## Financial Mathematics for Actuaries

## Chapter 2

Annuities

## Learning Objectives

1. Annuity-immediate and annuity-due
2. Present and future values of annuities
3. Perpetuities and deferred annuities
4. Other accumulation methods
5. Payment periods and compounding periods
6. Varying annuities

### 2.1 Annuity-Immediate

- Consider an annuity with payments of 1 unit each, made at the end of every year for $n$ years.
- This kind of annuity is called an annuity-immediate (also called an ordinary annuity or an annuity in arrears).
- The present value of an annuity is the sum of the present values of each payment.

Example 2.1: Calculate the present value of an annuity-immediate of amount $\$ 100$ paid annually for 5 years at the rate of interest of $9 \%$.

Solution: Table 2.1 summarizes the present values of the payments as well as their total.

Table 2.1: Present value of annuity

| Year | Payment (\$) | Present value (\$) |
| :--- | :---: | :---: |
|  |  |  |
| 1 | 100 | $100(1.09)^{-1}=91.74$ |
| 2 | 100 | $100(1.09)^{-2}=84.17$ |
| 3 | 100 | $100(1.09)^{-3}=77.22$ |
| 4 | 100 | $100(1.09)^{-4}=70.84$ |
| 5 | 100 | $100(1.09)^{-5}=64.99$ |
|  |  | 388.97 |

- Suppose the rate of interest per period is $i$, and we assume the compound-interest method applies.
- Let $a_{\bar{n}{ }_{i}}$ denote the present value of the annuity, which is sometimes denoted as $a_{\bar{n}}$ when the rate of interest is understood.
- As the present value of the $j$ th payment is $v^{j}$, where $v=1 /(1+i)$ is the discount factor, the present value of the annuity is (see Appendix A. 5 for the sum of a geometric progression)

$$
\begin{align*}
a_{\bar{n}} & =v+v^{2}+v^{3}+\cdots+v^{n} \\
& =v \times\left[\frac{1-v^{n}}{1-v}\right] \\
& =\frac{1-v^{n}}{i} \\
& =\frac{1-(1+i)^{-n}}{i} . \tag{2.1}
\end{align*}
$$



Figure 2.1: Time diagram of $n$-payment annuity-immediate

- The accumulated value of the annuity at time $n$ is denoted by $s_{\bar{n}} i_{i}$ or $s_{\bar{n}}$.
- This is the future value of $a_{\bar{n} \mid}$ at time $n$. Thus, we have

$$
\begin{align*}
s_{\bar{n}} & =a_{\bar{n} \mid} \times(1+i)^{n} \\
& =\frac{(1+i)^{n}-1}{i} \tag{2.2}
\end{align*}
$$

- If the annuity is of level payments of $P$, the present and future values of the annuity are $P a_{\bar{n} \bar{\eta}}$ and $P s_{\bar{n}}$, respectively.

Example 2.2: Calculate the present value of an annuity-immediate of amount $\$ 100$ paid annually for 5 years at the rate of interest of $9 \%$ using formula (2.1). Also calculate its future value at time 5.

Solution: From (2.1), the present value of the annuity is

$$
100 a_{\overline{5}\rceil}=100 \times\left[\frac{1-(1.09)^{-5}}{0.09}\right]=\$ 388.97
$$

which agrees with the solution of Example 2.1. The future value of the annuity is

$$
(1.09)^{5} \times\left(100 a_{\overline{5}\rceil}\right)=(1.09)^{5} \times 388.97=\$ 598.47
$$

Alternatively, the future value can be calculated as

$$
100 s_{5 \overline{7}}=100 \times\left[\frac{(1.09)^{5}-1}{0.09}\right]=\$ 598.47
$$

Example 2.3: Calculate the present value of an annuity-immediate of amount $\$ 100$ payable quarterly for 10 years at the annual rate of interest
of $8 \%$ convertible quarterly. Also calculate its future value at the end of 10 years.

Solution: Note that the rate of interest per payment period (quarter) is $(8 / 4) \%=2 \%$, and there are $4 \times 10=40$ payments. Thus, from (2.1) the present value of the annuity-immediate is

$$
100 a_{\overline{40}\rceil_{0.02}}=100 \times\left[\frac{1-(1.02)^{-40}}{0.02}\right]=\$ 2,735.55,
$$

and the future value of the annuity-immediate is

$$
2735.55 \times(1.02)^{40}=\$ 6,040.20
$$

- A common problem in financial management is to determine the installments required to pay back a loan. We may use (2.1) to calculate the amount of level installments required.

Example 2.4: A man borrows a loan of $\$ 20,000$ to purchase a car at annual rate of interest of $6 \%$. He will pay back the loan through monthly installments over 5 years, with the first installment to be made one month after the release of the loan. What is the monthly installment he needs to pay?

Solution: The rate of interest per payment period is $(6 / 12) \%=0.5 \%$. Let $P$ be the monthly installment. As there are $5 \times 12=60$ payments, from (2.1) we have

$$
\begin{aligned}
20,000 & =P a_{\overline{60 \mid 0.005}} \\
& =P \times\left[\frac{1-(1.005)^{-60}}{0.005}\right] \\
& =P \times 51.7256
\end{aligned}
$$

so that

$$
P=\frac{20,000}{51.7256}=\$ 386.66
$$

- The example below illustrates the calculation of the required installment for a targeted future value.

Example 2.5: A man wants to save $\$ 100,000$ to pay for his son's education in 10 years' time. An education fund requires the investors to deposit equal installments annually at the end of each year. If interest of $7.5 \%$ is paid, how much does the man need to save each year in order to meet his target?

Solution: We first calculate $s_{\overline{10}}$, which is equal to

$$
\frac{(1.075)^{10}-1}{0.075}=14.1471
$$

Then the required amount of installment is

$$
P=\frac{100,000}{s_{\overline{10}}}=\frac{100,000}{14.1471}=\$ 7,068.59
$$

### 2.2 Annuity-Due

- An annuity-due is an annuity for which the payments are made at the beginning of the payment periods
- The first payment is made at time 0 , and the last payment is made at time $n-1$.
- We denote the present value of the annuity-due at time 0 by $\ddot{a}_{\bar{n}\rangle_{i}}$ (or $\left.\ddot{a}_{\bar{n}}\right)$, and the future value of the annuity at time $n$ by $\ddot{s}_{\bar{n} i_{i}}\left(\right.$ or $\left.\ddot{s}_{\bar{n}}\right)$.
- The formula for $\ddot{a}_{\bar{n}}$ can be derived as follows

$$
\begin{align*}
\ddot{a}_{\bar{n}} & =1+v+\cdots+v^{n-1} \\
& =\frac{1-v^{n}}{1-v} \\
& =\frac{1-v^{n}}{d} . \tag{2.3}
\end{align*}
$$



Figure 2.2: Time diagram of $n$-payment annuity-due

- Also, we have

$$
\begin{align*}
\ddot{s}_{\bar{n}} & =\ddot{a}_{\bar{n} \mid} \times(1+i)^{n} \\
& =\frac{(1+i)^{n}-1}{d} . \tag{2.4}
\end{align*}
$$

- As each payment in an annuity-due is paid one period ahead of the corresponding payment of an annuity-immediate, the present value of each payment in an annuity-due is $(1+i)$ times of the present value of the corresponding payment in an annuity-immediate. Hence,

$$
\begin{equation*}
\ddot{a}_{\bar{n}}=(1+i) a_{\bar{n}} \tag{2.5}
\end{equation*}
$$

and, similarly,

$$
\begin{equation*}
\ddot{s}_{\bar{n} \bar{\prime}}=(1+i) s_{\bar{n}} . \tag{2.6}
\end{equation*}
$$

- As an annuity-due of $n$ payments consists of a payment at time 0 and an annuity-immediate of $n-1$ payments, the first payment of which is to be made at time 1 , we have

$$
\begin{equation*}
\ddot{a}_{\bar{n}}=1+a_{\overline{n-1}} . \tag{2.7}
\end{equation*}
$$

- Similarly, if we consider an annuity-immediate with $n+1$ payments at time $1,2, \cdots, n+1$ as an annuity-due of $n$ payments starting at time 1 plus a final payment at time $n+1$, we can conclude

$$
\begin{equation*}
s_{\overline{n+1}}=\ddot{s}_{\bar{n}}+1 \tag{2.8}
\end{equation*}
$$

Example 2.6: A company wants to provide a retirement plan for an employee who is aged 55 now. The plan will provide her with an annuityimmediate of $\$ 7,000$ every year for 15 years upon her retirement at the
age of 65 . The company is funding this plan with an annuity-due of 10 years. If the rate of interest is $5 \%$, what is the amount of installment the company should pay?

Solution: We first calculate the present value of the retirement annuity. This is equal to

$$
7,000 a_{\overline{15}}=7,000 \times\left[\frac{1-(1.05)^{-15}}{0.05}\right]=\$ 72,657.61
$$

This amount should be equal to the future value of the company's installments $P$, which is $P \ddot{s}_{\overline{10} \mid}$. Now from (2.4), we have

$$
\ddot{s}_{\overline{10}}=\frac{(1.05)^{10}-1}{1-(1.05)^{-1}}=13.2068
$$

so that

$$
P=\frac{72,657.61}{13.2068}=\$ 5,501.53
$$

### 2.3 Perpetuity, Deferred Annuity and Annuity Values at Other Times

- A perpetuity is an annuity with no termination date, i.e., $n \rightarrow \infty$.
- An example that resembles a perpetuity is the dividends of a preferred stock.
- To calculate the present value of a perpetuity, we note that, as $v<1$, $v^{n} \rightarrow 0$ as $n \rightarrow \infty$. Thus, from (2.1), we have

$$
\begin{equation*}
a_{\infty}=\frac{1}{i} . \tag{2.9}
\end{equation*}
$$

- For the case when the first payment is made immediately, we have, from (2.3),

$$
\begin{equation*}
\ddot{a}_{\bar{\infty} \mid}=\frac{1}{d} . \tag{2.10}
\end{equation*}
$$

- A deferred annuity is one for which the first payment starts some time in the future.
- Consider an annuity with $n$ unit payments for which the first payment is due at time $m+1$.
- This can be regarded as an $n$-period annuity-immediate to start at time $m$, and its present value is denoted by ${ }_{m \mid} a_{\left.\bar{n}\right|_{i}}$ ( or ${ }_{m \mid} a_{\bar{n} \mid}$ for short). Thus, we have

$$
\begin{aligned}
m_{m} a_{\bar{n} \mid} & =v^{m} a_{\bar{n}} \\
& =v^{m} \times\left[\frac{1-v^{n}}{i}\right] \\
& =\frac{v^{m}-v^{m+n}}{i} \\
& =\frac{\left(1-v^{m+n}\right)-\left(1-v^{m}\right)}{i}
\end{aligned}
$$

$$
\begin{equation*}
=a_{\overline{m+n}}-a_{\bar{m}\rceil} \tag{2.11}
\end{equation*}
$$

- To understand the above equation, see Figure 2.3.
- From (2.11), we have

$$
\begin{align*}
a_{\overline{m+n}} & =a_{\bar{m}}+v^{m} a_{\bar{n}} \\
& =a_{\bar{n}}+v^{n} a_{\bar{m}} \tag{2.12}
\end{align*}
$$

- Multiplying the above equations throughout by $1+i$, we have

$$
\begin{align*}
\ddot{a}_{\overline{m+n}} & =\ddot{a}_{\bar{m} \mid}+v^{m} \ddot{a}_{\bar{n}} \\
& =\ddot{a}_{\bar{n}}+v^{n} \ddot{a}_{\bar{m} \mid} . \tag{2.13}
\end{align*}
$$

- We also denote $v^{m} \ddot{a}_{\bar{n}}$ as ${ }_{m \mid} \ddot{a}_{\bar{n}\rceil}$, which is the present value of a $n$ payment annuity of unit amounts due at time $m, m+1, \cdots, m+n-1$.


Present value
of annuity

$$
a_{\bar{m}\rceil} \longleftrightarrow
$$

$$
\begin{aligned}
& a_{\overline{m+n}} \longleftrightarrow \\
& v^{m} a_{\bar{n} \ll-----------------------} a_{\bar{n} \mid}
\end{aligned}
$$

Figure 2.3: Illustration of equation (2.11)

- If we multiply the equations in (2.12) throughout by $(1+i)^{m+n}$, we obtain

$$
\begin{align*}
s_{\overline{m+n}} & =(1+i)^{n} s_{\bar{m}}+s_{\bar{n}\rceil} \\
& =(1+i)^{m} s_{\bar{n}\rceil}+s_{\bar{m}\rceil} \tag{2.14}
\end{align*}
$$

- See Figure 2.4 for illustration.
- It is also straightforward to see that

$$
\begin{align*}
\ddot{s}_{\overline{m+n}} & =(1+i)^{n} \ddot{s}_{\bar{m} \mid}+\ddot{s}_{\bar{n}} \\
& =(1+i)^{m} \ddot{s}_{\bar{n}}+\ddot{s}_{\bar{m}\rceil} . \tag{2.15}
\end{align*}
$$

- We now return to (2.2) and write it as

$$
s_{\overline{m+n}}=(1+i)^{m+n} a_{\overline{m+n}},
$$



Future value of annuity


Figure 2.4: Illustration of equation (2.14)
so that

$$
\begin{equation*}
v^{m} s_{\overline{m+n}}=(1+i)^{n} a_{\overline{m+n}} \tag{2.16}
\end{equation*}
$$

for arbitrary positive integers $m$ and $n$.

- How do you interpret this equation?


Figure 2.5: Illustration of equation (2.16)

### 2.4 Annuities Under Other Accumulation Methods

- We have so far discussed the calculations of the present and future values of annuities assuming compound interest.
- We now extend our discussion to other interest-accumulation methods.
- We consider a general accumulation function $a(\cdot)$ and assume that the function applies to any cash-flow transactions in the future.
- As stated in Section 1.7, any payment at time $t>0$ starts to accumulate interest according to $a(\cdot)$ as a payment made at time 0 .
- Given the accumulation function $a(\cdot)$, the present value of a unit payment due at time $t$ is $1 / a(t)$, so that the present value of a $n$ -
period annuity-immediate of unit payments is

$$
\begin{equation*}
a_{\bar{n} \mid}=\sum_{t=1}^{n} \frac{1}{a(t)} . \tag{2.17}
\end{equation*}
$$

- The future value at time $n$ of a unit payment at time $t<n$ is $a(n-t)$, so that the future value of a $n$-period annuity-immediate of unit payments is

$$
\begin{equation*}
s_{\bar{n}\rceil}=\sum_{t=1}^{n} a(n-t) . \tag{2.18}
\end{equation*}
$$

- If (1.35) is satisfied so that $a(n-t)=a(n) / a(t)$ for $n>t>0$, then

$$
\begin{equation*}
s_{\bar{n} \mid}=\sum_{t=1}^{n} \frac{a(n)}{a(t)}=a(n) \sum_{t=1}^{n} \frac{1}{a(t)}=a(n) a_{\bar{n} \mid} . \tag{2.19}
\end{equation*}
$$

- This result is satisfied for the compound-interest method, but not the simple-interest method or other accumulation schemes for which equation (1.35) does not hold.

Example 2.7: $\quad$ Suppose $\delta(t)=0.02 t$ for $0 \leq t \leq 5$, find $a_{5}$ and $s_{5\rceil}$.
Solution: We first calculate $a(t)$, which, from (1.26), is

$$
\begin{aligned}
a(t) & =\exp \left(\int_{0}^{t} 0.02 s d s\right) \\
& =\exp \left(0.01 t^{2}\right)
\end{aligned}
$$

Hence, from (2.17),

$$
a_{\overline{5}}=\frac{1}{e^{0.01}}+\frac{1}{e^{0.04}}+\frac{1}{e^{0.09}}+\frac{1}{e^{0.16}}+\frac{1}{e^{0.25}}=4.4957
$$

and, from (2.18),

$$
s_{\overline{5}\rceil}=1+e^{0.01}+e^{0.04}+e^{0.09}+e^{0.16}=5.3185
$$

Note that $a(5)=e^{0.25}=1.2840$, so that

$$
a(5) a_{\overline{5}\rceil}=1.2840 \times 4.4957=5.7724 \neq s_{5\rceil} .
$$

- Note that in the above example, $a(n-t)=\exp \left[0.01(n-t)^{2}\right]$ and

$$
\frac{a(n)}{a(t)}=\exp \left[0.01\left(n^{2}-t^{2}\right)\right]
$$

so that $a(n-t) \neq a(n) / a(t)$ and (2.19) does not hold.

Example 2.8: Calculate $a_{\overline{3}}$ and $s_{\overline{3}}$ if the nominal rate of interest is $5 \%$ per annum, assuming (a) compound interest, and (b) simple interest.

Solution: (a) Assuming compound interest, we have

$$
a_{\overline{3} \mid}=\frac{1-(1.05)^{-3}}{0.05}=2.723
$$

and

$$
s_{\overline{3} \mid}=(1.05)^{3} \times 2.72=3.153
$$

(b) For simple interest, the present value is

$$
a_{\overline{3} \mid}=\sum_{t=1}^{3} \frac{1}{a(t)}=\sum_{t=1}^{3} \frac{1}{1+r t}=\frac{1}{1.05}+\frac{1}{1.1}+\frac{1}{1.15}=2.731
$$

and the future value at time 3 is

$$
s_{\overline{3} \mid}=\sum_{t=1}^{3} a(3-t)=\sum_{t=1}^{3}(1+r(3-t))=1.10+1.05+1.0=3.150
$$

At the same nominal rate of interest, the compound-interest method generates higher interest than the simple-interest method. Therefore, the future value under the compound-interest method is higher, while its present value is lower. Also, note that for the simple-interest method, $a(3) a_{\overline{3}\rceil}=1.15 \times 2.731=3.141$, which is different from $s_{\overline{3}\rceil}=3.150$.

### 2.5 Payment Periods, Compounding Periods and Continuous Annuities

- We now consider the case where the payment period differs from the interest-conversion period.

Example 2.9: Find the present value of an annuity-due of $\$ 200$ per quarter for 2 years, if interest is compounded monthly at the nominal rate of $8 \%$.

Solution: This is the situation where the payments are made less frequently than interest is converted. We first calculate the effective rate of interest per quarter, which is

$$
\left[1+\frac{0.08}{12}\right]^{3}-1=2.01 \%
$$

As there are $n=8$ payments, the required present value is

$$
200 \ddot{a}_{\left.\overline{8}\right|_{0.0201}}=200 \times\left[\frac{1-(1.0201)^{-8}}{1-(1.0201)^{-1}}\right]=\$ 1,493.90
$$

Example 2.10: Find the present value of an annuity-immediate of $\$ 100$ per quarter for 4 years, if interest is compounded semiannually at the nominal rate of $6 \%$.

Solution: This is the situation where payments are made more frequently than interest is converted. We first calculate the effective rate of interest per quarter, which is

$$
\left[1+\frac{0.06}{2}\right]^{\frac{1}{2}}-1=1.49 \%
$$

Thus, the required present value is

$$
100 a_{\left.\overline{16}\right|_{0.0149}}=100 \times\left[\frac{1-(1.0149)^{-16}}{0.0149}\right]=\$ 1,414.27
$$

- It is possible to derive algebraic formulas to compute the present and future values of annuities for which the period of installment is different from the period of compounding.
- We first consider the case where payments are made less frequently than interest conversion, which occurs at time $1,2, \cdots$, etc.
- Let $i$ denote the effective rate of interest per interest-conversion period. Suppose a $m$-payment annuity-immediate consists of unit payments at time $k, 2 k, \cdots, m k$. We denote $n=m k$, which is the number of interest-conversion periods for the annuity.
- Figure 2.6 illustrates the cash flows for the case of $k=2$.
- The present value of the above annuity-immediate is (we let $w=v^{k}$ )

$$
\begin{align*}
v^{k}+v^{2 k}+\cdots+v^{m k} & =w+w^{2}+\cdots+w^{m} \\
& =w \times\left[\frac{1-w^{m}}{1-w}\right] \\
& =v^{k} \times\left[\frac{1-v^{n}}{1-v^{k}}\right] \\
& =\frac{1-v^{n}}{(1+i)^{k}-1} \\
& =\frac{a_{\bar{n}}}{s_{\bar{k}\rceil}} \tag{2.20}
\end{align*}
$$

and the future value of the annuity is

$$
\begin{equation*}
(1+i)^{n} \frac{a_{\bar{n}\rceil}}{s_{\bar{k}\rceil}}=\frac{s_{\bar{n}}}{s_{\bar{k}\rceil}} . \tag{2.21}
\end{equation*}
$$



Figure 2.6: Payments less frequent than interest conversion $(k=2)$

- We now consider the case where the payments are made more frequently than interest conversion.
- Let there be $m n$ payments for an annuity-immediate occurring at time $1 / m, 2 / m, \cdots, 1,1+1 / m, \cdots, 2, \cdots, n$, and let $i$ be the effective rate of interest per interest-conversion period. Thus, there are $m n$ payments over $n$ interest-conversion periods.
- Suppose each payment is of the amount $1 / m$, so that there is a nominal amount of unit payment in each interest-conversion period.
- Figure 2.7 illustrates the cash flows for the case of $m=4$.
- We denote the present value of this annuity at time 0 by $a_{\bar{n}\rceil_{i}}^{(m)}$, which can be computed as follows (we let $w=v^{\frac{1}{m}}$ )

$$
a_{\bar{n}\rceil_{i}}^{(m)}=\frac{1}{m}\left(v^{\frac{1}{m}}+v^{\frac{2}{m}}+\cdots+v+v^{1+\frac{1}{m}}+\cdots+v^{n}\right)
$$

Cash flow


Payment period ( $m=4$ )

$$
0 \quad \frac{1}{m} \quad \frac{2}{m} \quad \frac{3}{m} \quad 1 \quad 1+\frac{1}{m} \quad \ldots \ldots . \quad n-\frac{1}{m} n
$$

Interest-conversion period

1

Figure 2.7: Payments more frequent than interest conversion ( $m=4$ )

$$
\begin{align*}
& =\frac{1}{m}\left(w+w^{2}+\cdots+w^{m n}\right) \\
& =\frac{1}{m}\left[w \times \frac{1-w^{m n}}{1-w}\right] \\
& =\frac{1}{m}\left[v^{\frac{1}{m}} \times \frac{1-v^{n}}{1-v^{\frac{1}{m}}}\right] \\
& =\frac{1}{m}\left[\frac{1-v^{n}}{(1+i)^{\frac{1}{m}}-1}\right] \\
& =\frac{1-v^{n}}{r^{(m)}} \tag{2.22}
\end{align*}
$$

where

$$
\begin{equation*}
r^{(m)}=m\left[(1+i)^{\frac{1}{m}}-1\right] \tag{2.23}
\end{equation*}
$$

is the equivalent nominal rate of interest compounded $m$ times per interest-conversion period (see (1.19)).

- The future value of the annuity-immediate is

$$
\begin{align*}
s_{\bar{n} i}^{(m)} & =(1+i)^{n} a_{\bar{n} \mid i}^{(m)} \\
& =\frac{(1+i)^{n}-1}{r^{(m)}} \\
& =\frac{i}{r^{(m)}} s_{\bar{n}\rceil} \tag{2.24}
\end{align*}
$$

- The above equation parallels (2.22), which can also be written as

$$
a_{\bar{n}\rangle_{i}}^{(m)}=\frac{i}{r^{(m)}} a_{\bar{n} \mid}
$$

- If the $m n$-payment annuity is due at time $0,1 / m, 2 / m, \cdots, n-1 / m$, we denote its present value at time 0 by $\ddot{a}_{\bar{n}}^{(m)}$, which is given by

$$
\begin{equation*}
\ddot{a} \frac{(m)}{n}=(1+i)^{\frac{1}{m}} a_{\bar{n}}^{(m)}=(1+i)^{\frac{1}{m}} \times\left[\frac{1-v^{n}}{r^{(m)}}\right] \tag{2.25}
\end{equation*}
$$

- Thus, from (1.22) we conclude

$$
\begin{equation*}
\ddot{a}_{\bar{n}\rceil}^{(m)}=\frac{1-v^{n}}{d^{(m)}}=\frac{d}{d^{(m)}} \ddot{a}_{\bar{n} \downarrow} . \tag{2.26}
\end{equation*}
$$

- The future value of this annuity at time $n$ is

$$
\begin{equation*}
\ddot{s}_{\bar{n}\rceil}^{(m)}=(1+i)^{n} \ddot{a}_{\bar{n}\rceil}^{(m)}=\frac{d}{d^{(m)}} \ddot{s}_{\bar{n} \mid} . \tag{2.27}
\end{equation*}
$$

- For deferred annuities, the following results apply

$$
\begin{equation*}
{ }_{q \mid} a_{\bar{n}\rceil}^{(m)}=v^{q} a_{\bar{n}}^{(m)} \tag{2.28}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.{ }_{q \mid} \ddot{a} \bar{n} \bar{\eta}\right)=v^{q} \ddot{a} \bar{n}(m) \tag{2.29}
\end{equation*}
$$

Example 2.11: $\quad$ Solve the problem in Example 2.9 using (2.20).

Solution: We first note that $i=0.08 / 12=0.0067$. Now $k=3$ and $n=24$ so that from (2.20), the present value of the annuity-immediate is

$$
\begin{aligned}
200 \times \frac{a_{\left.\overline{24}\right|_{0.0067}}}{s_{\left.\overline{3}\right|_{0.0067}}} & =200 \times\left[\frac{1-(1.0067)^{-24}}{(1.0067)^{3}-1}\right] \\
& =\$ 1,464.27
\end{aligned}
$$

Finally, the present value of the annuity-due is

$$
(1.0067)^{3} \times 1,464.27=\$ 1,493.90
$$

Example 2.12: $\quad$ Solve the problem in Example 2.10 using (2.22) and (2.23).

Solution: Note that $m=2$ and $n=8$. With $i=0.03$, we have, from
(2.23)

$$
r^{(2)}=2 \times[\sqrt{1.03}-1]=0.0298
$$

Therefore, from (2.22), we have

$$
a_{\overline{8}\rceil_{0.03}^{(2)}}^{\left(1-(1.03)^{-8}\right.} \frac{1.0298}{0 .}=7.0720
$$

As the total payment in each interest-conversion period is $\$ 200$, the required present value is

$$
200 \times 7.0720=\$ 1,414.27
$$

- Now we consider (2.22) again. Suppose the annuities are paid continuously at the rate of 1 unit per interest-conversion period over $n$ periods. Thus, $m \rightarrow \infty$ and we denote the present value of this continuous annuity by $\bar{a}_{\bar{n}}$.
- As $\lim _{m \rightarrow \infty} r^{(m)}=\delta$, we have, from (2.22),

$$
\begin{equation*}
\bar{a}_{\bar{n} \mid}=\frac{1-v^{n}}{\delta}=\frac{1-v^{n}}{\ln (1+i)}=\frac{i}{\delta} a_{\bar{n}\rceil} . \tag{2.30}
\end{equation*}
$$

- The present value of a $n$-period continuous annuity of unit payment per period with a deferred period of $q$ is given by

$$
\begin{equation*}
{ }_{q} \mid \bar{a}_{\bar{n} \mid}=v^{q} \bar{a}_{\bar{n}\rceil}=\bar{a}_{\overline{q+n} \mid}-\bar{a}_{\bar{q}\rceil} . \tag{2.31}
\end{equation*}
$$

- To compute the future value of a continuous annuity of unit payment per period over $n$ periods, we use the following formula

$$
\begin{equation*}
\bar{s}_{\bar{n}}=(1+i)^{n} \bar{a}_{\bar{n}}=\frac{(1+i)^{n}-1}{\ln (1+i)}=\frac{i}{\delta} s_{\bar{n} \mid} . \tag{2.32}
\end{equation*}
$$

- We now generalize the above results to the case of a general accumulation function $a(\cdot)$. The present value of a continuous annuity
of unit payment per period over $n$ periods is

$$
\begin{equation*}
\bar{a}_{\bar{n} \mid}=\int_{0}^{n} v(t) d t=\int_{0}^{n} \exp \left(-\int_{0}^{t} \delta(s) d s\right) d t \tag{2.33}
\end{equation*}
$$

- To compute the future value of the annuity at time $n$, we assume that, as in Section 1.7, a unit payment at time $t$ accumulates to $a(n-t)$ at time $n$, for $n>t \geq 0$.
- Thus, the future value of the annuity at time $n$ is

$$
\begin{equation*}
\bar{s}_{\bar{n} \overline{ }}=\int_{0}^{n} a(n-t) d t=\int_{0}^{n} \exp \left(\int_{0}^{n-t} \delta(s) d s\right) d t \tag{2.34}
\end{equation*}
$$

### 2.6 Varying Annuities

- We consider annuities the payments of which vary according to an arithmetic progression.
- Thus, we consider an annuity-immediate and assume the initial payment is $P$, with subsequent payments $P+D, P+2 D, \cdots$, etc., so that the $j$ th payment is $P+(j-1) D$.
- We allow $D$ to be negative so that the annuity can be either stepping up or stepping down.
- However, for a $n$-payment annuity, $P+(n-1) D$ must be positive so that negative cash flow is ruled out.
- We can see that the annuity can be regarded as the sum of the following annuities: (a) a $n$-period annuity-immediate with constant
amount $P$, and (b) $n-1$ deferred annuities, where the $j$ th deferred annuity is a $(n-j)$-period annuity-immediate with level amount $D$ to start at time $j$, for $j=1, \cdots, n-1$.
- Thus, the present value of the varying annuity is

$$
\begin{align*}
P a_{\bar{n}\rceil}+D\left[\sum_{j=1}^{n-1} v^{j} a_{\overline{n-j}}\right] & =P a_{\bar{n} \mid}+D\left[\sum_{j=1}^{n-1} v^{j} \frac{\left(1-v^{n-j}\right)}{i}\right] \\
& =P a_{\bar{n} \mid}+D\left[\frac{\left(\sum_{j=1}^{n-1} v^{j}\right)-(n-1) v^{n}}{i}\right] \\
& =P a_{\bar{n} \mid}+D\left[\frac{\left(\sum_{j=1}^{n} v^{j}\right)-n v^{n}}{i}\right] \\
& =P a_{\bar{n} \mid}+D\left[\frac{a_{\bar{n}\rceil}-n v^{n}}{i}\right] . \tag{2.35}
\end{align*}
$$



Figure 2.8: Increasing annuity-immediate

- For a $n$-period increasing annuity with $P=D=1$, we denote its present and future values by $(I a)_{\bar{n} \mid}$ and $(I s)_{\bar{n} \mid}$, respectively.
- It can be shown that

$$
\begin{equation*}
(I a)_{\bar{n} \mid}=\frac{\ddot{a}_{\bar{n} \mid}-n v^{n}}{i} \tag{2.36}
\end{equation*}
$$

and

$$
\begin{equation*}
(I s)_{\bar{n} \mid}=\frac{s_{\overline{n+1}}-(n+1)}{i}=\frac{\ddot{s}_{\bar{n}}-n}{i} \tag{2.37}
\end{equation*}
$$

- For an increasing $n$-payment annuity-due with payments of $1,2, \cdots, n$ at time $0,1, \cdots, n-1$, the present value of the annuity is

$$
\begin{equation*}
(I \ddot{a})_{\bar{n} \overline{1}}=(1+i)(I a)_{\bar{n}\rceil} . \tag{2.38}
\end{equation*}
$$

- This is the sum of a $n$-period level annuity-due of unit payments and a ( $n-1$ )-payment increasing annuity-immediate with starting and incremental payments of 1 .
- Thus, we have

$$
\begin{equation*}
(I \ddot{a})_{\bar{n} \overline{1}}=\ddot{a}_{\bar{n} \mid}+(I a)_{\overline{n-1}} . \tag{2.39}
\end{equation*}
$$

- For the case of a $n$-period decreasing annuity with $P=n$ and $D=$ -1 , we denote its present and future values by $(D a)_{\bar{n}}$ and $(D s)_{\bar{n}}$, respectively.
- Figure 2.9 presents the time diagram of this annuity.
- It can be shown that

$$
\begin{equation*}
(D a)_{\bar{n}}=\frac{n-a_{\bar{n}}}{i} \tag{2.40}
\end{equation*}
$$

and

$$
\begin{equation*}
(D s)_{\bar{n} \mid}=\frac{n(1+i)^{n}-s_{\bar{n} \mid}}{i} \tag{2.41}
\end{equation*}
$$



Figure 2.9: Decreasing annuity-immediate with $P=n$ and $D=-1$

- We consider two types of increasing continuous annuities. First, we consider the case of a continuous $n$-period annuity with level payment (i.e., at a constant rate) of $\tau$ units from time $\tau-1$ through time $\tau$.
- We denote the present value of this annuity by $(I \bar{a})_{\bar{n} \mid}$, which is given by

$$
\begin{equation*}
(I \bar{a})_{\bar{n} \mid}=\sum_{\tau=1}^{n} \tau \int_{\tau-1}^{\tau} v^{s} d s=\sum_{\tau=1}^{n} \tau \int_{\tau-1}^{\tau} e^{-\delta s} d s \tag{2.42}
\end{equation*}
$$

- The above equation can be simplified to

$$
\begin{equation*}
(I \bar{a})_{\bar{n} \mid}=\frac{\ddot{a}_{\bar{n} \mid}-n v^{n}}{\delta} \tag{2.43}
\end{equation*}
$$

- Second, we may consider a continuous n-period annuity for which the payment in the interval $t$ to $t+\Delta t$ is $t \Delta t$, i.e., the instantaneous rate of payment at time $t$ is $t$.
- We denote the present value of this annuity by $(\bar{I} \bar{a})_{\bar{n}\rceil}$, which is given by

$$
\begin{equation*}
(\bar{I} \bar{a})_{\bar{n} \mid}=\int_{0}^{n} t v^{t} d t=\int_{0}^{n} t e^{-\delta t} d t=\frac{\bar{a}_{\bar{n}}-n v^{n}}{\delta} \tag{2.44}
\end{equation*}
$$

- We now consider an annuity-immediate with payments following a geometric progression.
- Let the first payment be 1 , with subsequent payments being $1+k$ times the previous one. Thus, the present value of an annuity with $n$ payments is (for $k \neq i$ )

$$
\begin{aligned}
v+v^{2}(1+k)+\cdots+v^{n}(1+k)^{n-1} & =v \sum_{t=0}^{n-1}[v(1+k)]^{t} \\
& =v \sum_{t=0}^{n-1}\left[\frac{1+k}{1+i}\right]^{t}
\end{aligned}
$$

$$
\begin{align*}
& =v\left[\frac{1-\left(\frac{1+k}{1+i}\right)^{n}}{1-\frac{1+k}{1+i}}\right] \\
& =\frac{1-\left(\frac{1+k}{1+i}\right)^{n}}{i-k} \tag{2.45}
\end{align*}
$$

Example 2.13: An annuity-immediate consists of a first payment of $\$ 100$, with subsequent payments increased by $10 \%$ over the previous one until the 10th payment, after which subsequent payments decreases by $5 \%$ over the previous one. If the effective rate of interest is $10 \%$ per payment period, what is the present value of this annuity with 20 payments?

Solution: The present value of the first 10 payments is (use the second
line of (2.45))

$$
100 \times 10(1.1)^{-1}=\$ 909.09
$$

For the next 10 payments, $k=-0.05$ and their present value at time 10 is

$$
100(1.10)^{9}(0.95) \times \frac{1-\left(\frac{0.95}{1.1}\right)^{10}}{0.1+0.05}=1,148.64
$$

Hence, the present value of the 20 payments is

$$
909.09+1,148.64(1.10)^{-10}=\$ 1,351.94
$$

Example 2.14: An investor wishes to accumulate $\$ 1,000$ at the end of year 5 . He makes level deposits at the beginning of each year for 5 years. The deposits earn a $6 \%$ annual effective rate of interest, which is credited
at the end of each year. The interests on the deposits earn $5 \%$ effective interest rate annually. How much does he have to deposit each year?

Solution: Let the level annual deposit be $A$. The interest received at the end of year 1 is $0.06 A$, which increases by $0.06 A$ annually to $5 \times 0.06 A$ at the end of year 5 . Thus, the interest is a 5 -payment increasing annuity with $P=D=0.06 A$, earning annual interest of $5 \%$. Hence, we have the equation

$$
1,000=5 A+0.06 A(I s)_{\overline{5} \mid 0.05}
$$

From (2.37) we obtain $(I s)_{\left.5\right|_{0.05}}=16.0383$, so that

$$
A=\frac{1,000}{5+0.06 \times 16.0383}=\$ 167.7206
$$

### 2.7 Term of Annuity

- We now consider the case where the annuity period may not be an integer. We consider $a_{\overline{n+k}}$, where $n$ is an integer and $0<k<1$. We note that

$$
\begin{align*}
a_{\overline{n+k}} & =\frac{1-v^{n+k}}{i} \\
& =\frac{\left(1-v^{n}\right)+\left(v^{n}-v^{n+k}\right)}{i} \\
& =a_{\bar{n} \mid}+v^{n+k}\left[\frac{(1+i)^{k}-1}{i}\right] \\
& =a_{\bar{n} \mid}+v^{n+k} s_{\bar{k} \mid} . \tag{2.46}
\end{align*}
$$

- Thus, $a_{\overline{n+k}}$ is the sum of the present value of a $n$-period annuityimmediate with unit amount and the present value of an amount $s_{\bar{k}}$
paid at time $n+k$.

Example 2.15: A principal of $\$ 5,000$ generates income of $\$ 500$ at the end of every year at an effective rate of interest of $4.5 \%$ for as long as possible. Calculate the term of the annuity and discuss the possibilities of settling the last payment.

Solution: The equation of value

$$
500 a_{\bar{n}\rceil_{0.045}}=5,000
$$

implies $a_{\bar{n}\rceil_{0.045}}=10$. As $a_{\overline{13}]_{0.045}}=9.68$ and $a_{\left.\overline{14}\right|_{0.045}}=10.22$, the principal can generate 13 regular payments. The investment may be paid off with an additional amount $A$ at the end of year 13, in which case

$$
500 s_{\overline{13}]_{0.045}+A=5,000(1.045)^{13}, ~}^{1 .}
$$

which implies $A=\$ 281.02$, so that the last payment is $\$ 781.02$. Alternatively, the last payment $B$ may be made at the end of year 14 , which is

$$
B=281.02 \times 1.045=\$ 293.67
$$

If we adopt the approach in (2.46), we solve $k$ from the equation

$$
\frac{1-(1.045)^{-(13+k)}}{0.045}=10
$$

which implies

$$
(1.045)^{13+k}=\frac{1}{0.55}
$$

from which we obtain

$$
k=\frac{\ln \left(\frac{1}{0.55}\right)}{\ln (1.045)}-13=0.58
$$

Hence, the last payment $C$ to be paid at time 13.58 years is

$$
C=500 \times\left[\frac{(1.045)^{0.58}-1}{0.045}\right]=\$ 288.32
$$

Note that $A<C<B$, which is as expected, as this follows the order of the occurrence of the payments. In principle, all three approaches are justified.

- Generally, the effective rate of interest cannot be solved analytically from the equation of value. Numerical methods must be used for this purpose.

Example 2.16: A principal of $\$ 5,000$ generates income of $\$ 500$ at the end of every year for 15 years. What is the effective rate of interest?

Solution: The equation of value is

$$
a_{\overline{15} i}=\frac{5,000}{500}=10
$$

so that

$$
a_{\overline{15} i}=\frac{1-(1+i)^{-15}}{i}=10
$$

A simple grid search provides the following results

| $i$ | $a_{\overline{15} i}$ |
| :---: | ---: |
| 0.054 | 10.10 |
| 0.055 | 10.04 |
| 0.056 | 9.97 |

A finer search provides the answer $5.556 \%$.

- The Excel Solver may be used to calculate the effective rate of interest in Example 2.16.
- The computation is illustrated in Exhibit 2.1.
- We enter a guessed value of 0.05 in Cell A1 in the Excel worksheet.
- The following expression is then entered in Cell A2: $\left(1-(1+\mathrm{A} 1)^{\wedge}(-15)\right) / \mathrm{A} 1$, which computes $a_{\overline{15} i}$ with $i$ equal to the value at A 1 .
- We can also use the Excel function RATE to calculate the rate of interest that equates the present value of an annuity-immediate to a given value. Specifically, consider the equations

$$
a_{\bar{n}\rceil_{i}}-A=0 \quad \text { and } \quad \quad \ddot{a}_{\bar{n}\rceil_{i}}-A=0
$$

Given $n$ and $A$, we wish to solve for $i$, which is the rate of interest per payment period of the annuity-immediate or annuity-due. The use of the Excel function RATE to compute $i$ is described as follows:


Exhibit 2.1: Use of Excel Solver for Example 2.16

```
Excel function: RATE(np,1,pv,type,guess)
np = n,
pv = - A,
type = 0 (or omitted) for annuity-immediate, 1 for annuity-due
guess = starting value, set to 0.1 if omitted
Output =i, rate of interest per payment period of the annuity
```

- To use RATE to solve for Example 2.16 we key in the following: $"=\operatorname{RATE}(15,1,-10) "$.

