INTEGRATION BY PARTS

Graham S McDonald

A self-contained Tutorial Module for learning the technique of integration by parts

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1. Theory

To differentiate a product of two functions of x, one uses the product rule:

$$\frac{d}{dx}\left(uv\right) = u\frac{dv}{dx} + \frac{du}{dx}v$$

where $u = u\left(x\right)$ and $v = v\left(x\right)$ are two functions of x. A slight rearrangement of the product rule gives

$$u\frac{dv}{dx} = \frac{d}{dx}\left(uv\right) - \frac{du}{dx}v$$

Now, integrating both sides with respect to x results in

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v\,dx$$

This gives us a rule for integration, called **INTEGRATION BY PARTS**, that allows us to integrate many products of functions of x. We take one factor in this product to be u (this also appears on the right-hand-side, along with $\frac{du}{dx}$). The other factor is taken to be $\frac{dv}{dx}$ (on the right-hand-side only v appears – i.e. the other factor integrated with respect to x).

2. Usage

We highlight here four different types of products for which integration by parts can be used (as well as which factor to label u and which one to label $\frac{dv}{dx}$). These are:

$$\text{(i)}\quad \int \underset{\substack{\uparrow \\ u}}{x^n} \cdot \underset{\substack{\uparrow \\ \frac{dv}{dx}}}{\left\{ \begin{array}{c} \sin bx \\ or \\ \cos bx \end{array} \right\}} dx \qquad \text{(ii)}\quad \int \underset{\substack{\uparrow \\ u}}{x^n} \cdot \underset{\substack{\uparrow \\ \frac{dv}{dx}}}{e^{ax}} dx$$

$$\text{(iii)}\quad \int \underset{\substack{\uparrow \\ \frac{dv}{dx}}}{x^r} \cdot \underset{\substack{\uparrow \\ u}}{\ln (ax)}\, dx \qquad \text{(iv)}\quad \int \underset{\substack{\uparrow \\ u}}{e^{ax}} \cdot \underset{\substack{\uparrow \\ \frac{dv}{dx}}}{\left\{ \begin{array}{c} \sin bx \\ or \\ \cos bx \end{array} \right\}} dx$$

where a, b and r are given constants and n is a positive integer.

3. Exercises

Click on Exercise links for full worked solutions (there are 14 exercises in total)

Exercise 1. $\int x \cos x \, dx$

Exercise 2. $\int x^2 \sin x \, dx$

Exercise 3. $\int x e^x dx$

Exercise 4. $\int x^2 e^{4x} dx$

Exercise 5. $\int x^2 \ln x \, dx$

Exercise 6. $\int (x+1)^2 \ln 3x \, dx$

Exercise 7. $\int e^{2x} \cos x \, dx$

Exercise 8. $\int e^{-x} \sin 4x \, dx$

Exercise 9. $\int_0^{1/2} x e^{2x} \, dx$

Exercise 10. $\int_0^{\pi/4} x \sin 2x \, dx$

Exercise 11. $\displaystyle\int_{1/2}^{1} x^4 \ln 2x \, dx$

Exercise 12. $\displaystyle\int_{0}^{\pi} 3x^2 \cos\left(\frac{x}{2}\right) \, dx$

Exercise 13. $\displaystyle\int x^3 e^x \, dx$

Exercise 14. $\displaystyle\int e^{3x} \cos x \, dx$

4. Final solutions

1. $x \sin x + \cos x + C$,

2. $-x^2 \cos x + 2x \sin x + 2 \cos x + C$,

3. $(x - 1)\, e^x + C$,

4. $\frac{1}{32} e^{4x} \left(8x^2 - 4x + 1\right) + C$,

5. $\frac{1}{9} x^3 \left(3 \ln x - 1\right) + C$,

6. $\frac{1}{3} \left(x + 1\right)^3 \ln 3x - \frac{1}{9} x^3 - \frac{1}{2} x^2 - x - \frac{1}{3} \ln x + C$,

7. $\frac{1}{5} e^{2x} \left(\sin x + 2 \cos x\right) + C$,

8. $-\frac{1}{17}e^{-x}\left(4\cos 4x - \sin 4x\right) + C$,

9. $\frac{1}{4}$,

10. $\frac{1}{4}$,

11. $\frac{1}{5}\ln 2 - \frac{31}{800}$,

12. $6\left(\pi^2 - 8\right)$,

13. $e^x\left(x^3 - 3x^2 + 6x - 6\right) + C$,

14. $\frac{1}{10}e^{3x}\left(\sin x + 3\cos x\right) + C$.

5. Standard integrals

$f(x)$	$\int f(x)dx$	$f(x)$	$\int f(x)dx$
x^n	$\frac{x^{n+1}}{n+1} \quad (n \neq -1)$	$[g(x)]^n g'(x)$	$\frac{[g(x)]^{n+1}}{n+1} \quad (n \neq -1)$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\frac{g'(x)}{g(x)}$	$\ln\lvert g(x)\rvert$
e^x	e^x	a^x	$\frac{a^x}{\ln a} \quad (a > 0)$
$\sin x$	$-\cos x$	$\sinh x$	$\cosh x$
$\cos x$	$\sin x$	$\cosh x$	$\sinh x$
$\tan x$	$-\ln\lvert\cos x\rvert$	$\tanh x$	$\ln\cosh x$
$\operatorname{cosec} x$	$\ln\left\lvert\tan\frac{x}{2}\right\rvert$	$\operatorname{cosech} x$	$\ln\left\lvert\tanh\frac{x}{2}\right\rvert$
$\sec x$	$\ln\lvert\sec x+\tan x\rvert$	$\operatorname{sech} x$	$2\tan^{-1} e^x$
$\sec^2 x$	$\tan x$	$\operatorname{sech}^2 x$	$\tanh x$
$\cot x$	$\ln\lvert\sin x\rvert$	$\coth x$	$\ln\lvert\sinh x\rvert$
$\sin^2 x$	$\frac{x}{2} - \frac{\sin 2x}{4}$	$\sinh^2 x$	$\frac{\sinh 2x}{4} - \frac{x}{2}$
$\cos^2 x$	$\frac{x}{2} + \frac{\sin 2x}{4}$	$\cosh^2 x$	$\frac{\sinh 2x}{4} + \frac{x}{2}$

$f(x)$	$\int f(x)\,dx$	$f(x)$	$\int f(x)\,dx$
$\frac{1}{a^2+x^2}$	$\frac{1}{a}\tan^{-1}\frac{x}{a}$	$\frac{1}{a^2-x^2}$	$\frac{1}{2a}\ln\left\lvert\frac{a+x}{a-x}\right\rvert \quad (0<\lvert x\rvert<a)$
	$(a>0)$	$\frac{1}{x^2-a^2}$	$\frac{1}{2a}\ln\left\lvert\frac{x-a}{x+a}\right\rvert \quad (\lvert x\rvert>a>0)$
$\frac{1}{\sqrt{a^2-x^2}}$	$\sin^{-1}\frac{x}{a}$	$\frac{1}{\sqrt{a^2+x^2}}$	$\ln\left\lvert\frac{x+\sqrt{a^2+x^2}}{a}\right\rvert \quad (a>0)$
	$(-a<x<a)$	$\frac{1}{\sqrt{x^2-a^2}}$	$\ln\left\lvert\frac{x+\sqrt{x^2-a^2}}{a}\right\rvert \quad (x>a>0)$
$\sqrt{a^2-x^2}$	$\frac{a^2}{2}\left[\sin^{-1}\left(\frac{x}{a}\right)\right.$	$\sqrt{a^2+x^2}$	$\frac{a^2}{2}\left[\sinh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{a^2+x^2}}{a^2}\right]$
	$\left.+\frac{x\sqrt{a^2-x^2}}{a^2}\right]$	$\sqrt{x^2-a^2}$	$\frac{a^2}{2}\left[-\cosh^{-1}\left(\frac{x}{a}\right)+\frac{x\sqrt{x^2-a^2}}{a^2}\right]$

6. Tips on using solutions

● When looking at the THEORY, INTEGRALS, FINAL SOLUTIONS, TIPS or NOTATION pages, use the Back button (at the bottom of the page) to return to the exercises

● Use the solutions intelligently. For example, they can help you get started on an exercise, or they can allow you to check whether your intermediate results are correct

● Try to make less use of the full solutions as you work your way through the Tutorial

7. Alternative notation

In this Tutorial, we express the rule for integration by parts using the formula:

$$\int u\frac{dv}{dx}dx = uv - \int \frac{du}{dx}v\,dx$$

But you may also see other forms of the formula, such as:

$$\int f(x)g(x)dx = F(x)g(x) - \int F(x)\frac{dg}{dx}dx$$

where

$$\frac{dF}{dx} = f(x)$$

Of course, this is simply different notation for the same rule. To see this, make the identifications: $u = g(x)$ and $v = F(x)$.

Full worked solutions

Exercise 1. We evaluate by `integration by parts`:

$$\int x\cos x\,dx = x\cdot\sin x - \int (1)\cdot\sin x\,dx, \text{ i.e. take } u = x$$

giving $\frac{du}{dx} = 1$ (by differentiation)

and take $\frac{dv}{dx} = \cos x$

giving $v = \sin x$ (by integration),

$$= x\sin x - \int \sin x\,dx$$

$$= x\sin x - (-\cos x) + C, \quad \text{where } C \text{ is an arbitrary}$$

$$= x\sin x + \cos x + C \quad \text{constant of integration.}$$

Return to Exercise 1

Exercise 2.

$$\int x^2 \sin x \, dx = x^2 \cdot (-\cos x) - \int (2x) \cdot (-\cos x) \, dx \,,$$

i.e. take $u = x^2$

giving $\dfrac{du}{dx} = 2x$

and take $\dfrac{dv}{dx} = \sin x$

giving $v = -\cos x$,

$$= -x^2 \cos x + \underbrace{2 \int x \cos x \, dx}$$

$\hookrightarrow$ we need to use integration by parts again!

$$
\begin{aligned}
&= -x^2 \cos x + 2\left\{ x \sin x - \int (1) \cdot \sin x \, dx \right\}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{as in question 1,} \\
&= -x^2 \cos x + 2x \sin x - 2 \int \sin x \, dx \\
&= -x^2 \cos x + 2x \sin x - 2 \cdot (-\cos x) + C \\
&= -x^2 \cos x + 2x \sin x + 2 \cos x + C \, .
\end{aligned}
$$

Return to Exercise 2

Exercise 3.

$$\int x\, e^x dx = x \cdot e^x - \int (1) \cdot e^x dx, \quad \text{i.e. take } u = x$$

$$\text{giving } \frac{du}{dx} = 1$$

$$\text{and take } \frac{dv}{dx} = e^x$$

$$\text{giving } v = e^x,$$

$$\begin{aligned} &= x\, e^x - \int e^x dx \\ &= x\, e^x - e^x + C \\ &= (x-1)\, e^x + C\,. \end{aligned}$$

Return to Exercise 3

Exercise 4.

$$\int x^2\, e^{4x} dx = x^2 \cdot \frac{1}{4} e^{4x} - \int 2x \cdot \frac{1}{4} e^{4x} dx\ ,$$

$$\text{i.e. take } u = x^2$$

$$\frac{du}{dx} = 2x$$

$$\frac{dv}{dx} = e^{4x}$$

$$v = \frac{1}{4} e^{4x}\ ,$$

$$= \frac{1}{4} x^2 e^{4x} - \frac{1}{2} \underbrace{\int x\, e^{4x} dx}$$

$\hookrightarrow$ now use integration by parts again

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{2}\left\{x \cdot \frac{1}{4}e^{4x} - \int 1 \cdot \frac{1}{4}e^{4x}dx\right\},$$

i.e. this time $u = x$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{4x}$$

$$v = \frac{1}{4}e^{4x},$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} - \frac{1}{2} \cdot \left(-\frac{1}{4}\right) \cdot \int e^{4x}dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{8}\int e^{4x}dx$$

$$= \frac{1}{4}x^2e^{4x} - \frac{1}{8}xe^{4x} + \frac{1}{8} \cdot \frac{1}{4}e^{4x} + C$$

$$= \frac{8}{32} x^2 e^{4x} - \frac{4}{32} x\, e^{4x} + \frac{1}{32} e^{4x} + C$$
$$= \frac{1}{32} e^{4x} \left(8x^2 - 4x + 1\right) + C \,.$$

Return to Exercise 4

Exercise 5.

$$\int x^2 \ln x \, dx = (\ln x) \cdot \left(\frac{1}{3}x^3\right) - \int \frac{1}{x} \cdot \left(\frac{1}{3}x^3\right) dx \, , \quad \text{i.e. } u = \ln x$$

$$\frac{du}{dx} = \frac{1}{x}$$

$$\frac{dv}{dx} = x^2$$

$$v = \frac{1}{3}x^3 \, ,$$

$$\begin{aligned}
&= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{3} \cdot \left(\frac{1}{3}x^3\right) + C \\
&= \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C \\
&= \frac{1}{9}x^3 \left(3 \ln x - 1\right) + C \, .
\end{aligned}$$

Return to Exercise 5

Exercise 6.

$$\int (x+1)^2 \ln 3x \, dx = (\ln 3x) \cdot \left(\frac{1}{3}\right)(x+1)^3 - \int \frac{1}{3x} \cdot (3) \cdot \frac{1}{3}(x+1)^3 dx$$

i.e. $u = \ln 3x$ gives $\frac{du}{dx} = \frac{1}{3x} \cdot \frac{d}{dx}(3x) = \frac{1}{3x} \cdot (3)$,

using the chain rule, and $\frac{dv}{dx} = (x+1)^2$ gives $v = \frac{1}{1} \cdot \frac{(x+1)^3}{3}$,

where we have used the result that if

$\frac{dv}{dx} = (ax+b)^n$ then $v = \frac{1}{a}\frac{(ax+b)^{n+1}}{(n+1)}$,

$$\begin{aligned}
\therefore \int (x+1)^2 \ln 3x \, dx &= \frac{1}{3}(x+1)^3 \ln 3x - \frac{1}{3}\int \frac{(x+1)^3}{x}\, dx \\
&= \frac{1}{3}(x+1)^3 \ln 3x - \frac{1}{3}\int \frac{x^3+3x^2+3x+1}{x}\, dx\,,
\end{aligned}$$

where we have used the binomial theorem,

or just multiplied out $(x+1)^3$,

$$\begin{aligned}
&= \frac{1}{3}(x+1)^3 \ln 3x - \frac{1}{3}\int x^2+3x+3+\frac{1}{x} dx \\
&= \frac{1}{3}(x+1)^3 \ln 3x - \frac{1}{3}\left[\frac{x^3}{3}+\frac{3}{2}x^2+3x+\ln x\right]+C \\
&= \frac{1}{3}(x+1)^3 \ln 3x - \frac{x^3}{9} - \frac{1}{2}x^2 - x - \frac{1}{3}\ln x + C\ .
\end{aligned}$$

Return to Exercise 6

Exercise 7.

$$\int e^{2x} \cos x \, dx$$

Set $u = e^{2x}$ and $\frac{dv}{dx} = \cos x$, to give $\frac{du}{dx} = 2e^{2x}$ and $v = \sin x$.

Let $I = \int e^{2x} \cos x \ dx$, since we will eventually get I on the right-hand-side for this type of integral

i.e. $I = e^{2x} \cdot \sin x - \int 2e^{2x} \cdot \sin x \, dx$

i.e. $I = e^{2x} \sin x - 2 \int e^{2x} \sin x \, dx.$

Use integration by parts again, with $u = e^{2x}$ and $\frac{dv}{dx} = \sin x$, giving $\frac{du}{dx} = 2e^{2x}$ and $v = -\cos x$

i.e. $I = e^{2x}\sin x - 2\left\{e^{2x}\cdot(-\cos x) - \int 2e^{2x}\cdot(-\cos x)\,dx\right\}$

i.e. $I = e^{2x}\sin x - 2\left\{-e^{2x}\cos x + 2\int e^{2x}\cos x\,dx\right\}$

i.e. $I = e^{2x}\sin x + 2e^{2x}\cos x - \underbrace{4\int e^{2x}\cos dx}_{=4I}$

i.e. $5I = e^{2x}\sin x + 2e^{2x}\cos x + C_1$

$\therefore\quad I = \frac{1}{5}e^{2x}(\sin x + 2\cos x) + C$, where $C = \frac{1}{5}C_1$ (another arbitrary constant).

Note It is customary to introduce the arbitrary constant after the last integration is performed, though strictly one could accommodate arbitrary constants arising from each $\int \frac{du}{dx}\cdot v\,dx$ (indefinite) integration and these would add up to give a single arbitrary constant in the final answer. Return to Exercise 7

Exercise 8.

$$\int e^{-x}\sin 4x\,dx. \qquad u = e^{-x}\,, \quad \text{giving} \quad \frac{du}{dx} = -e^{-x}\,,$$

$$\frac{dv}{dx} = \sin 4x, \quad \text{giving} \quad v = \frac{1}{4}\cdot(-\cos 4x)$$

$$\text{i.e.} \quad v = -\frac{1}{4}\cos 4x\,,$$

$$\text{Let} \quad I = \int e^{-x}\sin 4x\,dx$$

$$\text{i.e.} \quad I = e^{-x}\cdot\left(-\frac{1}{4}\cos 4x\right) - \int(-e^{-x})\cdot\left(-\frac{1}{4}\cos 4x\right)dx$$

$$\text{i.e.} \quad I = -\tfrac{1}{4}e^{-x}\cos 4x - \tfrac{1}{4}\underbrace{\int e^{-x}\cos 4x\,dx}_{\downarrow}$$

$$u = e^{-x}\,, \quad \tfrac{du}{dx} = -e^{-x}$$

$$\tfrac{dv}{dx} = \cos 4x\,, \quad v = \tfrac{1}{4}\sin 4x \quad ,$$

$$\text{i.e.} \quad I = -\tfrac{1}{4}e^{-x}\cos 4x - \tfrac{1}{4}\left\{e^{-x}\cdot\left(\tfrac{1}{4}\sin 4x\right) - \int\left(-e^{-x}\right)\cdot\left(\tfrac{1}{4}\sin 4x\right)dx\right\}$$

$$\therefore I = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{4}\left\{\frac{1}{4}e^{-x}\sin 4x + \frac{1}{4}\int e^{-x}\sin 4x\,dx\right\}$$

$$\text{i.e. } I = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x - \frac{1}{16}I$$

$$\text{i.e. } \left(1+\frac{1}{16}\right)I = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x$$

$$\left(\frac{17}{16}\right)I = -\frac{1}{4}e^{-x}\cos 4x - \frac{1}{16}e^{-x}\sin 4x$$

$$\text{i.e. } I = -\frac{4}{17}e^{-x}\cos 4x - \frac{1}{17}e^{-x}\sin 4x$$

$$= -\frac{1}{17}e^{-x}\left(4\cos 4x - \sin 4x\right) + C\,.$$

Return to Exercise 8

Exercise 9.

$$\int_0^{1/2} xe^{2x}dx = \left[x \cdot \left(\frac{1}{2}e^{2x}\right)\right]_0^{1/2} - \int_0^{1/2} 1 \cdot \left(\frac{1}{2}e^{2x}\right)dx, \quad \text{i.e. } u = x$$

$$\frac{du}{dx} = 1$$

$$\frac{dv}{dx} = e^{2x}$$

$$v = \frac{1}{2}e^{2x},$$

$$= \frac{1}{2}\left[xe^{2x}\right]_0^{1/2} - \frac{1}{2}\int_0^{1/2} e^{2x}dx$$

$$= \frac{1}{2}\left[xe^{2x}\right]_0^{1/2} - \frac{1}{2}\left[\frac{1}{2}e^{2x}\right]_0^{1/2}$$

$$
\begin{aligned}
&= \frac{1}{2}\left[\frac{1}{2}e^{2\cdot\frac{1}{2}} - 0.e^{0}\right] - \frac{1}{4}\left[e^{2\cdot\frac{1}{2}} - e^{0}\right] \\
&= \frac{1}{2}\cdot\frac{1}{2}e^{1} - \frac{1}{4}e^{1} + \frac{1}{4}e^{0} \\
&= 0 + \frac{1}{4}e^{0} = \frac{1}{4}\,.
\end{aligned}
$$

Return to Exercise 9

Exercise 10.

$$\int_0^{\pi/4} x \sin 2x \, dx = \left[x \cdot \left(-\frac{1}{2} \cos 2x \right) \right]_0^{\pi/4} - \int_0^{\pi/4} 1. \left(-\frac{1}{2} \cos 2x \right) dx,$$

$$u = x$$
$$\frac{dv}{dx} = \sin 2x$$
$$\frac{du}{dx} = 1$$
$$v = -\frac{1}{2} \cos 2x \ ,$$

$$= \left[-\frac{1}{2} x \cos 2x \right]_0^{\pi/4} + \frac{1}{2} \int_0^{\pi/4} \cos 2x \, dx$$

$$
\begin{aligned}
&= -\frac{1}{2}\left[x\cos 2x\right]_0^{\pi/4} + \frac{1}{2}\left[\frac{1}{2}\sin 2x\right]_0^{\pi/4} \\
&= -\frac{1}{2}\left[x\cos 2x\right]_0^{\pi/4} + \frac{1}{4}\left[\sin 2x\right]_0^{\pi/4} \\
&= -\frac{1}{2}\left\{\frac{\pi}{4}\cdot\cos\frac{\pi}{2} - 0\cdot\cos 0\right\} + \frac{1}{4}\left\{\sin\frac{\pi}{2} - \sin 0\right\} \\
&= -\frac{1}{2}\left\{0-0\right\} + \frac{1}{4}\left\{1-0\right\}, && \text{since } \cos\frac{\pi}{2} = 0, \\
&= \frac{1}{4}. && \sin\frac{\pi}{2} = 1, \\
& && \text{and } \sin 0 = 0,
\end{aligned}
$$

Return to Exercise 10

Exercise 11. Note that if a logarithm function is involved then we choose that factor to be u.

$$\text{i.e.} \int_{1/2}^{1} x^4 \ln 2x \, dx = \left[(\ln 2x)\cdot\left(\frac{x^5}{5}\right)\right]_{1/2}^{1} - \int_{1/2}^{1} \left(\frac{1}{x}\right)\cdot\left(\frac{x^5}{5}\right) dx$$

$$= \frac{1}{5}\left[x^5 \ln 2x\right]_{1/2}^{1} - \frac{1}{5}\int_{1/2}^{1} x^4 dx$$

$$= \frac{1}{5}\left[x^5 \ln 2x\right]_{1/2}^{1} - \frac{1}{5}\left[\frac{x^5}{5}\right]_{1/2}^{1}$$

$$= \frac{1}{5}\left[x^5 \ln 2x\right]_{1/2}^{1} - \frac{1}{25}\left[x^5\right]_{1/2}^{1}$$

$$= \frac{1}{5}\left\{(1\cdot\ln 2) - \left(\frac{1}{2}\right)^5 \cdot \ln 1\right\} - \frac{1}{25}\left\{1^5 - \left(\frac{1}{2}\right)^5\right\}$$

$$
\begin{aligned}
&= \frac{1}{5}\ln 2 - \frac{1}{25}\left\{1 - \frac{1}{32}\right\}, \quad \text{since } \ln 1 = 0 \text{ and } 2^5 = 32, \\
&= \frac{1}{5}\ln 2 - \frac{1}{25}\cdot\frac{31}{32} \\
&= \frac{1}{5}\ln 2 - \frac{31}{800}.
\end{aligned}
$$

Return to Exercise 11

Exercise 12. For definite integrals, we can either use integration by parts in the form:

$$\int_a^b u\frac{dv}{dx}dx = [uv]_a^b - \int_a^b \frac{du}{dx}\cdot v\,dx$$

or we can work out $\int u\frac{dv}{dx}dx$, i.e. without the limits, first and then apply the limits to the final result. We will do that here. So, to work out $\int_0^\pi 3x^2\cos\left(\frac{x}{2}\right)dx$, we will consider the indefinite integral first:

$$\begin{aligned}\int 3x^2\cos\left(\frac{x}{2}\right)dx &= 3x^2\left(\frac{1}{\left(\frac{1}{2}\right)}\right)\sin\left(\frac{x}{2}\right) - \int 6x\cdot\left(\frac{1}{\left(\frac{1}{2}\right)}\sin\left(\frac{x}{2}\right)\right)dx \\ &= 6x^2\sin\left(\frac{x}{2}\right) - 12\int x\sin\left(\frac{x}{2}\right)dx \\ &\qquad\qquad \text{(use integration by parts, again)} \\ &= 6x^2\sin\left(\frac{x}{2}\right) - 12\left\{x\left(-2\cos\left(\frac{x}{2}\right)\right) - \int -2\cos\left(\frac{x}{2}\right)dx\right\}\end{aligned}$$

$$
\begin{aligned}
&= 6x^2 \sin\left(\frac{x}{2}\right) - 12\left\{-2x\cos\left(\frac{x}{2}\right) + 2\int \cos\left(\frac{x}{2}\right) dx\right\} \\
\text{i.e.} \int 3x^2 \cos\left(\frac{x}{2}\right) dx &= 6x^2 \sin\left(\frac{x}{2}\right) + 24x\cos\left(\frac{x}{2}\right) - 24\int \cos\left(\frac{x}{2}\right) dx \\
&= 6x^2 \sin\left(\frac{x}{2}\right) + 24x\cos\left(\frac{x}{2}\right) - 24 \cdot 2 \cdot \sin\left(\frac{x}{2}\right) + C.
\end{aligned}
$$

On the next page, we will evaluate the definite integral ...

So,

$$
\begin{aligned}
\int_0^{\pi} 3x^2 \cos\left(\frac{x}{2}\right) dx &= \left[6x^2 \sin\left(\frac{x}{2}\right) + 24x \cos\left(\frac{x}{2}\right) - 48 \sin\left(\frac{x}{2}\right)\right]_0^{\pi} \\
&= \left\{6\pi^2 \sin\left(\frac{\pi}{2}\right) - 0\right\} + 24\left\{\pi \cos\left(\frac{\pi}{2}\right) - 0\right\} \\
&\qquad - 48\left\{\sin\left(\frac{\pi}{2}\right) - 0\right\}, \\
&\qquad \text{since } \sin 0 = 0, \\
&= 6\pi^2 - 48, \text{ since } \sin\left(\frac{\pi}{2}\right) = 1 \text{ and } \cos\left(\frac{\pi}{2}\right) = 0, \\
&= 6\left(\pi^2 - 8\right).
\end{aligned}
$$

Return to Exercise 12

Exercise 13.

$$\begin{aligned}
\int x^3 e^x dx &= x^3 e^x - \int 3x^2 e^x dx \\
&= x^3 e^x - 3\left\{x^2 e^x - \int 2x e^x dx\right\}, \qquad \text{using integration by parts,} \\
&= x^3 e^x - 3x^2 e^x + 6\int x e^x dx \\
&= x^3 e^x - 3x^2 e^x + 6\left\{x e^x - \int e^x dx\right\}, \text{using integration by parts,} \\
&= x^3 e^x - 3x^2 e^x + 6x e^x - 6e^x + C \\
&= e^x\left(x^3 - 3x^2 + 6x - 6\right) + C.
\end{aligned}$$

Return to Exercise 13

Exercise 14.

$$
\begin{aligned}
\text{Let } \; I &= \int e^{3x} \cos x \, dx \\
&= e^{3x} \sin x - \int 3e^{3x} \sin x \; dx \\
&= e^{3x} \sin x - 3 \int e^{3x} \sin x \, dx \\
&= e^{3x} \sin x - 3 \left\{ e^{3x} \cdot (-\cos x) - \int 3e^{3x} \cdot (-\cos x) \, dx \right\} \\
&= e^{3x} \sin x + 3e^{3x} \cos x \; - 9 \int e^{3x} \cos x \; dx \\
&= e^{3x} \sin x + 3e^{3x} \cos x - 9I \\
\text{i.e.} \quad 10I &= e^{3x} \sin x + 3e^{3x} \cos x + C_1 \\
\text{i.e.} \quad I &= \frac{1}{10} e^{3x} \left(\sin x + 3 \cos x \right) + C, \text{where } C = \frac{C_1}{10}.
\end{aligned}
$$

Return to Exercise 14