

An Introduction to Financial Mathematics

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1 Introduction

A wealthy acquaintance when recently asked about his profession reluctantly answered that he is a middleman in drug trade and has made a fortune helping drugs reach European markets from Latin America. When pressed further, he confessed that he was actually a ‘quant’ in a huge wall street bank and used mathematics to price complex derivative securities. He lied simply to appear respectable! There you have it. Its not fashionable to be a financial mathematician these days. On the plus side, these quants or financial mathematicians on the wall street are sufficiently rich that they can literally afford to ignore fashions.

On a serious note, it is important to acknowledge that financial markets serve a fundamental role in economic growth of nations by helping efficient allocation of investment of individuals to the most productive sectors of the economy. They also provide an avenue for corporates to raise capital for productive ventures. Financial sector has seen enormous growth over the past thirty years in the developed world. This growth has been led by the innovations in products referred to as financial derivatives that require great deal of mathematical sophistication and ingenuity in pricing and in creating an insurance or hedge against associated risks. This chapter briefly discusses some such popular derivatives including those that played a substantial role in the economic crisis of 2008. Our primary focus are the key underlying mathematical ideas that are used to price such derivatives. We present these in a somewhat simple setting.

Brief history: During the industrial revolution in Europe there existed great demand for setting up huge industrial units. To raise capital, entrepreneurs came together to form joint partnerships where they owned ‘shares’ of the newly formed company. Soon there were many such companies each with many shares held by public at large. This was facilitated by setting up of stock exchanges where these shares or stocks could be bought or sold. London stock exchange was first such institution, set up in 1773. Basic financial derivatives such as futures have been around for some time (we do not discuss futures in this chapter; they are very similar to the forward contracts discussed below). The oldest and the largest futures and options exchange, The Chicago Board of Trade (CBOT), was established in 1848. Although, as we discuss later, activity in financial derivatives took off in a major way beginning the early 1970’s.

Brief introduction to derivatives (see, e.g., [16] for a comprehensive overview): A derivative is a financial instrument that derives its value from an ‘underlying’ more basic asset. For instance, consider a forward contract, a popular derivative, between two parties: One party agrees to purchase from the other a specified asset at a particular time in future for a specified price. For instance, Infosys, expecting income in dollars in future (with its expenses in rupees) may enter into a forward contract with ICICI bank that requires it to purchase a specified amount of rupees, say Rs. 430 crores, using specified amount of dollars, say, \$ 10 crore, six months from now. Here, the fluctuations in more basic underlying exchange rate gives value to the forward contract.

Options are popular derivatives that give buyer of this instrument an option but not an obligation to engage in specific transactions related to the underlying assets. For instance, a call option allows the buyer of this instrument an option but not an obligation to purchase an underlying asset at a specified *strike* price at a particular time in future, referred to as *time to maturity*. Seller of the option on the other hand is obligated to sell the underlying asset to the buyer at the specified price if the buyer exercises the option. Seller of course receives the option price upfront for selling this derivative. For instance, one may purchase a call option on the Reliance stock, whose current value is, say, Rs. 1055, that gives the owner the option to purchase certain number of Reliance stocks, each at price Rs. 1100, three months from now. This option is valuable to the buyer at its time of maturity if the stock then is worth more than Rs. 1100. Otherwise this option is not worth exercising and has value zero. In the earlier example, Infosys may instead prefer to purchase a call option that allows it the option to pay \$10 crore to receive Rs. 430 crore six months from now. Infosys would then exercise this option if each dollar gets less than Rs. 43 in the market at the option’s time to maturity.

Similarly, a put option gives the buyer of the instrument the option but not an obligation to sell an asset at a specified price at the time to maturity. These options are referred to as European options if they can be exercised only at the time to maturity. American options allow an *early exercise feature*, that is, they can be exercised at any time up to the time to maturity. There exist variants such as Bermudan options that can be exercised at a finite number of specified dates. Other popular options such as interest rate swaps, credit debt swaps (CDS’s) and collateralized debt obligations (CDOs) are discussed later in the chapter. Many more exotic options are not discussed in this chapter (see, e.g, Hull [16], Shreve [30]).

1.1 The no-arbitrage principle

Coming up with a fair price for such derivatives securities vexed the financial community right up till early seventies when Black Scholes [3] came up with their famous formula for pricing European options. Since then, the the literature on pricing financial derivatives has seen a huge explosion and has played a major role in expansion of financial derivatives market. To put things in perspective, from a tiny market in the seventies, the market of financial derivatives has grown in notional amount to about \$600 trillion in 2007. This compared to the world GDP of order \$45 trillion. Amongst financial derivatives, as of 2007, interest rate based derivatives constitute about 72% of the market, currencies about 12%, and equities and commodities the remaining 16% (See, e.g., Baaquie [1]). Wall street employs thousands of PhDs that use quantitative methods or ‘rocket science’ in derivatives pricing and related

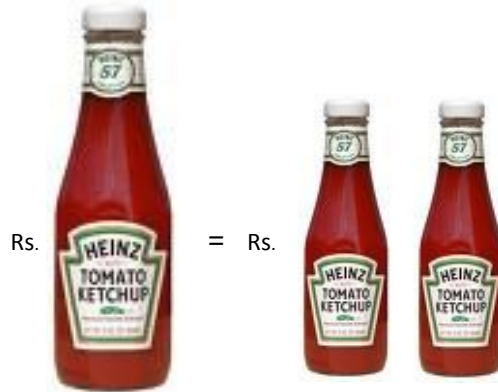


Figure 1: No Arbitrage Principle: Price of two liter ketchup bottle equals twice the price of a one liter ketchup bottle, else ARBITRAGE, that is, profits can be made without any risk.

activities.

‘No-arbitrage pricing principle’ is the key idea used by Black and Scholes to arrive at their formula. It continues to be foundational for financial mathematics. Simply told, and as illustrated in Figure 1, this means that price of a two liter ketchup bottle should be twice the price of a one liter ketchup bottle, otherwise by following the sacred mantra of *buy low and sell high* one can create an arbitrage, that is, instantaneously produce profits while taking zero risk. The no arbitrage principle precludes such free lunches and provides a surprisingly sophisticated methodology to price complex derivatives securities. This methodology relies on replicating pay-off from a derivative in every possible scenario by continuously and appropriately trading in the underlying more basic securities (transaction costs are assumed to be zero). Then, since the derivative and this trading strategy have identical payoffs, by the no-arbitrage principle, they must have the same price. Hence, the cost of creating this trading strategy provides the price of the derivative security.

In spite of the fact that continuous trading is an idealization and there always are small transaction costs, this pricing methodology approximates the practice well. Traders often sell complex risky derivatives and then dynamically trade in underlying securities in a manner that more or less cancels the risk arising from the derivative while incurring little transactional cost. Thus, from their viewpoint the price of the derivative must be at least the amount they need to cancel the associated risk. Competition ensures that they do not charge much higher than this price.

In practice one also expects no-arbitrage principle to hold as large banks typically have strong groups of arbitragers that identify and quickly take advantage of such arbitrage opportunities (again, by buying low and selling high) so that due to demand and supply the prices adjust and these opportunities become unavailable to common investors.

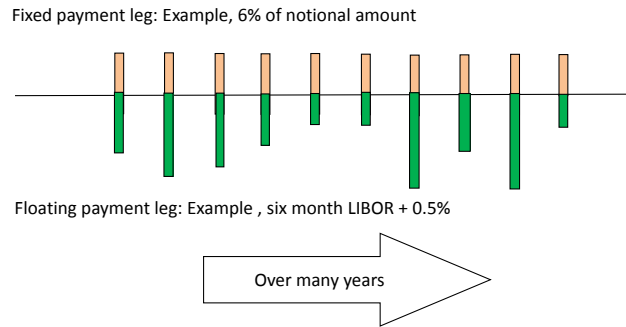


Figure 2: Example illustrating interest rate swap cash-flows.

1.2 Popular derivatives

Interest rate swaps and swaptions, options on these swaps, are by far the most popular derivatives in the financial markets. The market size of these instruments was about \$310 trillion in 2007. Figure 2 shows an example of cash flows involved in an interest rate swap. Typically, for a specified duration of the swap (e.g., five years) one party pays a fixed rate (fraction) of a pre-specified notional amount at regular intervals (say, every quarter or half yearly) to the other party, while the other party pays variable floating rate at the same frequency to the first party. This variable rate may be a function of prevailing rates such as the LIBOR rates (London Interbank Offered Rates; inter-bank borrowing rate amongst banks in London). This is used by many companies to match their revenue streams to liability streams. For instance, a pension fund may have fixed liabilities. However, the income they earn may be a function of prevailing interest rates. By entering into a swap that pays at a fixed rate they can reduce the variability of cash-flows and hence improve financial planning.

Swaptions give its buyer an option to enter into a swap at a particular date at a specified interest rate structure. Due to their importance in the financial world, intricate mathematical models have been developed to accurately price such interest rate instruments. Refer to, e.g., [4], [5] for further details.

Credit Default Swap is a financial instrument whereby one party (A) buys protection (or insurance) from another party (B) to protect against default by a third party (C). Default occurs when a debtor C cannot meet its legal debt obligations. A pays a premium payment at regular intervals (say, quarterly) to B up to the duration of the swap or until C defaults. During the swap duration, if C defaults, B pays A a certain amount and the swap terminates. These cash flows are depicted in Figure 3. Typically, A may hold a bond of C that has certain nominal value. If C defaults, then B provides protection against this default by purchasing this *much devalued* bond from A at its higher nominal price. CDS's were initiated in early

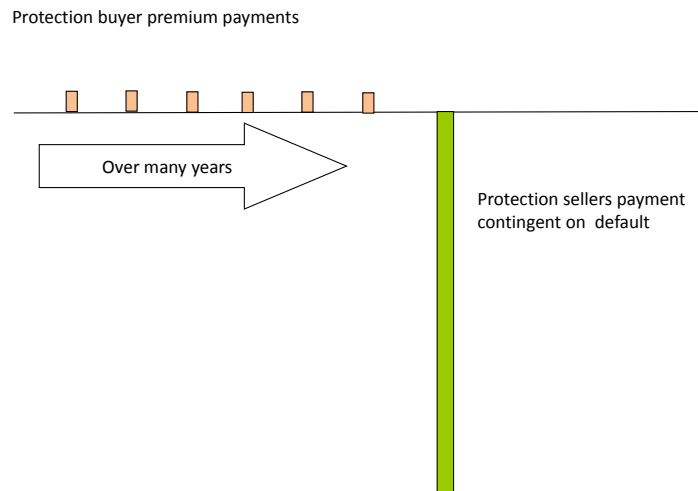


Figure 3: CDS cash flow

90's but the market took-off in 2003. By the year 2007, the amount protected by CDS's was of order \$60 trillion. Refer to [10], [20] and [28] for a general overview of credit derivatives and the associated pricing methodologies for CDSs as well as for CDOs discussed below.

Collateralized Debt Obligation is a structured financial product that became extremely popular over the last ten years. Typically CDO's are structures created by banks to offload many loans or bonds from their lending portfolio. These loans are packaged together as a CDO and then are sold off to investors as CDO tranche securities with varying levels of risk. For instance, investors looking for safe investment (these are typically the most sought after by investors) may purchase the super senior tranche securities (which is deemed very safe and maybe rated as AAA by the rating agencies), senior tranche (which is comparatively less safe and may have a lower rating) securities may be purchased by investors with higher appetite for risk (the amount they pay is less to compensate for the additional risk) and so on. Typically, the most risky equity tranche is retained by the bank or financial institution that creates this CDO. The cash flows generated when these loans are repayed are allocated to the security holders based on the seniority of the associated tranches. Initial cash flows are used to payoff the super senior tranche securities. Further generated cash-flows are used to payoff the senior tranche securities and so on. If some of the underlying loans default on their payments, then the risky tranches are the first to absorb these losses. See Figures 4 and 5 for a graphical illustration.

Pricing CDOs is a challenge as one needs to accurately model the dependencies between somewhat rare but catastrophic events associated with many loans defaulting together. Also note that more sophisticated CDOs, along with loans and bonds, may include other debt instruments such as CDS's and tranches from other CDO's in their portfolio.

As the above examples indicate, the key benefit of financial derivatives is that they help companies reduce risk in their cash-flows and through this improved risk management, aid

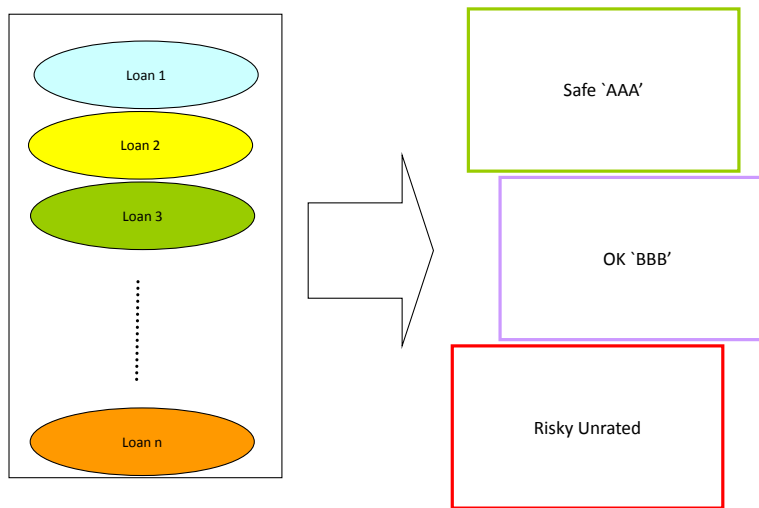


Figure 4: Underlying loans are tranching into safe and less safe securities that are sold to investors

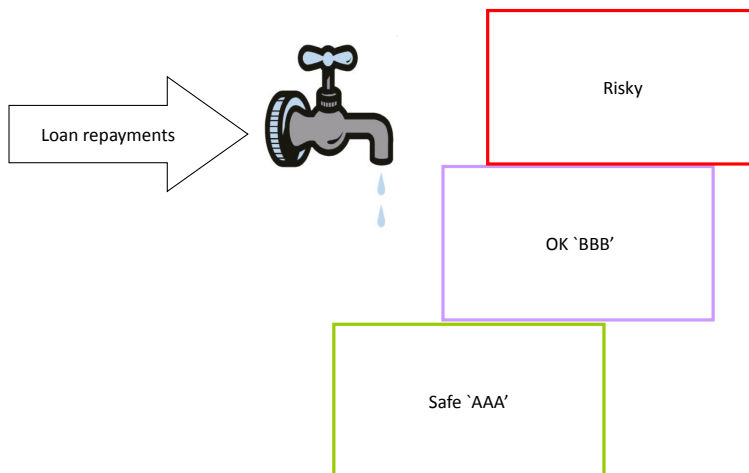


Figure 5: Loan repayments are first channeled to pay the safest tranche, then the less safe one and so-on. If there are defaults on the loans then the risky tranches are the first to suffer losses

in financial planning, reduce the capital requirements and therefore enhance profitability. Currently, 92% of the top Fortune 500 companies engage in derivatives to better manage their risk.

However, derivatives also make speculation easy. For instance, if one believes that a particular stock price will rise, it is much cheaper and logistically efficient to place a big bet by purchasing call options on that stock, then through acquiring the same number of stock, although the upside in both the cases is the same. This flexibility, as well as the underlying complexity of some of the exotic derivatives such as CDOs, makes derivatives risky for the overall economy. Warren Buffet famously referred to financial derivatives as time bombs and financial weapons of mass destruction. CDOs involving housing loans played a significant role in the economic crisis of 2008. (See, e.g. Duffie [9]). The key reason being that while it was difficult to precisely measure the risk in a CDO (due extremal dependence amongst loan defaults), CDOs gave a false sense of confidence to the loan originators that since the risk was being diversified away to investors, it was being mitigated. This in turn prompted acquisition of far riskier sub-prime loans that had very little chance of repayment.

In the remaining paper we focus on developing underlying ideas for pricing relatively simple European options such as call or put options. In Section 2, we illustrate the no-arbitrage principle for a two-security-two-scenario-two-time-period Binomial tree toy model. In Section 3, we develop the pricing methodology for pricing European options in more realistic continuous-time-continuous-state framework. From a probabilistic viewpoint, this requires concepts of Brownian motion, stochastic Ito integrals, stochastic differential equations, Ito's formula, martingale representation theorem and the Girsanov theorem. These concepts are briefly introduced and used to develop the pricing theory. They remain fundamental to the modern finance pricing theory. We end with a brief conclusion in Section 4.

Financial mathematics is a vast area involving interesting mathematics in areas such as risk management (see, e.g., Frey et. al. [22]), calibration and estimation methodologies for financial models, econometric models for algorithmic trading as well as for forecasting and prediction (see, e.g., [6]), investment portfolio optimization (see, e.g., Meucci [23]) and optimal stopping problem that arises in pricing American options (see, e.g., Glasserman [15]). In this chapter, however, we restrict our focus to some of the fundamental derivative pricing ideas.

2 Binomial Tree Model

We now illustrate how the no-arbitrage principle helps price options in a simple Binomial-tree or the 'two-security-two scenario-two-time-period' setting. This approach to price options was first proposed by Cox, Ross and Rubinstein [7] (see [29] for an excellent comprehensive exposition of Binomial tree models).

Consider a simplified world consisting of two securities: The risky security or the stock and the risk free security or an investment in the safe money market. We observe these at time zero and at time Δt . Suppose that stock has value S at time zero. At time Δt two scenarios *up* and *down* can occur (see Figure 6 for a graphical illustration): Scenario *up* occurs with probability $p \in (0, 1)$ and scenario *down* with probability $1 - p$. The stock takes value $S \exp(u\Delta t)$ in scenario *up* and $S \exp(d\Delta t)$ otherwise, where $u > d$. Money market

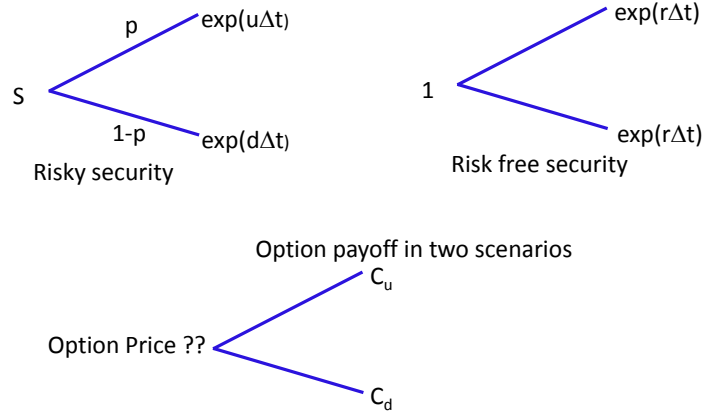


Figure 6: World comprising a risky security, a risk-free security and an option that needs to be priced. This world evolves in future for one time period where the risky security and the option can take two possible values. No arbitrage principle that provides a unique price for the option that is independent of $p \in (0, 1)$, where $(p, 1 - p)$ denote the respective probabilities of the two scenarios.

account has value 1 at time zero that increases to $\exp(r\Delta t)$ in both the scenarios at time Δt . Assume that any amount of both these assets can be borrowed or sold without any transaction costs.

First note that the no-arbitrage principle implies that $d < r < u$. Otherwise, if $r \leq d$, borrowing amount S from the money market and purchasing a stock with it, the investor earns at least $S \exp(d\Delta t)$ at time Δt where his liability is $S \exp(r\Delta t)$. Thus with zero investment he is guaranteed sure profit (at least with positive probability if $r = d$), violating the no-arbitrage condition. Similarly, if $r \geq u$, then by short selling the stock (borrowing from an owner of this stock and selling it with a promise to return the stock to the original owner at a later date) the investor gets amount S at time zero which he invests in the money market. At time Δt he gets $S \exp(r\Delta t)$ from this investment while his liability is at most $S \exp(u\Delta t)$ (the price at which he can buy back the stock to close the short position), thus leading to an arbitrage.

Now consider an option that pays C_u in the up scenario and C_d in the down scenario. For instance, consider a call option that allows its owner an option to purchase the underlying stock at the strike price K for some $K \in (S \exp(d\Delta t), S \exp(u\Delta t))$. In that case, $C_u = S - K$ denotes the benefit to option owner in this scenario, and $C_d = 0$ underscores the fact that option owner would not like to purchase a risky security at value K , when he can purchase it from the market at a lower price $S \exp(d\Delta t)$. Hence, in this scenario the option is worthless to its owner.

2.1 Pricing using no-arbitrage principle

The key question is to determine the fair price for such an option. A related problem is to ascertain how to cancel or hedge the risk that the seller of the option is exposed to by taking appropriate positions in the underlying securities. Naively, one may consider the expectation $pC_u + (1 - p)C_d$ suitably discounted (to account for time value of money and the risk involved) to be the correct value of the option. As we shall see, this approach may lead to an incorrect price. In fact, in this simple world, the answer based on the no-arbitrage principle is straightforward and is independent of the probability p . To see this, we construct a portfolio of the stock and the risk free security that exactly replicates the payoff of the option in the two scenarios. Then the value of this portfolio gives the correct option price.

Suppose we purchase $\alpha \in \Re$ number of stock and invest amount $\beta \in \Re$ in the money market, where α and β are chosen so that the resulting payoff matches the payoff from the option at time Δt in the two scenarios. That is,

$$\alpha S \exp(u\Delta t) + \beta \exp(r\Delta t) = C_u$$

and

$$\alpha S \exp(d\Delta t) + \beta \exp(r\Delta t) = C_d.$$

Thus,

$$\alpha = \frac{C_u - C_d}{S(\exp(u\Delta t) - \exp(d\Delta t))},$$

and

$$\beta = \frac{C_d \exp((u - r)\Delta t) - C_u \exp((d - r)\Delta t)}{\exp(u\Delta t) - \exp(d\Delta t)}.$$

Then, the portfolio comprising α number of risky security and amount β in risk free security exactly replicates the option payoff. The two should therefore have the same value, else an arbitrage can be created. Hence, the value of the option equals the value of this portfolio. That is,

$$\alpha S + \beta = \exp(-r\Delta t) \left[\frac{\exp(r\Delta t) - \exp(d\Delta t)}{\exp(u\Delta t) - \exp(d\Delta t)} C_u + \frac{\exp(u\Delta t) - \exp(r\Delta t)}{\exp(u\Delta t) - \exp(d\Delta t)} C_d \right]. \quad (1)$$

Note that this price is independent of the value of the physical probability vector $(p, 1 - p)$ as we are matching the option pay-off over each probable scenario. Thus, p could be .01 or .99, it will not in any way affect the price of the option. If by using another methodology, a different price is reached for this model, say a price higher than (1), then an astute trader would be happy to sell options at that price and create an arbitrage for himself by exactly replicating his liabilities at a cheaper price.

2.1.1 Risk neutral pricing

Another interesting observation from this simple example is the following: Set $\hat{p} = \frac{\exp(r\Delta t) - \exp(d\Delta t)}{\exp(u\Delta t) - \exp(d\Delta t)}$, then, since $d < r < u$, $(\hat{p}, 1 - \hat{p})$ denotes a probability vector. The value of the option may be re-expressed as:

$$\exp(-r\Delta t) (\hat{p}C_u + (1 - \hat{p})C_d) = \hat{E}(\exp(-r\Delta t)C) \quad (2)$$

where \hat{E} denotes the expectation under the probability $(\hat{p}, 1 - \hat{p})$ and $\exp(-r\Delta t)C$ denotes the discounted value of the random pay-off from the option at time 1, discounted at the risk free rate. Interestingly, it can be checked that

$$S = \exp(-r\Delta t) (\hat{p} \exp(u\Delta t)S + (1 - \hat{p}) \exp(d\Delta t)S)$$

or $\hat{E}(S_1) = \exp(r\Delta t)S$, where S_1 denotes the random value of the stock at time 1. Hence, under the probability $(\hat{p}, 1 - \hat{p})$, stock earns an annualized continuously compounded rate of return r . The measure corresponding to these probabilities is referred (in more general setups that we discuss later) as the *risk neutral* or the *equivalent martingale* measure. Clearly, these are the probabilities the risk neutral investor would assign to the two scenarios in equilibrium (in equilibrium both securities should give the same rate of return to such an investor) and (2) denotes the price that the risk neutral investor would assign to the option (blissfully unaware of the no-arbitrage principle!). Thus, the no-arbitrage principle leads to a pricing strategy in this simple setting that the risk neutral investor would in any case follow. As we observe in Section 3, this result generalizes to far more mathematically complex models of asset price movement, where the price of an option equals the mathematical expectation of the payoff from the option discounted at the risk free rate under the risk neutral probability measure.

2.2 Some extensions of the binomial model

Simple extensions of the binomial model provide insights into important issues directing the general theory. First, to build greater realism in the model, consider a two-security-three scenario-two-time-period model illustrated in Figure 7. In this setting it should be clear that one cannot replicate most options exactly without having a third security. Such a market where all options cannot be exactly replicated by available securities is referred to as *incomplete*. Analysis of incomplete markets is an important area in financial research as empirical data suggests that financial markets tend to be incomplete. (See, e.g., [21], [11]).

Another way to incorporate three scenarios is to increase the number of time periods to three. This is illustrated in Figure 8. The two securities are now observed at times zero, $0.5\Delta t$ and Δt . At times zero and at $0.5\Delta t$, the stock price can either go up by amount $\exp(0.5u\Delta t)$ or down by amount $\exp(0.5d\Delta t)$ in the next $0.5\Delta t$ time. At time Δt , the stock price can take three values. In addition, trading is allowed at time $0.5\Delta t$ that may depend upon the value of stock at that time. This additional flexibility allows replication of any option with payoffs at time Δt that are a function of the stock price at that time (more generally, the payoff could be a function of the path followed by the stock price till time Δt). To see this, one can repeat the argument for the two-security-two scenario-two-time-period case to determine how much money is needed at node (a) in Figure 8 to replicate the option pay-off at time Δt . Similarly, one can determine the money needed at node (b). Using these two values, one can determine the amount needed at time zero to construct a replicating portfolio that exactly replicates the option payoff along every scenario at time Δt . This argument easily generalizes to arbitrary n time periods (see Hull [16], Shreve [29]). Cox, Ross and Rubinstein (1979) analyze this as n goes to infinity and show that the resultant risky security price process converges to geometric Brownian motion (exponential

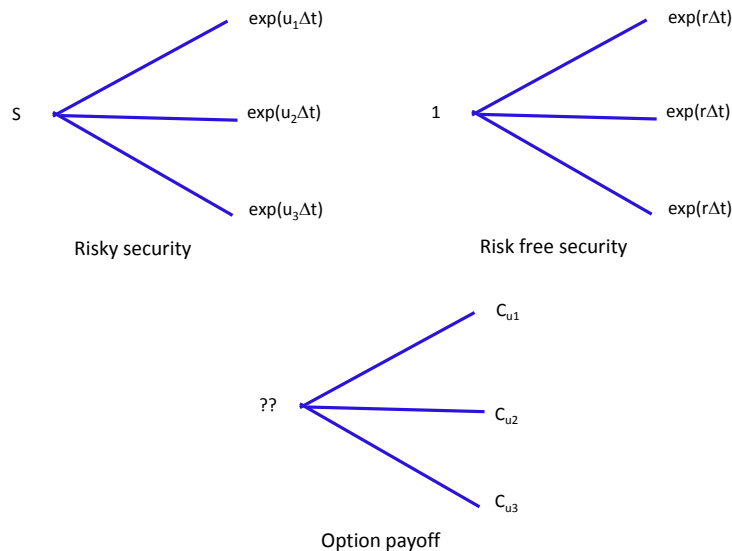


Figure 7: World comprising a risky security, a risk-free security and an option that needs to be priced. This world evolves in future for one time period where the risky security and the option can take three possible values. No arbitrage principle typically does not provide a unique price for the option in this case.

of Brownian motion; Brownian motion is discussed in section 3), a natural setting for more realistic analysis.

3 Continuous Time Models

We now discuss the European option pricing problem in the continuous-time-continuous-state settings which has emerged as the primary regime for modeling security prices. As discussed earlier, the key idea is to create a dynamic portfolio that through continuous trading in the underlying securities up to the option time to maturity, exactly replicates the option payoff in every possible scenario. In this presentation we de-emphasize technicalities to maintain focus on the key concepts used for pricing options and to keep the discussion accessible to a broad audience. We refer the reader to Shreve [30], Duffie [8], Steele [31] for simple and engaging account of stochastic analysis for derivatives pricing; Also see, [11]).

First in Section 3.1, we briefly introduce Brownian motion, perhaps the most fundamental continuous time stochastic process that is an essential ingredient in modeling security prices. We discuss how this is crucial in driving stochastic differential equations used to model security prices in Section 3.2. Then in Section 3.3, we briefly review the concepts of stochastic Ito integral, quadratic variation and Ito’s formula, necessary to appreciate the stochastic differential equation model of security prices. In Section 3.4, we use a well known technique to arrive at a replicating portfolio and the Black Scholes partial differential equation for options with simple payoff structure in a two security set-up.

Modern approach to options pricing relies on two broad steps: First we determine a probability measure under which discounted security prices are martingales (martingales

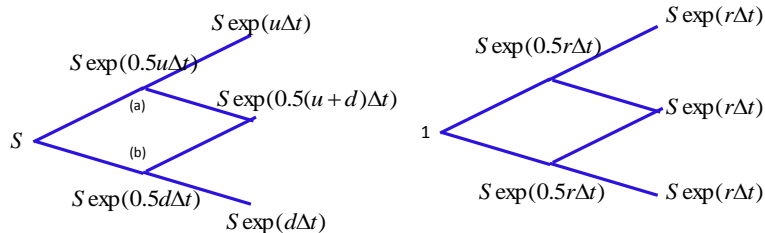


Figure 8: World comprising a risky security, a risk-free security and an option that needs to be priced. This world evolves in future for two time periods $0.5\Delta t$ and Δt . Trading is allowed at time $0.5\Delta t$ that may depend upon the value of stock at that time. This allows exact replication and hence unique pricing of any option with payoffs at time Δt .

correspond to stochastic processes that on an average do not change. These are precisely defined later). This, as indicated earlier, is referred to as the equivalent martingale or the risk neutral measure. Thereafter, one uses this measure to arrive at the replicating portfolio for the option. First step utilizes the famous Girsanov theorem. Martingale representation theorem is crucial to the second step. We review this approach to options pricing in Section 3.5. Sections 3.4 and 3.5 restrict attention to two security framework: one risky and the other risk-free. In Section 3.6, we briefly discuss how this analysis generalizes to multiple risky assets.

3.1 Brownian motion

Louis Bachelier in his dissertation [2] was the first to use Brownian motion to model stock prices with the explicit purpose of evaluating option prices. This in fact was the first use advanced mathematics in finance. Paul Samuelson [27] much later proposed geometric Brownian motion (exponential of Brownian motion) to model stock prices. This, even today is the most basic model of asset prices.

The stochastic process $(W_t : t \geq 0)$ is referred to as standard Brownian motion (see, e.g., Revuz and Yor [26]) if

1. $W_0 = 0$ almost surely.
2. For each $t > s$, the increment $W_t - W_s$ is independent of $(W_u : u \leq s)$. This implies that $W_{t_2} - W_{t_1}$ is independent of $W_{t_4} - W_{t_3}$ and so on whenever $t_1 < t_2 < t_3 < t_4$.

3. For each $t > s$, the increment $W_t - W_s$ has a Gaussian distribution with zero mean and variance $t - s$.
4. The sample paths of $(W_t : t \geq 0)$ are continuous almost surely.

Technically speaking, $(W_t : t \geq 0)$ is defined on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathcal{P})$ where $\{\mathcal{F}_{t \geq 0}\}$ is a filtration of \mathcal{F} , that is, an increasing sequence of sub-sigma algebras of \mathcal{F} . From an intuitive perspective, \mathcal{F}_t denotes the information available at time t . The random variable W_t is \mathcal{F}_t measurable for each t (i.e., the process $\{W_t\}_{t \geq 0}$ is adapted to $\{\mathcal{F}\}_{t \geq 0}$). Heuristically this means that the value of $(W_s : s \leq t)$ is known at time t . In this chapter we take \mathcal{F}_t to denote the sigma algebra generated by $(W_s : s \leq t)$. This has ramifications for the martingale representation theorem stated later. In addition, the filtration $\{\mathcal{F}\}_{t \geq 0}$ satisfies the *usual conditions*. See, e.g., Karatzas and Shreve [18].

The independent increments assumption is crucial to modeling risky security prices as it captures *the efficient market hypothesis* on security prices ([14]), that is, any change in security prices is essentially due to arrival of new information (independent of what is known in past). All past information has already been incorporated in the market price.

3.2 Modeling Security Prices

In our model, the security price process is observed till $T > 0$, time to maturity of the option to be priced. In particular, in the Brownian motion defined above the time index is restricted to $[0, T]$. The stock price evolution is modeled as a stochastic process $\{S_t\}_{0 \leq t \leq T}$ defined on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0 \leq t \leq T}, \mathcal{P})$ and is assumed to satisfy the stochastic differential equation (SDE)

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t, \quad (3)$$

where μ_t and σ_t maybe deterministic functions of (t, S_t) satisfying technical conditions such that a unique solution to this SDE exists (see, e.g., Oksendal [24], Karatzas and Shreve [18], Steele [31] for these conditions and a general introduction to SDEs). In (3), the simplest and popular case corresponds to both μ_t and σ_t being constants.

Note that $\frac{S_{t_1+t_2} - S_{t_1}}{S_{t_1}}$ denotes the return from the security over the time period $[t_1, t_2]$. Equation (3) may be best viewed through its discretized Euler's approximation at times t and $t + \delta t$:

$$\frac{S_{t+\delta t} - S_t}{S_t} = \mu_t \delta t + \sigma_t (W_{t+\delta t} - W_t).$$

This suggests that μ_t captures the drift in the instantaneous return from the security at time t . Similarly, σ_t captures the sensitivity to the independent noise $W_{t+\delta t} - W_t$ present in the instantaneous return at time t . Since, μ_t and σ_t maybe functions of (t, S_t) , independent of the past security prices, and Brownian motion has independent increments, the process $\{S_t\}_{0 \leq t \leq T}$ is Markov. Heuristically, this means that, given the asset value S_s at time s , the future $\{S_t\}_{s \leq t \leq T}$ is independent of the past $\{S_t\}_{0 \leq t \leq s}$.

Equation (3) is a heuristic differential representation of an SDE. A rigorous representation is given by

$$S_t = S_0 + \int_0^t \mu_s S_s ds + \int_0^t \sigma_s S_s dW_s. \quad (4)$$

Here, while $\int_0^t \mu_s S_s ds$ is a standard Lebesgue integral defined path by path along the sample space, the integral $\int_0^t \sigma_s S_s dW_s$ is a stochastic integral known as Ito integral after its inventor Ito [17]. It fundamentally differs from Lebesgue integral as it can be seen that Brownian motion does not have bounded variation. We briefly discuss this and related relevant results in the subsection below.

3.3 Stochastic Calculus

Here we summarize some useful results related to stochastic integrals that are needed in our discussion. The reader is referred to Karatzas and Shreve [18], Protter [25] and Revuz and Yor [26] for a comprehensive and rigorous analysis.

3.3.1 Stochastic integral

Suppose that $\phi : [0, T] \rightarrow \mathfrak{R}$ is a bounded continuous function. One can then define its integral $\int_0^T \phi(s) dA_s$ w.r.t. to a continuous process $(A_t : 0 \leq t \leq T)$, of finite first order variation, as the limit

$$\lim_{n \rightarrow \infty} \sum_{i=0}^{n-1} \phi(iT/n) (A_{(i+1)T/n} - A_{iT/n}).$$

(In our analysis, above and below, for notational simplification we have divided T into n equal intervals. Similar results are true if the intervals are allowed to be unequal with the caveat that the largest interval shrinks to zero as $n \rightarrow \infty$.) The above approach to defining integrals fails when the integration is w.r.t. to Brownian motion $(W_t : 0 \leq t \leq T)$. To see this informally, note that in

$$\sum_{i=0}^{n-1} |W_{(i+1)T/n} - W_{iT/n}|$$

the terms $|W_{(i+1)T/n} - W_{iT/n}|$ are i.i.d. and each is distributed as \sqrt{T}/n times $N(0, 1)$, a standard Gaussian random variable with mean zero and variance 1. This suggests that, due to the law of large numbers, this sum is close to $\sqrt{nT} E|N(0, 1)|$ as n becomes large. Hence, it diverges almost surely as $n \rightarrow \infty$. This makes definition of path by path integral w.r.t. Brownian motion difficult.

Ito defined the stochastic integral $\int_0^T \phi(s) dW_s$ as a limit in the L^2 sense¹. Specifically, he considered adapted random processes $(\phi(t) : 0 \leq t \leq T)$ such that

$$E \int_0^T \phi(s)^2 ds < \infty.$$

For such functions, the following Ito's isometry is easily seen

$$E \left[\sum_{i=0}^{n-1} (\phi(iT/n) (W_{(i+1)T/n} - W_{iT/n}))^2 \right] = E \left(\sum_{i=0}^{n-1} \phi(iT/n)^2 T/n \right).$$

¹A sequence of random variables $(X_n : n \geq 1)$ such that $EX_n^2 < \infty$ for all n is said to converge to random variable X (all defined on the same probability space) in the L^2 sense if $\lim_{n \rightarrow \infty} E(X_n - X)^2 = 0$.

To see this, note that ϕ is an adapted process, the Brownian motion has independent increments, so that $\phi(iT/n)$ is independent of $W_{(i+1)T/n} - W_{iT/n}$. Therefore, the expectation of the cross terms in the expansion of the square on the LHS can be seen to be zero. Further, $E[\phi(iT/n)^2(W_{(i+1)T/n} - W_{iT/n})^2]$ equals $E[\phi(iT/n)^2]T/n$.

This identity plays a fundamental role in defining $\int_0^T \phi(s)dW_s$ as an L^2 limit of the sequence of random variables $\sum_{i=0}^{n-1} \phi(iT/n)(W_{(i+1)T/n} - W_{iT/n})$ (see, e.g. Steele [31], Karatzas and Shreve [18]).

3.3.2 Martingale property and quadratic variation

As is well known, martingales capture the idea of a fair game in gambling and are important to our analysis. Technically, a stochastic process $(Y_t : 0 \leq t \leq T)$ on a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0 \leq t \leq T}, \mathcal{P})$ is a martingale if it is adapted to the filtration $\{\mathcal{F}\}_{0 \leq t \leq T}$, if $E|Y_t| < \infty$ for all t , and:

$$E[Y_t | \mathcal{F}_s] = Y_s$$

almost surely for all $0 \leq s < t \leq T$.

The process $(I(t) : 0 \leq t \leq T)$,

$$I(t) = \sum_{i=0}^{k-1} \phi(iT/n)(W_{(i+1)T/n} - W_{iT/n}) + \phi(kT/n)(W_t - W_{kT/n})$$

where k is such that $t \in [kT/n, (k+1)T/n)$, can be easily seen to be a continuous zero mean martingale using the key observation that for $t_1 < t_2 < t_3$,

$$E[\phi(t_2)(W_{t_3} - W_{t_2}) | \mathcal{F}_{t_1}] = E(E[\phi(t_2)(W_{t_3} - W_{t_2}) | \mathcal{F}_{t_2}] | \mathcal{F}_{t_1})$$

and this equals zero since

$$E[\phi(t_2)(W_{t_3} - W_{t_2}) | \mathcal{F}_{t_2}] = \phi(t_2)E[(W_{t_3} - W_{t_2}) | \mathcal{F}_{t_2}] = 0.$$

Using this it can also be shown that the limiting process $(\int_0^t \phi(s)dW_s : 0 \leq t \leq T)$ is a zero mean martingale if $(\phi(t) : 0 \leq t \leq T)$ is an adapted process and

$$E \int_0^T \phi(s)^2 ds < \infty.$$

Quadratic variation of any process $(X_t : 0 \leq t \leq T)$ may be defined as the L^2 limit of the sequence $\sum_{i=0}^{n-1} (X_{(i+1)T/n} - X_{iT/n})^2$ when it exists. This can be seen to equal $\int_0^T \phi(s)^2 ds$ for the process $(\int_0^t \phi(s)dW_s : 0 \leq t \leq T)$. In particular the quadratic variation of $(W_t : 0 \leq t \leq T)$ equals a constant T .

3.3.3 Ito's formula

Ito's formula provides a key identity that highlights the difference between Ito's integral and ordinary integral. It is the main tool for analysis of stochastic integrals. Suppose that

$f : \mathfrak{R} \rightarrow \mathfrak{R}$ is twice continuously differentiable and $E \int_0^t f'(W_s)^2 ds < \infty$. Then, Ito's formula states that

$$f(W_t) = f(W_0) + \int_0^t f'(W_s) dW_s + \frac{1}{2} \int_0^t f''(W_s) ds. \quad (5)$$

Note that in integration with respect to processes with finite first order variation, the correction term $\frac{1}{2} \int_0^t f''(W_s) ds$ would be absent.

To see why this identity holds, re-express $f(W_t) - f(W_0)$ as

$$\sum_{i=0}^{n-1} (f(W_{(i+1)t/n}) - f(W_{it/n})).$$

Expanding the summands using the Taylor series expansion and ignoring the remainder terms, we have

$$\sum_{i=0}^{n-1} f'(W_{it/n})(W_{(i+1)t/n} - W_{it/n}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(W_{it/n})(W_{(i+1)t/n} - W_{it/n})^2.$$

The first term converges to $\int_0^t f'(W_s) dW_s$ in L^2 as $n \rightarrow \infty$. To see that the second term converges to $\frac{1}{2} \int_0^t f''(W_s) ds$ it suffices to note that

$$\sum_{i=0}^{n-1} f''(W_{it/n}) ((W_{(i+1)t/n} - W_{it/n})^2 - t/n) \quad (6)$$

converges to zero as $n \rightarrow \infty$. This is easily seen when $|f''|$ is bounded as then (6) is bounded from above by

$$\sup_x |f''(x)| \sum_{i=0}^{n-1} ((W_{(i+1)t/n} - W_{it/n})^2 - t/n).$$

The sum above converges to zero in L^2 as the quadratic variation of $(W(s) : 0 \leq s \leq t)$ equals t .

In the heuristic differential form (5) may be expressed as

$$df(W_t) = f'(W_t) dW_t + \frac{1}{2} f''(W_t) dt.$$

This form is usually more convenient to manipulate and correctly derive other equations and is preferred to the rigorous representation.

Similarly, for $f : [0, T] \times \mathfrak{R} \rightarrow \mathfrak{R}$ with continuous partial derivatives of second order, we can show that

$$df(t, W_t) = f_t(t, W_t) dt + f_x(t, W_t) dW_t + \frac{1}{2} f_{xx}(t, W_t) dt \quad (7)$$

where f_t denotes the partial derivative w.r.t. the first argument of $f(\cdot, \cdot)$ and f_x and f_{xx} denote the first and the second order partial derivatives with respect to its second argument. Again, the rigorous representation for (8) is

$$f(t, W_t) = f(0, W_0) + \int_0^t f_t(s, W_s) ds + \int_0^t f_x(s, W_s) dW_s + \frac{1}{2} \int_0^t f_{xx}(s, W_s) ds.$$

3.3.4 Ito processes

The process $(X_t : 0 \leq t \leq T)$ of the (differential) form

$$dX_t = \alpha_t dt + \beta_t dW_t$$

where $(\alpha_t : 0 \leq t \leq T)$ and $(\beta_t : 0 \leq t \leq T)$ are adapted processes such that $E(\int_0^T \beta_t^2 dt) < \infty$ and $E(\int_0^T |\alpha_t| dt) < \infty$, are referred to as Ito processes. Ito's formula can be generalized using essentially similar arguments to show that

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)\sigma_t^2 dt. \quad (8)$$

3.4 Black Scholes partial differential equation

As in the Binomial setting, here too we consider two securities. The risky security or the stock price process satisfies (3). The money market $(R_t : 0 \leq t \leq T)$ is governed by a *short rate* process $(r_t : 0 \leq t \leq T)$ and satisfies the differential equation

$$dR_t = r_t R_t dt$$

with $R_0 = 1$. Here short rate r_t corresponds to instantaneous return on investment in the money market at time t . In particular, Rs. 1 invested at time zero in the money market equals $R_t = \exp(\int_0^t r_s ds)$ at time t . In general r_t may be random and the process $(r_t : 0 \leq t \leq T)$ may be adapted to $\{\mathcal{F}\}_{0 \leq t \leq T}$, although typically when short time horizons are involved, a deterministic model of short rates is often used. In fact, it is common to assume that $r_t = r$, a constant, so that $R_t = \exp(rt)$. In our analysis, we assume that $(r_t : 0 \leq t \leq T)$ is deterministic to obtain considerable simplification.

Now consider the problem of pricing an option in this market that pays a random amount $h(S_T)$ at time T . For instance, for a call option that matures at time T with strike price K , we have $h(S_T) = \max(S_T - K, 0)$. We now construct a replicating portfolio for this option. Consider the recipe process $(b_t : 0 \leq t \leq T)$ for constructing a replicating portfolio. We start with amount P_0 . At time t , let P_t denote the value of the portfolio. This is used to purchase b_t number of stock (we allow b_t to take non-integral values). The remaining amount $P_t - b_t S_t$ is invested in the money market. Then, the portfolio process evolves as:

$$dP_t = (P_t - b_t S_t)r_t dt + b_t dS_t. \quad (9)$$

Due to the Markov nature of the stock price process, and since the option payoff is a function of S_T , at time t , the option price can be seen to be a function of t and S_t . Denote this price by $c(t, S_t)$. By Ito's formula (8), (assuming $c(\cdot, \cdot)$ is sufficiently smooth):

$$dc(t, S_t) = c_t(t, S_t)dt + c_x(t, S_t)dS_t + \frac{1}{2}c_{xx}(t, S_t)\sigma_t^2 dt,$$

with $c(T, S_T) = h(S_T)$. This may be e-expressed as:

$$dc(t, S_t) = \left(c_t(t, S_t) + c_x(t, S_t)S_t\mu_t + \frac{1}{2}c_{xx}(t, S_t)\sigma_t^2 S_t^2 \right) dt + c_x(t, S_t)\sigma_t S_t dW_t \quad (10)$$

Our aim is to select $(b_t : 0 \leq t \leq T)$ so that P_t equals $c(t, S_t)$ for all (t, S_t) . To this end, (9) can be re-expressed as

$$dP_t = ((P_t - b_t S_t)r_t + b_t \mu_t S_t) dt + b_t \sigma_t S_t dW_t. \quad (11)$$

To make $P_t = c(t, S_t)$ we equate the drift (terms corresponding to dt) as well as the diffusion terms (terms corresponding to dW_t) in (10) and (11). This requires that $b_t = c_x(t, S_t)$ and

$$c_t(t, S_t) + \frac{1}{2} c_{xx}(t, S_t) S_t^2 \sigma_t^2 - c(t, S_t) r_t + c_x(t, S_t) S_t r_t = 0.$$

The above should hold for all values of $S_t \geq 0$. This specifies the famous Black-Scholes partial differential equation (pde) satisfied by the option price process:

$$c_t(t, x) + c_x(t, x) x r_t + \frac{1}{2} c_{xx}(t, x) x^2 \sigma_t^2 - c(t, x) r_t = 0 \quad (12)$$

for $0 \leq t \leq T$, $x \geq 0$ and the the boundary condition $c(T, x) = h(x)$ for all x .

This is a parabolic pde that can be solved for the price process $c(t, x)$ for all $t \in [0, T)$ and $x \geq 0$. Once this is available, the replicating portfolio process is constructed as follows. $P_0 = c(0, S_0)$ denotes the initial amount needed. This is also the value of the option at time zero. At this time $c_x(0, S_0)$ number of risky security is purchased and the remaining amount $c(0, S_0) - c_x(0, S_0) S_0$ is invested in the money market. At any time t , the number of stocks held is adjusted to $c_x(t, S_t)$. Then, the value of the portfolio equals $c(t, S_t)$. The amount $c(t, S_t) - c_x(t, S_t) S_t$ is invested in the money market. These adjustments are made at each time $t \in [0, T)$. At time T then the portfolio value equals $c(T, S_T) = h(S_T)$ so that the option is perfectly replicated.

3.5 Equivalent martingale measure

As in the discrete case, in the continuous setting as well, under mild technical conditions, there exists another probability measure referred to as the risk neutral measure or the equivalent martingale measure under which the discounted stock price process is a martingale. This then makes the discounted replicating portfolio process a martingale, which in turn ensures that if an option can be replicated, then the discounted option price process is a martingale. The importance of this result is that it brings to bear the well developed and elegant theory of martingales to derivative pricing leading to deep insights into derivatives pricing and hedging (see Harrison and Kreps [12] and Harrison and Pliska [13] for seminal papers on this approach).

Martingale representation theorem and Girsanov theorem are two fundamental results from probability that are essential to this approach. We state them in a simple one dimensional setting.

Martingale Representation Theorem: If the stochastic process $(M_t : 0 \leq t \leq T)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{0 \leq t \leq T}, \mathcal{P})$ is a martingale, then there exists an adapted process $(\nu(t) : 0 \leq t \leq T)$ such that $E(\int_0^T \nu(t)^2 dt) < \infty$ and

$$M_t = M_0 + \int_0^t \nu(s) dW_s$$

for $0 \leq t \leq T$.

As noted earlier, under mild conditions a stochastic integral process is a martingale. The above theorem states that the converse is also true. That is, on this probability space, every martingale is a stochastic integral.

Some preliminaries to help state the Girsanov theorem: Two probability measures \mathcal{P} and \mathcal{P}^* defined on the same space are said to be equivalent if they assign positive probability to same sets. Equivalently, they assign zero probability to same sets. Further if \mathcal{P} and \mathcal{P}^* are equivalent probability measures, then there exists an almost surely positive Radon-Nikodym derivative of \mathcal{P}^* w.r.t. \mathcal{P} , call it Y , such that $\mathcal{P}^*(A) = E_{\mathcal{P}}YI(A)$ (here, the subscript on E denotes that the expectation is with respect to probability measure \mathcal{P} and $I(\cdot)$ is an indicator function). Furthermore, if Z is a strictly positive random variable almost surely with $E_{\mathcal{P}}Z = 1$ then the set function $\mathcal{Q}(A) = E_{\mathcal{P}}ZI(A)$ can be seen to be a probability measure that is equivalent to \mathcal{P} . Girsanov Theorem specifies the new distribution of the Brownian motion $(W_t : 0 \leq t \leq T)$ under probability measures equivalent to \mathcal{P} . Specifically, consider the process

$$Y_t = \exp\left(\int_0^t \nu_s dW_s - \frac{1}{2} \int_0^t \nu_s^2 ds\right).$$

Let $X_t = \int_0^t \nu_s dW_s - \frac{1}{2} \int_0^t \nu_s^2 ds$. Then $Y_t = \exp(X_t)$ so that using Ito's formula

$$dY_t = Y_t \nu_t dW_t,$$

or in its meaningful form

$$Y_t = 1 + \int_0^t Y_s \nu_s dW_s.$$

Under technical conditions the stochastic integral is a mean zero martingale so that $(Y_t : 0 \leq t \leq T)$ is a positive mean 1 martingale. Let $\mathcal{P}^\nu(A) = E_{\mathcal{P}}[Y_T I(A)]$.

Girsanov Theorem: Under the probability measure \mathcal{P}^ν , under technical conditions on $(\nu_t : 0 \leq t \leq T)$, the process $(W_t^\nu : 0 \leq t \leq T)$ where

$$W_t^\nu = W_t - \int_0^t \nu_s ds$$

(or $dW_t^\nu = dW_t - \nu_t dt$ in the differential notation) is a standard Brownian motion. Equivalently, $(W_t : 0 \leq t \leq T)$ is a standard Brownian motion plus the drift process $(\int_0^t \nu_s ds : 0 \leq t \leq T)$.

3.5.1 Identifying the equivalent martingale measure

Armed with the above two powerful results we can now return to the process of finding the equivalent martingale measure for the stock price process and the replicating portfolio for the option price process. Recall that the stock price follows the SDE

$$dS_t = \mu_t S_t dt + \sigma_t S_t dW_t.$$

We now allow this to be a general Ito process, that is, $\{\mu_t\}$ and $\{\sigma_t\}$ are adapted processes (not just deterministic functions of t and S_t). The option payoff H is allowed to be an \mathcal{F}_T

measurable random variable. This means that it can be a function of $(S_t : 0 \leq t \leq T)$, not just of S_T .

Note that $R_t^{-1}S_t$ has the form $f(t, S_t)$ where S_t is an Ito's process. Therefore, using the Ito's formula, the discounted stock price process satisfies the relation

$$d(R_t^{-1}S_t) = -r_t R_t^{-1}S_t + R_t^{-1}dS_t = R_t^{-1}S_t((\mu_t - r_t)dt + \sigma_t dW_t). \quad (13)$$

It is now easy to see from Girsanov theorem that if $\sigma_t > 0$ almost surely, then under the probability measure \mathcal{P}^ν with $\nu_t = \frac{r_t - \mu_t}{\sigma_t}$ the discounted stock price process satisfies the relation

$$d(R_t^{-1}S_t) = R_t^{-1}S_t\sigma_t dW_t^\nu$$

(this can be seen by replacing dW_t by $dW_t^\nu + \nu_t dt$ in (13). This operation in differential notation can be shown to be technically valid). This being a stochastic integral is a martingale (modulo technical conditions), so that \mathcal{P}^ν is the equivalent martingale measure.

It is easy to see by applying the Ito's formula (Note that $S_t = R_t X_t$ where $X_t = R_t^{-1}S_t$ satisfies the SDE above) that

$$dS_t = r_t S_t dt + S_t \sigma_t dW_t^\nu. \quad (14)$$

Hence, under the equivalent martingale measure \mathcal{P}^ν , the drift of the stock price process changes from $\{\mu_t\}$ to $\{r_t\}$. Therefore, \mathcal{P}^ν is also referred to as the risk neutral measure.

3.5.2 Creating the replicating portfolio process

Now consider the problem of creating the replicating process for an option with pay-off H . Define V_t for $0 \leq t \leq T$ so that

$$R_t^{-1}V_t = E_{\mathcal{P}^\nu}[R_T^{-1}H|\mathcal{F}_t].$$

Note that $V_T = H$.

Our plan is to construct a replicating portfolio process $(P_t : 0 \leq t \leq T)$ such that $P_t = V_t$ for all t . Then, since $P_T = H$ we have replicated the option with this portfolio process and P_t then denotes the price of the option at time t , i.e.,

$$P_t = E_{\mathcal{P}^\nu} \left[\exp \left(- \int_t^T r_s ds \right) H | \mathcal{F}_t \right]. \quad (15)$$

Then, the option price is simply the conditional expectation of the discounted option payoff under the risk neutral or the equivalent martingale measure.

To this end, it is easily seen from the law of iterated conditional expectations that for $s < t$,

$$R_s^{-1}V_s = E_{\mathcal{P}^\nu} (E_{\mathcal{P}^\nu}[R_T^{-1}V_T|\mathcal{F}_t]|\mathcal{F}_s) = E_{\mathcal{P}^\nu}[R_t^{-1}V_t|\mathcal{F}_s],$$

that is, $(R_t^{-1}V_t : 0 \leq t \leq T)$ is a martingale. From the martingale representation theorem there exists an adapted process $(w_t : 0 \leq t \leq T)$ such that

$$d(R_t^{-1}V_t) = w_t dW_t^\nu. \quad (16)$$

Now consider a portfolio process $(P_t : 0 \leq t \leq T)$ with the associated recipe process $(b_t : 0 \leq t \leq T)$. Recall that this means that we start with wealth P_0 . At any time t the portfolio value is denoted by P_t which is used to purchase b_t units of stock. The remaining amount $P_t - b_t S_t$ is invested in the money market. Then, the portfolio process evolves as in (9). The discounted portfolio process

$$d(R_t^{-1}P_t) = R_t^{-1}b_t S_t ((\mu_t - r_t)dt + \sigma_t dW_t) = R_t^{-1}b_t S_t \sigma_t dW_t^\nu. \quad (17)$$

Therefore, under technical conditions, the discounted portfolio process being a stochastic integral, is a martingale under \mathcal{P}^ν .

From (16) and (17) its clear that if we set $P_0 = V_0$ and $b_t = \frac{w_t R_t}{S_t \sigma_t}$ then $P_t = V_t$ for all t . In particular we have constructed a replicating portfolio. In particular, under technical conditions, primarily that $\sigma_t > 0$ almost surely, for almost every t every option can be replicated so that this market is complete.

3.5.3 Black Scholes pricing

Suppose that the stock price follows the SDE

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ and $\sigma > 0$ are constant. Furthermore the short rate r_t equals a constant r . From above and (14), it follows that under the equivalent martingale measure \mathcal{P}^ν

$$dS_t = r S_t dt + S_t \sigma dW_t^\nu. \quad (18)$$

Equivalently,

$$S_t = S_0 \exp(\sigma W_t^\nu + (r - \sigma^2/2)t). \quad (19)$$

The fact that S_t given by (19) satisfies (18) can be seen applying Ito's formula to (19). Since (18) has a unique solution, it is given by (19). Since, $\{W_t^\nu\}$ is the standard Brownian motion under \mathcal{P}^ν , it follows that $\{S_t\}$ is geometric Brownian motion and that for each t , $\log S_t$ has a Gaussian distribution with mean $(r - \sigma^2/2)t + \log S_0$ and variance $\sigma^2 t$.

Now suppose that we want to price a call option that matures at time T and has a strike price K . Then, $H = (S_T - K)^+$ (note that $(x)^+ = \max(x, 0)$). The option price equals

$$E([\exp(N((r - \sigma^2/2)T + \log S_0, \sigma^2 T)) - K]^+) \quad (20)$$

where $N(a, b)$ denotes a Gaussian distributed random variable with mean a and variance b . (20) can be easily evaluated to give the price of the European call option. The option price at time t can be inferred from (15) to equal

$$E([\exp(N((r - \sigma^2/2)(T - t) + \log S_t, \sigma^2(T - t))) - K]^+).$$

Call this $c(t, S_t)$. It can be shown that this function satisfies the Black Scholes pde (12) with $h(x) = (x - K)^+$.

3.6 Multiple assets

Now we extend the analysis to n risky assets or stocks driven by n independent sources of noise modeled by independent Brownian motions $(W_t^1, \dots, W_t^n : 0 \leq t \leq T)$. Specifically, we assume that n assets $(S_t^1, \dots, S_t^n : 0 \leq t \leq T)$ satisfy the SDE

$$dS_t^i = \mu_t^i S_t^i dt + \sum_{j=1}^n \sigma_t^{ij} S_t^i dW_t^j \quad (21)$$

for $i = 1, \dots, n$. Here we assume that each μ^i and σ^{ij} is an adapted process and satisfy restrictions so that the integrals associated with (21) are well defined. In addition we let $R_t = \exp(\int_0^t r(s) ds)$ as before. Above, the number of Brownian motions may be taken to be different from the number of securities, however the results are more elegant when the two are equal. This is also a popular assumption in practice.

The key observation from essentially repeating the analysis as for the single risky asset is that for the equivalent martingale measure to exist, the matrix $\{\sigma_t^{ij}\}$ has to be invertible almost everywhere. This condition is also essential to create a replicating portfolio for any given option (that is, for the market to be complete).

It is easy to see that

$$dR_t^{-1} S_t^i = S_t^i \left((\mu_t^i - r_t) dt + \sum_{j=1}^n \sigma_t^{ij} dW_t^j \right)$$

for all i . Now we look for conditions under which the equivalent martingale measure exists. Using the Girsanov theorem, it can be shown that if \mathcal{P}^ν is equivalent to \mathcal{P} then, under it,

$$dW_t^{\nu,j} = dW_t^j - \nu_t^j dt$$

for each j , are independent standard Brownian motions for some adapted processes $(\nu^j : j \leq n)$. Hence, for \mathcal{P}^ν to be an equivalent martingale measure (so that each discounted security process is a martingale under it) a necessary condition is

$$\mu_t^i - r_t = \sum_{j=1}^n \sigma_t^{ij} \nu_j$$

for each i everywhere except possibly on a set of measure zero. This means that the matrix $\{\sigma_t^{ij}\}$ has to be invertible everywhere except possibly on a set of measure zero.

Similarly, given an option with payoff H , we now look for a replicating portfolio process under the assumption that equivalent martingale measure \mathcal{P}^ν exists. Consider a portfolio process $(P_t : 0 \leq t \leq T)$ that at any time t purchases b_t^i number of security i . The remaining wealth $P_t - \sum_1^n b_t^i S_t^i$ is invested in the money market. Hence,

$$dP_t = \left((P_t - \sum_1^n b_t^i S_t^i) r_t + \sum_1^n b_t^i \mu_t^i S_t^i \right) dt + \sum_{i=1}^n b_t^i S_t^i \sum_{j=1}^n \sigma_t^{ij} dW_t^j \quad (22)$$

and

$$dR_t^{-1}P_t = R_t^{-1} \sum_{i=1}^n b_t^i S_t^i \sum_{j=1}^n \sigma_t^{ij} dW_t^{\nu,j}.$$

Since this is a stochastic integral, under mild technical conditions it is a martingale.

Again, since the equivalent martingale measure \mathcal{P}^ν exists we can define V_t for $0 \leq t \leq T$ so that

$$R_t^{-1}V_t = E_{\mathcal{P}^\nu}[R_T^{-1}H|\mathcal{F}_t].$$

Note that $V_T = H$.

As before, $(R_t^{-1}V_t : 0 \leq t \leq T)$ is a martingale. Then, from a multi-dimensional version of the martingale representation theorem (see any standard text, e.g., Steele [31]), the existence of an adapted process $(w_t \in \mathfrak{R}^n : 0 \leq t \leq T)$ can be shown such that

$$d(R_t^{-1}V(t)) = \sum_{j=1}^n w_t^j dW_t^{\nu,j}. \quad (23)$$

Then, to be able to replicate the option with portfolio process $(P_t : 0 \leq t \leq T)$ we need that $P_0 = V_0$ and

$$w_t^j = R_t^{-1} \sum_{i=1}^n b_t^i S_t^i \sigma_t^{ij}$$

for each j almost everywhere. This again can be solved for $(b_t \in \mathfrak{R}^n : 0 \leq t \leq T)$ if the transpose of the matrix $\{\sigma_t^{ij}\}$ is invertible for almost all t . We refer the reader to standard texts, e.g., Karatzas and Shreve [19], Duffie [8], Shreve [30], Steele [31] for a comprehensive treatment of options pricing in the multi-dimensional settings.

4 Conclusion

In this chapter we introduced derivatives pricing theory emphasizing its history and its importance to modern finance. We discussed some popular derivatives. we also described the no-arbitrage based pricing methodology, first in the simple binomial tree setting and then for continuous time models.

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