Exact equations (Sect. 2.6).

- Exact differential equations.
- The Poincaré Lemma.
- Implicit solutions and the potential function.
- Generalization: The integrating factor method.

Exact differential equations.

Definition
Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$ and continuously differentiable functions $M, N : R \to \mathbb{R}$, denoted as $(t, u) \mapsto M(t, u)$ and $(t, u) \mapsto N(t, u)$, the differential equation in the unknown function $y : (t_1, t_2) \to \mathbb{R}$ given by

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0$$

is called *exact* iff for every point $(t, u) \in R$ holds

$$\partial_t N(t, u) = \partial_u M(t, u)$$

Recall: we use the notation: $\partial_t N = \dfrac{\partial N}{\partial t}$, and $\partial_u M = \dfrac{\partial M}{\partial u}$.

Exact differential equations.

Example

Show whether the differential equation below is exact,

$$2ty(t)\,y'(t) + 2t + y^2(t) = 0.$$

Solution: We first identify the functions N and M,

$$\big[2ty(t)\big]\,y'(t) + \big[2t + y^2(t)\big] = 0 \quad\Rightarrow\quad \begin{cases} N(t,u) = 2tu, \\ M(t,u) = 2t + u^2. \end{cases}$$

The equation is exact iff $\partial_t N = \partial_u M$. Since

$$N(t,u) = 2tu \quad\Rightarrow\quad \partial_t N(t,u) = 2u,$$

$$M(t,u) = 2t + u^2 \quad\Rightarrow\quad \partial_u M(t,u) = 2u.$$

We conclude: $\partial_t N(t,u) = \partial_u M(t,u)$. ◁

Remark: The ODE above is not separable and non-linear.

Exact differential equations.

Example

Show whether the differential equation below is exact,

$$\sin(t)y'(y) + t^2 e^{y(t)} y'(t) - y'(t) = -y(t)\cos(t) - 2te^{y(t)}.$$

Solution: We first identify the functions N and M, if we write

$$\big[\sin(t) + t^2 e^{y(t)} - 1\big]\,y'(t) + \big[y(t)\cos(t) + 2te^{y(t)}\big] = 0,$$

we can see that

$$N(t,u) = \sin(t) + t^2 e^u - 1 \quad\Rightarrow\quad \partial_t N(t,u) = \cos(t) + 2te^u,$$

$$M(t,u) = u\cos(t) + 2te^u \quad\Rightarrow\quad \partial_u M(t,u) = \cos(t) + 2te^u.$$

The equation is exact, since $\partial_t N(t,u) = \partial_u M(t,u)$. ◁

Exact differential equations.

Example

Show whether the linear differential equation below is exact,

$$y'(t) = -a(t)\,y(t) + b(t), \qquad a(t) \neq 0.$$

Solution: We first find the functions N and M,

$$y' + a(t)y - b(t) = 0 \quad \Rightarrow \quad \begin{cases} N(t,u) = 1, \\ M(t,u) = a(t)\,u - b(t). \end{cases}$$

The differential equation is not exact, since

$$N(t,u) = 1 \quad \Rightarrow \quad \partial_t N(t,u) = 0,$$

$$M(t,u) = a(t)u - b(t) \quad \Rightarrow \quad \partial_u M(t,u) = a(t).$$

This implies that $\partial_t N(t,u) \neq \partial_u M(t,u)$. $\triangleleft$

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The Poincaré Lemma.

Remark: The coefficients N and M of an exact equations are the derivatives of a potential function ψ.

Lemma (Poincaré)

Given an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, the continuously differentiable functions $M, N : R \to \mathbb{R}$ satisfy the equation

$$\partial_t N(t,u) = \partial_u M(t,u)$$

iff there exists a twice continuously differentiable function $\psi : R \to \mathbb{R}$, called potential function, *such that for all $(t,u) \in R$ holds*

$$\partial_u \psi(t,u) = N(t,u), \qquad \partial_t \psi(t,u) = M(t,u).$$

Proof: ($\Leftarrow$) Simple: $\left.\begin{aligned} \partial_t N = \partial_t \partial_u \psi, \\ \partial_u M = \partial_u \partial_t \psi, \end{aligned}\right\} \Rightarrow \partial_t N = \partial_u M.$

($\Rightarrow$) Difficult: Poincaré, 1880.

The Poincaré Lemma.

Example

Show that the function $\psi(t,u) = t^2 + tu^2$ is the potential function for the exact differential equation

$$2ty(t)\, y'(t) + 2t + y^2(t) = 0.$$

Solution: We already saw that the differential equation above is exact, since the functions M and N,

$$\left.\begin{aligned} N(t,u) &= 2tu, \\ M(t,u) &= 2t + u^2 \end{aligned}\right\} \quad \Rightarrow \quad \partial_t N = 2u = \partial_u M.$$

The potential function is $\psi(t,u) = t^2 + tu^2$, since

$$\partial_t \psi = 2t + u^2 = M, \qquad \partial_u \psi = 2tu = N. \qquad \triangleleft$$

Remark: The Poincaré Lemma only states necessary and sufficient conditions on N and M for the existence of ψ.

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Implicit solutions and the potential function.

Theorem (Exact differential equations)

Let $M, N : R \to \mathbb{R}$ be continuously differentiable functions on an open rectangle $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$. If the differential equation

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0 \qquad (1)$$

is exact, then every solution $y : (t_1, t_2) \to \mathbb{R}$ must satisfy the algebraic equation

$$\psi(t, y(t)) = c,$$

where $c \in \mathbb{R}$ and $\psi : R \to \mathbb{R}$ is a potential function for Eq. (1).

Proof: $0 = N(t, y)\, y' + M(t, y) = \partial_y \psi(t, y)\, \dfrac{dy}{dt} + \partial_t \psi(t, y))$.

$$0 = \frac{d}{dt}\psi(t, y(t)) \quad \Leftrightarrow \quad \psi(t, y(t)) = c. \qquad \square$$

Implicit solutions and the potential function.

Example

Find all solutions y to the equation

$$\left[\sin(t) + t^2 e^{y(t)} - 1\right] y'(t) + y(t)\cos(t) + 2te^{y(t)} = 0.$$

Solution: Recall: The equation is exact,

$$N(t,u) = \sin(t) + t^2 e^u - 1 \quad \Rightarrow \quad \partial_t N(t,u) = \cos(t) + 2te^u,$$

$$M(t,u) = u\cos(t) + 2te^u \quad \Rightarrow \quad \partial_u M(t,u) = \cos(t) + 2te^u,$$

hence, $\partial_t N = \partial_u M$. Poincaré Lemma says the exists ψ,

$$\partial_u \psi(t,u) = N(t,u), \quad \partial_t \psi(t,u) = M(t,u).$$

These are actually equations for ψ. From the first one,

$$\psi(t,u) = \int \left[\sin(t) + t^2 e^u - 1\right] du + g(t).$$

Implicit solutions and the potential function.

Example

Find all solutions y to the equation

$$\left[\sin(t) + t^2 e^{y(t)} - 1\right] y'(t) + y(t)\cos(t) + 2te^{y(t)} = 0.$$

Solution: $\psi(t,u) = \int \left[\sin(t) + t^2 e^u - 1\right] du + g(t)$. Integrating,

$$\psi(t,u) = u\sin(t) + t^2 e^u - u + g(t).$$

Introduce this expression into $\partial_t \psi(t,u) = M(t,u)$, that is,

$$\partial_t \psi(t,u) = u\cos(t) + 2te^u + g'(t) = M(t,u) = u\cos(t) + 2te^u,$$

Therefore, $g'(t) = 0$, so we choose $g(t) = 0$. We obtain,

$$\psi(t,u) = u\sin(t) + t^2 e^u - u.$$

So the solution y satisfies $y(t)\sin(t) + t^2 e^{y(t)} - y(t) = c.$ $\triangleleft$

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- **Generalization: The integrating factor method.**

Remark:
Sometimes a non-exact equation can we transformed into an exact equation multiplying the equation by an integrating factor. Just like in the case of linear differential equations.

Generalization: The integrating factor method.

Theorem (Integrating factor)
Let $M, N : R \to \mathbb{R}$ be continuously differentiable functions on $R = (t_1, t_2) \times (u_1, u_2) \subset \mathbb{R}^2$, with $N \neq 0$. If the equation

$$N(t, y(t))\, y'(t) + M(t, y(t)) = 0$$

is not exact, that is, $\partial_t N(t,u) \neq \partial_u M(t,u)$, and if the function

$$\frac{1}{N(t,u)}\big[\partial_u M(t,u) - \partial_t N(t,u)\big]$$

does not depend on the variable u, then the equation

$$\mu(t)\big[N(t, y(t))\, y'(t) + M(t, y(t))\big] = 0$$

is exact, where $\dfrac{\mu'(t)}{\mu(t)} = \dfrac{1}{N(t,u)}\big[\partial_u M(t,u) - \partial_t N(t,u)\big].$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

Solution: The equation is not exact:

$$N(t,u) = t^2 + tu \quad\Rightarrow\quad \partial_t N(t,u) = 2t + u,$$

$$M(t,u) = 3tu + u^2 \quad\Rightarrow\quad \partial_u M(t,u) = 3t + 2u,$$

hence $\partial_t N \neq \partial_u M$. We now verify whether the extra condition in Theorem above holds:

$$\frac{\big[\partial_u M(t,u) - \partial_t N(t,u)\big]}{N(t,u)} = \frac{1}{(t^2 + tu)}\big[(3t + 2u) - (2t + u)\big]$$

$$\frac{\big[\partial_u M(t,u) - \partial_t N(t,u)\big]}{N(t,u)} = \frac{1}{t(t+u)}\,(t+u) = \frac{1}{t}.$$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\big[t^2 + t\,y(t)\big]\,y'(t) + \big[3t\,y(t) + y^2(t)\big] = 0.$$

Solution: $\dfrac{\big[\partial_u M(t,u) - \partial_t N(t,u)\big]}{N(t,u)} = \dfrac{1}{t}.$

We find a function μ solution of $\dfrac{\mu'}{\mu} = \dfrac{\big[\partial_u M - \partial_t N\big]}{N}$, that is

$$\frac{\mu'(t)}{\mu(t)} = \frac{1}{t} \quad\Rightarrow\quad \ln(\mu(t)) = \ln(t) \quad\Rightarrow\quad \mu(t) = t.$$

Therefore, the equation below is exact:

$$\big[t^3 + t^2\,y(t)\big]\,y'(t) + \big[3t^2\,y(t) + t\,y^2(t)\big] = 0.$$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\left[t^2 + t\,y(t)\right] y'(t) + \left[3t\,y(t) + y^2(t)\right] = 0.$$

Solution: $\left[t^3 + t^2\,y(t)\right] y'(t) + \left[3t^2\,y(t) + t\,y^2(t)\right] = 0.$

This equation is exact:

$$\tilde{N}(t,u) = t^3 + t^2 u \quad \Rightarrow \quad \partial_t \tilde{N}(t,u) = 3t^2 + 2tu,$$

$$\tilde{M}(t,u) = 3t^2 u + tu^2 \quad \Rightarrow \quad \partial_u \tilde{M}(t,u) = 3t^2 + 2tu,$$

that is, $\partial_t \tilde{N} = \partial_u \tilde{M}$. Therefore, there exists ψ such that

$$\partial_u \psi(t,u) = \tilde{N}(t,u), \qquad \partial_t \psi(t,u) = \tilde{M}(t,u).$$

From the first equation above we obtain

$$\partial_u \psi = t^3 + t^2 u \quad \Rightarrow \quad \psi(t,u) = \int \left(t^3 + t^2 u\right) du + g(t).$$

Generalization: The integrating factor method.

Example

Find all solutions y to the differential equation

$$\left[t^2 + t\,y(t)\right] y'(t) + \left[3t\,y(t) + y^2(t)\right] = 0.$$

Solution: $\psi(t,u) = \int \left(t^3 + t^2 u\right) du + g(t).$

Integrating, $\psi(t,u) = t^3 u + \dfrac{1}{2}\,t^2 u^2 + g(t).$

Introduce ψ in $\partial_t \psi = \tilde{M}$, where $\tilde{M} = 3t^2 u + tu^2$. So,

$$\partial_t \psi(t,u) = 3t^2 u + tu^2 + g'(t) = \tilde{M}(t,u) = 3t^2 u + tu^2,$$

So $g'(t) = 0$ and we choose $g(t) = 0$. We conclude that a potential function is $\psi(t,u) = t^3 u + \dfrac{1}{2}\,t^2 u^2$.

And every solution y satisfies $t^3\,y(t) + \dfrac{1}{2}\,t^2\left[y(t)\right]^2 = c.$ $\lhd$