GRADUATE RECORD EXAMINATIONS®

Math Review
Large Print (18 point) Edition
Chapter 1: Arithmetic

The GRE® Math Review consists of 4 chapters: Arithmetic, Algebra, Geometry, and Data Analysis. This is the Large Print edition of the Arithmetic Chapter of the Math Review. Downloadable versions of large print (PDF) and accessible electronic format (Word) of each of the 4 chapters of the Math Review, as well as a Large Print Figure supplement for each chapter are available from the GRE® website. Other downloadable practice and test familiarization materials in large print and accessible electronic formats are also available. Tactile figure supplements for the 4 chapters of the Math Review, along with additional accessible practice and test familiarization materials in other formats, are available from ETS Disability Services Monday to Friday 8:30 a.m. to 5 p.m. New York time, at 1-609-771-7780, or 1-866-387-8602 (toll free for test takers in the United States, U.S. Territories, and Canada), or via email at stassd@ets.org.

The mathematical content covered in this edition of the Math Review is the same as the content covered in the standard edition of the Math Review. However, there are differences in the presentation of some of the material. These differences are the result of adaptations made for presentation of the material in accessible formats. There are also slight differences between the various accessible formats, also as a result of specific adaptations made for each format.

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Overview of the Math Review

The Math Review consists of 4 chapters: Arithmetic, Algebra, Geometry, and Data Analysis.

Each of the 4 chapters in the Math Review will familiarize you with the mathematical skills and concepts that are important to understand in order to solve problems and reason quantitatively on the Quantitative Reasoning measure of the GRE® revised General Test.

The material in the Math Review includes many definitions, properties, and examples, as well as a set of exercises (with answers) at the end of each review chapter. Note, however, that this review is not intended to be all-inclusive—there may be some concepts on the test that are not explicitly presented in this review. If any topics in this review seem especially unfamiliar or are covered too briefly, we encourage you to consult appropriate mathematics texts for a more detailed treatment.

Overview of this Chapter

This is the Arithmetic Chapter of the Math Review.

The review of arithmetic begins with integers, fractions, and decimals and progresses to real numbers. The basic arithmetic operations of addition, subtraction, multiplication, and division are discussed, along with exponents and roots. The chapter ends with the concepts of ratio and percent.

1.1 Integers

The **integers** are the numbers 1, 2, 3, and so on, together with their negatives, -1, -2, -3, . . . , and 0. Thus, the set of integers is $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$.

The positive integers are greater than 0, the negative integers are less than 0, and 0 is neither positive nor negative. When integers are added, subtracted, or multiplied, the result is always an integer; division of integers is addressed below. The many elementary number facts for these operations, such as

$7 + 8 = 15$, $78 - 87 = -9$, $7 - (-18) = 25$, and $(7)(8) = 56$, should be familiar to you; they are not reviewed here. Here are three general facts regarding multiplication of integers.

Fact 1: The product of two positive integers is a positive integer.

Fact 2: The product of two negative integers is a positive integer.

Fact 3: The product of a positive integer and a negative integer is a negative integer.

When integers are multiplied, each of the multiplied integers is called a **factor** or **divisor** of the resulting product. For example, $(2)(3)(10) = 60$, so 2, 3, and 10 are factors of 60. The integers 4, 15, 5, and 12 are also factors of 60, since $(4)(15) = 60$ and $(5)(12) = 60$. The positive factors of 60 are 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, and 60. The negatives of these integers are also factors of 60, since, for example, $(-2)(-30) = 60$. There are no other factors of 60. We say that 60 is a **multiple** of each of its factors and that 60 is **divisible** by each of its divisors. Here are five more examples of factors and multiples.

Example A: The positive factors of 100 are 1, 2, 4, 5, 10, 20, 25, 50, and 100.

Example B: 25 is a multiple of only six integers: 1, 5, 25, and their negatives.

Example C: The list of positive multiples of 25 has no end: 0, 25, 50, 75, 100, 125, 150, etc.; likewise, every nonzero integer has infinitely many multiples.

Example D: 1 is a factor of every integer; 1 is not a multiple of any integer except 1 and -1.

Example E: 0 is a multiple of every integer; 0 is not a factor of any integer except 0.

The **least common multiple** of two nonzero integers a and b is the least positive integer that is a multiple of both a and b. For example, the least common multiple of 30 and 75 is 150. This is because the positive multiples of 30 are 30, 60, 90, 120, 150, 180, 210, 240, 270, 300, etc., and the positive multiples of 75 are 75, 150, 225, 300, 375, 450, etc. Thus, the *common* positive multiples of 30 and 75 are 150, 300, 450, etc., and the least of these is 150.

The **greatest common divisor** (or **greatest common factor**) of two nonzero integers *a* and *b* is the greatest positive integer that is a divisor of both *a* and *b*. For example, the greatest common divisor of 30 and 75 is 15. This is because the positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15, and 30, and the positive divisors of 75 are 1, 3, 5, 15, 25, and 75. Thus, the ***common*** positive divisors of 30 and 75 are 1, 3, 5, and 15, and the greatest of these is 15.

When an integer *a* is divided by an integer *b*, where *b* is a divisor of *a*, the result is always a divisor of *a*. For example, when 60 is divided by 6 (one of its divisors), the result is 10, which is another divisor of 60. If *b* is ***not*** a divisor of *a*, then the result can be viewed in three different ways. The result can be viewed as a fraction or as a decimal, both of which are discussed later, or the result can be viewed as a **quotient** with a **remainder**, where both are integers. Each view is useful, depending on the context. Fractions and decimals are useful when the result must be viewed as a single number, while quotients with remainders are useful for describing the result in terms of integers only.

Regarding quotients with remainders, consider two positive integers a and b for which b is ***not*** a divisor of a; for example, the integers 19 and 7. When 19 is divided by 7, the result is greater than 2, since $(2)(7) < 19$, but less than 3, since $19 < (3)(7)$. Because 19 is 5 more than $(2)(7)$, we say that the result of 19 divided by 7 is the quotient 2 with remainder 5, or simply "2 remainder 5." In general, when a positive integer a is divided by a positive integer b, you first find the greatest multiple of b that is less than or equal to a. That multiple of b can be expressed as the product qb, where q is the quotient. Then the remainder is equal to a minus that multiple of b, or $r = a - qb$, where r is the remainder. The remainder is always greater than or equal to 0 and less than b.

Here are three examples that illustrate a few different cases of division resulting in a quotient and remainder.

Example A: 100 divided by 45 is 2 remainder 10, since the greatest multiple of 45 that's less than or equal to 100 is $(2)(45)$, or 90, which is 10 less than 100.

Example B: 24 divided by 4 is 6 remainder 0, since the greatest multiple of 4 that's less than or equal to 24 is 24 itself, which is 0 less than 24. In general, the remainder is 0 if and only if a is divisible by b.

Example C: 6 divided by 24 is 0 remainder 6, since the greatest multiple of 24 that's less than or equal to 6 is $(0)(24)$, or 0, which is 6 less than 6.

Here are five more examples.

Example D: 100 divided by 3, is 33 remainder 1, since $100 = (33)(3) + 1$.

Example E: 100 divided by 25 is 4 remainder 0, since $100 = (4)(25) + 0$.

Example F: 80 divided by 100 is 0 remainder 80, since $80 = (0)(100) + 80$.

Example G: When you divide 100 by 2, the remainder is 0.

Example H: When you divide 99 by 2, the remainder is 1.

If an integer is divisible by 2, it is called an **even integer**; otherwise it is an **odd integer**. Note that when a positive odd integer is divided by 2, the remainder is always 1. The set of even integers is $\{\ldots, -6, -4, -2, 0, 2, 4, 6, \ldots\}$, and the set of odd integers is $\{\ldots, -5, -3, -1, 1, 3, 5, \ldots\}$. Here are six useful facts regarding the sum and product of even and odd integers.

Fact 1: The sum of two even integers is an even integer.

Fact 2: The sum of two odd integers is an even integer.

Fact 3: The sum of an even integer and an odd integer is an odd integer.

Fact 4: The product of two even integers is an even integer.

Fact 5: The product of two odd integers is an odd integer.

Fact 6: The product of an even integer and an odd integer is an even integer.

A **prime number** is an integer greater than 1 that has only two positive divisors: 1 and itself. The first ten prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, and 29. The integer 14 is not a prime

number, since it has four positive divisors: 1, 2, 7, and 14. The integer 1 is not a prime number, and the integer 2 is the only prime number that is even.

Every integer greater than 1 either is a prime number or can be uniquely expressed as a product of factors that are prime numbers, or **prime divisors**. Such an expression is called a **prime factorization**. Here are six examples of prime factorizations.

Example A: $12 = (2)(2)(3) = \left(2^2\right)(3)$

Example B: $14 = (2)(7)$

Example C: $12 = (2)(2)(3) = \left(2^2\right)(3)$

Example D: $338 = (2)(13)(13) = (2)\left(13^2\right)$

Example E: $800 = (2)(2)(2)(2)(2)(5)(5) = \left(2^5\right)\left(5^2\right)$

Example F: $1{,}155 = (3)(5)(7)(11)$

An integer greater than 1 that is not a prime number is called a **composite number**. The first ten composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16, and 18.

1.2 Fractions

A **fraction** is a number of the form $\frac{a}{b}$, where a and b are integers and $b \neq 0$. The integer a is called the **numerator** of the fraction, and b is called the **denominator**. For example, $\frac{-7}{5}$ is a fraction in which -7 is the numerator and 5 is the denominator. Such numbers are also called **rational numbers**.

If both the numerator a and denominator b are multiplied by the same nonzero integer, the resulting fraction will be equivalent to $\frac{a}{b}$. For example,

$$\frac{-7}{5} = \frac{(-7)(4)}{(5)(4)} = \frac{-28}{20} \text{ and}$$

$$\frac{-7}{5} = \frac{(-7)(-1)}{(5)(-1)} = \frac{7}{-5}.$$

A fraction with a negative sign in either the numerator or denominator can be written with the negative sign in front of the fraction; for example, $\dfrac{-7}{5} = \dfrac{7}{-5} = -\dfrac{7}{5}$.

If both the numerator and denominator have a common factor, then the numerator and denominator can be factored and reduced to an equivalent fraction. For example,

$$\frac{40}{72} = \frac{(8)(5)}{(8)(9)} = \frac{5}{9}.$$

To add two fractions with the same denominator, you add the numerators and keep the same denominator. For example,

$$-\frac{8}{11} + \frac{5}{11} = \frac{-8+5}{11} = \frac{-3}{11} = -\frac{3}{11}.$$

To add two fractions with different denominators, first find a **common denominator**, which is a common multiple of the two denominators. Then convert both fractions to equivalent

fractions with the same denominator. Finally, add the numerators and keep the common denominator. For example, to add the fractions $\frac{1}{3}$ and $-\frac{2}{5}$, use the common denominator 15:

$$\frac{1}{3} + \frac{-2}{5} = \left(\frac{1}{3}\right)\left(\frac{5}{5}\right) + \left(\frac{-2}{5}\right)\left(\frac{3}{3}\right)$$
$$= \frac{5}{15} + \frac{-6}{15} = \frac{5 + (-6)}{15} = -\frac{1}{15}.$$

The same method applies to subtraction of fractions.

To multiply two fractions, multiply the two numerators and multiply the two denominators. Here are two examples.

Example A: $\left(\frac{10}{7}\right)\left(\frac{-1}{3}\right) = \frac{(10)(-1)}{(7)(3)} = \frac{-10}{21} = -\frac{10}{21}$

Example B: $\left(\frac{8}{3}\right)\left(\frac{7}{3}\right) = \frac{56}{9}$

To divide one fraction by another, first **invert** the second fraction—that is, find its **reciprocal**—then multiply the first fraction by the inverted fraction. Here are two examples.

Example A: $\dfrac{17}{8} \div \dfrac{3}{4} = \left(\dfrac{17}{8}\right)\left(\dfrac{4}{3}\right) = \left(\dfrac{4}{8}\right)\left(\dfrac{17}{3}\right) = \left(\dfrac{1}{2}\right)\left(\dfrac{17}{3}\right) = \dfrac{17}{6}$

Example B: $\dfrac{\dfrac{3}{10}}{\dfrac{7}{13}} = \left(\dfrac{3}{10}\right)\left(\dfrac{13}{7}\right) = \dfrac{39}{70}$

An expression such as $4\dfrac{3}{8}$ is called a **mixed number**. It consists of an integer part and a fraction part; the mixed number $4\dfrac{3}{8}$ means $4 + \dfrac{3}{8}$. To convert a mixed number to an ordinary fraction, convert the integer part to an equivalent fraction and add it to the fraction part. For example,

$$4\dfrac{3}{8} = 4 + \dfrac{3}{8} = \left(\dfrac{4}{1}\right)\left(\dfrac{8}{8}\right) + \dfrac{3}{8}$$

$$= \dfrac{32}{8} + \dfrac{3}{8} = \dfrac{35}{8}.$$

Note that numbers of the form $\frac{a}{b}$, where either a or b is not an integer and $b \neq 0$, are fractional expressions that can be manipulated just like fractions. For example, the numbers $\frac{\pi}{2}$ and $\frac{\pi}{3}$ can be added together as follows.

$$\frac{\pi}{2} + \frac{\pi}{3} = \left(\frac{\pi}{2}\right)\left(\frac{3}{3}\right) + \left(\frac{\pi}{3}\right)\left(\frac{2}{2}\right)$$
$$= \frac{3\pi}{6} + \frac{2\pi}{6} = \frac{5\pi}{6}$$

And the number $\dfrac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}}$ can be simplified as follows.

$$\frac{\frac{1}{\sqrt{2}}}{\frac{3}{\sqrt{5}}} = \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{5}}{3}\right) = \frac{\sqrt{5}}{3\sqrt{2}}$$

1.3 Exponents and Roots

Exponents are used to denote the repeated multiplication of a number by itself; for example, $3^4 = (3)(3)(3)(3) = 81$ and $5^3 = (5)(5)(5) = 125$. In the expression 3^4, 3 is called the **base**, 4 is called the **exponent**, and we read the expression as "3 to the fourth power." So 5 to the third power is 125.

When the exponent is 2, we call the process **squaring**. Thus, 6 squared is 36, $6^2 = (6)(6) = 36$, and 7 squared is 49, $7^2 = (7)(7) = 49$.

When negative numbers are raised to powers, the result may be positive or negative. For example, $(-3)^2 = (-3)(-3) = 9$, while $(-3)^5 = (-3)(-3)(-3)(-3)(-3) = -243$. A negative number raised to an even power is always positive, and a negative number raised to an odd power is always negative. Note that without the parentheses, the expression -3^2 means "the negative of '3 squared' "; that is, the exponent is applied before the negative sign. So $(-3)^2 = 9$, but $-3^2 = -9$.

Exponents can also be negative or zero; such exponents are defined as follows.

- The exponent zero: For all nonzero numbers a, $a^0 = 1$. The expression 0^0 is undefined.

- Negative exponents: For all nonzero numbers a, $a^{-1} = \dfrac{1}{a}$, $a^{-2} = \dfrac{1}{a^2}$, $a^{-3} = \dfrac{1}{a^3}$, etc. Note that $(a)\left(a^{-1}\right) = (a)\left(\dfrac{1}{a}\right) = 1$.

A **<u>square root</u>** of a nonnegative number n is a number r such that $r^2 = n$. For example, 4 is a square root of 16 because $4^2 = 16$. Another square root of 16 is –4, since $(-4)^2 = 16$. All positive numbers have two square roots, one positive and one negative. The only square root of 0 is 0. The symbol $\sqrt{n}$ is used to denote the ***nonnegative*** square root of the nonnegative number n. Therefore, $\sqrt{100} = 10$, $-\sqrt{100} = -10$, and $\sqrt{0} = 0$. Square roots of negative numbers are not defined in the real number system.

Here are four important rules regarding operations with square roots, where $a > 0$ and $b > 0$.

Rule 1: $\left(\sqrt{a}\right)^2 = a$

Example A: $\left(\sqrt{3}\right)^2 = 3$

Example B: $\left(\sqrt{\pi}\right)^2 = \pi$

Rule 2: $\sqrt{a^2} = a$

Example A: $\sqrt{4} = 2$

Example B: $\sqrt{\pi^2} = \pi$

Rule 3: $\sqrt{a}\sqrt{b} = \sqrt{ab}$

Example A: $\sqrt{3}\sqrt{10} = \sqrt{30}$

Example B: $\sqrt{24} = \sqrt{4}\sqrt{6} = 2\sqrt{6}$

Rule 4: $\dfrac{\sqrt{a}}{\sqrt{b}} = \sqrt{\dfrac{a}{b}}$

Example A: $\dfrac{\sqrt{5}}{\sqrt{15}} = \sqrt{\dfrac{5}{15}} = \sqrt{\dfrac{1}{3}}$

Example B: $\dfrac{\sqrt{18}}{\sqrt{2}} = \sqrt{\dfrac{18}{2}} = \sqrt{9} = 3$

A square root is a root of order 2. Higher-order roots of a positive number n are defined similarly. For orders 3 and 4, the **<u>cube root</u>** $\sqrt[3]{n}$ and **<u>fourth root</u>** $\sqrt[4]{n}$ represent numbers such that when they are raised to the powers 3 and 4, respectively, the result is n. These roots obey rules similar to those above (but with the exponent 2 replaced by 3 or 4 in the first two rules).

There are some notable differences between odd-order roots and even-order roots (in the real number system):

- For odd-order roots, there is ***exactly one*** root for ***every*** number n, even when n is negative.

- For even-order roots, there are ***exactly two*** roots for every ***positive*** number n and ***no*** roots for any ***negative*** number n.

For example, 8 has exactly one cube root, $\sqrt[3]{8} = 2$, but 8 has two fourth roots: $\sqrt[4]{8}$ and $-\sqrt[4]{8}$; and -8 has exactly one cube root, $\sqrt[3]{-8} = -2$, but -8 has no fourth root, since it is negative.

1.4 Decimals

The decimal number system is based on representing numbers using powers of 10. The place value of each digit corresponds to a power of 10. For example, the digits of the number 7,532.418 have the following place values.

For the digits before the decimal point:

7 is in the thousands place

5 is in the hundreds place

3 is in the tens place

2 is in the ones, or units, place

For the digits after the decimal point:

4 is in the tenths place

1 is in the hundredths place

8 is in the thousandths place.

That is,

the number 7,532.418 can be written as

$$7(1,000) + 5(100) + 3(10) + 2(1) + 4\left(\frac{1}{10}\right) + 1\left(\frac{1}{100}\right) + 8\left(\frac{1}{1,000}\right),$$

or alternatively it can be written as

$$7\left(10^3\right) + 5\left(10^2\right) + 3\left(10^1\right) + 2\left(10^0\right) + 4\left(10^{-1}\right) + 1\left(10^{-2}\right) + 8\left(10^{-3}\right).$$

If there are a finite number of digits to the right of the decimal point, converting a decimal to an equivalent fraction with integers in the numerator and denominator is a straightforward process. Since each place value is a power of 10, every decimal can be converted to an integer divided by a power of 10. Here are three examples:

Example A: $2.3 = 2 + \dfrac{3}{10} = \dfrac{23}{10}$

Example B: $90.17 = 90 + \dfrac{17}{100} = \dfrac{9,000 + 17}{100} = \dfrac{9,017}{100}$

Example C: $0.612 = \dfrac{612}{1,000} = \dfrac{153}{250}$

Conversely, every fraction with integers in the numerator and denominator can be converted to an equivalent decimal by dividing the numerator by the denominator using long division (which is not in this review). The decimal that results from the long division will either **terminate**, as in $\frac{1}{4} = 0.25$ and $\frac{52}{25} = 2.08$, or the decimal will **repeat** without end, as in $\frac{1}{9} = 0.111\ldots$, $\frac{1}{22} = 0.0454545\ldots$, and $\frac{25}{12} = 2.08333\ldots$. One way to indicate the repeating part of a decimal that repeats without end is to use a bar over the digits that repeat. Here are four examples of fractions converted to decimals.

Example A: $\frac{3}{8} = 0.375$

Example B: $\frac{259}{40} = 6 + \frac{19}{40} = 6.475$

Example C: $-\frac{1}{3} = -0.\overline{3}$

Example D: $\frac{15}{14} = 1.0\overline{714285}$

Every fraction with integers in the numerator and denominator is equivalent to a decimal that terminates or repeats. That is, every rational number can be expressed as a terminating or repeating decimal. The converse is also true; that is, every terminating or repeating decimal represents a rational number.

Not all decimals are terminating or repeating; for instance, the decimal that is equivalent to $\sqrt{2}$ is 1.41421356237..., and it can be shown that this decimal does not terminate or repeat. Another example is 0.010110111011110111110..., which has groups of consecutive 1's separated by a 0, where the number of 1's in each successive group increases by one. Since these two decimals do not terminate or repeat, they are not rational numbers. Such numbers are called **irrational numbers**.

1.5 Real Numbers

The set of **real numbers** consists of all rational numbers and all irrational numbers. The real numbers include all integers, fractions, and decimals. The set of real numbers can be represented by a number line called the **real number line**. Arithmetic Figure 1 below is a number line.

$$-\sqrt{5} \quad -\frac{3}{2} \quad -0.4 \quad \frac{1}{2} \quad \sqrt{2} \quad 2.6$$

$$-3 \qquad -2 \qquad -1 \qquad 0 \qquad 1 \qquad 2 \qquad 3$$
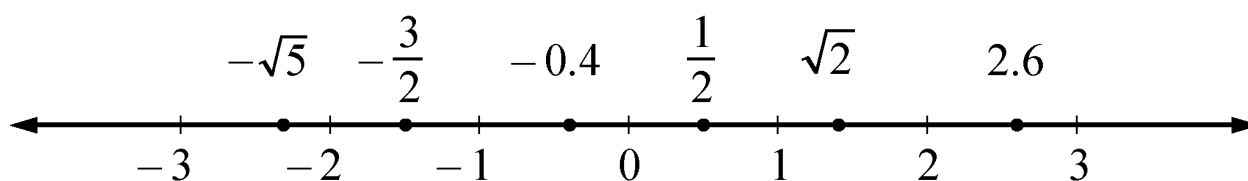
Arithmetic Figure 1

Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number. On the number line, all numbers to the left of 0 are negative and all numbers to the right of 0 are positive. As shown in Arithmetic Figure 1, the negative numbers

$-0.4, -1, -\dfrac{3}{2}, -2, -\sqrt{5}$, and -3 are to the left of 0 and the

positive numbers $\dfrac{1}{2}, 1, \sqrt{2}, 2, 2.6$, and 3 are to the right of 0.

Only the number 0 is neither negative nor positive.

A real number x is **<u>less than</u>** a real number y if x is to the left of y on the number line, which is written as $x < y$. A real number y is **<u>greater than</u>** x if y is to the right of x on the number line, which is written as $y > x$. For example, the number line in Arithmetic Figure 1 shows the following three less than/greater than relationships.

  Relationship 1: $-\sqrt{5} < -2$

  Relationship 2: $\dfrac{1}{2} > 0$

  Relationship 3: $1 < \sqrt{2} < 2$

To say that a real number x is between 2 and 3 on the number line means that $x > 2$ and $x < 3$, which can also be written as the double inequality $2 < x < 3$. The set of all real numbers that are between 2 and 3 is called an **interval**, and the double inequality $2 < x < 3$ is often used to represent that interval. Note that the endpoints of the interval, 2 and 3, are not included in the interval. Sometimes one or both of the endpoints are to be included in an interval. The following inequalities represent four types of intervals, depending on whether the endpoints are included.

Interval type 1: $2 < x < 3$

Interval type 2: $2 \leq x < 3$

Interval type 3: $2 < x \leq 3$

Interval type 4: $2 \leq x \leq 3$

There are also four types of intervals with only one endpoint, each of which consists of all real numbers to the right or to the left of the endpoint, perhaps including the endpoint. The following inequalities represent these types of intervals.

Interval type 1: $x < 4$

Interval type 2: $x \leq 4$

Interval type 3: $x > 4$

Interval type 4: $x \geq 4$

The entire real number line is also considered to be an interval.

The distance between a number x and 0 on the number line is called the **absolute value** of x, written as $|x|$. Therefore, $|3| = 3$ and $|-3| = 3$ because each of the numbers 3 and -3 is a distance of 3 from 0. Note that if x is positive, then $|x| = x$; if x is negative, then $|x| = -x$; and lastly, $|0| = 0$. It follows that the

absolute value of any nonzero number is positive. Here are three examples.

Example A: $\left|\sqrt{5}\right| = \sqrt{5}$

Example B: $|-23| = -(-23) = 23$

Example C: $|-10.2| = 10.2$

Here are 12 general properties of real numbers that are used frequently. In each property a, b, and c are real numbers.

Property 1: $a + b = b + a$ and $ab = ba$.

For example, $8 + 2 = 2 + 8 = 10$ and
$(-3)(17) = (17)(-3) = -51$.

Property 2: $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.

For example, $(7 + 3) + 8 = 7 + (3 + 8) = 18$ and
$\left(7\sqrt{2}\right)\sqrt{2} = 7\left(\sqrt{2}\sqrt{2}\right) = (7)(2) = 14$.

Property 3: $a(b + c) = ab + bc$.

For example, $5(3 + 16) = (5)(3) + (5)(16) = 95$.

Property 4: $a + 0 = a$, $(a)(0) = 0$, and $(a)(1) = a$.

Property 5: If $ab = 0$, then either $a = 0$ or $b = 0$ or both.

For example, if $-2b = 0$, then $b = 0$.

Property 6: Division by 0 is not defined; for example, $5 \div 0$, $\dfrac{-7}{0}$, and $\dfrac{0}{0}$ are undefined.

Property 7: If both a and b are positive, then both $a + b$ and ab are positive.

Property 8: If both a and b are negative, then $a + b$ is negative and ab is positive.

Property 9: If a is positive and b is negative, then ab is negative.

Property 10: $|a + b| \leq |a| + |b|$. This is known as the **<u>triangle inequality</u>**.

For example, if $a = 5$ and $b = -2$, then
$|5 + (-2)| = |5 - 2| = |3| = 3$ and $|5| + |-2| = 5 + 2 = 7$.
Therefore, $|5 + (-2)| \leq |5| + |-2|$.

Property 11: $|a||b| = |ab|$.

For example, $|5||-2| = |(5)(-2)| = |-10| = 10$.

Property 12: If $a > 1$, then $a^2 > a$. If $0 < b < 1$, then $b^2 < b$.

For example, $5^2 = 25 > 5$, but $\left(\frac{1}{5}\right)^2 = \frac{1}{25} < \frac{1}{5}$.

1.6 Ratio

The **ratio** of one quantity to another is a way to express their relative sizes, often in the form of a fraction, where the first quantity is the numerator and the second quantity is the denominator. Thus, if *s* and *t* are positive quantities, then the ratio of *s* to *t* can be written as the fraction $\frac{s}{t}$. The notation "*s* to *t*" or "*s* : *t*" is also used to express this ratio. For example, if there are 2 apples and 3 oranges in a basket, we can say that the ratio of the number of apples to the number of oranges is $\frac{2}{3}$ or that it is 2 to 3 or that it is 2 : 3. Like fractions, ratios can be reduced to lowest terms. For example, if there are 8 apples and 12 oranges in a basket, then the ratio of the numbers of apples to oranges is still 2 to 3. Similarly, the ratio 9 to 12 is equivalent to the ratio 3 to 4.

If three or more positive quantities are being considered, say *r*, *s*, and *t*, then their relative sizes can also be expressed as a ratio with the notation "*r* to *s* to *t*". For example, if there are 5 apples, 30 pears, and 20 oranges in a basket, then the ratio of the

numbers of apples to pears to oranges is 5 to 30 to 20. This ratio can be reduced to 1 to 6 to 4 by dividing each number by the greatest common divisor of 5, 30, and 20, which is 5.

A **proportion** is an equation relating two ratios; for example, $\frac{9}{12} = \frac{3}{4}$. To solve a problem involving ratios, you can often write a proportion and solve it by **cross multiplication**.

Example 1.6.1: To find a number x so that the ratio of x to 49 is the same as the ratio of 3 to 21 you can write

$$\frac{x}{49} = \frac{3}{21}.$$

Then cross multiply to get $21x = (3)(49)$, and solve for x to get $x = \frac{(3)(49)}{21} = 7$.

1.7 Percent

The term **percent** means *per hundred*, or *hundredths*. Percents are ratios that are often used to represent *parts* of a *whole*, where the whole is considered as having 100 parts. Here are three examples of percents.

Example A: 1 percent means 1 part out of 100 parts, or $\frac{1}{100}$.

Example B: 32 percent means 32 parts out of 100 parts, or $\frac{32}{100}$.

Example C: 50 percent means 50 parts out of 100 parts, or $\frac{1}{2}$.

Note that the *part* is the numerator of the ratio and the *whole* is the denominator. Percents are often written with the % symbol; fractional and decimal equivalents are often used as well but without the % symbol, as in the following five examples.

Example A: $1\% = \frac{1}{100} = 0.01$

Example B: $100\% = \dfrac{100}{100} = 1$

Example C: $32\% = \dfrac{32}{100} = 0.32$

Example D: $50\% = \dfrac{50}{100} = 0.5$

Example E: $0.3\% = \dfrac{0.3}{100} = 0.003$

Be careful not to confuse 0.01 with 0.01%. The percent symbol matters. For example, $0.01 = 1\%$ but $0.01\% = \dfrac{0.01}{100} = 0.0001$.

- To compute a *percent*, given the *part* and the *whole*, divide the part by the whole. The result will be the decimal equivalent, so multiply the result by 100 to convert to percent.

Example 1.7.1: If the whole is 20 and the part is 13, you can find the percent as follows.

$$\frac{\text{part}}{\text{whole}} = \frac{13}{20} = 0.65 = 65\%.$$

Example 1.7.2: What percent of 150 is 12.9 ?

Solution: Here the whole is 150 and the part is 12.9, so

$$\frac{\text{part}}{\text{whole}} = \frac{12.9}{150} = 0.086 = 8.6\%.$$

- To find the *part* that is a certain *percent* of a *whole*, you can either multiply the *whole* by the decimal equivalent of the percent or set up a proportion to find the part.

Example 1.7.3: To find 30% of 350, multiply 350 by the decimal equivalent of 30%, or 0.3 as follows.

$x = (350)(0.3) = 105$

To use a proportion, you need to find the number of parts of 350 that yields the same ratio as 30 out of 100 parts. You want a number x that satisfies the proportion

$$\frac{\text{part}}{\text{whole}} = \frac{30}{100}$$

$$\frac{x}{350} = \frac{30}{100}$$

Solving for x yields $x = \dfrac{(30)(350)}{100} = 105$, so 30% of 350 is 105.

- Given the ***percent*** and the ***part***, you can calculate the ***whole***. To do this you can either use the decimal equivalent of the percent or you can set up a proportion and solve it.

Example 1.7.4: 15 is 60% of what number?

Solution: Use the decimal equivalent of 60%. Because 60% of some number z is 15, multiply z by the decimal equivalent of 60%, or 0.6.

$0.6z = 15$

Now solve for z by dividing both sides of the equation by 0.6 as follows.

$$z = \frac{15}{0.6} = 25$$

Using a proportion, look for a number z such that

$$\frac{\text{part}}{\text{whole}} = \frac{60}{100}$$

$$\frac{15}{z} = \frac{60}{100}$$

Hence, $60z = (15)(100)$, and therefore,

$$z = \frac{(15)(100)}{60} = \frac{1,500}{60} = 25. \text{ That is, 15 is 60\% of 25.}$$

Although the discussion about percent so far assumes a context of a **part** and a **whole**, it is not necessary that the part be less than the whole. In general, the whole is called the **base** of the percent. When the numerator of a percent is greater than the base, the percent is greater than 100%. For example, 15 is 300% of 5, since

$$\frac{15}{5} = \frac{300}{100},$$

and 250% of 16 is $\left(\dfrac{250}{100}\right)(16) = (2.5)(16) = 40$. Note that the decimal equivalent of 250% is 2.5.

It is also not necessary for the part to be related to the whole at all, as in the question, "a teacher's salary is what percent of a banker's salary?"

When a quantity changes from an initial positive amount to another positive amount, for example, an employee's salary that is raised, you can compute the amount of change as a percent of the initial amount. This is called **percent change**. If a quantity increases from 600 to 750, then the **percent increase** is found by dividing the amount of increase, 150, by the base, 600, which is the initial number given:

$$\frac{\text{amount of increase}}{\text{base}} = \frac{750 - 600}{600} = \frac{150}{600} = \frac{25}{100} = 0.25 = 25\%.$$

We say the percent increase is 25%. Sometimes this computation is written as

$$\left(\frac{750 - 600}{600}\right)(100\%) = \left(\frac{150}{600}\right)(100\%) = 25\%.$$

If a quantity doubles in size, then the percent increase is 100%. For example, if a quantity changes from 150 to 300, then the percent increase is

$$\frac{\text{change}}{\text{base}} = \frac{300 - 150}{150} = \frac{150}{150} = 100\%.$$

If a quantity decreases from 500 to 400, calculate the **percent decrease** as follows.

$$\frac{\text{change}}{\text{base}} = \frac{500 - 400}{500} = \frac{100}{500} = \frac{20}{100} = 0.20 = 20\%$$

The quantity decreased by 20%.

When computing a percent *increase*, the base is the *smaller* number. When computing a percent *decrease*, the base is the *larger* number. In either case, the base is the initial number, before the change.

Example 1.7.5: An investment in a mutual fund increased by 12% in a single day. If the value of the investment before the increase was $1,300, what was the value after the increase?

Solution: The percent increase is 12%. Therefore, the value of the increase is 12% of $1,300, or, using the decimal equivalent, the increase is $(0.12)($1,300) = 156. Thus, the value of the investment after the change is

$1,300 + $156 = $1,456$.

Because the final result is the sum of the initial investment—100% of $1,300—and the increase—12% of $1,300—the final result is $100\% + 12\% = 112\%$ of $1,300. Thus, another way to get the final result is to multiply the value of the investment by the decimal equivalent of 112%, which is 1.12:

$$(\$1,300)(1.12) = \$1,456.$$

A quantity may have several successive percent changes. The base of each successive percent change is the result of the preceding percent change.

Example 1.7.6: The monthly enrollment at a preschool decreased by 8% during one month and increased by 6% during the next month. What was the cumulative percent change for the two months?

Solution: If E is the enrollment before the first month, then the enrollment as a result of the 8% decrease can be found by multiplying the base E by the decimal equivalent of $100\% - 8\% = 92\%$, which is 0.92:

$0.92E$.

The enrollment as a result of the second percent change—the 6% increase—can be found by multiplying the *new* base $0.92E$ by the decimal equivalent of $100\% + 6\% = 106\%$, which is 1.06:

$(1.06)(0.92)E = 0.9752E$.

The percent equivalent of 0.9752 is 97.52%, which is 2.48% less than 100%. Thus, the cumulative percent change in the enrollment for the two months is a 2.48% decrease.

Arithmetic Exercises

1. Evaluate the following.

(a) $15 - (6 - 4)(-2)$

(b) $(2 - 17) \div 5$

(c) $(60 \div 12) - (-7 + 4)$

(d) $(3)^4 - (-2)^3$

(e) $(-5)(-3) - 15$

(f) $(-2)^4 (15 - 18)^4$

(g) $(20 \div 5)^2 (-2 + 6)^3$

(h) $(-85)(0) - (-17)(3)$

2. Evaluate the following.

(a) $\dfrac{1}{2} - \dfrac{1}{3} + \dfrac{1}{12}$

(b) $\left(\dfrac{3}{4} + \dfrac{1}{7}\right)\left(\dfrac{-2}{5}\right)$

(c) $\left(\dfrac{7}{8} - \dfrac{4}{5}\right)^2$

(d) $\left(\dfrac{3}{-8}\right) \div \left(\dfrac{27}{32}\right)$

3. Which of the integers 312, 98, 112, and 144 are divisible by 8 ?

4.

(a) What is the prime factorization of 372 ?

(b) What are the positive divisors of 372 ?

5.

(a) What are the prime divisors of 100 ?

(b) What are the prime divisors of 144 ?

6. Which of the integers 2, 9, 19, 29, 30, 37, 45, 49, 51, 83, 90, and 91 are prime numbers?

7. What is the prime factorization of 585 ?

8. Which of the following statements are true?

(a) $-5 < 3.1$

(b) $\sqrt{16} = 4$

(c) $7 \div 0 = 0$

(d) $0 < \left| -\dfrac{1}{7} \right|$

(e) $0.3 < \dfrac{1}{3}$

(f) $(-1)^{87} = -1$

(g) $\sqrt{(-3)^2} < 0$

(h) $\dfrac{21}{28} = \dfrac{3}{4}$

(i) $-|-23| = 23$

(j) $\dfrac{1}{2} > \dfrac{1}{17}$

(k) $(59)^3 (59)^2 = 59^6$

(l) $-\sqrt{25} < -4$

9. Find the following.

(a) 40% of 15

(b) 150% of 48

(c) 0.6% of 800

(d) 15 is 30% of which number?

(e) 11 is what percent of 55 ?

10. If a person's salary increased from $200 per week to $234 per week, what was the percent increase in the person's salary?

11. If an athlete's weight decreased from 160 pounds to 152 pounds, what was the percent decrease in the athlete's weight?

12. A particular stock is valued at $40 per share. If the value increases by 20 percent and then decreases by 25 percent, what will be the value of the stock per share after the decrease?

13. If the ratio of the number of men to the number of women on a committee of 20 members is 3 to 2, how many members of the committee are women?

14. The integer a is even and the integer b is odd. For each of the following integers, indicate whether the integer is even or odd.

(a) $a + 2b$

(b) $2a + b$

(c) ab

(d) a^b

(e) $(a + b)^2$

(f) $a^2 - b^2$

15. When the positive integer n is divided by 3, the remainder is 2 and when n is divided by 5, the remainder is 1. What is the least possible value of n ?

Answers to Arithmetic Exercises

1.

(a) 19

(b) −3

(c) 8

(d) 89

(e) 0

(f) 1,296

(g) 1,024

(h) 51

2.

(a) $\dfrac{1}{4}$

(b) $-\dfrac{5}{14}$

(c) $\dfrac{9}{1,600}$

(d) $-\dfrac{4}{9}$

3. 312, 112, and 144

4.

(a) $372 = \left(2^2\right)(3)(31)$

(b) The positive divisors of 372 are 1, 2, 3, 4, 6, 12, 31, 62, 93, 124, 186, and 372.

5.

(a) $100 = \left(2^2\right)\left(5^2\right)$, so the prime divisors are 2 and 5.

(b) $144 = \left(2^4\right)\left(3^2\right)$, so the prime divisors are 2 and 3.

6. 2, 19, 29, 37, and 83

7. $585 = \left(3^2\right)(5)(13)$

8.

(a) True

(b) True

(c) False; division by 0 is undefined.

(d) True

(e) True

(f) True

(g) False; $\sqrt{(-3)^2} = \sqrt{9} = 3 > 0$

(h) True

(i) False; $-|-23| = -23$

(j) True

(k) False; $(59)^3 (59)^2 = 59^{3+2} = 59^5$

(l) True

9.

(a) 6

(b) 72

(c) 4.8

(d) 50

(e) 20%

10. 17%

11. 5%

12. $36 per share

13. 8 women

14.

(a) $a + 2b$ is even.

(b) $2a + b$ is odd.

(c) ab is even.

(d) a^b is even.

(e) $(a + b)^2$ is odd.

(f) $a^2 - b^2$ is odd.

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