell would equal the total energy L divided by the surface area of the shell,

$$F = \frac{L}{4\pi r^2}.$$

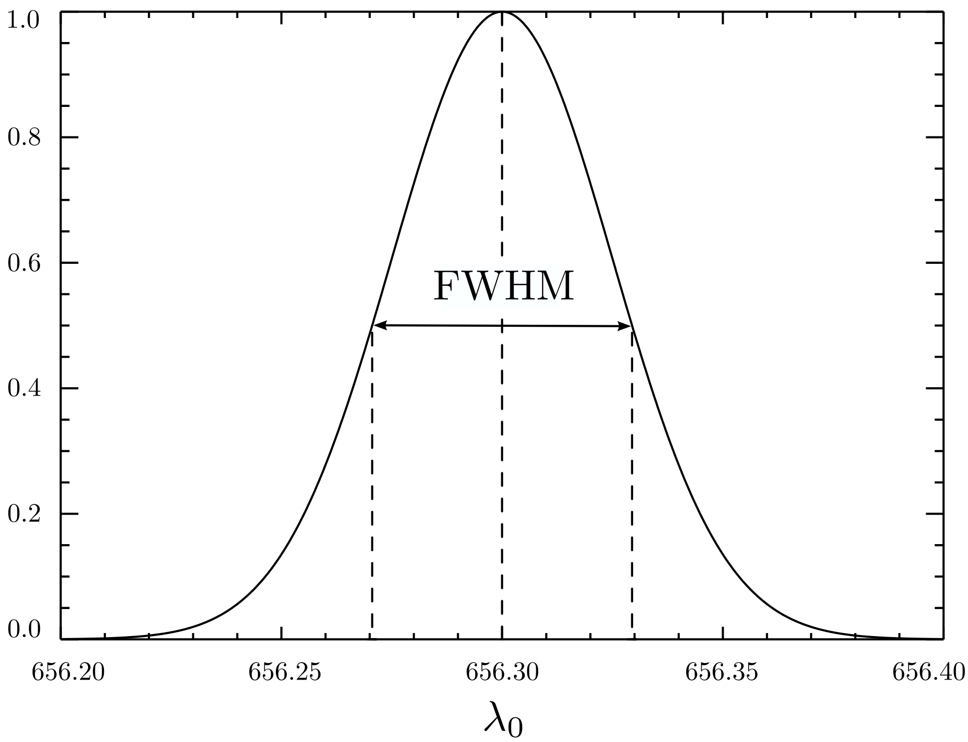

Figure 12: A Gaussian profile: The horizontal line shows the Full Width at Half Maximum (FWHM) which is where the curve has fallen to half of its maximum value. This is an emission line, an absorption line would look equal, just upside down.

Thus, the larger the distance r, the larger the surface area of the shell $4\pi r^2$ and the smaller the energy received per unit area (flux F). If we have two stars with observed fluxes F_1 and F_2 and magnitudes m_1 and m_2, we have learned that if $F_1 = F_2$ then $m_1 = m_2$ (agree?). We have also learned that if $F_1 = 100F_2$ then $m_2 - m_1 = 5$ (remember that in Hipparchus' system $m = 1$ stars were the brightest and $m = 6$ stars were the faintest).

The magnitude scale is logarithmic, thus we obtain the following general relation between magnitude and flux

$$\frac{F_1}{F_2} = 100^{(m_2 - m_1)/5},$$

or

$$m_1 - m_2 = -2.5 \log_{10}\left(\frac{F_1}{F_2}\right).$$

(Check that you can go from the previous equation to this one!) Given the difference in flux between two stars, we can now find the difference in magnitude.

We have so far discussed the *apparent magnitude* m of a star which depends on the distance r. If you change the distance to the star, the flux and hence the magnitude changes. We can also define *absolute magnitude M* which only depends on the total luminosity L of the star. The absolute magnitude M does not depend on the distance. It is defined as the star's apparent magnitude if we had moved the star to a distance of exactly 10 parsec (pc) (remember that 1pc=3.26ly). We can find the relation between apparent and absolute magnitude of a star,

$$\frac{F_r}{F_{r=10pc}} = \frac{L/(4\pi r^2)}{L/(4\pi(10\text{ pc})^2)} = \left(\frac{10\text{ pc}}{r}\right)^2 = 100^{(M-m)/5},$$

giving

$$m - M = 5 \log_{10}\left(\frac{r}{10\text{ pc}}\right).$$

(here we used a distance of $r = 10$ pc to calculate the flux for the absolute magnitude, this comes directly from the definition of absoluter magnitude: read it again if you did not understand this point). With this new more precise definition, stars can have magnitudes lower than 1. The brightest star in the sky, Sirius, has apparent magnitude -1.47 (note that the logarithmic dependence actually gives the brightest stars negative apparent magnitude). The planet Venus at maximum brightness has apparent magnitude -4.7 and the Sun has magnitude -26.7. The faintest object in the sky visible with the Hubble Space Telescope has apparent magnitude of about 30, about 100^5 times fainter than the faintest star visible with the naked eye. Originally the zero point of the magnitude scale was defined to be the star Vega. This has now been slightly changed with a more technical definition (outside the scope of this course).

Note: In order to define the magnitude we use the flux which we receive on Earth, the *received flux*. In some situations you will also need the *emitted flux*, the flux measured on the surface of the star emitting the radiation. It is important to keep these apart as they are calculated in a different manner (what is the difference?).

6 Problems

Problem 1 (20–30 min.)

At very large ($h\nu \gg kT$) and very small ($h\nu \ll kT$) frequencies, Planck's law can be written in a simpler form. The first limit is called the Wien limit and the second limit is called the Rayleigh-Jeans limit or simply the Rayleigh-Jeans law.

1. Show that Planck's law can be written as

$$B(\nu) = \frac{2h\nu^3}{c^2} e^{-h\nu/(kT)}$$

   in the Wien limit.

2. Show that Planck's law can be written as

$$B(\nu) = \frac{2kT}{c^2} \nu^2$$

   in the Rayleigh-Jeans limit. What kind of astronomer do you think uses Rayleigh-Jeans' law regularly?

Problem 2 (60–90 min.)

Now we will deduce Wien's displacement law by finding the peak in $B(\lambda)$.

1. Use the expression in the text for $B(\lambda)$ and take the derivative with respect to λ. After taking the derivative, eliminate λ everywhere using

$$x = \frac{hc}{kT\lambda}.$$

2. To find the peak in $B(\lambda)$, we need to set the derivative equal to zero. Show that this gives us the following equation

$$\frac{xe^x}{e^x - 1} = 5.$$

3. We now want to solve this equation numerically. We see that all we need to do is to find a value for x such that the expression on the

left hand side equals 5. The easiest way to do this is to try a lot of different values for x in the expression on the left hand side. When the expression on the left hand side has got a value very close to 5, we have found x.

(a) The solution to x will be in the range $x = [1, 10]$. Define an array x in Python with 1000 elements going from 1.0 as the lowest value to 10.0 as the highest value. Make a plot of the expression on the left hand side as a function of the array x. Can you see by eye at which value for x the curve crosses 5? Then you have already solved the equation.

(b) To make it slightly more exact, we try to find which x gives us the closest possible value to 5. We define the difference Δ between our expression and the value 5 which we want for this expression

$$\Delta = \left(\frac{xe^x}{e^x - 1} - 5\right)^2,$$

where we have taken the square to get the absolute value. Define an array in Python which contains the value of Δ for all the values of x. Plot Δ as a function of x. By eye, for which value of x do you find the minimum?

(c) Use Python to find the exact value of x (from the 1000 values defined above) which gives the minimum Δ.

(d) Now use the definition of x to obtain the constant in Wien's displacement law. Do you get a value close to the value given in the text?

Problem 3 (optional 1 hour–90 min.)

Here we will assume that the Sun is a perfect black body. You will need some radii and distances. By searching i.e. Wikipedia for 'Sun' or 'Saturn' you will find these data.

1. The surface temperature of the Sun is about $T = 5778$ K. At which wavelength λ does the Sun radiate most of its energy?
2. Plot $B(\lambda)$ for the Sun. What kind of electromagnetic radiation dominates?
3. What is the total energy emitted per time per surface area (flux) from the solar surface?
4. Repeat the previous exercise, but now by numerical integration. Do the integration with the box method (rectangular method) and Simpson's method. Compare the results. Compare your answer to the analytical value found in the previous exercise.
5. Use this flux to find the luminosity L (total energy emitted per time) of the Sun? (Here you need the solar radius).

6. What is the flux (energy per time per surface area) that we receive from the Sun? (Here you need the Sun-Earth distance). (See figure 13).

7. Spacecrafts are often dependent on solar energy. What is the flux received from the Sun by the spacecraft Cassini-Huygens orbiting Saturn? (Here you need the distance Sun-Saturn).

8. Assume that the efficiency of solar panels is 12%, i.e. that the electric energy that solar panels can produce is 12% of the energy that they receive. How many square meters of solar panel does the Cassini-Huygens spacecraft need in order to keep a 40 W light bulb glowing?

Problem 4 (optional 30–60 min.)

We will now study a simple climate model. You will need the results from question 1-6 in the previous problem.

1. We assume that the Earth's atmosphere is transparent for all wavelengths. How much energy per second arrives at the surface of Earth? The flux that you calculated in the previous exercises is the flux received by an area located at the earth's surface with orientation perpendicular to the distance-vector between the Sun and the Earth. **Hint** - Since the Earth is a sphere, the flux is not at all constant over the surface. However, we do not need to calculate the density for each square meter (fortunately). We can just look at the size of the effective absorption area (shadow area) which is shown in figure 14. Since the distance between the Sun and the Earth is so large, we can assume that the rays arriving at earth are travveling in the same direction (parallell). The radius r of the shaddow area is then equal to Earth's radius. The rest should be straight forward.

2. You are now going to estimate Earth's temperatur by using a simple climate model that just takes into account the radiation from the Sun and the Earth. We still assume that the atmosphere is transparent for all wavelengths. The model says that the Earth is a blackbody with a constant temperature. This means that it absorbs all incoming radiation and emits the same amount (in energy/time) in *all* directions. Calculate the Earth's temperature by using the simple climate model. You will need the result from the previous question and Stefan Boltzmann's law. Compare your result to the empirical average value $T_{\text{ave}} \approx 290$ K? Why do you think there is a difference?

Problem 5 (20–30 min.)

Here you will deduce a general expression for the flux per wavelength, $F(\lambda) = dE/(dA\,dt\,d\lambda)$, that we receive from a star with radius R at a

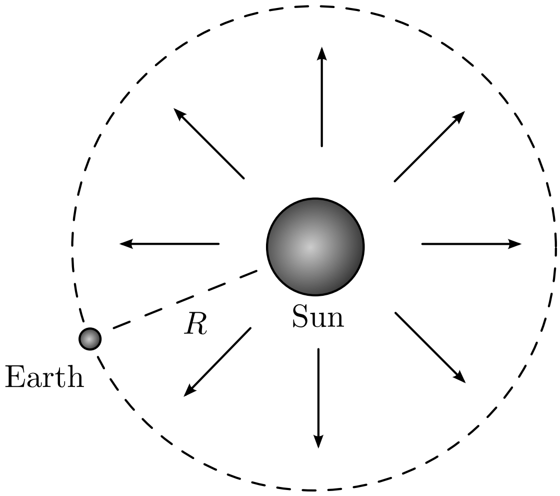

Figure 13: Radiation from the Sun. The flux is constant on a spherical surface with center at the Sun's center of mass.

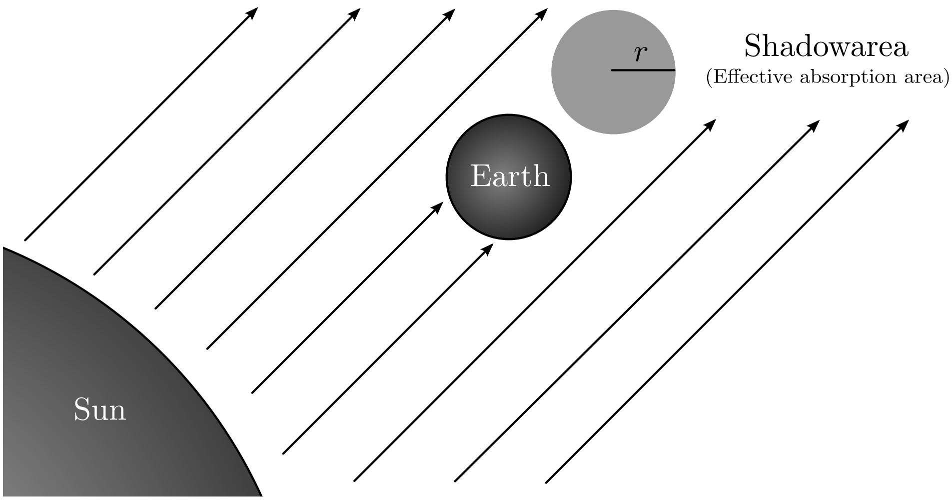

Figure 14: Shaddowarea (effective absorption area).

distance r with surface temperature T. Assume that the star is a perfect black body. You can solve this problem in two steps,

1. Find the luminosity per wavelength $L(\lambda) = dE/(d\lambda\, dt)$, i.e. the energy per time per wavelength, emitted from the star. The intensity $B(\lambda)$ is defined as $dE/(dA\, d\Omega\, d\lambda\, dt\, cos\theta)$. You need to integrate over solid angle and area to obtain the expression for the luminosity. **Hint:** Look at the derivation of Stefan-Boltzmann's law in the text.
2. Find the flux $F(\lambda)$ using $L(\lambda)$.
3. Does the expression for the flux peak at the same wavelength as for Planck's law? Can we simply use the maximum wavelength from flux measurements to obtain λ_{max} to be used in Wien's displacement law?

Problem 6 (4–4.5 hours)

I have produced a set of simulated spectra for a star. You will find the spectra in 10 files at

`http://folk.uio.no/frodekh/AST1100/lecture6`

These files show spectra taken of the same star at 10 different moments. The filename indicates the time of observation given in days from the first observation taken at $t = 0$. The first column of the file is the wavelength of observation in nm, the second column is the flux relative to the continuum flux around the spectral absorption line $H\alpha$ at $\lambda_0 = 656.3$ nm. Due to the Doppler effect, the exact position of the spectral line is different from λ_0. You will also see that this difference changes in time. As we have seen before, real life observations are noisy. It is not so easy to see exactly at which wavelength the center of the spectral line is located.

1. Plot each of the spectra as a function of wavelength. Can you see the absorption line?
2. Make a bye-eye estimate of the position of the center of the spectral line for each observation. Use the Doppler formula to convert this into relative velocity of the star with respect to Earth for each of the 10 observations (neglect the fact that the velocity of the Earth changes with time).
3. Now we will make a more exact estimate of the spectral line position using a least squares fit. As discussed in the text, we can model the spectral line as a Gaussian function (see equation(3)),

   $$F^{\text{model}}(\lambda) = F_{\text{max}} + (F_{\text{min}} - F_{\text{max}})e^{-(\lambda-\lambda_{\text{center}})^2/(2\sigma^2)}.$$

   When $\lambda = \lambda_{\text{center}}$, the model gives $F^{\text{model}}(\lambda) = F_{\text{min}}$. When λ is far from λ_{center} the model becomes $F^{\text{model}}(\lambda) = F_{\text{max}}$ as expected (check!). Thus the flux in this wavelength range if there hadn't been any spectral line would equal $F_{\max}$. The flux at the wavelength for which the absorption is maximal is $F_{\min}$. The spectra are normalized to the continuum radiation meaning that $F_{\max} = 1$. We are left with three unknown parameters, $F_{\min}$, σ and λ_{center}. The first parameter gives the flux at the center of the spectral line, the second parameter is a measure of the width of the line and the third parameter gives the central wavelength of the spectral line. In order to estimate the speed of the star with the Doppler effect, all we need is λ_{center}. But in order to get the best estimate of this parameter, we need to find the best fitting model to the spectral line, so we need to estimate all parameters in order to find the one that interests us. Again we will estimate the parameters using the method of least squares. We wish to minimize

$$\Delta(F_{\min}, \sigma, \lambda_{\text{center}}) = \sum_{\lambda} (F^{\text{obs}}(\lambda) - F^{\text{model}}(\lambda, F_{\min}, \sigma, \lambda_{\text{center}}))^2,$$

where $F^{\text{obs}}(\lambda)$ is the observed flux from the file and the sum is performed over all wavelengths available.

(a) For each spectrum, plot the spectrum as a function of wavelength and identify the range of possible values for each of the three parameters we are estimating. Define three arrays `fmin`, `sigma` and `lambdacenter` in Python which contain the range of values for each of $F_{\min}$, σ and λ_{center} where you think you will find the true values. Do not include more values of the parameters than necessary, but make sure that the true value of the parameter must be within the range of values that you select. Do not use more than 50 values for each parameter, preferably less. **Hint:** The FOR loop over λ might be easier if you use indices instead of actual values for λ. That is, the FOR loop runs over index i in the array, and then you find the lambda value which corresponds to this index to use in the expression for Δ.

(b) Define a 3-dimensional array `delta` where you calculate Δ for all the combinations of parameters which you found reasonable.

(c) Find for which combination of the parameters $F_{\min}$, σ and λ_{center} that Δ is minimal. These are your best estimates.

(d) Repeat this procedure for all 10 spectra and obtain 10 values for the Doppler velocity v_r.

4. Make an array of the 10 values you have obtained for the velocities and plot it as a function of time.

5. Assume that the change of velocity with time indicates the presence of a planet around the star (is there something in your observations which indicates this?). The mass of the star was found to be 0.8 solar masses. Find the minimum mass of this planet (find v_r and

the period 'by eye' looking at the velocity curve). Is this really a planet? **Hint:** Remember that you need to subtract the peculiar velocity (velocity of the center of mass of the system), found by taking the mean of the velocity.

Problem 7 (10–15 min.)

In the text you find the apparent magnitudes of Sirius, Vega and the Sun. Look up the distances to these objects (again, wikipedia is a useful source of information) and calculate the absolute magnitude. Which of these three stars is actually the brightest?

Problem 8 (15–20 min.)

1. Use the flux calculated in Problem 3.6 to check that the apparent magnitude of the Sun used in the text is correct. In order to calibrate the magnitude you also need to know that the star Vega has been defined to have zero apparent magnitude (actually with newer definitions it has magnitude 0.03) and that the absolute magnitude of Vega is 0.58. You also need to know the luminosity of Vega: Look it up in Wikipedia. All other quantities that you may need (for instance the distance to Vega) should be calculated using these numbers.
2. The faintest objects observed by the Hubble Space Telescope (HST) have magnitude 30. Assume that this is the limit for HST. How far away can a star with the same luminosity as the Sun be for HST to see it? (here you will need the luminosity of the Sun calculated in problem 3.5)
3. Assume that the luminosity of a galaxy equals the luminosity of 2×10^{11} Suns. How far away can we see a galaxy using HST?

AST1100 Lecture Notes

7–8 The special theory of relativity: Basic principles

1 Simultaneity

We all know that 'velocity' is a relative term. When you specify velocity you need to specify velocity *with respect to something*. If you sit in your car which is not moving (with respect to the ground) you say that your velocity is zero with respect to the ground. But with respect to the Sun you are moving at a speed of 30 km/s. From the point of view of an observer passing you in his car with a velocity of 100 km/h with respect to the ground, your speed is −100 km/h (see figure 1). Even though you are not moving with respect to the ground, you are moving backwards at a speed of 100 km/h with respect to the passing car.

In the following we will use the expression *'frame of reference'* to denote a system of observers having a common velocity. All observers in the same frame of reference have zero velocity with respect to each other. An observer always has velocity zero with respect to his own frame of reference. An observer on the ground measures the velocity of the passing car to be 100 km/h with respect to his frame of reference. On the other hand, the driver of the car measures the velocity of the ground to be moving at −100 km/h with respect to his frame of reference. We will also use the term *'rest frame'* to denote the frame of reference in which a given object has zero velocity. In our example we might say: In the rest frame of the passing car, the ground is moving backwards with 100 km/h.

You are observing a truck coming towards you with a speed of $v_{\text{truck}}^{\text{ground}} = -50$ km/h with respect to the ground (see figure 2, velocities are defined to be positive to the right in the figure). From your frame of reference, which is the same frame of reference as the ground, the speed of the truck is $|v_{\text{truck}}^{\text{ground}}| = 50$ km/h towards you. Now you start driving your car in the direction of the truck with a speed of $v_{\text{car}}^{\text{ground}} = +50$ km/h with respect to the ground (see again figure 2). From your frame of reference you observe the ground to be moving backwards with a velocity of $v_{\text{ground}}^{\text{car}} = -50$ km/h. Again, from your frame of reference you now ob-

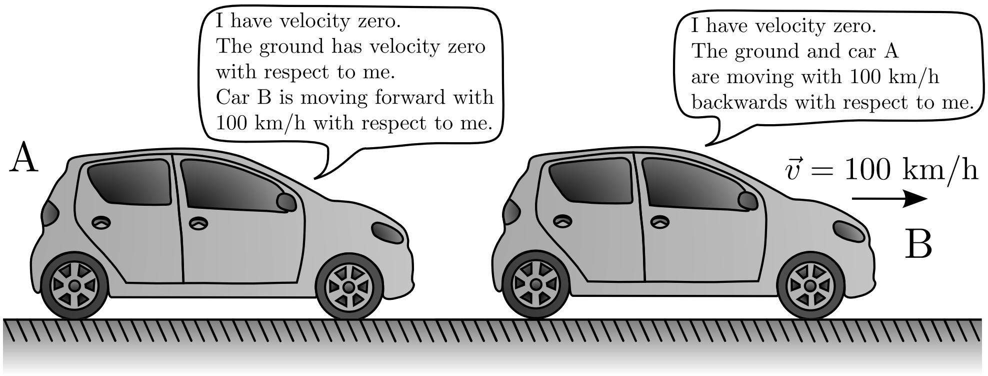

Figure 1: Velocities are relative.

serve the velocity of the approaching truck to be $v_{\text{truck}}^{\text{car}} = v_{\text{truck}}^{\text{ground}} - v_{\text{car}}^{\text{ground}} = (-50 \text{ km/h}) - (50 \text{ km/h}) = -100$ km/h (whereas from the frame of reference of an observer on the ground, the truck still has $v_{\text{truck}}^{\text{ground}} = -50$ km/h). Now you make a turn so that you drive in the opposite direction: Now your velocity is -50 km/h with respect to the ground, but now you are driving in the same direction as the truck. You are now moving in the same direction as the truck with exactly the same speed with respect to the ground. From your frame of reference (which is now the same frame of reference as the truck) the truck is not moving.

So far, so good. This was just stating some obvious facts from everyday life in a difficult way. Now, replace the truck with a beam of light (a photon) and the car with the Earth. The situation is depicted in figure 3. You observe the speed of light from a distant star at two instants: One at the 1st of January, another at the 1st of July. In January you are moving away from the photons approaching you from the distant star. In July you are moving towards the photons arriving from the star. If the speed of light with respect to the distant star is c, then in January you expect to measure the speed of the light beam from the star to be $c - v$ where $v = 30$ km/h is the speed of the Earth with respect to the same star (we assume that the star does not move with respect to the Sun, so this is also the orbital speed of the Earth). In July you expect to measure the speed of light from the star to be $c + v$, just as for the truck in the example above: The speed of the light beam seen from your frame of reference is supposed to be different depending on whether you move towards it or away from it.

In 1887 Michelson and Morley performed exactly this experiment which is now famous as the 'Michelson-Morley experiment'. The result however, was highly surprising: They measured exactly the same speed of light in both cases. The speed of light seemed to be the same independently of the frame of reference in which it is measured. This has some quite absurd consequences: Imagine that you see the truck driving at the speed of light (or very close to the speed of light, no material particle can ever travel at the speed of light). You are accelerating your car, trying to pass the truck.

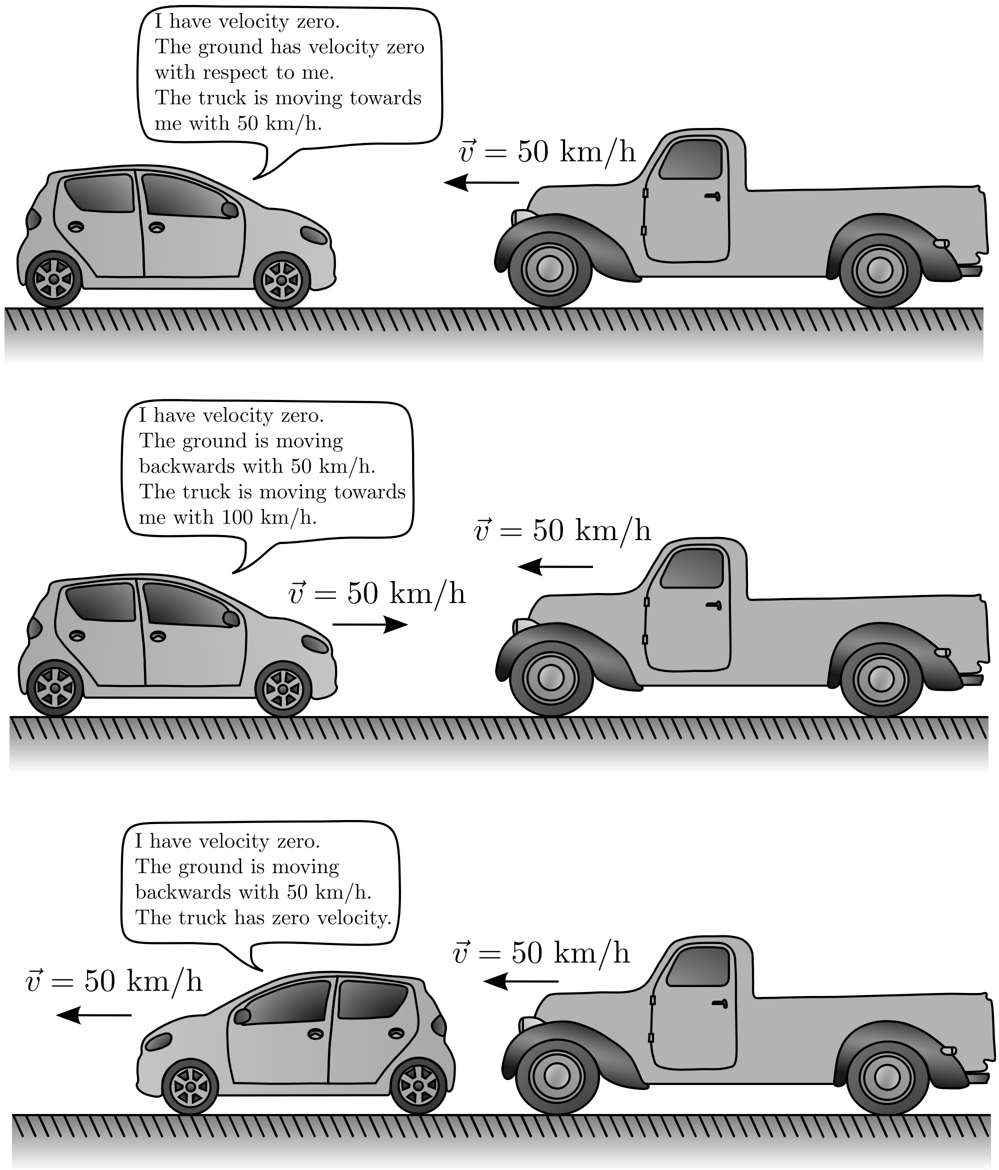

Figure 2: The velocity of the truck seen from the car depends on the velocity of the car.

But no matter at which speed you drive, you see the truck moving with the speed of light with respect to your frame. Even when you reach half the speed of light, you still see the truck moving with velocity c. But how is this possible? An observer at rest with respect to the ground measures the truck moving with the speed of light as well, not with the velocity $c + c/2 = 3c/2$ as you would expect given that it moves with velocity c with respect to something moving with velocity $c/2$.

This was one of the first signs showing that something was wrong with classical physics. The fact that the speed of light seemed to be constant in all frames of reference led to several contradictions. We have already seen one example of such a contradiction. We will now look at another one which might shed some light on the underlying reason for these contradictions. In figure 4 we show the situation. Observer O is standing on the ground (at rest with respect to the ground), observer P is standing in the middle of a train of length L moving with velocity v with respect to the ground. Observer O sees two lightnings striking the front and the rear of the train simultaneously. We call the two events A and B (An event is a point in space and time, a point with a space and time coordinate): Event A is the lightning striking the front, event B is the lightning striking the rear. Events A and B are simultaneous. The light from these two lightnings start traveling from the front and back end of the train towards observer P. The beam approaching observer P from the front is called beam 1 and the beam approaching from the rear is called beam 2. Both observers synchronize their clocks to $t = 0$ at the instant when the lightnings strike the train. Both observers have also defined their own coordinate systems x (observer on the ground) and x' (observer in the train) which is such that the position of observer P is at $x = x' = 0$ in both coordinate systems at the instant $t = 0$ when the lightenings strike. Thus the lightnings hit the train at the points $x = x' = L/2$ and $x = x' = -L/2$ as seen from both observers. We will now look how each of these observers experience these events:

From the point of view of observer O standing on the ground:

The frame of reference of observer O on the ground is often referred to as the *laboratory frame* . It is the frame of reference which we consider to be at rest. At what time $t = t_C$ does observer P see beam 1 (we call this event C)? To answer this question, we need to have an expression for the x-coordinate of observer P and the x-coordinate of beam 1 at a given time t. Observer P moves with constant velocity v so his position at time t is $x_P = vt$. Beam 1 moves in the negative x-direction with the speed of light c starting from $x_1 = L/2$ at $t = 0$. The expression thus becomes $x_1 = L/2 - ct$. Observer P sees beam 1 when $x_1 = x_P$ at time t_C. Equating these two expressions, we find

$$t_C = \frac{L/2}{c + v}. \tag{1}$$

At what time $t = t_D$ does observer P see beam 2 (we call this event D)?

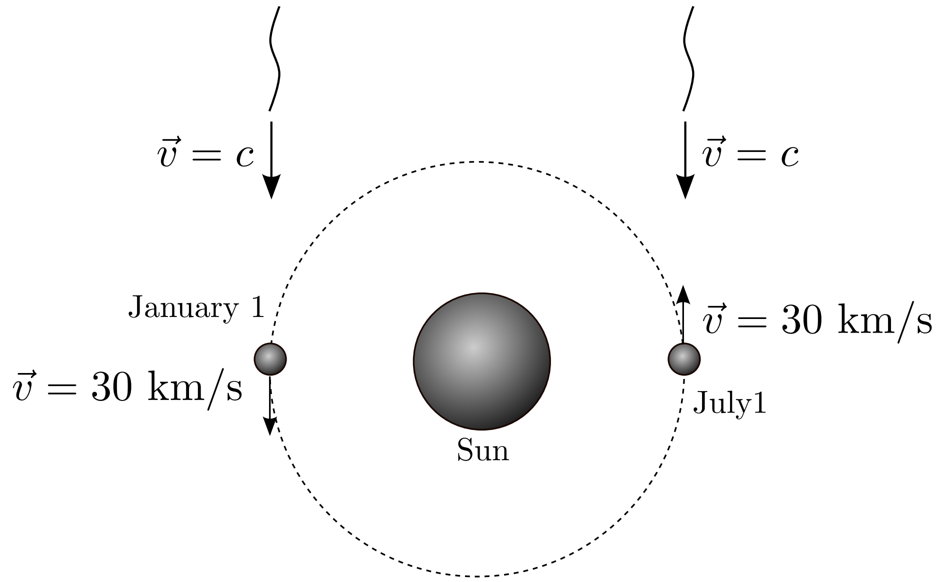

Figure 3: The velocity of the starlight is measured when the Earth has velocity 30 km/s towards and away from the light beam.

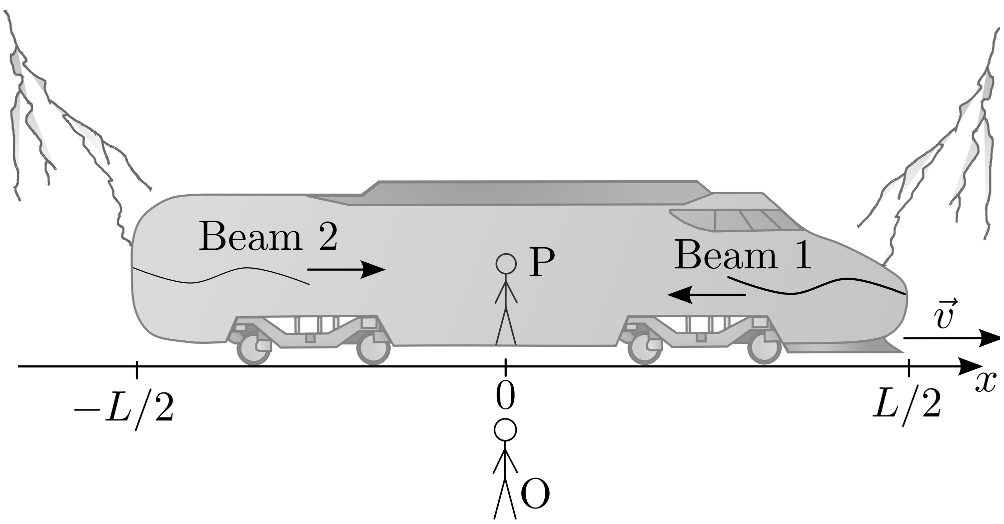

Figure 4: Event A: Lightning strikes the front part of the train. Event B: Lightning strikes the rear part of the train. These two events are observed by observer O on the ground and observer P in the train. The train has length L.

Following exactly the same line of thought as above, we find

$$t_D = \frac{L/2}{c - v}. \tag{2}$$

So according to observer O in the laboratory frame, $t_C < t_D$ and observer P should see the light beam from the lightning in front before the light from the back. This sounds reasonable: Observer P is moving towards beam 1 and away from beam 2 and should therefore see beam 1 first.

From the point of view of observer P standing in the train:

At what time $t = t_C$ does observer P see beam 1? We have just agreed on the fact that the speed of light is independent of the frame of reference. The result is that the speed of light is c also for the observer in the train. Seen from the frame of reference of observer P, observer P himself is at rest and the ground is moving backwards with speed v. Thus from this frame of reference, observer P is always standing at the origin $x'_P = 0$ (the coordinate system x' moves with observer P). The expression for x'_1 is the same as seen from observer O:$x'_1 = L/2 - ct$ (convince yourself that this is the case!). Again we need to set $x'_1 = x'_P$ giving

$$t_C = \frac{L/2}{c}$$

At what time $t = t_D$ does observer P see beam 2? Again we follow the same procedure and obtain

$$t_D = \frac{L/2}{c}$$

As calculated from the frame of reference of observer P, the two beams hit observer P at exactly the same time.

So not only are the exact times t_C and t_D different as calculated from the two frames of reference, but there is also an even stronger contradiction: Observer P should be hit by the two beams simultaneously as calculated from the frame of reference of observer P himself, but as calculated from the laboratory frame, beam 1 hits observer P before beam 2. What does really happen? Do the beams hit observer P simultaneously or not? Well, let's ask observer P himself:

So observer P, two lightnings struck your train simultaneously at the front and rear end. Did you see these two lightnings simultaneously or did you see one flash before the other?
Observer P: Sorry? I think you are not well informed. The two lightnings did not happen simultaneously. There was one lightning which struck the front part and then shortly afterwards there was another one striking the rear. So clearly I saw the flash in the front first.
Observer O: No, no, listen, the lightnings did strike the train simultaneously, there was no doubt about that. But you were moving in the direction of beam 1 and therefore it appeared to you that the front was hit by the lightning first.

Observer P: So you didn't watch very carefully I see. It is impossible that the two lightnings struck at the same time. Look, I was standing exactly in the middle of the train. The speed of light is always the same, no matter from which direction it arrives. Beam 1 and beam 2 had to travel exactly the same distance $L/2$ with exactly the same speed c. If the beams were emitted simultaneously I MUST have seen the two flashes at the same time. But I didn't....beam 1 arrived before beam2, and so event A must have happened before event B

So beam 1 did indeed hit observer P before beam 2. And indeed, observer P has got a point: From observer P the two lightnings could not have occurred at the same time. Asking observer O one more time he says that he is absolutely certain that the two lightnings struck simultaneously. Who is right?

We have arrived at one of the main conclusions that Einstein reached when he was discovering the theory of relativity: *simultaneity is relative.* If two events happen at the same time or not depends on who you ask. It depends on your frame of reference. In the example above, the two lightnings were simultaneous for the observer at rest on the ground, but not for the observer moving with velocity v. This has nothing to do with the movement of the light beams, it is simply time itself which is different as seen from two different frames of reference. Simultaneity is a relative term in exactly the same way as velocity is: When you say that two events are simultaneous you need to specify that they are simultaneous with respect to some frame of reference.

In order to arrive at the conclusion of the relativity of simultaneity, Einstein excluded an alternative: Couldn't it be that the laws of physics are different in different frames of reference? If the laws of physics in the train were different from those in the laboratory frame, then simultaneity could still be absolute. The problem then is that we need to ask the question 'Physics is different in frames which move with respect to *which* frame of reference?'. In order to ask this question, velocity would need to be absolute. If velocity is relative, then we can just exchange the roles: The observer in the train is at rest and the observer on the ground is moving. Then we would need to change the laws of physics for the observer on the ground. This would lead to contradictions. In order to arrive at the theory of relativity, Einstein postulated the *Principle of Relativity* . The principle of relativity states that all laws of physics, both the mathematical form of these laws as well as the physical constants, are the same in all *free float frames.* In the lectures on general relativity we will come back to a more precise definition of the free float frame. For the moment we will take a free float frame to be a frame which is not accelerated, i.e. a frame in which we do not experience fictive forces. You can deduce the laws of physics in one free float frame and apply these in any other free float frame. Imagine two space ships, one is moving with the velocity $v = 1/2c$ with respect to the other. If you close all windows in these spaceship there is no way, by performing experiments inside these spaceships, that you can tell which is which. All free float frames are equivalent, there is no way to tell which one is at rest and which one is moving. Each observer in a free float frame can define himself to be at rest.

2 Invariance of the spacetime interval

We have seen that two events which are simultaneous in one frame of reference are not simultaneous in another frame. We may conclude that time itself is relative. In the same way as we needed two coordinate systems x and x' to specify the position in space relative to two different frames, we need two time coordinates t and t' to specify the time of an event as seen from two different frames. We are used to think of time as a quantity which has the same value for all observers but we now realize that each frame of reference has its own measure of time. Clocks are not running at the same pace in all frames of reference. Observers which are moving with respect to each other will measure different time intervals between the same events. Time is not absolute and for this reason simultaneity is not absolute.

Look at figure 5. It shows two points A and B and two coordinate systems (x, y) and (x', y') rotated with respect to each other. The two points A and B are situated at a distance $\Delta x_{AB} = L$ and at the same y-coordinate $\Delta y_{AB} = 0$ in the (x, y) system. In the rotated (x', y') system however, there is a non-zero difference in the y-coordinate, $\Delta y_{AB} \neq 0$. Now, replace y with t. Do you see the analogy with the example of the train above?

If we replace y with t and y' with t', then the two points A and B are the events A and B in *spacetime*. Our diagram is now a *spacetime diagram* showing the position of events in space x and time t, rather than a coordinate system showing the position of a point in space (x, y). Consider the two coordinate systems (x, t) and (x', t') as measurements in two different frames of reference, the lab frame and the frame of observer P. We see that in the (x, t) system, the two events are simultaneous $\Delta t_{AB} = 0$ whereas in the (x', t') system, the events take place at two different points in time.

We are now entering deep into the heart of the special theory of relativity: We need to consider time as the *fourth dimension*. And moreover, we need to treat this fourth dimension similar (but not identical) to the three spatial dimensions. That is, we need to talk about distances in space and distances in time. But, you might object, we measure distances in space in meters and time intervals in seconds. Can they really be similar? Yes they can, and you will soon get rid of the bad habit of measuring space and time in different units. From now on you will either measure both space *and* time in meters, or both time *and* space in seconds. By the time you have finished this course you will, without thinking about it, ask the lecturer how many meters the exam lasts or complain to your friends about how small your room in the dormitory is, giving them the size in

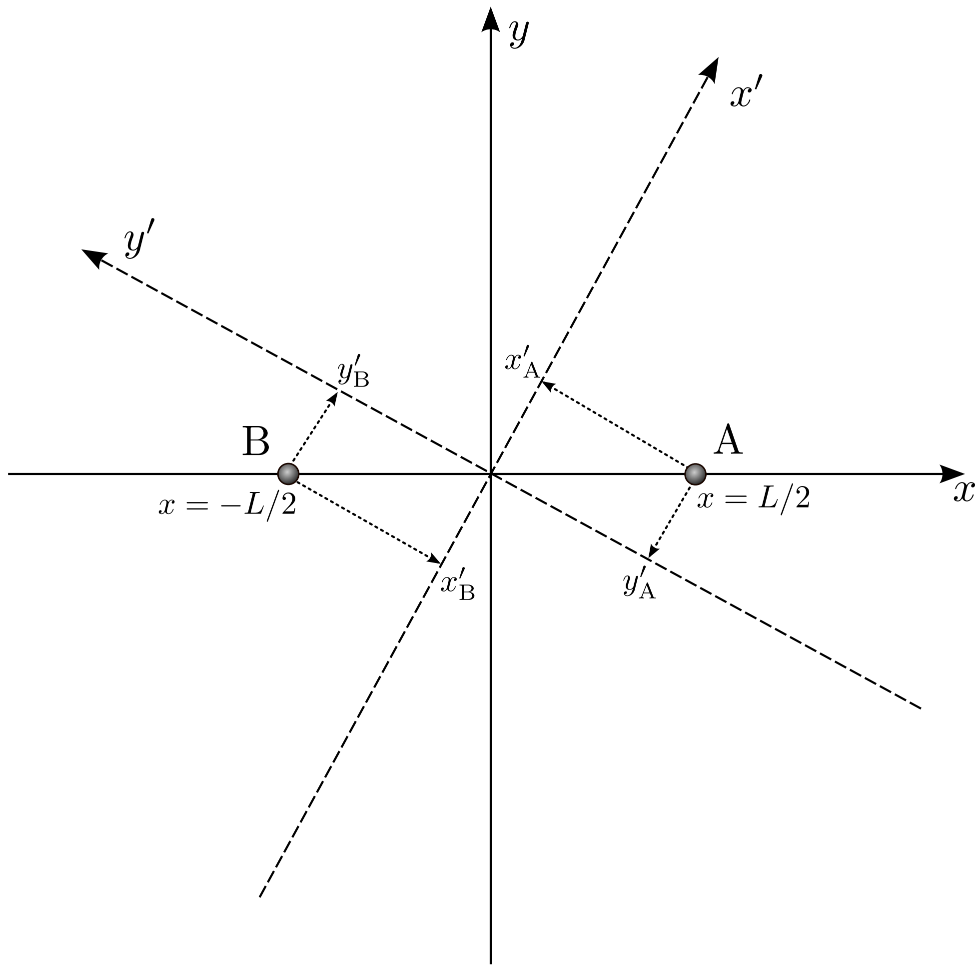

Figure 5: The position of two points A and B measured in two different coordinate systems rotated with respect to each other.

square seconds.

How do you convert from meters to seconds and vice versa? The conversion factor is given by the universal factor c, the speed of light. If you have a time interval measured in seconds, multiply it by c and you have the time interval in meters. If you have a distance in space measured in meters, divide it by c and you obtain the distance measured in seconds:

$$x = ct, \quad t = x/c.$$

From now on we will drop the factor c and suppose that distances in space and time are measured in the same units. When you put numbers in your equations you need to take care that you always add quantities with the same units, if you need to add two quantities with different units, the conversion factor is always a power of c.

Measuring time in meters might seem strange, but physically you can think about it this way: Since the conversion factor is the speed of light, a time interval measured in meters is simply the distance that light travels in the given time interval. If the time interval between two events is 2 meters, it means that the time interval between these events equals the time it takes for light to travel 2 meters. We might say that the time interval between these events is 2 meters of light travel time. Similarly for measuring distances in seconds: If the spatial distance between two events is 10 seconds, it means that the distance equals the distance that light travels in 10 seconds. The distance is 10 light seconds. Actually you are already accustomed to measure distances in time units: You say that a star is 4 light years away, meaning that the distance equals the distance that light travels in four years. Note also one more effect of measuring space and time in the same units: Velocities will be dimensionless. Velocity is simply distance divided by time, if both are measured in meters, velocity becomes dimensionless. We can write this as $v_{\text{dimensionless}} = dx/(cdt) = v/c$ (to convert dt to units of length we need to multiply it by c, thus cdt). If the velocity $v = dx/dt = c$ is just the speed of light, we get $v_{\text{dimensionless}} = 1$. From now on we will just write v for $v_{\text{dimensionless}}$. Note that some books use β to denote dimensionless velocity, here we will use v since we will always use dimensionless velocities when working with the theory of relativity. The absolute value of velocity v is now a factor in the range $v = [0, 1]$ being the velocity relative to the velocity of light.

This was the first step in order to understand the foundations of special relativity. Here comes the second: Let us, for a moment, return to the spatial coordinate systems (x, y) and (x', y') in figure 5. Clearly the coordinates of the points A and B are different in the two coordinate systems. But there is one thing which is identical in all coordinate systems: The distance between points A and B. If we call this distance Δs_{AB} we can write this distance in the two coordinate systems as

$$\begin{aligned}(\Delta s_{AB})^2 &= (\Delta x_{AB})^2 + (\Delta y_{AB})^2 \\ (\Delta s'_{AB})^2 &= (\Delta x'_{AB})^2 + (\Delta y'_{AB})^2\end{aligned}$$

(check that you understand why!). The distance between A and B has to be equal in the two coordinate systems, so

$$(\Delta s_{AB})^2 = (\Delta s'_{AB})^2.$$

Is this also the case in spacetime? Can we measure intervals between events in spacetime? This is now, at least in theory, possible since we measure space and time separations in the same units. In a spatial (x, y, z) system we know the geometrical relation,

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2,$$

from Euclidean geometry: The square of the distance between two points (called the *line element*) is simply the sum of the squares of the coordinate distances between these two points. But do the rules of Euclidean geometry apply to spacetime? No, not entirely. The geometry of spacetime is called *Lorentz geometry*. The distance between two events (line element) in Lorentz spacetime Δs^2, is given by

The spacetime interval

$$(\Delta s)^2 = (\Delta t)^2 - (\Delta x^2 + \Delta y^2 + \Delta z^2).$$

Note the minus sign. This minus sign is the only thing which distinguishes space from time. The square of the spacetime distance between two events equals the square of the time separation between these events *minus* the square of the spatial separations between the events. And in the same way as the distance between two points in space is the same in all coordinate systems, the distance in spacetime, *the spacetime interval* is the same in all frames of reference. We say that the spacetime interval is *invariant*. A quantity is invariant if it has the same value in all frames of reference. We already know another invariant quantity: the speed of light.

So, that was it. We're done. Now you know what the special theory of relativity is all about. Congratulations! You now see that we may write the special theory of relativity in two sentences: Measuring space and time intervals in the same units, you can calculate the spacetime interval between two events using the formula for the line element in Lorentz geometry. This spacetime interval between two events is invariant, it has the same value as measured from all frames of reference. We will now see what this means in practice. But before you continue, take a walk, go for a coffee or simply take half an hour in fresh air. Your brain will need time to get accustomed to this new concept.

3 An example

A train is moving along the x-axis of the laboratory frame. The coordinate system of the laboratory frame is (x, y) and of the train, (x', y'). In the train a light signal is emitted directly upwards along the y-axis (event A). Three meters above, it is reflected in a mirror (event B) and finally returns to the point where it was emitted (event C). In the train frame it takes the light beam 3 meters of time to reach the mirror and 3 meters of time to return to the point where it was emitted. The total up-down trip (event A to event C) took 6 meters of time in the frame of the train (light travels with a speed of $v = 1$, one meter per meter of light travel time). From event A to event C, the train had moved 8 meters along the x-axis in the laboratory frame. Because of the movement of the train, the light beam moved in a pattern as shown in figure 6 seen from the lab frame.

1. *Use the figure to find the total distance d traveled by the light beam in the laboratory frame.* Dividing the triangle into two smaller triangles (see the figure), we find from one triangle that the distance traveled from the emission of the light beam to the mirror is $d/2 = \sqrt{(4\text{ m})^2 + (3\text{ m})^2} = 5$ m and similarly for the return path. Thus, the total distance traveled by the light beam from event A to event C is $d = 10$ m.

2. *What was the total time it took for the light beam from event A to event C in the laboratory frame?* We have just seen that in the laboratory frame, the light beam traveled 10 meters from event A to event C. Since light travels at the speed of one meter per meter of time, it took 10 meters of time from event A to event C. In the frame of the train, it took only 6 meters of time.

3. *What is the speed of the train?* The train moved 8 meters in 10 meters of time, so the speed is $v = 8/10 = 4/5$, 4/5 the speed of light.

4. *What is the spacetime interval $\Delta s'$ between event A and event C with respect to the train frame?* In the train frame, event A and event C

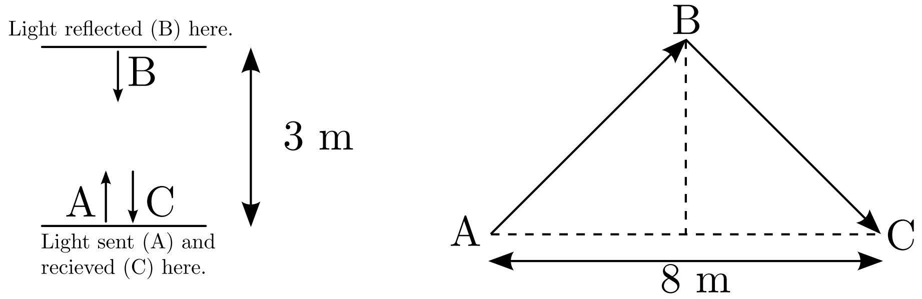

Figure 6: The light emitted (event A) upwards in the train is reflected (event B) and received (event C) at the same place (in the train frame) as it was emitted.

happened at the same point, so $\Delta x' = 0$. It took 6 meters of time from event A to event C, so $\Delta t' = 6$ m. The spacetime interval is thus $\Delta s' = \sqrt{(6 \text{ m})^2 - 0} = 6$ m.

5. *What is the spacetime interval Δs between event A and event C with respect to the laboratory frame?* In the laboratory frame, the distance between the events were $\Delta x = 8$ m and the time interval was $\Delta t = 10$ m. The spacetime interval is thus $\Delta s = \sqrt{(10 \text{ m})^2 - (8 \text{ m})^2} = 6$ m, exactly the same as $\Delta s'$ in the train frame.

6. *Was there an easier way to answer the previous question?* Oh... uhm, yes, you're right, the spacetime interval is the same in all frames of reference so I should immediately had answered $\Delta s = \Delta s' = 6$ m without any calculation... much easier!

Indeed much easier... remember that this will be very useful when calculating distances and intervals with respect to frames moving close to the speed of light.

4 Observer O and P revisited

Armed with the knowledge of the invariance of the spacetime interval we now return to observer O and P in order to sort out exactly what happened for each of the observers. We know that with respect to the laboratory frame, the two lightnings struck simultaneously (events A and B were simultaneous) at points $x = \pm L/2$ at the time $t = 0$ when observer P was at the origin $x_P = 0$. But at what time did the two lightnings strike with respect to observer P in the train? We have learned that with respect to the frame of reference following the train, the events A and B were *not* simultaneous. But in the reference frame of observer P, at what time t'_A and t'_B did the two lightnings strike? The two observers exchange a signal at $t = 0$ such that their clocks are both synchronized to $t = t' = 0$ at the instant when observer P is at the origin in both coordinate systems $x_P = x'_P = 0$. Did event A and B happen before or after $t' = 0$ on observer P's wristwatch? (It is common to talk about *wristwatches* when referring to the time measured in the rest frame of a moving object, i.e. the time measured by observers moving with the object. This wristwatch time is also called *proper time*).

We know that an event is characterized by a position x and a time t in each of the frames of reference. Let's collect what we know about the position and time of event A, B and the event when observer P passes $x = x' = 0$ which we call event P:

Event P:

$$x = 0 \quad t = 0$$
$$x' = 0 \quad t' = 0$$

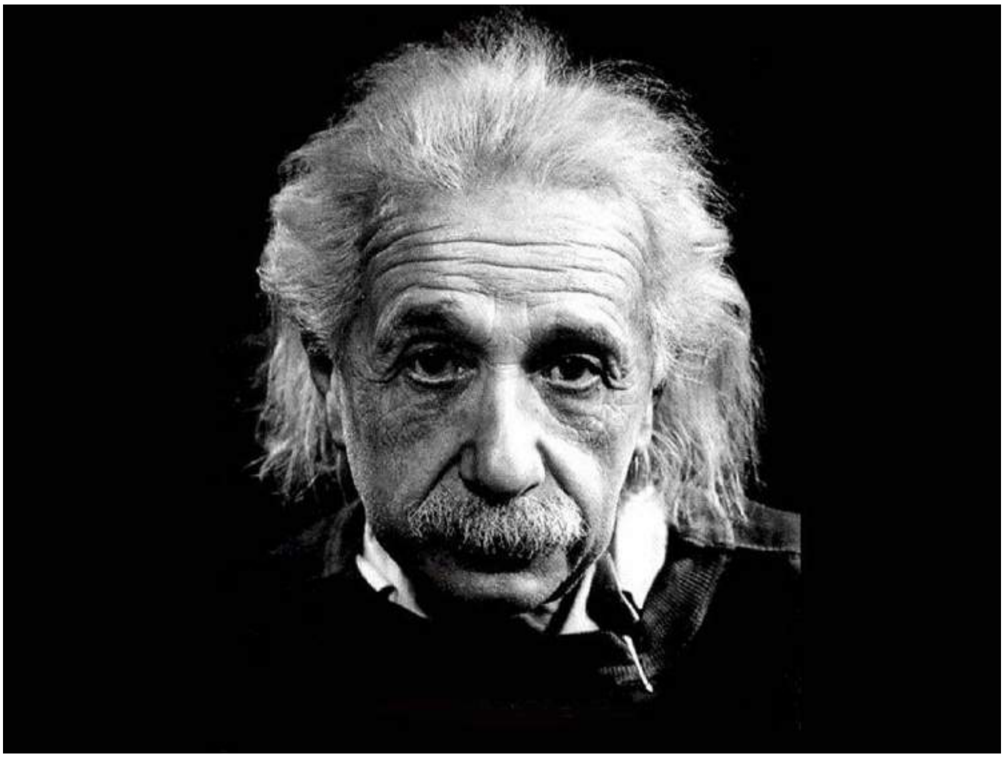

Figure 7: Info-figure: Near the beginning of his career, Albert Einstein (18791955) thought that Newtonian mechanics was no longer enough to reconcile the laws of classical mechanics with the laws of the electromagnetic field. This led to the development of his special theory of relativity (1905). It generalizes Galileo's principle of relativity that all uniform motion is relative, and that there is no absolute and well-defined state of rest from mechanics to all the laws of physics. Special relativity incorporates the principle that the speed of light is the same for all inertial observers regardless of the state of motion of the source. This theory has a wide range of consequences that have been experimentally verified, including length contraction, time dilation and relativity of simultaneity, contradicting the classical notion that the duration of the time interval between two events is equal for all observers. On the other hand, it introduces the spacetime interval, which is invariant.

Event A:

$$x = L/2 \quad t = 0$$
$$x' = L_0/2 \quad t' = t'_A$$

Event B:

$$x = -L/2 \quad t = 0$$
$$x' = -L_0/2 \quad t' = t'_B$$

Note that the length of the train is L_0 for observer P and L for observer O. We have already seen that observers in different frames of reference only agree on the length of the spacetime interval, *not* on lengths in space or intervals in time separately. For this reason, we do expect L and L_0 to be different. Look also at figure 5, the distance Δx_{AB} between the points A and B differ between the two coordinate systems, in the system (x, y) it is $\Delta x_{AB} = L$, but in the system (x', y') it is $\Delta x'_{AB} = x'_B - x'_A \equiv L'$. The length of the train in the rest frame of the train, L_0, is called *the proper length*. We will later come back to why it is given a particular name.

We want to find at which time t'_A and t'_B observed from the wristwatch of observer P, did events A and B happen? Did they happen before or after event P? For observer O all these events were simultaneous at $t = 0$, the moment in which the two observers exchanged a signal to synchronize their clocks. For observer P, could these events possibly had happened before they happened for observer O? Or did they happen later than for observer O?

In order to solve such problems, we need to take advantage of the fact that we know that the spacetime interval between events is invariant. Let's start with the spacetime interval between events A and B.

Spacetime interval AB: From each of the frames of reference it can be written as

$$\Delta s_{AB}^2 = \Delta t_{AB}^2 - \Delta x_{AB}^2,$$
$$\Delta (s'_{AB})^2 = (\Delta t'_{AB})^2 - (\Delta x'_{AB})^2.$$

(note that the y and z coordinates are always 0, so $\Delta y = \Delta y' = 0$ and $\Delta z = \Delta z' = 0$). In order to calculate the spacetime interval, we need the space and time intervals Δx_{AB}^2, Δt_{AB}^2, $(\Delta x'_{AB})^2$ and $(\Delta t'_{AB})^2$ separately. In both frames, the spatial distance between the two events equals the length of the train in the given frame of reference. So $\Delta x_{AB} = L$ and $\Delta x'_{AB} = L_0$. For observer O the events were simultaneous $\Delta t_{AB} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AB} = t'_A - t'_B$. Setting the two expressions for the spacetime interval equal we obtain,

$$L^2 = L_0^2 - (t'_A - t'_B)^2. \tag{3}$$

(check that you obtain this as well!). We have arrived at one equation connecting observables in one frame with observables in the other. We need more equations to solve for t'_A and t'_B. Let's study the spacetime interval between events A and P.

Spacetime interval AP: From each of the frames of reference it can be written as

$$\Delta s_{AP}^2 = \Delta t_{AP}^2 - \Delta x_{AP}^2$$
$$\Delta (s'_{AP})^2 = (\Delta t'_{AP})^2 - (\Delta x'_{AP})^2$$

In order to calculate the spacetime interval, we need the space and time intervals Δx_{AP}^2, Δt_{AP}^2, $(\Delta x'_{AP})^2$ and $(\Delta t'_{AP})^2$ separately. In both frames, the spatial distance between the two events equals half the length of the train in the given frame of reference. So $\Delta x_{AP} = L/2$ and $\Delta x'_{AP} = L_0/2$. For observer O the events were simultaneous $\Delta t_{AP} = 0$, whereas for observer P the events happened with a time difference $\Delta t'_{AP} = t'_A - 0 = t'_A$. Setting the two expressions for the spacetime interval equal we obtain,

$$(L/2)^2 = (L_0/2)^2 - (t'_A)^2. \tag{4}$$

Note that we have three unknowns, t'_A, t'_B and L. We need one more equation and therefore one more spacetime interval. The spacetime interval between B and P does not give any additional information, so we need to find one more event in order to find one more spacetime interval. We will use event C, the event that beam 1 hits observer P.

Spacetime interval CP: Again, we need

$$\Delta s_{CP}^2 = \Delta t_{CP}^2 - \Delta x_{CP}^2,$$
$$\Delta (s'_{CP})^2 = (\Delta t'_{CP})^2 - (\Delta x'_{CP})^2.$$

In the first section we calculated the time t_C when beam 1 hit observer P in the frame of observer O. The results obtained in the laboratory frame were correct since the events A and B really were simultaneous in this frame. As we have seen, the results we got for observer P were wrong since we assumed that events A and B were simultaneous in the frame of observer P as well. Now we know that this was not the case. We have $\Delta t_{CP} = t_C - 0 = t_C = L/2/(v+1)$ (from equation 1, note that since we measure time and space in the same units $c = 1$). As event C happens at the position of observer P, we can find the position of event C by taking the position of observer P at time t_C giving $\Delta x_{CP} = v \Delta t_{CP} = vL/2/(v+1)$. In the frame of observer P, event C clearly happened at the same point as event P so $\Delta x'_{CP} = 0$. The time of event C was just the time t'_A of event A plus the time $L/2$ it took for the light to travel the distance $L/2$ giving $\Delta t'_{CP} = t'_A + L_0/2$. Equating the line elements we have

$$\frac{L^2/4}{(v+1)^2}(v^2 - 1) = -(t'_A + L_0/2)^2 \tag{5}$$

Now we have three equations for the three unknowns. We eliminate L from equation (5) using equation (4). This gives a second order equation in t'_A with two solutions, $t'_A = -L_0/2$ or $t'_A = -vL_0/2$.

The first solution is unphysical: The time for event C is in this case $t'_C = t'_A + L_0/2 = 0$ so observer P sees the lightning at $t' = 0$. Remember that at $t = t' = 0$ observer O and observer P are synchronizing their clocks, so at this moment, and only this moment, their watch show the same time. This means that observer P sees flash A at the same moment as observer O sees the lightning. Thus at $t = t' = 0$, observer O would see the lightning hit the front of the train, but at the same time he would see it hit observer P.

Disregarding the unphysical solution we are left with

$$t'_A = -v\frac{L_0}{2}.$$

Thus event A happened for observers in the train before it happened for observers on the ground. Now we can insert this solution for t'_A in equation 4 and obtain L,

$$L = L_0\sqrt{1 - v^2} \equiv L_0/\gamma, \tag{6}$$

with $\gamma \equiv 1/\sqrt{1 - v^2}$. So the length of the train is smaller in the frame of observer O than in the rest frame of the train. We will discuss this result in detail later, first let's find t'_B. Substituting for t'_A and L in equation (3) we find

$$t'_B = v\frac{L_0}{2} = -t'_A.$$

So event B happened later for observers in the train than for observers on the ground. To summarize: Event A and B happened simultaneously at $t = t' = 0$ for observers on the ground. For observers in the train event A had already happened when they synchronize the clocks at $t = 0$, but event B happens later for the observers in the train. Note also that the time t'_A and t'_B are symmetric about $t' = 0$. If you look back at figure 5 we see that the analogy with two coordinate systems rotated with respect to each other is quite good: If we replace y by t we see that for the events which were simultaneous $\Delta y_{AB} = 0$ in the (x, y) frame, event A happens before $y = 0$ and event B happens after $y = 0$ in the rotated system (x', y'). But we need to be careful not taking the analogy too far: The geometry of the two cases are different. The spatial (x, y) diagram has Euclidean geometry whereas the spacetime diagram (x, t) has Lorentz geometry. We have seen that this simply means that distances are measured differently in the two cases (one has a plus sign the other has a minus sign in the line element).

We have seen that for observer P event A happens before event P when they synchronize their clocks. But does he also see the lightning before event P? As discussed above, this would be unphysical, so this is a good consistency check:

$$t'_C = t'_A + \frac{L_0}{2} = -v\frac{L_0}{2} + L_0/2 = L_0/2(1 - v),$$

which is always positive for $v < 1$. Thus observer P sees the flash after event P. When does observer P see the second flash (event D) measured on the wristwatch of observer P? Again we have $t'_D = t'_B + L_0/2$ giving

$$t'_D = L_0/2(1+v),$$

so the time interval between event C and D measured on the wristwatch of a passenger on the train is

$$\Delta t' = t'_D - t'_C = vL_0$$

How long is this time interval as measured on the wristwatch of observer O? We already have t_C and t_D from equations (1) and (2). Using these we get the time interval measured from the ground,

$$\Delta t = vL_0/\sqrt{1-v^2}$$

So we can relate a time interval in the rest frame of the train with a time interval on the ground as

$$\Delta t = \frac{\Delta t'}{\sqrt{1-v^2}} = \gamma \Delta t'. \tag{7}$$

Note that I have skipped index CD here since this result is much more general: It applies to any two events taking place at the position of observer P. This is easy to see. Look at figure 8. We define an observer O which is at rest in the laboratory frame using coordinates (x, t) and an observer P moving with velocity v with respect to observer O. In the frame of reference of observer P we use coordinates (x', t').

We now look at two ticks on the wristwatch of observer P. Observer P himself measures (on his wrist watch) the time between two ticks to be $\Delta t'$ whereas observer O measures the time intervals between these two ticks on P's watch to be Δt (measured on observer O's wrist watch). In the coordinate system of observer P, the wristwatch does not move, hence the space interval between the two events (the two ticks) is $\Delta x' = 0$. For observer O, observer P and hence his wristwatch is moving with velocity v. So observer O measures a space interval of $\Delta x = v\Delta t$ between the two events. The spacetime interval in these two cases becomes

$$\begin{aligned}(\Delta s)^2 &= \Delta t^2 - \Delta x^2 = \Delta t^2 - (v\Delta t)^2 = (\Delta t)^2(1-v^2) \\ (\Delta s')^2 &= (\Delta t')^2.\end{aligned}$$

Spacetime intervals between events are always equal from all frames of reference so we can equate these two intervals and we obtain equation (7).

Going back to the example with the train: If the train moves at the speed $v = 4c/5$ then we have $\Delta t = 5/3\Delta t' \approx 1.7\Delta t'$. When observer O on the ground watches the wristwatch of observer P, he notes that it takes 1.7 hours on his own wristwatch before one hours has passed on the wristwatch of observer P. If observer P in the train is jumping up and down every

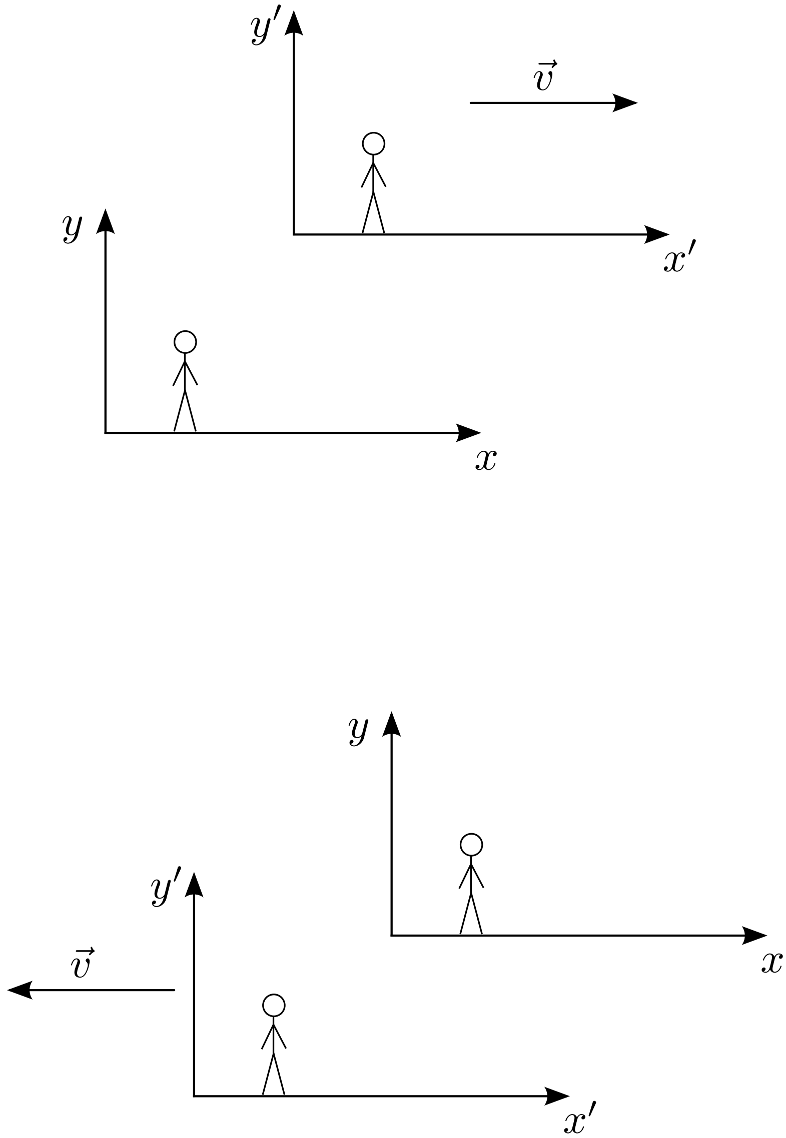

Figure 8: Two reference frames: (x, y) coordinates are used for the system defined to be at rest and (x', y') coordinates are used for the system defined to be moving. In the upper figure, observer O is in the laboratory frame with observer P in the frame moving with velocity v. In the lower figure, the two systems have exchanged roles and $v \to -v$. All equations derived in the above system will be valid for the system below by exchanging $v \to -v$.

second on his own wristwatch, it takes 1.7 seconds for each jump as seen from the ground. For observers on the ground it looks like everything is in slow-motion inside the train.

How does it look for the observers in the train? Remember that velocity is relative. Being inside the train, we define ourselves as being at rest. From this frame of reference it is the ground which is moving at the speed $-v$. Everything has been exchanged: Since we now define the train to be at rest, the coordinate system (x,t) is now for the train whereas the coordinate system (x',t') is for the ground which is moving at velocity $-v$ (see figure 8). Note the minus sign: The motion of the ground with respect to the train is in the opposite direction than the motion of the train with respect to the ground. We can now follow exactly the same calculations as above for two events happening at the position of observer O instead of observer P. For instance we watch two ticks on the clock of observer O. Then we find again formula (7) but with the meaning of Δt and $\Delta t'$ interchanges. Assuming again a speed of $v = -4c/5$ (note again the minus sign), observer P sees that it takes 1.7 hours on his wristwatch for one hour to pass on the wristwatch of observer O. It is the opposite result with respect to the above situation. While observers on the ground observe everything in the train in 'slow-motion', the observers on the train observe everything on the ground in 'slow-motion'. This is a consequence of the principle of relativity: There is no way to tell whether it is the train which is moving or the ground which is moving. We can define who is it rest and who is moving, the equations of motion that we obtain will then refer to one observer at rest and one observer in motion. When we change the definition, the roles of the observers in the equation will necessarily also change. Thus, if we define the ground to be at rest and the train to be moving and we deduce that observers on the ground will see the persons in the train in 'slow-motion', the opposite must also be true: If we define the train to be at rest and the ground to be moving, then the observers on the train will observe the observers on the ground in 'slow-motion'. Confused? Welcome to special relativity!

Consider two observers, both with their own wristwatch, one at rest in the laboratory frame (observer O) another moving with velocity v with respect to the laboratory frame (observer P). Going back to equation (7) we now know that if $\Delta t'$ is the interval between two ticks on the wristwatch of observer P, then Δt is the time interval between the same two ticks of observer P's watch measured on observer O's wristwatch. Using equation 7 we see that the shortest time interval between two ticks is always the time measured directly in the rest frame of the wristwatch producing the ticks. Any other observer moving with respect to observer P will measure a longer time interval for the ticks on observer P's wristwatch. This is of course also valid for observer O: The shortest time interval between two ticks on observer O's wristwatch is the time that observer O himself measures. The wristwatch time is called the *proper time* and is denoted τ.

Note that the proper time between two events (two ticks on a wristwatch) also equals the spacetime interval between these events. This is easy to see: consider again the ticks on observer P's wristwatch. In the rest frame of observer P, the wristwatch is not moving and hence the spatial distance between the two events (ticks) is $\Delta x = 0$. The time interval between these two events is just the proper time $\Delta\tau$. Consequently we have for the spacetime interval $\Delta s^2 = (\Delta t')^2 - (\Delta x')^2 = \Delta\tau^2 - 0 = \Delta\tau^2$.

Proper time

$$\Delta s^2 = \Delta\tau^2$$

in the rest frame.

Now, let's return to another result, the length of the train L as measured by observer O on the ground. Again, the result in equation 6 can be shown in a similar manner to be more general. The length L_0 can be the length of any object in the rest frame of this object. We see from equation 6 that any observer which is not at rest with respect to the object will observe the length L which is always smaller than the length L_0. The length of an object measured in the rest frame of the object is called the *proper length* of the object. An observer in any other reference frame will measure a smaller length of the object. The proper length L_0 is the longest possible length of the object. This also means that an observer in the moving train will measure the shorter length L for another identical train being at rest with respect to the ground (being measured to have length L_0 by observers on the ground).

5 The Lorentz transformations

Given the spacetime position (x, t) for an event in the laboratory frame, what are the corresponding coordinates (x', t') in a frame moving with velocity v along the x-axis with respect to the laboratory frame? So far we have found expressions to convert time intervals and distances from one frame to the other, but not coordinates. The transformation of spacetime coordinates from one frame to the other is called the Lorentz transformation. In the exercises you will deduce the expressions for the Lorentz transformations. Here we state the results. We start by the equations converting coordinates (x', y', z', t') in the frame moving along the x-axis to coordinates (x, y, z, t) in the laboratory frame,

The Lorentz transformations

$$\begin{aligned} t &= v\gamma x' + \gamma t', \\ x &= \gamma x' + v\gamma t', \\ y &= y', \\ z &= z'. \end{aligned}$$

To find the inverse transformation, we have seen that we can exchange the roles of the observer at rest and the observer in motion by exchanging the coordinates and let $v \to -v$ (see figure 8),

The Lorentz transformations (cont.)

$$\begin{aligned} t' &= -v\gamma x + \gamma t, \\ x' &= \gamma x - v\gamma t, \\ y' &= y, \\ z' &= z. \end{aligned}$$

Here

$$\gamma = \frac{1}{\sqrt{1-v^2}}.$$

6 List of expressions you should know by now

Laboratory frame	$\rightarrow$	page 4
Principle of relativity	$\rightarrow$	page 7
Free float frame	$\rightarrow$	page 7
Space time diagram	$\rightarrow$	page 8
Line element	$\rightarrow$	page 11
Lorentz geometry	$\rightarrow$	page 11
Spacetime interval	$\rightarrow$	page 11
Invariance	$\rightarrow$	page 11
Proper time	$\rightarrow$	page 13
Proper length	$\rightarrow$	page 15

7 Problems

Problem 1 (10–15 min.)

We have seen the effect of Lorentz contraction, namely that a stick of proper length L_0 (measured in the rest frame of the stick) moving at a speed v along the x-axis in the laboratory frame, is measured to have a shorter length $L = L_0/\gamma$ in the laboratory frame. But what happens to the size of the stick in y and z directions measured from the laboratory frame? Do we correspondingly measure the stick to become thinner? We will now investigate this:

To check this possibility, imagine two identical cylinders A and B which are hollow such that if one cylinder becomes smaller (smaller radius) than the other, it might pass inside the larger cylinder (see figure 9). The axis of both cylinders are aligned with the x-axis at $y = z = 0$. Thus, the axis

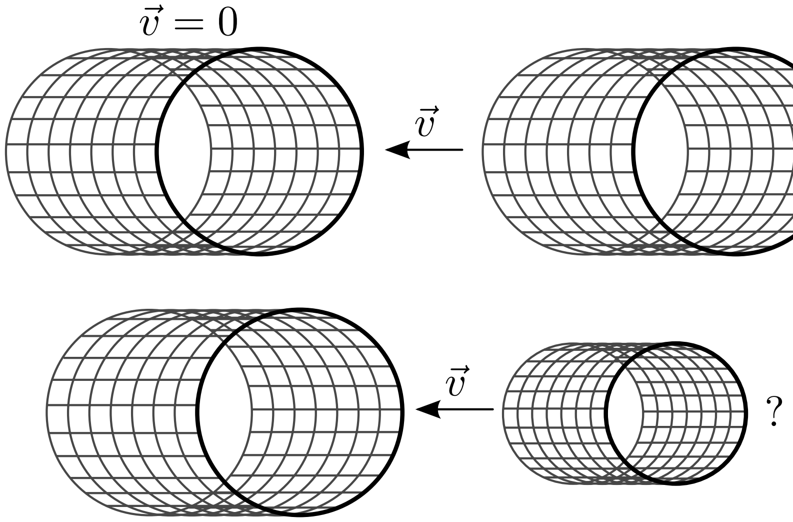

Figure 9: Does a moving cylinder become thinner as well as contracted seen from the laboratory frame? In problem 1 we study this more closely.

of both cylinders are exactly along the x-axis. Cylinder A is at rest in the laboratory frame, cylinder B is moving with velocity v along the x-axis, approaching cylinder A.

1. We know that the length of cylinder B as measured from the laboratory frame shrinks. Assume that the same effect takes place in the y and z directions such that the radius of cylinder B gets smaller measured in the laboratory frame. What happens when the two cylinders meet?
2. Now, look at exactly the same situation but from the point of view of an observer sitting on cylinder B. What happens when the two cylinders meet?
3. Can you give a good arguments to explain why $y = y'$ and $z = z'$ in the Lorentz transformations? (Note: this transformation is for movements along the x-axis. If there are movements along the y and z axes as well, the Lorentz transformation will look different and much more complicated. This is outside the scope of this course.)

Problem 2 (10–20 min.)

A proton and an electron separated by a distance L_0 are at rest in a train.

1. What is the electric field E' from the proton at the location of the electron? (as measured in the rest frame of the train)
2. The train moves with velocity v with respect to the laboratory frame. Show that the electric field E as measured in the laboratory frame can be written as $E = E'/(1 - v^2)$.
3. Based on this result, can you now use the principle of relativity to find general qualitative arguments showing that the electric feld

must be a relative quantity depending on the frame of reference in which it is measured?

Problem 3 (20 min.–1 hour)

When high energy cosmic ray protons collide with atoms in the upper atmosphere, so-called muon particles are produced. These muon particles have a mean life time of about 2 μs (2×10^{-6} s) after which they decay into other types of particles. They are typically produced about 15 km above the surface of the Earth. We will now study a cosmic ray muon approaching the surface with the velocity of $0.999c$.

1. How long time does it take for a muon to arrive at the surface of the Earth as measured from the Earth frame?
2. Ignore relativistic effects: Do you expect many muons to survive to the surface of the Earth before decaying? (compare with the mean life time)
3. From relativity, we know that from the rest frame of the muon, the time it takes to reach the surface of the Earth is different. We will now use invariance of the spacetime interval to find the time it takes in the frame of the muon to reach the surface of the Earth.
   (a) Find the space and time distances Δx and Δt in the Earth frame and use these to obtain the spacetime interval Δs. Give all the answers in seconds.
   (b) What is $\Delta x'$, the spatial distance traveled by the muon in the muon rest frame?
   (c) Use invariance of the line element to obtain the travel time $\Delta t'$ in the muon rest frame. Will we detect muons at the surface of the Earth?
4. The diameter of the galaxy is about 100 000 light years, thus even with the speed of light it would take 100 000 years to pass the galaxy. How long time does it take to transverse the galaxy in the reference frame of a cosmic ray particle traveling at the speed of $v = 0.999999999999c$? (Use again invariance of the spacetime interval). Does this give some hope for future long distance space travel?

Problem 4 (1–2 hours)

You have devised a clock which works the following way: It consists of two mirrors a distance L_0 apart. A light ray is emitted along the positive x-axis at one of the ends and then reflected back and forth between the two mirrors. Each time it hits one of the mirrors it gives a 'tick'. See figure 10.

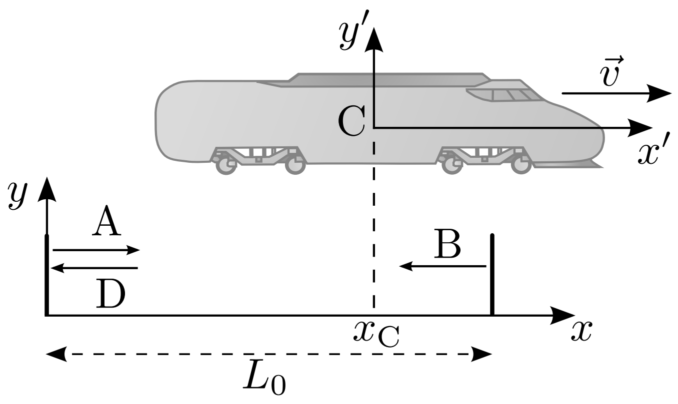

Figure 10: The situation in problem 4: A light beam is emitted when $x = x' = 0$ and $t = t' = 0$ (event A). Then the beam is reflected in the right mirror (event B) and reflected again in the left mirror (event D). This picture is taken from the laboratory frame at event B $t = t_B$ (the position of event A and D are just marked, they are not happening at this moment). Event C happens at the same time as event B in the laboratory frame. The position of event C in the laboratory frame is the position $x = x_C$ of the origin of the train frame.

1. How long does it take between each tick in the reference frame of the clock?

2. Now we observe the clock from a passing train. The clock is at rest in the laboratory frame with coordinates (x, t) and we observe it from the train moving with velocity v along the positive x-axis of the laboratory frame. We use coordinates (x', t') for the train frame (see figure 10). Event A is the emission of light at the left mirror. This is the reference event occurring at $x = x' = 0$ and $t = t' = 0$. Event B is when the light ray hits the opposite mirror. We also introduce event C which takes place at the position of the middle point of the train (where $x' = 0$) at the same time as event B seen from the laboratory frame. We want to find out how long time $\Delta t'_{AB}$ it took for the light beam to reach the right mirror in the train frame. Write a list of events A, B and C and write the position (x, t) and (x', t') in the two frames for all three events. The only unknowns here are x'_B, t'_B and t'_C. All the other coordinates should be expressed in terms of the known quantities, L_0 and v.

3. Write the spacetime intervals Δs_{AB} and $\Delta s'_{AB}$ between events A and B in the two frames. Show that invariance of the interval gives $x'_B = t'_B$. Could you have guessed this using physical arguments without any calculations?

4. Write the spacetime intervals Δs_{AC} and $\Delta s'_{AC}$ between events A and C in the two frames. Show that invariance of the interval gives $t'_C = L_0/\gamma$.

5. Write the spacetime intervals Δs_{BC} and $\Delta s'_{BC}$ between events B

and C in the two frames. Show that invariance of the interval gives $t'_B = L_0\gamma(1-v)$.

6. Now define event D which is when the light ray returns to the first mirror at $x = 0$. Use invariance of the spacetime interval for appropriate events to find at what time t'_D event D happened in the train frame.

7. In the frame of the train, how long time did it take from the light was emitted to the first 'tick'? And how long time did it take from the first tick to the second tick? Compare this to the results in the lab frame. Is this a useful clock in the frame of reference of the train?

Problem 5 (30 min.–1 hour)

Quasars are one of the most powerful sources of energy in the universe. They are smaller than galaxies, but emit about 100 times as much energy as a normal galaxy. The engine in a quasar is believed to be a black hole. Jets of plasma are ejected into space from areas close to the black hole.

1. In a Quasar called 3C273 at a distance of 2.6×10^9 light years from Earth, such a jet was observed during a period of three years. During this period it was found to have moved an angular distance of 2×10^{-3} arc seconds transversally on the sky. Show that the physical speed of the jet was $v = 8.4c$, more than eight times the speed of light.

2. We will now look at the physics of this process in order to understand what is going on. In figure 11 you can see the jet and two events A and B which are the events that photons were emitted as the jet moved through space. The photons emitted in event B were observed three years later than the photons emitted in point A. Here v is the real physical speed of the jet and θ is the angle between the direction of the jet and the line of sight. Show that the time interval $\Delta t_{\text{observed}}$ between the reception of photons (observations) from these two events is
$$\Delta t_{\text{observed}} = \Delta t(1 - v\cos\theta),$$
where Δt is the real time interval (in the Earth frame) between these two events. **Hint:** No theory of relativity is needed in this calculation, all quantities you need are taken in the same frame of reference.

3. Show that the apparent transversal speed of the jet can be written as
$$v_{\text{observed}} = \frac{v\sin\theta}{1 - v\cos\theta}.$$

4. Assume that $\theta = 45°$. For which range of real speeds v do we observe an apparent speed v_{observed} which is larger than the speed of light?

5. The theory of relativity says that no signal can travel faster than the speed of light. Is this principle violated?

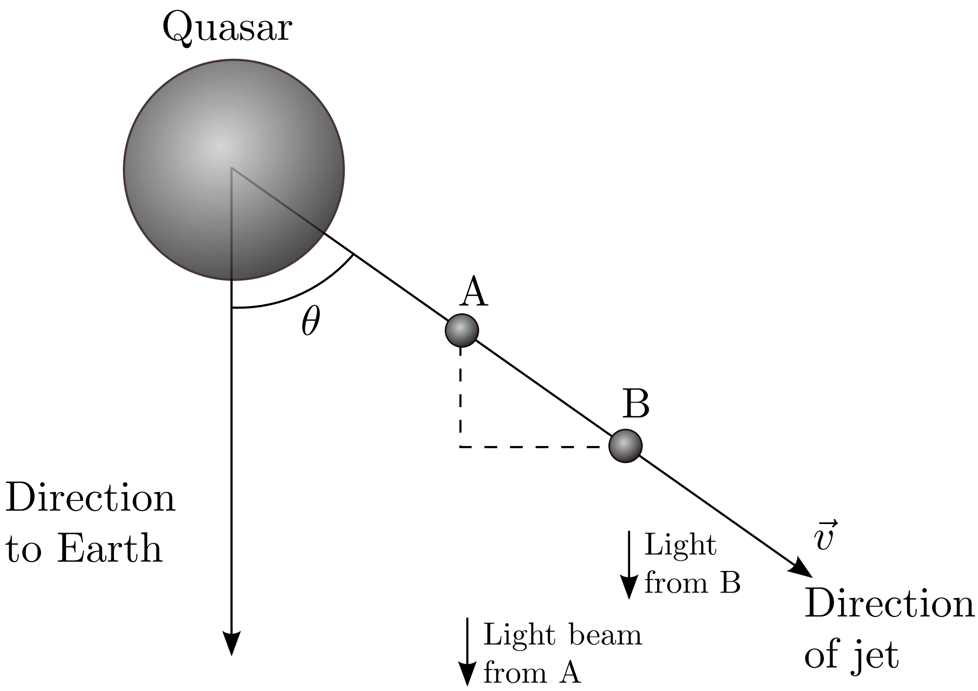

Figure 11: The quasar ejecting matter at an angle θ with the line of sight. The speed of the ejected matter is v. We define two events A and B which are the emission of photons from the ejected matter at the points A and B. At event A, the ejected matter passes point A and emits photons towards Earth. Three years later, the ejected matter passes point B and again emits photons.

The effect we have seen here, an apparent speed of an object which exceeds the speed of light, is called *superluminal motion*.

Problem 6 (30 min.–2 hours)

In this exercise we will deduce the Lorentz transformations. We start by noting that the transformation equations must be linear in x and t. This is because the inverse transformation needs to have the same form as the original transformation by the principle of relativity: We can exchange the definition of who is at rest and who is moving only if the transformation is linear such that if $x \propto x'$ then $x' \propto x$. For instance if we had a coordinate transformation $x \propto (x')^2$, the inverse transformation would read $x' \propto \sqrt{x}$. These two equations would be completely different and the principle of relativity would be violated: The two observers would have completely different equations for transforming from one system to the other. Thus we can write the Lorentz transformations on the form

$$t = f(v)x' + g(v)t', \tag{8}$$
$$x = h(v)x' + k(v)t', \tag{9}$$
$$y = y',$$
$$z = z',$$

where $f(v)$, $g(v)$, $h(v)$ and $k(v)$ are unknown functions of v. Note that the motion is along the x-axis, so no transformation is needed for the other two spatial dimensions. And again, by the principle of relativity, the inverse transformation must be obtained by exchanging the roles of the observers $(x, y) \leftrightarrow (x', y')$ and the velocity $v \to -v$ (see again figure 8),

$$t' = f(-v)x + g(-v)t, \tag{10}$$
$$x' = h(-v)x + k(-v)t, \tag{11}$$
$$y' = y,$$
$$z' = z.$$

We need to solve for our unknown functions of v, namely $f(v)$, $g(v)$, $h(v)$ and $k(v)$.

1. Consider two events A and B. Event A happens at $x = x' = 0$ at $t = t' = 0$. Event B happens at (x, t) in the laboratory frame and at the origin $x' = 0$ at time t' in the moving frame (which moves with velocity v with respect to the laboratory frame). Write the time intervals Δt_{AB} and $\Delta t'_{AB}$ in terms of the coordinates x, t, x', t'. Then use equation (7) to find a relation between t and t'. You see that this relation already resembles one of equations (8)–(11) with one term missing. Look at at your coordinates and compare with the equations (8)–(11) and you will realize that the missing term vanishes. Show that

$$g(v) = \gamma.$$

2. We will still study the same two events. At what position x in the laboratory frame does event B happen? Express the answer in terms of t and v. Then use the previous result to elimintate t and write this in terms of t' and v. This gives you a relation between x and t'. You would need either an x' or t to obtain one of the relations above (equations 8–11), but show that one of these vanishes. Then show that
$$k(v) = v\gamma.$$

3. We will now study two different events A and B. Event A is again $x = x' = 0$ and $t = t' = 0$. But event B now happens at the position $x' = L_0$ in the moving frame and $x = L$ in the laboratory frame. In the laboratory frame, the two events happen at the same time. Use equation 6 to obtain a relation between x and x'. Look again at the Lorentz transformation equations (equations 8–11): Your expression needs either a t or a t' but one of these vanishes. You can thus conclude that
$$h(-v) = \gamma = h(v)$$

4. Now we are only missing $f(v)$ in order to have deduced the full Lorentz transformations. Consider two other events A and B: Event A is again for $x = x' = 0$ at $t = t' = 0$ and event B is at position (x, t) in the laboratory frame and (x', t') in the moving frame. Use equations (8)–(9) to show that the spacetime interval between A and B for the two frames can be written
$$\Delta s^2 = (f(v)x' + \gamma t')^2 - (\gamma x' + v\gamma t')^2$$
$$(\Delta s')^2 = (t')^2 - (x')^2$$
Show that invariance of the spacetime interval gives
$$f(v) = \gamma v.$$

The Lorentz transformations have been deduced.

Problem 7 (20 min.–1 hour)

We will now return to the clock in problem 4 and solve this using the Lorentz transformations instead of the spacetime interval. We want to find at what time t'_B does the light hit the right mirror and at what time t'_D it has returned to the left mirror. Using the Lorentz transformations we will only need events A, B and D.

1. Again, write up the coordinates (x, t) and (x', t') for these three events. The following are unknown: x'_B, t'_B, x'_D and t'_D.

2. Use the Lorentz transformations to find t'_B and t'_D. You do not need to find x'_B and x'_D.

3. Use the Lorentz transformations to find the time (in the train frame) of the next two ticks of the clock. Are the intervals consistent with the first two ticks?

AST1100 Lecture Notes

9–10 The special theory of relativity: Four vectors and relativistic dynamics

1 Worldlines

In the spacetime diagram in figure 1 we see the path of a particle (or any object) through spacetime. We see the different positions (x, t) in space and time that the particle has passed through. Such a path showing the points in spacetime that an object passed is called a *worldline*. We will now study two events A and B (on the worldline of a particle) which are separated by a small spacetime interval Δs. These events could be the particle emitting two flashes of light or the particle passing through two specific points in space. The corresponding space and time intervals between these two events in the laboratory frame are called Δt and Δx. From the figure you see that $\Delta t > \Delta x$. You can see that this also holds for every small spacetime interval along the path. This has to be this way: The speed of the particle at a given instant is $v = \Delta x/\Delta t$. If $\Delta x = \Delta t$ then $v = 1$ and the particle travels at the speed of light. That $\Delta t > \Delta x$ simply means that the particle travels at a speed $v < c$ which it must. The worldline of a photon would thus be a line at 45° with the coordinate axes. The worldline of any material particle will therefore always make less than 45° with the time axis.

Events which are separated by spacetime distances such that $\Delta t > \Delta x$ are called *timelike events*. Timelike events may be causally connected since a particle with velocity $v < c$ would have the possibility to travel from one of the events to the other event. There is a possibility that the second event could have been caused by the first event since it is possible for a signal to travel between the events. Timelike events have positive line elements,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 > 0.$$

Events for which $\Delta t = \Delta x$ are called *lightlike events*. Only a particle traveling at the speed of light ($v = \Delta x/\Delta t = 1$) could travel from the first

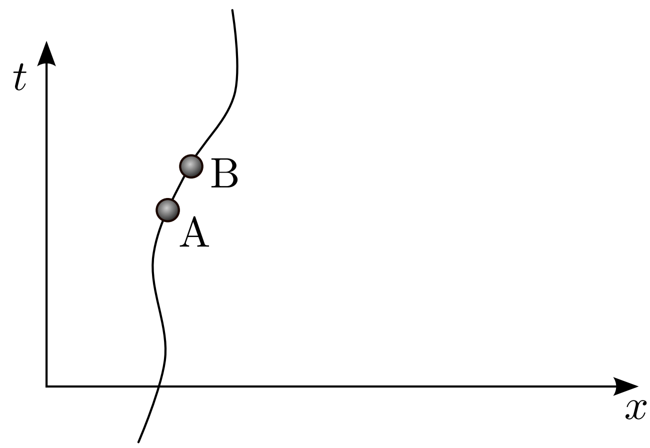

Figure 1: The worldline, the trajectory of a particle in a spacetime diagram. Two events A and B along the path of the particle have been marked.

event to the second. Lightlike events have zero spacetime interval,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = 0.$$

Note one consequence of this: Remember that the proper time interval $\Delta\tau^2$ equals the spacetime interval Δs^2. Thus, photons always have $\Delta\tau = 0$, the wristwatch attached to a photon would not change. Photons and other particles traveling at the speed of light do not feel the effect of time.

Events for which $\Delta x > \Delta t$ are called *spacelike events*. For these events, the spatial component of the distance is larger than the time component. No worldline could ever connect two spacelike events as it would require a particle to travel faster than light. Thus, spacelike events are not causally connected. The first event could not have caused the second. The spacetime interval for spacelike events is negative,

$$\Delta s^2 = \Delta t^2 - \Delta x^2 < 0.$$

In figure 2 we see two events A and B and three different worldlines between these events. These events could be a car passing position x_A and position x_B in the laboratory frame. In the spacetime diagram we see three worldlines each corresponding to a car. The straight worldline must correspond to a car driving with constant speed $v = \Delta x/\Delta t =$ constant. The two other worldlines must correspond to cars accelerating (changing their speed and thereby changing the slope of the worldline) along the way from x_A to x_B, but all cars reach point x_B at the same time (event B). All cars also passed point x_A at the same time (event A). Same time here means 'same time' for all frames of reference: all the cars meet at event A and B, so if they meet simultaneously in one frame of reference they must meet simultaneously in all other frames of reference (did you get this? If not, read the sentences again!).

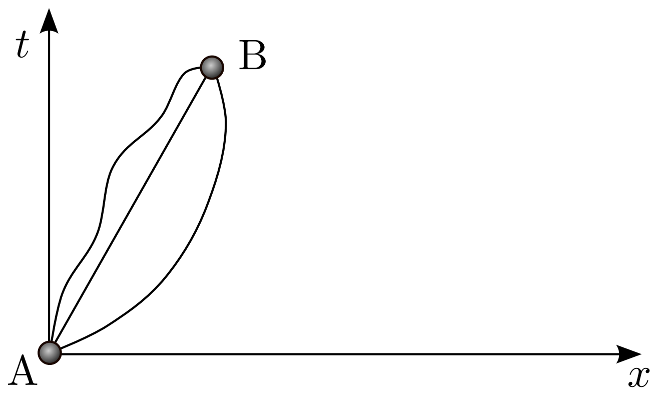

Figure 2: Different worldlines connecting the two events A and B.

We will now ask a question which answer may seem obvious in this case, but which might not be so obvious in other situations. The question is: Given a particle (or a car) going from event A to event B. If this particle is in free float (in special relativity this means that no forces act on the particle), which worldline will the particle take between event A and event B? Looking back at figure 2 we see three possible worldlines, but in fact there is an infinite number of possible worldlines connecting the two events. The obvious answer in this case is that it will follow a straight line in spacetime, i.e. the straight worldline corresponding to constant velocity. This is just a modern way of saying Newton's first law: A body which is not under the influence of external forces will continue moving with constant velocity. But is there a deeper principle behind? In the theory of relativity there is, and this principle is called the *principle of maximal aging*. This is a fundamental principle in the special as well as in the general theory of relativity.

The principle of maximal aging says that a particle in free float (no forces act on the particle) will follow the worldline which corresponds to the longest possible proper time interval between the two events. We remember that proper time is the wristwatch time, the time measured on the clock attached to the particle. So let different particles take different paths in spacetime between the two events. Attach a wristwatch to each of the particles. At event B, you look at the time interval between event A and B measured on the wristwatch of each of the particles. The particle which measures the longest proper time, i.e. the particle with the wristwatch which made most ticks during the trip from event A to event B, is the particle taking the path that a particle in free-float would take.

How do we calculate the proper time interval that a given particle takes from event A to event B? The clue is to remember that the proper time interval $\Delta\tau$ between two events equals the spacetime interval, or the total length of the path in spacetime Δs taken between the two events. For the worldline of a particle with constant velocity, we know that the distance in spacetime traveled from event A to event B is just $\Delta s = \sqrt{\Delta t^2 - \Delta x^2}$ where Δx and Δt are space and time intervals measured in an arbitrary frame of reference. To measure the total spacetime interval along the worldline of a particle which does not move with constant velocity, we need to break the path up into small path lengths ds. This path length is so small that we can assume the velocity to be constant during the time it takes to travel this interval in spacetime. We can thus write $ds = \sqrt{dt^2 - dx^2}$ where dx and dt are the corresponding small space and time displacement measured in the arbitrary frame of reference. To obtain the total length of the path in spacetime traveled between two events A and B, we need to integrate all these tiny spacetime intervals ds giving

$$\Delta s = \int_A^B \sqrt{dt^2 - dx^2}. \tag{1}$$

This equals measuring the length s of a curved path between two points A and B in the x-y plane:

$$\Delta s = \int_A^B \sqrt{dx^2 + dy^2}.$$

Note again a huge difference here: The minus sign in the spacetime interval. We know from Euclidean geometry that the shortest path s between two points A and B in the plane, is the straight line. The minus sign in the line element for Lorentz geometry gives rise to the opposite result (which we will not derive here): The *longest* path s between two events A and B in spacetime is the straight worldline. Therefore, if we measure the length of the spacetime path for all the three worldlines in figure 2 using the integral in (1), we find that the longest path in spacetime is the straight worldline, i.e. the worldline of the car driving with constant velocity. Remember again that the length of the spacetime interval Δs equals the total proper time $\Delta\tau$ measured on the wristwatch of the particle. So the longest proper time interval between two events is measured on the particle taking the straight line in spacetime, i.e. the particle which has constant velocity. We have just deduced Newton's first law from the principle of maximal aging. When we come to the general theory of relativity, we will see that the spacetime geometry and hence the form of the line elements Δs is different in a gravitational field. We will need the principle of maximal aging to tell us how a free float particle is moving in this case.

2 Four-vectors

We are used to vectors in three-dimensional space giving the position of a point in space,

$$\vec{x} = (x_1, x_2, x_3),$$

where I have used (x_1, x_2, x_3) instead of (x, y, z) for the components in the three spatial dimensions. A 4-vector is similarly defined to give the position of an *event* in four dimensional spacetime,

$$x = (x_0, x_1, x_2, x_3),$$

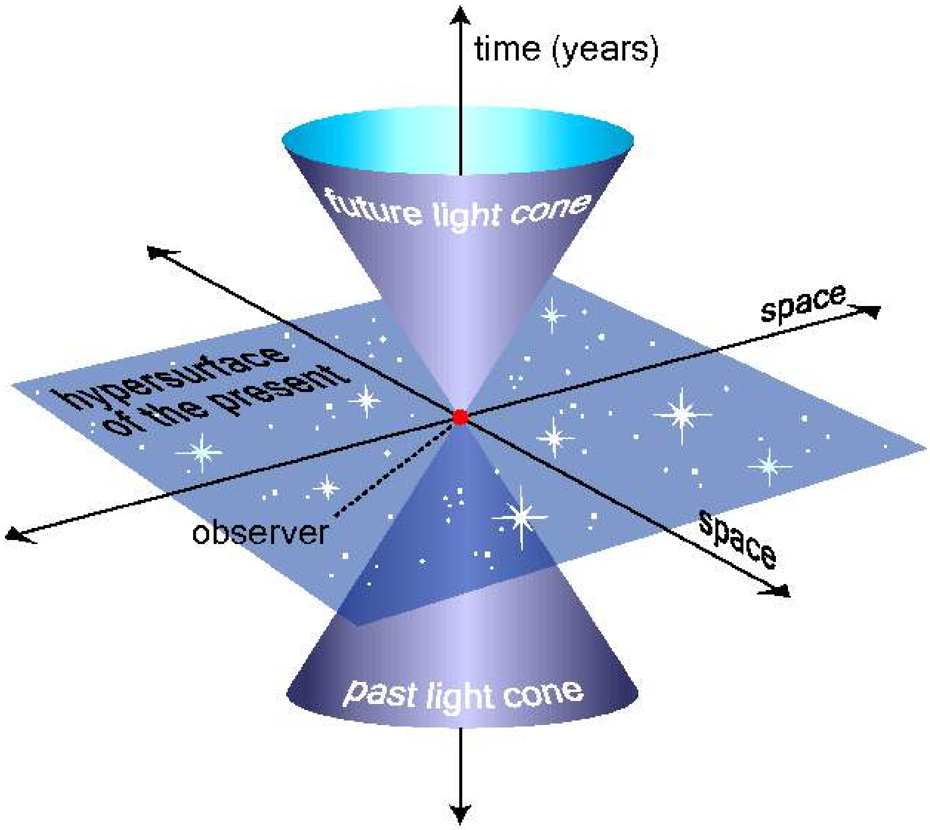

Figure 3: Info-figure: An example of a light cone, the three-dimensional surface of all possible light rays arriving at and departing from a point in spacetime. Here it is depicted with one spatial dimension suppressed. In general, there are three types of curves in spacetime: 1) Time-like curves, with a speed less than the speed of light. These curves must fall within a cone defined by light-like curves. 2) Light-like curves, having at each point the speed of light. They form a cone in spacetime, dividing it into two parts. 3) Space-like curves, falling outside the light cone. (Figure: Wikipedia)

or if you wish (t, x, y, z). For components of a normal three dimensional vector, we use Latin letters, typically i and j, for the indices: The components of $\vec{x}$ are x_i where i goes from 1 to 3. For the components of a 4-vector, we use Greek indices, typically μ and ν. The components of a four-vector x are x_μ where μ run from 0 to 3, 0 being the time component. If we wish to separate the time and space part of a four-vector we might also write it as $x = (t, x_i)$ where x_i refers to all three spatial components.

The four-vector x_μ points to an event in spacetime for a given frame of reference. We have already learned that in order to transform spacetime coordinates from one frame of reference to another, we need the Lorentz transformations. Thus, we may write the transformation of a four-vector x_μ in one frame of reference to x'_μ in another frame of reference by a matrix multiplication,

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -v\gamma & 0 & 0 \\ -v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

Compare with the expression for the Lorentz transformation in the previous lecture notes. Check that the matrix multiplication gives you the correct equations. (Compare this equation with matrices which are used to rotate between coordinate systems in two spatial dimensions, do you see a similarity? Remember the analogy used in the previous lecture notes between a coordinate change in the (x, y) plane and the (x, t) diagram).

We can write this matrix equation as

$$x'_\mu = \sum_{\nu=0}^{3} c_{\mu\nu} x_\nu,$$

where $c_{\mu\nu}$ is the matrix above. This is the equation which transforms any four-vector from one frame of reference to another. We will now write this equation using the so-called Einstein conventions. This is just a rule which will save you from a lot of writing. Instead of writing the sum symbol, we simply say that when two factors in a term contain the same index, there is an implicit sum over this index. If the index is Latin, then there is a sum over the three spatial dimensions, if the index is Greek, there is a sum over the three spatial dimensions plus time. Using this convention we can write the previous equation simply as

$$x'_\mu = c_{\mu\nu} x_\nu \tag{2}$$

It can be shown that four-vectors follow the normal rules for summations and subtractions (see exercises). We will now look at the scalar product. For three dimensional vectors, the scalar product can be written as,

$$\vec{x} \cdot \vec{y} = \sum_{i=1}^{3} x_i y_i = x_i y_i,$$

where the Einstein convention was used in the last expression. We can also define a scalar product for four-vectors. Instead of writing a dot between the vectors, one usually writes the scalar product with one upper index and one lower index,

$$x^\mu y_\mu = x_0 y_0 - x_i y_i.$$

One index μ is written high and the other low to show that this is the scalar product and *not* a normal sum. Note that the scalar product is defined with a minus sign in front of the spatial part. If we had written both indices low, this would mean,

$$x_\mu y_\mu = x_0 y_0 + x_i y_i,$$

using the Einstein summation convention. This is different from the scalar product. It should be clear where the minus sign comes from, consider a spacetime interval Δx_μ (a spacetime interval is an interval between two points x_μ^1 and x_μ^2 in time and space such that $\Delta x_\mu = x_\mu^1 - x_\mu^2 = (\Delta t, \Delta x, \Delta y, \Delta z)$). The scalar product of a spacetime interval with itself gives,

$$\Delta x^\mu \Delta x_\mu = \Delta t^2 - \Delta x^2 = \Delta s^2$$

(assuming $\Delta y = \Delta z = 0$). The result is the *scalar* Δs^2. A scalar is a quantity which is invariant, which has the same value in all frames of reference. We already knew that the spacetime interval Δs^2 is a scalar (where did we learn this?). For infinitesimal distances between events, we may write this as,

$$ds^2 = dx^\mu dx_\mu.$$

We learned above that a four vector is a vector which transforms according to the Lorentz transformation (equation 2) when changing from one frame of reference to another frame of reference having velocity v with respect to the first. This has an important consequence: You cannot choose 4 numbers on random, put them together and call it a 4-vector! The numbers entering in a four-vector need to be physical quantities which are such that the 4-vector transforms accoring to equation 2. We thus need to take care when performing mathematical operations with 4-vectors: The result may not necessarily be a 4-vector.

As an example we will now investigate what happens with a 4-vector when multiplying it with some number. Say that you for some reason need to multiply a spacetime distance $\Delta x_\mu = (\Delta t, \Delta x, \Delta y, \Delta z)$ with the corresponding time interval Δt forming

$$\Delta y_\mu = \Delta t \Delta x_\mu.$$

Is Δy_μ a 4-vector? We can easily check this by checking whether it transforms according to equation 2 when changing frame of reference. We know that Δx_μ follows this transformation. We also now that $\Delta t' = (1/\gamma)\Delta t$ when changing frame of reference. We thus have for $\Delta y'_\mu$ in a new frame of reference

$$\Delta y'_\mu = \Delta t' \Delta x'_\mu = (1/\gamma)\Delta t c_{\mu\nu} \Delta x_\nu = (1/\gamma) c_{\mu\nu} \Delta y_\nu.$$

Because of the factor $1/\gamma$ we see that Δy_μ does not transform according to equation 2 and Δy_μ is therefore NOT a 4-vector. We thus cannot multiply a 4-vector with a time interval and obtain a 4-vector.

A four-vector which is multiplied by a scalar however, is itself a four-vector. If instead of multiplying Δx_μ with Δt, we multiply it with the corresponding spacetime interval Δs we get

$$\Delta y_\mu = \Delta s \Delta x_\mu.$$

Transforming to a different frame of reference we have again $\Delta x'_\mu = c_{\mu\nu}\Delta x_\nu$ since Δx_μ is a four-vector and $\Delta s' = \Delta s$ since Δs is a scalar. We thus have

$$\Delta y'_\mu = \Delta s' \Delta x'_\mu = \Delta s c_{\mu\nu} \Delta x_\nu = c_{\mu\nu} \Delta y_\mu$$

which does follow equation 2. In this case Δy_μ is a four-vector. We thus have generally that when A_μ is a four vector and f is a scalar, the product

$$B_\mu = f A_\mu,$$

is a 4-vector. In the exercises you will show that the results of summing or subtracting 4-vectors are 4-vectors.

3 Four-velocity

Can we define a four dimensional velocity V_μ, that is, a four dimensional vector showing the direction of motion in spacetime of a particle with coordinates x_μ? By analogy to normal three dimensional velocity, the four-velocity V_μ should be the the rate of change of x_μ. A natural choice would be dx_μ/dt, but this is not a four-vector: As we discussed above, Δt or dt is not a scalar, it has different values in different frames of reference. Thus dx_μ/dt does not transform as a 4-vector, i.e. you cannot use the Lorentz transformation to transform it from one frame of reference to another. But in order to have velocity, we need the rate of change with respect to some time interval Δt. Which measure of time can we use?

Remember that proper time τ is a scalar, it is defined as the time observed on the wristwatch of an observer. All observers will measure the same time interval $\Delta\tau$ between two events (how do they measure $\Delta\tau$?). Consider the example with the train and observer P who is jumping up and down. Measured on the wrist watch of observer P, one jump takes one second, thus one second of proper time for the frame of reference of the train. According to observer O's wristwatch, the jump takes 1.7 seconds, but this is not the proper time for the train (remember the definition of proper time!). But observer O can take his binoculars and read of the time between each jump on observer P's wristwatch. He will then find, in agreement with observer P, that in proper time units for the train, each jump takes one second.

Note that proper time needs to be defined with respect to some frame of reference (in this case the train), but once this is defined, everybody agrees on the proper time interval between two events taking place at the same spot in that frame. In the case of four-velocity, there is no doubt about which proper time we are speaking about: Four-velocity is the velocity of a particle or an object (for instance a train) and the proper time $\Delta\tau$ which we use to define four velocity is the time measured in the rest frame of this object. So four-velocity can be defined as

$$V_\mu = \frac{dx_\mu}{d\tau}.$$

Let us find the length (absolute value) of the four-velocity (the square root of the scalar product of the vector with itself). The square of the length is (as for normal vectors) given by

$$V_\mu V^\mu = \frac{dx_\mu}{d\tau}\frac{dx^\mu}{d\tau} = \frac{dx_\mu dx^\mu}{d\tau^2} = \frac{ds^2}{d\tau^2} = \frac{d\tau^2}{d\tau^2} = 1.$$

(did you understand every step here?) Taking the square root of this we still get 1. The length of the four-velocity is thus always one. Remember that a velocity of one means the velocity of light. All particles move with the velocity of light in spacetime! For each proper time interval $\Delta\tau$ a particle moves an equal interval Δs in spacetime.

We can write the four-velocity in terms of normal 3-velocity as

$$V_\mu = (\frac{dt}{d\tau}, \frac{dx_i}{d\tau}) = (\frac{dt}{d\tau}, \frac{dt}{d\tau}\frac{dx_i}{dt}) = \frac{dt}{d\tau}(1, \vec{v}) = \gamma(1, \vec{v})$$

where we have used that $\Delta t/\Delta\tau = dt/d\tau = \gamma$ from the previous lecture notes (go back and check how you derived this, it is important!). Now we are ready to answer a question that has bothered us all the time since we learned about the Lorentz transformations: We know how to transform between coordinates (x, t) and (x', t') in two different frames of reference. But how do you transform a velocity v_x from one frame to the other? Say that you stand on the ground and look at a passing airplane. You measure the velocity of the airplane along the x-axis to be v_x. A car is passing you on the street with velocity v_{rel} along the same x-axis and you note that the driver is also watching the airplane. You start to wonder which velocity v'_x that the driver is measuring for the airplane. The situation is depicted in figure 4. In normal non-relativistic physics you know that the answer should read $v'_x = v_x - v_{\mathrm{rel}}$, but we have learned that this does not work for velocities close to the velocities of light (for instance, look back at the Michelson-Morley experiment). Assuming that there are no motions in the y and z direction, we can now write the four velocity of the airplane from our laboratory frame as $V_\mu = \gamma(1, v_x)$ and from the car as $V'_\mu = \gamma'(1, v'_x)$ where $\gamma = 1/\sqrt{1 - v_x^2}$ and $\gamma' = 1/\sqrt{1 - (v'_x)^2}$. We know that four-velocity is a four-vector and that four-vectors by definition transform from one frame of reference to the other under the Lorentz transformation,

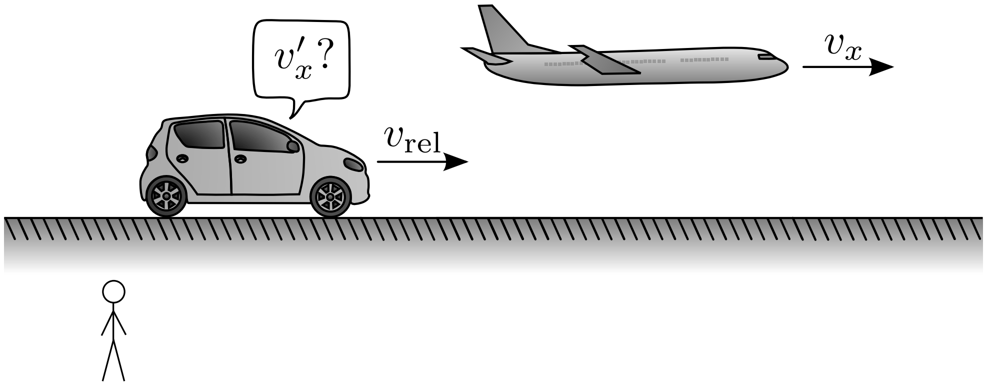

Figure 4: The observer on the ground measuring a velocity v_x for the airplane, wondering which velocity v'_x the driver of the car measures for the same airplane.

$$V'_\mu = c_{\mu\nu} V_\nu,$$

or written in terms of matrices as

$$\begin{pmatrix} \gamma' \\ \gamma' v'_x \\ \gamma' v'_y \\ \gamma' v'_z \end{pmatrix} = \begin{pmatrix} \gamma_{\rm rel} & -v_{\rm rel}\gamma_{\rm rel} & 0 & 0 \\ -v_{\rm rel}\gamma_{\rm rel} & \gamma_{\rm rel} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma \\ \gamma v_x \\ \gamma v_y \\ \gamma v_z \end{pmatrix}$$

where $\gamma_{\rm rel} = 1/\sqrt{1 - v_{\rm rel}^2}$.

From this matrix equation, we obtain two equations for the velocity v_x and v'_x,

$$\gamma' = (\gamma_{\rm rel} - v_{\rm rel}\gamma_{\rm rel} v_x)\gamma$$
$$\gamma' v'_x = (-v_{\rm rel}\gamma_{\rm rel} + \gamma_{\rm rel} v_x)\gamma.$$

Dividing the second equation by the first, we obtain

$$v'_x = \frac{v_x - v_{\rm rel}}{1 - v_{\rm rel} v_x}, \tag{3}$$

which is the Lorentz transformation for velocities. Note that when the speed of the airplane approaches the speed of light, $v_x \to 1$ then $v'_x \to 1$ showing that the laboratory observer and the observer in the car will both measure the speed of light for the airplane. This solves the weird result obtained by Michelson and Moreley: The speed of light is the same from all frames of reference.

4 Relativistic momentum and energy

What about momentum and energy? We have learned that the velocity v of an object as measured from two different frames of reference transform according to the Lorentz transformation (equation 3). This must necessarily have consequences for how we measure momentum $p = mv$ and energy $E = 1/2mv^2$ from two different frames of reference. There must be some corresponding Lorentz transformations for momentum and energy. We have learned a simple and easy recipe for finding the transformation equations between different frames: Construct a four-vector and use the transformation properties for four-vectors. This worked for velocity so let's try with momentum and energy.

We start with momentum. In order to construct a four-vector P_μ for momentum, let's try a form which is as similar as possible to the Newtonian form $\vec{p} = m\vec{v}$. Rest mass (the mass measured in the rest frame of the object) is a scalar quantity, so

$$P_\mu = mV_\mu$$

is a four-vector. Using that $V_\mu = \gamma(1, \vec{v})$, we can write momentum as

$$P_\mu = m\gamma(1, \vec{v}) = \gamma(m, \vec{p}),$$

where $\vec{p}$ is the Newtonian momentum. Taking the spatial part of this equation we see that relativistic momentum can be written in three dimensions simply as

$$\vec{p}_{\text{relativistic}} = \gamma m\vec{v}, \tag{4}$$

where $\vec{v}$ is the normal 3-velocity of an object. What is the meaning of the time component $P_0 = \gamma m$ of the momentum 4-vector? In order to investigate this let us write it in the Newtonian limit. For $v << 1$ (velocity much lower than the velocity of light) we can make a Taylor expansion in v,

$$P_0(v) = P_0(v = 0) + \frac{dP_0}{dv}(v = 0)v + \frac{1}{2}\frac{d^2P_0}{dv^2}(v = 0)v^2,$$

where the derivatives taken at $v = 0$ are (check it!) $P_0(v = 0) = m$, $dP_0/dv(v = 0) = 0$ and $d^2P_0/dv^2(v = 0) = m$. We get

$$P_0 = m + \frac{1}{2}mv^2.$$

The last term is just the expression for Newtonian kinetic energy. The first term is the *rest energy* of a particle, converted to normal units it can be written as the more well known $E = mc^2$. The rest energy is the energy of a particle at rest, it is the energy in the mass of the particle. Thus, the time component of the momentum four-vector is relativistic energy,

$$E_{\text{relativistic}} = m\gamma, \tag{5}$$

which in the Newtonian limit reduces to the Newtonian kinetic energy plus an energy term which did not exist in Newtonian physics, the energy of the mass of the particle. So the 4-vector P_μ is not just a momentum 4-vector, it is *the momentum-energy 4-vector* which time component is energy and space component is momentum. It means that energy and momentum are related in the same way as space and time are. In the same manner as we talk about spacetime, indicating that space and time are basically two aspects of the same thing, we can call energy and momentum collectively as *momenergy*. The four-vector P_μ is simply the *momenergy four-vector*.

What is the length of the momenergy four-vector? Using that $P_\mu = mV_\mu$ we have for the square of the length

$$P_\mu P^\mu = m^2 V_\mu V^\mu = m^2.$$

The length is the square root of m^2 which is m. The length of the momenergy four-vector is an invariant and it is thus simply the mass. We have seen that we can write $P_\mu = \gamma(m, \vec{p})$ giving (using equations 4 and 5)

$$P_\mu = (E_{\text{relativistic}}, \vec{p}_{\text{relativistic}}).$$

From now on we will drop the subscript 'relativistic' and always refer to the relativistic energy and relativistic momentum using E and p. But how can we be so sure? How can we know that this is the correct expression for energy and momentum? What is the criterion for a quantity to be energy or momentum? We know that energy and momentum are conserved quantities. The total energy and momentum of particles after a collision should always be the same as the total energy and momentum before the collision. So this is easy to check: Measure the total energy and momentum of particles before and after a collision, if they are the same we have found the correct expressions for momenergy. This has been tested thousands of times in particle accelerators with particles moving close to the speed of light. It turns out that the Newtonian energy and momentum are *not* conserved in these collisions. The relativistic energy and momentum defined as we have done above however, *are* conserved.

By now we have got used to measure time and space in the same units and therefore we have also got used to add these quantities $\Delta x + \Delta t$ without hesitating. We see that the result of measuring time and space in the same units is that momentum and energy are also measured in the same units, the units of mass. We remember that since space and time are measured in the same units, the speed v is a dimensionless number. The factor γ is clearly also dimensionless, so the momentum $p = m\gamma v$ can be measured in the units of mass (kg). The same goes for energy $E = m\gamma$, which also has dimension mass. So both energy and momentum are measured in kg and these quantities can therefore be added, just as we can add intervals in time and distances in space. The momenergy four-vector is $P_\mu = (E, \vec{p})$, taking the scalar product we have (remembering the result above that the length of P_μ is just m),

$$P_\mu P^\mu = E^2 - p^2 = m^2,$$

we can thus write energy in terms of momentum as

$$E = \sqrt{m^2 + p^2}.$$

A photon is massless, so for photons this relation is just

$$E = p,$$

or by using normal units $E = pc$ which is a more known form of this expression.

We return to the above example with the airplane and the passing car. You measure the relativistic energy and momentum of the airplane from the laboratory frame (the ground) and you wonder what energy and momentum the driver of the car measures for the same airplane. The momenergy four-vector is a four-vector which means that it can be transformed from one frame of reference to the other by the Lorentz transformation,

$$P'_\mu = c_{\mu\nu} P_\nu,$$

or in matrix form (remember that there were no movements in the y and z direction)

$$\begin{pmatrix} E' \\ p'_x \\ p'_y \\ p'_z \end{pmatrix} \begin{pmatrix} \gamma_{\text{rel}} & -v_{\text{rel}}\gamma_{\text{rel}} & 0 & 0 \\ -v_{\text{rel}}\gamma_{\text{rel}} & \gamma_{\text{rel}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ p_x \\ p_y \\ p_z \end{pmatrix}$$

Giving the following transformation equations for momentum and energy

$$E' = \gamma_{\text{rel}} E - v_{\text{rel}}\gamma_{\text{rel}} p_x$$
$$p'_x = \gamma_{\text{rel}} p_x - v_{\text{rel}}\gamma_{\text{rel}} E$$

where v_{rel} is the relative velocity between the two frames of reference, the observer on the ground and the car (see figure 5).

We will now use these equations to answer the following question: What energy and momentum (E', p'_x) does a person passing you in his car with a velocity v (relative to you) measure that you have? From your frame of reference in which you are at rest, your momentum is by definition zero $p = 0$ and you energy equals your mass $E = m$. We will now transform these quantities to the driver of the car measuring your energy and momentum to be E' and p'. The relative velocity of the car with respect to you is simply $v_{\text{rel}} = v$. Then the energy and momentum that the driver in the car measures that you have is simply (using the equations above, check that you get the same result),

$$E' = \gamma E \qquad p'_x = -v\gamma E$$

Note that $\gamma > 1$ so the driver in the car measures, not only a larger absolute momentum, but also larger energy.

From the point of view of Newtonian mechanics this was to be expected: with respect to the driver you have a non-zero velocity and kinetic energy,

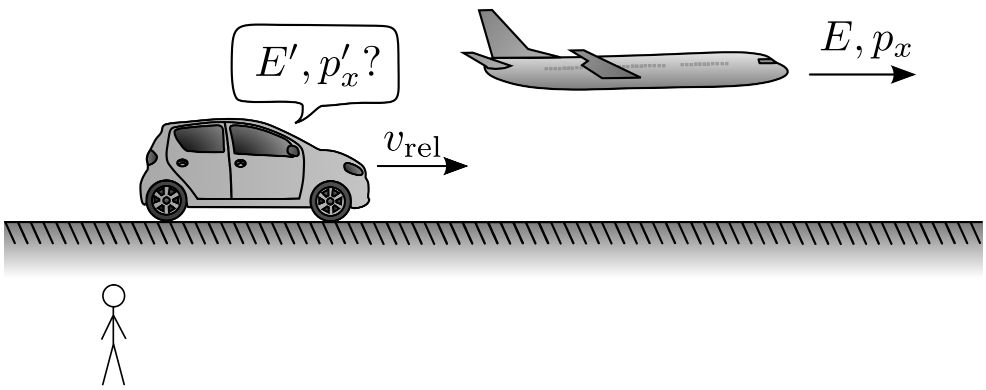

Figure 5: The observer on the ground measuring a velocity v_x for the airplane, wondering which velocity v'_x the driver of the car measures for the same airplane.

thus both your momentum and energy are clearly larger with respect to him than with respect to your rest frame. But from the point of view of geometry it might seem strange: In your rest frame the four-vector P_μ only has a time component and no space component. In the frame of the driver, both the time and space component of the vector are larger than in your frame. But the length of the momenergy vector P_μ is always the same, equal to m. Going back to normal 3D geometry this would not be possible. Imagine a vector $\vec{a} = (f, g, 0)$ and another vector $\vec{b} = (2f, h, 0)$. If the length of these vectors are the same, then we have that $h < g$. We see that from normal geometry you would expect that if the length of a vector is constant, then if you increase one component of the vector the other should decrease. The reason for this discrepancy with normal geometry is that spacetime has Lorentz geometry whereas 3D space has Euclidean geometry. Lorentz geometry has a minus sign in the definition of the scalar product (which also defines the length of the vector) making such an effect possible.

Now you know the basics of the special theory of relativity and you have got the necessary preparations to start studying the general theory of relativity. In the general theory of relativity we will study how masses curve spacetime, making the expression for the line element Δs different close to a large mass. This change in the line element changes the dynamics and gives rise to what we in Newtonian terms call the force of gravity.

5 List of expressions you should know by now

Worldline	$\rightarrow$	page 1
Timelike	$\rightarrow$	page 1
Lightlike	$\rightarrow$	page 1
Spacelike	$\rightarrow$	page 2
Principle of maximal aging	$\rightarrow$	page 3
Wristwatch time	$\rightarrow$	page 3
Scalar	$\rightarrow$	page 7
Four vector	$\rightarrow$	page 7
Four velocity	$\rightarrow$	page 9
Momenergy	$\rightarrow$	page 12

6 Problems

Problem 1 (2–3 hours)

Before embarking on the problems with four vectors and relativistic dynamics, we have one more important case to study from the previous lecture. This is the so-called 'twin paradox'. This long and detailed exercise is very important to gain some basic understanding for the underlying physics of many of the so-called paradoxes in the theory of relativity.

You are an astronaut traveling to the star Rigel, 800 light years from Earth. You start at $x = x' = 0$ and $t = t' = 0$ where (x, t) are Earth frame coordinates and (x', t') are spaceship coordinates. You travel in your spaceship at a velocity of $v = 0.99995$. We assume that Earth and Rigel do not move with respect to each other and that they therefore are in the same frame of reference.

1. Event A is you departing from Earth. Event B is you arriving at Rigel. In the Earth frame it took $800/0.99995 \approx 800.04$ years to arrive at Rigel. We know that for you it took a factor $\gamma = 1/\sqrt{1-v^2}$ less ($\Delta t = \gamma\Delta t'$, Δt is measured in Earth frame, $\Delta t'$ is measured in spaceship frame). How long time did it take for you (on your wristwatch) to arrive at Rigel?

2. After arriving at Rigel, you make the necessary scientific measurements (which takes very little time and can therefore be ignored) and start the return flight. You fly with exactly the same speed $v = 0.99995$ towards Earth. Event C is when you arrive back on Earth. Use the same arguments (or symmetry arguments) to find the time Δt and $\Delta t'$ it took from Rigel and back to Earth in the two frames of reference.

3. If you have done your calculations correct, here is a summary of the situation: In the Earth frame, it took you 1600.08 years to go to

Rigel and return. On your wristwatch it took you 16 years to go to Rigel and back. So while hundreds of generations have passed on Earth, you return only 16 years older.

4. We will now make the same calculations again, but just switch the frames: The laboratory frame (x, t) is now the frame of the spaceship and the moving frame (x', t') is the Earth frame. Because of the principle of relativity we are allowed to switch the roles and we should arrive at exactly the same result using the same laws of physics. From this point of view, this is what is happening: You sit in you spaceship which is now the laboratory frame defined to be at rest and at $x = x' = 0$ at $t = t' = 0$ (event A), the Earth starts moving away from you with velocity $v = 0.99995$ and Rigel starts approaching you with the same velocity. After a time Δt Rigel arrives at your position (event B). We know from previous calculations that the trip took 8 years in your frame of reference which is now the laboratory frame. Using again that $\Delta t = \gamma \Delta t'$ (and make sure not to confuse Δt and $\Delta t'$) show that the clocks on Earth at the moment when Rigel arrives at your position show 0.08 years. Only 0.08 years had passed on Earth during the 8 year (on your watch) trip to Rigel.

5. Now, this might look like a paradox, but we will show further down that it is not. No matter how strange this might sound, it is consistent. The paradox is still to come. After making your investigations of Rigel, Rigel departs and Earth approaches you again at the speed of $v = 0.99995$. Making the same calculations again you will find that it takes the Earth 8 years to return to you. Let's again check carefully how long it takes on the Earth clocks for Earth to return at your position: At the moment you have finished your investigations, the Earth clocks show $t' = 0.08$ years and your clock shows $t = 8$ years. It takes Earth again $\Delta t = 8$ years to arrive at your position. We have as always that $\Delta t = \gamma \Delta t'$. How long did it take for Earth to return to your position measured on Earth clocks?

6. If you made the last calculation correct, this is now the situation: It took you 16 years from Earth departed to Earth returned. However, on Earth clocks it took 0.16 years. So while you are 16 years older, only two months have passed on Earth. Above we found that 1600 years had passed on Earth. Now, this is a paradox!

7. Clearly we made an error somewhere in the calculations. Or maybe we simply forgot some basic principles from special relativity? It appears at first sight that the two roles are equal,that we can choose whether we consider the Earth frame as the laboratory frame or the spaceship frame as the laboratory frame. But are the two roles really identical? What is the difference between the two observers, the Earth observer and the spaceship observer?

8. Don't read on until you have found an answer to the previous ques-

tion. Here comes the solution: The difference is that whereas the Earth observer always stays in the same frame of reference, the spaceship observer changes frames of reference: The spaceship needs to accelerate at Rigel in order to change direction and return to Earth. The Earth does not undergo such an acceleration. The expression $\Delta t = \gamma \Delta t'$ was derived for constant velocity (look back at its derivation). It is not valid when the velocity is changing. In order to solve this problem properly one needs to either use general relativity which deals with accelerations or we can view the acceleration as an infinite number of different free float frames, frames with constant velocity, and apply special relativity to each of these frames. We will not do the exact calculation here, but we will do some considerations giving you some more understanding of what is happening. We will now consider three frames of reference. The Earth frame (x, t), the outgoing spaceship frame (x', t') and the returning spaceship frame (x'', t''). Instead of spaceships we will look at it as elevators going between Earth and Rigel. There are boxes going in both directions. At $x = x' = 0$ and $t = t' = 0$ you jump into one of these boxes leaving for Rigel. There are other observers in other boxes before you and after you. The situation is depicted in figure 6. In the following use the Lorentz transformations to transform between the coordinate systems. We write the distance between Earth and Rigel in the Earth frame as L_0. Event A happens at $x_A = x'_A = 0$ and $t_A = t'_A = 0$. Event B is again the moment when you arrive at Rigel. At what time t_B in the Earth frame do you arrive at Rigel? (express the answer in terms of L_0 and v)

9. Use the Lorentz transformations to find t'_B, the time on your wristwatch when you arrive at Rigel. Insert numbers and check that you still find that the trip takes 8 years for you.

10. We now define event B'. At the same time as you arrive at Rigel (in your frame of reference which is now the frame of the outgoing elevator),another observer in another box in your elevator (thus another observer in your frame of reference using clocks synchronized with yours) passes the Earth at position $x_{B'} = 0$. Event B' is the event that he looks out and checks what time it is on Earth. So event B' takes place at the position of the Earth, but at the same time as you arrive at Rigel (same time in the outgoing reference frame). Show that the time $t_{B'}$ he reads on the Earth clocks can be written as $t_{B'} = L_0/v - vL_0$. Insert numbers. **Hint:** You first need to find the position $x'_{B'}$ of event B' in the outgoing elevator frame.

11. Insert numbers in your previous result. Explain the result which we found earlier when using the spaceship as the laboratory frame: Namely that when Rigel arrived at the spaceship, we calculated that on the Earth clocks only 0.08 years had passed. Why is this not a surprise? Those who were surprised earlier, do you know understand which error you made when you got surprised? Which basic principle

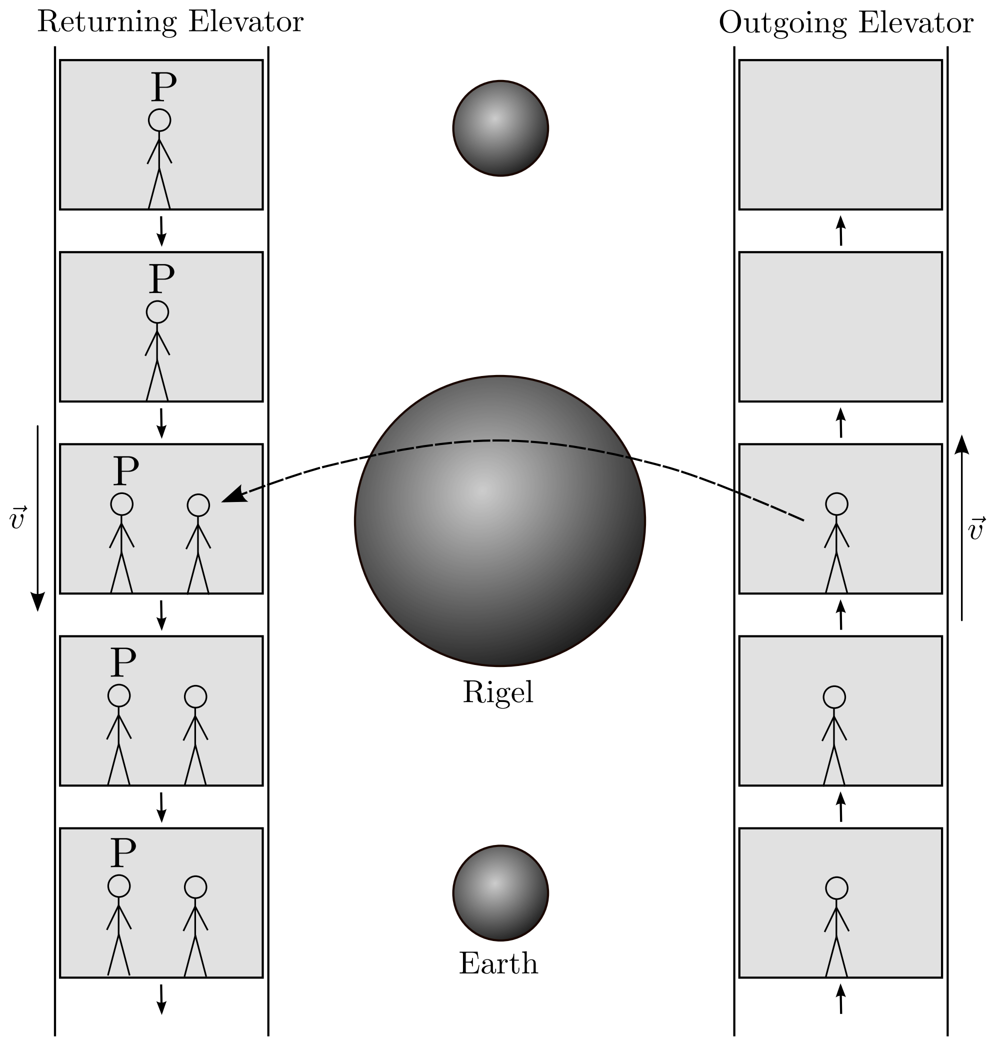

Figure 6: The elevator between Earth and Rigel.

of relativity had you forgotten?

12. We learned in the previous question that even if the Earth clocks were observed at the same moment as the spaceship/elevator arrived at Rigel (in the outgoing frame), these two events (the observation of the Earth clocks and the arrival at Rigel) were *not* simultaneous in the Earth frame. For you, sitting in the outgoing elevator, only 0.08 years have passed on Earth when you arrive at Rigel. For observers on the Earth on the other hand, you arrived at Rigel when 800 years had passed. At Rigel you meet a box in the returning elevator. You jump over to the box in the returning elevator at event B where you meet person P who has been traveling in the elevator from far away. Actually, at the same time (in the Earth frame) as you started your journey from Earth, person P started his journey from the other side of Rigel. We call the event when person P started his journey for event D. Event A and event D are simultaneous in the Earth frame. In order for you and person P to meet at event B, person P must have started on a planet a distance $2L_0$ from Earth (a distance L_0 from Rigel) as measured in the Earth frame. In that way you both cover a distance L_0 with the same speed v and therefore you can both meet at Rigel at time L_0/v as measured on Earth clocks. We call the coordinate system of the returning elevator (x'', t''). The clocks in the system of the returning elevator are set to zero at the moment when person P starts his journey. In the following, we will use spacetime intervals instead of the Lorentz transformation: The reason for this is that the returning elevator is not synchronized with the Earth frame at $x = 0, t = 0$. This was assumed when we deduced the form of the Lorentz transformation which we use in this course. Therefore, we will now again use invariance of the spacetime interval to obtain our answers. We will first check what the wristwatch of person P shows when he meets you at event B. In analogy to your own travel, it should intuitively show the same as your wristwatch: Both of you started at $t = 0$ on Earth clocks as well as on your own wristwatch. Both of you travel a distance L_0 (as measured in the Earth frame) at velocity v. But we have learned not to trust our intuition when working with relativity, so let's check. We will now consider the spacetime interval Δs_{BD} in order to find t''_B, the time on the wristwatch of person P at event B. Write down the space and time intervals Δx_{BD}, Δt_{BD}, $\Delta x''_{BD}$ and $\Delta t''_{BD}$. Show that invariance of the spacetime interval gives

$$\frac{L_0^2}{v^2} - L_0^2 = (t''_B)^2,$$

which gives $t''_B = L_0/(v\gamma)$, exactly as we thought. Your wristwatches agree at event B. Reassuring to see that our intiution still gives som reasonable results every now and then.

13. We will now try to find out what the time is on Earth for persons in the returning elevator. In the frame of the outgoing elevator, we used

a person who was situated in an elevator box at the same position of the Earth and looked out at the clocks on Earth exactly at the same time as event B happened (in the frame of the outgoing elevator). We called this event B' (looking at the clocks on Earth). We found that only 0.08 years had passed on Earth when you arrived at Rigel. We will now make the same check from the returning elevator. A person in an elevator box of the returning elevator being at the position of the Earth exactly at the same time as event B happens (now from the frame of the returning elevator) looks at the clocks on Earth. We call this event B" (the person in the box at the position of the Earth looking at the Earth clocks). We will now try to find out what he saw, i.e. which time $t_{B''}$ he observed on the Earth clocks. For this we will use spacetime interval $\Delta s_{DB''}$. Show that the space and time intervals from each frame are the following:

$$\begin{aligned}
\Delta x_{DB''} &= 2L_0 \\
\Delta t_{DB''} &= t_{B''} \\
\Delta x''_{DB''} &= L_0/\gamma \\
\Delta t''_{DB''} &= L_0/(\gamma v)
\end{aligned}$$

You might be a bit surprised by one of these results, but if you have doubts, do the following: Make one drawing for event D and one for event B". Show the position of the zero-point (the position of person P is the zero point of the x'' axis) of each of the x-axes in both plots and find the distances between events. Did it make it clearer?

14. Use invariance of the spacetime interval to show that

$$t_{B''} = \frac{L_0}{v} + L_0 v$$

Setting in for numbers this gives you $t''_{B''} = 1600.00$ years. Surprised? What has happened? You are still at event B, you made a very fast jump so almost no time has passed since you were in the outgoing elevator. But just before the jump, only 0.08 years had passed on Earth since you started your journey. Now, less than the fraction of a second later, 1600 years have passed on Earth. So in the short time that your jump lasted, 1599.92 years passed on Earth! This is were the solution to the twin paradox is hidden: When you jump, you change reference frame: You are accelerated. Special relativity is not valid for accelerated frames (actually one could solve this looking at the acceleration as an infinite sum of reference frames with different constant velocities). When you are accelerated, you experience fictive forces. This does not happen at Earth, the Earth does not experience the same acceleration. This is the reason for the asymmetry: If you speed had been constant, you and Earth could exchange roles and you would get consistent results. But since you are accelerated in the jump while the Earth is not, there is no symmetry here, you and the Earth cannot switch roles.

15. Let's summerize the situation: In your frame, you started your journey at $t = t' = 0$ and arrived at Rigel at $t' = 8$ years. In the Earth frame you arrived at Rigel after 800.04 years of travel. In your frame, the clocks on Earth show 0.08 years when you arrive at Rigel. Only 0.08 years have passed on Earth at the time you arrive at Rigel, seen from your frame. Then you jump to the returning elevator. Your watch still shows $t' = t'' = 8$ years. But now you have switched frame of reference. Now suddenly 1600 years have passed on Earth, clocks on Earth went from 0.08 years to 1600 years during the jump, as seen from your frame. As seen from Earth, the clock showed 800.04 years during your jump.

16. Seen from the Earth, you need 800.04 years to return, so the total time of your travel measured in the frame of reference of the Earth is $t = 1600.08$ years. In your own frame, the return trip took 8 years (by symmetry to the outgoing trip), so the total travel time for yourself is 16 years. But according to your frame of reference, the Earth clocks again aged 0.08 years during your return trip (by symmetry to the outgoing trip). When you were at Rigel, the observer in your frame of reference saw that the Earth clocks showed 1600 years. In your frame, 0.08 years passed on Earth during your return trip. So consistenly you find the Earth clocks to show 1600.08 years when you set your feet on the Earth again. This is also what we find making the calculation in the Earth frame $800.04 \times 2 = 1600.08$. But hundreds of generations have passed, and you have only aged 16 years. But after all these strange findings I'm sure you find this pretty normal by now. Everything clear? Read through one more time.

Problem 2 (30–45 min.)

You are in the laboratory frame watching two cars passing from position $x = 0$ at $t = 0$ (event 1) and arriving simultaneously at position $x = L$ some time $t = T_L$ (event 2) later (all coordinates taken in the laboratory frame). Car A moves with constant velocity $v_A = c/2$ whereas car B accelerates from $v = 0$ at $x = 0$ and accelerates such that it reaches $x = L$ simultaneously with car A. In the following you will draw some spacetime diagrams. We are not interested in exact numbers in this exercise, only roughly correct relative distances and slopes on the worldlines showing that you have understood the basic principles.

1. Make a spacetime diagram in the laboratory frame showing the worldlines of yourself and the two cars.

2. Make a spacetime diagram in the reference frame of car A showing the three same worldlines.

3. Make a spacetime diagram in the reference frame of car B showing the three same worldlines.

4. Return to the first spacetime diagram, the diagram for the laboratory frame. The wristwatch of the driver of car A makes exactly 10 ticks from event 1 to event 2. The first tick happens at event 1 and the last tick happens at event 2. Draw a dot on the worldline of car A at roughly the position of each of the ticks. The important point here is to have correct relative spacings between each tick.
5. The driver of car B has an identical wristwatch making ticks with exactly the same frequency in the rest frame of the watch. Use the principle of maximal aging to judge whether driver B will experience more or less ticks on his watch from event 1 to event 2.
6. Again, draw a dot on the worldline of car B at the positions where the wristwatch of the driver makes a tick. Again, the exact position is not important, but the relative distances between the dots should be correct. **Hint:** For each dot you draw, look at the slope of the worldline.

Problem 3 (10–30 min.)

A four vector is defined to be a vector in spacetime which transforms from one frame of reference to another (from x_μ to x'_μ) using the Lorentz transformation

$$x'_\mu = c_{\mu\nu} x_\nu.$$

To check if a four dimensional vector is a four-vector, you need to check whether this relation is true or not. We will now test if four-vectors follow the normal rules of addition, that the sum of two four-vectors is really a four-vector. Assume you have two four-vectors A_μ and B_μ. You sum the two to make a vector D_μ,

$$D_\mu = A_\mu + B_\mu.$$

You now need to show that the result, D_μ, is also a 4-vector. Use the transformation properties of A_μ and B_μ to obtain these vectors in a different frame A'_μ and B'_μ. Find an expression for the sum of the two vectors, D'_μ, in the other frame expressed by D_μ in the laboratory frame and show that D_μ is indeed a four vector.

Problem 4 (90 min.–2 hours)

A free neutron has a mean life time of about 12 minutes after which it disintegrates into a proton, an electron and a neutrino. We will ignore the neutrino here, assuming that the only products of disintegration are a proton and an electron. A neutron moves along the positive x axis in the laboratory frame with a velocity $v = 0.99$. It disintegrates spontaneously and a proton and an electron is seen to continue in the same direction as the neutron. Use tables to find the mass of the electron, proton and neutron. We will try to calculate the speed of the proton and the electron in the lab-frame. The easiest way to do this is in the rest frame of the neutron where the neutron has a very simple expression for energy and momentum. In the lab frame this would have been a lot more work since all three particles have velocities.

1. In the rest frame of the original neutron (which has now disintegrated), what was the total energy and momentum of the neutron before disintegration? Write the answer in terms of a momenergy four-vector P'_μ(neutron).

2. In the rest frame of the original neutron, write the momenergy four-vector P'_μ(proton) of the proton expressed in terms of the proton mass m_p and the unknown proton velocity v'_p in the neutron rest frame.

3. Still in the neutron frame, write the expression for the momenergy four-vector P'_μ(electron) in terms of the electron mass m_e and the unknown electron velocity v'_e measured in the neutron frame.

4. Use conservation of momenergy

$$P'_\mu(\text{neutron}) = P'_\mu(\text{proton}) + P'_\mu(\text{electron}),$$

to find the velocity of the proton and the electron in the rest frame of the original neutron. (insert numbers). **Hint:** This can be ugly if you don't do it right: Write the momentum part of the equation in terms of γ-factors only, then substitute for one of the γ from the energy part of the eqaution. Then you will avoid second order equations. Note that there are two possible solutions here: see if you understand why. Choose one of the solutions and continue with that in the rest of this exercise.

5. Use the transformation properties for four-vectors

$$P'_\mu(\text{electron}) = c_{\mu\nu} P_\nu(\text{electron}),$$

to find the energy and momentum of the electron and proton in the laboratory frame. (insert numbers:what units do your results have if you keep $c = 1$).

6. Use the numbers you have obtained for energy or momentum to obtain the speed of the electron and proton in the laboratory frame.

7. As an independent check (and to see an alternative way of doing it), use the relativistic formula for addition of velocities to obtain the speed of the proton and electron in the lab frame, using only the speed you have obtained for the proton in the neutron frame as well as the speed of the neutron seen from the lab frame.

8. For those who like long and ugly calculations only: Do everything from the beginning, but use only the lab-frame to obtain the same results. Do you see the advantage of using 4-vectors and change of frames?

Problem 5 (90 min.–2 hours)

An electron and a positron (the anti particle of the electron having the same mass) are approaching each other with the same velocity $v = 0.995$ in opposite directions in the laboratory frame. In the collision, both particles are annihilated and two photons are produced. One photon travels in the positive x direction, the other in the negative x direction. Use tables to find the mass of an electron.

1. What is the velocity of the positron in the rest frame of the electron?
2. Write down the momenergy four-vectors P_μ(electron) and P_μ(positron) of the positron and the electron in the laboratory frame (use numbers).
3. Use the transformation properties of four-vectors to write down the momenergy four-vectors P'_μ(electron) and P'_μ(positron) of the positron and the electron in the rest frame of the electron (again use numbers).
4. Show that the momenergy four-vector of a photon traveling in the positive x-direction can be written

   $$P_\mu^\gamma = (E, E, 0, 0),$$

   where E is the energy of the photon.
5. Use conservation of momenergy in the laboratory frame to argue that the two photons must have the same energy seen from the laboratory frame.
6. What is the energy of the photons and thereby the wavelength in the laboratory frame?
7. Use transformation properties for four-vectors to show that the energy E' of a photon in a frame moving with velocity v with respect to the laboratory frame (where the photon has energy E) is

   $$E' = E\gamma(1 - v)$$

8. What is the energy of each of the two photons in the rest frame of the electron?
9. Use the expression for E' in terms of E to derive the relativistic Doppler formula

   $$\frac{\Delta\lambda}{\lambda} = \left(\sqrt{\frac{1+v}{1-v}} - 1\right)$$

10. Show that the relativistic Doppler formula is consistent with the normal Doppler formula for low velocities. **Hint:** Make a Taylor expansion of $f(v) = \sqrt{(1+v)/(1-v)}$ for small v.

AST1100 Lecture Notes

11–12 The cosmic distance ladder

How do we measure the distance to distant objects in the universe? There are several methods available, most of which suffer from large uncertainties. Particularly the methods to measure the largest distances are often based on assumptions which have not been properly verified. Fortunately, we do have several methods available which are based on different and independent assumptions. Using cross-checks between these different methods we can often obtain more exact distance measurements.

Why do we want to measure distances to distant objects in the universe? In order to understand the physics of these distant objects, it is often necessary to be able to measure how large they are (their physical extension) or how much energy that they emit. When looking through a telescope, what we observe is not the physical extension or the real energy that the object emits, what we observe is the *appaerent magnitude* and the *angular extension* of an object. We have seen several times during this course that in order to convert these to *absolute magnitudes* (and thereby luminosity/energy) and *physical sizes* we need to know the distance (look back to the formula for converting appaerent magnitude to absolute magnitude as well as the small angle formula for angular extension of distant objects). In cosmology it is important to make 3D maps of the structure in the universe in order to understand how these structures originated in the Big Bang. To make such 3D maps, again knowledge of distances are indispensable.

There are 4 main classes of methods to measure distances:

1. Triognometric parallax (or simply parallax)
2. Methods based on the Hertzsprung-Russel diagram: main sequence fitting
3. Distance indicators: Cepheid stars, supernovae and the Tully-Fisher relation
4. The Hubble law for the expansion of the universe.

We will now look at each of these in turn.

1 Parallax

Shut your left eye. Look at an object which is close to you and another object which is far away. Note the position of the close object with respect to the distant object. Now, open you left eye and shut you right. Look again at the position of the close object with respect to the distant. Has it changed? If the close object was close enough and the distant object was distant enough, then the answer should be yes. You have just experienced *parallax*. The apparent angular shift of the position of the close object with respect to the distant is called the *parallax angle* (actually the parallax angle is defined as half this angle). The further away the close object is, the smaller is the parallax angle. We can thus use the parallax angle to measure distance.

In figure (1) we show the situation: It is the fact that your two eyes are located at different positions with respect to the close object that causes the effect. The larger distance between two observations (between the 'eyes'), the larger the parallax angle. The closer the object is to the two points of observation, the larger the parallax angle. From the figure we see that the relation between parallax angle p, baseline B (B is defined as half the distance between your eyes or between two observations) and distance d to the object is

$$\tan p = \frac{B}{d}.$$

For small angles, $\tan p \approx p$ (when the angle p is measured in radians) giving,

$$B = dp, \tag{1}$$

which is just the small angle formula that we encountered in the lectures on extrasolar planets. For distant objects we can use the Sun-Earth distance as the baseline by making two observations half a year apart as depicted in figure 2. In this case the distance measured in AU can be written (using equation (1) with $B = 1$ AU)

$$d = \frac{1}{p}\text{ AU} \approx \frac{206265}{p''}\text{ AU}.$$

Here p'' is the angle p measured in arcseconds instead of radians (I just converted from radians to arcseconds, check that you get the same result!). For a **par**allax of one arc**sec**ond (par-sec), the distance is thus 206265 AU which equals 3.26 ly. This is the definition of one *parsec* (pc). We can thus also write

$$d = \frac{1}{p''}\text{ pc}.$$

The Hipparcos satellite measured the parallax of 120 000 stars with a precision of $0.001''$. This is far better than the precision which can be achieved by a normal telescope. A large number of observations of each star combined with advanced optical techniques allowed for such high precision even with a relatively small telescope. With such a precision,

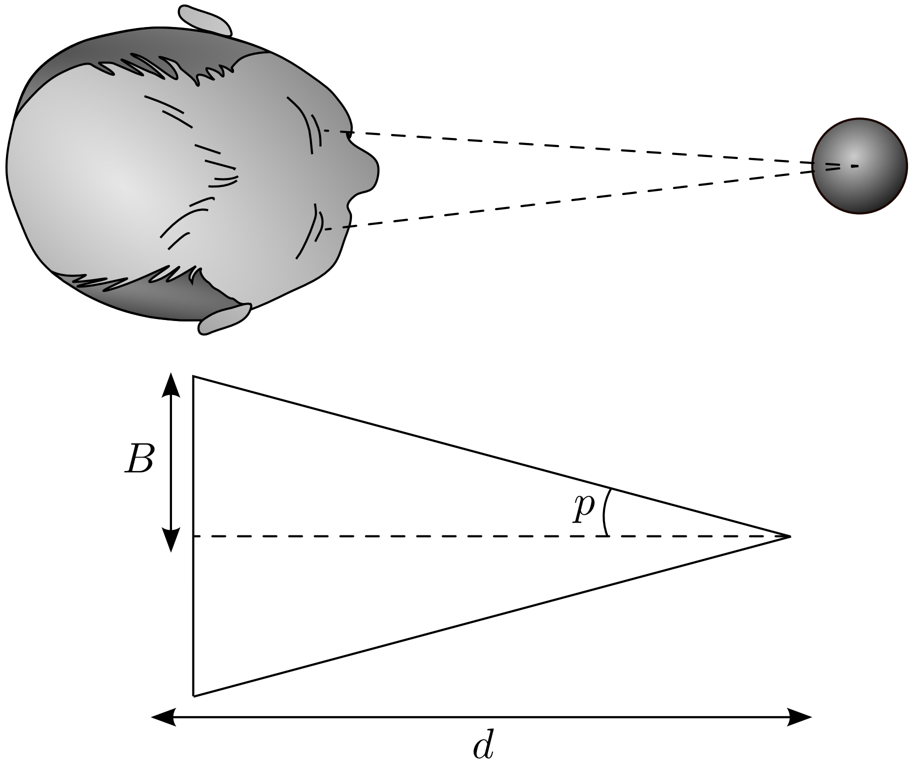

Figure 1: Definition of parallax: above is a face seen from above looking at an object at distance d. Below is the enlarged triangle showing the geometry.

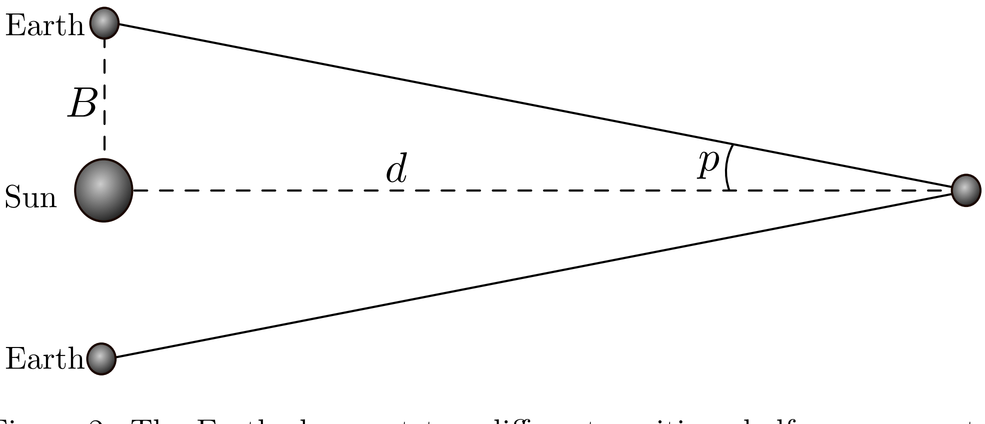

Figure 2: The Earth shown at two different positions half a year apart. The parallax angle p for a distant object at distance d is defined with respect to the Earth-Sun distance as baseline B.

distances of stars out to about 1000 pc (= 1 kpc) could be measured. The diameter of the Milky Way is about 30 kpc so only the distance to stars in our vicinity can be measured using parallax.

2 The Hertzsprung-Russell diagram and distance measurements

You will encounter the Hertzsprung-Russell (HR) diagram on several occations during this course. Here you will only get a short introduction and just enough information in order to be able to use it for the estimation of distances. In the lectures on stellar evolution, you will get more details.

There are many different versions of the HR-diagram. In this lecture we will study the HR-diagram as a plot with surface temperature of stars on the x-axis and absolute magnitude on the other. In figure 3 you see a typical HR-diagram: Stars plotted according to their surface temperature (or color) and absolute magnitude. The y-axis shows both the luminosity and the absolute magnitude M of the stars (remember: these are just two different measures of the same property, check that you understand this). Note that the temperature increases towards the left: The red and cold stars are plotted on the right hand side and the warm and blue stars on the left.

We clearly see that the stars are not randomly distributed in this diagram: There is an almost horizontal line going from the left to right. This line is called the *main sequence* and the stars on this line are called *main sequence stars*. The Sun is a typical main sequence star. In the upper right part of the diagram we find the so-called giants and super-giants, cold stars with very large radii up to hundreds of times larger than the Sun. Among these are the *red giants*, stars which are in the final phase of their lifetime. Finally, there are also some stars found in the lower part of the diagram. Stars with relatively high temperatures, but extremely low luminosities. These are *white dwarfs*, stars with radii similar to the Earth. These are dead and compact stars which have stopped energy production by nuclear fusion and are slowly becoming colder and colder.

In the lectures on stellar evolution we will come back to why stars are not randomly distributed in a HR-diagram and why they follow certain lines in this diagram. Here we will use this fact to measure distances. The HR-diagram in figure 3 has been made from stars with known distances (the stars were so close that their distance could be measured with parallax). For these stars, the absolute magnitude M (thus the luminosity, total energy emitted per time interval) could be calculated using the apparent magnitude m and distance r,

$$M = m - 5\log_{10}\left(\frac{r}{10\ \text{pc}}\right). \tag{2}$$

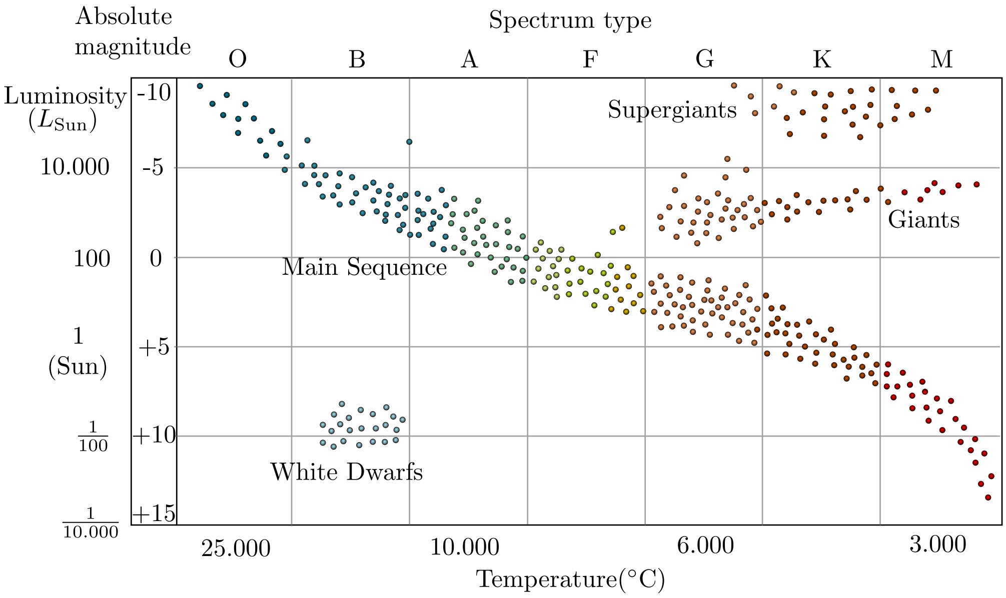

Figure 3: The Hertzsprung-Russell diagram.

HR-diagrams are often made from stellar clusters, a collection of stars which have been born from the same cloud of gas and which are still gravitationally bound to each other. The advantage with this is that all stars have very similar age. This makes it easier to predict the distribution of the stars in the HR-diagram based on the theory of stellar evolution. Another advantage with clusters is that all stars in the cluster have roughly the same distance to us. For studies of the main sequence, so-called *open clusters* are used. These are clusters containing a few thousand stars and are usually located in the galactic disc of the Milky Way and other spiral galaxies.

Now, consider that we have observed a few hundred stars in an open cluster which is located so far away that parallax measurements are impossible. We have measured the surface temperature (how?) and the apparent magnitude of all stars. We now make an HR-diagram where we, as usual, plot the surface temperature on the x-axis. However, we do not know the distance to the cluster and therefore the absolute magnitudes are unknown. We will have to use the apparent magnitudes on the y-axis. It turns out that this is not so bad at all: Since the cluster is far away, the distance to all the stars in the cluster is more or less the same. Looking at equation (2), we thus find,

$$M - m = -5\log_{10}\left(\frac{r}{10\text{ pc}}\right) = \text{constant},$$

for all stars in the cluster. The HR-diagram with apparent magnitude instead of absolute magnitude will thus show the same pattern of stars as the HR-diagram with absolute magnitudes on the y-axis. The only difference is a constant shift $m - M$ in the magnitude of all stars given by the distance of the cluster. Thus, by finding the shift in magnitude between the observed HR-diagram with apparent magnitudes and the HR-diagram in figure (3) based on absolute magnitudes, the distance to the cluster can be found. This method is called *main sequence fitting*.

Example

We observe a distant star cluster with unknown distance, measure the temperature and apparent magnitude of each of the stars in the cluster and plot these results in a diagram shown in figure 4 (lower plot) (note: spectral class is just a different measure of temperature, we will come to this in later lectures). In the same figure (upper plot) you see the HR-diagram taken from a cluster with a known distance (measured by parallax). Since the distance is known, the apparent magnitudes could be converted to absolute magnitudes and for this reason we plot absolute magnitude on the y-axis for this diagram. We know that the main sequence is similar in all clusters since stars evolve similarly. For this reason, we know that the two diagrams should be almost identical. We find that by shifting all the observed stars in the lower diagram upwards by 2 magnitudes (to higher luminosities but lower magnitudes), the two diagrams will look almost identical (look at the figure and check that you agree!). Thus, there is a difference between the apparent magnitude and the absolute magnitude of $M - m = -2$ and the distance is found by

$$-2 = -5 \log_{10} \left(\frac{r}{10 \text{ pc}} \right),$$

giving $r = 25$ pc.

Main sequence fitting can be used out to distances of about 7 kpc, still not reaching out of our galaxy. We now see why we use the phrase 'cosmic distance ladder'. The parallax method reaches out to about 1000 pc. After that, main sequence fitting needs to be used. But in order to use main sequence fitting, we needed a calibratet HR-diagram like figure 3. But in order to obtain such a diagram, the parallax method needed to be used on nearby clusters. So we need to go step by step, first the parallax method which we use to calibrate the HR-diagram to be used for the main sequence fitting at larger distances. Now we will continue one more step up the ladder. We use stars in clusters which distance is calibrated with main sequence fitting in order to calibrate the *distance indicators* to be used for larger distances.

3 Distance indicators

Again the method is based on equation (2). We can always measure the apparent magnitude m of a distant object. From the equation, we see that all we need in order to obtain the distance is the absolute magnitude. If

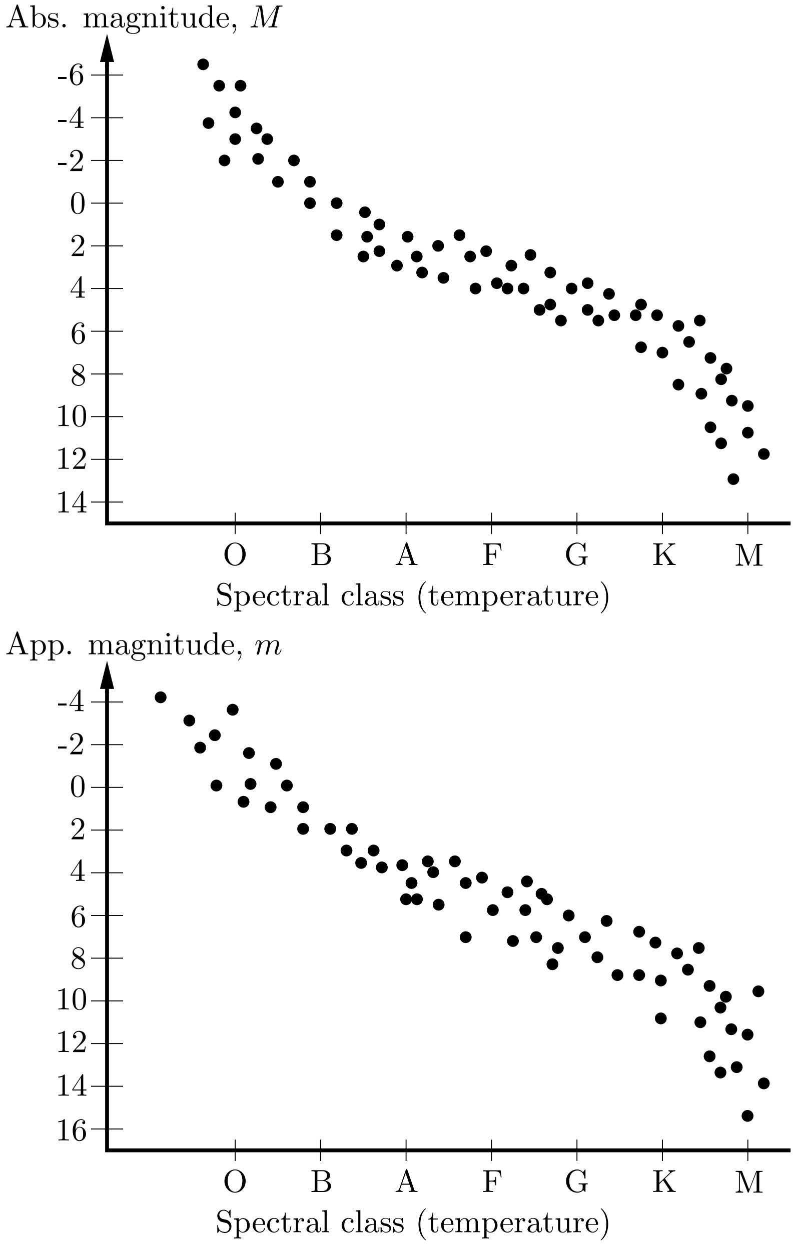

Figure 4: The HR-diagrams for the example exercise (note: spectral class is just a different measure of temperature). The upper plot shows the HR-diagram of a cluster with a known distance. Since the distance is known, we have been able to convert the apparent magnitudes to absolute magnitudes and we therefore plot absolute magnitudes on the y-axis. The lower plot is the HR-diagram of a cluster with unknown distance. Because of the unknown distance, we only have information about the apparent magnitude of the stars and therefore we now have apparent magnitude on the y-axis.

we know the absolute magnitude (luminosity) for an object, we can find its distance. But how do we know the absolute magnitude? There are a few classes of objects, called *standard candles*, which reveal their absolute magnitude in different ways. Examples of these 'standard candles' can be Cepheid stars or supernova explosions.

Another class of distance indicators are the so-called 'standard rulers'. The basis for the distance determination with standard rulers is the small-angle formula,

$$d = \theta r,$$

where d is the physical length of an object, r is the distance to the object and θ is the apparent angular extension (length) of the object. We can often measure the angular extension of an observed object. All that we need in order to find the distance is the physical length d. There are some objects for which we know the physical length. These objects are called *standard rulers*. For instance a special kind of galaxy which has been shown to always have the same dimensions could be used as a standard ruler.

3.1 Cepheid stars as distance indicators

Several stars show periodic changes in their apparent magnitudes. This was first thought to be caused by dark spots on a rotating star's surface: When the dark spots were turned towards us, the star appeared fainter, when the spots were turned away from us, the star appeared brighter. Today we know that these periodic variations in the star's magnitude is due to pulsations. The star is pulsating and therefore periodically changing its radius and surface temperature.

The Milky Way has two small satellite galaxies orbiting it, the Large and the Small Magellanic Cloud (LMC and SMC). They contain $10^9 - 10^{10}$ stars, less than one tenth of the number of stars in the Milky Way and are located at a distance of about 160 000 ly (LMC) and 200 000 ly (SMC) from the Sun. In 1908, Henrietta Leavitt at Harvard University discovered about 2400 of these pulsating stars in the SMC. The pulsation period of these stars were found to be in the range between 1 and 50 days. These stars were called *Cepheids* named after one of the first pulsating stars to be discovered, δ Cephei. She found a relationship between the stars' apparent magnitude and pulsation period. The shorter/longer the pulsation period, the fainter/brighter the star. Since all these stars were in the SMC they were all at roughly the same distance to us. We have seen above that for stars at the same distance, there is a constant difference $M - m$ in apparent and absolute magnitude. So the stars with a larger/smaller apparent magnitude also had a larger/smaller absolute magnitude. Since absolute magnitude is a measure of luminosity, what she had found was a *period-luminosity relation*.

Pulsating stars with higher luminosity were thus found to be pulsating with longer periods, pulsating stars with low luminosity were found to be pulsating with short periods. We can now reverse the argument: By measuring the period one can obtain the luminosity. There was one problem however: The method could not be calibrated as the distance to the SMC was unknown and therefore also the constant in $m - M =$ constant was unknown. Without this constant one cannot find M. One had to find Cepheids in our vicinity for which the distance was known. Only in that way could this constant and thus the relation between period and absolute magnitude be established.

Today the distance to several Cepheids in our galaxy are known by other methods. One of the most recent measurements of the constants in the period-luminosity relation came from the parallax measurements of several Cepheids by the Hipparcos satellite. The relation was found to be

$$M_V = -2.81 \log_{10} P_d - 1.43,$$

where P_d is the period in days. Here M_V is the absolute magnitude in the *Visual* part of the electromagnetic spectrum instead of the normal magnutide M which is based on the flux integrated over all wavelengths λ. Before describing in detail the difference between M and M_V, we will end our discussion on the Cepheid stars.

When pulsating stars were first used to measure distances one did not know that there are three different types of pulsating stars with different period-luminosity relations:

1. The classical Cepheids which belong to a class of giants, are very luminous stars. These are the most useful distance indicators for large distances because of their high luminosity.

2. W Virginis stars, or type II Cepheids are pulsating stars which on average have lower luminosity than the classical Cepheids.

3. RR Lyrae stars are small stars which usually have less mass than the Sun. Their luminosity is much lower than the luminosity of classical Cepheids and RR Lyrae stars are therfore less useful for distance determination at large distances. The advantage with RR Lyrae stars however, is that they are much more numerous than classical Cepheids.

When Edwin Hubble tried to estimate the distance to our neighbour galaxy Andromeda, he obtained a distance of about one million light years whereas the real distance is about twice as large. The reason for this error was that he observed W Virginis stars in Andromeda and applied the period-luminosity relation for classical Cepheids, thinking that they were the same. In this course we will mainly discuss the classical Cepheids.

Since Cepheids are very lumious (about 10^3 to 10^4 times higher luminosity than the Sun) they can be observed in distant galaxies. In order to determine the distance of a whole galaxy it suffices to find Cepheid stars in that galaxy and determine their distance. In this manner, the distance to several galaxies out to about 30 Mpc has been measured. Beyond 30 Mpc other methods need to be applied.

At the moment we will use the period-luminosity relation for Cepheids to determine distances without questioning why it works. When we come to the lectures on stellar structure we will study the physics behind these pulsations and see if we can deduce the period-luminosity relation theoretically by doing physics in the interior of stars.

We have now learned about our first distance indicator: We can find the absolute magnitude M_V at visual wavelength of Cepheids by observing their plusation period. Having the absolute magnitude M_V we can find the distance. We will now look at a different approach to find M_V for a distant object, but first we will discuss some extended definitions of magnitudes.

3.2 Magnitudes and color indices

Looking back at the definition of absolute magnitude, we see that we can write the absolute magnitude M as

$$M = M^{\mathrm{ref}} - 2.5 \log_{10} \left(\frac{F(10 \text{ pc})}{F^{\mathrm{ref}}(10 \text{ pc})} \right) = M^{\mathrm{ref}} - 2.5 \log_{10} \left(\frac{L}{L^{\mathrm{ref}}} \right),$$

where M_{ref} and F_{ref} are the absolute magnitude and flux (observed flux if the distance had been 10 pc) of a reference star used for calibration (as we have seen before, the star Vega with its magnitude defined to be 0, has often been used for this purpose). The flux is here the total flux of the star integrated over all wavelengths

$$F = \int_0^\infty F(\lambda) d\lambda. \tag{3}$$

The magnitude M which is based on flux integrated over all wavelengths is called *the bolometric magnitude*.

The *visual magnitude* M_V on the other hand, is based on the flux over a wavelength region defined by a *filter function* $S_V(\lambda)$. The filter function is a function which is centered at $\lambda = 550$ nm with an effective bandwidth of 89 nm. The flux F_V which is used instead of F to define visual magnitude can be written as

$$F_V = \int_0^\infty F(\lambda) S_V(\lambda) d\lambda.$$

Compare with expression (3): The main difference is that a limited wavelength range is selected by $S_V(\lambda)$. The magnitude is then defined as

$$M_V = M_V^{\mathrm{ref}} - 2.5 \log_{10} \left(\frac{F_V}{F_V^{\mathrm{ref}}} \right).$$

As for the bolometric magnitude, the relation between absolute and apparent visual magnitude is also given by

$$M_V - m_V = -5\log_{10}\left(\frac{r}{10\ \mathrm{pc}}\right).$$

The concept of the visual magnitude originates from the fact that detectors normally do not observe the flux over all wavelengths. Instead detectors are centered on a given wavelength and integrate over wavelengths around this center wavelength in a given *bandwidth*. There are three of these filters which are in common use:

- U-filter (ultraviolet), $\lambda_0 = 365$ nm, $\Delta\lambda_{FWHM} = 68$ nm
- B-filter (blue), $\lambda_0 = 440$ nm, $\Delta\lambda_{FWHM} = 98$ nm
- V-filter (visual), $\lambda_0 = 550$ nm, $\Delta\lambda_{FWHM} = 89$ nm

The absolute magnitudes M_V, M_B and M_U are used to define *color indices*. These color indices $(U - B)$ and $(B - V)$ are defined as

$$U - B = M_U - M_B = m_U - m_B,$$
$$B - V = M_B - M_V = m_B - m_V.$$

Note that these indices are written as a difference in apparent or absolute magnitudes: The color indices are independent of distance and will therefore give the same results if they are obtained using apparent magnitudes or absolute magnitudes (check that you can show this mathematically!). These indices are used to measure several properties of a star related to its color. The period-luminosity relation for a Cepheid can be improved using information about its color in terms of the $(B - V)$ color index as

$$M_V = -3.53\log_{10} P_d - 2.13 + 2.13(B - V).$$

For Cepheids, the $B-V$ color index is usually in the range 0.4 to 1.1. Thus, a more exact M_V and thereby a more exact distance (using relation (2)) can be obtained using the additional distance independent information contained in the color of the star. It suffices to observe the star with three color filters instead of one to obtain this additional information.

3.3 Supernovae as distance indicators

One of the most energetic events in the Universe are the Supernova explosions. In such an explosion, one star might emit more energy than the total energy emitted by all the stars in a galaxy. For this reason, supernova explosions can be seen at very large distances. The last confirmed supernova in the Milky way was seen in 1604 and was studied by Kepler. It reached an appaerent magnitude of about -2.5, similar to Jupiter at its brightest. There have been other reports of supernovae in the Milky way during the last 2000–3000 years, both in Europe and Asia. Some of these were so bright that they were seen clearly in the sky during daylight. Written material from Europe, Asia and the middle East all report about a supernova in 1006 which was so bright that one could use it to read at night time. The nearest supernova in modern times, called SN1987A, was observed in 1987 in the Large Magellanic Cloud at a distance of 51 kpc. It was visible by the naked eye from the southern hemisphere.

Supernovae can be classified as type I or type II,

1. Type I supernovae: These explosions show no hydrogen lines. There are three sub types, defined according to their spectra: Type Ia, Ib and Ic.

2. Type II supernovae: These are explosions with strong hydrogen lines. Type II supernova have several properties in common with type Ib and Ic.

It is now clear that supernovae of type Ib, Ic and II are *core collapse supernovae*. This is a star ending its life in a huge explosion, leaving behind a neutron star or a black hole. In the lectures on stellar evolution we will come back to the details of a core collapse supernova. Type Ia supernovae are usually brighter. These have the property which is desirable for a standard candle: Their luminosity is relatively constant and there is a receipe for finding their exact luminosity. The origin of type Ia supernovae are still under discussion, but according to the most popular hypothesis, the explosion occurs in a white dwarf star which has a binary companion. A white dwarf star is the result of one of the possible ways that a star can end its life: in the form of a very compact star consisting mainly of carbon and oxygen which are the end products from the nuclear fusion processes taking place in the final phase of a star's life. If a white dwarf is part of a binary star system (two stars orbitting a common center of mass), the white dwarf may start accreating material from the other star. At a certain point, the increased pressure and temperature from the accreted material may reignite fusion processes in the core of the white dwarf. This is the cause of the explosion. We will again defer details about the process to later lectures.

It can be shown that this explosion occurs when the mass of the white dwarf is close to the so-called *Chandrasekhar limit* which is about $1.4M_\odot$. Since the mass of the exploding star is always very similar, the luminosity of the explosions will also be very similar. The absolute magnitude of a type Ia supernova is $M_V \approx M_B \approx= -19.3$ with a spread of about 0.3 magnitudes. A more exact estimate of the absolute magnitude of a supernova may be obtained by the light curve. After reaching maximum magnitude, the supernova fades off during days, weeks or months. By observing the rate at which the supernova fades, one can determine the absolute magnitude of the supernova at its brightest.

Again, here we will only use the fact that the absolute magnitude of type

Ia supernovae can be obtained from its light curve in order to determine distances. More details about the physical processes giving rise to the explosion and to the fact that the light curve can be used to obtain the luminosity will be presented in later lectures. Supernovae can be used to determine distances to galaxies beyond 1000 Mpc.

3.4 The Tully-Fisher relation

The Tully-Fisher relation is a relation between the width of the 21 cm line of a galaxy and its absolute magnitude. As we remember, the 21 cm radiation is radiation from neutral hydrogen (look back at the lecture on electromagnetic radiation). Spiral galaxies have large quantities of neutral hydrogen and therefore emit 21 cm radiation from the whole disc. The 21 cm line is wide because of Doppler shifts: Hydrogen gas at different distances from the center of the galaxy orbits the center at different speeds giving rise to several different Doppler shifts. We also remember that the rotation curve for galaxies towards the edge of the galaxy was flat. So, gas clouds orbiting the galactic center at large distances all have the same orbital velocity $v_{\max}$ and thus the same Doppler shift. There are therefore many more gas clouds with velocity $v_{\max}$ than with any other velocity. The flux at the wavelength corresponding to the Doppler shift

$$\frac{\Delta\lambda_{\max}}{\lambda_0} = \frac{v_{\max}}{c},$$

is therefore larger than for instance at a wavelength of 21 cm itself. The result is a peak in the flux of the spectral line at either side of 21 cm at the wavelength $21 \pm \Delta\lambda_{\max}$ cm. The wavelength of this peak is a measure of the maximal velocity in the rotation curve:

$$v_{\max} = c\frac{\Delta\lambda_{\max}}{\lambda_0}.$$

We have seen in a previous lecture that the maximum velocity is related to the total mass of the galaxy. The higher the maximum velocity, the higher the mass (why?). If we assume that a higher total mass also means a higher content of lumious matter and therefore a higher luminosity, it is not difficult to immagine that a relation can be found between the maximal speed measured from the 21 cm line and the luminosity, or absolute magnitude of the galaxy. The relation can be written as

$$M_B = C_1 \log_{10} v_{\max} + C_2,$$

where M_B is the absolute magnitude at blue wavelenghts and C_1 and C_2 are constants depending on the type of spiral galaxy. The constant C_1 is normally in the range -9 to -10 and C_2 in the range 2.7 to 3.3. The Tully-Fisher relation can be used as a distance indicator out to distances beyond 100 Mpc.

3.5 Other distance indicators

Some other distance indicators:

- The globular cluster luminosity function: Globular clusters are clusters of stars containing a few 100 000 stars. These clusters are usually orbiting a galaxy. A galaxy has typically a few hundred globular clusters orbiting. It has been found that the luminosity function, i.e. the percentage of globular clusters with a given luminosity, is similar for all galaxies. By finding this luminosity function for galaxies with a known distance, the globular clusters can be used as distance indicators for other galaxies.
- The planetary nebula luminosity function: Planetary nebulae (which have nothing to do with planets) are clouds composed of gas which dying stars ejected at the end of their lifetime. The planetary nebulae have a known luminosity function which can be used as distance indicators for distant galaxies.
- The brightest galaxies in clusters: It has been found that the brightest galaxies in clusters of galaxies have a very similar absolute magnitude in all clusters. They can therefore be used as distance indicators to clusters of galaxies.

4 The Hubble law

At the top of the distance ladder, we find the Hubble law. Edwin Hubble discovered in 1926 that all remote galaxies are moving away from us. The furter away the galaxy, the faster it was moving away from us. This has later been found to be due to the expansion of the Universe: The galaxies are not moving away from us, the space between us and distant galaxies is expanding inducing a Doppler shift similar to that induced by a moving galaxy. Waves emitted by an object moving away from us have larger wavelengths than in the rest frame of the emitter. Thus, light from distant galaxies are red shifted. By measuring the red shift of distant galaxies, we can measure their velocities, or in reality the speed with which the distance is increasing due to the expanstion of space. From this velocity we can find their distance using the Hubble law

$$v = H_0 r,$$

where r is the distance to the galaxy, $H_0 \approx 71$ km/s/Mpc is the Hubble constant and v is the velocity measured by the redshift.

$$v = c\frac{\Delta\lambda}{\lambda}.$$

The Hubble law is only valid for large distances. We will come back to the Hubble law and its consequences in the lectures on cosmology.

5 Uncertainties in distance measurements

There are several uncertainties connected with distance measurements. One of the main problems is caused by *interstellar extinction*. Our galaxy contains huge clouds of dust between the stars. Light which passes through these dust clouds loose flux as

$$F(\lambda) = F_0(\lambda)e^{-\tau(\lambda)}, \tag{4}$$

where $F(\lambda)$ is the observed flux and $F_0(\lambda)$ is the flux we would have observed had there not been any dust clouds between us and the emitting object. Finally, the quantity $\tau(\lambda)$ is called the *optical depth* and is given by

$$\tau(\lambda) = \int_0^r dr'n(r')\sigma(\lambda, r').$$

Here $n(r)$ is the number density of dust grains at distance r from us and $\sigma(\lambda, r)$ is a measure of the probability for a photon to be scattered on a dust grain. The optical depth is simply an integral along the line of sight from us to the emitting object of the density of dust grains times the probability of scattering. The larger the density of dust grains or the larger the probability of scattering, the larger the optical depth. The optical depth is a measure of how many photons which are scattered away during the trip from the radiation source to us. If the scattering probability is constant along the line of sight (this depends on properties of the dust grains), we can write the optical depths as

$$\tau(\lambda) = \sigma(\lambda)\int_0^r dr'n(r') = N(r)\sigma(\lambda),$$

where $N(r)$ is the total number of dust grains that the photons encounter during the trip from the emitter at distance r.

Interstellar extinction increases the apparent magnitude (decreases the flux) of an object. Photons are scattered away from the line of sight and the objects appear dimmer. Taking this into account we need to correct our formula for the relation between the apparent and the absolute magnitude

$$m(\lambda) = M(\lambda) + 5\log_{10}\left(\frac{r}{10 \text{ pc}}\right) + A(\lambda),$$

where $A(\lambda)$ is the total extinction at wavelength λ and $m(\lambda)$ and $M(\lambda)$ are the apparent and absolute magnitudes based on the flux at wavelength λ only. Using the formula for the difference between two apparent magnitudes in lecture 6, we can write the change in apparent magnitude due to extinction as

$$m(\lambda) - m_0(\lambda) = -2.5\log_{10}\left(\frac{F(\lambda)}{F_0(\lambda)}\right) = -2.5\log_{10}(e^{-\tau(\lambda)}) = 1.086\tau(\lambda),$$

where also equation (4) was used (check that you can deduce this formula!). Here $m_0(\lambda)$ and $F_0(\lambda)$ is the apparent magnitude and flux we would have had if there hadn't been any extinction. Thus, we see that

$$m(\lambda) = M(\lambda) + 5\log_{10}\left(\frac{r}{10\ \text{pc}}\right) + 1.086\tau(\lambda).$$

Clearly, if we use a distance indicator and do not take into account interstellar extinction, we obtain the wrong distance. It is often difficult to know the exact optical depth from scattering on dust grains. This is an important source of error in distance measurements. Note that the extinction does not only increase the apparent magnitude of an object, it also changes the color. We have seen that the optical depth $\tau(\lambda)$ depends on wavelength λ. The scattering on dust grains is larger on smaller wavelength. Thus, it affects red light less than blue light with the result that the light from the object appears redder. This is called *interstellar reddening*.

Another source of error in the measurement of large distances in the Universe is the fact that objects observed at a large distance are also observed at an earlier phase in the history of the universe. The light from an object at a distance of 1000 Mpc or 3260 million light years has travelled for 3260 million years or roughly one fourth of the lifetime of the Universe. Thus, we observe this object as it was 3260 millions years ago. The universe has been evolving all the time since the Big Bang until today. We do not know if the galaxies and stars at this early epoch had the same properties as they have today. Actually, we have good reasons to believe that they did not. We will come to this later. This could imply that for instance the relation between light curve and absolute magnitudes of supernovae were different at that time than today. Using relations obtained from obervations of the present day universe to observations in the younger universe may lead to errors in measurements of the distance.

6 Problems

Problem 1 (30 min.–1 hour)

1. A star is observed to change its angular position with respect to very distant stars by $1''$ in half a year. Assuming that the star does not have any peculiar velocity with respect to us, what is the parallax angle for the star? And its distance?

2. What is the parallax angle for our nearest star Proxima Centauri at a distance of 4.22 ly? (Assume again that the observations are made with a distance of half a year).

3. An open star cluster is observed to have red stars (surface temperature $3000 - 4000$ K) with apparent magnitudes in the range

$m = [10, 12]$, yellow stars (about 6000 K) in the apparent magnitude range $m = [6, 9]$ and a few hotter white stars (10000 K) in the apparent magnitude range $m = [1, 5]$. What is the distance to the cluster? Use the diagram in figure 3.

4. A supernova explosion of type Ia is detected today in a distant galaxy. Its apparent visual magnitude at maximum was $m_V = 20$. You still need to wait a few days to obtain the light curve and thereby the exact absolute magnitude. But you can already find an approximate distance. In which distance range do you expect to find the supernova?

5. A distant galaxy is measured to have the center of its 21 cm line ($\lambda_0 = 21.2$ cm) shifted to $\lambda = 29.7$ cm. What is the distance of the galaxy?

6. If the dust optical depth to the open cluster discussed in the above problem is $\tau = 0.2$, what is the real distance to the cluster. How large error did you do not taking into account galactic extinction?

7. What about the supernova: Let us assume that the dust optical depth to the supernova was $\tau = 1$. How large error did you get in your distance measurement?

AST1100 Lecture Notes

13–14 Stars and stellar birth

1 The Hertzsprung-Russell diagram revisited

We have already encountered the Hertzsprung-Russell (HR) diagram, the diagram where stars are plotted according to their temperature and luminosity. There are several versions of this diagram, differing mainly in the units plotted on the axes. The most used units on the x-axis are:

- Temperature
- B-V color index
- spectral classes

We have so far seen temperature on the x-axis. The temperature of a star is directly related to its color and one can therefore also use the B–V color index (see the lecture on cosmic distances) on the x-axis. There is also another another possibility: *spectral classes*. Stars are classified according to their spectral class which consists of a letter and a number. This historical classification is based on the strength of different spectral lines found in the spectra of the stars. It turned out later that these spectral classes are strongly related to the temperature of the star: The temperature of the star determines the state of the different atoms and therefore the possible spectral lines which can be created.

The letters used in the spectral classification are, in the order of decreasing temperature, O, B, A, F, G, K, M. The warmest O stars have surface temperatures around 40 000 K, the coldest M stars have surface temperatures down to about 2 500 K. Each of these classes are divided into 10 subclasses using a number from 0 to 9. So the warmest F stars are called F0 and the coldest F stars are called F9.

Normally observational astronomers tend to use either spectral class or color index which are quantities related to the observed properties of the star. Theoretical astrophysicists on the other hand, tend to use temperature which is more important when describing the physics of the star.

Also the y-axis in an HR-diagram have different units. We have already seen luminosity and absolute magnitude which are two closely related quantities. In addition one can use *luminosity classes*. It turns out that stars which have the same spectral class but different luminosities also have some small differences in the spectral lines. These differences have been shown to depend on the luminosity of the star. There are 6 luminosity classes, numbered with Roman numerals from I to VI. The most luminous stars have luminosity class I. Using this classification, the Sun is a G2V star.

Before we start to discuss the diagram in more detail, let us try to understand what it is telling us. We know that the flux of a star with temperature T can be expressed using the Stefan-Boltzmann law as $F = \sigma T^4$. To obtain the luminosity L, we need to integrate this flux over the full area $4\pi R^2$ of the surface of the star giving (why?, check that you understand this!),

$$L = 4\pi R^2 \sigma T^4.$$

Looking at the HR-diagram (see figure 1), we see that there are some spectral classes for which there are stars with many different luminosities. For instance stars with spectral class K0 have a range in luminosity from 0.5 to 1000 solar luminosities. If we fix T in the relation above (remember: fixed T means fixed spectral class) , we see than higher luminosity simply means larger radius. So for a fixed temperature, the higher the star is located in the HR-diagram the larger radius it has. This also means that we can find lines of constant radius in the diagram. Fixing the radius to a constant we get

$$R^2 = \frac{L}{4\pi\sigma T^4} = \text{constant},$$

so that for stars located along lines following $L \propto T^4$ in the diagram, the radius is the same. In figure 2 some of these lines have been plotted. Note that these lines go from the upper left to the lower right, a bit similar to the main sequence. So main sequence stars are stars which have a certain range of radii. The fact that most of the stars are located on the main sequence means that the physics of stars somehow prohibits smaller and larger radii (look at the figure again and check that you understand) . We will come to this in some more detail later.

Now it is clear why the stars which are situated above the main sequence are called giants or super giants and the stars well below the main sequence are called dwarfs. Main sequence stars usually have radii in the range $0.1R_\odot$ to about $10R_\odot$. Giant stars fall in the range between $10R_\odot$ to about $100R_\odot$ whereas super giants may have radii of several 100 solar radii. The masses of stars range from $0.08M_\odot$ for the least massive stars up to about $100M_\odot$ for the most massive stars. We will later discuss theoretical arguments explaining why there is a lower and an upper limit of star masses.

We will now start to look at the evolution of stars, from birth to death. Stars start out as huge clouds of gas contracting due to their own gravity.

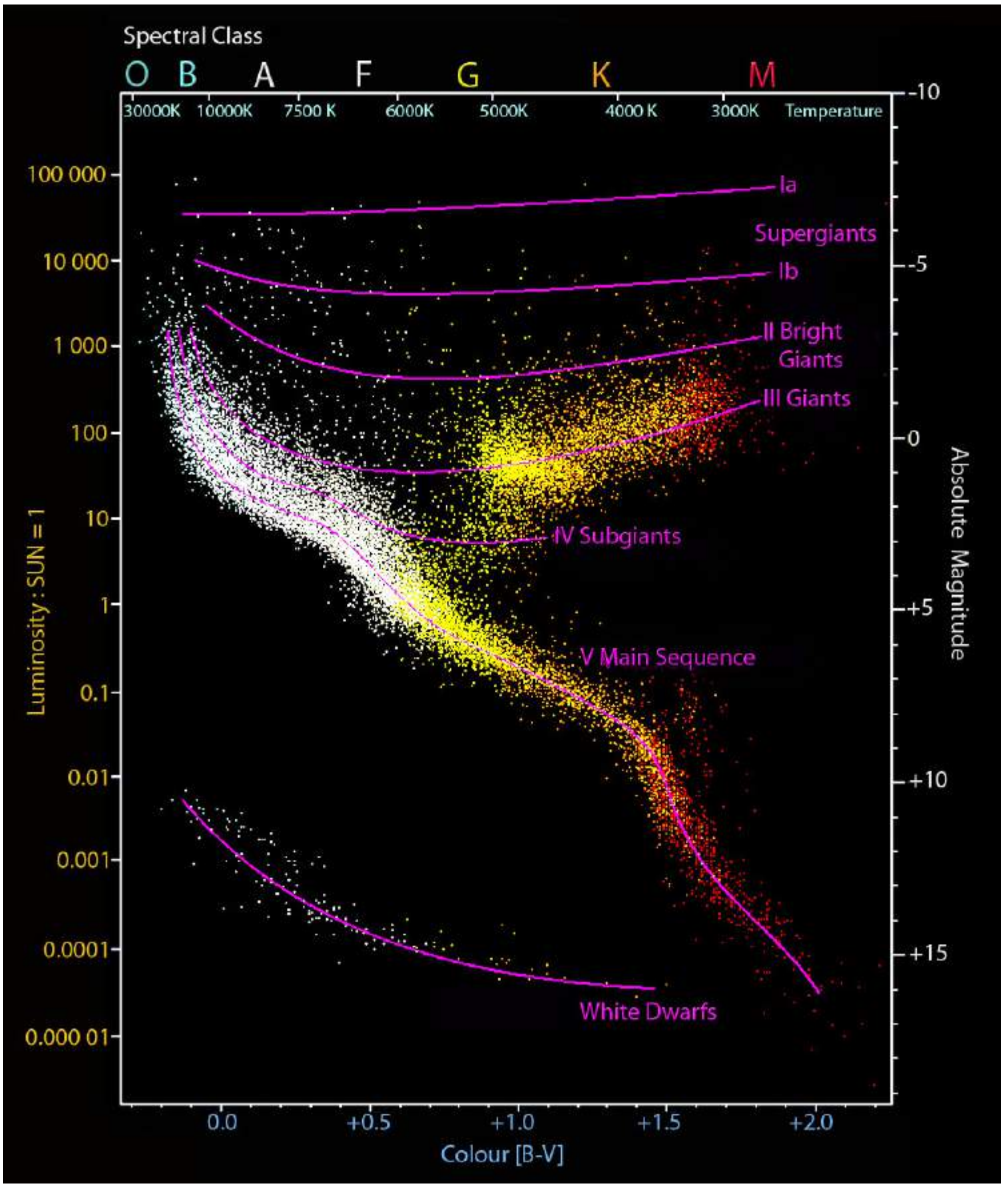

Figure 1: HertzsprungRussell diagram with 22 000 stars plotted from the Hipparcos catalog and 1000 from the Gliese catalog of nearby stars. Stars tend to fall only into certain regions of the diagram. The most predominant is the diagonal, going from the upper-left (hot and bright) to the lower-right (cooler and less bright), called the main sequence. White dwarfs are found in the lower-left, while subgiants, giants, and supergiants are located above the main sequence. The Sun is found on the main sequence at absolute magnitude 4.8 (relative luminosity 1) and BV color index 0.66 (temperature 5780 K, spectral type G2). (Figure:Wikipedia)

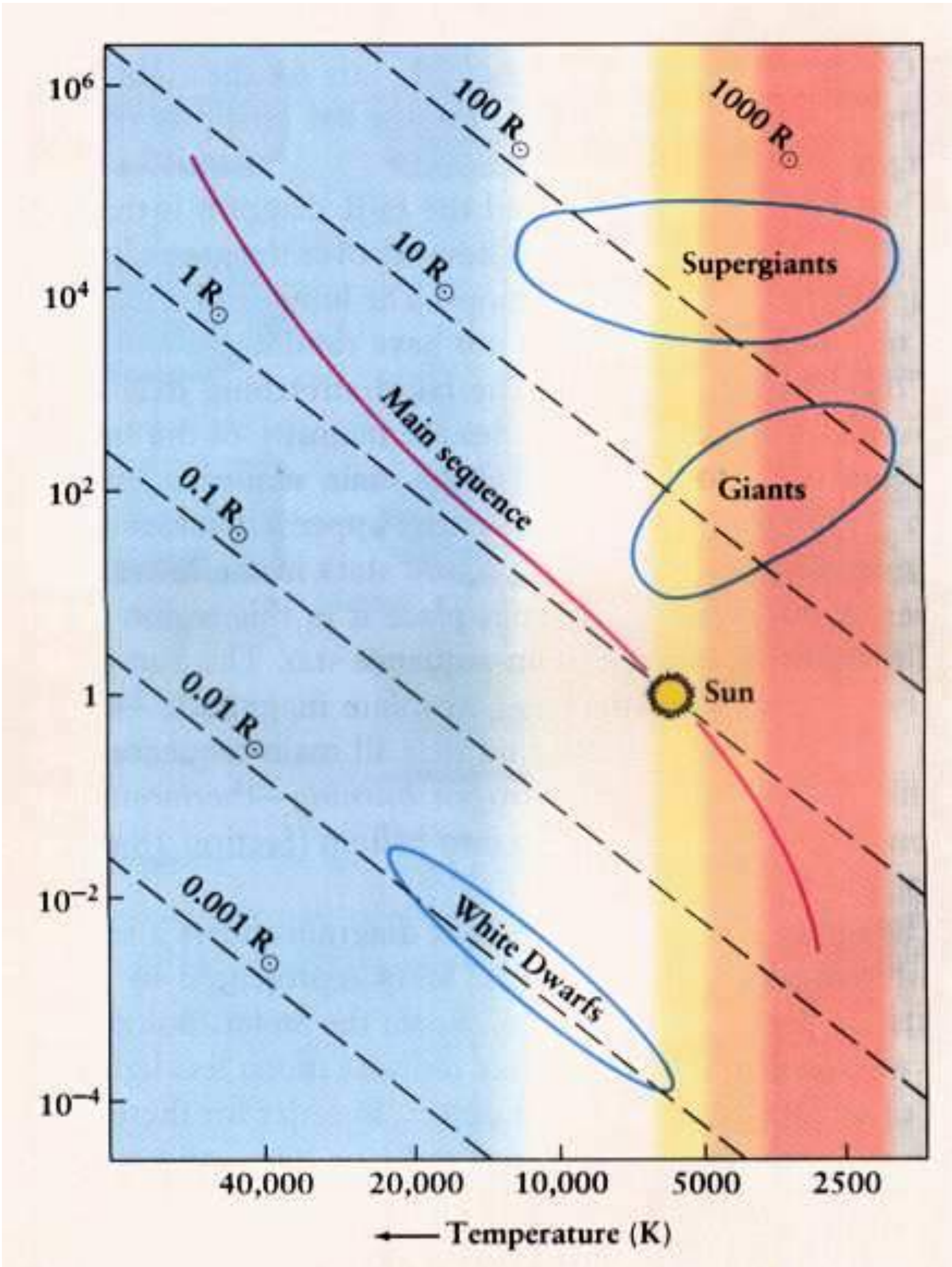

Figure 2: HR-diagram with constant radii lines plotted.
From `http://astro.wsu.edu/worthey/astro/html/im-Galaxy/`

Thus a star starts out on the far right side of the HR-diagram, with a very low temperature. Then, as it contracts, the radius decreases and the temperature increases. It moves leftwards and finally after nuclear reactions have begun, the star settles on the main sequence. Where it settles on the main sequence depends on the mass of the star. As we will show later, the larger the mass, the higher the luminosity and the higher the surface temperature. So the more massive stars settles on the left side of the main sequence whereas the less massive stars settles on the right side of the main sequence. Stars spend the largest part of their lives on the main sequence. During the time on the main sequence they move little in the HR-diagram. Towards the end of their lives, when the hydrogen in the core has been exhausted, the stars increase their radii several times becoming giants or supergiants. The surface temperature goes down, but due to the enormous increase in radius the luminosity increases. After a short time as a giant, the star dies: Low mass stars die silently, blowing off the outer layers and leaving behind a small white dwarf star in the lower part of the HR-diagram. The more massive stars die violently in a supernova explosion leaving behind a so-called neutron star or a black hole. We will now discuss the physics behind each of these steps in turn. Beginning here with star birth: a gas of cloud contracting.

2 The Jeans criterion

A star forms from a cloud of gas, a so-called *molecular cloud*, undergoing gravitational collapse. These molecular clouds consist mainly of atomic and molecular hydrogen, but also contain dust and even more complex organic molecules. The question is whether a cloud will start collapsing or not. In the lectures on the virial theorem we saw that the condition for stability is $2K + U = 0$. If the kinetic energy is larger compared to the potential energy, the system does not stabilize, the gas pressure is larger than the gravitational forces and the cloud expands. On the other hand, if the potential energy is dominating, the cloud is gravitationally bound and undergoes collapse. For a cloud to collapse we thus have the condition (why?),

$$2K < |U|.$$

In the lectures on the virial theorem, we found an expression for the potential energy of the cloud:

$$U \propto \frac{3GM^2}{5R},$$

where M is the mass of the cloud and R is the radius. From thermodynamics, we learn that the kinetic energy of a gas is given by

$$K = \frac{3}{2}NkT,$$

where N is the number of particles in the gas, k is the Boltzmann constant and T is the temperature. We can write N as

$$N = \frac{M}{\mu m_H}, \tag{1}$$

where $\overline{m} = \mu m_H$ is the mean mass per gas particle. The *mean molecular weight*

$$\mu = \frac{\overline{m}}{m_H},$$

is simply the mean mass per particle measured in units of the hydrogen mass m_H (check now that expression 1 for N makes sense to you! This is important!). So the condition $2K < |U|$ becomes simply

$$\frac{3MkT}{\mu m_H} < \frac{3GM^2}{5R}.$$

We can write this as a criterion on the mass

$$M > \frac{5kT}{G\mu m_H} R.$$

This minimum mass is called the *Jeans mass* M_J which we can write in terms of the mean density of the cloud as

The Jeans mass

$$M_J = \left(\frac{5kT}{G\mu m_H}\right)^{3/2} \left(\frac{3}{4\pi\rho}\right)^{1/2},$$

where we used $\rho = M/((4/3)\pi R^3)$ assuming constant density throughout the cloud. Thus, clouds with a larger mass than the Jeans mass $M > M_J$ will have $2K < |U|$ and therefore start a gravitational collapse. We can also write this in terms of a criterion on the radius of the cloud. Using again the expression for the density we have the *Jeans length* (check again that you can deduce this expression from the expression above).

The Jeans length

$$R_J = \left(\frac{15kT}{4\pi G\mu m_H \rho}\right)^{1/2}.$$

A cloud with a larger radius than the Jeans length $R > R_J$ will undergo gravitational collapse. The Jeans criterion for the collapse of a cloud is a good approximation in the absence of rotation, turbulence and magnetic fields. In reality however, all these factors do contribute and far more complicated considerations are needed in order to calculate the exact criterion.

The collapsing cloud will initially be in free fall, a period when the photons generated by the converted potential energy are radiated away without heating the cloud (the density of the cloud is so low that the photons can easily escape without colliding with the atoms/molecules in the gas). The initial temperature of the cloud of about $T = 10 - 100$ K will not increase. After about one million years, the density of the cloud has increased and the photons cannot easily escape. They start heating the cloud and potential energy is now radiated away as thermal radiation. In the lectures on the virial theorem we made an approximate calculation of the time it would take the Sun to collapse to its present size assuming a constant luminosity. We found a collapse time of about 10 million years. Proper calculations show that this process would take about 40 million years for a star similar to the Sun. The contracting star is called a *protostar*.

When the core of the collapsing protostar has reached sufficiently high temperatures, thermonuclear fusion begins in the center. The luminosity starts to get dominated by the energy produced by nuclear fusion rather than converted potential energy from the gravitational collapse. The protostar keeps contracting until hydrostatic equilibrium is reached and the star has entered the main sequence.

3 Settling on the main sequence: Hydrostatic equilibrium

In figure 4 we show a mass element with mass dm inside a star at a radius r from the center. We know that gravity pulls this element towards the center. But a main sequence star does not change its radius with time, so there must be a force working in the opposite direction keeping this mass element stable at radius r. This force is the pressure. In a main sequence star, the pressure forces must exactly equal the force of gravity, otherwise the star would change its radius. This fact, called *hydrostatic equilibrium*, gives us an invaluable source of information about a star's interior. We can't observe the interior of a star directly, but the equation of hydrostatic equilibrium together with other thermodynamic relations combined with observations of the star's surface allow detailed computer modeling of the interior of stars. Here we will deduce this important equation.

In figure 5 we have zoomed in on the mass element dm. Because of the symmetry of the problem (the fact that gravitation only works radially), we can assume spherical symmetry, i.e. that density, pressure and temperature are all only a function only of the distance r from the center. We show the forces of pressure pushing on the mass element from above and below, as well as the force of gravity. Assuming that the element is infinitesimally small, there are no gravitational forces pushing on the sides and the pressure forces on the sides will be equal since the distance r from the center is the same on both sides. The forces on the sides must therefore sum up to zero. We will now look at a possible radial movement

Figure 3: Info-figure: A close-up of one of the famous "Pillars of Creation" in the Eagle Nebula (M16), a nearby star-forming region some 2000 pc away in the constellation Serpens. This pillar of cool interstellar hydrogen gas and dust is roughly 4 light-years long and protrudes from the interior wall of a dark molecular cloud. As it is slowly eroded away by strong ultraviolet light from nearby stars, small globules of even denser gas buried within the pillar are uncovered. These globules are most easily seen at the top of the pillar. They are dense enough to collapse under their own gravity, forming young stars and possibly planetary systems. This color image is constructed from three separate images taken through filters specially designed to isolate the light from different gases. Red shows emission from singly-ionized sulfur atoms, green shows emission from hydrogen, and blue shows light emitted by doubly-ionized oxygen atoms. (Figure: NASA, ESA, STScI, J. Hester and P. Scowen)

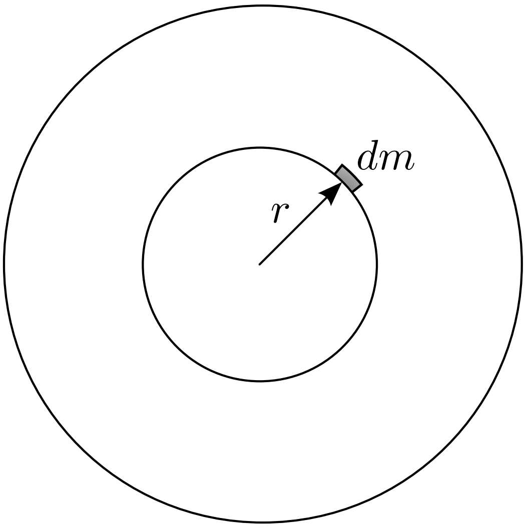

Figure 4: The mass element dm inside a main sequence star is not moving.

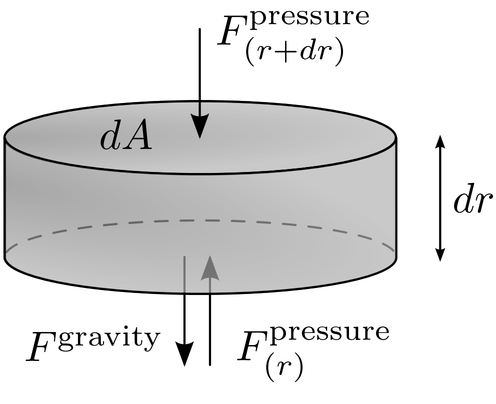

Figure 5: The mass element dm inside a main sequence star is not moving: The forces add to zero.

of the mass element. Newton's second law on the mass element gives

$$dm\frac{d^2r}{dt^2} = -F^{\text{grav}} - F^{\text{pressure}}(r+dr) + F^{\text{pressure}}(r),$$

where all forces are defined to be positive. The minus sign on the two first forces show that they push towards the center in negative r direction. The area of the upper and lower sides of the element is dA. Pressure is defined as force per area, so

$$P = \frac{F^{\text{pressure}}}{dA},$$

giving

$$dm\frac{d^2r}{dt^2} = -G\frac{M(r)dm}{r^2} - P(r+dr)dA + P(r)dA,$$

(check that you understand where each term comes from here) where $M(r)$ is the total mass inside radius r:

$$M(r) = \int_0^r dr' 4\pi(r')^2\rho(r') \qquad (2)$$

The infinitesimal difference in pressure between r and $r+dr$ is $dP = P(r+dr) - P(r)$. We have

$$\frac{dm}{dA}\frac{d^2r}{dt^2} = -\frac{dm}{dA}\frac{GM(r)}{r^2} - dP$$

We write the mass of the element as the density $\rho(r)$ at radius r times the volume $dA\,dr$ of the mass element $dm = \rho dA\,dr$. Dividing by dr on both sides gives

$$\rho\frac{d^2r}{dt^2} = -G\frac{\rho M(r)}{r^2} - \frac{dP}{dr}.$$

(Did you understand all parts of the deduction?) For a main sequence star, the radius is not changing so the mass element cannot have any acceleration in r direction giving $d^2r/dt^2 = 0$. This gives the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho(r)g(r),$$

where $g(r)$ is the local gravitational acceleration

$$g(r) = G\frac{M(r)}{r^2}.$$

The equation of hydrostatic equilibrium tells us how the pressure $P(r)$ must change as a function of radius in order for the star to be stable. In the following we will study what kind of pressure we might experience inside a star and which effect it has.

From thermodynamics we learn that the gas pressure in an *ideal gas* can be written as

Ideal gas law

$$P = \frac{\rho k T}{\mu m_H}.$$

An ideal gas is a gas where the atoms or molecules of which the gas consists do not interact with each other. This is not the case in real gases but often a good approximation. In the stellar interior, there is a high density of photons traveling in all possible directions. The photons behave like the atoms or molecules in a gas. So we may consider the collection of photons as a *photon gas*. This photon gas also has a pressure in the same way as a normal gas has. Thermodynamics tells us that the pressure of a photon gas is given by

$$P_{\mathrm{r}} = \frac{1}{3} a T^4,$$

where $a = 7.56 \times 10^{-16}$ J/m^{3}K^4 is the *radiation constant*.

4 Problems

Problem 1 (10–20 min.)

Look at the HR-diagram in figure 1. Assume that you observe a main sequence star with spectral class G0. The apparent magnitude of the star is $m = 1$.

1. Roughly what luminosity and absolute magnitude would you expect the star to have? (use the diagram)
2. Using this result, can you give a rough approximation of the distance?
3. Looking again at the HR-diagram. Roughly what is the minimum and maximum absolute magnitude you would expect the star to have?
4. What is the range of distances the star could have?

This method for measuring distances is called *spectroscopic parallax* (although it has nothing to do with normal parallax). I have not included this method in the lectures on distance measurements. From the answer to the last question you will understand why it is not a very exact method.

Problem 2 (30–45 min.)

A Giant Molecular Cloud (GMC) has typically a temperature of $T = 10$ K and a density of about $\rho = 3 \times 10^{-17}$ kg/m^3. A GMC has been observed at a distance of $r = 200$ pc. It's angular extension on the sky is $3.5'$. Assume the cloud to be spherical with uniform density.

1. What is the actual radius of the cloud?
2. What is the mass of the cloud?
3. Is the mass larger than the Jeans mass? Is the cloud about to collapse and form a protostar?
4. A supernova explodes in the vicinity of the star emitting a pressure wave which passes through the cloud. If an external pressure is pushing the cloud together, could this possibly lead to a decrease in the minimum mass required for collapse (give arguments in terms of K and U)? Argue why a decrease in minimum mass is more probable than an increase. (Hint: does K really increase for all particles when you compress the cloud?).
5. Could the supernova thus have contributed to the collapse of a cloud which has a mass less than the Jeans mass?
6. The galaxy has a fairly uniform distribution of hydrogen in the galactic disc. If a pressure wave is moving around the center of the disc in a spiral like shape, would this explain why we observe galaxies as spirals and not as a disc?

Problem 3 (2–3 hours)

We will now assume a very simple model of the Sun in order to show how one can use the equation of hydrostatic equilibrium to understand stellar interiors and the nuclear reactions taking place in the stellar cores. We will assume that the density of the Sun $\rho = \rho_0$ is uniform throughout.

1. Find an expression for the total mass $M(r)$ inside a radius r.
2. We will now assume that the only pressure in the Sun is the gas pressure from an ideal gas. We ignore the radiation pressure. Insert this expression for $M(r)$ into the equation of hydrostatic equilibrium and show that it can be written as
$$\frac{dT}{dr} = -\frac{4\pi}{3}G\rho_0 r\frac{\mu m_H}{k}$$
3. Integrate this equation from the core at $r = 0$ to the surface of the Sun at $r = R$ and show that the temperature T_c in the core of the Sun can be written
$$T_C = T(R) + \frac{2\pi}{3}GR^2\rho_0\frac{\mu m_H}{k}.$$
4. Assume that the Sun consists entirely of protons with a mass of 1.67×10^{-27} kg. Use the solar mass of 2×10^{30} kg, the solar radius of 700 000 km and the surface temperature of the Sun $T = 5780$ K to obtain the density ρ_0 and thereby the core temperature T_C. (By doing this calculation properly taking into account variations of the

density with distance from the core, one obtains a core temperature of about 15 million Kelvin)

5. In the coming lectures, we will learn that hydrogen can fuse to Helium by two different processes, the pp-chain and the CNO-cycle. The pp chain is more efficient at temperatures below 20 million Kelvin whereas the CNO-cycle starts dominating at temperatures above 20 million Kelvin. Use your result for the core temperature of the Sun to decide which of these processes produces most of the energy in the Sun.

6. Write ρ_0 in terms of the mass M and the radius R of the Sun. We have seen that the surface temperature of the Sun is much smaller than the core temperature and might therefore be neglected. Show that the core temperature of a star depends on the mass and radius as
$$T_C \propto \frac{M}{R}$$

7. In later lectures we will discuss in detail the evolution of a star. We will learn that when the Hydrogen in the core of a star has been exhausted, the nuclear fusion processes cease. In this case the pressure forces cannot sustain the force of gravity and the radius of the core starts shrinking. It will continue shrinking until some other force can oppose the force of gravity. If Helium, an element which is now found in large abundances in the core, starts to fuse to heavier elements this would create a photon pressure high enough to sustain gravity. A temperature of at least 100 million degrees Kelvin is needed in order for this fusion process to start. By how much does the core radius of the Sun need to shrink in order for Helium fusion to start?

8. In the last case, the radiation pressure is giving the dominant contribution to the forces of pressure. Show that in this case, the temperature of the core can be written as
$$T_C = \left(T(R)^4 + \frac{2\pi G}{a} \rho_0^2 R^2 \right)^{1/4},$$
again assuming a constant density.

Problem 4 (2–3 hours)

We will now assume a slightly more realistic model of the Sun. Assume that the density of the Sun as a function of distance r from the core can be written as
$$\rho(r) = \frac{\rho_C}{1 + (r/R)^2},$$
where ρ_C is the density in the core of the Sun and R is the radius at which the density has fallen by a factor 1/2 (check this by inserting $r = R$

in the expression). In this exercise we will use our knowledge about the minimum temperature which is needed to obtain nuclear reactions in order to calculate the density in the solar core.

1. We will now find an expression for the total mass $M(r)$ inside a radius r using this density profile. In order to perform the integral in equation (2) we make the substitution $x = r/R$ and integrate over x instead of r. Show that $M(r)$ can be written

$$M(r) = 4\pi\rho_C R^3 \int_0^{r/R} dx \frac{x^2}{1+x^2}$$

2. In order to perform such integrals, the Mathematica package is very useful. Not everybody has access to Mathematica, but a free web interface exists for performing integrals.

   Go to `http://integrals.wolfram.com/index.jsp`,

   type `x^2/(1+x^2)` and click "Compute online with *Mathematica*",

   and you get a nice and easy answer. Using this result, together with the assumption of pure ideal gas pressure, show that the equation of hydrostatic equilibrium can now be written

$$\frac{d}{dr}(\rho(r)T(r)) = -\frac{\mu m_H}{k} 4\pi G\rho_C^2 R^3 \frac{r/R - \arctan(r/R)}{r^2} \frac{1}{1+(r/R)^2}.$$

3. We now need to integrate this equation from radius 0 to an arbitrary radius r. Again the substitution $x = r/R$ is useful. Show that the equation of hydrostatic equilibrium now reads

$$\rho(r)T(r) - \rho_C T_C = -\frac{\mu m_H}{k} 4\pi G\rho_C^2 R^2 \int_0^{r/R} dx \left(\frac{1}{x(1+x^2)} - \frac{\arctan(x)}{x^2(1+x^2)}\right)$$

4. To solve this integral you need to use the 'Integrator' and type the following: `1/(x(1+x^2))` and `arctan(x)/(x^2(1+x^2))`.

   Using these results, show that the core temperature T_c can be written

$$T_C = T(r)/(1+x^2) + \frac{\mu m_H}{k} 4\pi G\rho_C R^2 \left(\frac{1}{2}(\arctan x)^2 + \frac{\arctan(x)}{x} - 1\right)$$

5. We will now try to obtain values for the central density ρ_C. In order to obtain that, we wish to get rid of x and r from the equation. When $x \to \infty$, that is, when going far from the center, show that the equation reduces to

$$T_C = \frac{\mu m_H}{k} 4\pi G\rho_C R^2 (\frac{\pi^2}{8} - 1)$$

6. Before continuing, we need to find a number for R, the distance from the center where the density has fallen by 1/2. Assume that considerations based on hydrodynamics and thermodynamics tell us that the core of the Sun extends out to about $0.2R_\odot$ and that the density has fallen to 10 percent of the central density at this radius. Using this information, show that

$$R = \frac{0.2R_\odot}{\sqrt{9}} \approx 0.067R_\odot.$$

7. We know that a minimum core temperature of about 15 million degrees is needed in order for thermonuclear fusion to be an efficient source of energy production. What is the minimum density in the center of the Sun? Assume the gas in the Sun to consist entirely of protons. Express the result in units of the mean density $\rho_0 = 1400$ kg/m^3 of the Sun. (More accurate calculations show that the core density of the Sun is about 100 times the mean density).

In the last two exercises we have used some very simplified models together with some rough assumptions and observed quantities to obtain knowledge about the density and temperature in the interior of the Sun. These exercises were made to show you the power of the equation of hydrostatic equilibrium: By combing this equation with the knowledge we have about the Sun from observations of its surface together with knowledge about nuclear physics, we are able to deduce several facts about the solar interior. In higher courses in astrophysics, you will also learn that there are more equations than the equation of hydrostatic equilibrium which must be satisfied in the solar interior. Most of these equations come from thermodynamics and fluid dynamics. In the real case, we thus have a set of equations for $T(r)$ and $\rho(r)$ enabling us to do *stellar model building*, without using too many assumptions we can obtain the density and temperature of stars at different distances from the center. These models have been used to obtain the understanding we have today of how stars evolve. Nevertheless many questions are still open and poorly understood. Particularly towards the end of a star's life, the density distribution and nuclear reactions in the stellar interior become very complicated and the equations become difficult to solve. But solving these equations is important in order to understand the details of supernova explosions.

AST1100 Lecture Notes

15–16: General Relativity Basic principles

1 Schwarzschild geometry

The general theory of relativity may be summarized in one equation, the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

where $G_{\mu\nu}$ is the Einstein tensor and $T_{\mu\nu}$ is the stress-energy tensor (A tensor is a matrix with particular properties in the same way as a 4-vector is a vector with specific properties). This equation is not a part of this course as tensor mathematics and linear algebra, not required for taking this course, are needed to understand it. I present it here anyway as it illustrates the basic principle of general relativity: The stress-energy tensor on the right hand side contains the energy content of spacetime, the Einstein tensor on the left hand side specifies the geometry of spacetime. Thus, general relativity says that the energy content in spacetime specifies its geometry.

What do we mean by geometry of spacetime? We have already seen two examples of such geometries, Euclidean geometry and Lorentz geometry. We have also seen that the geometry is specified by the spacetime interval (also called *line element*) Δs which tells us how distances are measured. Thus, by inserting the energy content as a function of spacetime coordinates on the right side, the left side gives us an expression for Δs, i.e. how to measure distances in spacetime in the presence of mass/energy. Thus, in the presence of a mass, for instance like the Earth, the geometry of spacetime is no longer Lorentz geometry and the laws of special relativity are no longer valid. This should be obvious: Special relativity tells us that a particle should follow a straight line in spacetime, i.e. a path with constant velocity. This is clearly not the case on Earth, objects do not keep a constant velocity, they accelerate with the gravitational acceleration.

You might object here: Special relativity says that a particle continues with constant velocity if it is not influenced by external forces, but here the force of gravity is at play. The answer to this objection is given by a very important concept of general relativity: *gravity is not a force*. What we experience as 'the force of gravity' is simply a result of the spacetime geometry in the vicinity of masses. The principle of maximal aging (go back and repeat it now!) tells us that a particle which is not influenced by external forces follows the longest path in spacetime, i.e. the path which gives the largest possible proper time. An object falls to the ground because the geometry of spacetime around a large mass like Earth is such, that when the object follows the path with the longest possible path length Δs, it falls to the ground. It does not continue in a straight line with constant velocity as it would in a spacetime with Lorentz geometry.

Few years after Einstein published the general theory of relativity, Karl Schwarzschild found a general solution to the Einstein equation for the geometry around an isolated spherically symmetric body. This is one of the very few analytic solutions to the Einstein equation that exists. Thus, the Schwarzschild solution is valid around a lonely star, planet or a black hole. The spacetime geometry resulting from this solution is called Schwarzschild geometry and is described by the line element:

The Schwarzschild line element

$$\Delta s^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)} - r^2\Delta\phi^2. \tag{1}$$

There are two things to note in this equation. First, we are using polar coordinates (r, ϕ) instead of Cartesian coordinates (x, y). This is a natural choice for a situation with a well defined center. These are not three dimensional coordinates: Symmetry allows us to describe the geometry on *any* plane passing through the center of the central massive body. Given two events with spacetime distance Δs as well as the position of the central mass, we have three points in space which define a plane on which we define the polar coordinates. Thus, the r coordinate is a 'distance' from the center, we will later come back to how we measure this distance. The ϕ coordinate is the normal ϕ angle used in polar coordinates. The line element for Lorentz geometry in polar coordinates can similarly be written as

Lorentz line element in polar coordinates

$$\Delta s^2 = \Delta t^2 - \Delta x^2 = \Delta t^2 - \Delta r^2 - r^2\Delta\phi^2.$$

The second thing to note in the equation for the Schwarzschild line element is the term $1-2M/r$. Here M/r must be dimensionless since it is added to a number. But we know that mass is measured in kilograms and distances in meters, so how can this term be dimensionless? Actually, there should have been a G/c^2 here, $G = 6.67\times10^{-11}$ m^3/kg/s^2 being the gravitational constant and $c = 3 \times 10^8$ m/s being the speed of light. We have that

$$\frac{G}{c^2} = 7.42 \times 10^{-28} \text{ m/kg}. \tag{2}$$

Since M/r has units kg/m, G/c^2 is clearly the constant which is missing here. We are now used to measure time intervals in units of meters. If we now decide to also measure mass in units of meters, equation (2) gives us a natural conversion factor.

$$\frac{M_{\mathrm{m}}}{M_{\mathrm{kg}}} = \frac{G}{c^2},$$

where M_{m} is mass measured in meters and M_{kg} is mass measured in kg.

Thus we have that

$$1 \text{ kg} = 7.42 \times 10^{-28} \text{ m}.$$

The equation gives us a conversion formula from kg to m. We see that measuring mass in meters equals setting $G/c^2 = 1$ everywhere in the formulas. This is equal to what happened when we decided to measure time in meters, we could set $c = 1$ everywhere. The reason for measuring mass in meters is pure laziness, it means that we don't need to write this factor all the time when doing calculations. Thus instead of writing $1 - 2M_{\mathrm{kg}}G/(rc^2)$ we write $1 - 2M/r$ where M is now mass measured in meters. All the physics is captured in the last expression, we have just got rid of a constant. From now on, all masses will be measured in units of meters and when we have the final answer we convert to normal units by multiplying or dividing by the necessary factors of G/c^2 and c in order to obtain the units that we wish.

2 The inertial frame

In the lectures on special relativity we defined inertial frames, or free-float frames, to be frames which are not accelerated, frames moving with constant velocity on which no external forces are acting. We can give a more general definition in the following way: To test if the room where you are sitting at the moment is an inertial frame, take an object, leave it at rest with zero velocity. If the object stays at rest with zero velocity, you are in an inertial frame. If you give the object a velocity v and the object continues in a straight line with velocity v, you are in an inertial frame. Clearly, a frame (a room) which is not accelerated on which no external forces work is an inertial frame according to this definition. But are there other examples? In general relativity we use the notion of a *local internal frame*, i.e. limited regions of spacetime which are inertial frames. An example of such a local inertial frame is a space craft in orbit around the Earth. Another example is an elevator for which all cables have broken so that it is freely falling. All freely falling frames can be local inertial frames. How do we now that? If an astronaut in the orbiting space craft

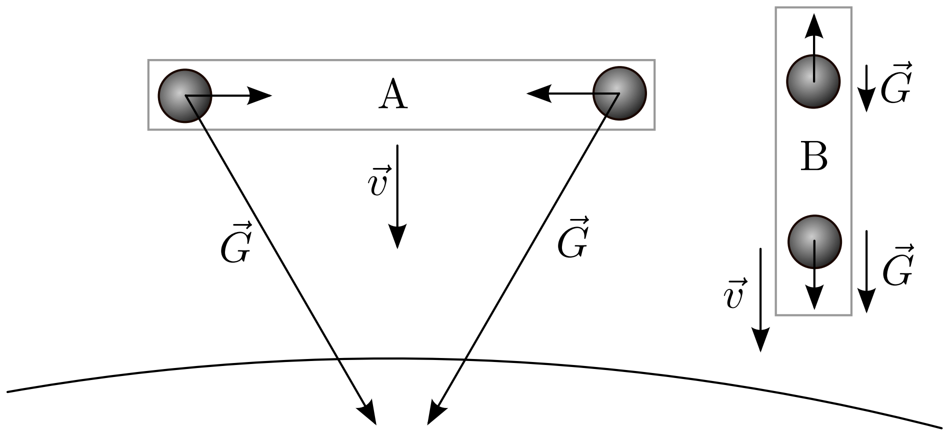

Figure 1: Two boxes in free fall: If they are large enough in either direction, the objects at rest inside the boxes will start moving. A local inertial frame needs to be small enough in space and time such that this motion cannot be measured.

takes an object and leaves it with zero velocity, it stays with zero velocity. This is why the astronauts experience weightlessness. If a person in a freely falling elevator takes an object and leaves it at rest, it stays at rest. Also the person in the elevator experiences weightlessness. Thus, they are both, within certain limits, in an inertial frame even though they are both accelerated. Note that an observer standing on the surface of Earth is in a local inertial frame for a very short period of time: If an observer on Earth leaves an object at rest, it will start falling, it will not stay at rest: An observer at the surface of Earth is not in a local inertial frame unless the time interval considered is so short that the effect of the gravitational acceleration is not measurable. The only thing that keeps the observer on the surface of the Earth from being in a local inertial frame is the ground which exerts an upward force on the observer. If suddenly a hole in the ground opens below him and he starts freely falling, he suddenly finds himself in an inertial frame with less strict time limits.

We now need to find out which limitations this inertial frame has. Local means that the inertial frame is limited in space and time, but we need to define these limits. In figure 1 we see two falling boxes, box A falling in the horizontal position, box B falling in the vertical position. Since the gravitational acceleration is directed towards the center of the Earth, two objects at rest at either side of box A will start moving towards the center of the box due to the direction of the acceleration. The shorter we make the box, the smaller this motion is. If we make the box so short that we cannot measure the horizontal motion of the objects, we say that the box is a local inertial frame. The same argument goes for time: If we wait long enough, we will eventually observe that the two objects have moved. The inertial frame is limited in time by the time it takes until the motion of the two objects can be measured. Similarly for box B: The object which is closer to the Earth will experience a stronger gravitational force than the object in the other end of the box. Thus, if the box is long enough, an observer in the box will observe the two objects to move away from each other. This is just the normal tidal forces: The gravitational attraction of the moon makes the oceans on either side of the Earth to move away from each other: we get high tides. But if the box is small enough, the difference in the gravitational acceleration is so small that the motion of the objects cannot be measured. Again, it is a question of time before a motion will be measured: The local inertial frame is limited in time. We have thus seen that a local inertial frame can be found if we define the frame so small in space and time that the gravitational acceleration within the frame (in space and time) is constant. In these frames, within the limited spatial extent and limited duration in time, an object which is left at rest will remain at rest in that frame. The stronger the gravitational field and the larger the variations in the gravitational field, the smaller in space and time we need to define our local inertial frame.

We have learned from special relativity that an inertial frame has Lorentz geometry. Within the local inertial frame, spacetime intervals are measured according to $\Delta s^2 = \Delta t^2 - \Delta x^2$ and the laws of special relativity are all valid within the limits of this frame. In general relativity, we can view spacetime around a massive object as being an infinite set of local inertial frames. When performing experiments within these limited frames, special relativity is all we need. When studying events taking place so far apart in space and time that they do not fit into one such local inertial frame, general relativity is needed. This is why only special relativity is needed for particle physicists making experiments in particle accelerators. The particle collisions take place in such a short time that the gravitational acceleration may be neglected: They take place in a local inertial frame.

We will now call spacetime where Lorentz geometry is valid for *flat spacetime*. This is because Lorentz geometry is similar to Euclidean geometry on a flat surface (except for a minus sign). We know that a curved surface, like the surface of the sphere, has spherical geometry not Euclidean geometry. In the same way, Schwarzschild geometry represents *curved spacetime*, the rules of Lorentz geometry are not valid and Schwarzschild geometry needs to be used. We say that the presence of matter curves spacetime. Far away from all massive bodies, spacetime is flat and special relativity is valid.

We can take the analogy even further: Since the surface of a sphere has spherical and not Euclidean geometry, the rules of Euclidean geometry may not be used. But if we focus on a very small part of the surface of a sphere, the surface looks almost flat and Euclidean geometry is a very good approximation. The surface of the Earth is curved and therefore has spherical geometry, but since a house is very small compared to the full surface of the Earth, the surface of the Earth appears to us to have flat geometry within the house. We use Euclidean geometry when measuring the area of the house. The same is the case for the curved space: Since spacetime is curved around a massive object, we need to use Schwarzschild geometry. But if we only study events which are within a small area in spacetime, spacetime looks flat and Lorentz geometry is a good approximation.

3 Three observers

In the lectures on general relativity we will use three observers, *the far-away observer*, *the shell observer* and *the freely falling observer*. We will also assume that the central massive body is a black hole. A black hole is the simplest possible macroscopic object in universe: it can be described by three parameters, mass, angular momentum and charge. Any black holes which have the same values for these three parameters are identical in the same way as two electrons are identical. A black hole is a region in space where the gravitational acceleration is so high that not even light can escape from it. A black hole can arise for instance when a massive star is dying: A star is balanced by two forces, the forces of gravity (which we no longer call forces) trying to pull the star together and the gas/radiation pressure trying to make the star expand. When all fuel in the stellar core is exhausted, the forces of pressure are not strong enough to withstand the forces of gravity and the star collapses. No forces can stop the star from shrinking to an infinitely small point. The gravitational acceleration just outside this point is so large that even light that tries to escape will fall back. The escape velocity is larger than the speed of light. This is a black hole. Note that the Schwarzschild line element becomes singular at $r = 2M$. This radius is called the *Schwarzschild radius* or *the horizon*. This is the 'point of no return', any object (or light) which comes inside this horizon cannot get out. At any point before the horizon a spaceship with strong engines could still escape. But after it has entered, no information can be transferred out of the horizon.

The *far-away observer* is situated in a region far from the central black hole where spacetime is flat. He does not observe any events directly, but gets information about time and position of events from clocks situated everywhere around the black hole. *The shell observers* live on the surface of shells constructed around the black hole. Also a spaceship which uses its engines to stay at rest at a fixed radius r could serve as a shell observer. The shell observers experience the gravitational attraction. When they leave an object at rest it falls to the surface of the shell.

There is one more observer which we already discussed in the previous section. This is the *freely falling observer*. The freely falling observer carries with him a wristwatch and registers the position and personal wristwatch time of events. The freely falling observer is living in a local inertial frame with flat spacetime. Thus for events taking place within short time intervals and short distances in space, the freely falling observer uses Lorentz geometry to make calculations.

4 The time and position coordinates for the three observers

Each of the observers have their own set of measures of time and space. The far away observer uses Schwarzschild coordinates (r, t) and shell observers use shell coordinates $(r_{\text{shell}}, t_{\text{shell}})$. For the freely falling observers, we will be viewing all events from the origin in his frame of reference (and we will therefore not need a position coordinate since it will always be zero) using his wristwatch time which will then always be the proper time τ. We will now discuss these different coordinate systems and how they are defined.

When the shell observer wants to measure his position r from the center of the black hole, he runs into problems: When he tries to lower a meter stick down to the center of the hole to measure r, the stick just disappears behind the horizon. He needs to find other means to measure his radial position. What he does is to measure the circumference of his shell. In Euclidean geometry, we know that the circumference of a circle is just $2\pi r$. So the shell observer measures the circumference of the shell and divides by 2π to obtain his coordinate r. In a non-Euclidean geometry, the radius measured this way does not correspond to the radius measured inwards. We *define* the *Schwarzschild coordinate* r in this way.

$$r = \frac{\text{circumference}}{2\pi}$$

The r in the expression for the Schwarzschild line element is the Schwarzschild coordinate r. Now the shell observers at shell r lower a stick to the shell observers at shell r'. The length of the stick is Δr_{shell}. They compare this to the difference in Schwarzschild coordinate $r - r'$ and find that $\Delta r_{\text{shell}} \neq \Delta r = r - r'$. This is what we anticipated, in Euclidean geometry these need to be equal, in Schwarzschild geometry they are not. We have obtained a second way to measure the radial distances between shells using shell distances Δr_{shell} (note that since the absolute shell coordinate r_{shell} cannot be measured it is meaningless, only relative shell coordinate differences Δr_{shell} between shells can be measured (did you understand why?)).

We have obtained two different measures of radial distances,

- the Schwarzschild coordinate r defined by the circumference of the shell. The far-away observer uses Schwarzschild coordinates to measure distances.
- the shell distances Δr_{shell} found by physical measurements between shells.This is the distance which the shell observers can measure directly with meter sticks and is therefore the most natural measure for these observers.

What about time coordinates? Again we have two meaures of time,

- The far-away observer uses *far-away time* t to measure time. This is the time t entering in the Schwarzschild line element. Far away time for an event is measured on a clock which has been synchronized with the clock of the far-away observer and which is located at the same location as the event (we will later describe how events can be timed which such clocks in practice).
- The shell observer uses local shell time t_{shell}: it is simply the wristwatch time of the shell observer, the time measured on a clock at rest at the specified shell. Note that shell observers at different shells may measure different times intervals Δt_{shell} and distances Δr_{shell} between two events depending on which shell they live on. Shell coordinates are local coordinates.

In order to relate time and space coordinates in the different frames we will now (as we did in special relativity) use the invariance of the space time interval (or line element) Δs. First we will find a relation between the more abstract far away-time t and the locally measurable shell time t_{shell}. The shell time is the wristwatch time, or proper time τ of the shell observers. We will use two events A and B which are two ticks on the clock of a shell observer. The shell observers are at rest at shell r, so , $\Delta r_{AB} = 0$ and $\Delta\phi_{AB} = 0$. Inserting this into the Schwarzschild line element (equation 1) using that $\Delta s_{AB} = \Delta\tau_{AB} = \Delta t_{\text{shell}}$ (the time period between A and B measured on shell clocks is by definition the same as the proper time period between A and B which we have learned is always equal to the invariant four dimensional line element between these events)

Shell time

$$\Delta t_{\text{shell}} = \sqrt{\left(1 - \frac{2M}{r}\right)}\Delta t. \tag{3}$$

(Are you sure you see how this expression comes about?) For shell observers outside the horizon ($r > 2M$), the local time goes slower by a factor $\sqrt{\left(1 - \frac{2M}{r}\right)}$ with respect to the far-away time. We also see that the smaller the distance r from the center, the slower the shell clock with respect to the far-away time. Thus, the further down we live in a gravitational field, the slower the clocks run. This has consequences for people living on Earth: Our clocks tick slower than the clocks in satellites in orbit around Earth. At the end of this lecture, we will look closer at this fact.

We have now found a relation between time intervals measured on shell clocks and time intervals measured on clocks synchronized with far-away clocks. How is the relation between distances measured with Schwarzschild coordinates and distances measured directly by shell observers? We can measure the length of a stick as the spatial distance between two events taking place at the same time at either end of the stick (see figure 2). For events taking place within short time intervals and short spatial extensions, the shell observer sees flat spacetime and can therefore use

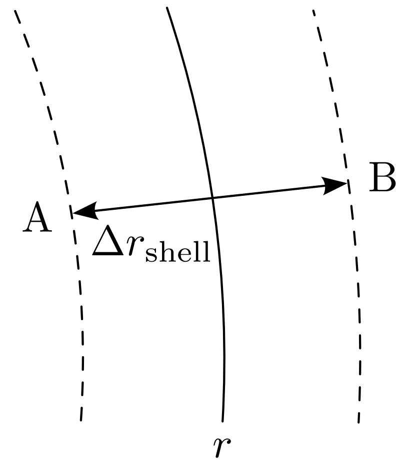

Figure 2: The shell observer at shell r measure the proper lenght of a stick by two simultaneous events A and B on either side of the stick.

Lorentz geometry: $\Delta s^2_{\rm shell} = \Delta t^2_{\rm shell} - \Delta r^2_{\rm shell}$ (we will look at a stick which is aligned with the radial direction, the events therefore take place at the same ϕ coordinate so $\Delta\phi = 0$). The far-away observer always needs to use the Schwarzschild line element (equation 1) instead of the Lorentz line element. Using invariance of the line element we have for two events A and B ($\Delta s_{\rm shell}(AB) = \Delta s(AB)$) taking place simultaneously on either side of the stick

$$\Delta t^2_{\rm shell} - \Delta r^2_{\rm shell} = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)},$$

where we have set $\Delta\phi = 0$. Check that you understand how to arrive at this expression. Now, we measure the length Δr of a stick in the radial direction by measuring the distance between the two simultaneous events A and B taking place at either end of the stick at spatial distance Δr. Since events which are simultaneous for shell observers at a given shell r also are simultaneous for the far-away observer (equation 3), $\Delta t_{\rm shell} = \Delta t = 0$. Inserting this, we get

$$\Delta r_{\rm shell} = \frac{\Delta r}{\sqrt{\left(1 - \frac{2M}{r}\right)}}. \tag{4}$$

for short distances Δr close to the shell. Thus, radial distances measured by the shell observers, lowering meter sticks from one shell to the other is always larger than the radial distances found by taking the difference between the Schwarzschild coordinate distances. What about a stick which is perpendicular to the radial direction? In this case, the observers will agree on the length of this stick, check that you can deduce this in the same manner as we deduced the relation for the radial stick.

There is a practical problem in all this: We said that the far-away time was measured by clocks located at the position of events (which can take place close to the central black hole) but which are synchronized with the far-away clocks. How can we synchronize clocks which are located deep in the gravitational field and which therefore run slower than the far-away clock? Let's imagine the clocks measuring far-away-time to be positioned at different shells around the black hole. The shell observers design the clocks such that they run faster by a factor $\sqrt{\left(1 - \frac{2M}{r}\right)}$. To synchronize all these clocks, the far-away observer sends a light signal to all the other clocks at the moment when he sets his clock to $t = 0$. The shell observers know the distance from the far-away observer to the far-away-time clocks and thus know the time t it took for the light signal to reach their clock. They had thus already set the clock to this time t and made a mechanism such that it started to run at the moment when the light signal arrived. In this way, all far-away-time clocks situated at different positions around the black hole are synchronized.

Another practical question: How does the far-away observer know the time and position of events. Each time an event happen close to one of the far-away-clocks close to the black hole, it sends a signal to the far-away observer telling the time and position this clock registered for the event. In this way, the far-away observer does not need to take into account the time it takes for the signal from the clock to arrive, the signal itself contains information with the correct far-away-time for the event recorded on the clock positioned at the same location where the event took place.

In the following we will describe events either as they are seen by the far-away observer using global Schwarzschild coordinates (r, t), by the shell observer using local coordinates $(r_{\text{shell}}, t_{\text{shell}})$ or the freely falling observer also using local coordinates. Before proceeding, make a drawing of all these observers, their coordinates and the relation between these different coordinates.

5 The principle of maximum aging revisited

In the lectures on special relativity we learned that the principle of maximum aging tells objects in free float to move along paths in spacetime which give the longest possible wristwatch time τ which corresponds to the longest possible spacetime interval s. We also used that for Lorentz geometry, the longest (in terms of s or τ) path between two points is the straight line, i.e. the path with constant velocity. We never proved the latter result properly. We will do this now, first for Lorentz geometry and then we will use the same approach to find the result for Schwarzschild geometry.

5.1 Returning for a moment to special relativity: deducing Netwon's first law

We will now show that the principle of maximum aging leads to Newton's first law when using Lorentz geometry.

Look at figure 3. We see the worldline of a particle going from position x_1 at time t_1 to position x_3 at time t_3 passing through position x_2 at time t_2. Say that the points x_1, x_2 and x_3 are fixed and known positions. We also say that the total time interval t_{13} it takes the object to go from x_1 to x_3 is fixed and known. What we do not know is the time interval t_{12} it takes the particle to go from point x_1 to point x_2. Remember that we do not know that the object will move with constant velocity, this is what we want to show. Thus we leave open the possibility that the particle will have a different speed between x_1 and x_2 than between x_2 and x_3. The time t_2 can be at any possible point between t_1 and t_3. In figure 3 we show some possible spacetime paths that the object could take. We now assume that the distances x_{12} and x_{23} are very short, so short that the object can be assumed to move with constant velocity between these two points, i.e. that the object is in a local inertial frame between x_1 and x_2 and in a (possibly different) intertial frame between x_2 and x_3. Therefore, the time intervals t_{12} and t_{23} to travel these two short paths also need to be short.

The total wristwatch time τ it takes the particle to move from x_1 to x_3 is

$$\tau_{13} = \tau_{12} + \tau_{23} = \sqrt{t_{12}^2 - x_{12}^2} + \sqrt{t_{23}^2 - x_{23}^2}, \tag{5}$$

where we used that $\Delta\tau = \Delta s = \sqrt{\Delta t^2 - \Delta x^2}$ for Lorentz geometry. According to the principle of maximal aging, we need to find the path, i.e. the t_{12}, which maximizes the total wristwatch time τ_{13}. We do this by setting the derivative of τ_{13} with respect to the free parameter t_{12} equal to zero, i.e. you look for the maximum point of the function $\tau_{13}(t_{12})$:

$$\frac{d\tau_{13}}{dt_{12}} = \frac{t_{12}}{\sqrt{t_{12}^2 - x_{12}^2}} + \frac{t_{23}}{\sqrt{t_{23}^2 - x_{23}^2}}\frac{dt_{23}}{dt_{12}}.$$

Since $t_{23} = t_{13} - t_{12}$ we have that $dt_{23}/dt_{12} = -1$ (remember that t_{13} is a fixed constant). Thus we have

$$\frac{t_{12}}{\sqrt{t_{12}^2 - x_{12}^2}} - \frac{t_{23}}{\sqrt{t_{23}^2 - x_{23}^2}} = 0$$

or written in terms of τ_{12} and τ_{23} we have

$$\frac{t_{12}}{\tau_{12}} = \frac{t_{23}}{\tau_{23}}.$$

Check that you understood every step in the deduction so far! This is only for three points x_1, x_2 and x_3 along the worldline of a particle. If we

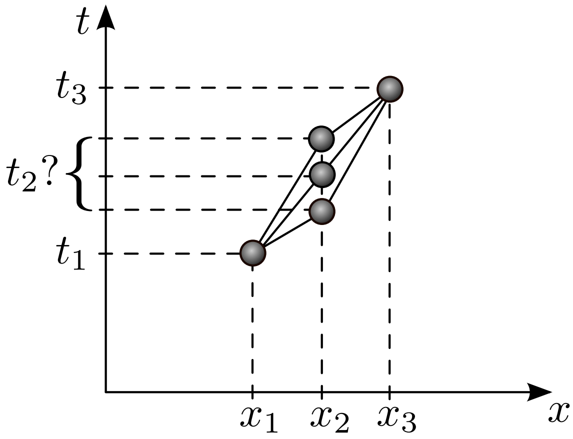

Figure 3: The motion of a particle in free float in Lorentz geometry. Points x_1, x_2, x_3 as well as the times t_1 and t_3 are fixed. For a particle at free float, at what time t_2 will it pass x_2? Which of the possible spacetime paths in the figure does the particle take? We use the principle of maximal aging to show that in Lorentz geometry, the particle follows the straight spacetime path.

continue to break up the worldline in small local inertial frames at points x_4, x_5, etc. we can do the same analysis between any three adjacent points along the curve. The result is that

$$\frac{dt}{d\tau} = \text{constant},$$

where I have written dt instead of t_{12} or t_{23} and $d\tau$ instead of τ_{12} or τ_{23}. Remember that we assumed these time interval to be very short. In this final expression we have taken the limit in which these time intervals are infinitesimally short. We also remember (do you?) from special relativity that

$$\frac{dt}{d\tau} = \frac{1}{\sqrt{1-v^2}} = \gamma.$$

So the principle of maximal aging has given us that γ = constant along a worldline. But γ only contains the velocity v of the object so it follows that v = constant. In Lorentz geometry, a free-float object will follow the spacetime path for which the velocity is constant. We can write this in a different way. In special relativity we had that

$$E = \gamma m$$

so we can write $\gamma = E/m$ from which follows that

$$\gamma = \frac{E}{m} = \text{constant}.$$

We have just deduced that energy is conserved, or more precisely energy per mass E/m is conserved. In the lectures on special relativity we learned that experiments tell us that the relativistic energy $E = \gamma m$ is conserved and not Newtonian energy. Here we found that the principle of maximal aging tells us that there is a quantity which is conserved along the motion of a particle. This quantity is the same as the quantity we call relativistic energy.

Is it possible that the principle of maximal aging can give us something more? We will now repeat the above calculations, but now we fix t_{12}, t_{23} and t_{13}. All times are fixed. We also fix x_1 and x_3, but leave x_2 free. The situation is shown in figure 4. Now the question is 'which point x_2 will the object pass through?'. We need to take the derivative of expression (5) with respect to the small interval x_{12} which is a free parameter.

$$\frac{d\tau_{13}}{dx_{12}} = \frac{-x_{12}}{\sqrt{t_{12}^2 - x_{12}^2}} - \frac{x_{23}}{\sqrt{t_{12}^2 - x_{12}^2}}\frac{dx_{23}}{dx_{12}}.$$

Again $x_{23} = x_{13} - x_{12}$ so that $dx_{23}/dx_{12} = -1$ and we have

$$\frac{x_{12}}{\tau_{12}} = \frac{x_{23}}{\tau_{23}},$$

we have found another constant of motion

$$\frac{dx}{d\tau} = \text{constant}$$

But we can write this as

$$\frac{dx}{d\tau} = \frac{dx}{dt}\frac{dt}{d\tau} = v\gamma.$$

We have that

$$v\gamma = \text{constant} = \frac{p}{m}.$$

(Go through this deduction in detail yourself and make sure you understand every step). We remember that $p = m\gamma v$, so the principle of maximal aging has given us the law of momentum conservation, or actually the law of conservation of momentum per mass p/m. We have seen that the principle of maximal aging seems to be more fundamental than the principles of energy and momentum conservation. It is sufficient to assume the principle of maximal aging. From that we can deduce the expressions for energy and momentum and also that these need to be conserved quantities.

5.2 Returning to general relativity: deducing and generalizing Newton's law of gravitation

Now, what about general relativity? We will see how the principle of maximal aging tells a particle to move in Schwarzschild spacetime. Look at figure 5. A particle travels from radius r_1 at time t_1 to radius r_3 at point t_3 passing through point r_2 at time t_2. We fix r_1, r_2 and r_3. We also fix the start and end times t_1 and t_3. We leave t_2 free. We will find

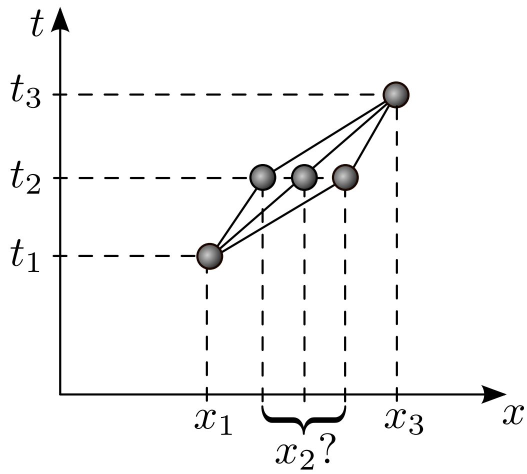

Figure 4: The motion of a particle in free float in Lorentz geometry. Points x_1, x_3 as well as the times t_1, t_2 and t_3 are fixed. For a particle in free float, which position x_2 will it pass at time t_2? Which of the possible spacetime paths in the figure does the particle take? We use the principle of maximal aging to show that in Lorentz geometry, the particle follows the straight spacetime path.

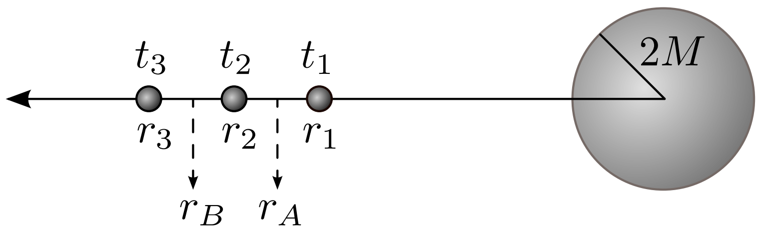

Figure 5: The motion of a particle in free float in Schwarzschild geometry. Points r_1, r_2, r_3 as well as the times t_1 and t_3 are fixed. For a particle in free float, at what time t_2 will it pass through r_2? We assume that the distances $r_2 - r_1$ and $r_3 - r_2$ are so small that we can assume the radial distance to be $r = r_A$ always in the former interval and $r = r_B$ always in the latter interval.

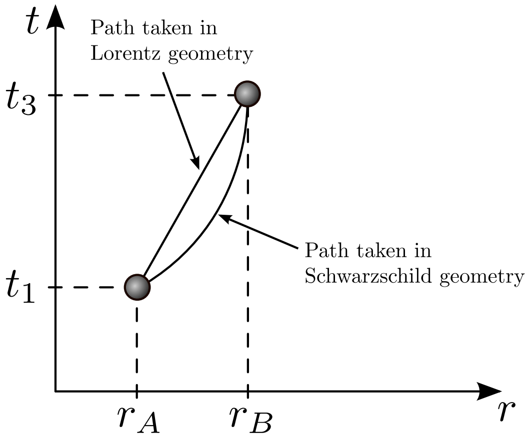

Figure 6: The motion of a particle in free float in Schwarzschild geometry. Which spacetime path will the particle take between points A and B?

at which time t_2 the particle passes through point r_2. Again we write the total proper time for the object from r_1 to r_3 as (using the Schwarzschild line element, equation 1, for $\Delta\tau$)

$$\tau_{13} = \tau_{12} + \tau_{23} = \sqrt{\left(1 - \frac{2M}{r_A}\right) t_{12}^2 - \frac{r_{12}^2}{\left(1 - \frac{2M}{r_A}\right)}} + \sqrt{\left(1 - \frac{2M}{r_B}\right) t_{23}^2 - \frac{r_{23}^2}{\left(1 - \frac{2M}{r_B}\right)}},$$

where r_A is the radius halfway between r_1 and r_2. We assume that r_{12} is so small that we can use the radius r_A for the full interval. In the same way, r_B is the radius halfway between r_2 and r_3 which we count as valid for the full interval r_{23}. Following the procedure above, we will now maximize the total proper time τ_{13} with respect to the free parameter t_{12}. We get

$$\frac{d\tau_{13}}{dt_{12}} = \frac{\left(1 - \frac{2M}{r_A}\right) t_{12}}{\tau_{12}} + \frac{\left(1 - \frac{2M}{r_B}\right) t_{23}}{\tau_{23}} \frac{dt_{23}}{dt_{12}}.$$

As above, $t_{23} = t_{13} - t_{12}$ giving $dt_{23}/dt_{12} = -1$. Thus we have that

$$\frac{\left(1 - \frac{2M}{r_A}\right) t_{12}}{\tau_{12}} = \frac{\left(1 - \frac{2M}{r_B}\right) t_{23}}{\tau_{23}}.$$

We find that

$$\left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} = \text{constant}, \quad (6)$$

where again we have taken the limit where t_{12}, t_{23}, τ_{12} and τ_{13} are so small that they can be expressed as infinitesimally small periods of time dt and $d\tau$. In the case with Lorentz geometry we used this constant of motion to find that the velocity had to be constant along the worldline of a freely floating particle. Now we want to investigate how this constant of motion tells us how a freely floating particle moves in Schwarzschild spacetime. First we need to find an expression for $dt/d\tau$. In special relativity we related this to the velocity of the particle using $dt/d\tau = \gamma$, but this was deduced using the line element of Lorentz geometry. Here we want to relate this to the local velocity that a shell observer at a given radius observes. The locally measured shell velocity as an object passes by a given shell is given by

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}}$$

We now use equation 3 (the equation connecting far-away-time and shell time, remember?) to write the constant of motion (equation 6) as

$$\left(1 - \frac{2M}{r}\right) \frac{\left(1 - \frac{2M}{r}\right)^{-1/2} dt_{\text{shell}}}{d\tau} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{dt_{\text{shell}}}{d\tau}$$
$$= \left(1 - \frac{2M}{r}\right)^{1/2} \gamma_{\text{shell}} = \left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{\sqrt{1 - v_{\text{shell}}^2}} = \text{constant}.$$

In the last transition we used the fact that the shell observer lives in a local inertial frame for a very short time. The shell observer makes the velocity measurement so fast that the gravitational acceleration could not be noticed and he could use special relativity assuming flat spacetime. So using his local time t_{shell}, the relation $dt_{\text{shell}}/d\tau = \gamma_{\text{shell}}$ from special relativity is valid. We have thus found a constant of motion:

$$\left(1 - \frac{2M}{r}\right)^{1/2} \frac{1}{\sqrt{1 - v_{\text{shell}}^2}} = \text{constant}.$$

Consider a particle moving from radius r_A to a higher radius r_B (see figure 6). This time, the distance between points A and B does not need to be small. As the object moves past shell r_A, the shell observers at this shell measure the local velocity v_A. As the object moves past shell r_B, the shell observers at this shell measure the local velocity v_B. Equating this constant of motion at the two positions A and B we find

$$\left(1 - \frac{2M}{r_A}\right)^{1/2} \frac{1}{\sqrt{1 - v_A^2}} = \left(1 - \frac{2M}{r_B}\right)^{1/2} \frac{1}{\sqrt{1 - v_B^2}}.$$

Squaring and reorganizing we find

$$(1 - v_B^2)\left(1 - \frac{2M}{r_A}\right) = (1 - v_A^2)\left(1 - \frac{2M}{r_B}\right).$$

We already see from this equation that if $r_B > r_A$ then $v_B < v_A$ (check!). Thus if the object is moving away from the central mass, the velocity is decreasing. If we have $r_B < r_A$ we see that the opposite is true: If the object is moving towards the central mass, the velocity is increasing. So the principle of maximum aging applied in Schwarzschild geometry gives a very different result than in Lorentz geometry. In Lorentz geometry we found that the velocity of a freely floating particle is constant. In Schwarzschild spacetime we find that the freely floating particle accelerates towards the central mass: If it moved outward it slows down, if it moved inwards it accelerates. This is just what we normally consider the 'force of gravity'. We see that here we have not included any forces at all: We have just said that the central mass curves spacetime giving it Schwarzschild geometry. By applying the principle of maximal aging, that an object moving through spacetime takes the path with longest possible wristwatch time τ, we found that the object needs to take a path in spacetime such that it accelerates towards the central mass. We see how geometry of spacetime gives rise to the 'force of gravity'. But in general relativity we do not need to introduce a force, we just need one simple principle: The principle of maximal aging.

We will now check if the acceleration we obtain in the limit of large radius r and low velocities v_{shell} is equal to the Newtonian expression. We now call the constant of motion K giving

$$\left(1-\frac{2M}{r}\right)\frac{1}{1-v_{\text{shell}}^2}=K.$$

Reorganizing this we have

$$v_{\text{shell}}=\sqrt{1-\frac{1}{K}\left(1-\frac{2M}{r}\right)} \tag{7}$$

We want to find the acceleration

$$g_{\text{shell}}=\frac{dv_{\text{shell}}}{dt_{\text{shell}}}$$

that a shell observer measures. Taking the derivative of equation 7 we get (check!)

$$\frac{dv_{\text{shell}}}{dt_{\text{shell}}}=\frac{1}{2v_{\text{shell}}}\frac{2M}{K}\left(-\frac{1}{r^2}\right)\frac{dr}{dt_{\text{shell}}}.$$

Using equation 4 and that $v_{\text{shell}}=dr_{\text{shell}}/dt_{\text{shell}}$ we obtain

$$g_{\text{shell}}\propto\sqrt{\left(1-\frac{2M}{r}\right)}\frac{M}{r^2}$$

Newton's law of gravitation is not valid close to the Schwarzschild horizon, so to take the Newtonian limit we need to consider this expressions for $r >> 2M$. In this limit the expression reduces to

$$g_{\text{shell}}\propto\frac{M}{r^2},$$

exactly the Newtonian expression for the gravitational acceleration. We find that far away from the Schwarzschild radius, general relativity reduces to Newton's law of gravitation. We can now return to figure 6 and look at the path marked Schwarzschild path. This is the spacetime path between A and B that a freely floating object needs to take in order to get the longest proper time τ. Looking at the slope of this path, we see that the object changes velocity during its trip from A to B. This is in sharp contrast to the results from special relativity with Lorenzt geometry where the path which gives longest possible proper time is the straight line with constant velocity.

We will now return to our constant of motion

$$\left(1-\frac{2M}{r}\right)^{1/2}\frac{1}{\sqrt{1-v_{\text{shell}}^2}}=\text{constant} \tag{8}$$

In special relativity we found that a constant of motion which we obtain in the same manner was just the energy per mass. We will now go to the Newtonian limit to see if the same is the case for Schwarzschild spacetime. We will use two Taylor expansions,

$$\sqrt{1-x}\approx 1-\frac{1}{2}x+...$$
$$\frac{1}{\sqrt{1-x^2}}\approx 1+\frac{1}{2}x^2,$$

both taken in the limit of $x << 1$. In the Newtonian limit we have that $2M/r << 1$ and $v << 1$. Applying this to equation (8) we have

$$\left(1-\frac{M}{r}\right)\left(1+\frac{1}{2}v^2\right)\approx 1+\frac{1}{2}v^2-\frac{M}{r}=\text{constant}$$

In the last expression we used that since both $2M/r$ and v are very small, the product of these small quantities is even smaller than the remaining terms and could therefore be omitted. Compare this to the Newtonian expression for the energy of a particle in a gravitational field

$$E=\frac{1}{2}mv^2-\frac{Mm}{r}.$$

We see again that the constant of motion was just energy per mass E/m where the expression now tells us how the gravitational potential looks like (have you noticed this: you have actually derived why the form of the Newtonian gravitational potential is the way it is). Note the additional term in the relativistic expression which is just the rest energy m. Again the principle of maximal aging has given us that energy is conserved and it has given us the relativistic expression for energy in a gravitational field.

Relativistic energy in a gravitational field

$$\frac{E}{m}=\left(1-\frac{2M}{r}\right)\frac{dt}{d\tau}=\text{constant}.$$

We also found that this expression for the energy equals the Newtonian expression for distances far from the Schwarzschild radius.

In the exercises you will use the principle of maximum aging to find that angular momentum per mass is conserved in Schwarzschild spacetime and that the expression for angular momentum per mass in Schwarzschild spacetime is

Angular momentum per mass in Schwarzschild spacetime

$$\frac{L}{m} = r^2 \frac{d\phi}{d\tau} = \gamma_{\text{shell}} r v_\phi = \text{constant}.$$

6 Freely falling

Armed with the expression for the conserved energy and angular momentum we will now start to look at motion around the black hole. First, we will leave an object at rest at a large distance from the central mass. We leave the object with velocity zero $v = 0$ at a distance so large that we can let $r \to \infty$. The energy per mass of the particle is then only the rest energy of the particle, $E = m$.

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right) \frac{dt}{d\tau} = 1$$

From this we have

$$d\tau = \left(1 - \frac{2M}{r}\right) dt.$$

Remember from the previous deduction that the proper time τ here refers to time measured on the wristwatch attached to the falling object. Inserting this into the Schwarzschild line element (equation 1) we obtain (remember $ds = d\tau$ always!)

$$d\tau^2 = \left(1 - \frac{2M}{r}\right)^2 dt^2 = \left(1 - \frac{2M}{r}\right) dt^2 - \frac{dr^2}{(1 - \frac{2M}{r})}$$

The object starts falling radially inwards towards the central mass. We want to find the velocity of the falling object at a given radius r. Reorganizing the previous equation we have (check!)

$$\left(\frac{dr}{dt}\right)^2 = \left(1 - \frac{2M}{r}\right)^2 \frac{2M}{r}.$$

To obtain the velocity $v = dr/dt$, we need to take the square root on both sides. This leaves a positive and a negative solution. Since this object is falling in the negative r direction, we choose the negative solution

$$v = -\left(1 - \frac{2M}{r}\right) \sqrt{\frac{2M}{r}}. \tag{9}$$

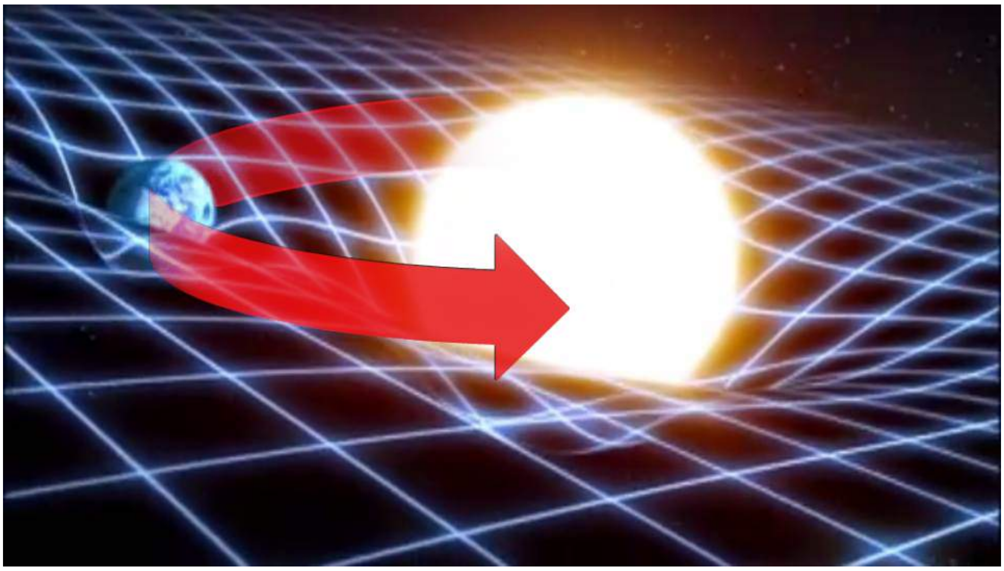

Figure 7: Info-figure: An example of a two-dimensional analogy of the warping of space and time by massive objects, often used in introductory texts on general relativity. General relativity was proposed by Einstein in 1916 and provides a unified description of gravity as a geometric property of space and time, or spacetime. The curvature of spacetime is directly related to the energy and momentum of whatever matter and radiation are present. Some predictions of general relativity differ significantly from those of classical physics, especially concerning the passage of time, the geometry of space, the motion of bodies in free fall, and the propagation of light. Examples of such differences include gravitational time dilation, gravitational lensing, the gravitational redshift of light, and the gravitational time delay. (Figure: WGBH Boston)

At large distance $r \to \infty$ the velocity goes to zero as expected. What happens when the object approaches the black hole? For large distances the factor $\sqrt{2M/r}$ is dominating. This factor increases with decreasing r, so the velocity increases just as expected. When we approach the Schwarzschild radius, the first factor $\left(1 - \frac{2M}{r}\right)$ starts dominating as the last factor now goes to one. In this case, the velocity is decreasing when r is decreasing. At the horizon the velocity reaches exactly zero. What we see is plotted in figure 8. When the object starts falling the velocity increases until a point where it starts decreasing. At the horizon the object stops. This result was obtained using Schwarzschild coordinates. Thus, this is the result that the far-away observer sees. This means that if we let a spaceship fall into a black hole, we, as far-away observers, would see the spaceship stopping at the horizon and it would stay there for ever. Remember also that time is going slower closer to the horizon,

$$\Delta t_{\text{shell}} = \sqrt{\left(1 - \frac{2M}{r}\right)} \Delta t$$

At the horizon $r \to 2M$, we observe that time stops. Thus, looking at the spaceship we would observe the persons in the spaceship to freeze at the horizon. Everything stops. In the exercises you will show that light from a central mass is red shifted. Thus we will also see a strongly redshifted light from the spaceship. Using the expression from the exercises you will see that light arriving from the horizon is infinitely red shifted. Thus you will not see any light from the horizon. You will only see the spaceship just before it reaches the horizon and then only as radio waves with a large wavelength (see problem 1).

What do the shell observers living at shells close to the horizon see? Using formulas (3) and (4) we can find the local velocity $v_{\text{shell}} = dr_{\text{shell}}/dt_{\text{shell}}$ as observed by the shell observers when the spaceship passes by the shell. We get

$$v_{\text{shell}} = \frac{dr_{\text{shell}}}{dt_{\text{shell}}} = \frac{\left(1 - \frac{2M}{r}\right)^{-1/2} dr}{\left(1 - \frac{2M}{r}\right)^{1/2} dt} = \frac{v}{\left(1 - \frac{2M}{r}\right)} = -\sqrt{\frac{2M}{r}},$$

where the expression (9) for the far-away velocity v was used. Shell observers closer and closer to the horizon will always observe a larger and larger local velocity. The shell observers on the shell just above the horizon $r = 2M$ sees that $v_{\text{shell}} \to -1$, that the velocity of the object approaches the speed of light as the spaceship approaches the horizon. We have seen a huge difference in results: The far-away observer sees that the object falls to rest at the horizon, the local observer close to the horizon sees the object approaching the speed of light. Already from special relativity we are used to the fact that observers in different frames measure different numbers, but this is a really extreme example. What do the freely falling observers in the spaceship see? For the freely falling observers nothing particular at all happens when they pass the horizon. The freely falling

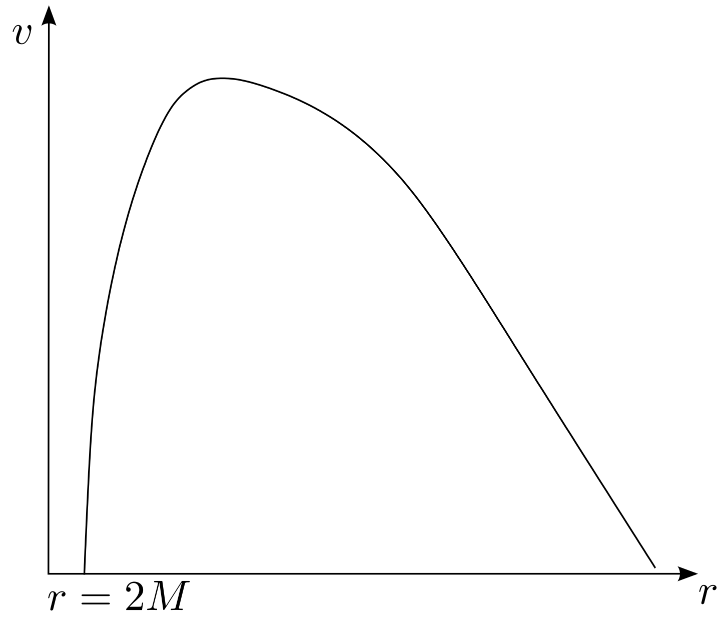

Figure 8: Schematic plot of the variation of velocity as a function of radial distance from the center for an object falling in from a huge distance.

observers are always moving from one local inertial frame to the other, but nothing special happens at $r = 2M$.

What velocity do local observers measure beyond the horizon? Do they measure a velocity larger than the velocity of light? In a coming lecture we will look a little bit more at motion beyond the horizon, but here we will look briefly at the Schwarzschild line element to see if we get some hints.

$$\Delta\tau^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{\left(1 - \frac{2M}{r}\right)}$$

Exactly at the horizon, the line element is singular. This is not a physical singularity, but what we call a *coordinate singularity*. By changing coordinate system, this singularity will go away and one can calculate Δs at the horizon without problems. One may understand this easier by looking at the analogy with the sphere: If a function on the sphere contains the expression $1/\theta$ (where θ is the polar angel being zero at the north pole) it will become singular on the north pole. By changing the coordinate system by defining the north pole at some other point on the sphere, the point of the previuos north pole will not be singular. In this case the function in itself is not singular on the point of the previous north pole, it is the coordinate system which makes the expression singular at this point.

We will now take a look at this line element when $r < 2M$. In this case we can write it as

$$\Delta\tau^2 = \frac{\Delta r^2}{\left|1 - \frac{2M}{r}\right|} - \left|1 - \frac{2M}{r}\right|\Delta t^2$$

Looking at the sign, the space and time coordinates interchange their roles. This does not directly mean that space and time interchange their roles, but space does attain one feature which we normally associate with time: An inevitable forward motion. In the same way as we always move forward in time, an observer inside the horizon will always move forwards towards the center. No matter how strong engines you have, you cannot stop this motion: you cannot be at rest inside the horizon, always moving forwards towards destruction at the center exactly as we always move forward in time. A consequence of this is that no shell observers can exist inside the horizon. You cannot construct a shell at rest, everything will always be moving. Inside the horizon we cannot measure a local shell velocity, so even if the shell velocity approaches the speed of light at the horizon it does not necessarily mean that we will have a local velocity larger than speed of light inside the horizon. More about this later.

7 An example: GPS, Global Positioning System

We have seen that general relativity becomes important for large masses and for distances close to the Schwarzschild radius $r \to 2M$. The question now is when we need to take into account general relativistic effects. Clearly this depends on the accuracy required for a given calculation. We will now see one example where general relativity is important in everyday life. The Global Positioning System (GPS) is used by a large number of people, from hikers in the mountain trying to find their position on the map to airplanes navigating with GPS in order to land even in dense fog. GPS is based on 24 satellites orbiting the Earth with a period of 12 hours at an altitude of about 20 000 kilometers. Each satellite sends a stream of signals, each signal containing information about their position $\vec{x}_{\mathrm{sat}}$ of the satellite at the time t_{sat} when the signal was sent. Your GPS receiver receives signals from three satellites (actually from four in order to increase the precision of the internal clock in your GPS receiver, but if your GPS receiver has an extremely accurate clock, only three satellites are strictly necessary: We will for simplicity use three satellites and assume that your GPS receiver contains an atomic clock in this illustration). The situation is illustrated in figure 9. Your GPS receiver contains a very accurate clock showing the time t when you receive the signal. This gives your GPS receiver three equations with the three coordinates of your position $\vec{x}$ as the three unknowns,

$$
\begin{aligned}
|\vec{x}-\vec{x}_{\mathrm{sat1}}| &= c(t-t_{\mathrm{sat1}}) \equiv c\Delta t_1,\\
|\vec{x}-\vec{x}_{\mathrm{sat2}}| &= c(t-t_{\mathrm{sat2}}) \equiv c\Delta t_2,\\
|\vec{x}-\vec{x}_{\mathrm{sat3}}| &= c(t-t_{\mathrm{sat3}}) \equiv c\Delta t_3.
\end{aligned}
$$

The GPS receiver receives the time t_{sat} when a signal was emitted from the satellite. Knowing that the signal travels with light speed c and reading off

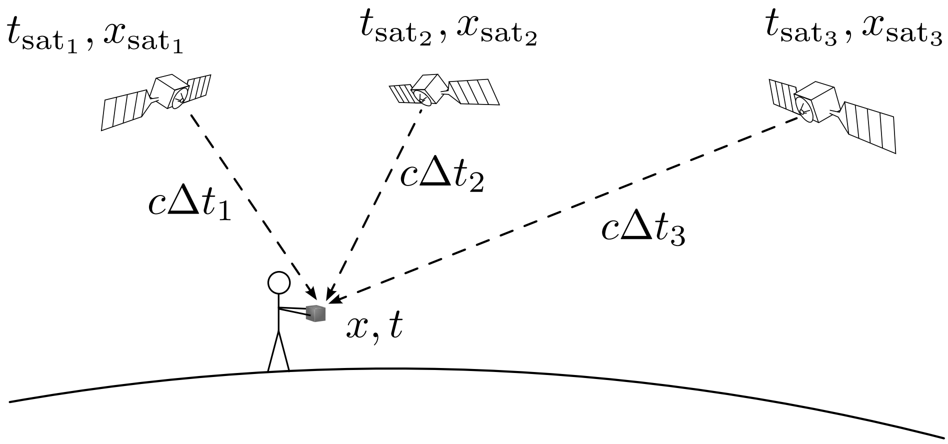

Figure 9: The GPS system.

the time of reception of the signal on the internal clock of the GPS receiver, the receiver can calculate the distance $c\Delta t$ that the signal has traveled. This distance is equal to the difference between your position $\vec{x}$ and the position $\vec{x}_{\text{sat}}$ of the satellite when the signal was emitted. Solving the three equations above, the GPS receiver solves for your position $\vec{x} = (x, y, z)$ normally expressed in terms of longitude, latitude and altitude. (Note that if a fourth equation were added using a signal from a fourth satellite, another unknown could be allowed: This is how the precision of your GPS clock is increased: your time t is solved from the four set of equations. Here we will assume that your GPS receiver has an atomic clock)

If we assume a simplified one dimensional case, i.e. that you only have a one dimensional position x, the solution would be

$$x = c\Delta t \pm x_{\text{sat}}.$$

We see that the precision of your calculated position x depends on the precision with which we can calculate the time difference $\Delta t = t - t_{\text{sat}}$. The signals move with velocity of 300 000 kilometers per second. If there is an inaccuracy of the order 1 μs $= 10^{-6}$ s, one microsecond, the inaccuracy in the calculated position would be of the order 3×10^8 m/s$\times10^{-6}$ s $= 300$ m. An inaccuracy of one microsecond corresponds to an inaccuracy of 300 meters in the position calculated by GPS. In such a case GPS would be useless for many of its applications and more seriously, the airplane missing the tarmac with 300 meters would crash!

We know that due to special relativity, the clocks in the satellite and the clocks on Earth (in your GPS receiver) run at different paces because of the relative motions of the satellites with respect to you. We also know from general relativity that the clocks in the satellite run at a different pace than your clock because of difference in distance from the center of attraction (center of Earth). If the clocks in the satellites and the clocks in the GPS receivers were synchronized at the moment when the satellites were launched into orbit, the question is how long does it take until the relativistic effects make the Earth and satellite clocks showing so different times that GPS has become useless. Relativistic effects are usually small so one could expect that it would take maybe thousands of years. If this were the case, we wouldn't need to worry. But remember that we require a precision better than 1 μs here. This could make relativistic effects important. Let's check.

We start by the gravitational effect. We consider two shells, shell 1 is the surface of the Earth situated at radial distance $r_1 = 6000$ km (approximately, we are only looking for orders of magnitude here, not exact numbers). Shell 2 is the orbit of the satellites at radial distance $r_2 = 6000 + 20000$ km. A time interval Δt_1 on the surface of the Earth is related to a time interval Δt of the far-away observer by (see equation 3)

$$\Delta t_1 = \sqrt{\left(1 - \frac{2M}{r_1}\right)}\Delta t.$$

Similarly, a time interval Δt_2 measured on the satellite clock is related to the far-away time Δt by

$$\Delta t_2 = \sqrt{\left(1 - \frac{2M}{r_2}\right)}\Delta t.$$

Dividing these two equations on each other we find that

$$\Delta t_1 = \sqrt{\frac{1 - \frac{2M}{r_1}}{1 - \frac{2M}{r_2}}}\Delta t_2.$$

This is the difference in clock pace between the satellite and Earth clocks taking into account only gravitational effects. We will first check the order of magnitude of these terms. What is the mass of the Earth measured in meters? We have

$$M_{\text{Earth}} = 6 \times 10^{24} \text{ kg} = 6 \times 10^{24} \times (7.42 \times 10^{-28} \text{ m}) = 0.0044 \text{ m}.$$

(in case you have forgotten: go back and check how to go from kg to meters). So the term $2M/r$ is of the order 10^{-8} for Earth, very much smaller than 1. Thus we can use Taylor expansions,

$$\sqrt{1-x} \approx 1 - \frac{1}{2}x + ...$$
$$1/\sqrt{1-x} \approx 1 + \frac{1}{2}x + ...$$

giving for $x = 2M/r$

$$\Delta t_1 = \Delta t_2 + \left(\frac{M}{r_2} - \frac{M}{r_1}\right)\Delta t_2,$$

where we have skipped terms of second order in small quantities (two x multiplied with each other) as these are much smaller than the terms of first order in x. We see that $\Delta t_1 < \Delta t_2$ as expected: Observers far away from the central mass see that clocks close to the central mass run slower. Observers far away from Earth will observe that it takes longer than one second on their wristwatch (Δt_2) for the clocks on Earth (Δt_1) to move one second forward.

Inserting numbers for r_1 and r_2 we obtain

$$\Delta t_1 \approx \Delta t_2 - 7 \times 10^{-10} \Delta t_2.$$

We see that after about one day $\Delta t_2 = 3600 \times 24$ s, the satellite clocks are 60 microseconds ahead of the Earth clocks. This corresponds to uncertainties in position measurements of the order 20 kilometers. Thus, one day after launching the satellites, GPS would be useless unless relativistic effects are taken into account!

In order to be sure about this, we need to also look at special relativistic effects. Seen from Earth, satellite clocks (which send time signals read from their own clocks to Earth) go slower (since they are moving with respect to the observers on the surface of the Earth). We have

$$\Delta t_1^{\mathrm{SR}} = \gamma \Delta t_2^{SR},$$

where SR stands for special relativity. In this case $\Delta t_1^{SR} > \Delta t_2^{SR}$ opposite of the general relativistic effect. We need to check whether this effect might be just large enough to cancel the general relativistic effect. From Kepler's 3rd law for the satellite we have (check that you can actually derive this),

$$\left(\frac{2\pi r_2}{v_{\phi 2}}\right)^2 = \frac{4\pi^2 r_2^3}{GM_{\mathrm{Earth,kg}}}$$

(using conventional units) we find that the orbital speed of the satellite is $v_{\phi 2} = 1.3 \times 10^{-5}$ (dimensionless velocity). In addition an observer at the surface of the Earth moves with velocity (due to Earth's rotation)

$$v = \frac{2\pi r_1}{24 \text{ h}} = 0.5 \text{ km/s}$$

or $v_{\phi 1} = 1.5 \times 10^{-6}$ in dimensionless units. The velocity of the satellite relative to the observer on the ground is thus approximately $v_\phi = v_{\phi 1} + v_{\phi 2} \approx 1.5 \times 10^{-5}$ giving $\gamma \approx 1 + 10^{-10}$. In one day we find that the satellite clocks run about 10 microseconds slow (actually aboout 9, check that you agree), by far not enough to cancel the general relativistic effect. Both effects need to be taken into account in order to make GPS of any use at all, and in order to not make your airplane crash when landing in fog.

We have so far used approximate general and special relativistic expressions separately. Using the Schwarzschild line element we may take both effects into account simultaneously and obtain a more accurate expression. Writing first the line element (between two clock ticks) for the observer on the surface of the Earth, we have

$$\Delta\tau^2 = \Delta t_1^2 = \left(1 - \frac{2M}{r_1}\right)\Delta t^2 - r_1^2 \Delta\phi_1^2,$$

where $\Delta r_1 = 0$ since the observer stays at the same radial distance. We can express this as

$$\left(\frac{\Delta t_1}{\Delta t}\right)^2 = \left(1 - \frac{2M}{r_1}\right) - v_{\phi 1}^2,$$

where $v_{\phi 1}$ is the tangential velocity of the Earth observer, $v_\phi = r d\phi/dt$ (did you get this transition?). Using the same arguments, we get the same expression for the satellite

$$\left(\frac{\Delta t_2}{\Delta t}\right)^2 = \left(1 - \frac{2M}{r_2}\right) - v_{\phi 2}^2,$$

where $v_{\phi 2}$ is the tangential velocity of the satellite. Dividing these two expressions on each other, we have

$$\left(\frac{\Delta t_1}{\Delta t_2}\right) = \sqrt{\frac{1 - \frac{2M}{r_1} - v_{\phi 1}^2}{1 - \frac{2M}{r_2} - v_{\phi 2}^2}}.$$

For low velocities and small $2M/r$ this expression reduces to the approximate expressions above. Note that we have not been very careful when measuring the tangential velocities: We did not specify tangential velocity with respect to which time, Earth time, Satellite time or far-away time. In turns out that taking into account these differences gives corrections to the correction which are so small that they can be ignored. We also did not specify whether the radial distances we used for Earth and the satellite were in Schwarzschild coordinates r or in shell distances r_{shell}. Also these differences are so small that they can be ignored in this case.

8 Exercise to be presented on the blackboard: Deriving the Scwarzschild line element.

The setting up Einstein's field equations for general relativity and solving them go beyond the scope of AST1100, but we can still make an argument for why the Schwarzschild line element is plausible as a description of the spacetime outside a spherical mass distribution. In the following you can assume that all gravitational fields are weak, and that all speeds are negligible compared to the speed of light in vacuum, c. A point mass m is dropped infinitely far from M with zero speed initially. (See figure 10).

a) Show that the velocity of the point mass in position $\vec{r}$ is given by

$$\vec{v} = -\sqrt{\frac{2GM}{r}}\vec{e}_r, \tag{10}$$

where $\vec{e}_r$ is the unit vector in the direction of $\vec{r}$.

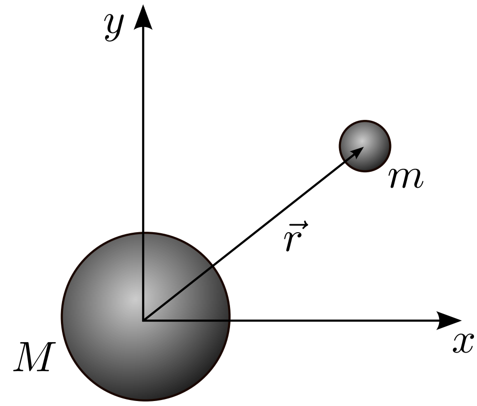

Figure 10: A sketch of the situation in this problem.

b) Consider a free-float system attached to the point mass with coordinates $(t_{\text{FF}}, x_{\text{FF}}, y_{\text{FF}}, z_{\text{FF}}) = (t_{\text{FF}}, \vec{x}_{\text{FF}})$. Explain briefly why the line element in this system is

$$ds_{\text{FF}}^2 = -c^2 dt^2 + d\vec{x}_{\text{FF}}^2.$$

c) Consider a rigid coordinate system fixed to the stars infinitely far from the mass distribution, $(t_{\text{r}}, x_{\text{r}}, y_{\text{r}}, z_{\text{r}})$. Explain why the transformation between this system and the free-float system in the previous point is given by

$$\begin{aligned} dt_{\text{FF}} &= dt_{\text{r}} \\ d\vec{x}_{\text{FF}} &= d\vec{x}_{\text{r}} = \vec{v} dt_{\text{r}}, \end{aligned}$$

where $\vec{v}$ is given by equation (10).

d) Show that the line element in the rigid reference frame is given by

$$ds_{\text{r}}^2 = -\left(c^2 - \frac{2GM}{r}\right) dt_{\text{r}}^2 + 2\sqrt{\frac{2GM}{r}} dr_{\text{r}} dt_{\text{r}} + d\vec{x}_{\text{r}}^2.$$

It can be shown that this line element is an *exact* solution of Einstein's field equations in the empty space outside the spherical mass distribution, expressed in the so-called Painlevé-Gullstrand coordinates. Adopting spherical spatial coordinates and dropping subscripts we have

$$ds^2 = -\left(c^2 - \frac{2GM}{r}\right) dt^2 + 2\sqrt{\frac{2GM}{r}} dr dt + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

We can write this line element in a more familiar form by transforming to a new time coordinate, t_{S}:

$$t_{\text{S}} = t - \left[\frac{2r}{c}\sqrt{\frac{2GM}{rc^2}} - \frac{4GM}{c^3}\tanh^{-1}\left(\sqrt{\frac{2GM}{rc^2}}\right)\right] = t_{\text{S}}(t, r).$$

e) Carry out the transformation of the line element. Do you recognize the result? (Hint: Use the fact that $dt_{\text{S}} = \frac{\partial t_{\text{S}}}{\partial t} dt + \frac{\partial t_{\text{S}}}{\partial r} dr$.)

9 Problems

Problem 1 (2–3 hours)

Imagine a shell observer at shell r is pointing a laser pen radially outwards from the central mass. The beam has wavelength λ. Here we will try to find the wavelength λ' observed by the far-away observer.

1. The frequency of the light emitted by the laser pen is $\nu = 1/\Delta t$. The frequency of the light received by the far-away observer is $\nu' = 1/\Delta t'$. Here Δt and $\Delta t'$ is the time interval between two peaks of electromagnetic waves. Show that the difference in time interval measured by the two observers is given by

$$\Delta t' = \frac{\Delta t}{\sqrt{\left(1 - \frac{2M}{r}\right)}}.$$

   Hint: Imagine that a clock situated at shell r ticks each time a peak of the electromagnetic wave passes.

2. Use this fact to show that the gravitational 'Doppler' formula, i.e. the formula which gives you the wavelength observed by the far-away observer for light emitted close to the central mass, is given by

$$\frac{\Delta\lambda}{\lambda} = \frac{\lambda' - \lambda}{\lambda} = \frac{1}{\sqrt{\left(1 - \frac{2M}{r}\right)}} - 1$$

3. Show that for distances $r >> 2M$ this can be written as

$$\frac{\Delta\lambda}{\lambda} = \frac{M}{r}$$

   Hint: Taylor expansion.

4. We will now study what wavelength of light that an observer far away from the Sun will observe for the light with the wavelength $\lambda_{\text{max}} = 500$ nm emitted from the solar surface.

   (a) Find the mass of the Sun in meters.

   (b) Find the ratio M/r for the surface of the Sun.

   (c) Find the redshift $\Delta\lambda/\lambda$ measured by a far-away observer.

   (d) Is the apparent color of the Sun changed due to the gravitational redshift?

   (e) For light coming from far away and entering the gravitational field of the Earth, an opposite effect is taking place. The light is blue shifted. Find the ratio M/r for the surface of the Earth.

   (f) Find the gravitational blue shift $\Delta\lambda/\lambda$ for light arriving at Earth. Does this change the apparent color of the Sun?

5. A quasar is one of the most powerful sources of energy in the universe. The quasars are thought to be powered by a so-called accretion disc: Hot gas circling and falling into a black hole. The gas reaches velocities close to the speed of light as it approaches the horizon, but since we only see the sum of the radiation coming from all sides of the black hole, we expect the Doppler effect due to the velocity of the gas to cancel out. Assume that we find evidence for an emission line at $\lambda = 2150$ nm in the radiation from a quasar. Assume also that we recognize this emission line to be a line which in the laboratory is measured to occur at $\lambda = 600$ nm. Give some arguments explaining why this observation supports the hypothesis of quasars having a black hole in the center and find from which distance r (expressed in terms of the black hole mass M) from the center, the radiation is emerging. Assume that the Doppler effect due to the quasar's movement with respect to us has been subtracted.

6. Imagine you are a shell observer living at a shell at $r = 2.01M$ very close to the horizon of a black hole of mass M. Can you use optical telescopes to observe the stars around you? What kind of telescope do you need?

Problem 2 (30 min.–1 hour)

In this exercise we will use the principle of maximal aging to deduce the law of conservation of angular momentum in general relativity. In the text you have seen three examples of this kind of derivation and here we will follow exactly the same procedure. Before embarking on this exercise, please read the examples in the text carefully.

1. Use figure 11 in this exercise: We will study the motion of an object which passes through three points ϕ_1, ϕ_2 and ϕ_3 at times t_1, t_2 and t_3. We fix t_1, t_2 and t_3 as well as ϕ_1 and ϕ_3. The free parameter here is ϕ_{12}, the angle between ϕ_1 and ϕ_2. We assume that between ϕ_1 and ϕ_2 the radius is $r = r_A$ (we assume the distance between these two points to be so small that r is constant) and between ϕ_2 and ϕ_3 we have $r = r_B$ (see again figure 11). Use the Schwarzschild line element to show that the proper time interval from ϕ_1 to ϕ_3 can be written as

$$\Delta\tau_{13} = \tau_{12} + \tau_{23} = \sqrt{\left(1 - \frac{2M}{r_A}\right) t_{12}^2 - r_A^2 \phi_{12}^2} + \sqrt{\left(1 - \frac{2M}{r_B}\right) t_{13}^2 - r_B^2 \phi_{13}^2}$$

2. Use the principle of maximal aging to show that

$$\frac{r_A^2 \phi_{12}}{\tau_{12}} = \frac{r_B^2 \phi_{23}}{\tau_{23}}$$

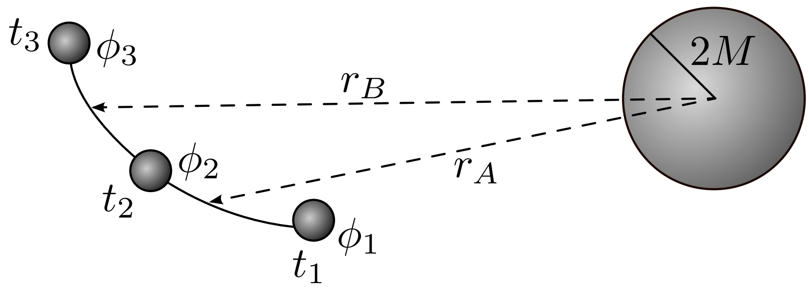

Figure 11: A sketch of problem 2.

3. Show that

$$L = r^2 \frac{d\phi}{d\tau}$$

is a conserved quantity.

Problem 3 (1–2 hours)

Assume that the crew on an airplane works on average 8 hours per day 365 days a year for 50 years. Assume that all this time, they are at a height of $\Delta r = 10$ km above the ground (assume the radius of the Earth to be $r = 6000$ km) moving at a velocity of $v_{\text{Airplane}} = 1000$ km/h with respect to the center of the Earth. We will here ignore the effect of acceleration during take-off and landing.

1. Show that proper time intervals $\Delta\tau$ for the crew at work can be written in terms of time intervals Δt_{Earth} measured on Earth clocks as

$$\frac{\Delta\tau}{\Delta t_{\text{Earth}}} = \sqrt{\frac{1 - \frac{2M}{r+\Delta r} - v_{\text{Airplane}}^2}{1 - \frac{2M}{r} - v_{\text{Earth}}^2}},$$

where v_{Earth} is the velocity of a person on the Earth with respect to the center of the Earth.

2. This expression may give numerical problems when using very small numbers. For this reason we will try a Taylor expansion. Calculate M/r as well as the velocity v_{Airplane} and v_{Earth} (use Earth's rotation period) in dimensionless units. Are these so small that we can Taylor expand the expression above in terms of $2M/r$, v_{Airplane} and v_{Earth}?

3. Show that the Taylor expansion of this expression, assuming that these three quantities are small, can be written as

$$\frac{\Delta\tau}{\Delta t_{\text{Earth}}} \approx 1 + \frac{1}{2}(v_{\text{Earth}}^2 - v_{\text{Airplane}}^2) + M\left(\frac{1}{r} - \frac{1}{r+\Delta r}\right).$$

Hint: Taylor expand first in

$$x = -(\frac{2M}{r+\Delta r} + v_{\text{Airplane}}^2),$$

then in

$$y = -(\frac{2M}{r} + v_{\text{Earth}}^2).$$

4. Use this expression to find how much shorter a crew member lives with respect to persons staying on Earth, taking into account only relativistic effects?

5. Would you now skip next year's vacation in the fear of getting old too fast?

Problem 4 (1–2 hours)

Study carefully the chapter on GPS.

1. Use the information in that chapter to find how long after the launch of the satellites the GPS system will be useless. Define useless to be when the uncertainty in the position measurements is of the order 1 kilometer.

2. Imagine that in the future when Mars has been terraformed (i.e. one has managed to grow plants there which produce oxygen and thus an atmosphere which we can breath) and a large part of the population of Earth has moved to Mars. It is finally decided to set up a GPS system on Mars in order to make it easier for the mountaineers boldly climbing the Olympus Mons (the highest mountain on Mars, 27 000 meters high) to find their position on the map. However, even with future high-tech, human errors are unavoidable and the people setting up the GPS forgot about relativistic corrections. In this exercise you will need to look up the mass, radius and orbital period of Mars. You also need to know that the GPS satellites have orbital periods of 12 hours (assume circular orbits).

   (a) At what height above the surface are the satellites orbiting?

   (b) How long after the launch of the satellites will the poor mountaineers get lost on Olympus Mons? (They are lost when they think they are 1 kilometer away from where they really are).

AST1100 Lecture Notes

17: General Relativity: Orbits

1 Schwarzschild step-by-step motion

In this lecture we will look at corrections to orbital motion due to general relativity. We have already learned that a body in the gravitational field of another body may go in elliptical orbits or escape to infinity following parabolic or hyperbolic trajectories depending on the total energy of the body. We have now obtained more accurate expressions for motion in gravitational fields and will check if these corrections may give rise to orbital behavior different from the Newtonian prediction. We will study the motion of a body in the gravitational field of a black hole. We might already anticipate a few differences to Newtonian gravity: If the body comes too close to the black hole (inside the Schwarzschild radius), it will be swallowed by the black hole without possibilities to get out. We will now check this in more detail.

In figure 1 we show a spaceship at position (r, ϕ, t) in Schwarzschild coordinates around a black hole of mass M. The spaceship has used all its fuel and can therefore not use its engine, it is falling freely. The astronauts in the spaceship are wondering whether the spaceship will pass the black hole so close to the center that they will be swallowed by the black hole or not. We will now study the motion of the spaceship step by step. We will ask the question, what is the new position (r, ϕ, t) in Schwarzschild coordinates of the spaceship after a time interval $\Delta\tau$ has passed on the wrist watches of the astronauts? We will look for the small increments Δr, $\Delta\phi$ and Δt for each small increment in astronaut proper time $\Delta\tau$. By increasing $\Delta\tau$ and thereby the other coordinates step by step, we will be able to follow the motion (r, ϕ) of the spaceship and check if it at some point will reach $r = 2M$ or not.

Knowing that the total energy per mass E/m is a constant of motion, we can rewrite the expression

$$\frac{E}{m} = \left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau},$$

for total energy per mass as

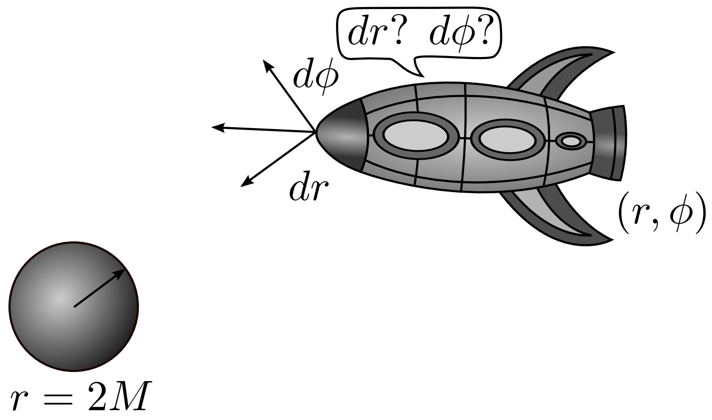

Figure 1: The spaceship is out of fuel. The engines stop. What will be the next movement in r and ϕ direction?

$$\Delta t = \frac{E/m}{(1 - \frac{2M}{r})}\Delta\tau. \tag{1}$$

Similarly we can use that the angular momentum per mass L/m is a constant of motion

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau}$$

to get

$$\Delta\phi = \frac{L/m}{r^2}\Delta\tau. \tag{2}$$

We have already obtained the displacements $\Delta\phi$ and Δt per proper time interval $\Delta\tau$. Now we need to find the radial displacement Δr. The Schwarzschild line element (see previous lecture) gives

$$\Delta s^2 = \Delta\tau^2 = \left(1 - \frac{2M}{r}\right)\Delta t^2 - \frac{\Delta r^2}{(1 - \frac{2M}{r})} - r^2\Delta\phi^2.$$

We insert the expressions (1) and (2) into the line element and obtain

$$\Delta\tau^2 = \left(1 - \frac{2M}{r}\right)\left(\frac{E/m}{(1 - \frac{2M}{r})}\right)^2\Delta\tau^2 - \frac{\Delta r^2}{(1 - \frac{2M}{r})} - r^2\left(\frac{L/m}{r^2}\right)^2\Delta\tau^2.$$

Reorganizing we find

$$\Delta r = \pm\sqrt{\left(\frac{E}{m}\right)^2 - \left[1 + \left(\frac{L/m}{r}\right)^2\right]\left(1 - \frac{2M}{r}\right)}\Delta\tau. \tag{3}$$

We now have three equations (1), (2) and (3) giving us the motion of the spaceship as observed by the far-away observer for each tick $\Delta\tau$ on the wristwatch of the astronauts. Note that these expressions in reality give the first order terms of a Taylor expansion in $\Delta\tau$. The second derivative terms are not included (and will not be treated in this course) and we can therefore not use them in this form to describe a full orbital motion. In orbital motion, when the radial velocity reaches zero, the spaceship will start moving outwards again (do you see that this is the case? Think about the motion of a planet). Radial velocity equal to zero means that the first derivative is zero and that the second derivatives (second order in the Taylor expansion) is needed in order to describe the next step. But we may use it up to the point where the radial velocity is zero. If the radius at this point is outside $r = 2M$ we are saved. If the radial velocity does not reach zero before $r = 2M$ the spaceship will fall into the black hole. In order to describe the full motion in a more complete manner we can either continue the Taylor expansion to higher orders or, much easier, we can consider the effective potential.

2 Effective potential

To explain the concept of effective potential, we will go to a well known example: An object sliding down a hill without friction. In figure (2) we see the situation. An object is located at horizontal position x and at height h. We can write the total (Newtonian) energy, kinetic plus potential, with the well known expression

$$E/m = \frac{1}{2}v^2 + gh(x) = \frac{1}{2}v^2 + V(x),$$

where g is the constant gravitational acceleration, $h(x)$ is the height of the hill at position x, $V(x)$ is the potential, v is the velocity of the object and m its mass. In figure 3 I have made the same plot as figure 2, but the function is now multiplied by g such that the y-axis now shows $gh(x)$ instead of only $h(x)$. Thus, as you see from the previous expression, the units on the y-axis is now energy per mass and the height of the hill is just the potential $V(x)$. When the velocity is zero $v = 0$, the height of the object in this plot directly gives us the total $E/m = gh(x)$ for the object (you can see this from the previous equation: if $v = 0$ then $E/m = V(x)$). Thus we can draw a horizontal line passing through this point, showing that this is the energy per mass of the object for all positions x (remember that E/m is constant). The object will have velocity zero at all points where the horizontal line intersects the hill curve (why?).

We have defined the height $h(x)$ to go to asymptotically to zero for large distances $x \to \infty$. Thus, at large distances the energy of the object consists of purely kinetic energy as the potential energy $gh(x) \to 0$. A total negative energy of the object corresponds to an object left at rest at $h(x) < 0$. This object can never reach infinity: We just learned that at infinity the energy of the object is purely kinetic, but kinetic energy

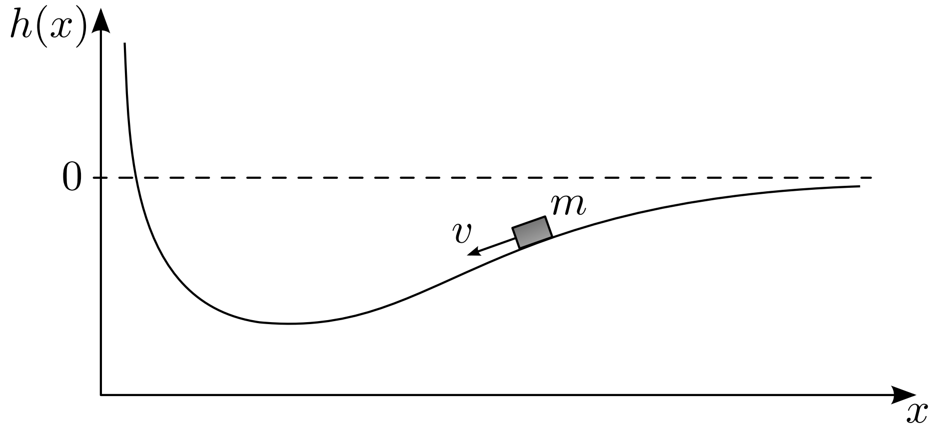

Figure 2: Object sliding down a frictionless hill with height $h(x)$.

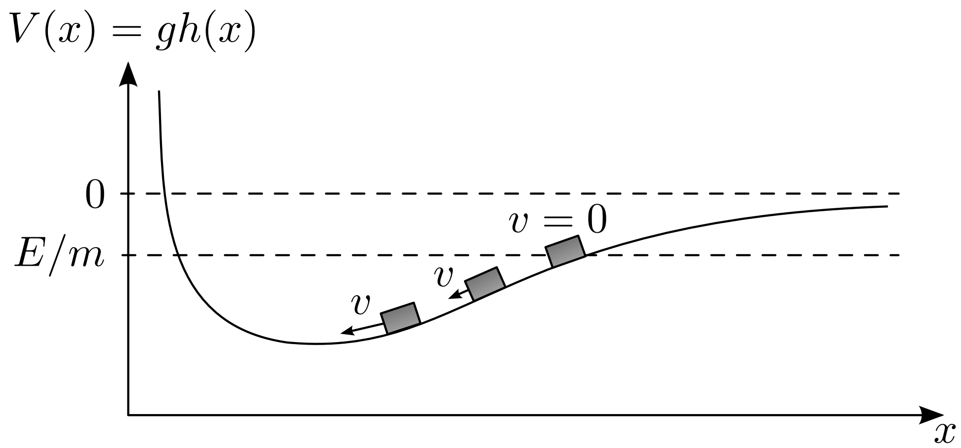

Figure 3: Object sliding down a frictionless hill with energy per mass $E/m = gh(x)$ deciding the future motion.

cannot be negative. So an object with a negative total energy is trapped in the 'valley' seen in the figures. Note also that an object with negative energy cannot move all the way in to $x = 0$, it can only reach up to the point E/m on the y-axis where the velocity will be zero: It will start oscillating back and forth between the two points where the horizontal line at E/m crosses the hill curve. The situation is different for an object with positive energy: Leave the object far out on the positive x axis with an initial velocity different from zero and so large that E/m is positive. By drawing a horizontal line at E/m you can find how far in the object will move before it has $v = 0$ from where it will move back an out to infinity. This object is not bound in the valley. The two situations are illustrated in figure 4 and 5.

This case was probably not new to you. We will now generalize this situation. We see (from equation 3) that the equation of motion for this object can be written as

$$A = B\dot{\vec{x}}^2 + V(x), \tag{4}$$

where A (equal to E/m in our example) and B (equal to 1/2 in our example) are constants (B being positive), $\vec{x}$ is the position vector of the object and $V(x)$ is the position dependent *potential.* If $V(x)$ has a 'valley' similar to figure 3 and $V(x) \to 0$ when $x \to \infty$, then the object with position $\vec{x}$ will move in the following way:

- With $A < 0$ (corresponding to $E < 0$ in our example) the object is trapped and will oscillate back and forth between two positions.
- With $A > 0$ (corresponding to $E > 0$ in our example), the object can escape out to any position.

We recognize the situation described here from a similar physical system: The two body problem. We remember for the two-body problem that an object with negative total energy was bound to orbital motion around the other object whereas objects with positive total energy could escape to infinity. Let's try to see the mathematical analogy. The total energy of an object with mass m close to a star of mass M is

$$E/m = \frac{1}{2}v^2 - G\frac{M}{r},$$

and angular momentum

$$L/m = r^2\dot{\phi} \tag{5}$$

using for the moment conventional units. We are only interested in the radial motion of the object, i.e. whether the object will be bound or whether it can escape to infinity. We are not interested in the details of the motion in ϕ direction. We can rewrite the equation for the energy as

$$E/m = \frac{1}{2}(\dot{r}^2 + r^2\dot{\phi}^2) - G\frac{M}{r} = \frac{1}{2}\dot{r}^2 + \left(\frac{1}{2}\frac{(L/m)^2}{r^2} - G\frac{M}{r}\right), \tag{6}$$

where equation 5 was used. Setting $A = E/m$, $B = 1/2$ and

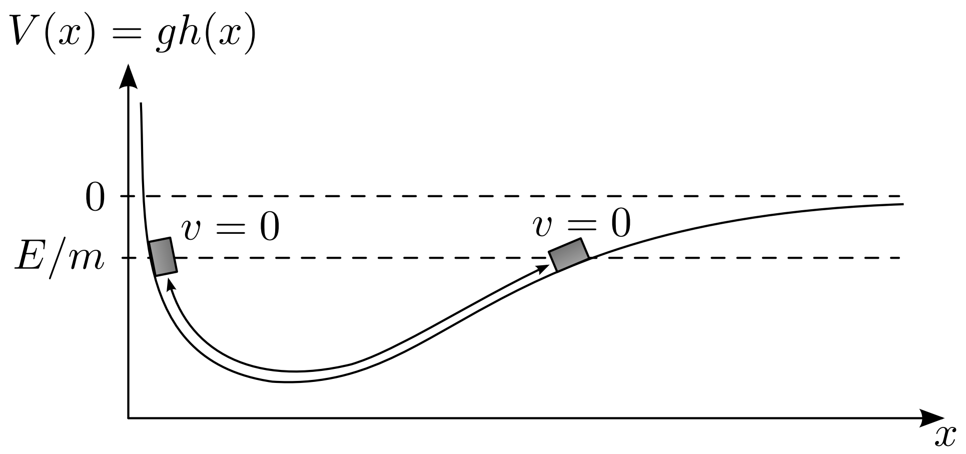

Figure 4: Bound object oscillating between two points on the hill.

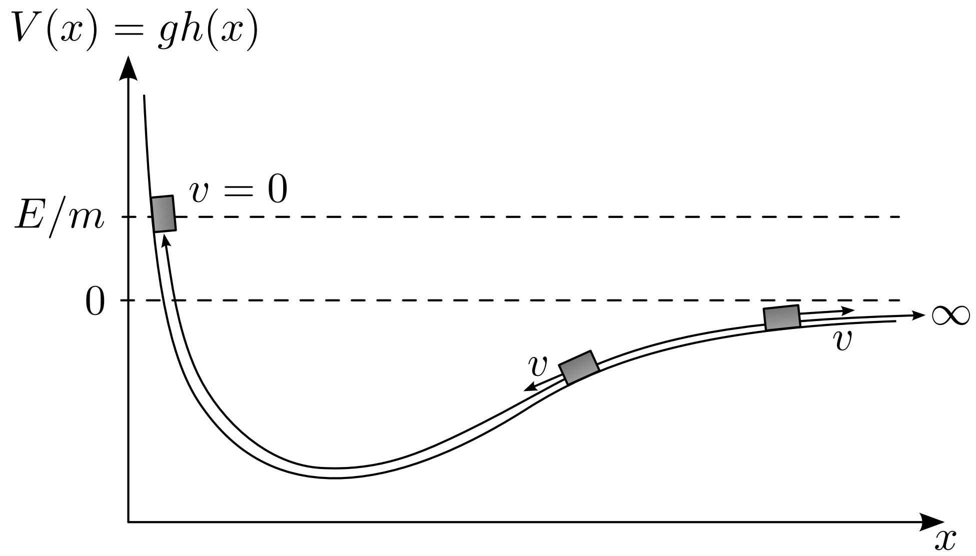

Figure 5: Free object: slides up to a maximum point and then escapes to infinity.

$$V_{\text{eff}}(r)/m = \frac{1}{2}\frac{(L/m)^2}{r^2} - G\frac{M}{r},$$

we see that equation 6 can be written on the form of equation 4. We will call the potential $V(r)$ for the *effective potential*. Thus the problem is mathematically identical to the problem of the object sliding down the hill. This means that also the results are identical. The r coordinate corresponds to position on the hill, and the effective potential corresponds to the shape of the hill. In figure 6 we can see the shape of the 'hill' or effective potential. The object falling in the gravitational field of a star is identical to the object sliding down the hill using the effective potential as the shape of the hill. Again we have the result that for $A = E/m < 0$, the object is bound and will oscillate between two r positions which we know (from earlier lectures) are $r = a(1-e)$ and $r = a(1+e)$. Here we have ignored the motion in ϕ direction, but we already know that this corresponds to an elliptical orbit. For $E/m = 0$, the object will reach zero velocity at in infinite distance $r \to \infty$. We already learned in previous lectures that this corresponds to the parabolic trajectory. Finally for $E/m > 0$, the object can move to infinite distances with arbitrary velocity corresponding to the hyperbolic trajectory. Even though the treatment with effective potential did not give us the exact shape of the orbit it did tell us the essentials using the radial motion only: The object can either oscillate between two radial positions or it can move out to infinity depending on the total energy E/m.

3 Orbital motion in Schwarzschild geometry

We will now turn to the relativistic case. We have seen that by looking just at the radial motion of an object in a gravitational field we can obtain essential information about the future motion of this object without going into details. Equation 3 can be written as

$$\left(\frac{dr}{d\tau}\right)^2 = \left(\frac{E}{m}\right)^2 - \left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]. \tag{7}$$

Again comparing to equation 4 we see that we can make the following substitutions: $A = (E/m)^2$, $B = 1$ and

$$\frac{V_{\text{eff}}(r)}{m} = \sqrt{\left(1 - \frac{2M}{r}\right)\left[1 + \frac{(L/m)^2}{r^2}\right]}$$

We have defined the effective potential such that the square of the effective potential appears in equation 7, different from the previous cases. This is just to have an effective potential with units energy. Note that A is now

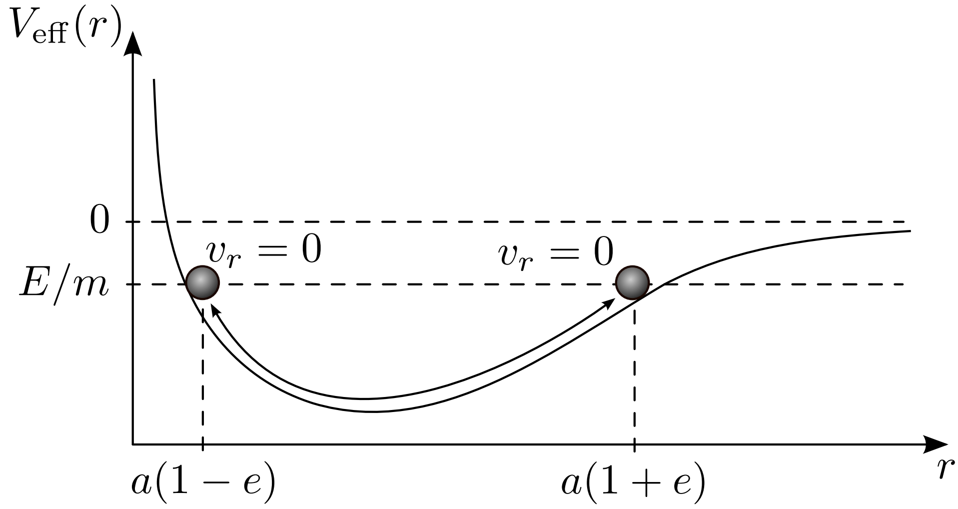

Figure 6: A bound object in elliptical orbit in a Newtonian effective potential.

the energy per mass E/m *squared* instead of just E/m as we had in the above examples. In equation 4 we only required A to be a constant, it is not required that it equals energy. So we still have exactly the same case as we had above and we can use the same argumentation. Note one more difference: The effective potential goes to $V(r) \to 1$ for large distances instead of $V(r) \to 0$ as above (see the plot of the effective potential in figure 7). The reason for this is that the rest energy for a particle in relativistic dynamics is $E/m = 1$. If the velocity of the object is zero at large distances then $E/m = 1$ whereas in Newtonian dynamics $V(r) \to 0$ because $E/m = 0$ at large distances. Remember that in Newtonian dynamics we do not consider the rest energy $E = m$. This makes one difference in our argumentation with respect to above. In the Newtonian case, the limiting energy deciding whether the object would be trapped in the potential and therefore stay in a bound orbit or if it would escape to infinity was $E/m = 0$. As we see, in relativistic dynamics this limit is $E/m = 1$. If $E/m < 1$ then the ball starts falling with zero velocity at some point on the hill below $E/m = 1$ and it can therefore never escape to $r \to \infty$, it will start orbiting. If however the energy $E/m > 1$ it has the possibility to escape to infinity as it will have a non-zero velocity as $r \to \infty$ (check that you understand this by looking at equation 7 and figure 7).

Looking at figure 7 we see one radical difference in the shape of the effective potential with respect to the Newtonian case. At a certain critical radius $r = r_{\text{crit}}$ the potential has a peak and thereafter it falls steeply downwards towards $r = 0$. This is not surprising: Any particle which passes inside the horizon at $r = 2M$ cannot escape. We see from figure 7 that even objects with energies larger than $E/m = 1$ (objects which are not bound in the classical sense) may be swallowed by the black hole. The objects with an energy E/m larger than the critical energy E_{crit}/m will pass too

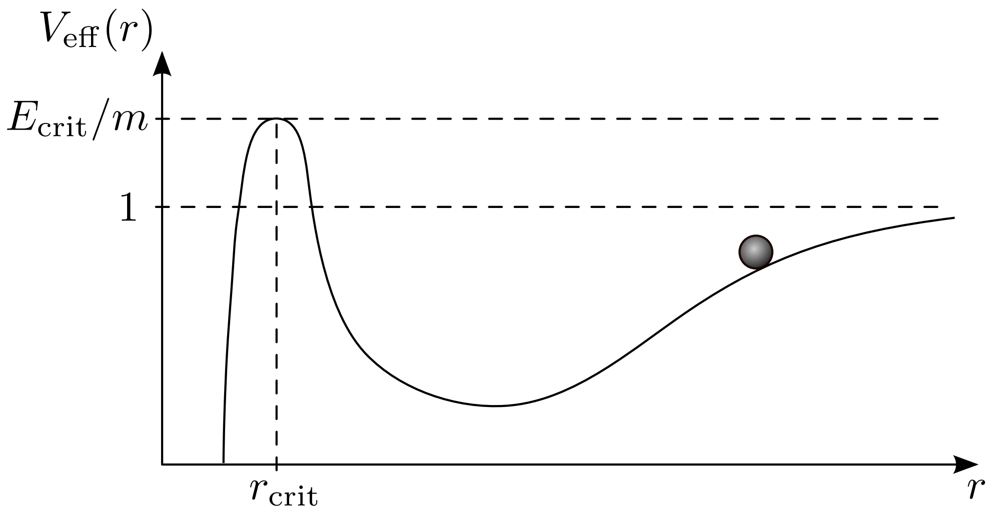

Figure 7: A bound object in elliptical orbit in a Schwarzschild effective potential.

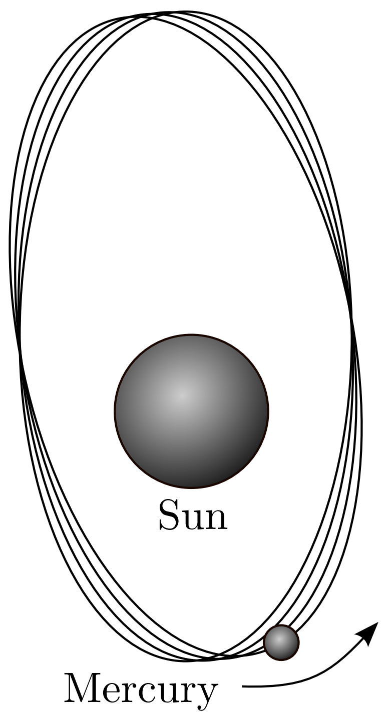

Figure 8: Perihelion precision of Mercury.

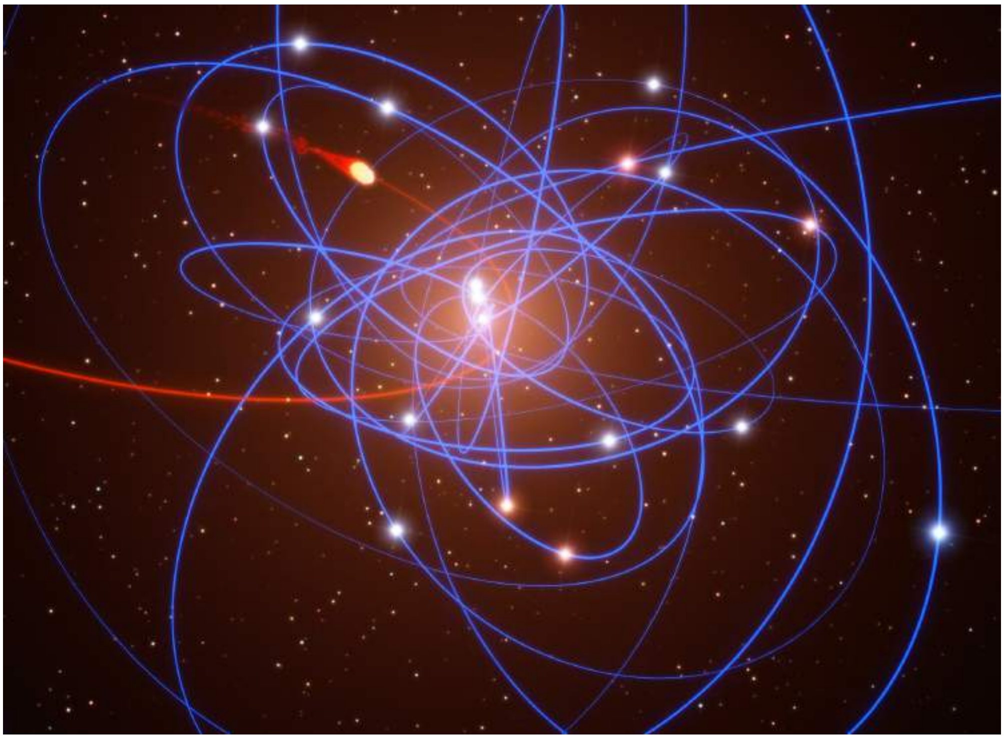

Figure 9: Info-figure: This simulated view based on real data shows stars orbiting the supermassive black hole at the center of the Milky Way along with blue lines marking their orbits. Also, a gas cloud (above center, with its orbit shown in red) has recently been observed approaching the black hole at more than 8 million km/h. The stars and the cloud are shown in their actual positions in 2011. Extremely precise measurements of the stellar orbits in the galactic center show that the supermassive black hole, formally known as Sgr A* (pronounced Sagittarius A star), has a mass of 4.1 million solar masses. The interstellar dust that fills the galaxy blocks our view of the Milky Way's central region in visible light, but astronomers use infrared wavelengths that can penetrate the dust to probe the region.(Figure: ESO)

close to the object, so close that $r < r_{\mathrm{crit}}$ and it is captured by the black hole. In the Newtonian case, this object would have a large enough energy to escape as $E/m > 1$. In the exercises you will derive an expression for E_{crit}. An object which enters the black hole with an energy $E = E_{\mathrm{crit}}$ equal the critical energy will make a few orbits around the black hole at $r = r_{\mathrm{crit}}$ before coincidences will make tiny changes to the energy of the object. These tiny changes may go in either direction, either the object will escape or the object will plunge into the black hole. We thus have three possibilities:

- $E/m < 1$ which gives orbits
- $1 < E/m < E_{\mathrm{crit}}/m$ for which the object can move to infinity
- $E/m > E_{\mathrm{crit}}/m$ for which the object will plunge into the black hole

There is one more important difference between the relativistic and the Newtonian effective potential. We will now consider a planet in orbit around a star. Because of the peak at $r = r_{\mathrm{crit}}$, the potential rises more steeply after the minimum than in the Newtonian case. A planet moving inwards in its orbit towards the star will thus have to climb up this steeper potential and will therefore slow down more close to the perihelion (the point in the orbit of a planet closest to the star). The radial velocity of the planet in the parts of the orbit close to the star is thus slower than in the Newtonian case. Since the planet then spends more time in the orbit close to the star, the planet now also has more time to move in the ϕ direction for which there is no slow-down. Thus, in general relativity the planet has moved more in the ϕ direction after passing close to the star than it would in the Newtonian case. How does this affect the orbit? The result is that the perihelion moves around the star. This is illustrated in figure 8. For each orbit, the perihelion moves a little bit in ϕ direction. In Newtonian physics, the perihelion stays at the same point. This ϕ motion of the perihelion is called *perihelion precession*.

Long before Einstein discovered the general theory of relativity, it was well known that Mercury, the planet closest to the Sun, had a strong perihelion precession. A large part of this precession could be attributed to the gravitational forces from other planets in the solar system. But the gravitational attraction from other planets was not able to explain the full precession. A little part remained and it turned out that general relativity accounts for exactly this difference.

4 Inside the horizon

In the previous lecture we studied an object falling into the black hole from rest at a large distance from the black hole. We found that the conserved energy gave

$$\left(1 - \frac{2M}{r}\right)\frac{dt}{d\tau} = 1.$$

Using this, we obtained the speed of the object as measured by the far-away observer

$$\frac{dr}{dt} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}}$$

and the speed of the object measured by the local shell observers as the object passes the shells

$$\frac{dr_{\text{shell}}}{dt_{\text{shell}}} = -\sqrt{\frac{2M}{r}}.$$

What is the velocity $dr/d\tau$ measured on the wristwatch time τ of the falling object? Using these three equations we can write

$$\frac{dr}{d\tau} = \frac{dr}{dt}\frac{dt}{d\tau} = -\left(1 - \frac{2M}{r}\right)\sqrt{\frac{2M}{r}}\left(1 - \frac{2M}{r}\right)^{-1} = -\sqrt{\frac{2M}{r}}. \quad (8)$$

Even when measuring velocity on the wristwatch of the object, the velocity approaches the speed of light at the horizon and gets larger than the speed of light inside the horizon. But who measures this velocity? Nobody! In this velocity measurement, length is measured by the far-away observer (who cannot measure anything after the object has entered the horizon) and time is measured on the wristwatch of the falling object. We also learned that inside the horizon there are no shell observers to measure the velocity since you cannot be at rest inside the horizon. A local observer sitting in an unpowered spaceship passing the object will always measure that the velocity is less than unity. Why? Because any freely falling observer is in a local inertial frame for a short moment when the spaceship passes nearby, even when inside the horizon. So for the freely falling observer special relativity applies (for a short moment when the spaceship passes nearby)) and he will always measure the velocity of the object as being less than the velocity of light.

How long will it take for the object to reach the singularity in the center from the moment it enters the horizon? We can integrate equation 8 to find the time measured on the wristwatch of the object

$$\tau = -\int_{2M}^{0} dr\sqrt{\frac{r}{2M}} = -\left[\frac{2}{3}\sqrt{\frac{r}{2M}}r\right]_{2M}^{0} = \frac{4M}{3}.$$

How long will it take for an observer falling into a black hole with one solar mass to go from the horizon to the singularity? Measured on the wristwatch of the observer it takes

$$\tau = \frac{4M_\odot}{3} = \frac{4 \times 2 \times 10^{30}\ \text{kg} \times 7.42 \times 10^{-28}\ \text{m/kg}}{3} \approx 2000\ \text{m} \approx 7\ \mu\text{s}$$

In problem 3, you will study how the astronaut in a spaceship inside the horizon experiences the world.

5 Problems

Problem 1 (2–3 hours)

A rocket is launched from shell $r = 20M$ around a black hole of mass M with velocity $v_{\text{shell}} = 0.993$ at an angle $\theta = 167°$ with the outward pointing vector from the black hole (see figure 10). Just after launch, there is a problem with the engines and they stop. The shell observers at shell $r = 20M$ need to make another rocket to rescue the astronauts, but this takes a long time. The astronauts are worried that they will be captured by the black hole. In this exercise we will try to find out whether the rocket will be captured by the black hole or not. The angular momentum of the rocket is L and the mass of the rocket is m. In this exercise you will need the following relation a couple of times

$$\frac{dx}{d\tau} = \frac{dx}{dt_{\text{shell}}}\frac{dt_{\text{shell}}}{d\tau},$$

where x can be any quantity.

1. First we need to find out the shape of the effective potential. Use the general relativistic expression for the effective potential to show that the minimum and the maximum of the effective potential are located at the following distances(measured in Schwarzschild coordinates) from the black hole

$$r_{\text{extremum}} = \frac{(L/m)^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{(L/m)^2}}\right).$$

   Which of these two solutions is the maximum of the potential?

2. Show that the angular momentum per mass for the rocket can be written as

$$\frac{L}{m} = r^2\frac{d\phi}{d\tau} = r\gamma_{\text{shell}}v_{\text{shell}}\sin\theta,$$

   where $\gamma_{\text{shell}} = 1/\sqrt{1 - v_{\text{shell}}^2}$. **Hint 1:** Remember that for short time intervals dt_{shell}, the shell observers can use special relativity. **Hint 2:** How could we write $dt/d\tau$ in special relativity?

3. Use the general relativistic expression for E/m to show that the total energy per mass of the rocket can be written as

$$\frac{E}{m} = \sqrt{1 - \frac{2M}{r}}\gamma_{\text{shell}}$$

4. Insert numbers in the expression for L/m and draw the potential (by hand using the information you have obtained from the previous exercises) having r in units of M on the x-axis and numbers for V_{eff}/m on the y-axis.

5. Will the rocket be captured by the black hole?

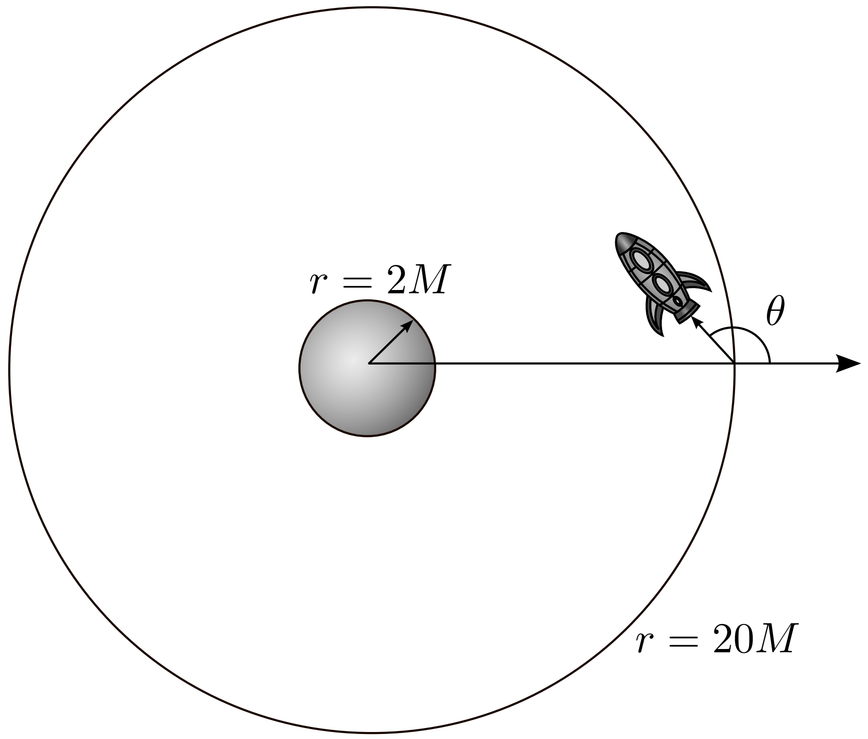

Figure 10: Rocket launched from shell $r = 20M$ inwards at an angle θ. Note: Figure not to scale.

6. If they are captured by the black hole, how long does it take (on the wristwatch of the astronauts) to reach the singularity from the moment they enter the horizon. For simplicity ignore the spin of the rocket. (give the answer in seconds assuming that this is the black hole in the center of the Milky way, $M \approx 4 \times 10^6 M_\odot$).

   Important hint: You cannot use the result given in the text. Check that you understand why and find the correct result. In the end you will need to do an ugly integral. Go to 'The Integrator' (`http://integrals.wolfram.com/index.jsp`) and type

   `1/sqrt(a+b/x).`

7. What will happen with the astronauts just before entering the singularity? Draw an astronaut and draw the gravitational forces (ok, let's cheat and use forces for a moment since they are easier to draw than spacetime geometry). Which shape will he/she have just before reaching the center?

Problem 2 (1–2 hours)

In this exercise we will make a python (or matlab or whatever) code to plot the orbit of the spaceship in the previous exercise. We will start at $r = 20M$ and evolve the position of the spaceship forward in time using equations (2), (3).

1. Define variables for $(L/m)/M$ and (E/m) and give them the values you found in the previous exercise. Define a variable for the distance from the black hole r/M and give it the initial value of 20. Finally define a variable ϕ which is the angular position with respect to the black hole. We give ϕ an initial value of zero. Define a variable which is the number of time steps we will use. Set the variable to 1000. Finally define a variable which is the proper time step $\Delta\tau = 0.01$.

2. Now, define two arrays both with size equal to the number of time steps (1000). The first array will contain the r position at each time step, the other will contain the ϕ position at each time step. Set the first element in both arrays to the current value of r and ϕ.

3. Make a FOR loop over all time steps. For each step, update r and ϕ with the increments Δr and $\Delta\phi$ until $r/M < 2$.

4. Finally we need to plot the orbit. Make two arrays x and y converting the arrays with r and ϕ values from polar to Cartesian coordinates. The black hole is at position $x = 0$ and $y = 0$. Now we have two arrays with the x and y position of the spaceship at different time steps. Now plot a dot at each step in the orbit.

5. Now we will overplot the Schwarzschild radius: Make an array with, say 100 elements, with the r position of the horizon $r = 2$. Make a corresponding array with equal number of elements having numbers going from 0 to 2π being the ϕ position. Then transform from polar to Cartesian coordinates exactly as you did in the previous step and plot the set of x and y positions you have obtained. Now you will see a circle showing the horizon.

6. How large angle $\Delta\phi$ did the spaceship revolve around the black hole before entering the horizon?

Problem 3 (1–2 hours)

We are in a spaceship inside the horizon falling towards the central singularity. We are trying to find a way to escape. In order to check all possibilities we send one light beam backwards away from the central singularity and one forward towards the central singularity. In order to study how these beams of light are moving we need to write the Schwarzschild line element in terms of our wristwatch time t' instead of Schwarzschild time t. We will make this change of coordinates already before entering the horizon as this allows us to use shell frames as a help. Assume in the following that we have velocity only in the radial direction. Assume also that we started falling freely with velocity $v = 0$ far away from the black hole.

1. Use the Lorentz transformations to show that time intervals measured on the wristwatch of the astronauts are related to time and

space intervals measured by shell observers as

$$dt' = -v_{\rm shell}\gamma_{\rm shell}dr_{\rm shell} + \gamma_{\rm shell}dt_{\rm shell},$$

where $v_{\rm shell}$ and $\gamma_{\rm shell}$ are based on the local velocity of the astronaut measured by the shell observer at the shell which the spaceship passes.

2. Use the expressions relating shell coordinates and Schwarzschild coordinates to show that

$$dt' = -\frac{v_{\rm shell}\gamma_{\rm shell}dr}{\sqrt{\left(1-\frac{2M}{r}\right)}} + \gamma_{\rm shell}\sqrt{\left(1-\frac{2M}{r}\right)}dt.$$

3. In the previous lecture, we deduced the shell velocity $v_{\rm shell}$ of a falling spaceship starting with $v = 0$ far from the black hole. Go back and check this expression. Insert it in the previous expression and show that

$$dt = dt' - \frac{\sqrt{2M/r}dr}{\left(1-\frac{2M}{r}\right)}.$$

4. Use this to substitute dt with dt' in the normal Schwarzschild line element and show that the Schwarzschild line element can be written

$$ds^2 = d\tau^2 = \left(1-\frac{2M}{r}\right)(dt')^2 - 2\sqrt{\frac{2M}{r}}dt'dr - dr^2 - r^2d\phi^2.$$

Note that this form of the Schwarzschild line element does not have a singularity at $r = 2M$.

5. We will now study the motion of the two light beams that we emit, one forwards and one backwards. We know that for light, proper time is not moving $d\tau = 0$. The light beams in this case are moving only radially so $d\phi = 0$. Show that the speed of the two beams can be written as

$$\frac{dr}{dt'} = -\sqrt{\frac{2M}{r}} \pm 1.$$

After entering the horizon, do the astronauts in the spaceship measure the speed of the light beams to be larger than the speed of light? (think twice before answering)

6. In figure 11 we show the worldline of the spaceship falling into the black hole. We have also plotted the world lines of light beams emitted from the spaceship at various points in the trajectory. Use the previous equation to explain why the world lines of the light beams look like they do in the figure.

7. What happens to the light beams... in which direction does each of them move? Suppose that the astronauts had a small rescue rocket which could accelerate to a velocity close to the speed of light. They

went into the rescue rocket and went in the direction opposite of the black hole. What would happen? How would their motion look like? Do you understand better why nothing can escape the black hole?

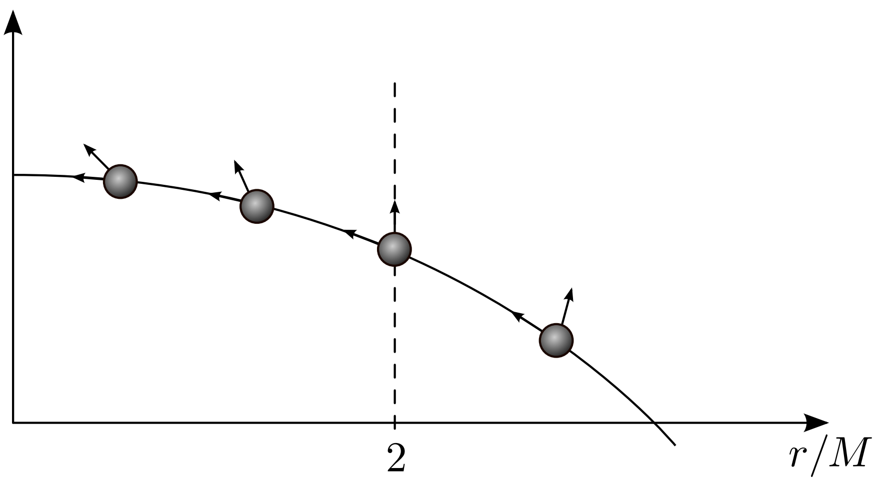

Figure 11: Worldline of the rocket (marked by a balls) and parts of the worldlines of the forward and backward light beam (arrows) at several points during the free fall into the black hole.

AST1100 Lecture Notes

18: General Relativity Gravitational lensing

1 Motion of light in Schwarzschild spacetime

There is one huge difference between Newton's and Einstein's theory of gravity. In the Einstein equation

$$G_{\mu\nu} = 8\pi T_{\mu\nu},$$

energy (not only mass but total *energy*) enters in the energy momentum tensor on the right hand side. This means that not only mass but also pure energy (for instance in the form of light or other kinds of radiation) give rise to curvature of spacetime as described by the left side of the equation. This means that light gives rise to a gravitational field. In the same manner, light is also affected by a gravitational field. We know that light follows a spacetime path such that $ds = 0$. If the geometry of spacetime is the Schwarzschild geometry, this line will necessarily be different than if the geometry is Lorentz geometry. Hence the general theory of relativity predicts light rays to be deflected in a gravitational field. We will now look at the step-by-step motion of a ray of light through Schwarzschild spacetime in the same way as we did for a particle in the previous lecture. There is however one difference: We cannot use the proper time τ as the time parameter as $\Delta\tau = 0$ always for light. We need to eliminate $\Delta\tau$ from the equation of motion for particles. We can do this by using the expression for relativistic energy

$$\frac{d\tau}{dt} = \frac{1 - \frac{2M}{r}}{E/m}.$$

Using this expression to eliminate $\Delta\tau$ in equations (2) and (3) from the previous lecture and taking the limit $m \to 0$ since light is massless, we obtain (check!):

$$\Delta r = \pm\left(1-\frac{2M}{r}\right)\sqrt{1-\left(1-\frac{2M}{r}\right)\frac{(L/E)^2}{r^2}}\Delta t \tag{1}$$

$$r\Delta\phi = \pm\frac{L/E}{r}\left(1-\frac{2M}{r}\right)\Delta t. \tag{2}$$

These equations can again be used to describe the trajectory (r,ϕ) of light as the far-away time t advances. We will use these equations to look at the speed of light in various cases. First we will emit a beam of light radially towards the center of the black hole. This is purely radial motion so $\Delta\phi = 0$ and the angular momentum is zero $L = 0$. Equation 1 then gives

$$v_r = \frac{dr}{dt} = -\left(1-\frac{2M}{r}\right).$$

We see that the speed of light is not one as we are used to. Surprise, surprise! Special relativity was constructed based on the fact that the speed of light is one for all observers. In general relativity this is no longer true: We see here that the speed of light as measured in Schwarzschild coordinates (r,t), the coordinates of the far-away observer, is different from one. And moreover as $r \to 2M$ the speed of light goes to zero. Light slows down to zero close to the horizon (for the far-away observer), just as material particles do.

Now, this was measurements made by the far-away observer who makes measurements based on observations made by different local observers. What speed of light does a shell observer on a shell close to the horizon measure? Does he also see that light slows down and eventually stops? This was not the case for material particles, we will now make the same calculations for light.

The shell observer measures the speed of the light beam as it passes his shell. He makes the measurement in a short time interval such that he can be considered to be in a local inertial frame. Then his geometry is Lorentz geometry

$$d\tau^2 = dt_{\text{shell}}^2 - dr_{\text{shell}}^2$$

(you can show this last expression simply by inserting the expressions relating dr and dr_{shell} as well as dt and dt_{shell} into the Schwarzschild line element) and he will necessarily measure

$$\frac{dr_{\text{shell}}}{dt_{\text{shell}}} = -1$$

We can thus change the principle of invariant speed of light to: A *local* observer, an observer who measures the speed of light directly, will always measure the speed of light to be one. The far-away observer who bases his measurement on the collection of observations from several different local observers will see a different speed of light.

We will also check what happens to a beam of light which moves tangentially. For this light $\Delta r = 0$ which inserted in equation 1 gives

$$\frac{L}{E} = \frac{r}{\sqrt{\left(1 - \frac{2M}{r}\right)}}.$$

Inserting this in equation 2 we get (check!)

$$v_\phi = r\frac{d\phi}{dt} = \sqrt{\left(1 - \frac{2M}{r}\right)}.$$

Also light moving tangentially has a speed different from one, but note the square root which is not present for the radial velocity. We have that $v_r = v_\phi^2$, light moves faster in the the tangential direction than in the radial direction. Also in this case $v_\phi \to 0$ at the horizon. Again, this was light speed measured in Schwarzschild coordinates (r, t) and therefore the light speed measured by the far-away observer. Local observers would again measure a tangential light speed of one.

2 Impact parameter

To study the motion of light in a gravitational field we need to define the *impact parameter* b. The impact parameter is used in many fields of physics and astrophysics, for instance to study colliding particles. In figure 1 we see a large central mass M (for instance a black hole) and a small particle far away from the central mass moving in any given direction. Draw a line passing through the particle going in the direction of motion of the particle. Then draw another line which is parallel to the first line but which passes through the center of the black hole. The distance between these two lines is called the impact parameter. It is important to note that the first line is drawn on the basis of the movement of the particle when the particle is so far away that it has not yet been influenced by the gravitational field. We will soon see that this impact parameter will decide the future motion of the photon.

We can calculate the angular momentum of the photon when it is still far away as

$$L = \vec{r} \times \vec{p} = rp\sin\theta = pb.$$

The angle θ is the angle between $\vec{r}$ pointing at the particle from the center of the black hole and $\vec{p}$ the momentum vector of the particle. The geometry is shown in figure 2 explaining why we can write $b = r\sin\theta$. Thus, the impact parameter of a particle can be written as the ratio between angular and linear momentum

$$b = \frac{L}{p}.$$

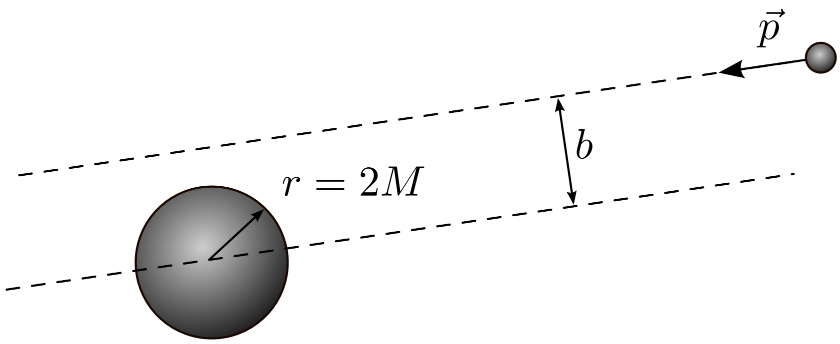

Figure 1: Defining the impact parameter.

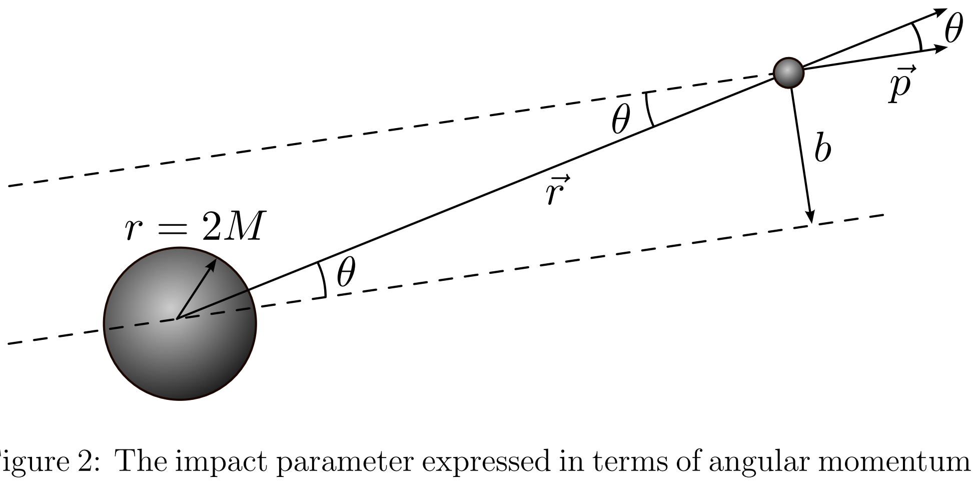

Figure 2: The impact parameter expressed in terms of angular momentum.

For a photon, we have that $p = E$ so that

$$b = \frac{L}{E}.$$

(valid for photons only). Using this, we can rewrite equation (1) and (2) using the impact parameter as (check!)

$$\frac{dr}{dt} = \pm\left(1 - \frac{2M}{r}\right)\sqrt{1 - \left(1 - \frac{2M}{r}\right)\frac{b^2}{r^2}} \tag{3}$$

$$\frac{rd\phi}{dt} = \pm\frac{b}{r}\left(1 - \frac{2M}{r}\right). \tag{4}$$

In the exercises you will show that the equations of motion for a photon can be written as

$$A = Bv_{\mathrm{r,shell}}^2 + V_{\mathrm{eff}}(r)^2,$$

where $A = B = 1/b^2$ and

$$V_{\mathrm{eff}}(r) = \frac{1}{r}\sqrt{\left(1 - \frac{2M}{r}\right)}.$$

We see again that we have an equation on the same form as equation (4) in the previous lecture. We know that we need to compare the value of the constant A (which usually contains the energy E/m, but which this time contains only the impact parameter) with the shape of the effective potential. For a material body we showed in the previous lecture that it was the energy E/m which appeared in the constant A and therefore it was the value of this energy which decided whether the particle would move in an orbit, escape to infinity or be swallowed by the black hole. For the photon, we see that it is the impact parameter alone and not the energy which decides its destiny.

In figure 3 we see the effective potential for light. The first thing which strikes us in this figure is that the potential does not exhibit a minimum as all the other potentials we have discussed so far. The consequence is that light cannot go in a stable orbit. If $1/b^2$ is lower than the peak in the figure, the light will approach the black hole, be deflected in some direction and escape to infinity (do you see why?). If $1/b^2$ is larger than the value at the peak in the figure, light will be captured by the black hole. In the exercises you will show that the peak in the potential is located at $r = 3M$ for which $1/b^2 = 1/(27M^2)$.

Light which approaches the black hole with $1/b^2$ equal to the value of the potential at the peak $1/(27M^2)$ will go in an unstable orbit at $r = 3M$. For this reason $r = 3M$ is called the *light sphere*. All the stars around a black hole radiate light in all possible directions with a huge range of impact parameters. There will always be light approaching the black hole with an impact parameter equal to the critical impact parameter

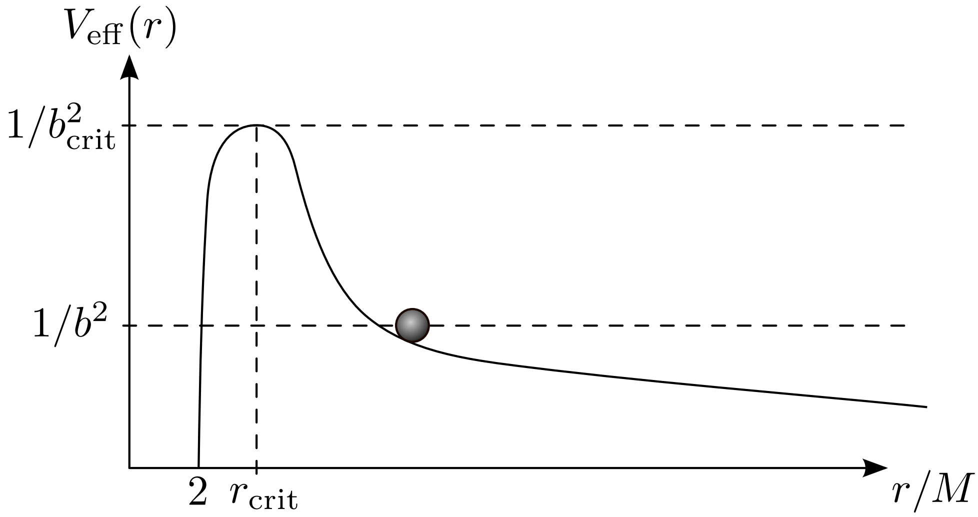

Figure 3: The effective potential for light.

$1/b^2_{\text{crit}} = 1/(27M^2)$ such that the light will orbit the black hole at the light sphere. A shell observer at the light sphere will see a ring around the black hole with several copies of images of the stars in the sky. The light will not stay in the light sphere for very long: Staying at the peak of the potential means being in an unstable orbit. Tiny fluctuations in the impact parameter will make the light either plunge into the black hole or escape. Coincidences will decide. This is exactly what we saw for material bodies approaching the black hole with an energy such that it balanced on the peak of the potential for a few revolutions and then either plunged or escaped.

3 Deflection of light

In figure 4 we see light approaching a star at a large distance with an impact parameter such that the light will pass the star, be deflected and then escape to infinity. The question is with how large an angle $\Delta\phi$ the light is deflected. If the light is significantly deflected by a star it would mean that we cannot trust the position of objects that we observe on the sky: If the light from distant galaxies is deflected by all the stars it passes on the way to Earth, the original direction of the light and hence of the galaxy would be lost. We need to calculate how large the deflection is to find out whether this could be a problem for astronomical observations or not.

In figure 5 we show the situation in detail: Light with impact parameter b is approaching a star of mass M. We have defined the ϕ coordinate such that $\phi = 0$ when the light is infinitely far away. If light had not been deflected by the gravitational field, it would continue in a straight line to infinity at $\phi = \pi$. But we know that the light is deflected an angle $\Delta\phi$ such that the light goes to infinity at $\phi = \pi + \Delta\phi$. We will now study light

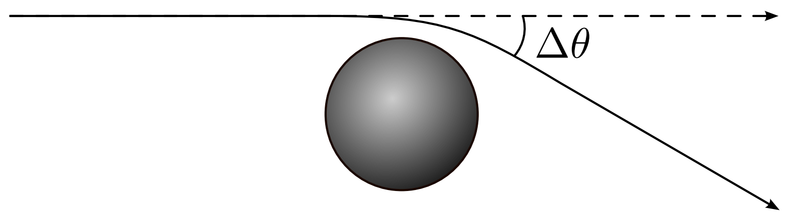

Figure 4: Deflection of light by a star. The dotted line is the direction light would have taken if no deflection had taken place.

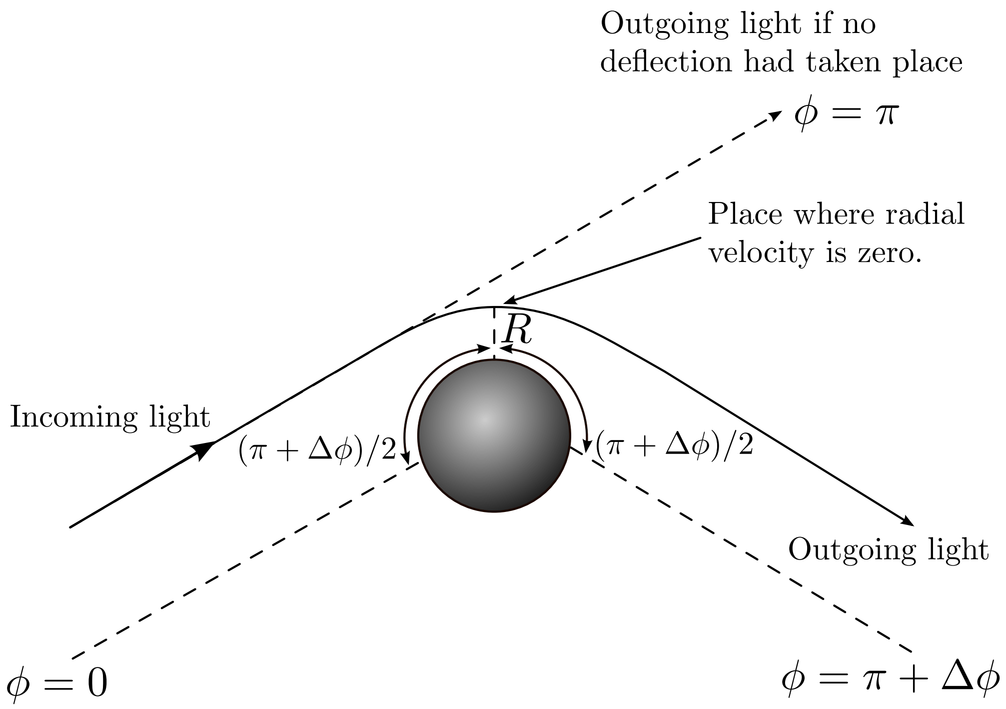

Figure 5: Deflection of light by a star. Symmetry makes the situation equal on either side of the point where the distance between the light beam and the star is minimal $r = R$ and the radial velocity of the beam is zero.

which has an impact parameter b such that it passes the star with radial shell velocity $v_{\text{r,shell}}$ equal to zero at a distance R from the star (see figure 5). In order to calculate the deflection $\Delta\phi$ for this beam of light we will use the equations of motion for light in Schwarzschild geometry given by equations (1) and (2). Dividing the two equations by each other we find

$$d\phi = \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)}}.$$

We need to integrate this equation to obtain the deflection $\Delta\phi$ from the particle arrives at $r = \infty, \phi = 0$ to $r = \infty, \phi = \pi + \Delta\phi$. Because of symmetry, it is sufficient to find the deflection $\Delta\phi/2$ occurring during the trip from $(r = \infty, \phi = 0)$ to $(r = R, \phi = \pi/2 + \Delta\phi/2)$ (see again figure 5). The symmetry of the problem tells us that this deflection equals the deflection occurring during the trip from $(r = R, \phi = \pi/2 + \Delta\phi/2)$ to $(r = \infty, \phi = \pi + \Delta\phi)$. The geometry of the problem is detailed in figure 5. We therefore need to perform the following integration (integrating the previous equation)

$$\int_0^{\pi/2+\Delta\phi/2} d\phi = \int_\infty^R \frac{dr}{r^2\sqrt{\frac{1}{b^2} - \frac{1}{r^2}\left(1 - \frac{2M}{r}\right)}}.$$

To make the integration easier we will make the substitution $u = R/r$ giving

$$\int_0^{\pi/2+\Delta\phi/2} d\phi = \frac{1}{R}\int_1^0 \frac{du}{\sqrt{\frac{1}{b^2} - \frac{u^2}{R^2}\left(1 - \frac{2M}{R}u\right)}}.$$

Before integrating there is one more information which we have not used: The fact that we know the impact parameter b. The radial shell velocity at $r = R$ is equal to zero. In problem 1 you will find an expression for the radial shell velocity of light (equation 7) as a function of distance and impact parameter. Setting the radial velocity to zero this equation gives us

$$\frac{1}{b^2} = \frac{1}{R^2}\left(1 - \frac{2M}{R}\right), \tag{5}$$

which we can insert in our integral allowing us to get rid of the impact parameter. The integral now reads

$$\frac{\pi}{2} + \frac{\Delta\phi}{2} = \int_0^1 \frac{du}{\sqrt{\left(1 - \frac{2M}{R}\right) - u^2\left(1 - \frac{2M}{R}u\right)}}.$$

For stars, we normally have $R >> 2M$ (check that this must be so for the Sun: find the radius of the Sun expressed in Solar masses. Remember that R must be larger than the radius of the Sun (why?)). Therefore we define $x = M/R << 1$ and we will try to Taylor expand the integrand in the small value x. The integrand can be written as

$$f(x) = (1 - 2x - u^2(1 - 2xu))^{-1/2} \approx f(0) + f'(0)x$$

Taking the derivative of $f(x)$ with respect to x we find

$$f(0) = \frac{1}{\sqrt{1-u^2}} \qquad f'(0) = \frac{1-u^3}{(1-u^2)^{3/2}}$$

Thus the integral can be written

$$\frac{\pi}{2} + \frac{\Delta\phi}{2} = \underbrace{\int_1^0 \frac{du}{\sqrt{1-u^2}}}_{\pi/2} + \frac{M}{R} \underbrace{\int_1^0 \left[\frac{1}{(1-u^2)^{3/2}} - \frac{u^3}{(1-u^2)^{3/2}}\right] du}_{2}\,.$$

The solution to these integrals can be found in tables of integrals. We get

$$\frac{\Delta\phi}{2} = \frac{2M}{R},$$

or

$$\Delta\phi = \frac{4M}{R}.$$

In the exercises you will see how close to a star light needs to pass for the deflection to be important. You will also show that light from stars which pass close to the surface of the Sun will be deflected significantly. Stars which we observe in a direction close to the surface of the Sun will thus be observed in the wrong position on the sky. The stars will be shifted due to the deflection of light. This is a good test of the theory of general relativity: We now have a formula to predict exactly by how large angle the position of a star on the sky will change when viewed close to the surface of the Sun. The problem is that the light from the Sun is so strong that we cannot see stars which have a position on the sky close to the Sun. The only possibility to observe these stars is during a total solar eclipse. During a solar eclipse in 1919, this effect was measured for the first time: Stars which were seen close to the surface of the Sun were measured to have shifted their position with exactly the angle predicted by general relativity. This was the discovery which made Einstein famous.

4 Gravitational lensing

The gravitational deflection of light is used today to study the most remote objects in the visible universe. In figure 6 we show a typical situation. A quasar (a black hole with gas falling into it producing strong radiation at several wavelengths, quasars are one of the most powerful radiation sources in the universe) is located at a distance d_S and a cluster of galaxies with mass M is located at distance d_L. The indices S and L refer to 'source' and 'lens'. The quasar is the source of light and the cluster of galaxies deflects this light similar to an optical lens. For this reason we call the cluster of galaxies for the 'lens' and the effect of light deflection is called *gravitational lensing*.

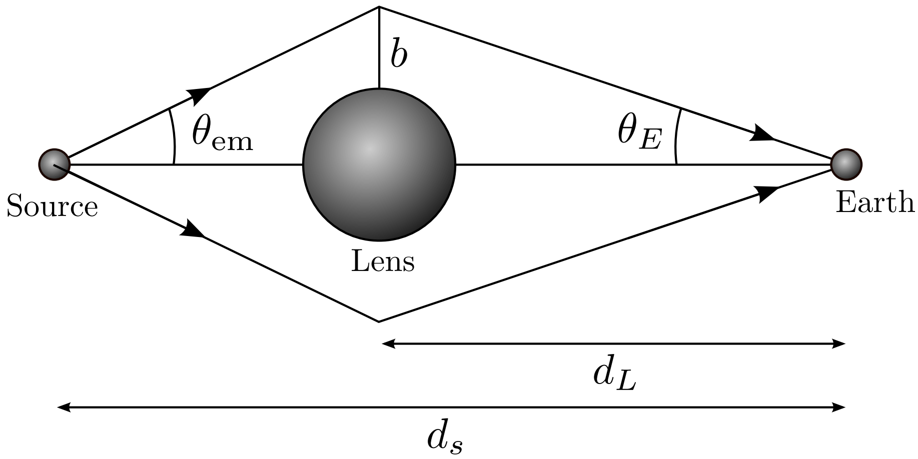

Figure 6: The source on the left (a quasar), the lens in the middle (a cluster of galaxies) and the Earth on the right receiving the radiation from the quasar from several angles.

The limiting angle $\theta_{\rm em}$ (see figure 6) is the angle that the light emitted from the quasar needs to have in order to reach Earth. Light emitted with a smaller angle will be deflected too much, light emitted with a larger angle will be deflected too little. Only light with angle $\theta_{\rm em}$ will be deflected in such a way that the light will reach us and we will see the quasar. The figure shows only a two dimensional plane, taking into account the three dimensional geometry of the problem, light emitted with an angle $\theta_{\rm em}$ will reach us from all direction the result being that we see the quasar as a ring of light around the cluster (see figure 9). We call this ring an *Einstein ring*. The angle θ_E is the observed angular radius of the Einstein ring (you find the angle both in figure 6 and 9 check that you understand the relation between the two figures). In the exercises, you will show that this angle can be written as

$$\theta_E = \sqrt{\frac{4M(d_S - d_L)}{d_L d_S}}, \tag{6}$$

which is called the *lensing formula*.

From spectroscopic measurements, the distances d_S and d_L of the quasar and the cluster are normally known. The angular radius of the Einstein ring can be measured by observations. Combining these numbers, the lensing formula can be used to find the mass of a cluster of galaxies. We remember from previous lectures that we can use the virial theorem to find the mass of clusters of galaxies. The mass estimates of clusters obtained using the lensing formula is based on assumptions very different from those used in the virial theorem approach. Thus we have two independent measurements of the mass of the cluster. These two ways of measuring mass are in good agreement taking into account the uncertainties in the two methods. Both methods tell us that there is far more dark than

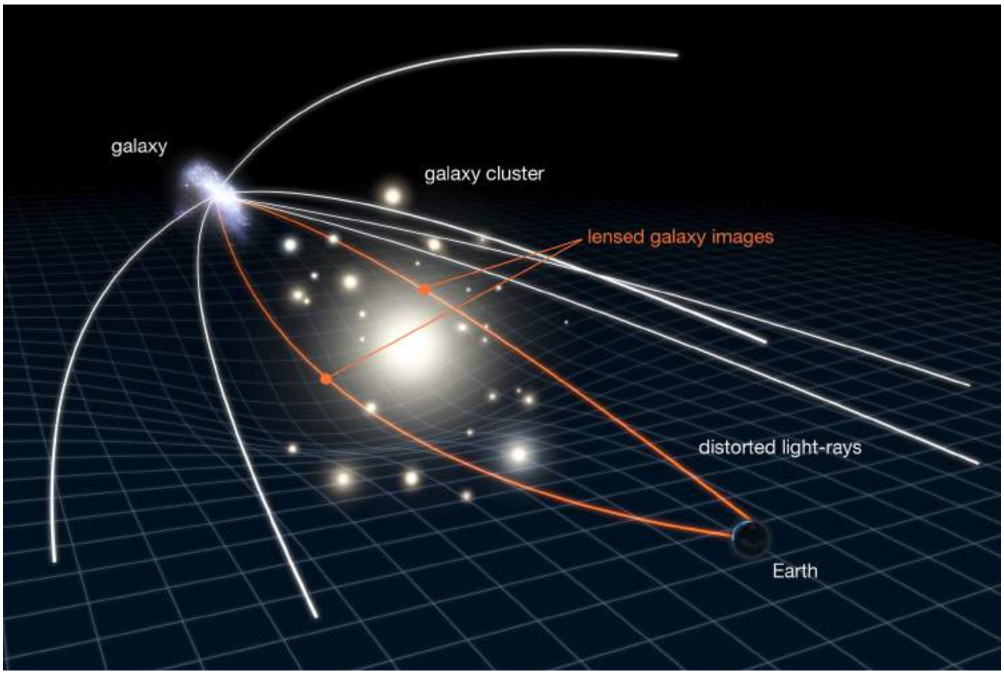

Figure 7: Info-figure:This illustration shows how gravitational lensing works. The gravity of a large galaxy cluster is so strong that it bends, brightens and distorts the light of a distant galaxy behind it. In this case observers on Earth see two images of the same object. Note that in reality, the distant galaxy is much farther away than it appears here. Gravitational lensing is an impressive astronomical tool; it can be used to detect exoplanets, learn about distant galaxies and galaxy clusters, and measure dark matter, dark energy and the age of the universe. Astronomer Fritz Zwicky postulated in 1937 that gravitational light bending could allow galaxy clusters to act as gravitational lenses. It was not until 1979 that this exotic phenomenon was confirmed observationally with the discovery of the "Double Quasar" QSO 0957+561. The Norwegian astronomer Sjur Refsdal made pioneering work on gravitational lensing and microlensing in the 1960s, 70s and 80s. (Figure: NASA, ESA & L. Calcada)

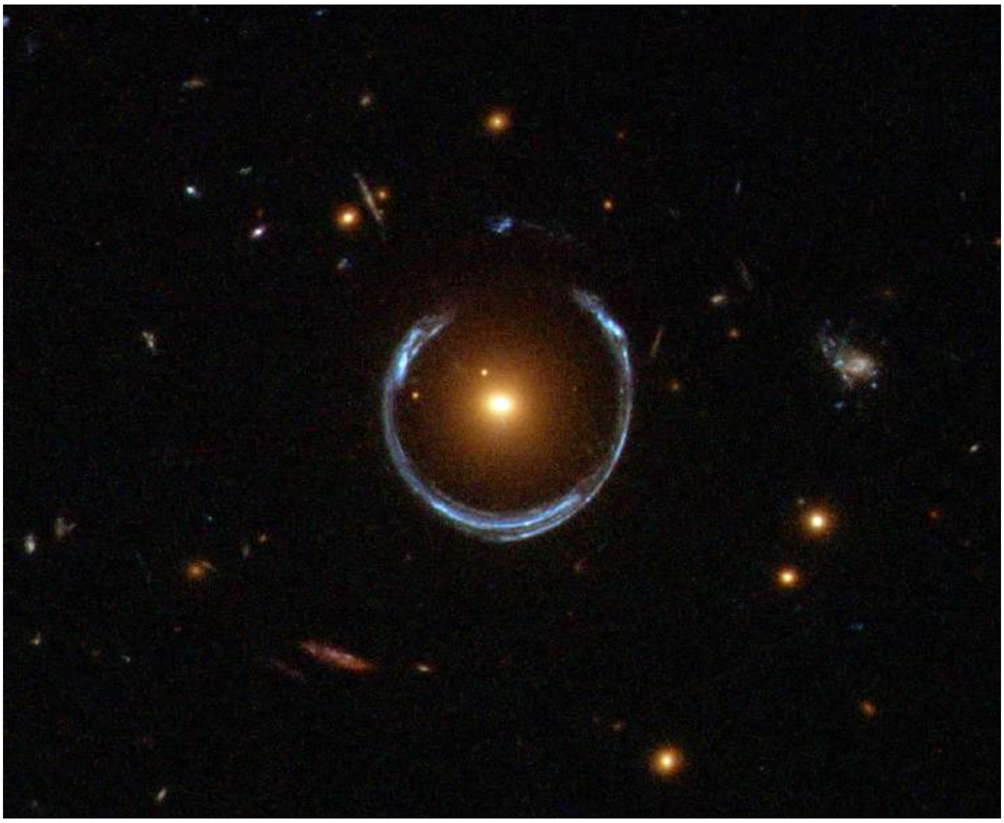

Figure 8: Info-figure: Believe it or not, this is a real picture of the sky, taken with the Hubble Space Telescope. The gravity of an unusually massive galaxy (the fuzzy yellow object in the middle) has gravitationally distorted the light from a much more distant blue galaxy. More typically, such light bending results in two discernible images of the distant galaxy, but here the lens alignment is so precise that the background galaxy is distorted into nearly a complete Einstein ring! The blue galaxy's redshift is approximately 2.4. This means we see it as it was only about 3 billion years after the Big Bang.(Figure: ESA/Hubble & NASA)

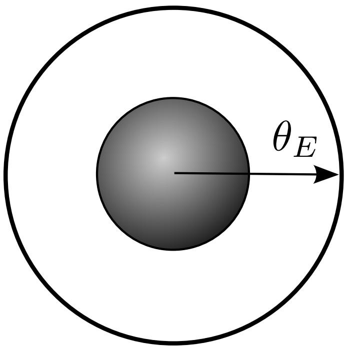

Figure 9: The cluster of galaxies in the middle and the Einstein ring being the lensed image of the quasar behind. The angular radius of the ring is θ_E.

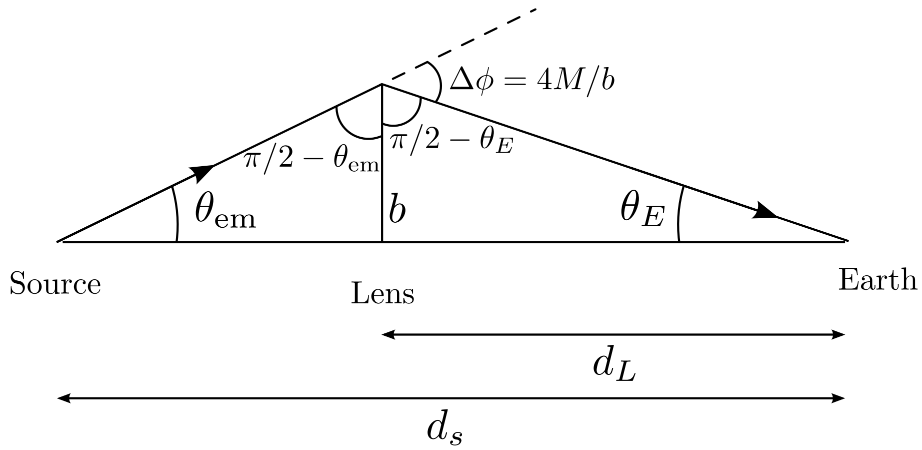

Figure 10: Detailed geometry of the situation in figure 6.

luminous matter in clusters of galaxies being another confirmation of the existence of dark matter.

To obtain an Einstein ring, the quasar needs to be exactly behind the center of the cluster of galaxy. Furthermore the cluster needs to have a spherical mass distribution. This is basically never the case, a complete Einstein ring is very rarely observed. What we rather see are small arcs around the cluster. By studying these arcs combined with more advanced theory of gravitational lensing, one can even infer the distribution of mass in the cluster of galaxies.

Finally I will mention another important use of gravitational lensing based on *microlensing*. The idea of microlensing is based on the following observation: The lens deflects light from the source towards Earth, light which otherwise would not have reached us. The lensing effect increases the total amount of photons from the quasar arriving to the Earth. Gravitational lensing does not only happen at the scale of clusters of galaxies. Even if an object passes in front of a star, gravitational lensing occurs. In this case, the Einstein ring is so small that it cannot be resolved on the sky. Only one effect of the lensing is directly observable: The fact that more light is directed towards us. The flux we receive from the star increases when the object is in front of the star. This is called microlensing.

Microlensing has been used to look for 'lumps' of dark matter in the Milky Way, so-called MAssive Compact Halo Objects (MACHO). If these MACHOs, lumps of dark matter orbiting the center of the Milky way, exist they should cause microlensing of stars in the LMC and SMC (Large and Small Magellanic Clouds). The LMC and SMC are dwarf galaxies orbiting the center of the Milky Way. The MACHOs are expected to have orbits between us and the Magellanic clouds. When the MACHOs pass in front of stars in the Magellanic clouds microlensing will increase the flux from these stars for a few days or weeks. An extensive search program is running looking for these microlensing events in the Magellanic clouds in order to get closer to the solution of the dark matter mystery.

5 Problems

Problem 1 (30 min.–1 hour)

Use the equations of motion for a photon (equation 3) to show that the radial light speed $dr_{\text{shell}}/dt_{\text{shell}}$ observed by the shell observer can be written as

$$\frac{1}{b^2}\left(\frac{dr_{\text{shell}}}{dt_{\text{shell}}}\right)^2 = \frac{1}{b^2} - \frac{\left(1-\frac{2M}{r}\right)}{r^2}. \tag{7}$$

Look at equation (4) and (7) from the previous lecture and show that we can define an effective potential for light (based on the shell velocity rather than the velocity dr/dt) as

$$V(r) = \sqrt{\frac{\left(1-\frac{2M}{r}\right)}{r^2}}.$$

Problem 2 (30 min.–1 hour)

1. By taking the derivatives of the effective potential for light, show that the potential has only one extremal point which is a maximum. Explain why this means that there are no stable orbits for light.
2. Show that this maximum occurs at $r = 3M$ and explain why we call this radius the light sphere.
3. Show that the criterion deciding whether light will escape a black hole or plunge into it is given by the critical impact parameter as

$$b_{\text{crit}} = 3\sqrt{3}M \approx 5.2M$$

Problem 3 (1–2 hours)

In 1919 a solar eclipse gave one of the first opportunities to check the validity of Einstein's theory. Stars which appear very close to the Sun can normally not be seen due to the much stronger light from the Sun. Only light from the stars which appear very close to the Sun on the sky would be significantly affected by the gravitational field of the Sun. The only possibility we have to see these stars is during a solar eclipse. The light from the Sun is blocked and the stars can be seen. If the light from these stars pass close to the surface of the Sun they will be deflected by the solar gravitational field. This deflection will shift the position of the star on the sky.

1. Draw the situation showing why the apparent position of such a star is shifted and whether it will be shifted towards the Sun or away from the Sun. Then show that the angular shift on the sky is given by $\Delta\phi = 4M/R$ (you can assume that you already know that light will be deflected by this angle, but it is not obvious that this equals the apparent angular shift on the sky, this is what you need to show by the figure/geometry). You will need a good drawing and some geometrical consideration to arrive at the answer.

2. Calculate the angular shift in position in arc seconds assuming that the light passes very close to the solar surface.

3. Repeat the previous calculation for the Moon. Is the same effect measurable for stars close to the Moon? (Here you need the mass of the Moon)

Problem 4 (60–90 min.)

In this exercise, we will deduce the lensing formula (equation 6). Go back and read what the different symbols in the lensing formula mean. Also go and check that you understand figures 6 and 9 well.

1. First use the fact that $R >> M$ to show that light with impact parameter b will pass the cluster at a distance $R \approx b$ from the center of the cluster at the closest point. (Hint: equation 5). This is the reason why the closest distance of the light beam to the cluster is given by b in figure 6.

2. Show that the deflection angle $\Delta\phi$ is given by

$$\Delta\phi \approx \frac{4M}{b}.$$

3. Only light emitted with an angle $\theta_{\rm em}$ will reach Earth. We just found out that this light will be deflected an angle $4M/b$ and will reach Earth in an angle θ_E. In figure 10 we show the geometry tin more detail. Make sure that you understand the figure and why all the different angels can be written the way they are written in this figure. Use the figure to show that

$$\theta_{\rm em} + \theta_E = \frac{4M}{b}. \qquad (8)$$

4. We will assume that the distances d_L and d_S as well as $d_L - d_S$ are much larger than the distance between the center of the cluster and the light beam at the closest given by b. If this is the case (as it always is in this situation), then the angles θ_E and $\theta_{\rm em}$ are so small that we can use the small angle formula. Show that

$$\theta_{\rm em} \approx \frac{b}{d_S - d_L}, \quad \theta_E \approx \frac{b}{d_L}.$$

5. Now you have enough information to show the lensing formula

$$\theta_E = \sqrt{\frac{4M(d_S - d_L)}{d_L d_S}},$$

6. An Einstein ring is observed around a cluster of galaxies. The radius of the Einstein ring is $3'$. The distance to the cluster has been estimated to be 10^9 light years. Using spectroscopy on the light from the Einstein ring it is recognized as a quasar and the distance to the quasar is estimated to be 10^{10} light years. What is the mass of the cluster of galaxies expressed in solar masses?

AST1100 Lecture Notes

19: Nuclear reactions in stellar cores

Before embarking on the details of thermonuclear reactions in stellar cores, we need to discuss a few topics...

1 Some particle physics

Nature is composed of three kinds of elementary particles: *leptons*, *quarks*, and *gauge bosons*. Nature also has four forces acting on these elementary particles: the strong and weak nuclear forces, the electromagnetic force and the force of gravity (from the point of view of general relativity the latter is not a force, from the point of view of particle physics, it is). Actually, it has been discovered that the weak nuclear force and the electromagnetic force are two aspects of the same thing. At higher energies they unify and are therefore together called the electroweak force.

The leptons can be divided in two groups, the 3 'heavy' (with much more mass than in the other group) leptons and 3 light leptons called neutrinos (with a very small mass). Each heavy lepton has a neutrino associated with it. In all there are thus 6 leptons

- the electron and the electron associated neutrino.
- the muon and the muon associated neutrino.
- the tau particle and the tau associated neutrino.

In collisions involving the electron, an electron (anti)neutrino is often created, in collisions involving the muon, a muon (anti)neutrino is often created and the same goes for the tau particle. Each lepton has *lepton number* +1 whereas an antilepton has lepton number -1. This is a property of the particle similar to charge: In the same way as the total charge is conserved in particle collisions, the total lepton number is also conserved.

There are also 6 kinds of quarks grouped in three generations. In the order of increasing mass these are

- the up (charge $+2/3e$) and down (charge $-1/3e$) quarks.
- the strange (charge $-1/3e$) and charm (charge $+2/3e$) quarks.

- the bottom (charge $-1/3e$) and top (charge $+2/3e$) quarks.

A quark has never been observed alone it is always connected to other quarks via the strong nuclear force. A particle consisting of two quarks is called a *meson* and a particle consisting of three quarks is called a *baryon*. Mesons and baryons together are called *hadrons*. A proton is a baryon consisting of three quarks, two up and one down quark. A neutron is another example of a baryon consisting of two down and one up quark.

In quantum theory, the forces of nature are carried by so-called gauge bosons. Two particles attract or repel each other through the interchange of gauge bosons. Normally these are *virtual gauge bosons*: Particles existing for a very short time, just enough to carry the force between two particles. The energy to create such a particle is borrowed from vacuum: The Heisenberg uncertainty relation

$$\Delta E \Delta t \leq \frac{h}{4\pi}, \quad (1)$$

allows energy ΔE to be borrowed from the vacuum for a short time interval Δt. The gauge bosons carrying the four forces are

- gluons in the case of the strong nuclear force
- W and Z bosons in the case of the weak nuclear force
- photons in the case of the electromagnetic force
- (gravitons in the case of the gravitational force: note that a quantum theory of gravity has not yet been successfully developed)

In quantum theory, the angular momentum or spin of a particle is quantized. Elementary particles can have integer spins or half integer spins. Particles of integer spins are called *bosons* (an example is the gauge bosons) and particles of half integer spin are called *fermions* (leptons and quarks are examples of fermions. Fermions and bosons have very different statistical properties, we will come to this in the next lecture.

Finally, all particles have a corresponding antiparticle: A particle having the same mass, but opposite charge. Antileptons also have opposite lepton number: -1. This is why a lepton is always created with an antineutrino in collisions. For instance, when a free neutron disintegrates (a free neutron only lives for about 12 minutes), it disintegrates into a proton and electron and an electron antineutrino. A neutron is not a lepton and hence has lepton number 0. Before the disintegration, the total lepton number is therefore zero. After the disintegration, the total lepton number is: 0 (for the proton) + 1 (for the electron) -1 (for the antineutrino) = 0, thus lepton number is conserved due to the creation of the antineutrino.

Now make a schematic summary of all the elementary particles and forces that have been observed in nature.

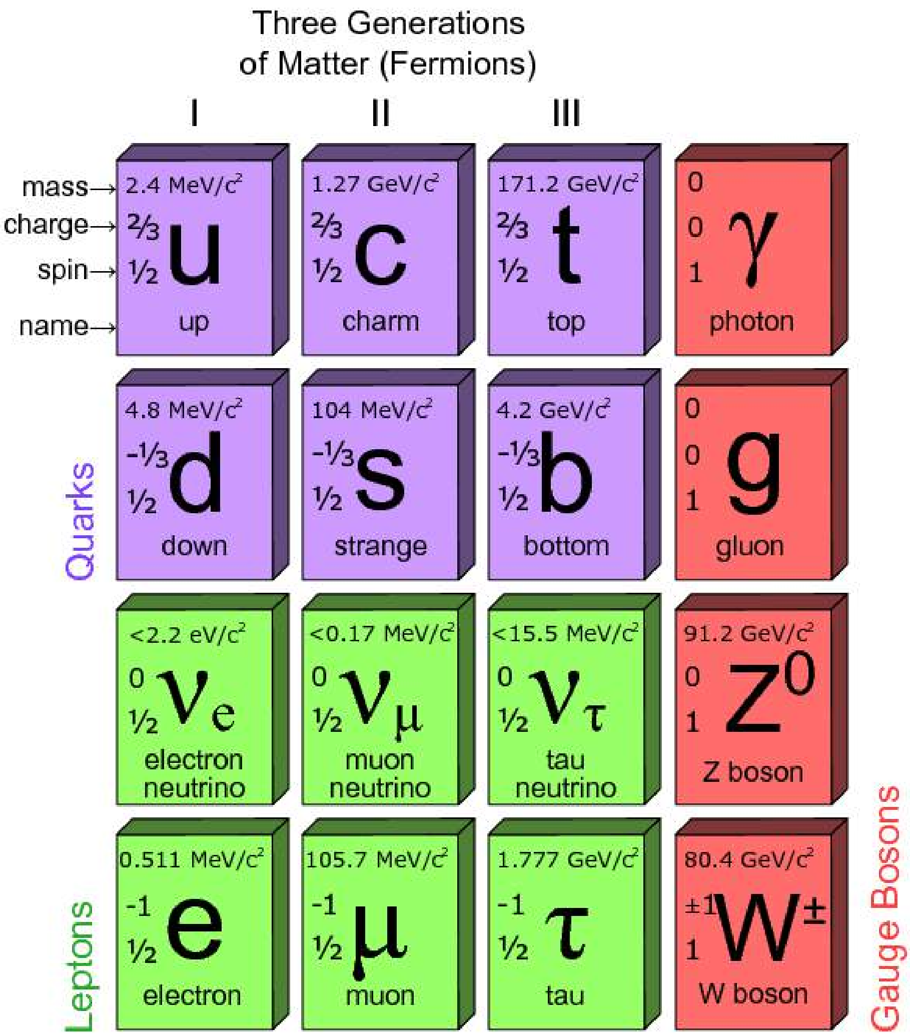

Figure 1: Info-figure: The Standard Model of particle physics is a theory concerning the electromagnetic, weak, and strong nuclear interactions, which mediate the dynamics of the known subatomic particles. The model includes 12 fundamental fermions and 4 fundamental bosons. The 12 elementary particles of spin 1/2 (6 quarks and 6 leptons) known as fermions are classified according to how they interact, or equivalently, by what charges they carry. Pairs from each classification are grouped together to form a generation, with corresponding particles exhibiting similar physical behavior. Fermions respect the Pauli exclusion principle, and each fermion has a corresponding antiparticle. Gauge bosons (red boxes) are defined as force carriers that mediate the strong, weak, and electromagnetic interactions. (Note that the masses of certain particles are subject to periodic reevaluation by the scientific community. The values in this graphic are as of 2008 and may have been adjusted since.) (Figure:Wikipedia)

2 Mass in special relativity

Another topic which we need to discuss before studying nuclear reactions is the notion of mass in the special theory of relativity. We have already seen that the scalar product of the momenergy four-vector equals the mass of a particle,

$$P_\mu P^\mu = E^2 - p^2 = m^2. \tag{2}$$

Imagine we have two particles with mass m_1 and m_2, total energy E_1 and E_2 and momenta p_1 and p_2. Assume that they have opposite momenta $p_1 = -p_2 = p$,

$$P_\mu^1 = (E_1, p), \quad P_\mu^2 = (E_2, -p)$$

with $E_1 = \sqrt{m_1^2 + p^2}$ and $E_2 = \sqrt{m_2^2 + p^2}$ These two particles could for instance constitute the proton and the neutron in a deuterium nucleus. The question now is, what is the total mass of the two-particle system (deuterium nucleus)? Let us form the momenergy four-vector for the nucleus

$$P_\mu = P_\mu^1 + P_\mu^2 = (E_1 + E_2, 0).$$

Using equation 2 we can now find the total mass of the two-particle system (the nucleus),

$$\begin{aligned} M^2 = P_\mu P^\mu = (E_1 + E_2)^2 &= E_1^2 + E_2^2 + 2E_1E_2 \\ &= m_1^2 + m_2^2 + 2p^2 + \sqrt{(m_1^2 + p_1^2)(m_2^2 + p_2^2)} \end{aligned}$$

where M is the total mass of the nucleus. We have two important observations: (1) Mass is *not* an additive quantity. The total mass of a system of particles is *not* the sum of the mass of the individual particles. (2) The mass of a system of particles depends on the total energy of the particles in the system. The energy of particles in an atomic nucleus includes the potential energy between the particles due to electromagnetic and nuclear forces.

Consider an atomic nucleus with mass M. This nucleus can be split into two smaller nuclei with masses m_1 and m_2. If total mass of the two nuclei m_1 and m_2 is smaller than the total mass of the nucleus, the rest energy is radiated away when the nucleus is divided. This is a nuclear fission process creating energy. Similarly if the total mass of m_1 and m_2 is larger than the total mass of the nucleus, then energy must be provided in order to split the nucleus. The same argument goes for nuclear fusion processes: Consider two nuclei with masses m_1 and m_2 which combine to form a larger nucleus of mass M. If M is smaller than the total mass of the nuclei m_1 and m_2 then the rest mass is radiated away and energy is 'created' in the fusion process. In some cases (particularly for large nuclei), the mass M is larger than the total mass of m_1 and m_2. In this case energy must be provided in order to combine the two nuclei to a larger nucleus. We will soon see that in order to produce atomic nuclei larger than iron, energy must always be provided.

3 Penetrating the Coloumb barrier

The strong nuclear force (usually referred to as the strong force) is active over much smaller distances than the electromagnetic force. The strong force makes protons attract protons and protons attract neutrons (and vice versa). For two atomic nuclei to combine to form a larger nucleus, the two nuclei need to be close enough to feel the attractive nuclear forces from each other. Atomic nuclei have positive charge and therefore repulse each other at larger distances due to the electromagnetic force. Thus for a fusion reaction to take place, the two nuclei need to penetrate the Coloumb barrier, the repulsive electromagnetic force between two equally charged particles. They need to get so close that the attractive strong force is stronger than the repulsive electromagnetic force. In figure 2 we show the combined potential from electromagnetic and nuclear forces of a nucleus. We clearly see the potential barrier at $r = R$. For a particle to get close enough to feel the attractive strong force it needs to have an energy of at least $E > E(R)$. We can make an estimate of the minimal temperature a gas needs in order to make a fusion reaction happen: The mean kinetic energy of a particle in a gas of temperature T is $E_K = (3/2)kT$ (see the exercises). The potential energy between two nuclei A and B can be written as

$$U = -\frac{1}{4\pi\epsilon_0}\frac{Z_A Z_B e^2}{r},$$

where ϵ_0 is the vacuum permittivity, Z_1 and Z_2 is the number of protons in each nucleus, e is the electric charge of a proton and r is the distance between the two nuclei. For nucleus A to reach the distance R (see figure 2) from nucleus B where the strong force starts to dominate, the kinetic energy must at least equal the potential energy at this point

$$\frac{3}{2}kT = \frac{1}{4\pi\epsilon_0}\frac{Z_A Z_B e^2}{R}.$$

The distance R is typically $R \sim 10^{-15}$ m. Considering the case of two hydrogen nuclei $Z = 1$ fusing to make helium $Z = 2$, we can solve this equation for the temperature and obtain $T \sim 10^{10}$ K. This temperature is much higher than the core temperature of the Sun $T_C \sim 15 \times 10^6$ K. Still this reaction is the main source of energy of the Sun. How can this be?

The secret is hidden in the world of quantum physics. Due to the Heisenberg uncertainty relation (equation 1), nucleus A can borrow energy ΔE from vacuum for a short period Δt. If nucleus A is close enough to nucleus B, the time Δt might just be enough to use the borrowed energy to penetrate the Coloumb barrier and be captured by the potential well of the strong force. This phenomenon is called *tunneling*. Thus, there is a certain probability that nucleus A spontaneously borrows energy to get close enough to nucleus B in order for the fusion reaction to take place.

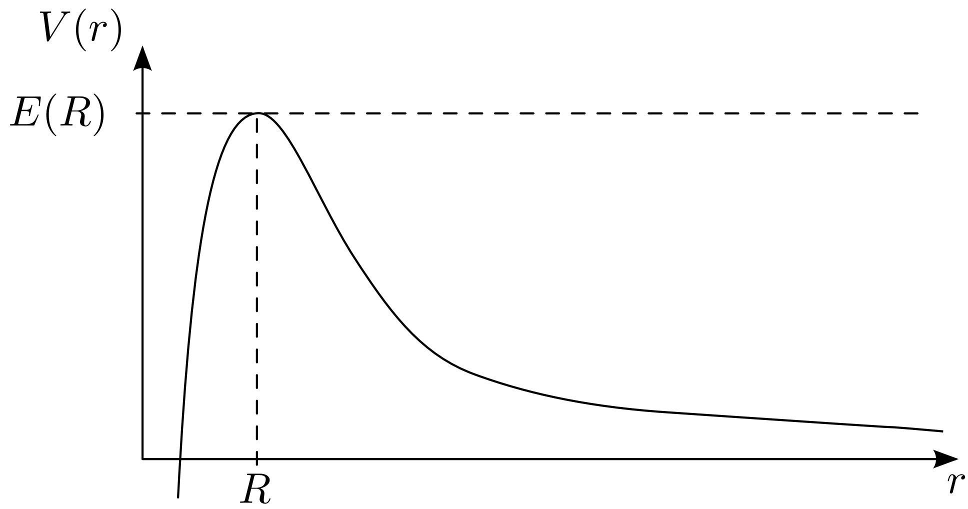

Figure 2: The repulsive Coloumb potential $V(r)$ as a function of distance between nuclei r. At small distances r we see the potential well from the attractive strong forces.

4 Nuclear reaction probabilities and cross sections

Quantum physics is based on probability and statistics. Nothing can be predicted with 100% certainty, only statistical probabilities for events to happen can be calculated. When nucleus A is at a certain distance from nucleus B we cannot tell whether it will borrow energy to penetrate the Coloumb barrier or not, we can only calculate the probability for the tunneling to take place. These probabilities are fundamental for understanding nuclear reactions in stellar cores. These probabilities are usually represented as *cross sections* σ.

The definition of the cross section is based on an imaginary situation which is a bit different from the real situation but gives an intuitive picture of the reaction probabilities and, most importantly, makes the calculations easier. It can be proven that the calculations made for this imaginary picture gives exact results for the real situation. Instead of the real situation where we have one nucleus A and one nucleus B passing each other at a certain distance (and we want to know the probability that they react), one imagines the nucleus B to be at rest and a number of nuclei of type A approaching it. One imagines nucleus B to have a finite two dimensional extension, like a disk, with area σ. Towards this disk there is a one dimensional flow of A particles (see figure 3). If a nucleus A comes within this disk, it is captured and fusion takes place, if not the nuclei do not fuse. It is important to understand that this is not really what happens: fusion can take place with any distance r between the nuclei. It might also well be that A is within the disk and the fusion reaction is not taking place. But in order to make calculations easier one makes this imaginary disk

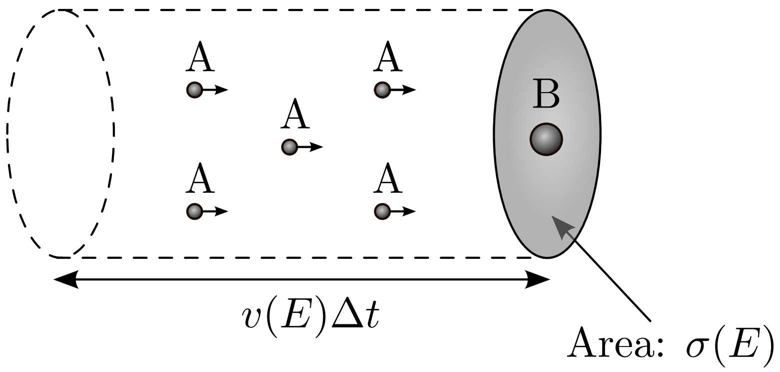

Figure 3: A particles streaming towards the disk with cross section $\sigma(E)$ around the B nucleus. A particles of energy E within the volume $v(E)\Delta t\sigma(E)$ will react with the nucleus B within time Δt.

with an effective cross section σ saying that any nucleus A coming within this disk will fuse. It can be shown that calculations made with this representation gives correct reaction rates even though the model does not give a 100% correct representation of the physical situation. Because of the simplified mathematics, the cross section σ is the most common way of representing a probability for a reaction or collision process to take place. You will now see how this imaginary picture is used to calculate reaction rates.

The disk cross section (tunneling probability) $\sigma(E)$ depends on the energy E of the incoming nucleus A. Thus the size of the immaginary disk (for the nucleus B at rest) depends on the energy E of the incoming particle A. We will now make calculations in the center of mass system. In problem 5 in the lectures on celestial mechanics, you showed that the total kinetic energy of a two-body system can be written as (ignoring gravitational forces)

$$E = \frac{1}{2}\hat{\mu}v^2,$$

where $\hat{\mu}$ is the reduced mass $\hat{\mu} = (m_1 m_2)/(m_1 + m_2)$. We showed that the two-body problem is equivalent to a system where a particle with mass $M = m_1 + m_2$ is at rest and a particle with the reduced mass $\hat{\mu}$ is moving with velocity v. In this case we imagine the nucleus B to be at rest and the particle A is approaching with velocity v.

We have deferred the full calculation of the reaction rate between A and B nuclei in a plasma using the cross-section to problem 4. In order to be able to do that calculation, we need to recall an expression which we have seen before. In the lectures on electromagnetic radiation we learned that the number density of particles with velocity between v and $v + dv$ in an ideal gas of temperature T with molecules of mass m can be written as

$$n(v)dv = n\left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}} 4\pi v^2 dv. \tag{3}$$

You will use this in problem 4 when you need to multiply with the number of A and B nuclei in the gas.

The result you will find in problem 4 is the energy produced per kilogram of gas per second from nuclear reactions between A and B nuclei, ε_{AB}. You will show that it is given by

$$\varepsilon_{AB} = \frac{\varepsilon_0}{\rho}\left(\frac{2}{kT}\right)^{3/2}\frac{n_A n_B}{\sqrt{\hat{\mu}\pi}}\int_0^\infty dE E e^{-E/kT}\sigma(E), \tag{4}$$

where ε_0 is the energy released in each nuclear reaction between an A and a B nucleus, ρ is the total density of the gas and n_A and n_B are number densities of A and B nuclei. We will not do the integral here but note that the solution can be Taylor expanded around given temperatures T as

$$\varepsilon_{AB} = \varepsilon_{0,\text{reac}} X_A X_B \rho^\alpha T^\beta,$$

where ρ is the density, X_A and X_B are the mass fractions of the two nuclei

$$X_A = \frac{n_A m_A}{nm} = \frac{\text{total mass in type A nuclei}}{\text{total mass}},$$

and α and β depend on the temperature T around which the expansion is made.

Here, $\varepsilon_{0,\text{reac}}$, α and β will depend on the nuclear reaction (calculated from the integral 4). The constant $\varepsilon_{0,\text{reac}}$ includes the energy per reaction ε_0 for the given reaction as well as several other constants. If we have $\varepsilon_{0,\text{reac}}$, α and β for different nuclear reactions, we can use this expression to find the nuclear reactions which are important for a given temperature T in a stellar core.

The energy release per mass per time, ε, can be written as luminosity per mass

$$\frac{dL}{dm} = \varepsilon$$

The luminosity at a shell at a distance r from the center of a star can therefore be written as

$$\frac{dL(r)}{dr} = 4\pi r^2 \rho(r)\varepsilon(r), \tag{5}$$

which is another of the equations used together with the equation of hydrostatic equilibrium in the stellar model building described in the exercises of lecture 13–14.

5 Stellar nuclear reactions

For main sequence stars the most important fusion reaction fuses four $^1_1$H atoms to $^4_2$He. When writing nuclei, ^A_ZX, A is the total number of nucleons (protons and neutrons), Z is the total number of protons and X is the chemical symbol. There are mainly two chains of reaction responsible for this process. One is the pp-chain,

$$\begin{aligned} {}^1_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^2_1\mathrm{H} + {}^0_0\bar{\mathrm{e}} + {}^0_0\nu \\ {}^2_1\mathrm{H} + {}^1_1\mathrm{H} &\rightarrow {}^3_2\mathrm{He} + {}^0_0\gamma \\ {}^3_2\mathrm{He} + {}^3_2\mathrm{He} &\rightarrow {}^4_2\mathrm{He} + 2 \times {}^1_1\mathrm{H} \end{aligned}$$

Here ${}^0_0\nu$ is the electron associated neutrino, ${}^0_0\gamma$ is a photon and the bar represents antiparticles: ${}^0_0\bar{\mathrm{e}}$ is the antiparticle of the electron called the positron. This is the pp-I chain, the most important chain reactions in the solar core. There are also other branches of the pp-chain (with the first two reactions equal) but these are less frequent. The pp-chain is most effective for temperatures around 15 millions Kelvin for which we can write the reaction rate for the full pp-chain as

$$\varepsilon_{\mathrm{pp}} \approx \varepsilon_{0,\mathrm{pp}} X_H^2 \rho T_6^4,$$

where $T = 10^6 T_6$ with T_6 being the temperature in millions of Kelvin. This expression is valid for temperatures close to $T_6 = 15$. For this reaction $\varepsilon_{0,\mathrm{pp}} = 1.08 \times 10^{-12}$ Wm^3/kg^2. The efficiency of the pp-chain is 0.007, that is only 0.7% of the mass in each reaction is converted to energy.

The other reaction converting four ${}^1_1$H to ${}^4_2$He is the CNO-cycle,

$$\begin{aligned} {}^{12}_6\mathrm{C} + {}^1_1\mathrm{H} &\rightarrow {}^{13}_7\mathrm{N} + {}^0_0\gamma \\ {}^{13}_7\mathrm{N} &\rightarrow {}^{13}_6\mathrm{C} + {}^0_0\bar{\mathrm{e}} + {}^0_0\nu \\ {}^{13}_6\mathrm{C} + {}^1_1\mathrm{H} &\rightarrow {}^{14}_7\mathrm{N} + {}^0_0\gamma \\ {}^{14}_7\mathrm{N} + {}^1_1\mathrm{H} &\rightarrow {}^{15}_8\mathrm{O} + {}^0_0\gamma \\ {}^{15}_8\mathrm{O} &\rightarrow {}^{15}_7\mathrm{N} + {}^0_0\bar{\mathrm{e}} + {}^0_0\nu \\ {}^{15}_7\mathrm{N} + {}^1_1\mathrm{H} &\rightarrow {}^{12}_6\mathrm{C} + {}^4_2\mathrm{He} \end{aligned}$$

with a total reaction rate

$$\varepsilon_{\mathrm{CNO}} = \varepsilon_{0,\mathrm{CNO}} X_H X_{\mathrm{CNO}} \rho T_6^{20},$$

where $\varepsilon_{0,\mathrm{CNO}} = 8.24 \times 10^{-31}$ Wm^3/kg^2 and

$$X_{\mathrm{CNO}} = \frac{M_{\mathrm{CNO}}}{M}$$

is the total mass fraction in C, N and O. These three elements are only catalysts in the reaction, the number of C, N and O molecules do not change in the reaction. This expression is valid for $T_6 \approx 15$. We see that when the temperature increases a little, the CNO cycle becomes much more effective because of the power 20 in temperature. In the exercises you will find how much. Thus, the CNO cycle is very sensitive to the temperature. Small changes in the temperature may have large influences on the energy production rate by the CNO cycle.

For stars with an even hotter core, also ${}^4_2$He may fuse to heavier elements. In the triple-alpha process three ${}^4_2$He nuclei are fused to form ${}^{12}_6$C.

$$ {}^4_2\mathrm{He} + {}^4_2\mathrm{He} \rightarrow {}^8_4\mathrm{Be} +^0_0 \gamma $$

$$ {}^8_4\mathrm{Be} + {}^4_2\mathrm{He} \rightarrow {}^{12}_6\mathrm{C}^* +^0_0 \gamma $$

Here the reaction rate can be written as

$$ \varepsilon_{3\alpha} = \varepsilon_{0,3\alpha}\rho^2 X_{He}^3 T_8^{41}. $$

Here $T = 10^8 T_8$, T_8 is the temperature in hundred millions of Kelvin and $\varepsilon_{0,3\alpha} = 3.86 \times 10^{-18}$ Wm^6/kg^3. This expression is valid near $T_8 = 1$. We see an extreme temperature dependence. When the temperature is high enough, this process will produce much more than the other processes.

For higher temperatures, even heavier elements will be produced for instance with the reactions

$$ {}^4_2\mathrm{He} + {}^{12}_6\mathrm{C} \rightarrow {}^{16}_8\mathrm{O} +^0_0 \gamma \tag{6} $$

$$ {}^{12}_6\mathrm{C} + {}^{12}_6\mathrm{C} \rightarrow {}^{24}_{12}\mathrm{Mg} +^0_0 \gamma \tag{7} $$

There is a limit to which nuclear reactions can actually take place: The mass of the resulting nucleus must be lower than the total mass of the nuclei being fused. Only in this way energy is produced. This is not always the case. For instance the reactions

$$ {}^{12}_6\mathrm{C} + {}^{12}_6\mathrm{C} \rightarrow {}^{16}_8\mathrm{O} + 2{}^4_2\mathrm{He} \tag{8} $$

and

$$ {}^{16}_8\mathrm{O} + {}^{16}_8\mathrm{O} \rightarrow {}^{24}_{12}\mathrm{Mg} + 2{}^4_2\mathrm{He} \tag{9} $$

require energy *input*, that is the total mass of the resulting nucleus is larger than the total mass of the input nuclei. It is extremely difficult to make such reactions happen: Only in extreme environments with very high temperatures is the probability for such reactions large enough to make the processes take place.

In figure 4 we show the mass per nucleon for the different elements. We see that we have a minimum for ${}^{56}_{26}$Fe. This means that for lighter elements (with less than 56 nucleons), the mass per nucleon decreases when combining nuclei to form more heavier elements. Thus, for lighter elements, energy is usually released in a fusion reaction (with some exceptions, see equation 8 and 9). For elements heavier than iron however, the mass per nucleus increases with increasing number of nucleons. Thus, energy input is required in order to make nuclei combine to heavier nuclei. The latter processes are very improbable and require very high temperatures.

We see that we can easily produce elements up to iron in stellar cores. But the Earth and human beings consist of many elements much heavier

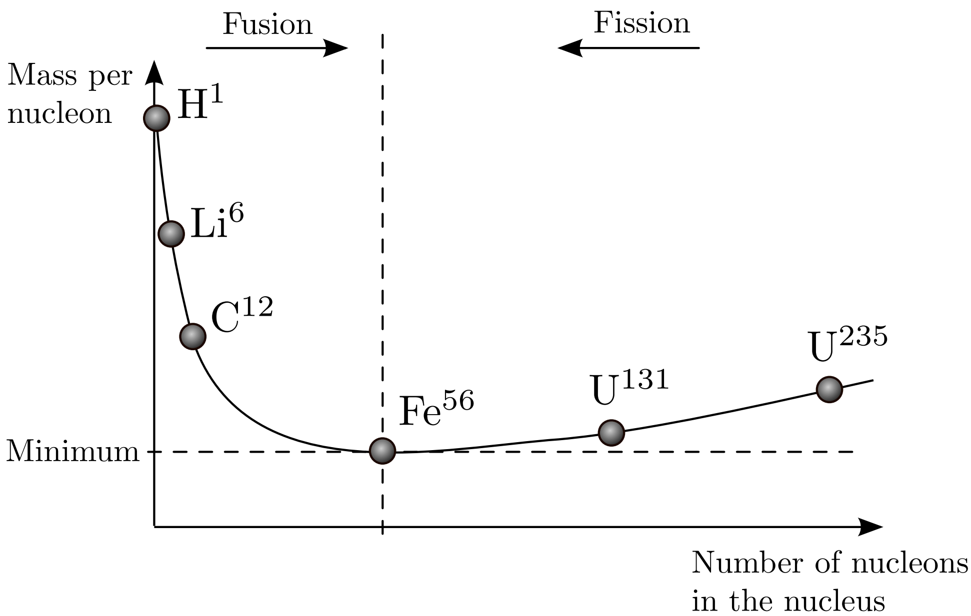

Figure 4: Schematic diagram of mass per nucleon as a function of the number of nucleons in the nucleus. Note that we are only illustrating the general trends. There are for instance a few light elements for which the mass per nucleon increases with increasing number of nucleons in the nucleus.

than iron. How were these produced? In the Big Bang only hydrogen and helium were produced so the heavier elements must have been created in nuclear reactions at a later stage in the history of the universe. We need situations were huge amounts of energy are available to produce these elements. The only place we know about where such high temperatures can be reached are supernova explosions. We will come back to this later.

6 The solar neutrino problem

If you look back at the chain reactions above you will see that neutrinos are produced in the pp-chain and the CNO cycle. We have learned in earlier lectures that neutrinos are particles which hardly react with matter. Unlike the photons which are continuously scattered on charged particles on they way from the core to the stellar surface, the neutrinos can travel directly from the core of the Sun to the Earth without being scattered even once. Thus, the neutrinos carry important information about the solar core, information which would have otherwise been impossible to obtain without being at the solar core. Using the chain reactions above combined with the theoretical reaction rates, we can calculate the number of neutrinos with a given energy we should observe here at Earth. This would be an excellent test of the theories for the composition of the stellar interiors as well as of our understanding of the nuclear reactions in the stellar cores. The procedure is as follows

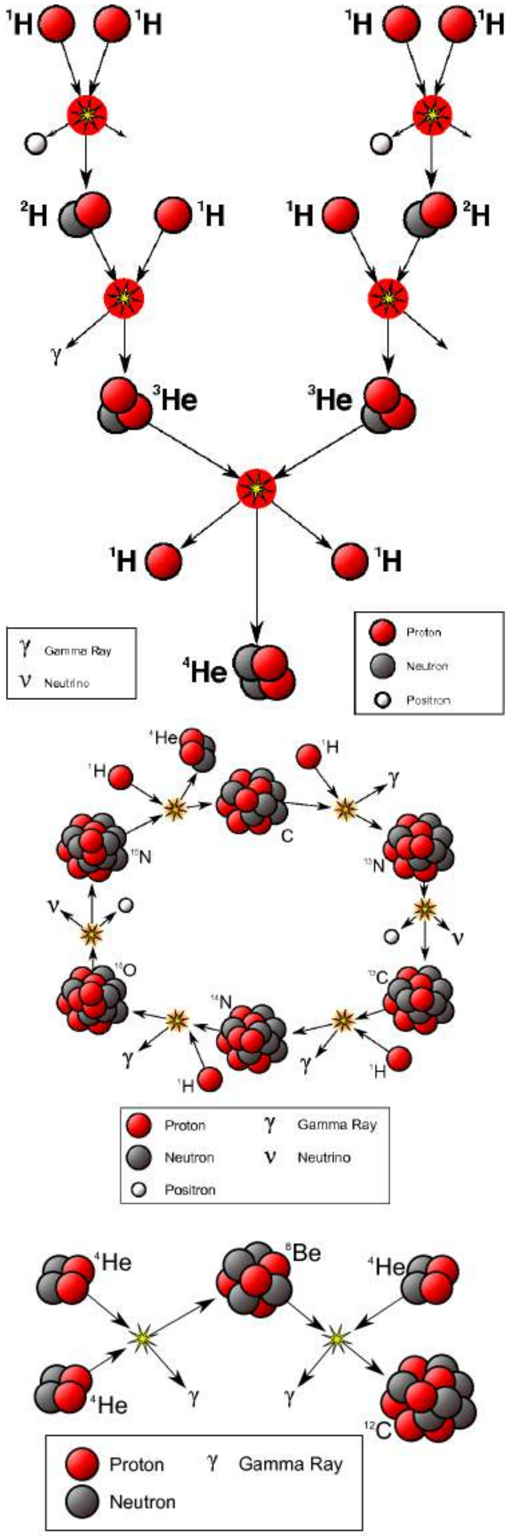

Figure 5: Info-figure: The protonproton (pp) chain reaction, The carbon-nitrogen-oxygen (CNO) cycle (the helium nucleus is released at the top-left step) and the triple-alpha process.(Figure:Wikipedia)

1. Stellar model building: Solve the coupled set of equations consisting of the equation of hydrostatic equilibrium, equation 5 as well as several equations from thermodynamics describing the transport of energy within the Sun. The solutions to these equations will give you the density $\rho(r)$ and temperature $T(r)$ of the Sun as a function of distance r from the center.

2. The temperature $T(r)$ at a given distance r combined with the above expressions for stellar reaction rates gives the number of neutrinos produced in the different kinds of chain reactions and what energies E these neutrinos should have.

3. Measure the flux of neutrinos for different energy ranges E that we receive on Earth and compare to theoretical predictions.

4. If there is agreement, it means we have obtained the correct model for the Sun. If the agreement is not satisfactory, we need to go back to the first step and make the stellar model building with different assumptions and different parameters.

For many years, there was a strong disagreement between the neutrino flux observed at Earth and the solar models. The observed number of neutrinos was much lower than predicted. Now the discrepancy is resolved and the solution led to an important discovery in elementary particle physics: It was discovered that the neutrinos have mass. It was previously thought that neutrions were massless like the photons. Elementary particle physics predicted that if the neutrinos have mass, they may oscillate between the three different types of neutrino. If neutrinos have mass, then an electron neutrino could spontaneously convert itself into a muon or tau neutrino. The first neutrino experiments were only able to detect electron neutrinos. The reason they didn't detect enough solar neutrinos was that they had converted themselves to different types of neutrinos on the way from the solar core to the Earth. Today neutrino detectors may also detect other kinds of neutrino and the observed flux is in much better agreement with the models. But it does not mean that the solar interior and solar nuclear reactions are completely understood. Modern neutrino detectors are now used to measure the flux of different kinds of neutrinos in different energy ranges in order to understand better the processes being the source of energy in the Sun as well as other stars.

But the neutrinos hardly react with matter, how are they detected? This is not an easy task and a very small fractions of all the neutrinos passing through the Earth are detected. One kind of neutrino detector consists of a tank of cleaning fluid C_2Cl_4, by the reaction

$$^{37}_{17}\text{Cl} + {^0_0}\nu \rightarrow {^{37}_{18}}\text{Ar} + {^{\ 0}_{-1}}\text{e}.$$

The argon produced is chemically separated from the system. Left to itself the argon can react with an electron (in this case with its own inner shell electron) by the converse process

$$^{37}_{18}\text{Ar} + {^{\ 0}_{-1}}\text{e} \rightarrow {^{37}_{17}}\text{Cl} + {^0_0}\nu.$$

The chlorine atom is in an excited electronic state which will spontaneously decay with the emission of a photon. The detection of such photons by a photomultiplier then is an indirect measurement of the solar neutrino flux.

7 Problems

Problem 1 (2–3 hours)

We will show that the mean kinetic energy of a particle in the gas is

$$K = \frac{3}{2}kT.$$

In statistics, if x is a stochastic random variable and we want to find the mean value of a function $f(x)$ of this random variable, we use the formula for the mean

$$< f(x) >= \int dx f(x) P(x),$$

where $P(x)$ is the probability distribution function describing the probability of finding a certain value for the random variable x. The probability distribution needs to be normalized such that

$$\int dx P(x) = 1.$$

All integrals over x are over all possible values of x.

Let's translate the last sentences into a more understandable language: physics. Our random variable x is simply the velocity v of particles in a gas. Why random? Because if you take a gas and choose randomly one particle in the gas, you do not know which value you will find for v, it is random. Thermodynamics gives us the *probability distribution* $P(x)$ of velocities. This probability distribution tells us the probability that our choosen gas particle has a given velocity v. In an ideal gas, the probability distribution is given by the Maxwell-Boltzmann distribution function in equation 3. Finally, the function $f(x)$ is any function of the velocity, of which we want the mean value. This could for instance be the kinetic energy $K(v) = (1/2)mv^2$. This is a function of the random variable v and we would indeed like to find the mean value of this function, that is, the mean kinetic energy of a particle in the gas. This mean kinetic energy would be the energy we would find if we measured the kinetic energy of a large number of particles in the gas and took the mean. It is that simple. So now we substitute x with v, $f(x)$ with $K(v)$ and the probability distribution $P(x)$ with $n(v)$. There is however one caveat: Above we mentioned that $P(x)$ needs to be normalized. The form of the Maxwell-Boltzmann distribution in equation 3 is not normalized. We call the normalized distribution $n_{\text{norm}}(v)$. Then we have

$$< K >= \int dv K(v) n_{\text{norm}}(v).$$

In the following you will need the following two integrals

$$\int_0^\infty dx e^{-x} x^{1/2} = \frac{\sqrt{\pi}}{2}$$

and

$$\int_0^\infty dx e^{-x} x^{3/2} = \frac{3\sqrt{\pi}}{4}.$$

1. First we need to find $n_{\text{norm}}(v)$. We write

$$n_{\text{norm}}(v) = \frac{1}{N} n(v).$$

   Use the normalization integral for $P(x)$ above to find N.

2. Now use the normalized distribution function to find the mean kinetic energy of a particle in an ideal gas.

3. Now we will check our result numerically: Note that the Maxwell-Boltzmann distribution function in equation 3 is the probability of finding a particle with absolute value v of the velocity. We now want to simulate gas particles using this distribution, but in order to create a realistic simulation we also need to take account the direction of the particles. The corresponding Maxwell-Boltzmann distribution function for the probability of finding a gas particle with velocity vector $\vec{v}$ can be written like this:

$$n(\vec{v}) = \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{1}{2}\frac{mv^2}{kT}},$$

   which is the expression we need to use to simulate particles (in a later lecture you will learn how to go from this exression for $n(\vec{v})$ to the expression for $n(v)$ in equation 3). Note that this distribution is already normalized. Looking at this expression, you see that this is a Gaussian distribution function which can be written on the form

$$P(\vec{v}) = \frac{1}{(2\pi\sigma^2)^{3/2}} e^{-(v_x^2+v_y^2+v_z^2)/(2\sigma^2)}$$

   (a) Comparing with the Maxwell-Boltzmann distribution for $\vec{v}$, what is σ here?

   (b) In Python there is a function `random.gauss(mean,sigma)` to produce random numbers with a Gaussian probability distribution. The v_x, v_y and v_z components of the velocity of gas particles are thus all distributed randomly with mean value 0 and standard deviation given by the σ which you just found. Now you will simulate 10000 gas particles with a temperature $T = 6000$ K (like on the solar surface), assuming that the atoms in the gas are hydrogen atoms. Now produce the random velocity components v_x, v_y and v_z of these particles using the `random.gauss` function in Python. Now you have an array which has the velocity in each direction for all your 10000 particles representing what you would really find if you had a look at the velocities of 10000 particles in a gas with this temperature. Now compute the kinetic energy for each of the particles and take the mean value over all your particles. Compare the number you get to what you obtain with the analytic expression you found above. Does it fit? If it does not fit precisely, and if you have the computer power to do it, repeat the code but now with 100000 particles. Does it fit better now? In most real situatuions, an analytic expression cannot be found and simulations like these have to be made.

Problem 2 (60–90 min.)

One of the solar standard models predict the following numbers for the solar core: $\rho = 1.5 \times 10^5$ kgm^{-3}, $T = 1.57 \times 10^7$ K, $X_{\mathrm{H}} = 0.33$, $X_{\mathrm{He}} = 0.65$ and $X_{\mathrm{CNO}} = 0.01$. We will assume that the expressions for energy production per kilogram given in the text are valid at the core temperature of the Sun. We will make this approximation even for the expression for the triple-alpha reaction which is supposed to be correct only for higher temperatures.

1. Calculate the total energy produced per kilogram in the Sun by the pp-chain, CNO-cycle and the triple-alpha process.

2. Find the ratio between the energy production of the pp-chain and the CNO-cycle and between the pp-chain and the triple-alpha process. The energy produced by the CNO cycle is only about 1% of the total energy production of the Sun. If you got a very different number in your ratio between the pp-chain and the CNO-cycle, can you find an explanation for this difference? What would you need to change in order to obtain a more correct answer?

3. Now repeat the previous question using a mean core temperature of about $T = 13 \times 10^6$ K. Use this temperature in the rest of this exercise.

4. At which temperature T does the CNO cycle start to dominate?

5. Assume for a moment that only the pp-chain is responsible for the total energy production in the Sun. Assume that all the energy production in the Sun takes place wihin a radius $R < R_E$ inside the solar core.Assume also that the density, temperature and mass fractions of the elements are constant within the radius R_E. So all the energy produced by the Sun is produced in a sphere of radius R_E in the center of the solar core. Use the above numbers and the solar luminosity $L_\odot = 3.8 \times 10^{26}$ W to find the size of this radius R_E within which all the energy production takes place. Express the result in solar radii $R_\odot \approx 7 \times 10^8$ m. The solar core extends to about $0.2R_\odot$. How well did your estimate of R_E agree with the radius of

the solar core?

6. If the CNO-cycle alone had been responsible for the total energy production of the Sun, what would the radius R_E had been? (again express the result in solar radii)

Problem 3 (30 min.–1 hour)

1. Go through all the nuclear reactions in the pp-chain and CNO cycle. For each line in the chain, check that total charge and total lepton number is conserved. (there might be some printing errors here, if you spot one where is it?)
2. After having checked all these reactions you should have gained some intuition about these reactions and the principles behind them. So much that you should be able to guess the missing numbers and particles in the following reactions

$$\begin{aligned} {}^{27}_{14}\mathrm{Si} &\rightarrow {}^{27}_{?}\mathrm{Al} + e^+ + ? \\ {}^{27}_{?}\mathrm{Al} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{24}_{12}\mathrm{Mg} + {}^{?}_{2}? \\ {}^{35}_{17}\mathrm{Cl} + {}^{1}_{1}\mathrm{H} &\rightarrow {}^{36}_{18}\mathrm{Ar} + ? \end{aligned}$$

Problem 4 (30–60 min.)

Before you can do this exercise, you need to read through section 4 again. In that section, we were considering a gas with a total number density of particles n per volume, a number density n_A per volume of A nuclei and a number density n_B per volume of B nuclei. We will now calculate the rate of reaction between A and B nuclei in the gas.

1. First we will try to find how many A nuclei with a given energy E will react with one B nucleus per time interval Δt. The answer is simple: All the A particles with energy E which are in such a distance from B that they will hit the disk with cross section $\sigma(E)$ around nucleus B within the time interval Δt (again, this is an imaginary situation: only one nucleus A can really react with B, the numbers we obtain are in reality probabilities). In figure 3 we illustrate the situation. Let $n_A(E)$ be the number density of A nuclei with energy E such that $n_A(E)dE$ is the number density of A nuclei with energies between E and $E+dE$. Then, show that the total number of nuclear reactions per nucleus B from A nuclei with energies in the interval E to $E+dE$ is given by
$$dN_A(E) = v(E)\, dt\, \sigma(E) n_A(E)\, dE. \qquad (10)$$
2. Show that the velocity of an A-nuclei can be written as
$$v(E) = \sqrt{\frac{2E}{\hat{\mu}}}$$

(What is μ here?)

3. Use equation 3 to show that the number of A nuclei with energies between E and $E + dE$ is given by

$$n_A(E)dE = \frac{2n_A}{\sqrt{\pi}(kT)^{3/2}} E^{1/2} e^{-\frac{E}{kT}} dE.$$

4. Show now that the total reaction rate per B nucleus, i.e. the number of A nuclei reacting with each B nucleus (independent of the energy of the B nucleus, remember that the B nucleus is at rest) is given by

$$\frac{dN_A(E)}{dt} = \frac{1}{\sqrt{\pi\hat{\mu}}}(\frac{2}{kT})^{3/2}\sigma(E) n_A E e^{-\frac{E}{kT}} dE,$$

5. To obtain the total reaction rate r_{AB} between A and B nuclei having the number of reactions per B nucleus, we thus need to multiply with the total density of B nuclei n_B and integrate over all energies E. Show that the reaction rate, the total number of nuclear reactions per unit of volume per unit of time, is given by

$$r_{AB} = \frac{dN}{dt} = \left(\frac{2}{kT}\right)^{3/2} \frac{n_A n_B}{\sqrt{\hat{\mu}\pi}} \int_0^\infty dE E e^{-E/kt} \sigma(E)$$

6. Use this expression to find the units of the reaction rate r_{AB} to check if you find the units that you would expect for r_{AB} being the total number of nuclear reactions per unit of volume per unit of time.

7. It is common to express the reaction rate using ε_{AB} which is the energy released per kilogram of gas per second. Assume that the energy released in each reaction between an A nucleus and a B nucleus is given by ε_0 (which is *not* the vacuum permittivity ϵ_0) and show that ε_{AB} can be written in terms of r_{AB} as

$$\varepsilon_{AB} = \frac{\varepsilon_0}{\rho} r_{AB}.$$

(Why does the density enter here?)

AST1100 Lecture Notes

20: Stellar evolution: The giant stage

1 Energy transport in stars and the life time on the main sequence

How long does the star remain on the main sequence? It will depend on the available hydrogen in the core. Note that as hydrogen is converted to helium the mean molecular weight μ (see lecture 13–14) increases. We remember that the pressure in an ideal gas can be written as

$$P = \frac{\rho k T}{\mu m_H}.$$

Thus as μ increases, P decreases provided ρ and T remain approximately constant. The result is that the hydrostatic equilibrium is lost. The battle between the gravitational forces and the pressure forces is won by gravitation and the stellar core starts contracting. The result of the contracting core is that the core density and temperature rise. At higher core temperatures, the nuclear reactions which are more effective at higher temperatures start to be more important. We will now make an estimate of how long time it takes until the hydrogen in the core is exhausted. At this point, the star leaves the main sequence and starts the transition to the giant stage.

Before continuing the discussion on energy production in the core we need to have a quick look at how the energy is transported from the core to the surface. Clearly the photons produced in the nuclear reactions in the core do not stream directly from the core and to the surface. The total luminosity that we observe does not come directly from the nuclear reactions in the core. The photons produced in the nuclear reactions scatter on the nuclei and electrons in the core transferring the energy to the particles in the core. Thus, the high temperature of the stellar core is a result of the energetic photons produced in the nuclear reactions. The high temperature plasma in the core emits thermal radiation. The photons resulting from this thermal radiation constitutes a dense photon gas in the core of the star. How is the energy, that is, the heat of the plasma or the photons in the photon gas, transported to the stellar surface? There are three possible ways to transport energy in a medium:

- By *radiation*: Photons from the photon gas traveling outwards. The photons cannot travel directly from the core, but will be continuously scattered in many different directions by collisions with other particles. After a large number of scatterings and direction changes it will eventually reach the surface and escape.
- By *convection*: Large masses of the hot gas may stream outwards while the cooler gas falls inwards. In this way, the heat and thereby the energy is transfered outwards. Convection is a much more efficient way of energy transport than radiation.
- By *conduction*: Heat is transfered directly outwards by particle collisions.

In stars, mostly the two former mechanisms for energy transport are at play. In solar mass stars, energy is transported from the core by radiation until a distance of about $r = 0.7R_\odot$ where convection starts to be the most important mechanism for energy transport out to the surface.

We will now make a very crude estimate of how long a star remains on the main sequence. In order to do this properly it is necessary to do stellar model building, i.e. solve the coupled set of equations of hydrostatic equilibrium, the equations of energy production and the equations of energy transport. This gives a model of the star in terms of density and temperature as a function of distance from the center. From this model, the proper life time of the star can be calculated. It turns out that the estimates and relations that we now will deduce using some very rough approximations give results close to the results obtained using the full machinery of stellar model building.

The outline of the method is the following: Find an expression for the luminosity of the star. We know that luminosity is energy radiated away per unit of time. If we assume how much energy the star has available to radiate away during its life time, we can divide this energy by the luminosity to find the life time (assuming constant luminosity which is a good assumption during the main sequence phase).

We will again consider the photon gas in the stellar core. You will in later courses in thermodynamics show that the energy density, i.e. energy per volume, of a photon gas goes as $\rho_E \propto T^4$ (actually $\rho_E = aT^4$ where a is the radiation constant that we encountered in lecture 13–14 for the pressure of a photon gas $P = 1/3aT^4$). The question is how long time it will take for the photos in the photon gas to reach the surface of the star. We will now assume that the only mechanism for energy transport is by radiation. A photon which starts out in the core will be scattered on particles and continuously change directions until it reaches the surface of the star (see figure 2). We assume that the photon travels a mean free path ℓ between each collision. After being scattered N times, the position $\vec{d}$ of the photon

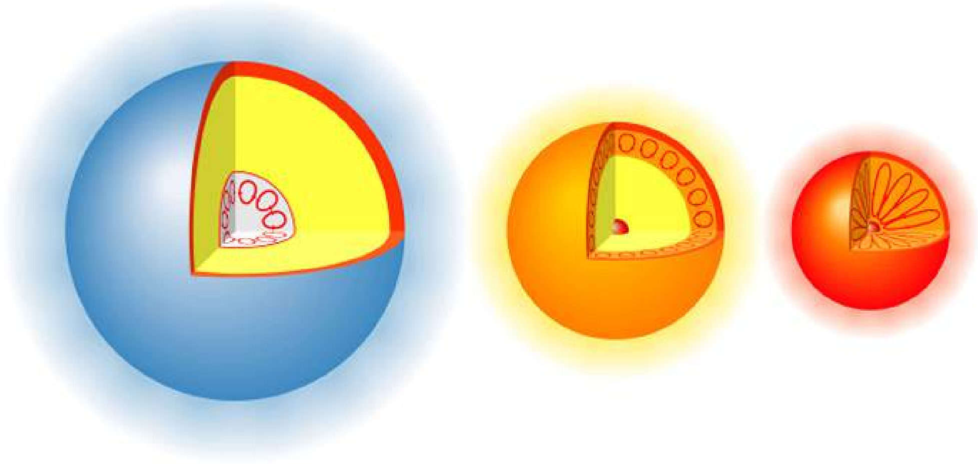

Figure 1: Info-figure: Stars produce energy by fusion in their deep interior because only there are the pressures and temperatures high enough to sustain thermonuclear reactions. However, most of the luminous energy of stars is radiated from the thin region at the surface that we call the photosphere. The two most important ways of transporting energy from the core to the surface in main sequence stars are by radiation and by convection. A low mass main sequence star (middle) will have convection in its outer layers and a radiation zone (yellow area) in the center, like the Sun. If the star is really low mass (right) it will have convection all the way in. A high mass star (left) will have convection only in its core.(Figure: B. Boroson)

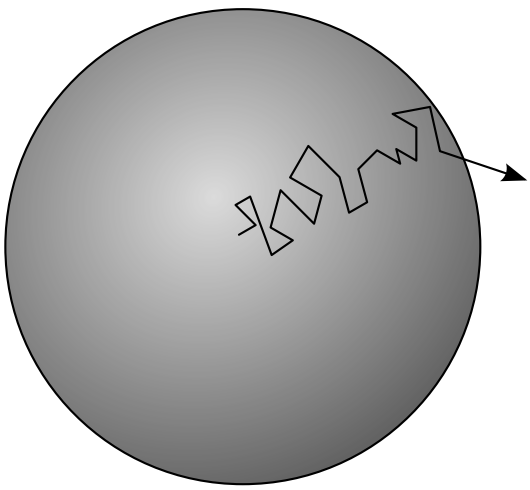

Figure 2: Energy transport by radiation: random walk of the photons from the core of the star to the surface.

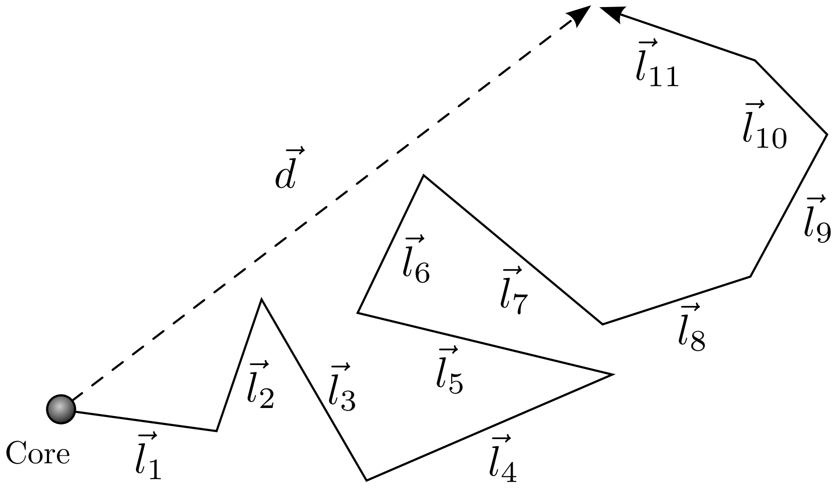

Figure 3: Random walk from the core. The position after N scatterings $\vec{l}_i$ is $\vec{d}$.

(see figure 3) is given by

$$\vec{d} = \sum_{i=1}^{N} \vec{l}_i,$$

where $\vec{l}_i$ is the displacement vector between each scattering i (see again figure 3). The total length Δr of the vector d is the total distance the photon has moved from the center. It is given by (check!)

$$\Delta r^2 = \vec{d} \cdot \vec{d} = \sum_{i,j} \vec{l}_i \cdot \vec{l}_j = N\ell^2 + \ell^2 \sum_{i \neq j} \cos\theta_{ij},$$

where θ_{ij} is the angle between two vectors $\vec{l}_i$ and $\vec{l}_j$. The directions of the scatterings are random, so $\cos\theta_{ij}$ will have values between -1 and 1. After many scatterings, the mean value of this term will approach zero and we have

$$\Delta r = \sqrt{N}\ell,$$

or writing this in terms on number of scatterings N to reach the surface we thus have $N = R^2/\ell^2$ where R is the radius of the star (check!).

The time Δt for a photon to reach the surface is then (note that the total distance traveled by the photon is $N\ell$)

$$\Delta t = \frac{N\ell}{c} = \frac{\ell}{c}\frac{R^2}{\ell^2} = \frac{R^2}{\ell c}.$$

If we assume that within a radius r of the star, the temperature T and energy density ρ_E of the photon gas is constant, the total energy content of the photon gas within radius r is

$$E = \frac{4}{3}\pi r^3 \rho_E \propto r^3 T^4,$$

where we used that $\rho_E \propto T^4$. We will now use a very rough model of the star: We assume the density and temperature of the star to be constant everywhere in the star. Then the energy content of the photon gas in the whole star is given by $E \propto R^3T^4$. If we assume that this energy is released within the time Δt it takes for the photons in the core to reach the surface, then the luminosity of the star can be written as

$$L \propto \frac{E}{\Delta t} \propto \frac{R^3T^4}{R^2/\ell} \propto RT^4\ell. \tag{1}$$

The mean free path ℓ depends on the density of electrons and the different nuclei in the core. If we assume that photons are only scattered on electrons, it can be shown that the mean free path $\ell \propto 1/\rho$ which does seem reasonable: The higher the density the lower the mean free path between each scattering. Since we assume constant density we have $\rho \propto M/R^3$. Inserting this in equation 1 we have

$$L \propto RT^4\ell \propto \frac{RT^4}{\rho} \propto \frac{R^4T^4}{M}. \tag{2}$$

Finally we will use the equation of hydrostatic equilibrium

$$\frac{dP}{dr} = -\rho g.$$

If we assume that the pressure can be written as $P \propto r^n$ where n is unknown then

$$\frac{dP}{dr} = nr^{n-1} = \frac{nr^n}{r} = \frac{nP}{r} \propto \frac{P}{r}.$$

The equation of hydrostatic equilibrium then yields

$$\frac{P}{R} \propto \rho g \propto \frac{M}{R^3}\frac{M}{R^2} \propto \frac{M^2}{R^5},$$

or $P \propto M^2/R^4$. We remember from lecture 13–14 than for an ideal gas $P \propto \rho T$. Inserting this in the previous equation gives

$$T \propto \frac{M}{R}.$$

Inserting this in equation 2 we get

$$L \propto \frac{R^4}{M}\left(\frac{M}{R}\right)^4 \propto M^3. \tag{3}$$

The luminosity is proportional to the mass of the star to the third power. A more exact calculation would have shown that

$$L \propto M^{\beta},$$

where β is usually between 3 and 4 depending on the exact details of the star. It turns out that most low or medium mass stars have $\beta \approx 4$. This is also supported by observations. Therefore we will in the following use $L \propto M^4$. Having the luminosity of the star, we can easily find the life time. Assume that a fraction p of the mass of the star is converted to energy. Then then total energy radiated away during the lifetime of the star is given by

$$E = pMc^2.$$

If we assume constant luminosity during the lifetime we have

$$L = \frac{pMc^2}{t_{\text{life}}} \propto M^4,$$

giving

$$t_{\text{life}} \propto \frac{1}{M^3}.$$

This can be the total life time of the star, or just the life time on the main sequence (in fact, for most stars the time on the main sequence is so much longer than other stages in a star's life so the time on the main sequence is roughly the same as the life time of the star). If we take p to be the fraction of mass converted to energy during the main sequence, then this is the expression for the time the star spends on the main sequence. We see that the life time of a star is strongly dependent on the mass of the star. The Sun is expected to live for about 10×10^9 years. A star with half the mass of the Sun will live 8 times longer (which is much longer than the age of the universe). A star with two times the mass of the Sun will live only 1/8 or roughly 10^9 years. The most massive stars only live for a few million years. We see from equation 3 that this can be explained by the fact that massive stars are much more luminous than less massive stars and therefore burn their fuel much faster. A star with two times the mass of the Sun will burn 16 times (equation 3) as much 'fuel' per time as the Sun, but it only has twice as much 'fuel'. It will therefore die much younger.

As the last expression is just a proportionality, we need to find the constant of proportionality, that is, we need to know the life time and mass of one star in order to use it for other stars. We know these numbers for the Sun and we will now use approximations to calculate this number. One can show that a star will leave the main sequence when about 10% of its hydrogen has been converted to helium. We discussed in the previous lecture that the efficiency of the pp-chain is 0.7%. So the total energy that will be produced of the Sun during its lifetime is therefore $0.1 \times Mc^2 \times 0.007$. Assuming that the solar luminosity 3.7×10^{26} W is constant during the time on the main sequence we have

$$t_{\odot}^{\text{mainsequence}} = \frac{0.1 \times 2 \times 10^{30}\ \text{kg} \times (3 \times 10^8\ \text{m/s})^2 \times 0.007}{3.7 \times 10^{26}\ \text{W}} \approx 10^{10}\ \text{years}.$$

We will now try to find a way to estimate the mass of a star. Remember that in the lectures on extrasolar planets, we needed to know the mass of the star by independent measurements in order to be able to estimate the mass of a planet orbiting it. In the above approximation we considered a star with constant density and temperature. The conditions we used are normally valid only for the core of the star. Thus, the approximations we made are more correct in the core of the star. We found that the temperature $T \propto M/R$. For main sequence stars, the core temperature is reasonably constant, there is not a large difference in core temperatures for different main sequence stars. Using this assumption we can write

$$T_c \propto \frac{M}{R} = \text{constant}.$$

We can write this as $R \propto M$. Now, we know that the luminosity of a star can be written in terms of the effective temperature as

$$L = 4\pi R^2 \sigma T_{\text{eff}}^4,$$

where $4\pi R^2$ is the area of the surface and $F = \sigma T_{\text{eff}}^4$ is the flux at the surface. Using $R \propto M$ and $L \propto M^4$ this gives

$$L \propto M^4 \propto R^2 T_{\text{eff}}^4 \propto T_{\text{eff}}^4 M^2,$$

so $M^4 \propto T_{\text{eff}}^4 M^2$ giving

$$M \propto T_{\text{eff}}^2 \tag{4}$$

and we have obtained a way to find the mass of a star from its temperature. In the exercises you will use this expression to find the temperature of stars with different masses.

2 From the main sequence to the giant stage

We will now follow a star during the transition from the main sequence to the giant stage. The exact sequence of events will be slightly different depending on the mass of the star. Here we will only discuss the general features and discuss a few main differences between low and high mass stars. In figure 4 we can follow the evolutionary path of the star in the HR diagram. The theories for stellar evolution are developed using computer models of stars obtained by solving the equations for stellar model building numerically. The chain of arguments that we will use below to describe stellar evolution are obtained by studying the outcome of computer simulations.

When the hydrogen in the core has been exhausted, the forces of pressure are not any longer strong enough to sustain the forces of gravity. The hydrostatic equilibrium is lost and the core starts contracting. During

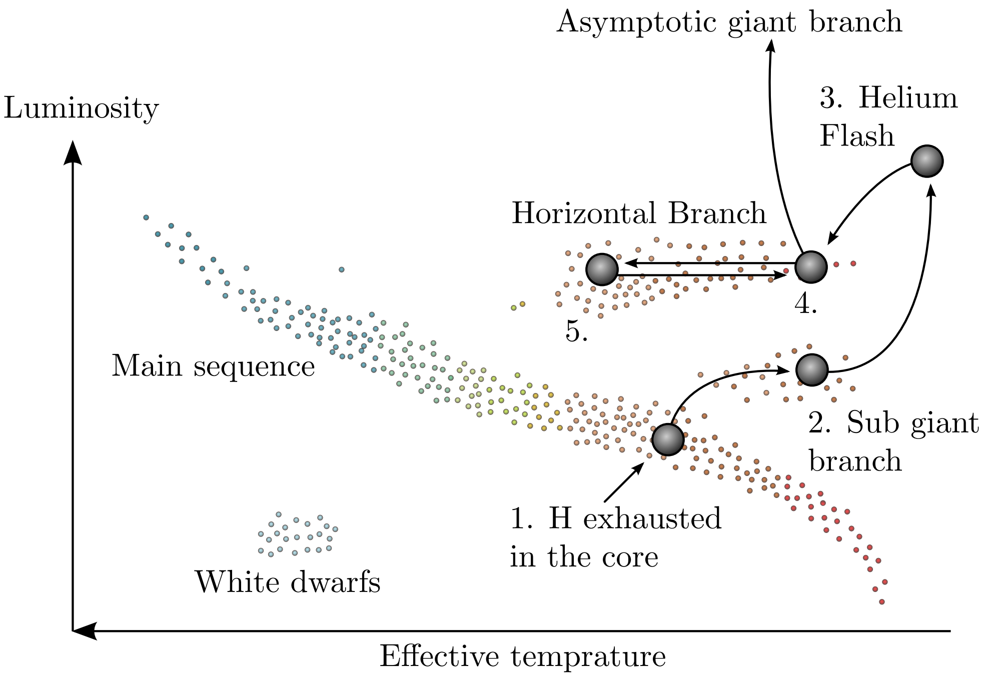

Figure 4: HR-diagram of the evolution of a star from the main sequence to the giant stage.

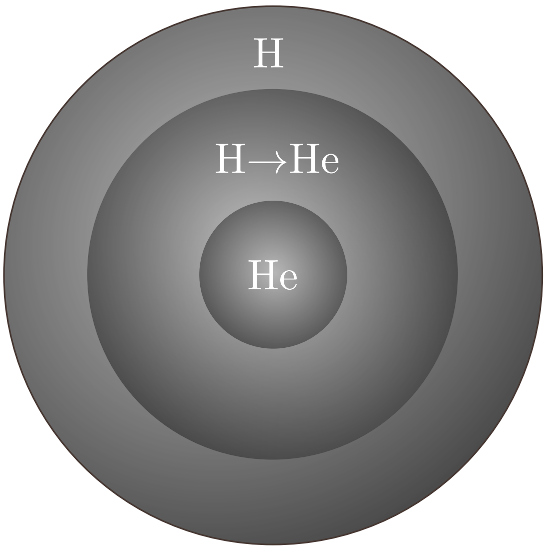

Figure 5: The structure of a subgiant and red giant. The core consists mainly of helium, but the core temperature is not high enough for helium burning. Hydrogen is burning to helium in a shell around the core. For red giants, convection transports material all the way from the core to the surface and the material is mixed (in the figure there is only hydrogen in the outer parts, for red giants the mixing due to convection will also transfer other elements all the way to the surface). The relative sizes of the shells are not to scale, this will depend on the exact evolutionary stage.

the core contraction, the temperature in and around the core increases. The temperature in the core is still not high enough to 'burn' helium (all energy production is by nuclear fusion, not by 'burning' in the classical sense but it is common practice to use the term 'burning' anyway), but the temperature in a shell around the core now reaches temperatures high enough to start hydrogen burning outside the core. The structure of the star is illustrated in figure 5. Because of the increased outward pressure due to hydrogen burning in the shell, the radius of the star starts increasing significantly. The star has become a *sub giant* of luminosity class IV (see the lecture on the HR diagram and luminosity classes). In figure 4 the star has left the main sequence and is now on the *sub giant branch* between point 1 and 2. The luminosity has been increasing slightly because the energy produced in the shell is higher than the energy previously produced in the core. But because of the increasing radius of the star, the surface temperature is dropping. Thus the star moves to the right and slightly upwards in the HR diagram.

When reaching point 2 in the HR-diagram, the radius of the star has been increasing so much that the surface temperature is close to 2500 K which is a lower possible limit. When reaching this limit, the dominant mechanism of energy transport in the star changes from being radiation to convection. Convection is much more efficient, the energy is released at a much larger rate and the luminosity increases rapidly. The star has now become a *red giant*. At the red giant stage, convection takes place all the way from the core to the surface. Material from the core is moved all the way to the surface. This allows another test of the theories of stellar evolution. By observing the elements on the surface of a red giant we also know the composition of elements in the core. The star is now on the red giant branch in the HR-diagram (figure 4). The structure of the star still resembles that of figure 5. The radius is between 10 and 100 times the original radius at the main sequence and the star has reached luminosity class III.

The next step in the evolution depends on the mass of the star. For stars more massive than $\sim 2M_\odot$, the temperature in the core (which is still contracting) will eventually reach temperatures high enough to start the triple-alpha process burning helium to carbon as well as other chains burning helium to oxygen. In low mass stars, something weird happens before the onset of helium burning. As the core is contracting the density becomes so high that a quantum mechanical effect sets in: there is no more space in the core for more electrons. Quantum physics sets an upper limit on the number of electrons within a certain volume with a certain momentum. This is called electron degeneracy. The core has become *electron degenerate*. In the next lecture we will discuss this effect in detail. At the moment all we need to know is that an electron degenerate core will have a new type of pressure: degeneration pressure. The degeneration pressure is now the outward force which battles the inward gravitational force in the equation of hydrostatic equilibrium. The degeneration pres-

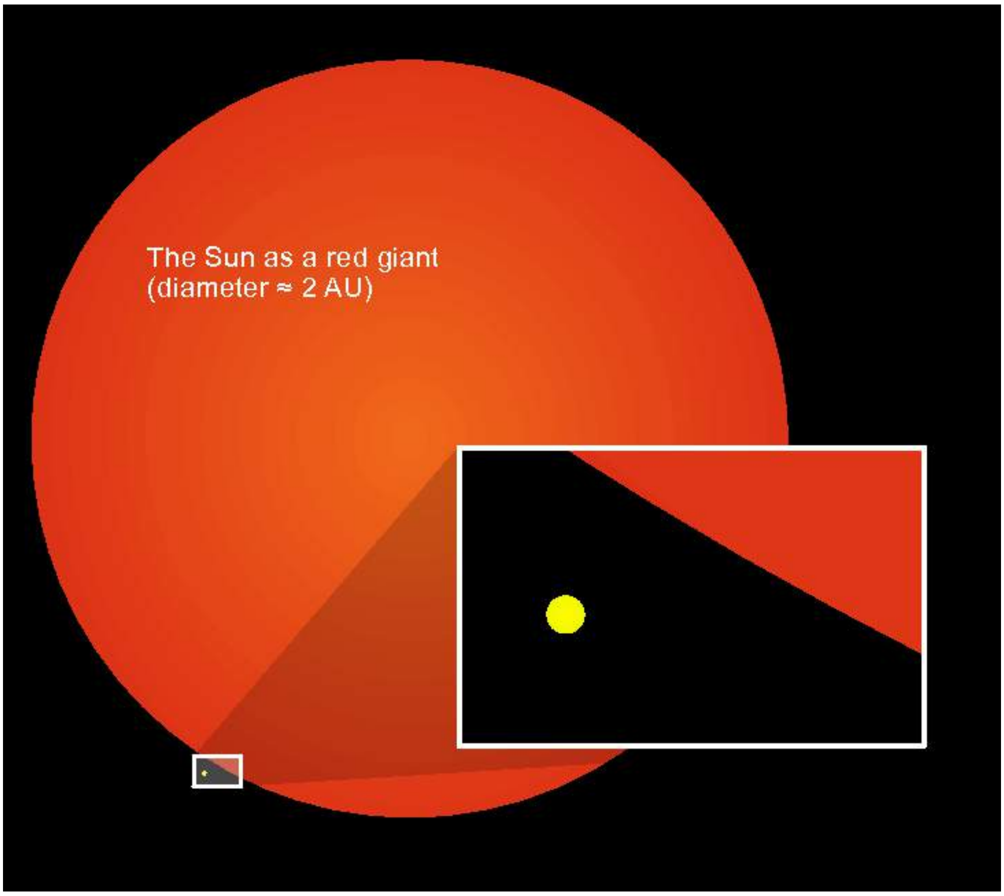

Figure 6: Info-figure: The size of the current Sun compared to its estimated size during its red giant phase in the future. The outer atmosphere of a red giant is inflated and tenuous, making the radius immense and the surface temperature low. Prominent bright red giants in the night sky include Aldebaran, Arcturus, and Mira, while the even larger Antares and Betelgeuse are red supergiants. (Figure: Wikipedia)

sure does not depend on temperature. Thus, even when the temperature of the core increases significantly, the core does not expand. The degenerate core is close to isothermal and when the temperature is high enough to start helium burning, this happens everywhere in the core at the same time. An enormous amount of energy is released in a very short time causing an explosive onset of the helium burning phase. This is called the *helium flash*. After a few seconds, a large part of the helium in the core has already been burned. The huge amounts of energy released breaks the electron degeneracy in the core and the gas starts to behave normally, i.e. the pressure is again dependent on the temperature allowing the core to expand. The onset of helium burning (which includes the helium flash for low mass stars and a less violent transition for high mass stars) is marked by 3 in figure 4.

The final result of the onset of helium burning is therefore the same for both low and high mass stars: The core will finally expand, pushing the hydrogen burning shells outward to larger radii where the gas will cool and the hydrogen burning will therefore cease in large parts of the shell. The energy produced in the helium burning is not enough to substitute the energy production in the shell and the total luminosity of the star will decrease. This is the case also for stars which undergo a helium flash. This is seen in the transition from 3 to 4 in figure 4. The star has now entered the *horizontal branch*. This stage is in a way similar to the main sequence: This is where the star burns its helium to carbon and oxygen in the core. Hydrogen burning is still taking place in parts of the shell. The structure of the star is shown in figure 7. Horizontal branch giants are called so because, as we will discuss now, they will move back and forth along a horizontal branch.

After the rapid expansion of the star after the onset of helium burning, the star starts contracting again in order to reach hydrostatic equilibrium. The result is an increasing effective temperature and the star moves to the left along the horizontal branch. After a while on the horizontal branch, the mean molecular weight in the core has increased so much that the forces of pressure in the core are lower than the gravitational forces and the core starts contracting. The temperature of the core increases and the energy released in this process makes the star expand: The effective temperature of the surface is decreasing and the star is moving to the right along the horizontal branch. At this point the helium in the core is exhausted and nuclear energy production ceases. The following scenario resemble the scenario taking place when the hydrogen was exhausted: The core which now mainly consists of carbon and oxygen starts to contract (due to the lack of pressure to sustain the gravitational forces after the energy production ceased). The core contraction heats a shell around the core sufficiently for the ignition of helium burning. Energy is now produced in a helium burning as well as hydrogen burning shell around the core. The radius of the star increases because of the increased pressure. Again we reach a stage of strong convective energy transport which (exactly as

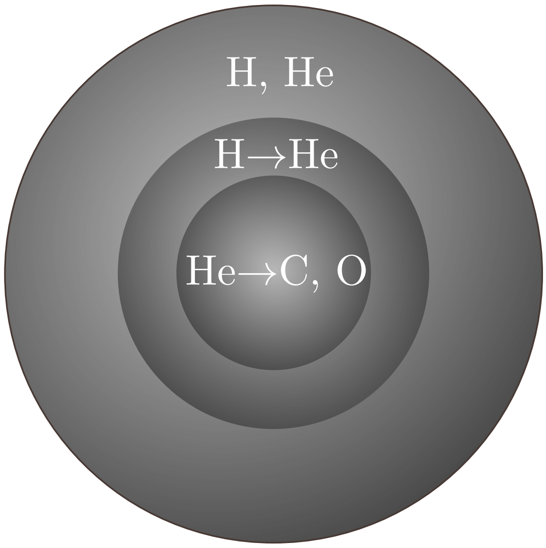

Figure 7: A horizontal branch giant. Helium is burning to carbon and oxygen in the core. Hydrogen is burning to helium in a shell around the core. The relative sizes of the shells are not to scale, this will depend on the exact evolutionary stage.

on the red giant branch) rises the luminosity. The star now moves to the asymptotic giant branch becoming a bright giant of luminosity class II or even a super giant of luminosity class I. The star now has a radius of up to 1000 times the original radius. The structure of the star is shown in figure 8.

Most stars follow an evolution similar to this. The stars with very high mass (more than $\sim 20M_{\odot}$) do not have a significant convective phase and do therefore not change their luminosity much during their evolution. They will mainly move left and right in the HR-diagram.

Open stellar clusters can be used to test the theories of stellar evolution. An open cluster is a collection of stars which were born roughly at the same time from the same cloud of gas. Observing different open clusters with different ages, we can obtain HR diagrams from different epochs of stellar evolution. We can use observed diagrams to compare with the predicted diagrams obtained using the above arguments. In figure 9 we see a schematic example of HR diagrams taken at different epochs (from clusters with different ages). We see that the most massive stars start to leave the main sequence earlier: This is because the life time of stars is proportional to $t \propto 1/M^3$. The most massive stars exhaust their hydrogen much earlier than less massive stars. As discussed above, the most massive stars do not have a phase with strong convection and do therefore not move vertically up and down but mostly left and right in the diagram. For this reason we do not see the red giant branch and the asymptotic branch for these stars. Only in the HR diagram of the oldest cluster has the intermediate mass stars started to leave the main sequence. For these stars we now clearly see all the different branches. Comparing such theoretical

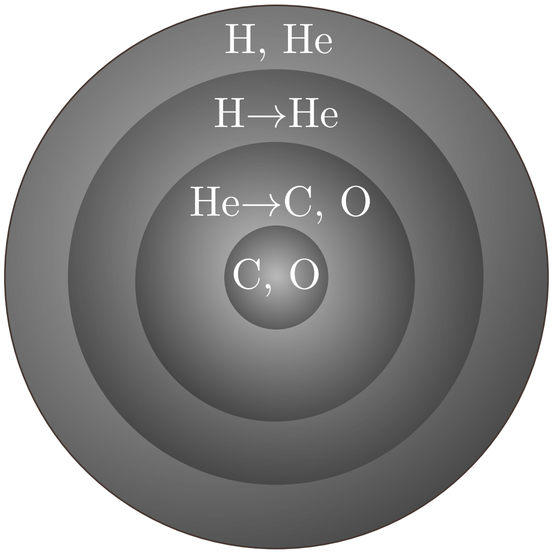

Figure 8: A bright/super giant. The core consists mainly of carbon and oxygen but the temperature is not high enough for these elements to burn. Around the core there is a shell where helium is fused to carbon and oxygen and another shell where hydrogen is fused to helium. In the outer parts the temperature is not high enough for fusion reactions to take place.

diagrams with diagrams for observed clusters has been one of the most important way to test and understand theories of stellar evolution.

Having reached the asymptotic giant branch, the star has almost ended its life cycle. The final stages will be discussed in more detail in the next lectures. First we will look at a typical feature of giant stars: pulsations.

3 Stellar pulsations

Some giant stars have been observed to be pulsating. We have already encountered one kind of pulsating stars: the Cepheids. The pulsating stars have been found to be located in narrow vertical bands, so-called instability strips, in the HR-diagram. The Cepehids for instance, are located in a vertical band about 600 K wide around $T_{\mathrm{eff}} \sim 6500$ K. The pulsations start during the core contraction and expansions starting when the star leaves the main sequence. They last only for a limited period when the star passes through an instability strip in the HR diagram. We remember that for Cepheids there is a relation between the pulsation period and the luminosity of the star allowing us to determine the distance to the star (see the lecture on the cosmic distance ladder). The period-luminosity relation for Cepheids can be written in terms of luminosity (instead of absolute magnitude) as

$$\langle L \rangle \propto P^{1.15}, \tag{5}$$

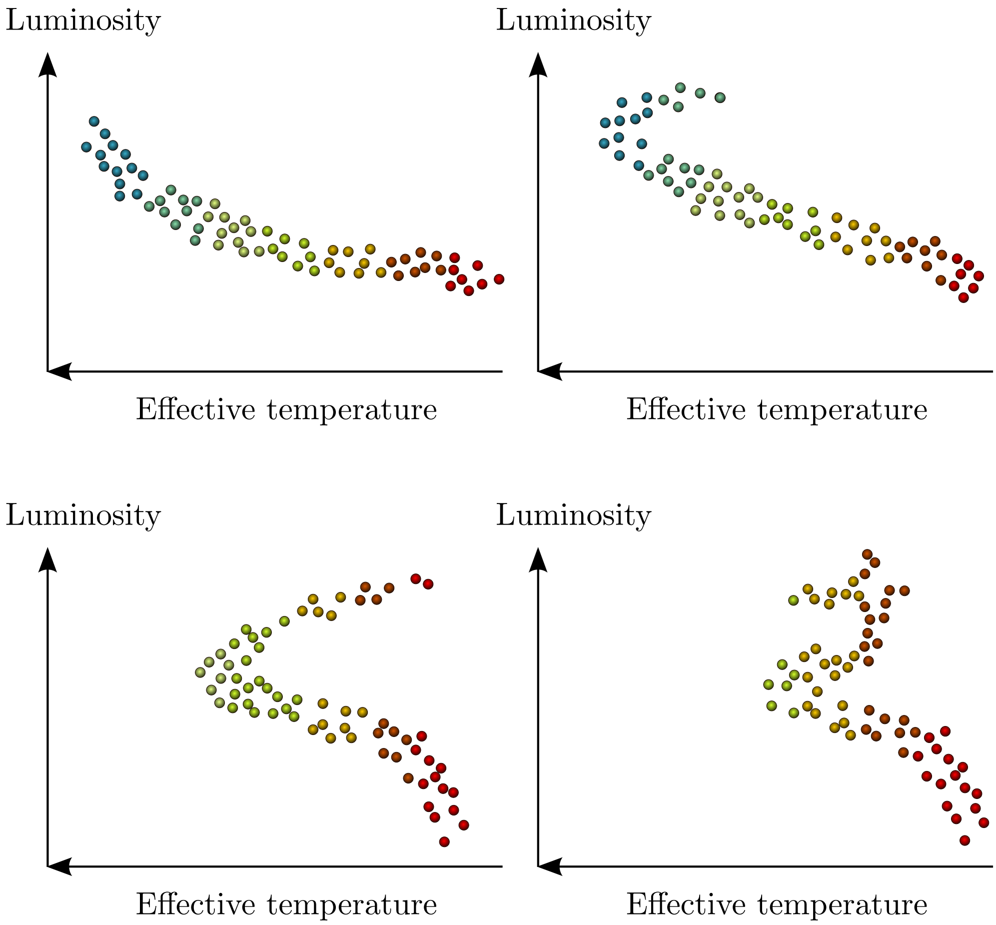

Figure 9: Schematic HR diagrams of open clusters of different ages:

Upper left: A cluster still in the process of forming. The less massive stars are still in the contracting phase and have not yet reached the main sequence.
Upper right: A cluster with an age of about 10^7 years. The most massive stars have started to leave the main sequence.
Lower left: A cluster of about 10^9 years. The low mass stars have now reached the main sequence.
Lower right: A cluster of about 10^{10} years. The medium mass stars have now started to leave the main sequence and we can clearly see the different branches discussed in the text.

where $\langle L \rangle$ is the mean luminosity and P is the pulsation period. We will now see if we can deduce this relation using physics in the stellar interior.

The pulsations are due to huge density waves, sound waves, traveling through the interior of the star. We can find an approximate expression for the pulsation period of a star by considering the time it takes for a sound wave to go from one end of the star to the other. We will for simplicity consider a star with radius R and constant density ρ. The pulsation period P is thus the time it takes for a sound wave to travel a distance $2R$. In thermodynamics you will learn that the sound speed (the so-called adiabatic sound speed) at a given distance r from the center of a star is given by

$$v_s(r) = \sqrt{\frac{\gamma P(r)}{\rho}},$$

where γ is a constant depending on the specific heat capacities for the gas. We have assumed constant density and therefore only need to find the pressure as a function of r. The equation of hydrostatic equilibrium can give us the pressure. We have

$$\frac{dP}{dr} = -g\rho = -\frac{GM(r)}{r^2}\rho = -\frac{4}{3}G\pi r\rho^2.$$

Integrating this expression from the surface where $P = 0$ and $r = R$ down to a distance r we get

$$P(r) = \frac{2}{3}\pi G\rho^2(R^2 - r^2).$$

We now have the necessary expressions in order to find the pulsation period of a Cepheid. At position r, the sound wave travels with velocity $v_s(r)$. It takes time dt to travel a distance dr, so

$$dt = \frac{dr}{v_s(r)}.$$

To find the pulsation period, we need to find the total time P it takes for the sound wave to travel a distance $2R$

$$P \approx 2\int_0^R \frac{dr}{v_s(r)} \approx 2\int_0^R \frac{dr}{\sqrt{\frac{2}{3}\gamma\pi G\rho(R^2 - r^2)}}$$

$$= \frac{1}{\sqrt{\frac{2}{3}\gamma\pi G\rho}}\left[-\tan^{-1}\frac{r\sqrt{R^2 - r^2}}{r^2 - R^2}\right]_0^R$$

Taking the limits in this expression, we find

$$P \approx \sqrt{\frac{3\pi}{2\gamma G\rho}} \propto \frac{1}{\sqrt{\rho}} \propto \left(\frac{R^{3/2}}{M^{1/2}}\right).$$

From equation 4 we see that $M^{1/2} \propto T_{\text{eff}}$ but since Cepheids are located along the instability strip in the HR-diagram their effective temperatures are roughly constant. So we have

$$P \propto R^{3/2}.$$

The luminosity of a star can be written as as $L = 4\pi R^2 \sigma T_{\text{eff}}^4$. Again we consider $T_{\text{eff}} \approx$ constant so $L \propto R^2$ or $R \propto L^{1/2}$. Inserting this into the previous expression for the pulsation period we have

$$P \propto L^{3/4},$$

or

$$L \propto P^{4/3} \propto P^{1.3}.$$

Comparing to the observed period-luminosity relation (equation 5), this agreement is excellent taking into account the huge simplifications we have made. We have shown that by assuming the pulsations to be caused by sounds waves in the stellar interiors, we obtain a period luminosity relation for Cepheids similar to what we observe.

4 Problems

Problem 1 (10–20 min.)

In the text there is a formula for estimating the effective temperature of a star with a given mass (or estimating the mass of a star with a given effective temperature).

1. Given the effective temperature (5780 K) and mass ($M_\odot$) of the Sun, find the effective temperature of a small star with $M = 0.5M_\odot$, an intermediate mass star $M = 5M_\odot$ and a high mass star $M = 40M_\odot$.
2. The star Regulus in the constellation Leo is a blue main sequence star. It is found to have a peak in the flux at a wavelength of about $\lambda = 240$ nm. Give an estimate of its mass expressed in solar masses.

Problem 2 (30–60 min.)

In the text we derived that the luminosity of a low/intermediate mass star is proportional to mass to the third power $L \propto M^3$. In this derivation you used the ideal gas law. For high mass stars, the radiation pressure is more important than the ideal gas pressure and the expression for radiation pressure (you need to find it in the text) needs to be used instead of the expression for the ideal gas pressure. Repeat the derivation for the mass-luminosity relation using radiation pressure instead of ideal gas pressure and show that for high mass stars $L \propto M$. How is the relation between the life time and the mass of a star for a high mass star compared to a low mass star?

Problem 3 (30–60 min.)

Read carefully the description for the evolution of a star from the main sequence to the giant stage. Take an A4-sheet. You are allowed to make some *simple* drawings and write a maximum of 10 words on the sheet. Make the drawings and words such that you can use it to be able to tell someone how a star goes from the main sequence to the giant stage, describing the logic of how the core contracts/expands and how the star moves in the HR-diagram depending on temperature, means of energy transport and nuclear reactions. Bring the sheet to the group and use it (and nothing else) to tell the story of stellar evolution to another student, then exchange your roles.

Problem 4 (10–20 min.)

Look at the HR-diagram for the oldest cluster in figure 9. Can you identify the different branches of stellar evolution?

Problem 5 (1–2 hours)

We will now study the phase when the hydrogen in the stellar core has been depleted. The energy production in the core stops and the core starts shrinking. The star reaches the sub giant branch and then the red giant branch while the core keeps shrinking. The core will keep shrinking until the temperature in the core is high enough for helium burning to start. We will try to find out how much the core shrinks before this takes place. For simplicity we will study a star with so high mass that the core does not become degenerate before helium burning sets in. We will assume the core density at the main sequence to be $\rho = 1.7 \times 10^5$ kg/m^3.

We imagine the stellar core as an 'independent' sphere of mass M_C, radius R_C, pressure P_C and temperature T_C. We assume the density and temperature to be the same everywhere in the core.

1. Use the equation of hydrostatic equilibrium to show that

$$P_C \propto \frac{M_C^2}{R_C^4}.$$

   This is done in the text, but try to find your own arguments before looking it up.

2. Then combine this with the ideal gas law to show that

$$T_C \propto \frac{M_C}{R_C}.$$

3. We assume that the core temperature of the star on the main sequence was $T_C = 18 \times 10^6$ K. Use the expressions for the nuclear energy production rates from the previous lecture to find out whether it was the pp-chain or the CNO cycle which dominated the energy production in the star while it was on the main sequence. Assume $X_{\mathrm{H}} = 0.5$ and $X_{\mathrm{CNO}} = 0.01$.

4. Now use the expressions for nuclear energy production to find at which temperature T the energy production rate of the triple-alpha process equals the energy production the star had on the main sequence (using the numbers in the previous question). To calculate the energy production rate from the triple-alpha process you need to find a reasonable number for X_{He} in the core at the onset of helium burning. Give some arguments for how you find this number. You also need a density ρ, but since the energy production rate is so much more sensitive to the temperature than to the density you can make the crude approximation that the core density is the same as it was on the main sequence. Use the temperature you find here as the criterion for the onset of helium burning (and therefore the criterion for when the star moves to the horizontal branch in the HR-diagram).

5. Use the equations and numbers we have derived in this exercise to find the radius R_C of the core at the moment when the energy production from helium fusion starts (has become significant). Express the result in terms of solar radii $R_\odot$. At the main sequence, the core radius was $R_C = 0.2R_\odot$. You have now found how much the core needs to contract in order to start helium fusion and therefore to move down to the horizontal branch.

6. When you calculated the temperature for the onset of helium burning you made a very rough approximation: You used the core density which the star had on the main sequence, whereas you should really use the much higher density in the core when the core temperature is high enough for helium burning. Now you have estimated the size of the core radius when helium burning starts. Use this to obtain the correct density when helium burning starts and go back to find a more correct temperature for the onset of helium burning. Was the error in your first crude estimate of the helium burning temperature large relative to the temperature?

AST1100 Lecture Notes

21: Quantum gases

In this lecture we will discuss a quantum phenomenon known as degeneration. A gas which is degenerate has very special properties. The helium flash was caused by a degenerate stellar core. We will now see that the final result of stellar evolution is also a star which consists of degenerate gas: a white dwarf or a neutron star. What gives the degenerate gas its peculiar properties is its *equation of state*. The equation of state is an equation relating the pressure of the gas to its density and temperature. We have already encountered two such equations of state, $P = \rho kT/(\mu m_H)$ for an ideal gas and $P = (1/3)aT^4$ for a photon gas (we will now use capital P to denote pressure in order to distinguish it from the momentum p). We will therefore start by studying how we can obtain an expression for the pressure of a gas.

1 Pressure

To calculate the pressure of a gas we need to consider the force that particles in a gas exert on a wall (real or imaginary). Pressure is defined as the force per area on the wall $P = F/A$ from the particles in the gas. In figure 1 we see particles within a cylinder of length Δx. The particles collide with the wall at the right end of the cylinder. The surface area of the part of the wall limiting the cylinder on the right end is A. We will assume that the collisions are elastic. In elastic collisions the total absolute value of the momentum of the particle is conserved.If we define the x-direction as the direction towards the wall and the y-direction as the direction along the wall (see again figure 1), then momentum p_y in y-direction is always conserved since there is no force working in that direction. For the absolute value of the momentum to be conserved, the absolute value of the momentum p_x in the x-direction must be conserved. This means that the momentum in the x-direction after the collision must the $-p_x$ where p_x is the momentum before the collision. This means simply that the incoming angle equals the outgoing angle (see figure 2).

Before the collision, a particle has momentum (p_x, p_y). After the collision the particle has momentum $(-p_x, p_y)$. The total change in momentum is just $2p_x$. The force exerted on the wall during a time interval Δt is

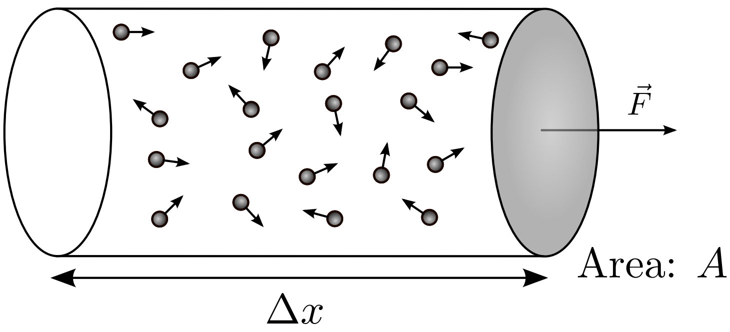

Figure 1: Pressure on the wall of area A is the total force exerted from the particles within the cylinder divided by the area.

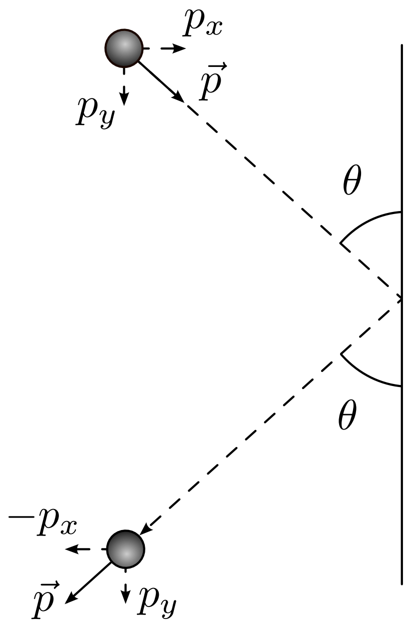

Figure 2: An elastic collision: the absolute value of the momentum is conserved.

according to Newton's second law

$$f = \frac{dp}{dt} \approx \frac{\Delta p}{\Delta t} = \frac{2p_x}{\Delta t}. \tag{1}$$

We will now consider a time interval Δt such that all the particles within the cylinder with velocity $|v_x|$ and momentum $|p_x|$ has collided with the wall within time Δt. The time it takes a particle with x-velocity v_x at a distance Δx from the wall to hit the wall is $\Delta t' = \Delta x / v_x$. Within the time $\Delta t'$, all the particles in the cylinder traveling in the direction towards the wall (with velocity v_x) have collided with the wall. But only half of the particles travel in the direction towards the wall. The other half travels in the opposite direction and had therefore already hit the wall within a time interval $\Delta t'$ earlier. So within the time

$$\Delta t = 2\Delta t' = \frac{2\Delta x}{v_x}$$

every single particle in the cylinder with velocity $|v_x|$ has collided with the wall. Inserting this in equation 1 we find that the force exerted by any single particle of velocity $|v_x|$ and momentum $|p_x|$ in the cylinder within the time Δt is

$$f = \frac{2p_x}{\Delta t} = \frac{v_x p_x}{\Delta x}.$$

The total velocity is given by $v^2 = v_x^2 + v_y^2 + v_z^2$, but on average the velocity components are equally distributed among all three dimensions such that the mean values $\langle v_x^2 \rangle = \langle v_y^2 \rangle = \langle v_z^2 \rangle$ giving that $v^2 = 3v_x^2$ or $v_x = v/\sqrt{3}$ (note: this will not be true for one single particle, but will be true when taking the average over many particles)on average. Exactly the same argument holds for the momentum giving $p_x = p/\sqrt{3}$. We thus have

$$p_x v_x = \frac{1}{3} pv.$$

such that

$$f = \frac{vp}{3\Delta x}.$$

We have a distribution function $n(p)$ which gives us the number density of particles in the gas with momentum p such that $n(p)dp$ is the number of particles with momentum between p and $p + dp$. We have already seen at least one example of such a distribution function: The Maxwell-Boltzmann distribution function for an ideal gas. We used this for instance to find the width of a spectral line as well as to find the number of particles with a certain energy in a stellar core when calculating nuclear reaction rates. We will here assume a general distribution function $n(p)$. Since $n(p)$ is a number density, i.e. number per unit volume, we can write the total particles in the cylinder $N(p)$ with momentum p as the density times the volume $A\Delta x$ of the cylinder

$$N(p) = n(p) A \Delta x.$$

The total force exerted on the wall of the cylinder by particles of momentum p is then

$$dF = \frac{pv}{3\Delta x} N(p) dp = \frac{1}{3}\, p\, v\, n(p)\, A\, dp,$$

or in terms of the pressure exerted by these particles

$$dP = \frac{dF}{A} = \frac{1}{3}\, p\, v\, n(p)\, dp. \tag{2}$$

We obtain the total pressure by integrating

$$P = \frac{1}{3} \int_0^\infty p\, v\, n(p)\, dp \tag{3}$$

which is the *pressure integral*. Given the distribution function $n(p)$ (for instance the Maxwell-Boltzmann distribution) and an expression relating v and momentum p (for instance $v = p/m$ for non-relativistic particles) we can integrate this equation to obtain the pressure in the gas.

2 Distribution functions

A statistical distribution function $n(p)$ describes how the momenta are distributed between the particles in a gas. It tells us the number density of particles having a specific momentum p. The density of particles with momentum between p and $p + dp$ is given by $n(p)dp$. By making substitutions (for instance $p = mv$), we can obtain the velocity distribution function $n(v)$ which we used to obtain the width of a spectral line in the lectures on electromagnetic radiation. Or by making the substitution $E = p^2/(2m)$ we can obtain the distribution function $n(E)$ giving the number density of particles having a certain energy E. We used the latter in the lecture on nuclear reactions.

The Maxwell-Boltzmann distribution function for an ideal gas is

$$n(\vec{p}) = n \left(\frac{1}{2\pi mkT}\right)^{3/2} e^{-p^2/(2mkT)},$$

where n is the total number of particles per volume. This is the density $n(\vec{p})$ of particles with momentum $\vec{p}$. Above we needed the number density $n(p)$ of particles with an absolute value of the momentum p. Thus, we need to integrate over all possible angles of the vector $\vec{p}$. We can imagine that we have a momentum space, i.e. a three dimensional space with axes p_x, p_y and p_z (see figure 3). All possible momentum vectors $\vec{p}$ are vectors pointing to a coordinate (p_x, p_y, p_z) in this momentum space. All particles which have an absolute value p of their momentum $\vec{p}$ are located on a spherical shell at distance p from the origin in this momentum space. Thus we may imagine a particle to have a position in the six dimensional

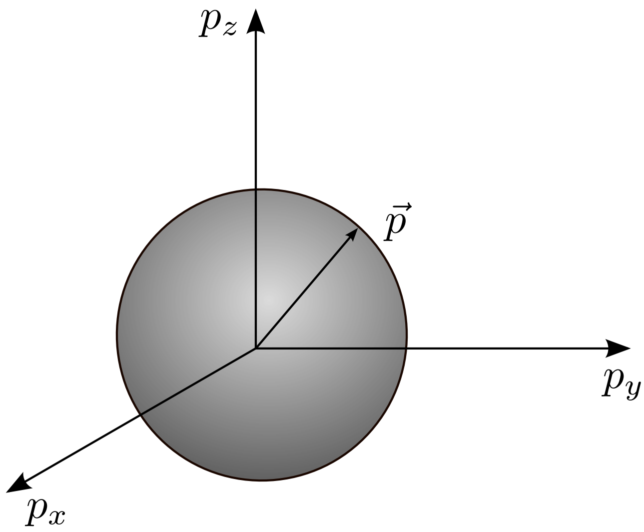

Figure 3: Momentum space: All particles with $|p|$ within $[p, p + \Delta p]$ are located on the thin shell of thickness Δp at radius $|p|$.

position-momentum space $(\vec{x}, \vec{p})$. All particles have a position in real space (x, y, z) and a position in momentum space (p_x, p_y, p_z). All particles with momentum between p and $p+dp$ are located on a thin shell of thickness dp at a distance p from the origin. The total volume of this shell is $4\pi p^2 dp$. Thus, to obtain the total number of particles within this momentum range, we need to multiply the distribution with the momentum space volume $4\pi p^2 dp$,

$$n(p)dp = n\left(\frac{1}{2\pi mkT}\right)^{3/2} e^{-p^2/(2mkT)} 4\pi p^2 dp.$$

This is the distribution function for absolute momenta p that we already know. Note that whereas $n(\vec{p})$ has dimensions number density per real volume per volume in momentum space, $n(p)dp$ has dimensions number density per real volume. The latter follows from the fact that we have simply multiplied $n(\vec{p})$ with a volume in momentum space $(4\pi p^2 dp)$ to obtain $n(p)dp$.

We have also, without knowing it, encountered another distribution function in this course. The Planck distribution. The Planck distribution is the number density of photons within a given frequency range

$$B(\nu) = \frac{2h\nu^3}{c^2}\frac{1}{e^{h\nu/(kT)} - 1}.$$

When you have taken courses in quantum mechanics and thermodynamics you will deduce two more general distribution functions. When taking quantum mechanical effects into account it can be shown that the distribution function for fermions (fermions were particle with half integer quantum spin like the electron, proton or neutron) and bosons (bosons were particles with integer quantum spin like the photon) can be written generally as

$$n(E) = \frac{g(E)}{e^{(E-\mu_C)/(kT)} \pm 1},$$

where μ_C is the *chemical potential* and $g(E)$ is the *density of states* which we will come back to later. Here the minus sign is for bosons and the plus sign for fermions. In the limit of low densities it turns out (we will not show it here) that the exponential part dominates and the distribution function becomes equal for fermions and bosons. In this case the chemical potential has such a form that we get back the Maxwell-Boltzmann distribution function (compare with the above expression). Note that the expression for bosons resembles the Planck function: the Planck function can be derived from the distribution function for bosons (you will do this in later courses).

3 Degenerate gases

In the core of stars, the fermions, i.e. the electrons, is the dominating species. Therefore we will here study the distribution function for fermions and use the + sign in the above equation. We will look at an approximation of the distribution function for an electron gas at low temperature. Of course, the temperature in the core of a star is not particularly low, but we will later show that the same approximation and results are valid even for high temperatures provided we are in the high density limit. In the low temperature limit it can be shown that the chemical potential μ_C equals the so-called *Fermi energy* E_F. We will later find an expression and physical interpretation for the Fermi energy, but first we will consider the distribution function for fermions (in our case, electrons) given by

$$n(E) = \frac{g(E)}{e^{(E-E_F)/(kT)} + 1} \tag{4}$$

where

$$g(E) = 4\pi \left(\frac{2m_e}{h^2}\right)^{3/2} E^{1/2}, \tag{5}$$

where m_e is the electron mass. The number of electrons per volume with an energy between E and $E+dE$ in a gas with temperature T is now given by $n(E)dE$. The energy E of the electron may be larger or smaller than the Fermi energy E_F. We will now measure the energy of the electron in units of the Fermi energy. We define $x = E/E_F$ such that $x < 1$ when the energy is less than the Fermi energy and $x > 1$ when the energy is larger than the Fermi energy. The distribution function as a function of x, the energy in units of the Fermi energy, can thus be written

$$n(x) = \frac{g(x)}{e^{(x-1)E_F/(kT)} + 1}.$$

In the low temperature limit, $T \to 0$, the factor $E_F/(kT)$ is a very large quantity. The energy x defines whether the number in the exponential is a large positive or a large negative quantity. If $x > 1$, i.e. that the energy is larger than the Fermi energy, then the number in the exponential is a large positive number and $n(x) \to 0$. For $x < 1$, i.e. the energy is less than the Fermi energy, the number in the exponential is a large negative number. Thus the exponential goes to zero and $n(x) \to g(x)$. So for very low temperatures, there is a sharp limit at $x = 1$. For $E < E_F$ we find $n(x) = g(x)$ whereas for $E > E_F$ we find $n(x) = 0$. In figure 4 we show $n(x)/g(x)$ for lower and lower temperatures.

The physical meaning of this is that for very low temperatures, all the electrons have energies up to the Fermi energy whereas no electrons have energies larger than the Fermi energy. The Fermi energy is a low temperature energy limit for the electrons. Even if we cool an electron gas down to zero temperature, there will still be electrons having energies all the way up to the Fermi energy. But if the temperature is zero, why don't all electrons have an energy close to zero? Why don't all electrons go and occupy the lowest possible energy state allowed by quantum mechanics (in quantum mechanics, a particle cannot have zero energy)? The reason for this is hidden in quantum physics: at low temperatures the gas of electrons start to behave like a quantum gas, a gas where quantum mechanical effects are important. The quantum mechanical effect which we see on play here is the Pauli exclusion principle: Two fermions cannot occupy the same energy state. To understand this principle we need to dig even deeper into the quantum theory. According to quantum mechanics momentum is quantized. This means that a particle cannot have an arbitrary momentum. The momentum in any direction can be written as

$$p_x = N_x \cdot p_0,$$

where N_x is an integer quantum number and p_0 is the lowest possible momentum. Thus, an electron can only have x-momenta p_0, $2p_0$, $3p_0$ etc. No values in between are allowed. So the total momentum of an electron (or any particle) can be written

$$p^2 = p_x^2 + p_y^2 + p_z^2 = p_0^2(N_x^2 + N_y^2 + N_z^2) \equiv p_0^2 N^2,$$

where (N_x, N_y, N_z) are the three quantum numbers defining the state of the electron. According to the Pauli exclusion principle only one electron can occupy the quantum state (N_x, N_y, N_z). No other electrons can have exactly the same combination of quantum numbers. We go back to the above image of a momentum space where a particle has a position (p_x, p_y, p_z) in a three dimensional momentum space in addition to a position in normal space. We can now write this position in terms of quantum numbers as $(p_x, p_y, p_z) = p_0(N_x, N_y, N_z)$. Since only one electron can have a given momentum $p_0(N_x, N_y, N_z)$, one could imagine the momentum space filled with boxes of volume $p_0 \times p_0 \times p_0$. Only one electron fits into each box. We remember that all electrons with momentum

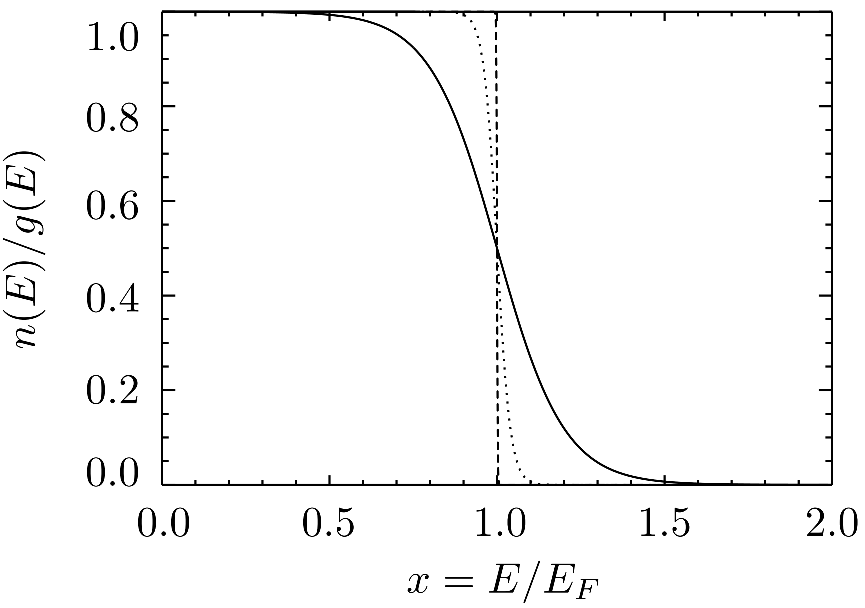

Figure 4: The number of electrons $n(E)$ divided by $g(E)$ for different energies E. The solid line is for a gas at temperature $T = 10$ K, the dotted line for a gas at temperature $T = 2$ K and the dashed line for $T = 0.1$ K. When the temperature approaches zero, there are less and less electrons with energy larger than the Fermi energy E_F.

lower than a given momentum p is within a sphere with radius p in this momentum space. All electrons with a higher momentum p are outside of this sphere. But inside the sphere of radius p, there is only room for $4/3\pi N^3$ boxes of size p_0^3 (total volume of the momentum space sphere $(4/3)\pi p^3 = (4/3)\pi p_0^3 N^3$ divided by volume of box p_0^3). If all these boxes are filled, no more electron may settle on a position inside this sphere, it has to remain outside of the sphere. When you lower the temperature of an electron gas, the electrons loose momentum and start to occupy the lowest possible momentum states, i.e. they all start to move towards the origin $(0, 0, 0)$ in momentum space. But when all start to move towards the origin in momentum space (see again figure 3), all the boxes around the origin are soon occupied, so the electrons need to remain with higher momenta at larger distances p from the origin. But if they need to remain with larger momenta, this means that they also have larger energy: The same argument therefore applies to energy. The energy states of the electrons are quantized so not all electrons may occupy the lowest energy state. For this reason we see that the distribution function for electrons at low temperatures is a step function: All electrons try to occupy the lowest possible energy state. The lowest energy states are filled up to the Fermi energy. If we call p_F the Fermi momentum, the momentum corresponding to the Fermi energy we can imagine that all electrons start to gather around the origin in momentum space out to the radius p_F. All electrons are packed together inside a sphere of radius p_F in momentum space. When you add more electrons to the gas, i.e. the density of electrons increases, the sphere in momentum space inside which all the electrons are packed also needs to expand and the Fermi momentum p_F increases. Thus the Fermi momentum and the Fermi energy are functions of the electron density n_e.

Having learned that for very low temperatures, the electrons are packed together in momentum space in a sphere of radius p_F we can find the total number density (per real space volume) n_e of electrons in the gas by summing up all the boxes of size p_0^3 inside this sphere. We know that all these boxes are occupied by one electron and that no electrons are outside this sphere (this is completely true only for $T = 0$). First we need to know the fermion distribution function $n(\vec{p})$ in terms of momentum rather than in terms of energy which we used above. The fermion distribution function for momentum can be written in the low temperature limit as

$$n(\vec{p}) = \frac{1}{e^{(p^2 - p_F^2)/(2mkT)} + 1} \frac{2}{h^3}.$$

This is the number density per volume in real space per volume in momentum space. Considering again the low temperature case, we see, using the same arguments as before, that $n(\vec{p}) \to 0$ for $p > p_F$ and $n(\vec{p}) \to 2/h^3$ for $p < p_F$. Thus $n(\vec{p})$ is a constant for $p < p_F$ and zero for $p > p_F$. In order to obtain the number density of electrons per real space volume we need to integrate this expression over the momentum space volume. So for $T \to 0$

$$n_e = \int_0^\infty n(\vec{p}) 4\pi p^2 dp = \int_0^{p_F} \frac{2}{h^3} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3$$

where we integrate over the sphere in momentum space in shells of thickness dp out to the Fermi momentum p_F where $n(\vec{p})$ is a constant $(2/h^3)$ for $p < p_F$ and is zero for $p > p_F$. Make sure you understood this derivation! We use this result to obtain an expression for the Fermi momentum

$$p_F = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}. \tag{6}$$

Using the non-relativistic expression for energy we can now find the Fermi energy expressed in terms of the electron number density n_e

$$E_F = \frac{p_F^2}{2m_e} = \frac{h^2}{8m_e} \left(\frac{3n_e}{\pi} \right)^{2/3}. \tag{7}$$

As we anticipated, the Fermi energy depends on the density of electrons. The higher the density, the larger the Fermi energy and the Fermi momentum in order to have space for all the electrons within the sphere of radius p_F. A gas where all particles are packed within this sphere so that the particles are fighting for a box in momentum space among the lowest energy states is called a *degenerate gas*. A partially degenerate gas is a gas where there are still a few vacant boxes among the lowest energy states such that some particles have energies larger than the Fermi energy. We now need to find a criterion for when a gas is degenerate.

When the temperature of a gas is high and the density low, the distribution function is the Maxwell-Boltzmann distribution function. We have previously learned that for a gas following the Maxwell-Boltzmann distribution function, the mean energy per particle is $< E >= (3/2)kT$. The gas starts to become degenerate when most of the particles have energies below the Fermi energy. The gas therefore starts to be degenerate when the mean energy of the particles go below the Fermi energy. For an electron gas we thus have the criterion

$$\frac{3}{2}kT < E_F = \frac{h^2}{8m_e}\left(\frac{3n_e}{\pi}\right)^{2/3},$$

or

$$\frac{T}{n_e^{2/3}} < \frac{h^2}{12m_e k}\left(\frac{3}{\pi}\right)^{2/3}. \tag{8}$$

As discussed above, this criterion is satisfied for very low temperatures, but we now see that it is also satisfied for very high densities. In the exercises you will estimate what kind of densities are needed in the stellar cores for the core to be degenerate.

Now take a deep breath, close your eyes and try to find out how much you have understood from this section. Then if this is not the 3rd time you read it, go back and read again with the goal of understanding a little bit more this time.

4 The pressure of a degenerate electron gas

When the density of electrons in the stellar core becomes high enough, most electrons have energies below the Fermi energy and the above criterion for degeneracy is satisfied. The core is electron degenerate. Now we will study the properties of a degenerate gas. The equation of state, the equation for the pressure as a function of density and temperature, is one of the most important properties describing how a gas behaves.

In order to find the pressure, we need to evaluate the pressure integral (equation 3) for the degenerate gas. First we need the density $n(p)dp$ of electrons per volume with momentum p in the interval $[p, p+dp]$. By now we have learned that $n(\vec{p})4\pi p^2 dp = n(p)dp$ such that for $p < p_F$ we have $n(p)dp = (2/h^3)4\pi p^2 dp$ and for $p > p_F$ we have $n(p) = 0$.

$$P = \frac{1}{3}\int_0^\infty p\,v\,n(p)\,dp = \frac{1}{3}\int_0^{p_F} \frac{p^2}{m_e}\frac{2}{h^3}4\pi p^2 dp = \frac{8\pi}{3m_e h^3}\frac{1}{5}p_F^5.$$

Inserting the expression for the Fermi momentum (equation 6), we find

$$P = \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20m_e} n_e^{5/3} \tag{9}$$

We see that the pressure of a degenerate gas does not depend on the temperature. If the temperature increases or decreases, the pressure does not change! This is very different from a normal gas. It means that the degenerate stellar core will not expand or contract as the temperature changes. The only exception being when the temperature increases so much that the condition (8) for degeneracy is no longer valid and the degeneracy is broken. In this case, the electrons have gained so much energy that they are not packed in the sphere of the lowest momentum states in momentum space. The gas is no longer degenerate and a normal equation of state which depends on the temperature needs to be used.

We have deduced the pressure of a degenerate gas using the non-relativistic expressions for energy. The temperature in the stellar cores are often so high that the velocities of the particles are relativistic. Repeating the above deductions using the relativistic expression, we would obtain

$$P = \frac{hc}{8} \left(\frac{3}{\pi}\right)^{1/3} n_e^{4/3}. \tag{10}$$

5 Summary

We have seen that if we compress a gas of fermions sufficiently, so that the degneracy condition (equation 8) is fulfilled, the fermions are packed together inside a sphere of radius p_F in momentum space. All the lowest energy states of the fermions are occupied up to the Fermi energy E_F. This typically happens when the temperature is very low so that the fermions fall down to the lowest possible energy states in momentum space. It might also happen for high temperatures if the density is high enough: In this case there are so many fermions present within a volume so all fermion states up to E_F are occupied even if the temperature is not particularly low.

A degenerate fermion gas has a degeneracy pressure which is independent of the temperature of the gas given by equation (9) for a non-relativistic gas (the particles have non-relativistic velocities) and by equation (10) for a relativistic gas. This pressure originates from the resistance against being squeezed further together in real and momentum space and only depends on the density of the gas. We obtained the expression for the pressure by inserting the distribution function for a degenerate gas in the pressure integral (equation 3). The distribution function for a degenerate gas took on a particular form: It is a step function being constant for energies below the Fermi energy and zero above. This was simply a consequence of the Pauli exclusion principle, one energy state cannot be occupied by two fermions at the same time. When the quantum states of lowest energy are occupied, the fermions need to occupy states of higher energy. For a completely degenerate gas, the Fermi energy E_F gives within which energy there is room for all fermions at a given density.

If the temperature increases sufficiently, the fermions gain enough energy to occupy states well outside the sphere of radius p_F in momentum space. Then there will be vacant low energy states, the condition of degeneracy is no longer fulfilled and the gas has become non-degenerate following a normal temperature-dependent equation of state.

6 Problems

Problem 1 (2–3 hours)

In the text we used the pressure integral to find the pressure of a degenerate electron gas. Study the derivation carefully and make sure you understand every step before embarking on this exercise.

1. Now we want to find the pressure in a "normal" ideal gas which follows the Maxwell-Boltzmann distribution function. Find the expression for the Maxwell-Boltzmann distribution function and use this in the pressure integral. Assume non-relativistic velocities. Remember also that the distribution function $n(p)$ used in the pressure integral needs to be normalized such that

$$\int_0^\infty n(p)dp = n,$$

   where n is the number density of particles per real space volume. You now have all the information you need to find the pressure of a gas following Maxwell-Boltzmann statistics so your task is simply: find P as a function of n and T. These integrals might be useful:

$$\int_0^\infty x^{3/2}e^{-x}dx = \frac{3\sqrt{\pi}}{4}$$
$$\int_0^\infty x^{1/2}e^{-x}dx = \frac{\sqrt{\pi}}{2}$$

   The answer you find for P should be familiar to you.

2. Now we will test the expression by making a computer simulation. We will now simulate a box of size 10 cm × 10 cm × 10 cm.

   (a) We will fill the box with hydrogen atoms with a temperature of $T = 6000$ K and compute the pressure that they exert on the walls of the box. We will put 10 millions particles in the box. You first need to make an array with the (x, y, z) position of each of the particles and another array with the (v_x, v_y, v_z) velocities of each of the particles. In order to find the velocities of the atoms, use the same algorithm that you used in problem 1 in the chapter on nuclear reactions to draw random velocities from a Maxwell-Boltzmann distribution. The position of the particles in the box is also random, but random with a uniform distribution which means that there is an equal probability of finding an atom at any position in the box. Use the function `random.uniform` in each direction to find a random position of the particles. Then choose one of the walls of the box and find out which particles will hit this wall within a time $\Delta t = 10^{-9}$ s in the future, as well as how many particles hit the wall a time Δt in the past. For each of these particles calculate the force they exert on the wall and sum up these forces to calculate the pressure.

(b) Now increase the temperaure of the gas, first to $T = 50000$ K, $T = 15 \times 10^6$ K and finally $T = 10^9$ K. For each of these temperaures, calculate the pressure on a wall of the box.

(c) Use the analytical expression you obtained for the pressure as a function of density and temperaure (you need to calculate the density of the gas in your box) and make a plot: Plot this analytical function P as a function of temperaure T from $T = 6000$ K to $T = 10^9$ K in a log-log plot. In the same plot, plot 4 points: the 4 values for the pressure that you obtained in the simulation above. Does the analyical expression match the simulated values well? If there are any discrepancies can you imagine why? How could you improve the precision?

Problem 2 (1–2 hours)

In the text we found the condition for a gas to be degenerate in terms of the temperature T of the gas and the number density n_e of electrons (number of electrons per volume). We will now try to rewrite this expression into a condition on the mass density ρ of the gas.

1. Assume that the gas is neutral, i.e. that there is an equal number of protons and electrons. Show that this gives

$$n_e = \frac{Z\rho}{Am_H},$$

where Z is the average number of protons per nucleus, A is the average number of nucleons per nucleus, m_H is the hydrogen mass and ρ is the total mass density.

2. Find the expression for the condition for degeneracy in terms of the total mass density ρ instead of n_e.

3. Find the minimum density a gas with temperature $T = 10^9$ K must have in order to be degenerate. A typical atom in the gas has the same number of protons and neutrons.

4. If you compress the whole Sun into a sphere with radius R and uniform density until it becomes degenerate, what would be the radius R of the degenerate compressed Sun (assume the temperature $T = 10^9$ K for the final stages of the Sun's life time)? This is basically what will happen at the end of the Sun's life time. Gravitation will compress it until it becomes a degenerate white dwarf star. A white dwarf star typically has a radius similar to the radius of the Earth. Does this fit well with your result?

5. What about Earth? To which radius would you need to press the Earth in order for it to become degenerate (assume again that the temperature will reach $T = 10^9$ K when compressing the Earth)?

Problem 3 (1–2 hours)

The number density per real space volume per momentum space volume of particles with momentum $\vec{p}$ is given by $n(\vec{p})$ found in the text. In order to find the number density per real space volume of particles with absolute momentum p we multiplied $n(\vec{p})$ with an infinitely small volume element $4\pi p^2 dp$ and obtained $n(p)dp$. Go back to the text and make sure that you understand this transition.

1. Now we will try to find the number density per real space volume of particles with energy E using the non-relativistic formula for energy $E = p^2/2m$. Start with $n(p)dp$, make the substitution and show that you arrive at equation 4 with $g(E)$ looking like equation in 5.

2. In the exercises in the previous lecture, we found that the mean kinetic energy of a particle in an ideal gas is $(3/2)kT$. Now we will try to find the mean kinetic energy in a degenerate gas. First of all, repeat what you did in the exercise in the previous lecture. Now you will repeat the same procedure, but use $n(E)$ and E directly,

$$< E >= \int_0^\infty P(E)EdE.$$

You will need to find out how $P(E)$ looks like. The answer is

$$< E >= \frac{3}{5}E_F.$$

Hint: Assume a very degenerate gas at very high density.

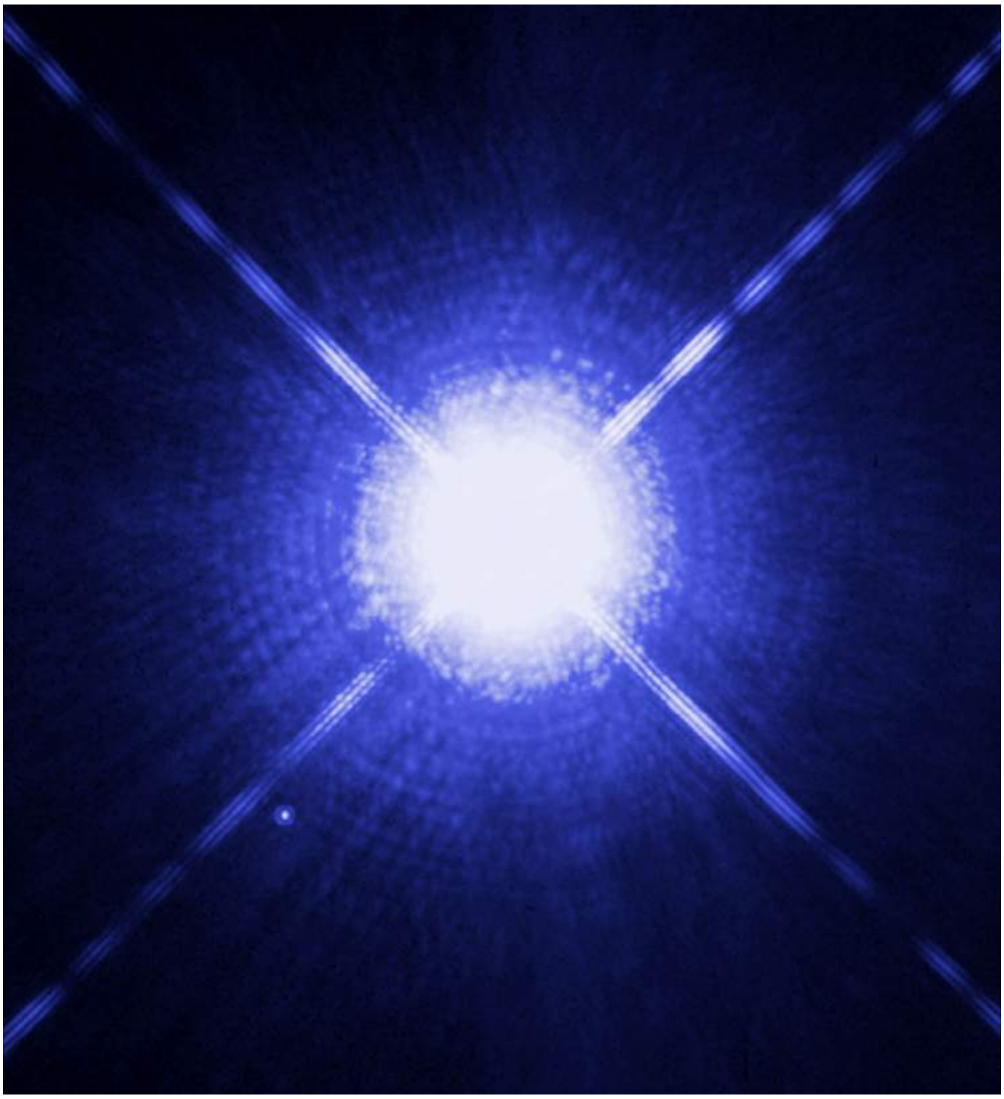

Figure 5: Info-figure for problem 2: This image shows Sirius A, the brightest star in our nighttime sky, along with its faint, tiny stellar companion, the white dwarf Sirius B, which is the tiny dot at lower left. Using the Hubble telescope's STIS spectrograph astronomers have been able to isolate the light from the white dwarf and disperse it into a spectrum. STIS measured that the light from Sirius B was stretched to longer, redder wavelengths due to the white dwarf's powerful gravitational pull. Based on those measurements, astronomers have calculated Sirius B's mass at 98 percent that of our Sun (its diameter is only 12 000 km). Analysis of the spectrum also yielded an estimate for its surface temperature: about 25 000 K. (Figure: NASA & ESA)

AST1100 Lecture Notes

22: The end state of stars

We will continue the discussion on stellar evolution from lecture 20. The star has reached the asymptotic giant branch having a radius of up to 1000 times the original radius. The core consists of carbon and oxygen but the temperature is not high enough for these elements to fuse to heavier elements. Helium fuses to carbon and oxygen in a shell around the core. Hydrogen fuses to helium in another shell further out. In the outer parts of the star, the temperature is too low for fusion reactions to take place. In the low and medium mass stars, convection has been transporting heavy elements from the core to the surface of the star allowing observers to study the composition of the core and test stellar evolutionary theories by studying the composition of elements on the stellar surface. The core is still contracting trying to reach a new hydrostatic equilibrium. The further evolution is now strongly dependent on the mass of the star.

1 Low mass stars

We will soon find out how we define low mass stars, but for the moment we will only say that a typical low mass star is our Sun. The core of the star, consisting mainly of carbon and oxygen contracts until the density of electrons is so high that the core becomes electron degenerate. In the more massive 'low mass stars' nuclear fusion may to some extent burn these elements to heavier elements like neon and magnesium. But eventually the core temperature is not high enough for further nuclear reactions and the core remains electron degenerate.

As the star contracts, the temperature in the outer parts of the star increases and the hydrogen burning again becomes more efficient than the helium burning in lower shells. The helium produced in the upper shells 'falls' down on lower shells where no helium burning takes place (helium burning takes place even further down in the hotter areas). After a while, the density in the lower helium rich shell becomes very high and partially degenerate. At a point, the temperature in the lower shell is high enough for an explosive ignition of helium and a helium shell flash occurs, similar to the helium core flash described in lecture 20, but less energetic. The flash lifts the hydrogen burning shell to larger distances from the center.

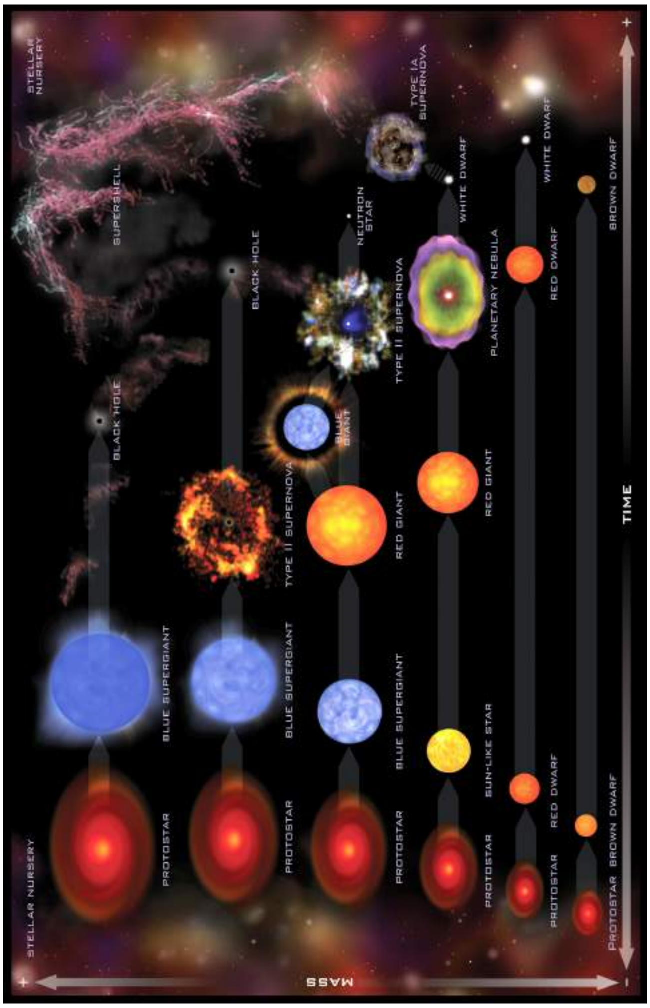

Figure 1: Info-figure: Many factors influence the rate of stellar evolution, the evolutionary path, and the nature of the final remnant. By far the most important factor is the initial mass of the star. This diagram illustrates in a general way how stars of different masses evolve and whether the final remnant will be a white dwarf, neutron star, or black hole. (Note that for the highest mass stars it is also possible for the supernova explosion to obliterate the star rather than producing a black hole. This alternative is not illustrated here.) Stellar evolution gets even more complicated when the star has a nearby companion. For example, excessive mass transfer from a companion star to a white dwarf may cause the white dwarf to explode as a Type Ia supernova. (Figure: NASA/CXC/M.Weiss)

Hydrogen burning ceases, the star contracts until the temperature again is high enough for hydrogen burning. The whole process repeats, the produced helium falls on to lower layers which finally start burning helium in another helium shell flash. The star is very unstable and the repeated helium flashes result in huge mass losses from the star. The outer layers of the star are blown away in the helium flashes (this is one of the theories describing the huge mass losses the star undergoes in this period). A huge cloud of gas and dust is remaining outside the core of the star. After a few millions years, all the outer layers of the star have been blown off and only the degenerate carbon/oxygen core remains. This star which now consist only of the remaining degenerate core is called a *white dwarf.* The surrounding cloud of gas which has been blown off is called a *planetary nebula* (these have nothing to do with planets)

As the star was blowing away the outer layers, the hotter inner parts of the star made up the surface. Thus, the surface temperature of the star was increasing, and the star was moving horizontally to the left from the asymptotic giant branch in the HR-diagram. Finally when the layers producing energy by nuclear fusion are blown off, the luminosity of the star decreases dramatically and the star ends up on the bottom of the HR-diagram as a white dwarf (see HR-diagram in figure 2). The degenerate white dwarf does not have any sources of energy production and will gradually cool off as the heat is lost into space. The white dwarf will move to the right in the HR-diagram becoming cooler and dimmer.

How large is a white dwarf star? We can use the equation of hydrostatic equilibrium to get an estimate of the radius R assuming uniform density of the white dwarf (look back at lecture 20 where we did a similar approximation in the equation of hydrostatic equilibrium):

$$\frac{P}{R} \approx \frac{GM}{R^2}\frac{M}{(4/3)\pi R^3} = \frac{3GM^2}{4\pi R^5} \qquad (1)$$

The pressure P is now the degeneration pressure. Inserting the expression for the degeneration pressure from the previous lecture in this equation gives

$$\left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20m_e}n_e^{5/3} = \frac{3GM^2}{4\pi R^4}. \qquad (2)$$

The electron number density n_e can be written in terms of the total gas density as

$$n_e = n_p = \frac{\rho_p}{m_H} = \frac{Z}{A}\frac{\rho}{m_H}.$$

Here Z is the number of protons per nucleus and A is the number of nucleons per nucleus. As the gas in total is neutral, the number of electrons in the gas equals the number n_p of protons. The number density, i.e., total number of protons per volume in the gas equals the mass density ρ_p of protons divided by the hydrogen mass (basically equal to the proton mass). The mass density of protons in a gas equals the mass density of the gas times the fraction of the mass in protons given by Z/A. A nucleus

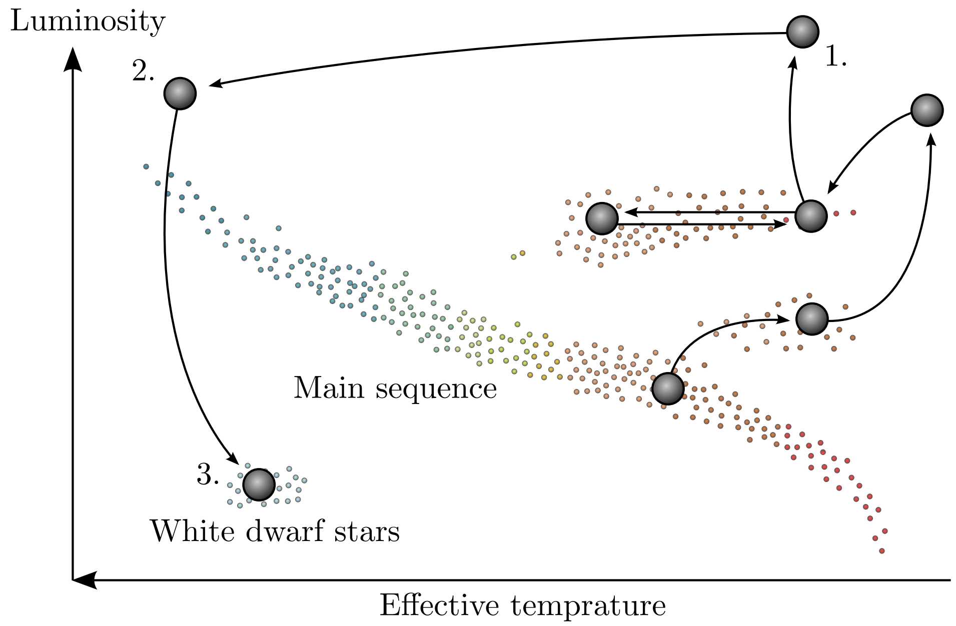

Figure 2: Motion in the HR-diagram for the last stages of stellar evolution. This is the path that a low mass $M < 8M_{\odot}$ star follows. From the asymptotic giant branch, the outer layers are blown off and the hotter inner parts become the new and hotter 'surface'. The effective temperature increases and the star moves from 1 to 2. As the layers were nuclear fusion takes place are blown off, the total luminosity decreases as the star is no more capable of producing energy. The star therefore falls down from 2 to 3 to the white dwarf area in the HR-diagram.

Figure 3: Info-figure: At around 215 pc away, the famous Helix Nebula in the constellation of Aquarius is one of the nearest planetary nebulae to Earth. The red ring measures roughly 1 pc across and spans about one-half the diameter of the full Moon. A forest of thousands of comet-like filaments, embedded along the inner rim of the nebula, points back toward the central star, which is a small, super-hot white dwarf. In general, a planetary nebula is an emission nebula consisting of an expanding glowing shell of ionized gas ejected during the asymptotic giant branch phase of certain types of stars late in their life. They have nothing to do with planet formation, but got their name because they look like planetary disks when viewed through a small telescope. Planetary nebulae have extremely complex and varied shapes, as revealed by modern telescopes. (Figure: NASA/ESA)

typically contains the same number of neutrons as protons such that the total number of nucleons is twice the number of protons and $Z/A = 0.5$. We will use this number in the calculations. Inserting this expression for n_e in the equation of hydrostatic equilibrium (equation 2) we have

$$\left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20m_e} \left(\frac{Z}{Am_H}\right)^{5/3} \frac{M^{5/3}}{(4/3\pi)^{5/3}R^5} = \frac{3GM^2}{4\pi R^4},$$

or

$$R_{\mathrm{WD}} \approx \left(\frac{3}{2\pi}\right)^{4/3} \frac{h^2}{20m_e G} \left(\frac{Z}{Am_H}\right)^{5/3} M^{-1/3}.$$

For $M = M_\odot$ (where M is the mass which remained in the degenerate core, the star originally had more mass which was blown off) the radius of the white dwarf is similar to the radius of the Earth. A white dwarf is thus extremely dense, one solar mass compressed roughly to the size of the Earth. There is another interesting relation to be extracted from this equation. Multiplying the mass on the left side we have

$$MR^3 \propto MV = \text{constant},$$

where V is the volume of the white dwarf. Thus, if the mass of a white dwarf increases, the volume decreases. A white dwarf gets smaller and smaller the more mass it gets. It shrinks by the addition of mass. This can be understood by looking at the degenerate equation of state: When more mass is added to the white dwarf, the gravitational inward forces increase. This has to be balanced by an increased pressure. From the equation of state we see that since there is no temperature dependence, the only way to increase the pressure is by increasing the density. The density is increased by shrinking the size. So a white dwarf must shrink in order to increase the density and thereby the degeneration pressure in order to sustain the gravitational forces from an increase in mass.

Can a white dwarf shrink to zero size if we just add enough mass? Clearly when the density increases, the Fermi energy increases and the energy of the most energetic electrons increases. Finally the energy of the most energetic electrons will be so high that the velocity of these electrons will be close to the speed of light. In this case, the relativistic expression for the degeneration pressure is needed. We remember that the relativistic expression for the degeneration pressure went as $P \propto \rho^{4/3}$ instead of $P \propto \rho^{5/3}$ in the non-relativistic case. Inserting the relativistic expression in the equation of hydrostatic equilibrium we thus expect to obtain a different relation between radius and mass. Inserting the relativistic expression in equation 1 instead we obtain

$$\left(\frac{3}{\pi}\right)^{1/3} \frac{hc}{8} \left(\frac{Z}{Am_H}\right)^{4/3} \frac{M^{4/3}}{(4/3\pi)^{4/3}R^4} = \frac{3GM^2}{4\pi R^4}.$$

We see that the radius cancels out of the equation and we are left with a number for the mass of the relativistic white dwarf:

$$M \approx \frac{3}{16\pi} 2^{-3/2} \left(\frac{hc}{G}\right)^{3/2} \left(\frac{Z}{Am_H}\right)^2 .$$

A more exact calculation taking into account non-uniform density would have given

$$M_{\mathrm{Ch}} \approx \frac{\sqrt{3/2}}{2\pi} \left(\frac{hc}{G}\right)^{3/2} \left(\frac{Z}{Am_H}\right)^2 \approx 1.4 M_\odot .$$

This is the *Chandrasekhar mass* M_{Ch} which gives the upper limit of the mass of a white dwarf. For a relativistic degenerate gas the pressure can only withstand the gravitational forces from a maximum mass of $M = 1.4M_\odot$. If the mass increases beyond that, the white dwarf will collapse. We will discuss this further in the next section. Having both the typical mass (or actually the upper bound on the mass) and the typical radius of a white dwarf we can find the typical density: a small needle made of white-dwarf material would weight about 50 kg.

So we are now closer to the definition of a 'low mass star'. A 'low mass star' is a star which has a mass low enough so that the remaining core after all mass losses has a mass less than the Chandrasekhar mass $1.4M_\odot$. The final result of stellar evolution for low mass stars is therefore a white dwarf. It turns out that stars which have up to about $8M_\odot$ when they reach the main sequence will have a core mass lower than the Chandrasekhar limit. We will now discuss the fate of stars with a main sequence mass larger than $8M_\odot$.

2 Intermediate and high mass stars

For stars of mass $M > 8M_\odot$, the evolution is different. The higher mass makes the pressure and thereby the temperature in the core higher than in the case of a low mass star. The forces of gravity are larger and therefore the pressure needs to by higher in order to maintain hydrostatic equilibrium. The carbon-oxygen core contracts, but before it gets degenerate the temperature is high enough for these elements to fuse to heavier elements. This sequence of processes which hydrogen and helium followed is repeated for heavier and heavier elements. When one element has been depleted, the core contracts until the temperature is high enough for the next fusion process to ignite while burning of the different elements takes place in shells around the core. This will continue until the core consists of iron $^{56}_{26}\mathrm{Fe}$. We learned in the lecture on nuclear reactions that in order to produce elements heavier than iron, energy needs to be added, no energy is released. For elements heavier than iron, the mass per nucleon increases when the number of nucleons in the core increases. This is why no energy can be released in further fusion reactions. Nuclear processes which need energy input are difficult to make happen: The quantum mechanical probability for a nucleus to tunnel through the Coloumb barrier only to loose energy in the fusion process is very small. Thus, when the stellar core consists of iron, no more nuclear processes take place and the core starts contracting again. At this point, the star might look like figure 4. There are several layers of elements which resulted from previous nuclear fusion processes around the iron core. Fusion processes are still taking place in these shells.

At this point the temperature in the core is extremely high $T \sim 10^9 - 10^{10}$ K containing a dense gas of high energy photons. The core continues contracting and no more nuclear fusion processes are available to produce a pressure to withstand the forces of gravity pushing the core to higher and higher densities. The energy of the photons is getting so high that they start splitting the iron atoms by the photo disintegration process

$$^{56}_{26}\mathrm{Fe} +^{0}_{0}\gamma \rightarrow 13^{4}_{2}\mathrm{He} + 4^{1}_{0}\mathrm{n}$$

and helium atoms are further split into single protons and neutrons

$$^{4}_{2}\mathrm{He} +^{0}_{0}\gamma \rightarrow 2^{1}_{1}\mathrm{p} + 2^{1}_{0}\mathrm{n}$$

reversing the processes which have been taking place in the core of the star for a full stellar life time. To split iron nuclei or other nuclei lighter than iron (with a few exceptions) requires energy. Again, by looking at the plot from the lecture on nuclear processes we see that nuclear fission processes only produce energy for elements heavier than iron. For lighter elements, the mass per nucleon increases when a core is split and energy is needed in the process. Thus, the photodisintegration processes take thermal energy from the core, energy which would contribute to the gas pressure preventing a rapid gravitational collapse of the core. When this energy is now taken away in the nuclear fission processes, the temperature and thereby the gas pressure goes down and the forces acting against gravity are even smaller. The core can now contract even faster. The result of the fast collapse is that the core is divided in two parts, the inner core which is contracting and the outer core being in free fall towards the rapidly contracting inner core.

The inner core becomes electron degenerate, but the degeneration pressure is not sufficient to withstand the weight from the mass around the core. But quantum physics absolutely forbids more than one electron to occupy one quantum state in momentum space, so how can the inner core continue contracting? Nature has found a solution: electron capture. Electrons and the free protons which are now available after the splitting of the nuclei combine to form neutrons

$$^{1}_{1}\mathrm{p} +^{1}_{-1}\mathrm{e} \rightarrow^{1}_{0}\mathrm{n} +^{0}_{0}\nu$$

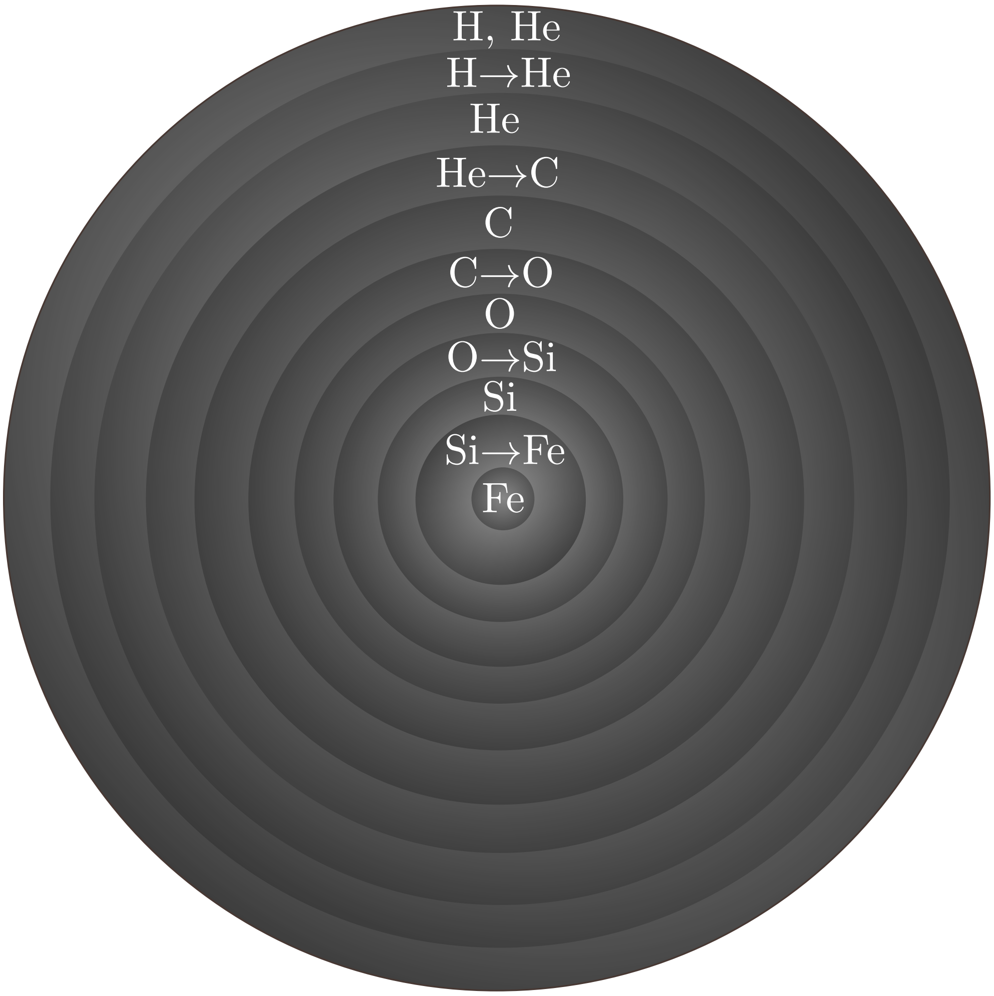

Figure 4: The structure of a star just hours before a supernova explosion. Energy production in the core has ceased as it now consists of Fe—the final product of nuclear fusion. Different elements are still burning in layers around the core.

The final result is an inner core consisting almost entirely of neutrons. The neutron core continues collapsing until it becomes degenerate. This time, it is the neutrons and not the electrons which are degenerate. The stellar core is now so dense that all the quantum states of the neutrons are occupied and the core cannot be compressed further. The neutron degeneration pressure withstands the forces of gravity. Why can the neutron degeneration pressure withstand the forces of gravity when the electron degeneration pressure could not? The answer can be found in the expression (9) in the previous lecture for the degeneration pressure. Even if the neutron mass is much larger than the electron mass which should lead to a smaller degeneration pressure, the density is now much higher than it was in the electron degenerate gas. The higher density makes the total degeneration pressure larger.

When the inner core consisting mainly of neutrons becomes degenerate, the collapse is suddenly stopped, the core bounces back and an energetic shock wave is generated. This shock wave travels outwards from the core but is blocked by the massive and dense 'iron cap', the outer core, which is in free fall towards the inner core. The energy of the shock wave heats the outer core till temperatures large enough for photon disintegration and electron capture processes to take place. Thus, almost all the energy of the shock wave is absorbed in these energy demanding processes. So the huge amounts of energy carried outwards by the shock wave could in principle have blown the whole star apart, but the wave is blocked by the outer core where the energy is absorbed in photo disintegration and electron capture processes.

The final part of the story is truly remarkable. We see that the electron capture process releases neutrinos. So parts of the energy 'absorbed' in this process is reemitted in the form of energetic neutrinos. A large amount of electron capture processes are now taking place and computer simulations show that a neutrino sphere is created, an immensely dense wall of neutrinos is traveling outwards. Normally, we do not need to take neutrinos into account when studying processes in the stellar interiors because neutrinos hardly interact with matter at all and just leaves the star directly without influencing the star in any way. Now the outer core is extremely dense increasing the reaction probability of processes where neutrinos are involved. There is a huge amount of energetic neutrinos trying to pass through the very dense outer core. The combination of extreme densities and extreme neutrino fluxes makes the 'impossible' (or actually improbable) possible: a large part of the neutrinos reacts with the matter in the outer core. About 5% of the energy in the neutrinos is heating the outer parts just enough to allow the shock wave to continue outwards. The shock wave lifts the outer parts of the star away from the core, making the star expand rapidly. In short time, most of the star has been blown away to hundreds of AU away from the remaining inner core. This is a *supernova* explosion. The luminosity of the explosion is about $10^9 L_\odot$ which is the luminosity of a normal galaxy. The total luminsity of the supernova is thus similar to the total luminosity of an entire galaxy. And the energy released in photons is just a fraction of the energy released in neutrinos. An enormous amount of energy is released over very short time scales. Where does the energy from this explosion originally come from? We will discuss this in the exercises. Now we will look at the corpse of the dead star.

3 The fate of intermediate and high mass stars

For stars with mass $M < 25M_\odot$, the neutron degeneration pressure is high enough to withstand the forces of gravity and thereby to maintain hydrostatic equilibrium. Normally only $2 - 3M_\odot$ have remained in the core, the rest of the star was blown away in the supernova explosion. The remaining $2 - 3M_\odot$ star consists almost entirely of neutrons produced in the electron capture process taking place in the last seconds before the supernova explosion. It is a *neutron star*. As you will show in the exercises, the density of the neutron star is similar to the density in atomic nuclei. The neutron star is a huge atomic nucleus. In the exercises you will also show that the typical radius of a neutron star is a few kilometers. The mass of 2-3 Suns are compressed into a sphere with a radius of a few kilometers. The density is such that if you make a small needle out of materials from a neutron star it would weight about 10^6 tons.

It is thought that neutron stars have solid outer crusts comprised of heavy nuclei (Fe) and electrons. Interior to this crust the material is comprised mostly of neutrons, with a small percentage of protons and electrons as well. At a sufficiently deep level the neutron density may become high enough to give rise to exotic physical phenomena such as *super-fluidity* and perhaps even a *quark-gluon plasma* where one could find free quarks. To model the physics of the interior of neutron stars, unknown particle physics is needed. These neutron stars are therefore macroscopic objects which can be used to understand details of microscopic physics.

For white dwarfs we found that there is a an upper limit to the mass of the dwarf. Repeating these calculations reveals that there is a similar upper limit to the mass of neutron stars. Depending somewhat on the less known physics in the interiors of neutron stars, one has found this upper limit to be somewhere between 2 and 3 solar masses. If the collapsing core has a mass larger than about 3 solar masses, the gravitational forces will be higher than the neutron degeneration pressure. In this case, no known physical forces can withstand the forces of gravity and the core continues to contract and becomes a black hole. Using current theories of quantum physics and gravitation, the core shrinks to an infinitely small point with infinite mass densities. Infinite results in physics is usually a sign of physical processes which are not well understood. When the collapsing

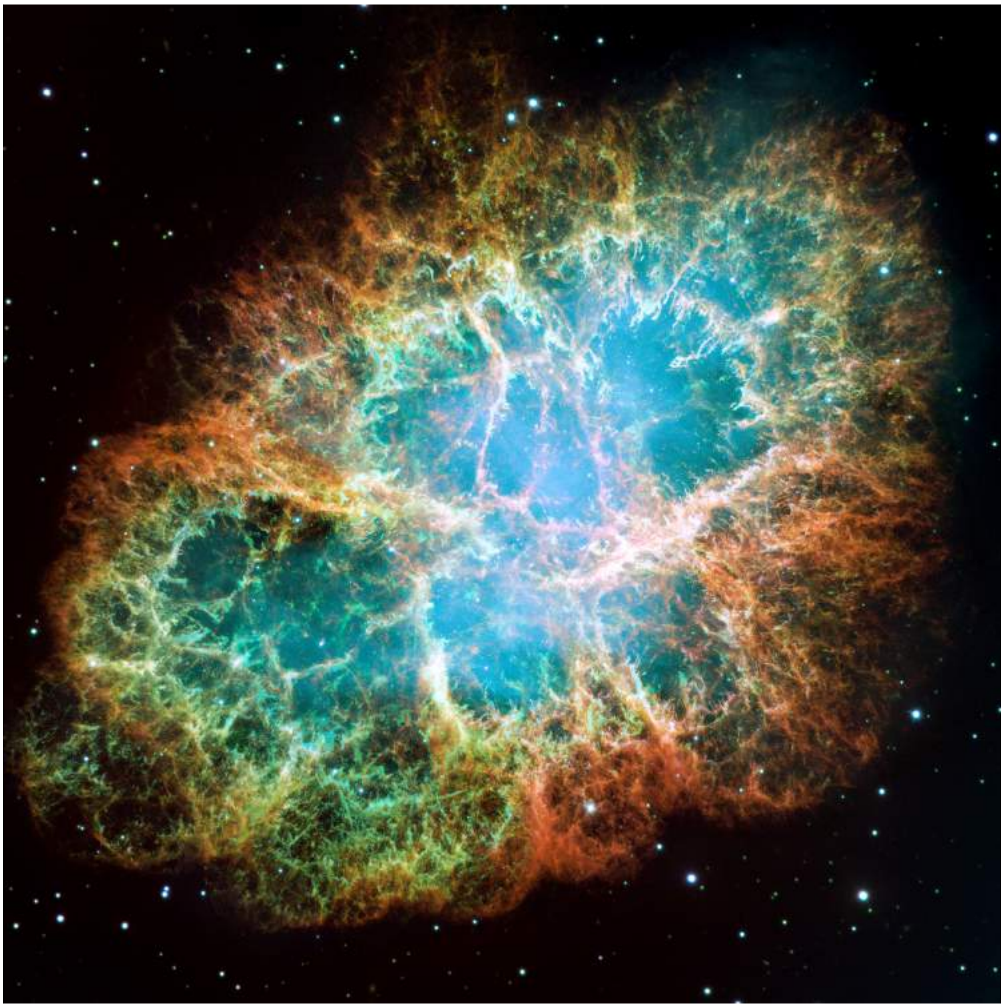

Figure 5: Info-figure: This dramatic image shows the Crab Nebula, a 3.4 pc wide expanding supernova remnant in the constellation of Taurus. The supernova explosion was recorded in 1054 by Chinese, Japanese, Arab, and (possibly) Native American observers. The orange filaments are the tattered remains of the star and consist mostly of hydrogen. The rapidly spinning and highly magnetized neutron star embedded in the center of the nebula is the dynamo powering the nebula's interior bluish glow. The blue light comes from electrons whirling at nearly the speed of light around magnetic field lines. The neutron star, like a lighthouse, ejects twin beams of radiation from gamma-rays to radio waves that appear to pulse 30 times a second due to the neutron star's rotation. (Figure: NASA/ESA)

core becomes sufficiently small, both the general theory of relativity for large masses as well as quantum physics for small scales are needed at the same time. These theories are at the moment not compatible and a more general theory is sought in order to understand what happens at the center of the black hole.

4 Pulsars

When the stellar core is contracting, the rotational velocity of the core increases because of conservation of angular momentum. In order to maintain the angular momentum when the radius decreases, the angular velocity needs to increase. In the exercises you will calculate the rotational speed of the Sun if it had been compressed to the size of a neutron star. After the formation of the neutron star the rotational period is typically less than a second.

In 1967, Jocelyn Bell discovered a source of radio emission which emitted regular radio pulses. The pulses where found to be extremely regular, with exactly the same period between each pulse. The period between each pulse turned out to be about one second. Later, several of these regular radio emitters have been discovered, most of which with a period of less than a second. At first, no physical explanation for the phenomenon was found and the first radio emitter was called LGM-1 (Little Green Men). Later the name *pulsar* has been adopted. Today about 1500 pulsars are known, the fastest is called the millisecond pulsar due to the extremely short pulsation period.

Today the leading theory trying to explain the regular radio pulses from pulsars is that the pulsars are rotating neutron stars. In the exercises you will show that in order to explain pulsars in terms of a rotating object, the radius of the object needs to be similar to the radius of a neutron star. For larger object to rotate sufficiently fast, the outer parts of the object would need to rotate with a velocity larger than the velocity of light. The physics behind the process leading to the radio pulses of pulsars is still an an active field of research and the details are not well understood. In short, the current theory says that the neutron star has a strong magnetic field (which is created during the collapse of the stellar core) with the axis of the magnetic field lines shifted with respect to the axis of rotation (see figure 6). When the star rotates, the field lines are sweeping out a cone around the rotation axis. Electrons in a hot electron gas around the neutron star are accelerated in the strong magnetic field lines from the neutron star. When electrically charged particles are accelerated, they emit electromagnetic radiation, synchrotron radiation. The electrons only feel the magnetic field when the magnetic poles (which are not aligned with the rotation axis) of the neutron star points almost directly in their direction. Thus, only the electrons in the gas where the magnetic field lines are pointing at the moment emit synchrotron radiation. This radiation is

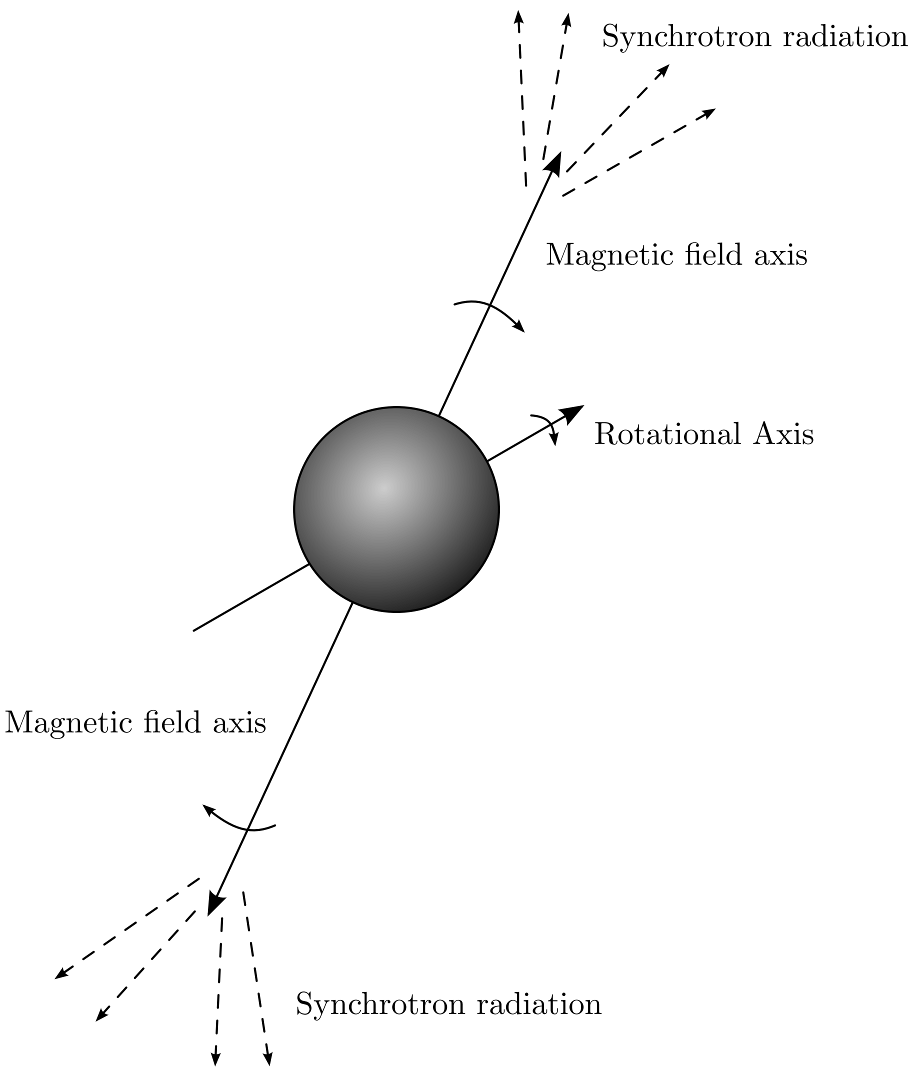

Figure 6: A rotating neutron star: The magnetic axis is not aligned with the rotation axis. Therefore the magnetic field lines are dragged around, producing synchrotron radiation as they accelerate electrons outside the neutron star. The synchrotron radiation is directed in the direction of the magnetic field lines. Every time the fields lines sweep over our direction we see a pulse of radiation.

emitted radially outwards from the neutron star. If the Earth is along the line of such an emission, observers will see a pulse of synchrotron radiation each time the magnetic poles of the neutron star points in the direction of the Earth. We will therefore receive a pulse of radiation once every rotation period.

5 Generations of stars

In the Big Bang, mainly hydrogen and helium were produced. We have learned that heavier elements are produced in nuclear fusion processes in the interior of stars. But fusion processes produce energy only when the nuclei involved are lighter than iron. Fusion processes which produce elements heavier than iron need energy input and are therefore extremely difficult to make happen. How can it be that the Earth consists of large amounts of elements heavier than iron? Human beings contain elements heavier than iron. Where did they come from?

We learned that in the final stages of the stellar evolutionary process for high mass stars, the temperature in the core is very high and high energy photons are able to split nuclei. At these high temperatures, the iron nuclei around the core have so high thermal energy that even nuclear processes requiring energy may happen. Even the heavy nuclei have high enough energy to break the Coloumb barrier and fuse to heavier elements. All the elements in the universe heavier than iron have been produced close to the core of a massive star undergoing a supernova explosion. When the shock wave blows off the material around the core, these heavier elements are transported to the interstellar material. We remember that a star started its life cycle as a cloud of interstellar gas contracting due to its own weight. So the elements produced in supernova explosions are being used in the birth of another star. Parts of these elements go into the planets which are formed in a disc around the protostar.

The first stars which formed in the universe are called *population III* stars. These stars contained no heavier elements (in astrophysics, all the heavier elements are called 'metals', even if they are not metals in the normal sense). These stars have never been observed directly but theories for the evolution of the universe predict that they must have existed. The next generation, produced in part from the 'ashes' of the the population III stars are called *population II* stars. These stars have small traces of metals but are generally also metal poor stars. Finally, the *population I* stars is the latest generation of stars containing a non-zero abundance of metals. The Sun is a population I star. The exact details of stellar evolution are different depending on whether it is a population I, II or III star: computer simulations show that the metal content (which is usually very small even in population I stars) plays an important role in stellar evolutionary processes.

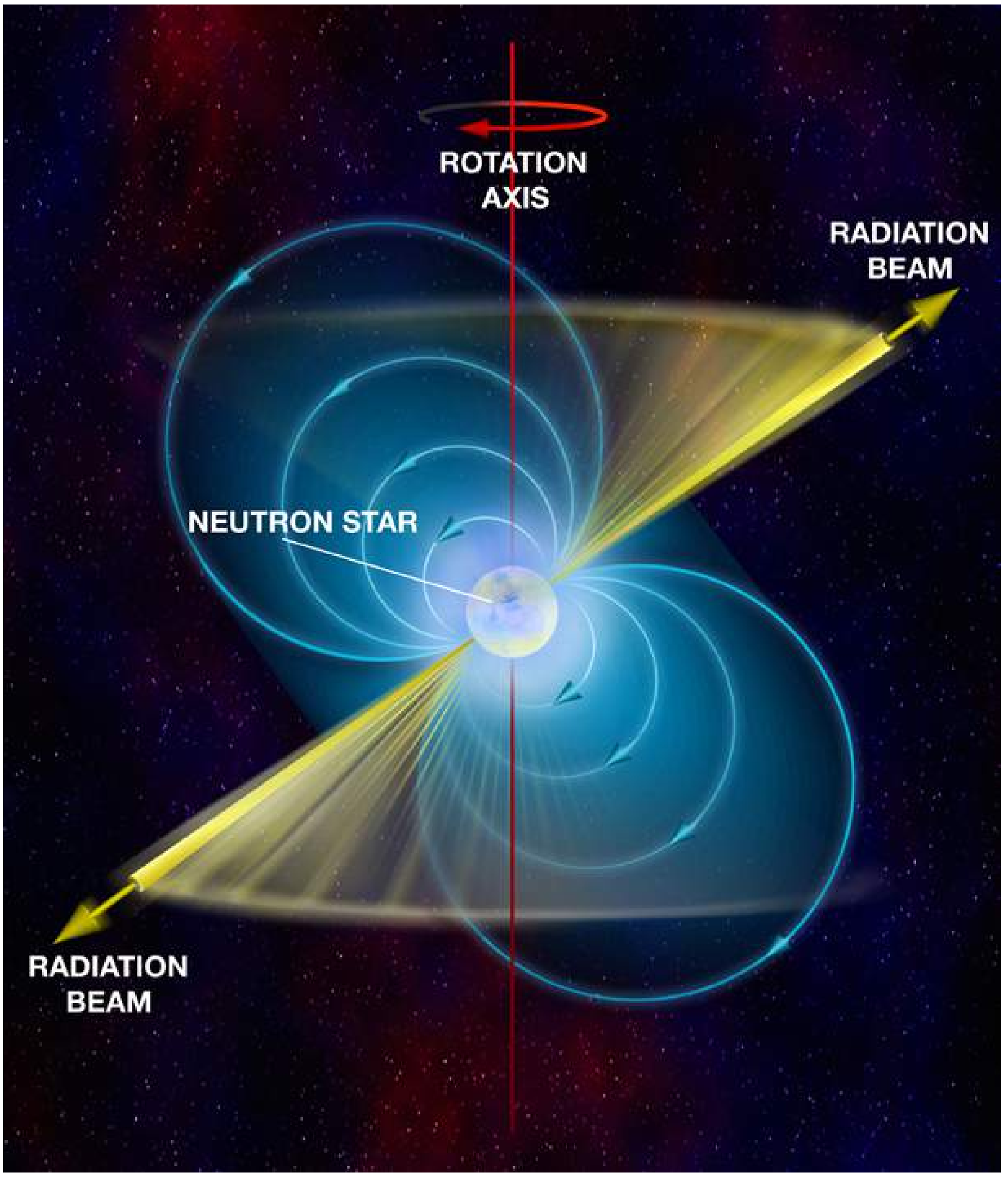

Figure 7: Info-figure: A pulsar is a highly magnetized, rotating neutron star that emits a beam of intense electromagnetic radiation. The magnetic axis is not aligned with the rotation axis, therefore the magnetic field lines are dragged around, producing synchrotron radiation as they accelerate electrons outside the neutron star. The synchrotron radiation is emitted in the direction of the magnetic axis and can only be observed when the beam of emission is pointing towards the Earth, much the way a lighthouse can only be seen when the light is pointed in the direction of an observer. The precise periods of pulsars make them useful tools. E.g., observations of a pulsar in a binary neutron star system (PSR B1913+16) were used to indirectly confirm the existence of gravitational waves, and the first exoplanets were discovered around a pulsar (PSR B1257+12). Certain types of pulsars rival atomic clocks in their accuracy in keeping time. (Figure: NRAO)

6 Type Ia supernovae

We learned in previous lectures that supernovae can be divided into type I and type II according to their spectra. The type I supernovae could further be divided into type Ia, Ib and Ic. The type I supernovae did not show any hydrogen lines in their spectra whereas type II supernovae show strong hydrogen lines. It is currently believed that type II as well as type Ib and Ic are core collapse supernovae discussed above. Type Ib and Ic do not have hydrogen lines because the outer hydrogen rich parts of the star have been blown off before the supernova explosion.

A type Ia supernova is believed to be a completely different phenomenon. There are several different theories trying to explain type Ia supernova, non of which are understood well. In one of the most popular theories, a type Ia supernova occurs in a binary system: A white dwarf star and a main sequence or giant star orbit a common center of mass. When the two stars are sufficiently close, the white dwarf starts accreting material from the other star. Material from the other star is accreted in a shell on the surface of the degenerate white dwarf. The temperature in the core of the white dwarf increases and nuclear fusion processes burning carbon and oxygen to heavier elements ignite. Since the white dwarf is degenerate, a process similar to the helium flash occur. Fusion processes start everywhere in the white dwarf at the same time and the white dwarf is completely destroyed in the following explosion. The exact details of these explosions are still studied along with other completely different theories in computer models. Hopefully, in the future, these models will be able to tell us exactly what happens in type Ia supernova explosions. As we will see in the lectures on cosmology, understanding type Ia supernovae is crucial for understanding the universe as a whole. We also remember that type Ia supernovae are used as standard candles to measure distances. It turns out that the luminosity is usually almost the same in most type Ia explosions. If these explosions really are white dwarf stars exploding, we can understand why the luminosity is almost the same in all supernovae of this kind: There is one common factor in all cases: the white dwarf stars usually have a mass close to the Chandrasekhar mass of $1.4M_\odot$.

7 An Example: SK 69^o202

Let us take as an example the blue giant that exploded as a supernova of type II in the Large Magellenic Cloud in 1987; the star SK 69^o202, later known as SN 1987a. The details of the star's life have been obtained through computer simulations. This originally $20M_\odot$ star's life may be summarized as follows:

1. H→He fusion for a period of roughly 10^7 yr with a core temperature $T_c \approx 40 \times 10^6$ K, a central density $\rho_c \approx 5 \times 10^3$ kg/m^3, and a radius $\approx 6\mathrm{R}_\odot$.

2. He→C, O fusion for a period of roughly 10^6 yr with a core temperature $T_c \approx 170 \times 10^6$ K, a central density $\rho_c \approx 9 \times 10^5$ kg/m^3, and a radius $\approx 500R_\odot$. The core mass is $6M_\odot$. The star is in this phase a red super-giant.

3. C→Ne, Na, Mg fusion for a period of roughly 10^3 yr with a core temperature $T_c \approx 700\times10^6$ K, a central density $\rho_c \approx 1.5\times10^8$ kg/m^3, and a radius $\approx 50R_\odot$. The core mass is $4M_\odot$. From this stage and onward the star loses more energy through the emission of neutrinos than from the emission of photons. In addition, from this point onwards the evolution of the core is very rapid and the outer layers do not have time to adjust to the changes happening below: the star's radius remains unchanged.

4. Ne→O, Mg fusion for a period of some few years with a core temperature $T_c \approx 1.5 \times 10^9$ K, a central density $\rho_c \approx 10^{10}$ kg/m^3.

5. O→S, Si fusion for a period of some few years with a core temperature $T_c \approx 2.1 \times 10^9$ K. The neutrino luminosity is at this stage 10^5 greater than the photon luminosity.

6. S, Si→"Fe" (actually a mix of Fe, Ni, Co) fusion for a period of a few days with a core temperature $T_c \approx 3.5\times10^9$ K, a central density $\rho_c \approx 10^{11}$ kg/m^3. Si "melts" into α-particles, neutrons and protons which again are built into "Fe" nuclei. The core mass is now roughly $1.4M_\odot$.

7. The core looses energy in the electron capture process and starts contracting rapidly. The energy in the core collapse is released and the star explodes as a supernova.

8 Problems

Problem 1 (20–30 min.)

Where does the huge amount of energy released in a supernova explosion originally come from? Does it come from nuclear processes or from other processes? Explain how the energy is released and how the energy is transferred between different types of energy until it is released as a huge flux of photons and neutrinos. Read the text carefully to understand the details in the processes and make a diagram of the energy flow.

Problem 2 (1–2 hours)

In this exercise we will study the properties of a neutron star.

1. In the text we find an approximate expression for the radius of a white dwarf star using the equation of hydrostatic equilibrium and the degeneration pressure for electrons. Go back to the text and

study how this was done in detail. Now, repeat the same exercise but for a neutron degenerate neutron star. What is the radius of a neutron star having $1.4M_\odot$? (The size of a neutron star is typically 10 km, your answer will not be correct but should be roughly this order of magnitude)

2. In the following, use a neutron star radius of 10 km which is more realistic. What is the typical density of a neutron star? Express the result in the following way: To which radius R would you need to compress Earth in order to obtain densities similar to neutron star densities?

3. Compare the density in the neutron star to the densities in an atomic nucleus: Use the density you obtained in the previous question for the neutron star density. For the density of an atomic nucleus: Assume that a Uranium atom has about 200 nucleons and has a spherical nucleus with radius of about 7 fm (1 fm= 10^{-15} m).

4. Express the radius of a $1.4M_\odot$ neutron star (use numbers from previous questions) in terms of the mass of the neutron star. How close are you to the Schwarzschild radius when you are at the surface of a neutron star? Is general relativity needed when modeling a neutron star?

5. The Sun rotates about its axis once every 25 days. Use conservation of angular momentum to find the rotation period if the Sun is compressed to

   (a) a typical white dwarf star

   (b) a typical neutron star

   Compare your answer to numbers given in the text for the rotational period of a neutron star. If you found a faster period, can you find some possible reasons why the observed rotational period is slower? **Hint 1:** The angular momentum for a solid object is given by $L = I\omega$ where I is the moment of inertia. **Hint 2:** Assume that the Sun is a solid sphere and remember that the moment of inertia of a solid sphere is $(2/5)MR^2$

6. We will assume that we do not know what pulsars are, but we suspect that they might be rotating objects. The larger the radius of a rotating object, the faster the velocity of an object on the surface of the rotating object. The fastest observed pulsar is the millisecond pulsar. Assume that its rotational period is 1 ms. What is the maximum radius R that the object can have without having objects at the surface of the object moving faster than light? What kind of astronomical objects could this possibly be? If you find several possibilities, try to find reasons to eliminate some of them.

Problem 3 (30–60 min.)

We learned in the lectures on distances in the universe that by observing the light curve of a supernova one could find the luminosity and thereby the absolute magnitude in order to obtain the distance. Here we will study a very simple model for a supernova in order to see if we can understand this relation between light curve and luminosity.

1. We have seen that in a supernova explosion, the outer shells of a star is basically lifted away from the central core. Assume that we can model the supernova in this way: the shell of gas simply expands spherically very fast outwards in all directions. It is equal to saying that the radius R of the star increases. Assume that we use Doppler measurements every day after the explosion of the supernova and find that the shell around the core expands with a constant velocity v. A time Δt after the explosion started we also measure the effective temperature of the shell by spectroscopic measurements and find it to have the temperature T. If we assume the shell to be a black body, show that the luminosity of the supernova at this point is given approximately by

$$L = 4\pi\sigma T^4 v^2 \Delta t^2.$$

   Which assumptions did you make in order to arrive at this expression?

2. A supernova is observed in a distant galaxy. Nobody had so far managed to measure the distance to this distant galaxy, but now there was a supernova explosion there and this allowed us to find the distance to the supernova and therefore also to the galaxy. You can now make all the assumptions from the previous question about the supernova. The supernova is observed every day after the explosion and the velocity of the shell is found to be moving at a constant velocity of 9500 km/s. After 42 days, the effective temperature is measured to be 6000 K. Find the luminosity of the supernova after 42 days expressed in solar luminosities ($L_{\odot} = 3.8 \times 10^{26}$W)

3. Find the absolute magnitude of the supernova after 42 days. (The absolute magnitude of the Sun is $M = 4.83$).

4. The apparent magnitude of the supernova after 42 days is $m = 10$. What is the distance to the supernova? Express your answer in Mpc.