Finite Wing Theory

To date we have considered airfoil theory, or said another way, the theory of infinite wings. Real wings are, of course, finite with a defined length in the "z-direction."

Basic Wing Nomenclature

Wing Span, b – the length of the wing in the z-direction

Wing Chord, c – equivalent to the airfoil chord length

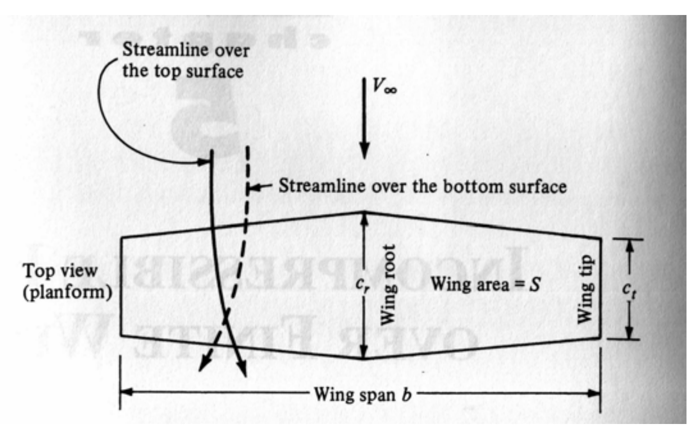
Wing Tip - the end of the wing in the span-wise direction

Wing Root – the center of the wing in the span-wise direction

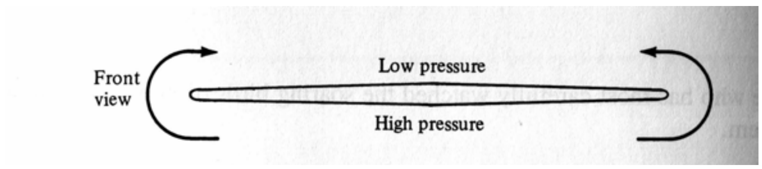
Wing Area, S

L', D', M' – two dimensional lift, drag and moment

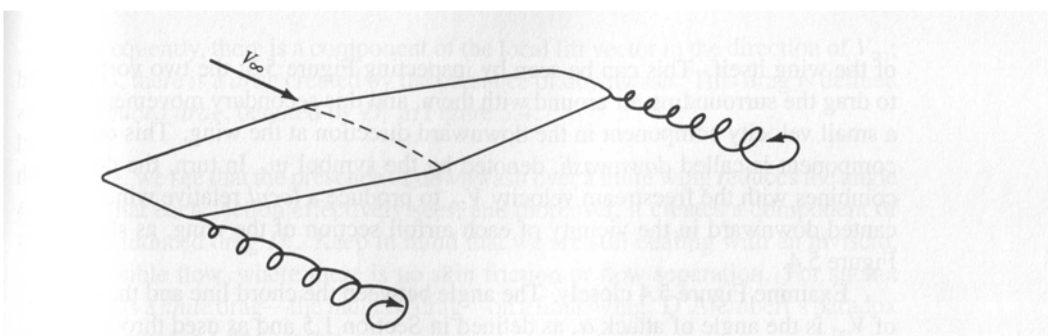
$\boldsymbol{C_L, C_D, C_M}$ - three dimensional lift drag and moment coefficients

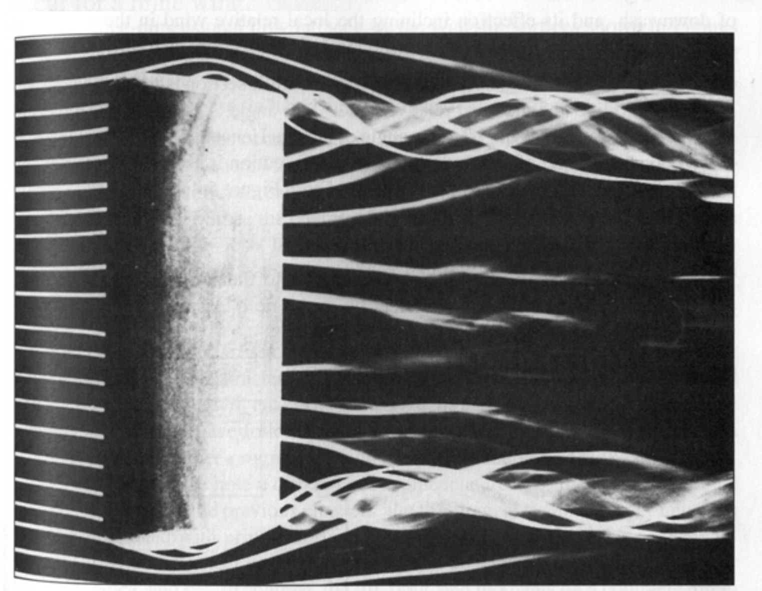


The flow over a finite wing is decidedly three dimensional, with considerable flow possible in the span-wise direction. This comes about because of the pressure difference between the top and bottom of the wing. As in two-dimensional fluid mechanics, the flow wants to move in the direction of a decreasing pressure gradient, i.e., it will usually travel from high to low pressure conditions. In effect the flow spills from the bottom to the top as shown in the figure below.

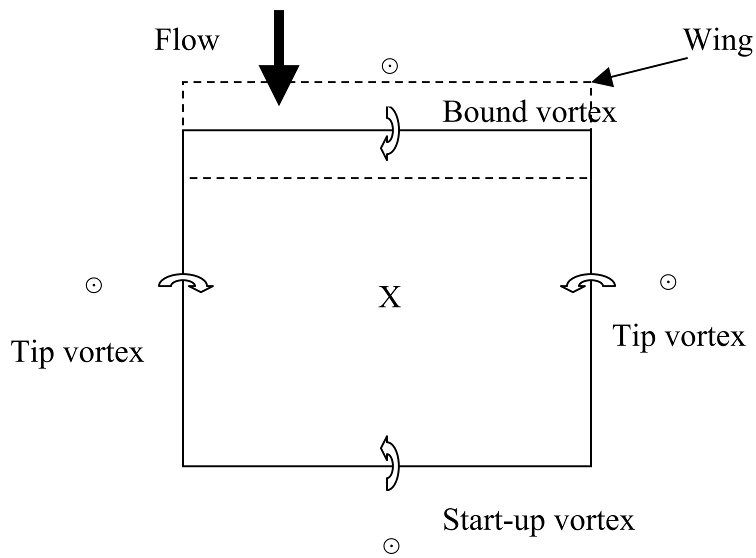


The span-wise rotation manifests itself as a wing tip vortex that continues downstream.





Interestingly the "sense" (orientation, rotation) of these wing tip vortices is consistent with taking the two-dimensional airfoil circulation, imagining that it exists off to infinity in both directions and bending it back at the wing tips. The idea is also consistent with Kelvin's theorem regarding the start-up vortex. Combining the two ideas one sees that a closed box-like vortex is formed. However, in much of the theory presented next the start-up vortex is ignored and we consider a *horseshoe vortex*.

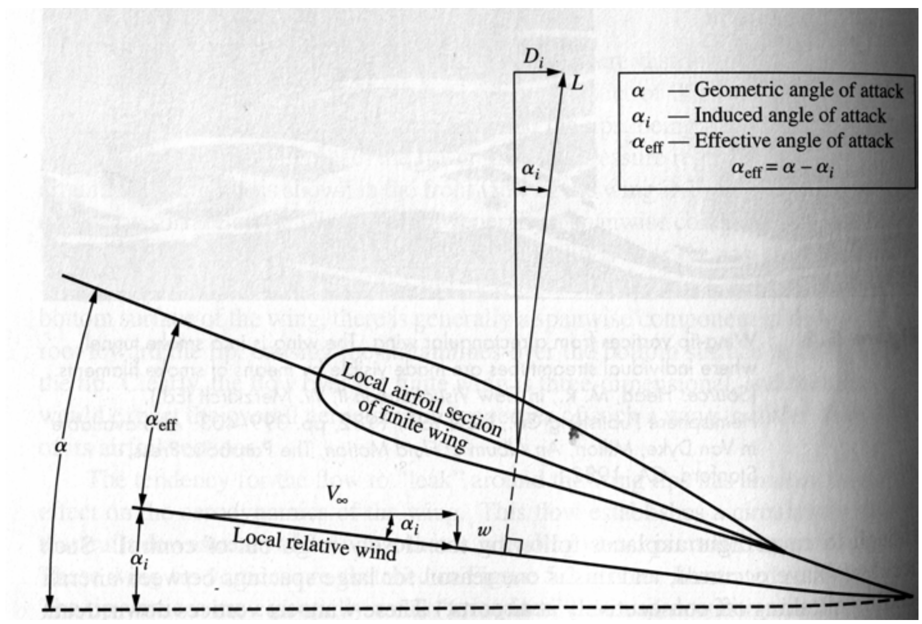


As shown in the figure, the vortices induce flow downward inside the box and upward outside the box. This flow is called the *induced velocity* or *downwash, w*. The strength of these vortices is directly related to the amount of lift generated on the wing. Aircraft inflight spacing is determined in part because of these wingtip vortices. An example is the Airbus aircraft that crashed at JFK a few days after 9/11. The spacing was too small and the Airbus's tail was buffeted by the wake vortices off a JAL 747 that was ahead of it in the flight path.

Angle of Attack

The idea that vortex motion induces downward flow changes the way we have to look at angle of attack as compared to the airfoil theory.

Geometric angle of attack, α - the angle between the airfoil chord line and the freestream velocity vector.



Induced angle of attack, α_i – the angle formed between the local relative wind and the undisturbed freestream velocity vector.

$$\tan \alpha_i = \frac{w}{V_\infty} \tag{5.1}$$

Effective angle of attack, α_{eff} – the angle formed between the airfoil chord and the local relative wind.

$$\alpha_{eff} = \alpha - \alpha_i \tag{5.2}$$

It is important to note that this also changes how we look at lift, *L,* and drag, *D*. This is because the actual lift is oriented perpendicular to the local relative wind (since that is the wind that it sees) not the freestream velocity direction. Because of that, when we go back to our original lift and drag directions (perpendicular and parallel to the freestream) we now see a reduction in the lift as compared to what we expect from the airfoil theory and an actual drag called the *induced drag,* even though the flow is still inviscid.

<u>What a Drag</u>

Induced drag, D_i – drag due to lift force redirection caused by the induced flow or downwash.

Skin friction drag, D_f – drag caused by skin friction.

Pressure drag, D_p – drag due to flow separation, which causes pressure differences between front and back of the wing.

Profile drag coefficient, C_d – sum of the skin friction and pressure drag. Can be found from airfoil tests. Note the notation.

$$C_d = \frac{D_f + D_p}{q_\infty S} \tag{5.3}$$

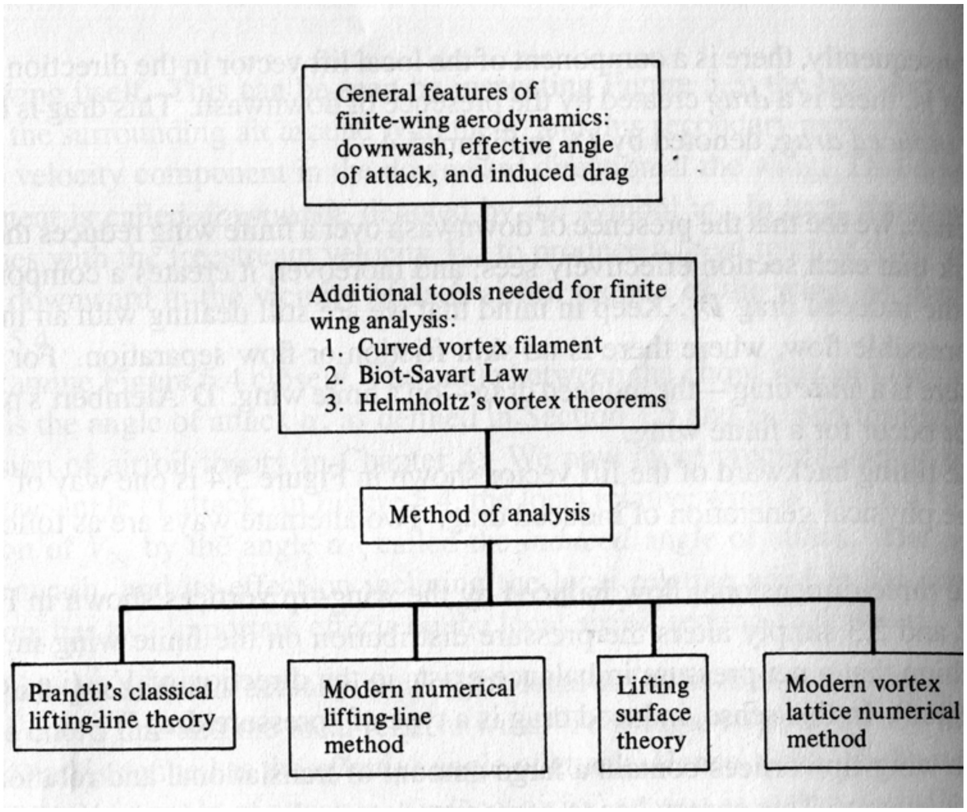
Induced drag coefficient, C_{D_i} - nondimensional induced drag

$$C_{D_i} = \frac{D_i}{q_\infty S} \tag{5.4}$$

Total drag coefficient, C_D

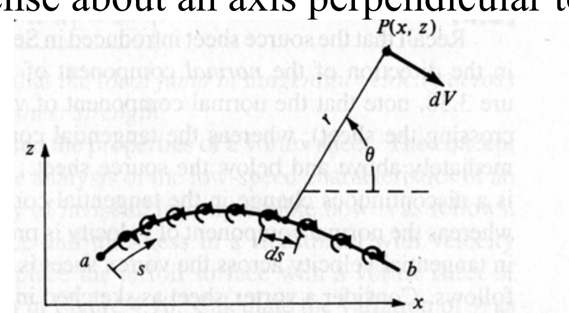
$$C_D = C_d + C_{D_i} \tag{5.5}$$

Airfoil data (C_d)   Finite wing theory (C_{D_i})



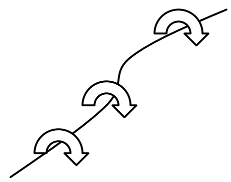
Biot-Savart Law

During our discussion of panel methods we developed the idea of a vortex sheet, essentially a line along which vorticity occurs that has a rotation sense about an axis perpendicular to the line.



vortex sheet

A similar but distinctly different idea is that of *vortex filament*, which is again a line of vorticity, but this time with rotation about the line itself.

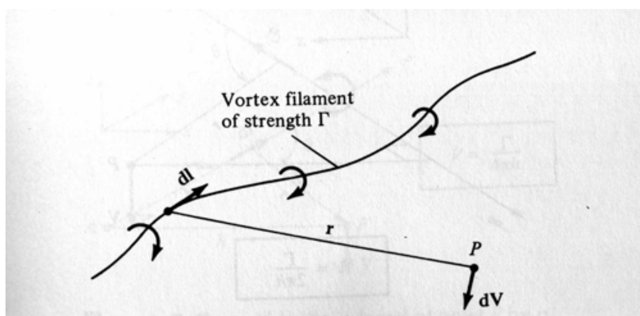


Vortex filament

The ***Biot-Savart Law*** defines the velocity induced by an infinitesimal length, ***dl***, of the vortex filament as

$$\boldsymbol{dV} = \frac{\Gamma}{4\pi}\frac{\boldsymbol{dl}\times\boldsymbol{r}}{\left|\boldsymbol{r}^3\right|} \tag{5.6}$$

where



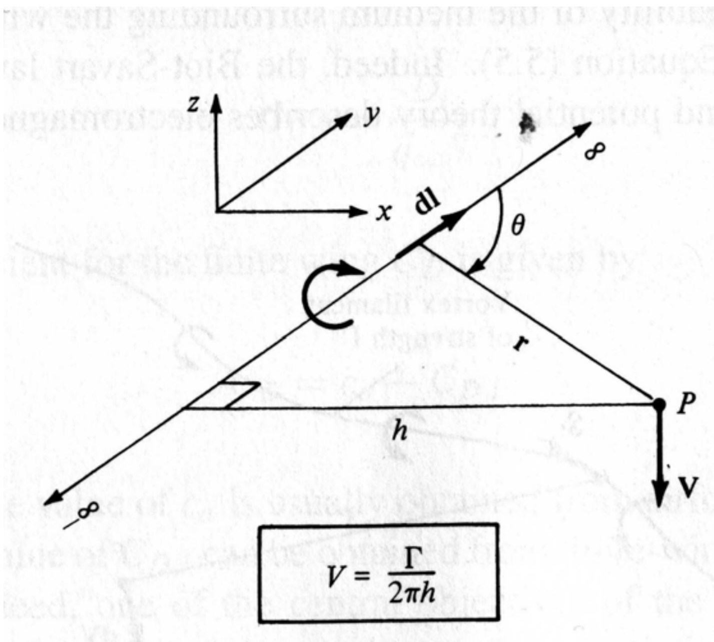
dl **–** infinitesimal length along the vortex filament
r – radius vector from ***dl*** to some point in space, P.
dV – infinitesimal induced velocity

Note that this velocity is perpendicular to both ***dl*** and ***r.***

If the vortex filament has infinite length the total induced velocity is found by integrating over its entire length

$$\boldsymbol{V} = \int_{-\infty}^{\infty}\frac{\Gamma}{4\pi}\frac{\boldsymbol{dl}\times\boldsymbol{r}}{\left|\boldsymbol{r}^3\right|} \tag{5.7}$$

Consider a straight vortex filament in the y-direction and a point, P, in the x-y plane. Equation (5.7) can be put into geometric functions by considering the figure below



$$\boldsymbol{V} = \int_{-\infty}^{\infty} \frac{\Gamma}{4\pi} \frac{\sin\theta}{\boldsymbol{r}^2} \boldsymbol{dl} \tag{5.8}$$

where θ is the angle formed by ***r*** and the filament. The geometry gives

$$\boldsymbol{r} = \frac{\boldsymbol{h}}{\sin\theta}, \boldsymbol{l} = \frac{\boldsymbol{h}}{\tan\theta}, \boldsymbol{dl} = \frac{\boldsymbol{h}}{\sin^2\theta} \boldsymbol{d\theta} \tag{5.9}$$

which gives

$$\boldsymbol{V} = -\frac{\Gamma}{4\pi \boldsymbol{h}} \int_{\pi}^{o} \sin \boldsymbol{\theta d\theta} \tag{5.10}$$

we get $\boldsymbol{l} \rightarrow \pm\infty$ if we have θ=*0* or π.
This leads to the simple result

$$\boldsymbol{V} = \frac{\Gamma}{2\pi \boldsymbol{h}} \tag{5.11}$$

same as the two dimensional theory.

If we have a "semi-infinite" filament we get

$$\boldsymbol{V} = \frac{\Gamma}{4\pi}\int_0^\infty \frac{\sin\theta}{\boldsymbol{r}^2}\boldsymbol{dl} = -\frac{\Gamma}{4\pi \boldsymbol{h}}\int_{\pi/2}^{0} \sin\theta \boldsymbol{d}\theta \tag{5.12}$$

$$\boldsymbol{V} = \frac{\Gamma}{4\pi \boldsymbol{h}}$$

<u>Helmholtz Theorem</u>

1. Strength of a vortex filament remains the same along the filament.
2. A vortex filament cannot end in a fluid, i.e., it must either extend to the boundaries or form a closed path.

<u>Additional Nomenclature</u>

Geometric twist – a twist of the wing about the span-wise axis that results in a change in the geometric angle of attack with span-wise position.

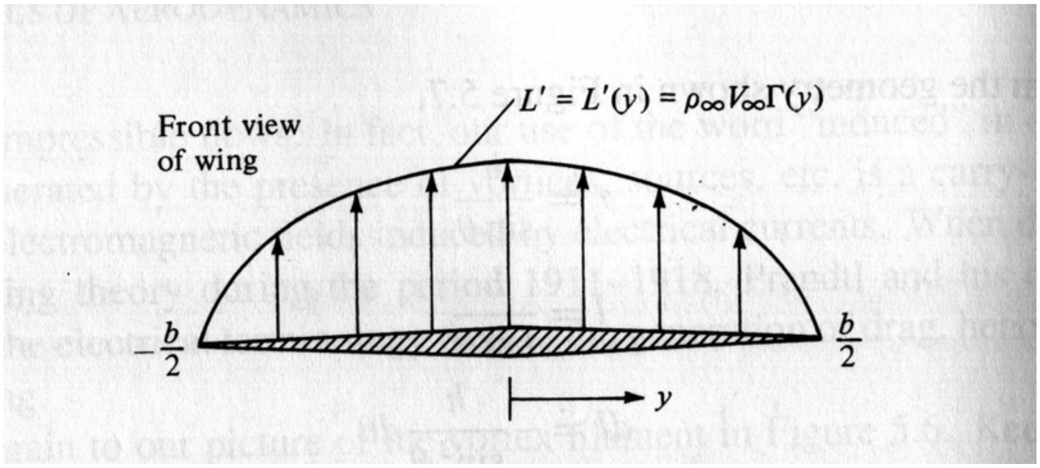
Washout – geometric twist such that $\alpha_{tip} < \alpha_{root}$

Washin - geometric twist such that $\alpha_{tip} > \alpha_{root}$

Aerodynamic twist – a wing with different airfoil sections along the span, so that the zero lift angle of attack changes with span-wise position.

Lift distribution – the local value of the lift force. This can change with span-wise position.

For example, since the pressure equalizes at the tip there is no lift there.

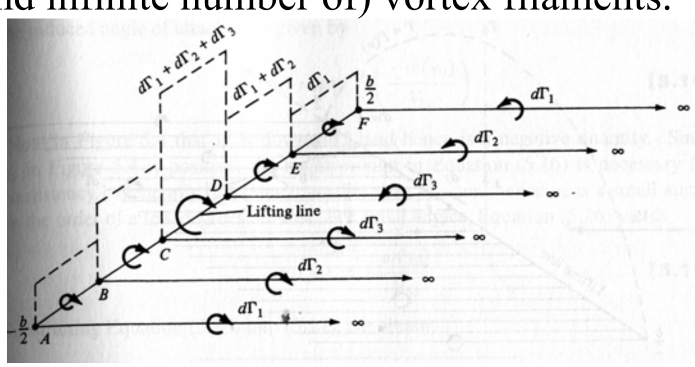


Lift per unit span, $\boldsymbol{L'}$ - akin to pressure, i.e., force per unit area.

$$\boldsymbol{L'} = \rho_\infty \boldsymbol{V}_\infty \Gamma(\boldsymbol{y}) \tag{5.13}$$

$$\boldsymbol{L} = \int_{-b/2}^{b/2} \boldsymbol{L'}(\boldsymbol{y})\boldsymbol{dy} \tag{5.14}$$

If the lift changes along the span it implies that there are multiple (perhaps and infinite number of) vortex filaments.



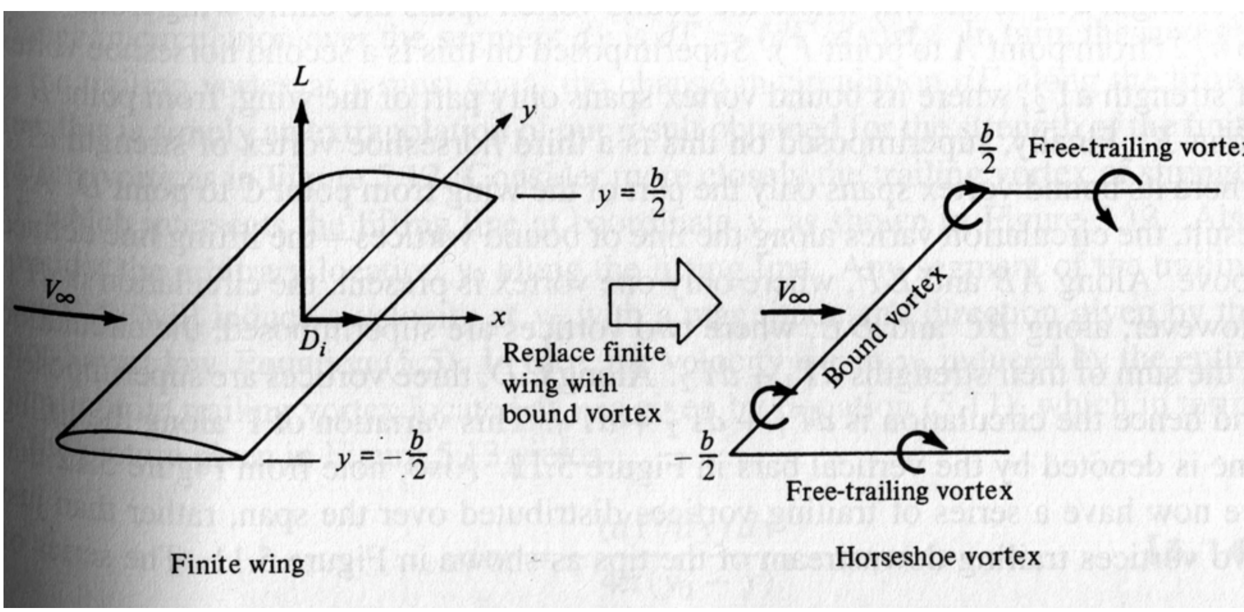
Prandtl's Lifting Line Theory

Prandtl's lifting line theory is centered about a fundamental integro-differential equation.

$$\alpha(y_o) = \frac{\Gamma(y_o)}{\pi V_\infty c(y_o)} + \alpha_{L=0}(y_o) + \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2} \frac{\left(d\Gamma/dy\right)dy}{y_o - y} \qquad (5.15)$$

which is used to find the circulation distribution about the wing. Equation (5.15) is useful if one knows the desired geometric angle of attack, the aerodynamic twist (i.e., $\alpha_{L=0}$), and the wing planform (i.e., local chord length). Two approaches are presented to make use of this equation. Equation (5.15) is developed from the idea of vortex filaments.

Prandtl's lifting line theory stems from the idea of replacing a wing with a bound vortex. Helmholtz theorem then requires there to exist trailing vortices at the wing tips



The Biot-Savart law allows us to determine the downwash along the wing and results in:

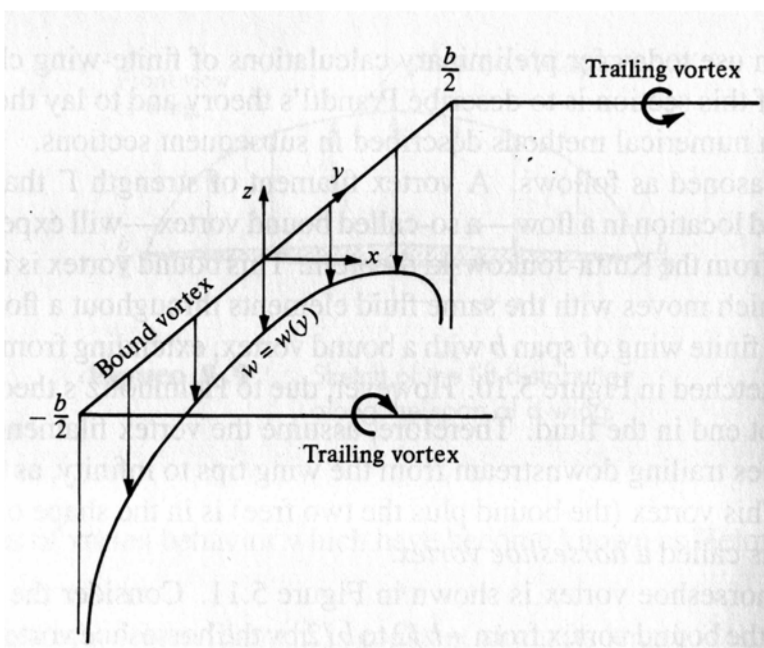
$$\boldsymbol{w}(\boldsymbol{y}) = -\frac{\Gamma}{4\pi\left(\boldsymbol{b}/2 + \boldsymbol{y}\right)} - \frac{\Gamma}{4\pi\left(\boldsymbol{b}/2 - \boldsymbol{y}\right)} \tag{5.16}$$

or

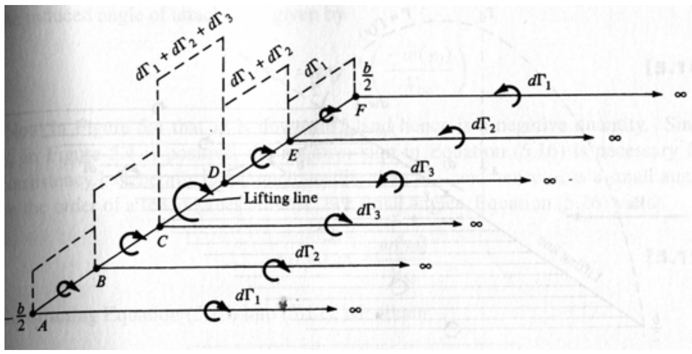
$$\boldsymbol{w}(\boldsymbol{y}) = -\frac{\Gamma}{4\pi}\frac{\boldsymbol{b}}{\left(\boldsymbol{b}/2\right)^2 - \boldsymbol{y}^2} \tag{5.17}$$

Note that the downwash is a negative number as you would expect from the coordinate system.

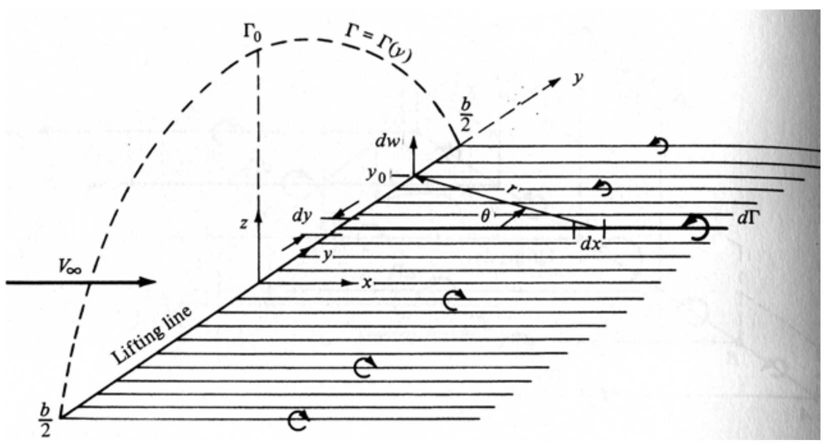
However, the single vortex filament case is not sufficient to describe the physical conditions on the wing, because of



The trouble is that the downwash at the wing tips is infinite instead of zero. Fortunately, this can be fixed if one considers a distribution of vortex filaments as shown below



It is important to note that the strength of the individual vortex filaments is equal to the jump in circulation at the point where the trailing vortex meets the bound vortex. This can be carried to the logical extreme by considering a continuous sheet of vortex filaments and their associated continuous change in circulation.



In that case the downwash induced at point y_o by the vortices at point y is given by

$$dw(y_o) = \frac{\left(d\Gamma / dy \right) dy}{4\pi (y_o - y)} \tag{5.18}$$

So that if one integrates from wingtip to wingtip

$$w(y_o) = -\frac{1}{4\pi}\int_{-b/2}^{b/2}\frac{\left(d\Gamma/dy\right)dy}{\left(y_o - y\right)} \tag{5.19}$$

Relationship between Γ distribution and downwash at y_o

Using Equation (5.1), but recognizing that *w* is a negative number

$$\alpha_i(y_o) = \tan^{-1}\left(\frac{-w(y_o)}{V_\infty}\right) \tag{5.1b}$$

for small angles Eq. (5.1b) gives

$$\alpha_i(y_o) = \frac{1}{4\pi V_\infty}\int_{-b/2}^{b/2}\frac{\left(d\Gamma/dy\right)dy}{\left(y_o - y\right)} \tag{5.20}$$

Recall the two dimensional lift coefficient for an airfoil

$$C_l = a_o\left\lfloor \alpha_{eff} - \alpha_{L=0}\right\rfloor = 2\pi\left\lfloor \alpha_{eff} - \alpha_{L=0}\right\rfloor \tag{5.21}$$

where

$\alpha_{eff} = \alpha_{eff}(y_o)$ because of downwash

$\alpha_{L=0} = \alpha_{L=0\,f}(y_o)$ because of aerodynamic twist

but

$$L' = \frac{1}{2}\rho_\infty V_\infty^2 c(y_o) C_l = \rho_\infty V_\infty \Gamma(y_o) \tag{5.22}$$

then

$$C_l = \frac{2\Gamma(y_o)}{V_\infty c(y_o)} \tag{5.23}$$

Combining Eqs. (5.21) & (5.23) gives

$$\alpha_{eff} = \frac{\Gamma(y_o)}{\pi V_\infty c(y_o)} + \alpha_{L=0} \tag{5.24}$$

which is clearly a function of y_o.

Recall notes Eq. (5.2)

$$\alpha_{eff} = \alpha - \alpha_i \tag{5.2}$$

and combine Eq. (5.2) with (5.24) & (5.20) to get

$$\alpha(y_o) = \frac{\Gamma(y_o)}{\pi V_\infty c(y_o)} + \alpha_{L=0}(y_o) + \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{\left(d\Gamma/dy\right)dy}{y_o - y} \tag{5.15}$$

Fundamental Equation of Prandtl's Lifting Line Theory

Once $\Gamma(y_o)$ is known L, C_L, D_i, C_{D_i} follow directly.

$$L'(y_o) = \rho_\infty V_\infty \Gamma(y_o) \tag{5.13}$$

$$L = \int_{-b/2}^{b/2} L'(y)dy \tag{5.14}$$

$$C_L = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y)\,dy \tag{5.25}$$

$$D_i = \int_{-b/2}^{b/2} L_i' \alpha_i \,dy \tag{5.26}$$

$$C_{D_i} = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y)\alpha_i(y)\,dy \tag{5.27}$$

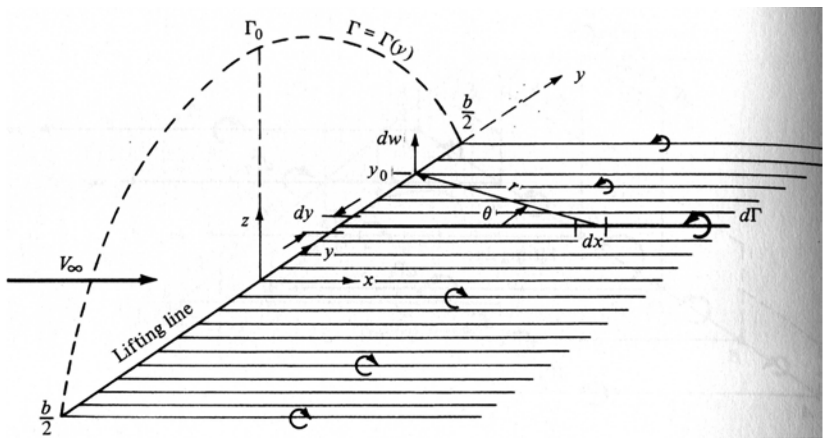
Two approaches can be taken from this point

1. *Direct* – A wing planform is given with a distribution of aerodynamic twist, Eq. (5.15) is solved and lift and drag information extracted.
2. *Inverse* – A lift distribution is proposed and the corresponding planform distribution developed.

Inverse Approach – Elliptic Lift Distribution

Prandtl's lifting line theory can be used in an inverse approach by assuming the form of the lift distribution and using Eq. (5.15) to determine the wing planform. The most famous example of this is the *elliptic lift distribution* which is found directly from an *elliptic circulation distribution.*

$$\Gamma(\boldsymbol{y}) = \Gamma_o \sqrt{1 - \left(\frac{2\boldsymbol{y}}{\boldsymbol{b}}\right)^2} \tag{5.28}$$



where we can use $\boldsymbol{L}'(\boldsymbol{y}) = \rho_\infty \boldsymbol{V}_\infty \Gamma(\boldsymbol{y})$ to show

$$\boldsymbol{L}'(\boldsymbol{y}) = \rho_\infty \boldsymbol{V}_\infty \Gamma_o \sqrt{1 - \left(\frac{2\boldsymbol{y}}{\boldsymbol{b}}\right)^2} \tag{5.29}$$

Recall that Eq. (5.19) requires $\frac{\boldsymbol{d}\Gamma}{\boldsymbol{dy}}$, so

$$\frac{\boldsymbol{d}\Gamma}{\boldsymbol{dy}} = -\frac{4\Gamma_o}{\boldsymbol{b}^2} \frac{\boldsymbol{y}}{\sqrt{1 - 4\boldsymbol{y}^2 / \boldsymbol{b}^2}} \tag{5.30}$$

so that the downwash becomes

$$w(y_o) = \frac{\Gamma_o}{\pi b^2} \int_{-b/2}^{b/2} \frac{y}{\sqrt{1 - 4y^2/b^2}} \frac{dy}{(y_o - y)} \tag{5.31}$$

we can again invoke geometry and use

$$y = \frac{b}{2}\cos\theta, \qquad dy = -\frac{b}{2}\sin\theta d\theta \tag{5.32}$$

where $\theta : \pi \to 0$ as $y : -\frac{b}{2} \to \frac{b}{2}$

So Eq. (5.31) becomes

$$w(\theta_o) = -\frac{\Gamma_o}{2\pi b} \int_{\pi}^{0} \frac{\cos\theta}{\cos\theta_o - \cos\theta} d\theta \tag{5.33}$$

or

$$w(\theta_o) = -\frac{\Gamma_o}{2\pi b} \int_{0}^{\pi} \frac{\cos\theta}{\cos\theta - \cos\theta_o} d\theta \tag{5.34}$$

which is a standard integral form

$$w(\theta_o) = -\frac{\Gamma_o}{2\pi b} \frac{\pi \sin n\theta_o}{\sin\theta_o} \qquad \text{with } n=1 \tag{5.35}$$

$$w(\theta_o) = -\frac{\Gamma_o}{2b} \tag{5.36}$$

Downwash is constant for an Elliptical Lift Distribution

However, this also implies:

$$\alpha_i = -\frac{w}{V_\infty} = \frac{\Gamma_o}{2bV_\infty} \tag{5.37}$$

Induced a.o.a. is constant for an Elliptical Lift Distribution

In the end we want to determine the lift and drag coefficient for the elliptic lift distribution and also the shape of this wing. To do this we go back to Eqs. (5.14) and (5.29)

$$L = \int_{-b/2}^{b/2} L'(y)dy = \rho_\infty V_\infty \Gamma_o \int_{-b/2}^{b/2} \sqrt{1-\left(\frac{2y}{b}\right)^2} dy$$

$$= \rho_\infty V_\infty \Gamma_o \int_{\pi}^{0} \sqrt{1-\cos^2\theta}\left(-\frac{b}{2}\right)\sin\theta d\theta$$

$$= \rho_\infty V_\infty \Gamma_o \frac{b}{2}\int_0^{\pi} \sqrt{\sin^2\theta}\sin\theta d\theta = \rho_\infty V_\infty \Gamma_o \frac{b}{2}\int_0^{\pi} \sin^2\theta d\theta \tag{5.38}$$

$$= \rho_\infty V_\infty \Gamma_o \frac{b}{2}\left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{\pi}\right] = \rho_\infty V_\infty \Gamma_o \frac{b}{2}\frac{\pi}{2}$$

$$L = \rho_\infty V_\infty \Gamma_o \frac{b}{4}\pi \tag{5.39}$$

We can next use the definition of C_L and Eq. (5.39) to give

$$L = \frac{1}{2}\rho_\infty V_\infty^2 S C_L = \rho_\infty V_\infty \Gamma_o \frac{b}{4}\pi \tag{5.40}$$

or

$$\Gamma_o = \frac{2V_\infty S C_L}{b\pi} \tag{5.41}$$

Then going back to Eq. (5.37) we find

$$\alpha_i = \frac{\Gamma_o}{2bV_\infty} = \frac{SC_L}{b^2\pi} \tag{5.42}$$

<u>Nomenclature</u>

Aspect Ratio, $AR = \frac{b^2}{S}$ (5.43)

So that

$$\alpha_i = \frac{C_L}{\pi AR} \tag{5.44}$$

We can then get induced drag from Eq. (5.27)

$$C_{D_i} = \frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)\alpha_i(y)dy \tag{5.27}$$

$$C_{D_i} = \frac{2\alpha_i}{V_\infty S} \int_{-b/2}^{b/2} \Gamma_o \sqrt{1 - \left(\frac{2y}{b}\right)^2} dy$$

$$= \frac{2\alpha_i \Gamma_o}{V_\infty S} \int_{\pi}^{0} \sqrt{1 - \cos^2\theta} \left(-\frac{b}{2}\right) \sin\theta d\theta$$

geometry

$$= \frac{\alpha_i \Gamma_o b}{V_\infty S} \int_0^{\pi} \sin^2\theta d\theta = \frac{\alpha_i \Gamma_o b}{V_\infty S} \left[\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta \Big|_0^{\pi} \right]$$

$$C_{D_i} = \frac{\alpha_i \Gamma_o b}{V_\infty S} \frac{\pi}{2} \qquad (5.45)$$

Which can be rewritten by substituting Eqs. (5.41) and (5.44) into (5.45)

$$C_{D_i} = \frac{C_L}{\pi AR} \frac{2V_\infty S C_L}{b\pi} \frac{b}{V_\infty S} \frac{\pi}{2}$$

$$C_{D_i} = \frac{C_L^2}{\pi AR} \qquad (5.46)$$

$C_{D_i} \propto C_L^2$ - a typical drag result

$C_{D_i} \propto \frac{1}{AR}$ - use high AR wing (long and thin)

But what's the geometry?!!!

The geometry can be found by going back to the lift coefficient and Eqs. (5.23), (5.21) and (5.2)

$$C_l = \frac{2\Gamma(y_o)}{V_\infty c(y_o)} \qquad (5.23)$$

$$C_l = 2\pi\lfloor\alpha_{eff} - \alpha_{L=0}\rfloor \qquad (5.21)$$

$$\alpha_{eff} = \alpha - \alpha_i \qquad (5.2)$$

Then using the idea that the induced a.o.a. is constant and if there is no aerodynamic twist we see from Eqs. (5.21) and (5.2) that

$$C_l = const. \qquad (5.47)$$

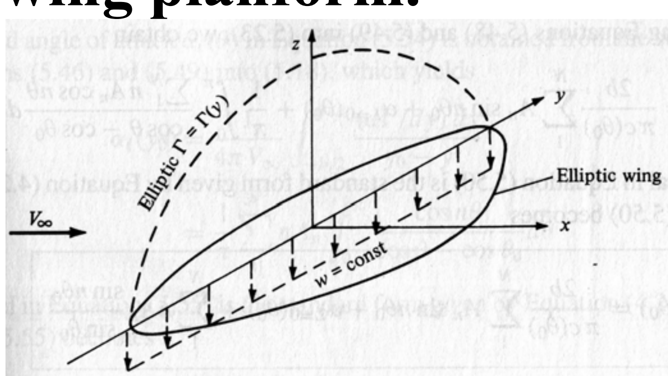
Elliptic Lift Distribution

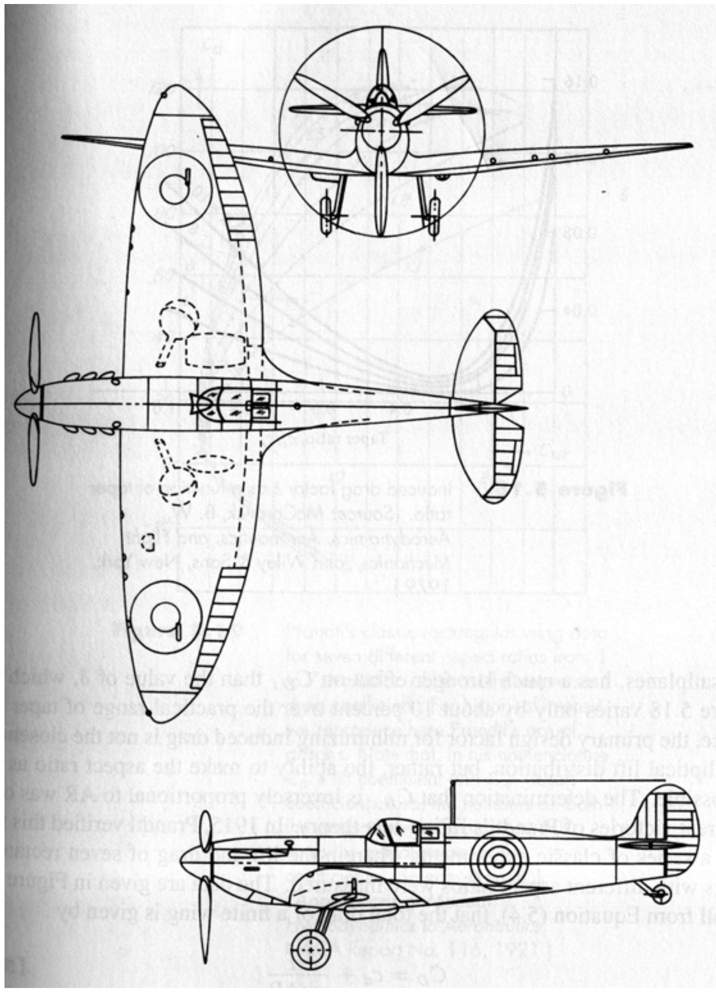
Combining Eqs. (5.47) and (5.23) gives

$$const. = \frac{2\Gamma(y_o)}{V_\infty c(y_o)} \Rightarrow c(y_o) = const._2\,\Gamma(y_o) \qquad (5.48)$$

With the end result:

An elliptic lift distribution is found from an elliptic wing planform.





Application of Prandtl's Lifting Line Theory for an Elliptic Wing

The previous section demonstrated that an elliptic wing planform develops an elliptic lift distribution but did not answer the question of how one can find the aerodynamic properties of an elliptic wing. Consider an elliptic wing

$$c(y) = c_r\sqrt{1-\left(\frac{2y}{b}\right)^2} \tag{5.49}$$

It is clear from Eq. (5.23) that C_l can be written in terms of Γ_o but Eqs. (5.21) & (5.2) also show that C_l depends on α_i, which in turn depends on C_L, which then depends on an integrated value of C_l. What's needed is a way to close the loop. To do this consider again Eqs. (5.23), (5.21) and (5.2)

$$C_l = \frac{2\Gamma_o}{V_\infty c_r} = 2\pi\left[\alpha_{eff} - \alpha_{L=0}\right]$$

$$= \frac{2\Gamma_o}{V_\infty c_r} = 2\pi\left[\alpha - \alpha_i - \alpha_{L=0}\right]$$

$$= 2\pi\left[\alpha - \frac{\Gamma_o}{2bV_\infty} - \alpha_{L=0}\right]$$

from which we see

$$\Gamma_o\left[\frac{2}{V_\infty c_r} + \frac{\pi}{V_\infty b}\right] = 2\pi\left[\alpha - \alpha_{L=0}\right]$$

$$\Gamma_o = \frac{2\pi V_\infty\left[\alpha - \alpha_{L=0}\right]}{\left[\dfrac{2}{c_r} + \dfrac{\pi}{b}\right]} \tag{5.50}$$

Therefore, given the root chord, span and airfoil shape Γ_o can be found. Upon rearranging Eq. (5.41)

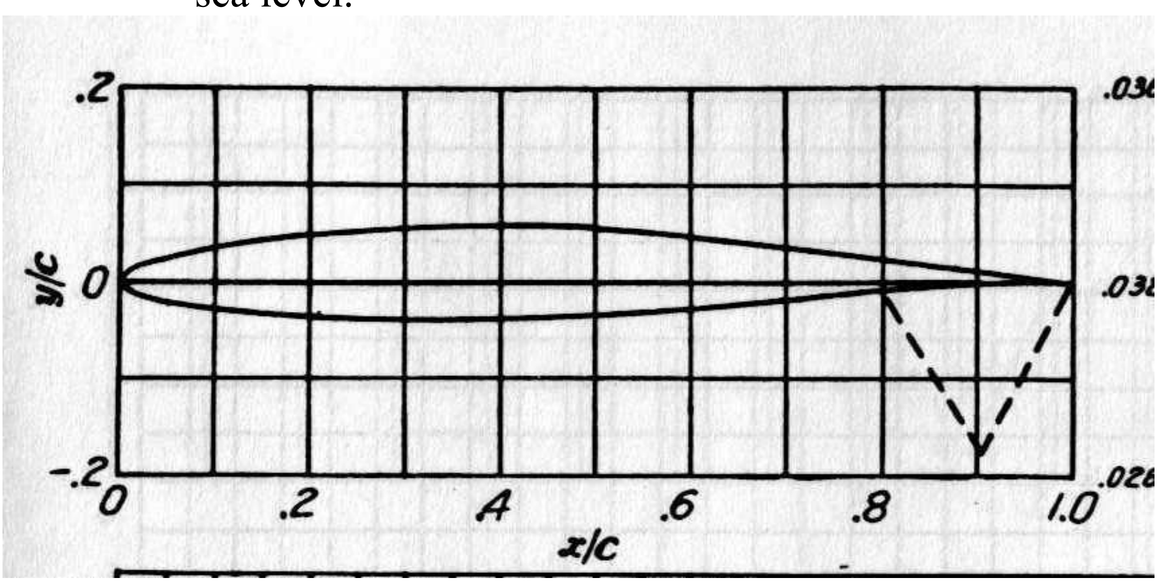
$$C_L = \frac{b\pi\Gamma_o}{2V_\infty S} \qquad (5.41b)$$

and from Eq. (5.46)

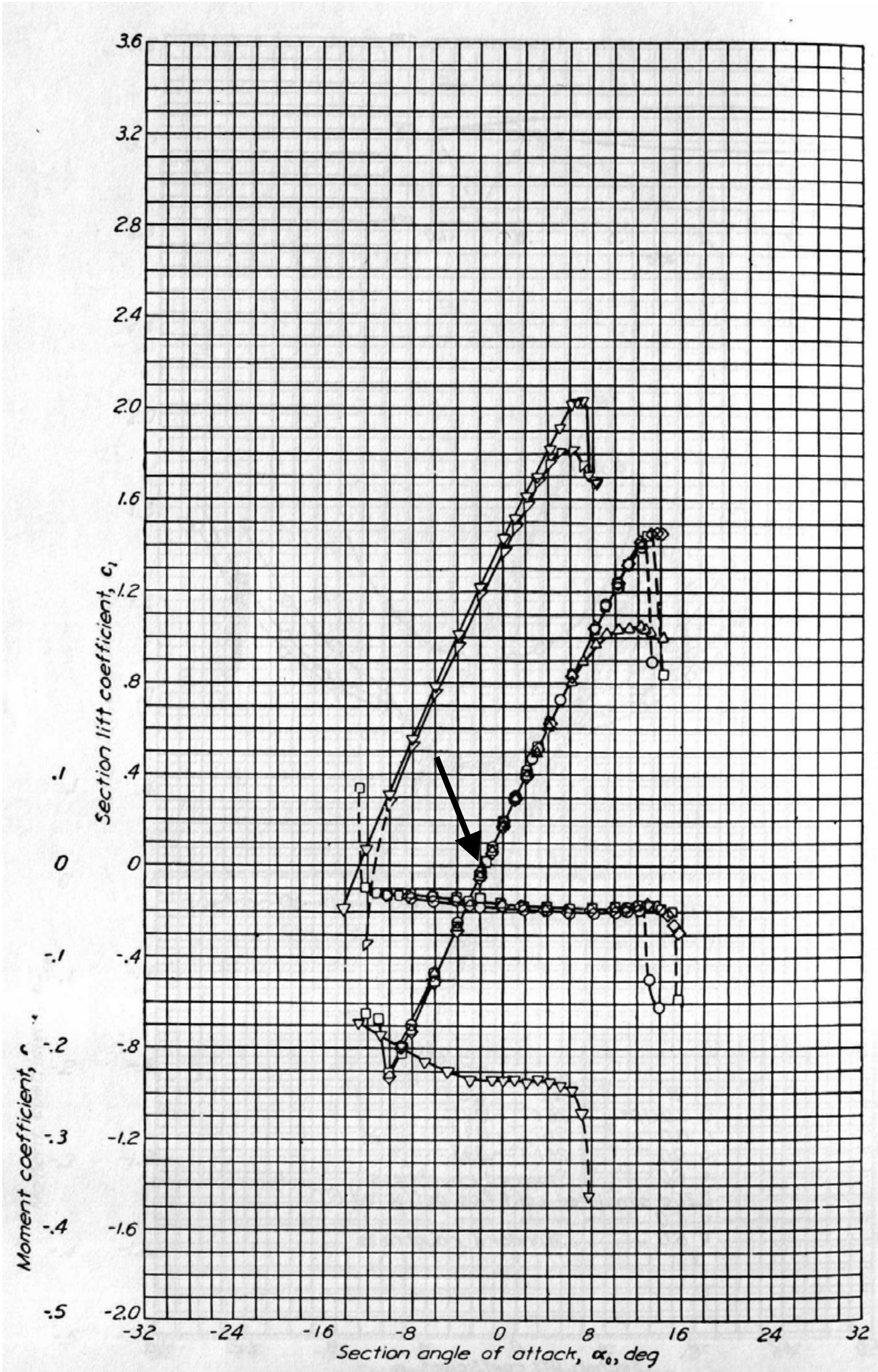
$$C_{D_i} = \frac{C_L^2}{\pi AR} \qquad (5.46)$$

Example Problem: Consider an elliptic wing with 10m span and 2.5m root chord. If the wing is made up of NACA 64-210 wing sections and is flying 50m/s at a geometric angle of attack of 8 degrees, compute

1. C_L and C_{D_i}
2. L and D_i
3. The acceleration of this wing if it has a mass of 1 Mg at sea level.



NACA 64-210



NACA 64-210 Wing Section

The chart shows that $\alpha_{L=0} \approx -1.8^o$, so that

$$\Gamma_o = \frac{2\pi(50\,m/s)\left[8^o - \left(-1.8^o\right)\right]}{\left[\dfrac{2}{2.5m} + \dfrac{\pi}{10m}\right]} = 48.23\frac{m^2 \cdot rad}{\sec}$$

$$S = \pi\frac{b}{2}\frac{c_r}{2} = \frac{\pi(10m)(2.5m)}{4} = 19.63m^2$$

$$AR = b^2/S = (10m)^2/19.63m^2 = 5.09$$

$$C_L = \frac{b\pi\Gamma_o}{2V_\infty S} = \frac{\pi(10m)\left(48.23\dfrac{m^2 \cdot rad}{\sec}\right)}{2(50\,m/s)(19.63m^2)}$$

$$C_L = 0.77$$

$$C_{D_i} = \frac{C_L^2}{\pi AR} = \frac{(0.77)^2}{\pi(5.09)} = 0.037$$

<u>Lift and Drag calculations</u>

$$L = \frac{1}{2}\rho_\infty V_\infty^2 S C_L$$

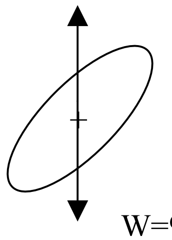
$$L = \frac{1}{2}\left(1.225\,kg/m^3\right)(50\,m/s)^2\left(19.63m^2\right)(0.77) = 23.1kN$$

$$D_i = \frac{1}{2}\rho_\infty V_\infty^2 S C_{D_i}$$

$$D_i = \frac{1}{2}\left(1.225\,kg/m^3\right)(50\,m/s)^2\left(19.63m^2\right)(0.037) = 1.1kN$$

So it can lift 2.38Mg

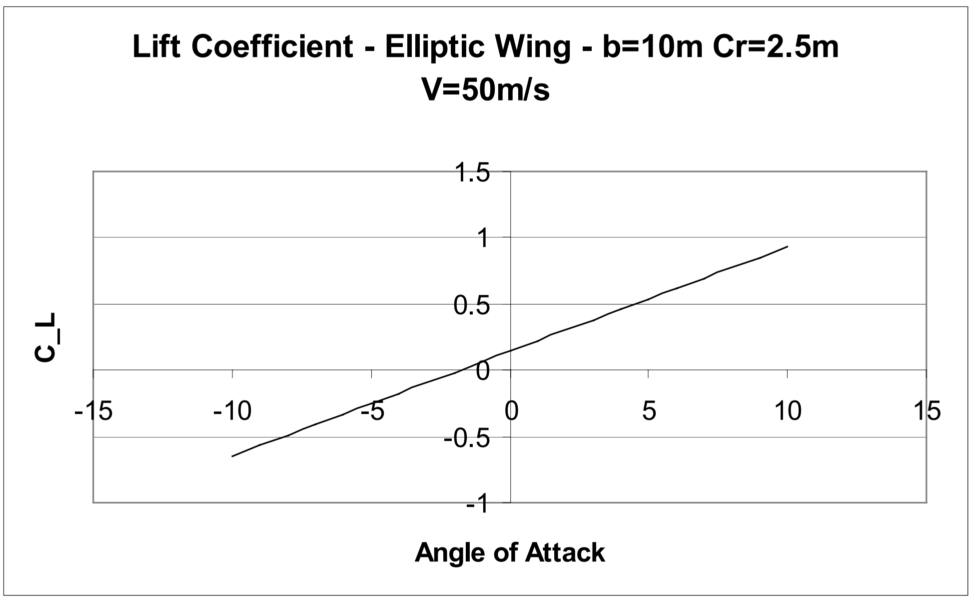
Acceleration calculations

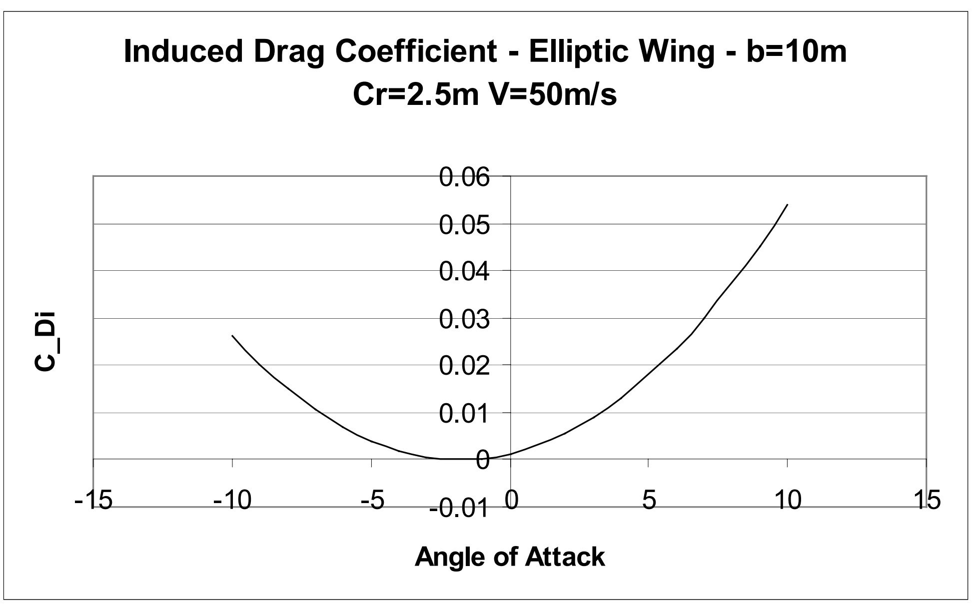


$$F_{net} = 23.1kN - 9.8kN = 13.3kN$$

$$\vec{a} \approx 1.36g$$

Exercise: Develop C_L and C_{D_i} over the range of a.o.a from $-10^o \Rightarrow +10^o$





Elliptical Wing Lessons: Design Considerations

Equations (5.50), (5.41b) & (5.46) give us a clear path for the aerodynamic analysis of an elliptical wing of given dimensions and airfoils. On the other hand, the analyst's job is to determine properties of a given wing, a designer's job is to decide the geometry itself so that it reaches a specific design objective. This is a distinctly different skill. The above equations can give the designer some insight if they are manipulated properly. Of course, one should be careful about drawing conclusions for a general from this analysis since it applies strictly to elliptic planforms, but it turns out that other wings behave similarly for many parameters, although their analysis is more complicated.

Start by again considering Eq. (5.50)

$$\Gamma_o = \frac{2\pi V_\infty[\alpha - \alpha_{L=0}]}{\left[\frac{2}{c_r} + \frac{\pi}{b}\right]} \qquad (5.50)$$

$$\boxed{S = \frac{\pi}{4} b c_r}$$

$$= \frac{2\pi V_\infty[\alpha - \alpha_{L=0}]}{\left[\frac{2b + \pi c_r}{c_r b}\right]} = \frac{2\pi c_r b V_\infty[\alpha - \alpha_{L=0}]}{2b + \pi c_r}$$

$$= \frac{8 S V_\infty[\alpha - \alpha_{L=0}]}{2b + \frac{4S}{b}} \qquad \boxed{c_r = \frac{4S}{\pi b}}$$

$$\Gamma_o = \frac{4 S V_\infty b[\alpha - \alpha_{L=0}]}{b^2 + 2S}$$

So that if S is a constant, Γ_o goes down as b goes up. However, this really doesn't tell us anything about lift and drag on the wing.

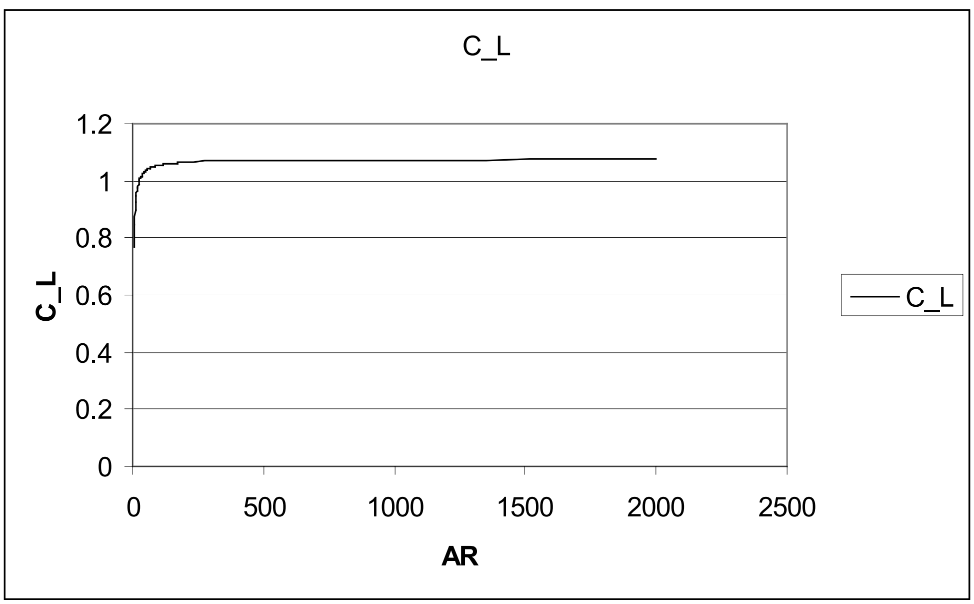
To get that information we need to consider Eq. (5.41b)

$$C_L = \frac{b\pi\Gamma_o}{2V_\infty S} \qquad (5.41b)$$

$$= \frac{b\pi}{2V_\infty S}\frac{4SV_\infty b[\alpha-\alpha_{L=0}]}{(b^2+2S)} = \frac{2\pi[\alpha-\alpha_{L=0}]}{1+2S/b^2}$$

$$C_L = \frac{2\pi[\alpha-\alpha_{L=0}]}{1+2/AR}$$

Therefore if keep the wing area the same C_L goes up with *b*. Note that this result drops back to the two-dimensional airfoil result as $b \to \infty$.



How then does this result affect the induced drag? Eq. (5.46) says:

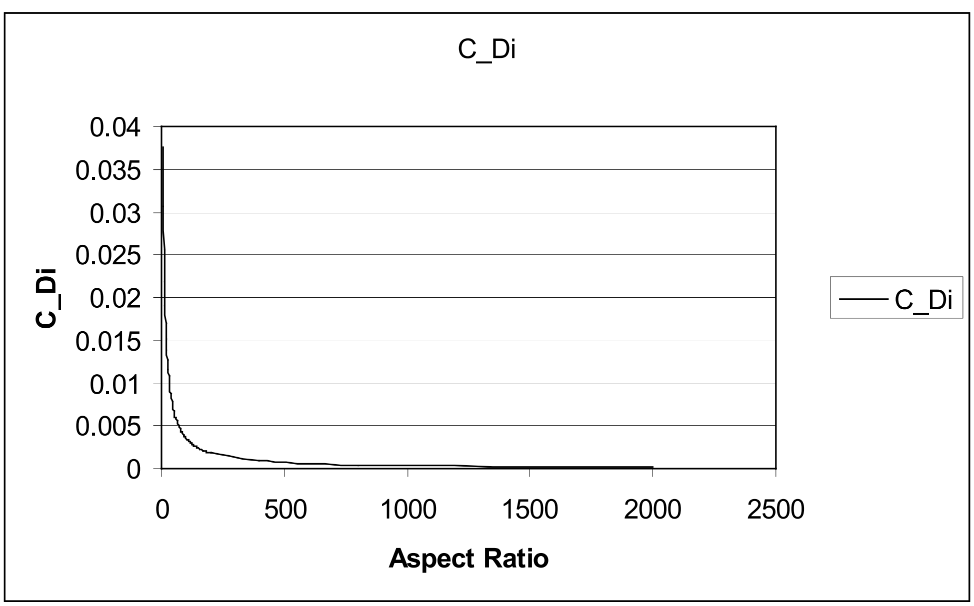
$$C_{D_i} = \frac{C_L^2}{\pi AR} = \frac{4\pi^2[\alpha - \alpha_{L=0}]^2}{\pi\left(1+\frac{2}{AR}\right)^2 AR}$$

$$= \frac{4\pi^2[\alpha - \alpha_{L=0}]^2}{\pi\left(1+\frac{4}{AR}+\frac{4}{AR^2}\right)AR}$$

$$C_{D_i} = \frac{4\pi[\alpha - \alpha_{L=0}]^2}{\left(AR+4+\frac{4}{AR}\right)}$$

which says that C_{D_i} decreases as b increases. This is because $C_L \rightarrow C_l$ as $b \rightarrow \infty$ and hence $C_{D_i} \rightarrow 0$.

Hence the compromise comes to life between what can be built and what is aerodynamically efficient.



General Lift Distribution

Our previous analysis for the elliptical wing utilized an elliptic circulation distribution shown in Eq. (5.28)

$$\Gamma(\boldsymbol{y}) = \Gamma_o \sqrt{1-\left(\frac{2\boldsymbol{y}}{\boldsymbol{b}}\right)^2} \tag{5.28}$$

This form was simplified by the geometric transformation

$$\boldsymbol{y} = \frac{\boldsymbol{b}}{2}\cos\boldsymbol{\theta}, \qquad \boldsymbol{dy} = -\frac{\boldsymbol{b}}{2}\sin\boldsymbol{\theta}\boldsymbol{d}\boldsymbol{\theta} \tag{5.32}$$

When combined, Eq. (5.28) becomes

$$\begin{aligned}\Gamma(\boldsymbol{y}) &= \Gamma_o \sqrt{1-\cos^2\boldsymbol{\theta}} \\ &= \Gamma_o \sin\boldsymbol{\theta}\end{aligned}$$

The utility of this transformation is apparent, however, its impact is much bigger because it suggests the use of a sine series to represent any circulation distribution. The basic idea being that the circulation can be written as

$$\Gamma(\boldsymbol{y}) = 2\boldsymbol{b}\boldsymbol{V}_\infty \sum_{\boldsymbol{n}=1}^{\boldsymbol{N}} \boldsymbol{A}_{\boldsymbol{n}} \sin\boldsymbol{n}\boldsymbol{\theta} \tag{5.51}$$

Why should this work?

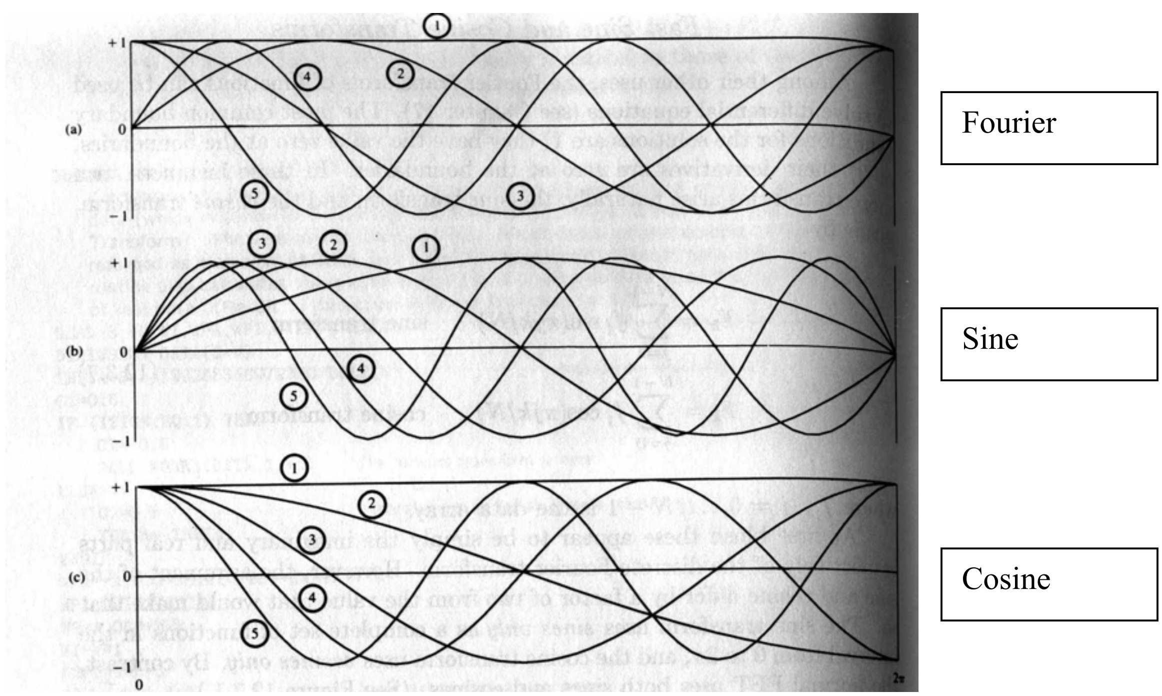
First off, let's assume that this series *is* a reasonable approximation of the actual circulation function and further assume that you can find values for the *N* constants, $\boldsymbol{A}_{\boldsymbol{n}}$. Then recall what was done for the elliptical wing planform; we started things out not knowing what is Γ_o, but by rearranging equations we were able to determine it in terms of the velocity, root chord, span and airfoil shape. In addition, all of this information was obtained at a single location, the wing root.

If we use equation (5.51) for a general airfoil, there are now N unknown coefficients needed to determine the circulation. These coefficients will be determined later by using conditions and geometry at N locations on the wing. However, before we do that it is important to recognize why Eq. (5.51) might be reasonable.

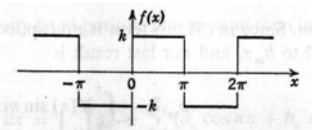
<u>Fourier Sine Series</u>

A very useful engineering tool is the Fourier Series, which is essentially a summation function composed of sines, cosines or both depending on the function to be represented. Mathematicians have proven that *any* function can be represented by infinite series of this form and practical experience shows that only a limited number of terms are needed to get a reasonable approximation.

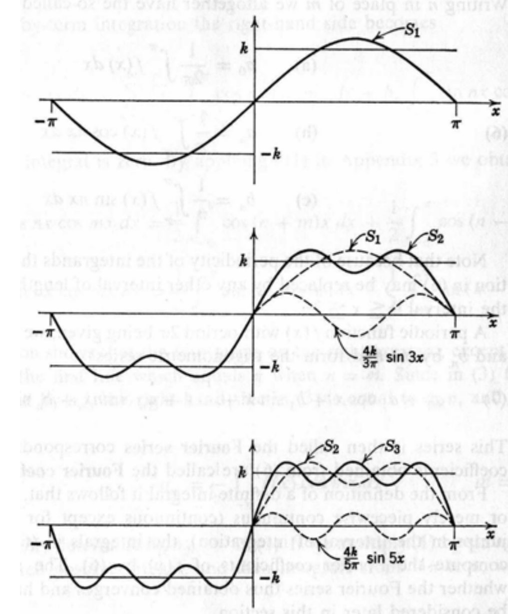
The functions themselves look as shown below for the first 5 terms



A severe example of its application for a square and a sawtooth wave are shown below



(a) The given function $f(x)$ (Periodic square wave)



(b) The first three partial sums of the corresponding Fourier series

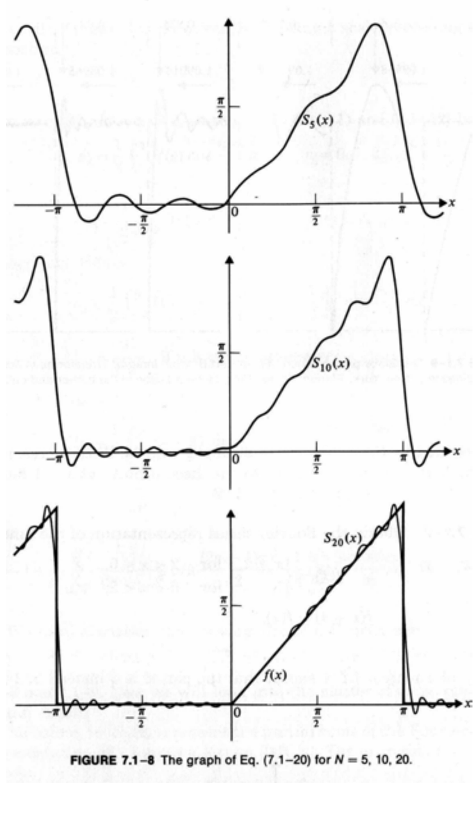


FIGURE 7.1–8 The graph of Eq. (7.1–20) for $N = 5, 10, 20$.

Dr. Orkwis has used this approximation approach to represent the wakes behind a turbine stator and found that only 4 terms were needed to get about 95% of the total energy. The approximation functions are generally good representations of the actual functions if the actual function is smooth. Discontinuous functions like square or saw tooths produce "ringing" or "Gibb's phenomenon" unless a very large number of terms are included. Fortunately, wing circulation distributions are usually quite smooth and require relatively few terms. Keep in mind that finite wing theory requires both $\Gamma(\boldsymbol{y})$ and $\dfrac{\boldsymbol{d}\Gamma}{\boldsymbol{dy}}$, so ringing can be a problem as the derivative can be badly distorted even though the function is well represented.

Application to Prandtl's Lifting Line Theory

As stated above, Eq. (5.51) must be differentiated to be used in the Prandtl Lifting Line Theory. We get

$$\frac{d\Gamma}{dy} = \frac{d\Gamma}{d\theta}\frac{d\theta}{dy} = 2bV_\infty \sum_{n=1}^{N} nA_n \cos n\theta \frac{d\theta}{dy} \tag{5.52}$$

Substitute the above into Eq. (5.15)

$$\alpha(\theta_o) = \frac{2b}{\pi c(\theta_o)}\sum_{n=1}^{N} A_n \sin n\theta_o + \alpha_{L=0}(\theta_o)\frac{d\theta}{dy} + \frac{1}{\pi}\int_{\pi}^{0} \frac{\sum_{n=1}^{N} nA_n \cos n\theta}{\cos\theta_o - \cos\theta} d\theta \tag{5.53}$$

or

$$\alpha(\theta_o) = \frac{2b}{\pi c(\theta_o)}\sum_{n=1}^{N} A_n \sin n\theta_o + \alpha_{L=0}(\theta_o)\frac{d\theta}{dy} + \frac{1}{\pi}\int_{0}^{\pi} \frac{\sum_{n=1}^{N} nA_n \cos n\theta}{\cos\theta - \cos\theta_o} d\theta$$

where the integral is the familiar standard form used earlier, so that

$$\alpha(\theta_o)=\frac{2b}{\pi c(\theta_o)}\sum_{n=1}^{N}A_n\sin n\theta_o+\alpha_{L=0}(\theta_o)\frac{d\theta}{dy}+\sum_{n=1}^{N}nA_n\frac{\sin n\theta_o}{\sin\theta_o} \qquad \boxed{(5.54)}$$

Which can be used to find the *N*, A_n's if Eq. (5.54) is evaluated at *N*, θ_o values. A system of equations is thus developed and easily solved. We should recognize that the original elliptic wing still lives in this equation if we compare Eq. (5.54) to Eq. (5.15) and note that *N=1*.

As before, everything follows once $\Gamma(\theta_o)$ is known. C_L comes from Eq. (5.25)

$$C_L=\frac{2}{V_\infty S}\int_{-b/2}^{b/2}\Gamma(y)dy \qquad (5.25)$$

$$\begin{aligned} C_L&=\frac{2}{V_\infty S}\int_{\pi}^{0}2bV_\infty\sum_{n=1}^{N}A_n\sin n\theta\left(-\frac{b}{2}\sin\theta\right)d\theta \\ &=\frac{2b^2}{S}\sum_{n=1}^{N}A_n\int_0^{\pi}\sin n\theta\sin\theta d\theta \end{aligned} \qquad (5.55)$$

and, of course, an integral table will reveal that

$$\int_0^{\pi}\sin n\theta\sin\theta d\theta=\begin{cases}\pi/2 & n=1\\ 0 & n\neq 1\end{cases} \qquad (5.56)$$

So no matter how many terms you have in the series, it is only the first one that matters for C_L, i.e.,

$$C_L=\frac{2b^2}{S}\pi A_1=A_1\pi AR \qquad (5.57)$$

Keep in mind though that A_1 is part of a system of equations and as such depends upon $A_2\ldots A_N$.

Next, for the induced drag we need α_i as indicated by Eq. (5.27). Again, Eq. (5.20) says

$$\alpha_i = \frac{1}{4\pi V_\infty} \int_{-b/2}^{b/2} \frac{\left(d\Gamma / dy \right)}{(y_o - y)} dy \qquad (5.20)$$

which by comparison to Eq. (5.53) is

$$\alpha_i = \frac{1}{\pi} \int_0^\pi \frac{\sum_{n=1}^{N} nA_n \cos n\theta}{\cos\theta - \cos\theta_o} d\theta$$

or from Eq. (5.54)

$$\alpha_i = \sum_{n=1}^{N} nA_n \frac{\sin n\theta_o}{\sin\theta_o} \qquad (5.58)$$

We then go back to Eq. (5.27)

$$C_{D_i} = \frac{2}{V_\infty S} \int_{-b/2}^{b/2} \Gamma(y) \alpha_i(y) dy \qquad (5.27)$$

$$C_{D_i} = \frac{2}{V_\infty S} \int_\pi^0 2bV_\infty \left(\sum_{n=1}^{N} A_n \sin n\theta \right) \left(\sum_{n=1}^{N} nA_n \frac{\sin n\theta}{\sin\theta} \right) \left(-\frac{b}{2} \sin\theta \right) d\theta$$

Note θ not θ_o

Upon further evaluation

$$C_{D_i} = 2AR\int_0^{\pi}\left(\sum_{n=1}^{N} A_n \sin n\theta\right)\left(\sum_{m=1}^{N} mA_m \sin m\theta\right)d\theta \tag{5.59}$$

Which is easily evaluated since

$$\int_0^{\pi} \sin m\theta \sin k\theta d\theta = \begin{cases} \pi/2 & m = k \\ 0 & m \neq k \end{cases} \tag{5.56}$$

Hence, the only time the integrals yield something is when $n=m$. So

$$C_{D_i} = 2AR\frac{\pi}{2}\sum_{n=1}^{N} nA_n^2 \tag{5.61}$$

or equivalently

$$C_{D_i} = \pi AR\left[A_1^2 + \sum_{n=2}^{N} nA_n^2\right]$$

$$= \pi ARA_1^2\left[1 + \sum_{n=2}^{N} n\left(\frac{A_n}{A_1}\right)^2\right]$$

which we can write as

$$C_{D_i} = \pi ARA_1^2\left[1+\delta\right]$$

$$C_{D_i} = \frac{C_L^2}{\pi AR}\left[1+\delta\right] \tag{5.62}$$

Eq. (5.62) should be compared with Eq. (5.46) for the elliptic wing. Yet another way to write Eq. (5.62) is

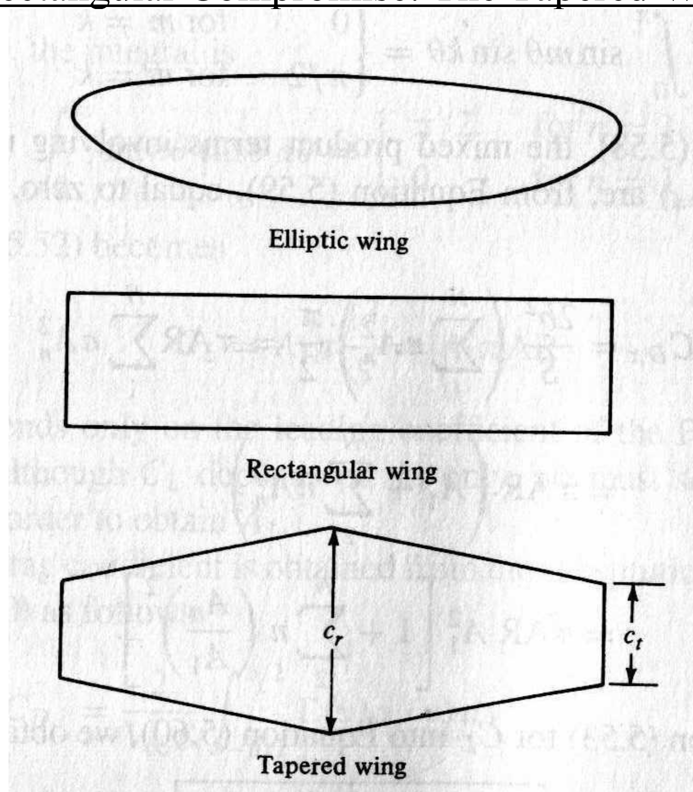
$$C_{D_i} = \frac{C_L^2}{\pi eAR} \tag{5.63}$$

Nomenclature

$$\textit{Span efficiency factor, } e \text{ – where } \boldsymbol{e} = \frac{1}{1+\delta} \qquad (5.64)$$

Clearly *e=1* for an elliptic wing.

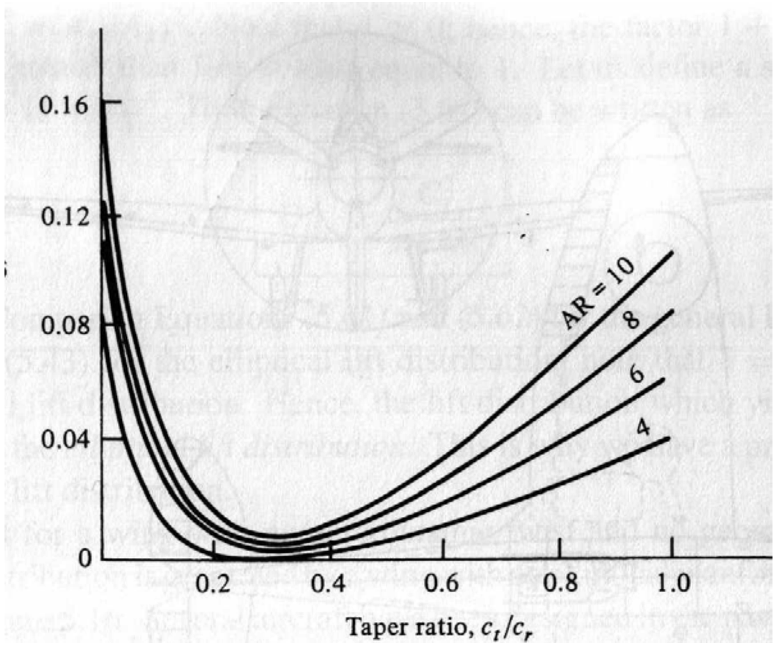
Elliptic vs. Rectangular Compromise: The Tapered Wing



Nomenclature

$$\textit{Taper Ratio} \equiv \frac{\boldsymbol{c}_t}{\boldsymbol{c}_r}$$

The idea is to match closely the elliptic wing planform, i.e., chord lengths of similar size, and thereby match the elliptic lift distribution.



Clearly, δ has a minimum for a taper ratio of about *0.3*.

Numerical Lifting Line Theory

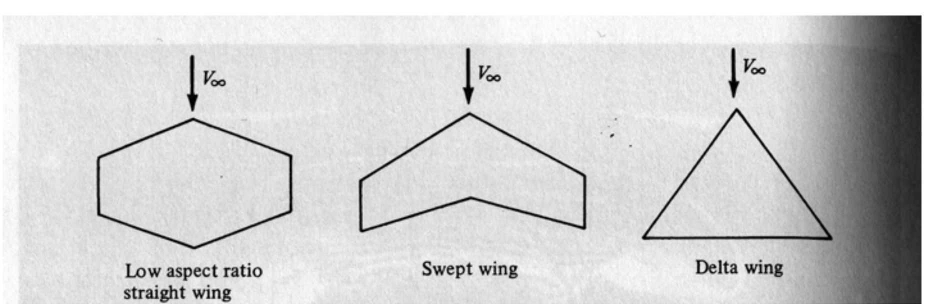
The lifting line theory assumes a linear lift curve slope, as such, it does not predict the nonlinear or stall regime. Anderson describes a numerical lifting line process

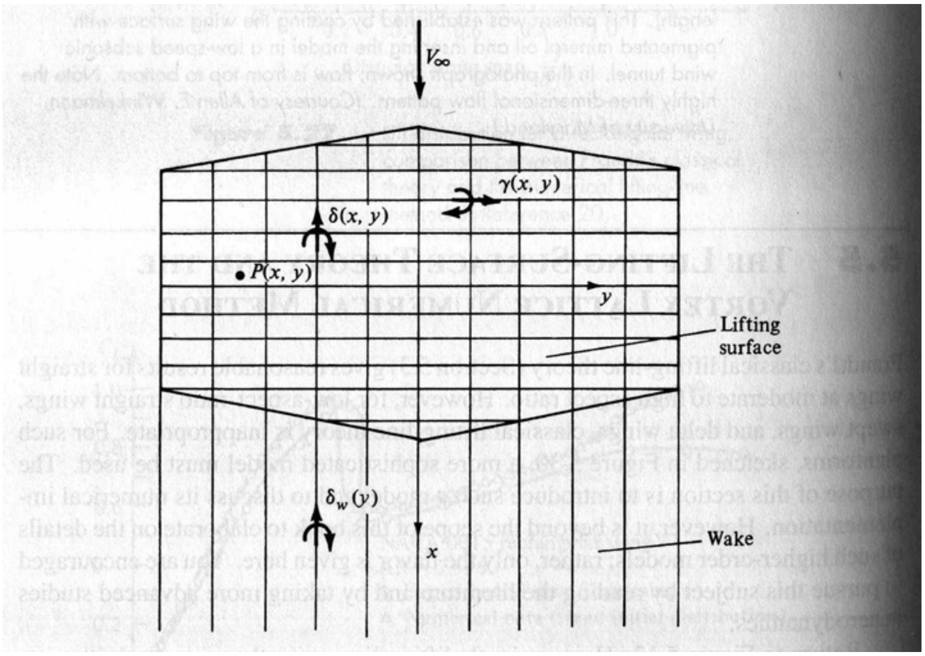
- Assume a $\Gamma(\boldsymbol{y})$
- Calculate α_i
- Calculate α_{eff}
- Use tabulated data for $\boldsymbol{C}_L$ to compute $\Gamma(\boldsymbol{y})$
- Iterate until convergence

The beauty of the technique is that it works in the nonlinear/stall regime, however, it requires significant table look-ups, which are slow.

Vortex Lattice Method

A disadvantage of the lifting line theory is that all of the action associated with the bound vortex occurs at the quarter chord point, such that only the lift and drag coefficients are computed but not the moment coefficient. Unfortunately, the moment coefficient is essential to the performance calculations. An answer is found in the *vortex lattice method* which not only provides the pressure distribution but also anchors the results to the actual geometry rather than implicitly through the $\alpha_{L=0}$. This is essential not only for calculating moments but for many practical wing planforms like delta wings.





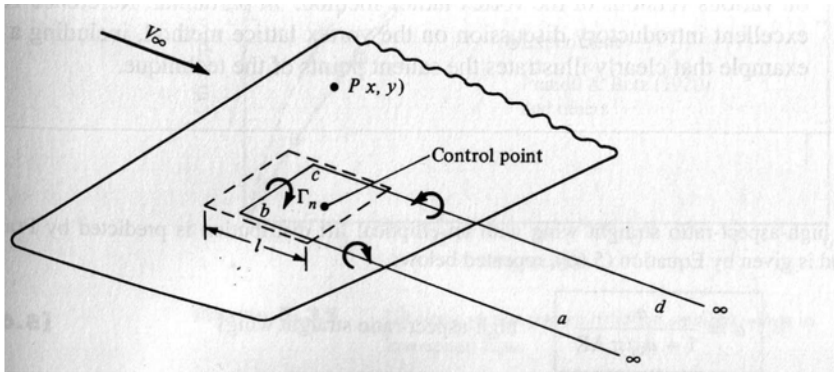
Essential ideas:

- Panel the wing with discrete spanwise, γ, and streamwise, δ, distribution of vorticies.
- Set a "control point" somewhere on this panel to apply the flow tangency condition.
- Biot-Savart to determine the induced velocity from all points.
- Solution of a system of equations determines the discrete vorticity distributions via the downwash equation:

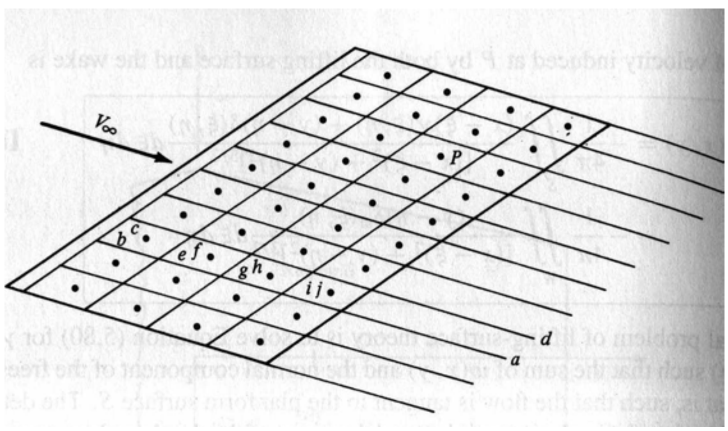
$$
\begin{aligned}
w(x,y) = &-\frac{1}{4\pi}\iint_S \frac{(x-\xi)\gamma(\xi,\eta)+(y-\eta)\delta(\xi,\eta)}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}\,d\xi\, d\eta \\
&-\frac{1}{4\pi}\iint_W \frac{(y-\eta)\delta_W(\xi,\eta)}{\left[(x-\xi)^2+(y-\eta)^2\right]^{3/2}}\,d\xi\, d\eta
\end{aligned}
\tag{5.65}
$$

Note that the wake is also included.

Another way to look at this is as a system of horseshoe vortices, each one applied over a single panel with a bound vortex at the quarter point of the panel. The control point is again defined to apply the surface tangency condition.



The wing is then tessellated with a vortex lattice system like that shown in the figure.



Once again the velocity contributions are determined from each horseshoe vortex and using the flow tangency condition and we are left with a system of equations for the unknown circulations.

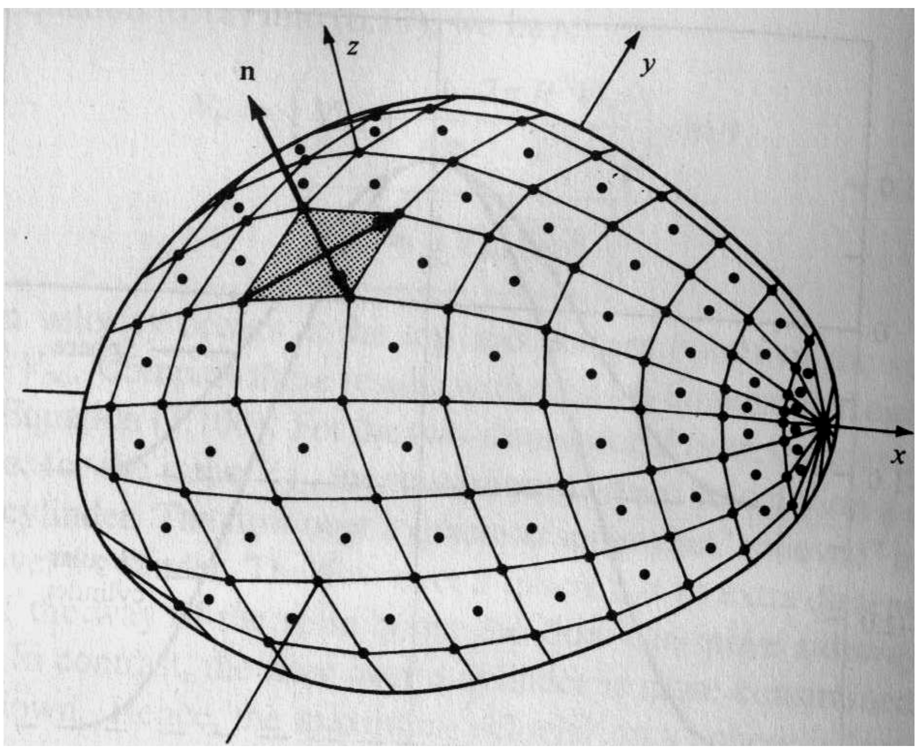
Note that this approach is applied in the plane of the airfoil, not along the airfoil so that we are really still making a thin airfoil assumption.

3D Panel Methods

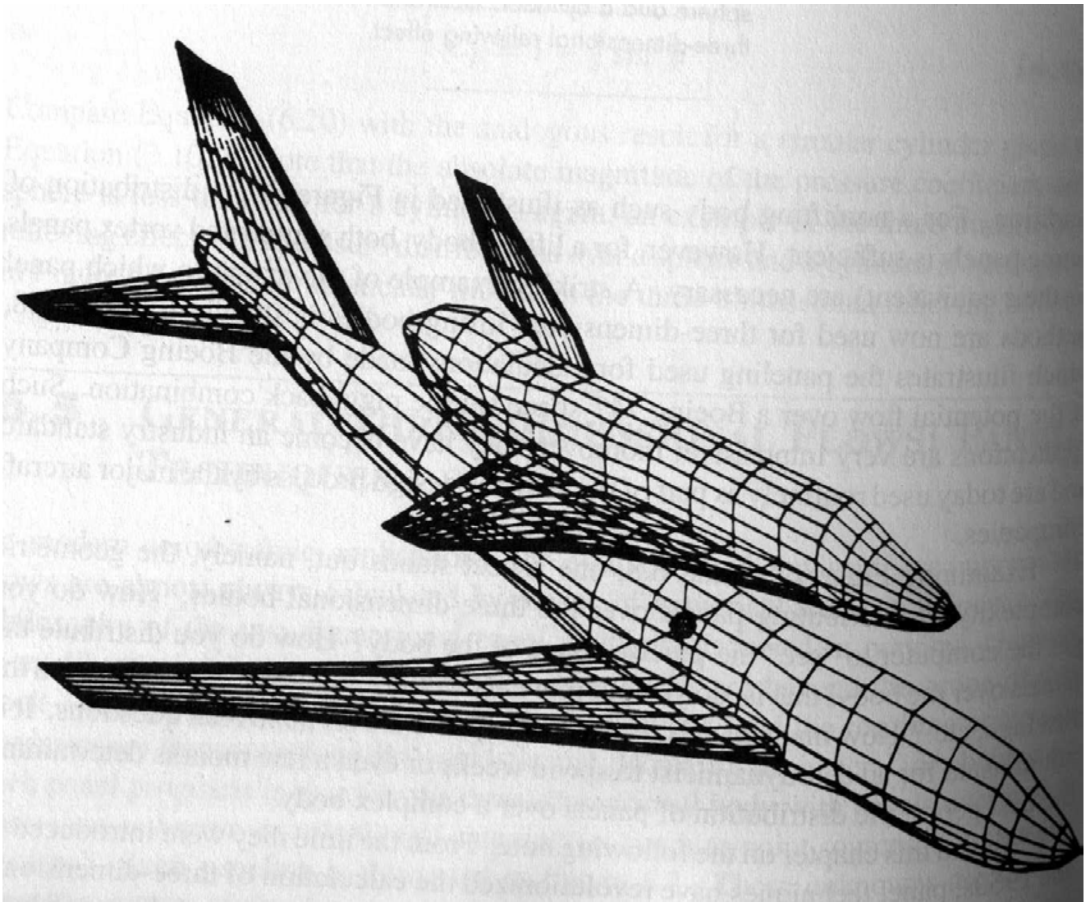
The next step in the hierarchy of techniques is a direct extension of the 2D source and vortex panel methods, a *3D panel method.* The text describes these methods briefly and develops the 3D source and doublet.

Basic Idea:

- Distribute sources, doublets or vortices on the surface of a body.
- Apply the flow tangency condition.
- Solve for the unknown source, doublet and vortex strengths.

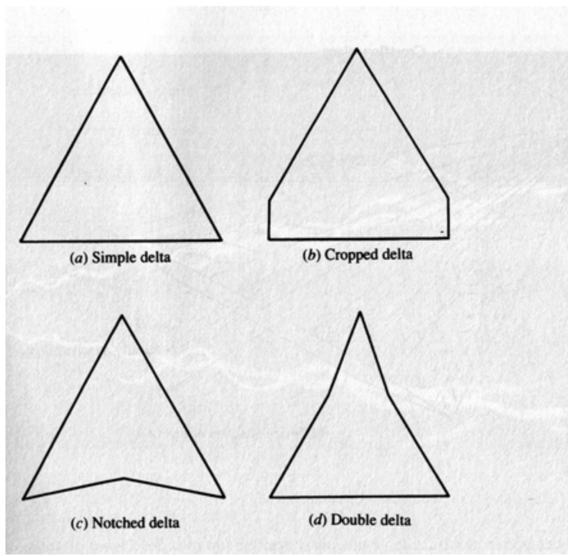


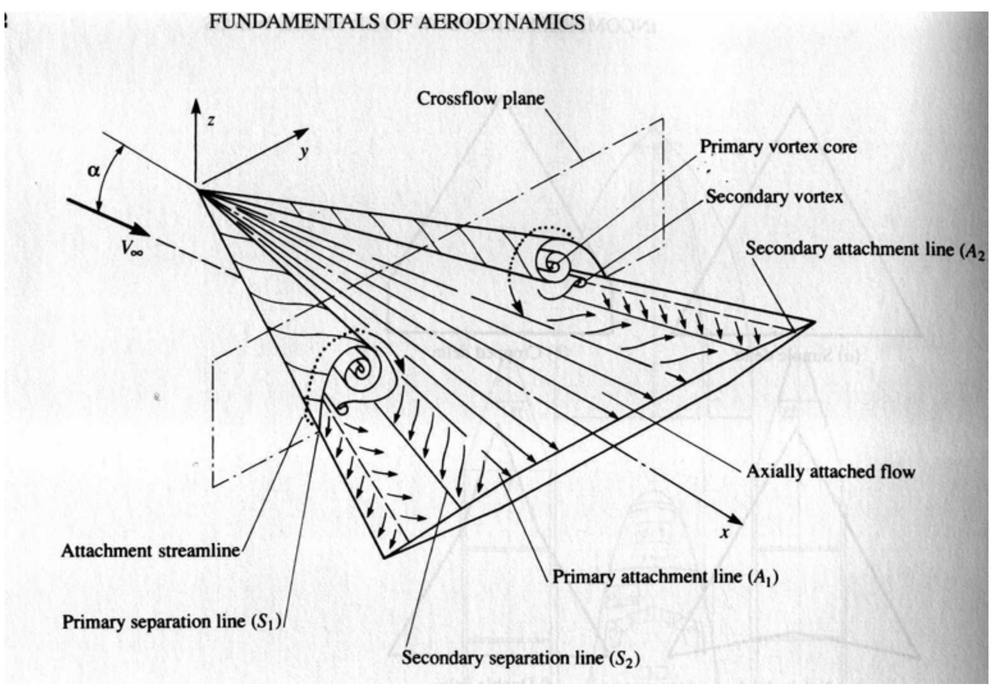
This approach is widely used in the industry for preliminary design considerations and allows us to apply the surface tangency conditions to all points on the wing. A large code is written for this purpose and generally takes a good deal of effort to define the geometry and apply the method.



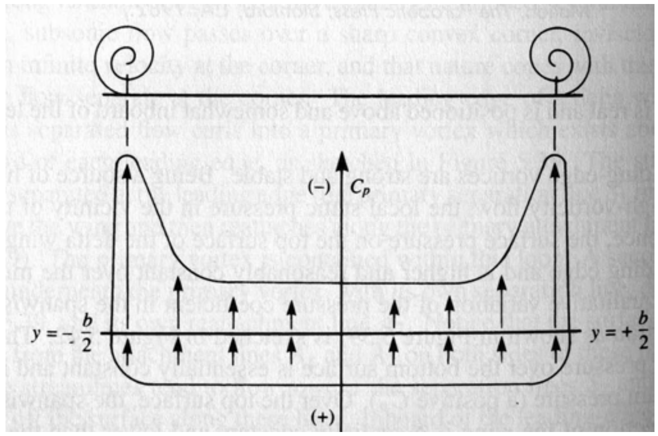
Separated Flow – Vortex Lift

While we are discussing the inviscid flow about a wing we need to consider what happens if the flow is massively separated. In most instances this is a bad thing as it will disrupt the orderly flow about a wing and lead to pressure drag. However, the case of a delta wing this is an entirely different issue as separated flow actually results in enhanced lift, *vortex lift*.





As illustrated in the figure, severe flow separation and reattachment occurs for the delta wing. The vortices account for lift on the wing because they cause the local pressure to drop considerably.

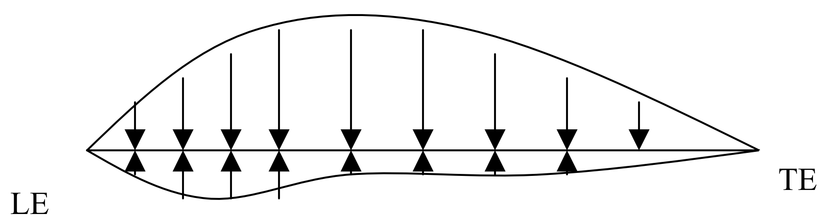


The methods we have discussed so far are not good candidates for predicting these flows; one has to resort to full computational fluid dynamic analysis, i.e., the field solution of the Euler or Navier-Stokes equations, which is beyond the scope of this class.

<u>Moments</u>

An advantage of the vortex lattice method, the 3D panel methods and CFD, as compared to the lifting line theory, is that the pressure distribution is computed and hence moments can be found. To recall the details of this process we need to go back to some basic mechanic.

Consider an airfoil as a beam with distributed loads defined from the pressure.



We recall that the pressure is a force per unit area and that a force is defined once the pressure is integrated over an area. In the case of an airfoil we only integrated in x, since they direction (as defined for a wing) has unit length. We can then use these ideas to come up with moments about a point on the wing.

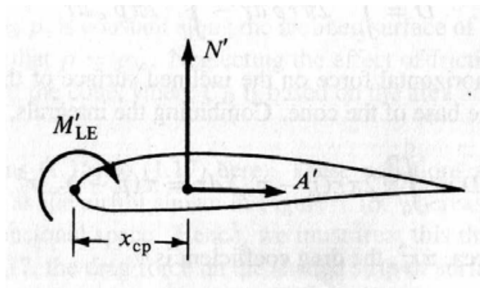
$$M_{LE} = \int_{LE}^{TE} \left(C_{P_{Top}} - C_{P_{Bot}} \right) x\,dx(1) \qquad (5.66)$$

However, the moment could as be represented in terms of the resultant force, R.

$$M_{LE} = R x_{CP} \qquad (5.67)$$

where x_{CP} is the point at which the resultant of the integrated forces acts to generate the same moment about the LE.

Let's return now to an airfoil:



Note that

1. The chord line is a straight line that connects the LE to the TE. (Not the mean camber line)
2. Axial force is assumed to act along the chord line, therefore no moment is developed because of it.
3. Normal force acts normal to the chord line at the *center of pressure.*
4. Moment is defined positive clockwise, therefore, a positive lift force results in a negative M'_{LE}.

$$x_{CP} = -\frac{M'_{LE}}{N'} \tag{5.68}$$

5. The moment coefficient about the center of pressure is zero.
6. Moments can be defined about the LE and the quarter chord point. If $\alpha \approx 0$ then $L' \approx N'$ and

$$M'_{LE} = -\frac{c}{4}L' + M'_{c/4} = -x_{CP}L' \tag{5.69}$$

This idea can be easily extended from the airfoil to the wing by turning Eq. (5.66) into a double integral.

$$\boldsymbol{M_{LE}} = \int_{-b/2}^{b/2} \int_{LE}^{TE} \left(\boldsymbol{C_{P_{Top}}} - \boldsymbol{C_{P_{Bot}}} \right) \boldsymbol{x\,dx\,dy} \qquad (5.70)$$