Euler's Formula and Trigonometry

Peter Woit
Department of Mathematics, Columbia University

September 10, 2019

These are some notes first prepared for my Fall 2015 Calculus II class, to give a quick explanation of how to think about trigonometry using Euler's formula. This is then applied to calculate certain integrals involving trigonometric functions.

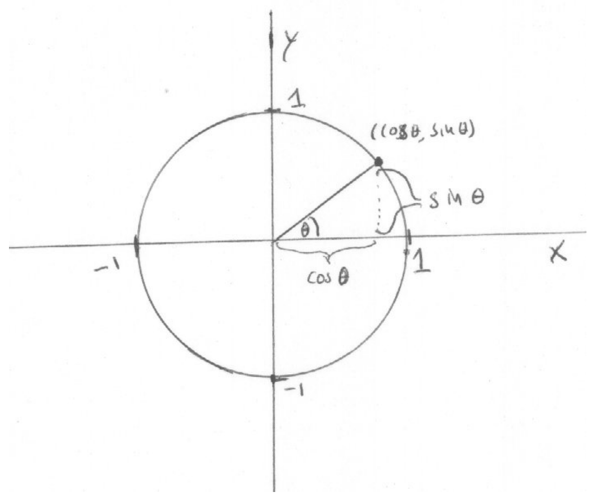
1 The sine and cosine as coordinates of the unit circle

The subject of trigonometry is often motivated by facts about triangles, but it is best understood in terms of another geometrical construction, the unit circle. One can define

Definition (Cosine and sine). *Given a point on the unit circle, at a counter-clockwise angle θ from the positive x-axis,*

- $\cos\theta$ *is the x-coordinate of the point.*
- $\sin\theta$ *is the y-coordinate of the point.*

The picture of the unit circle and these coordinates looks like this:



Some trigonometric identities follow immediately from this definition, in particular, since the unit circle is all the points in plane with x and y coordinates satisfying $x^2 + y^2 = 1$, we have

$$\cos^2\theta + \sin^2\theta = 1$$

Other trignometric identities reflect a much less obvious property of the cosine and sine functions, their behavior under addition of angles. This is given by the following two formulas, which are not at all obvious

$$\begin{aligned} \cos(\theta_1 + \theta_2) &= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 \\ \sin(\theta_1 + \theta_2) &= \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 \end{aligned} \tag{1}$$

One goal of these notes is to explain a method of calculation which makes these identities obvious and easily understood, by relating them to properties of exponentials.

2 The complex plane

A complex number c is given as a sum

$$c = a + ib$$

where a, b are real numbers, a is called the "real part" of c, b is called the "imaginary part" of c, and i is a symbol with the property that $i^2 = -1$. For any complex number c, one defines its "conjugate" by changing the sign of the imaginary part

$$\bar{c} = a - ib$$

The length-squared of a complex number is given by

$$c\bar{c} = (a + ib)(a - ib) = a^2 + b^2$$

which is a real number.

Some of the basic tricks for manipulating complex numbers are the following:

- To extract the real and imaginary parts of a given complex number one can compute

$$\begin{aligned}\mathrm{Re}(c) =& \frac{1}{2}(c+\overline{c})\\ \mathrm{Im}(c) =& \frac{1}{2i}(c-\overline{c})\end{aligned} \tag{2}$$

- To divide by a complex number c, one can instead multiply by

$$\frac{\overline{c}}{c\overline{c}}$$

  in which form the only division is by a real number, the length-squared of c.

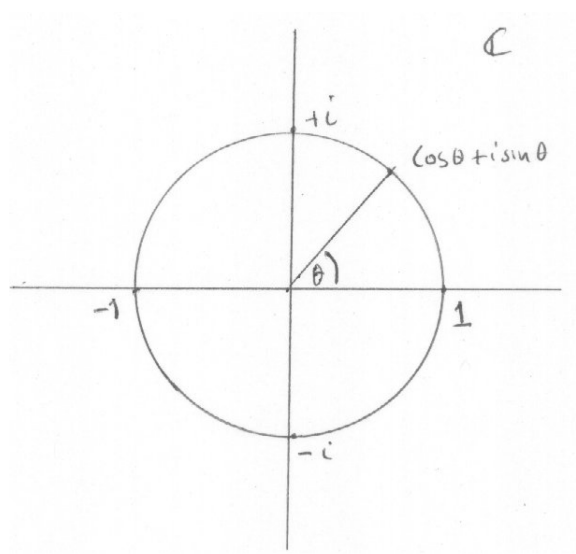
Instead of parametrizing points on the plane by pairs (x, y) of real numbers, one can use a single complex number

$$z = x + iy$$

in which case one often refers to the plane parametrized in this way as the "complex plane". Points on the unit circle are now given by the complex numbers

$$\cos\theta + i\sin\theta$$

These go around the circle once starting at $\theta = 0$ and ending up back at the same point when $\theta = 2\pi$. Now the picture is



A remarkable property of complex numbers is that, since multiplying two of them gives a third, they provide something new and not at all obvious: a consistent way of multiplying points on the plane. We will see in the next section that multiplication by a point on the unit circle of angle θ will have an interesting geometric interpretation, as counter-clockwise rotation by an angle θ.

3 Euler's formula

The central mathematical fact that we are interested in here is generally called "Euler's formula", and written

$$\boxed{e^{i\theta} = \cos\theta + i\sin\theta}$$

Using equations 2 the real and imaginary parts of this formula are

$$\boxed{\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}) \qquad \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})}$$

(which, if you are familiar with hyperbolic functions, explains the name of the hyperbolic cosine and sine).

In the next section we will see that this is a very useful identity (and those of a practical bent may want to skip ahead to this), but first we should address the question of what exactly the left-hand side means. The notation used implies that it is "the number e raised to the power $i\theta$" and a striking example of this is the special case of $\theta = \pi$, which says

$$e^{i\pi} = -1$$

which relates three fundamental constants of mathematics (e, i, π) although these seem to have nothing to do with each other. The problem though is that the idea of multiplying something by itself an imaginary number of times does not seem to make any sense.

To understand the meaning of the left-hand side of Euler's formula, it is best to recall that for real numbers x, one can instead write

$$e^x = \exp(x)$$

and think of this as a function of x, the exponential function, with name "exp". The true signficance of Euler's formula is as a claim that the definition of the exponential function can be extended from the real to the complex numbers, preserving the usual properties of the exponential. For any complex number $c = a + ib$ one can apply the exponential function to get

$$\exp(a + ib) = \exp(a)\exp(ib) = \exp(a)(\cos b + i\sin b)$$

The trigonmetric addition formulas (equation 1) are equivalent to the usual property of the exponential, now extended to any complex numbers $c_1 = a_1+ib_1$ and $c_2 = a_2 + ib_2$, giving

$$
\begin{aligned}
e^{c_1+c_2} =& e^{a_1+a_2}e^{i(b_1+b_2)}\\
=& e^{a_1+a_2}(\cos(b_1+b_2)+i\sin(b_1+b_2))\\
=& e^{a_1+a_2}((\cos b_1\cos b_2 - \sin b_1\sin b_2)+i(\sin b_1\cos b_2+\cos b_1\sin b_2))\\
=& e^{a_1}(\cos b_1 + i\sin b_1)e^{a_2}(\cos b_2+i\sin b_2)\\
=& e^{c_1}e^{c_2}
\end{aligned}
$$

It is possible to show that $e^{i\theta} = \cos\theta + i\sin\theta$ has the correct exponential property purely geometrically, without invoking the trigonometric addition formulas. One can do this by showing that multiplication of a point $z = x + iy$ in the complex plane by $e^{i\theta}$ rotates the point about the origin by a counterclockwise angle θ. It then follows that multiplication by the product of $e^{i\theta_1}$ and $e^{i\theta_2}$ will be counterclockwise rotation by an angle $\theta_1 + \theta_2$, implying the correct exponential property

$$e^{i\theta_1}e^{i\theta_2} = e^{i(\theta_1+\theta_2)}$$

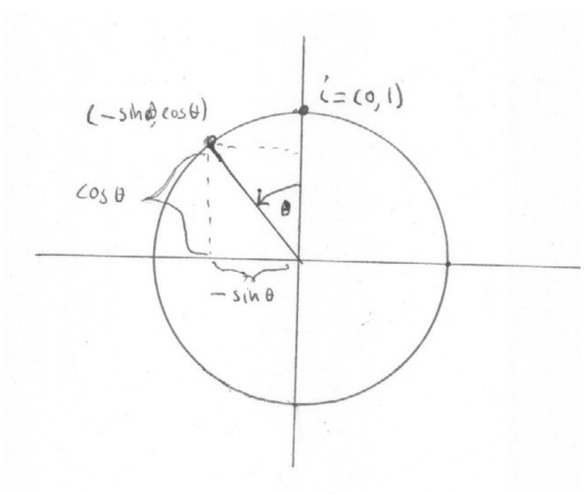
To show that multiplication by $e^{i\theta}$ will give a rotation by θ, one can argue as follows. One can easily see that multiplication by $e^{i\theta}$ rotates the point $z = 1$ along the unit circle by an angle θ, taking (in terms of real coordinates)

$$(1,0) \to (\cos\theta, \sin\theta)$$

This is also true for the point $z = i$, which gets taken to $i(\cos\theta + i\sin\theta) = -\sin\theta + i\cos\theta$. In terms of real coordinates on the plane, this is

$$(0,1) \to (-\sin\theta, \cos\theta)$$

and the rotation looks like this:



An arbitrary point on the plane is a linear combination of the points $(1,0)$ and $(0,1)$, and one can see that multiplication by $e^{i\theta}$ will act as rotation by θ on any such linear combination, knowing that it does so for the cases of $(1,0)$ and $(0,1)$.

Two other ways to motivate an extension of the exponential function to complex numbers, and to show that Euler's formula will be satisfied for such an extension are given in the next two sections.

3.1 $e^{i\theta}$ as a solution of a differential equation

The exponential functions $f(x) = \exp(cx)$ for c a real number has the property

$$\frac{d}{dx}f = cf$$

One can ask what function of x satisfies this equation for $c = i$. Using the derivatives of the cosine and sine one finds

$$\frac{d}{dx}(\cos x + i\sin x) = -\sin x + i\cos x = i(\cos x + i\sin x)$$

so $\cos x + i\sin x$ has the correct derivative to be the desired extension of the exponential function to the case $c = i$.

3.2 $e^{i\theta}$ and power series expansions

By the end of this course, we will see that the exponential function can be represented as a "power series", i.e. a polynomial with an infinite number of terms, given by

$$\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$

There are similar power series expansions for the sine and cosine, given by

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots$$

and

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} + \cdots$$

Euler's formula then comes about by extending the power series for the exponential function to the case of $x = i\theta$ to get

$$\exp(i\theta) = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + \cdots$$

and seeing that this is identical to the power series for $\cos\theta + i\sin\theta$.

4 Applications of Euler's formula

4.1 Trigonometric identities

Euler's formula allows one to derive the non-trivial trigonometric identities quite simply from the properties of the exponential. For example, the addition formulas can be found as follows:

$$\begin{aligned}\cos(\theta_1+\theta_2) =&\mathrm{Re}(e^{i(\theta_1+\theta_2)})\\ =&\mathrm{Re}(e^{i\theta_1}e^{i\theta_2})\\ =&\mathrm{Re}((\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2))\\ =&\cos\theta_1\cos\theta_2-\sin\theta_1\sin\theta_2\end{aligned}$$

and

$$\begin{aligned}\sin(\theta_1+\theta_2) =&\mathrm{Im}(e^{i(\theta_1+\theta_2)})\\ =&\mathrm{Im}(e^{i\theta_1}e^{i\theta_2})\\ =&\mathrm{Im}((\cos\theta_1+i\sin\theta_1)(\cos\theta_2+i\sin\theta_2))\\ =&\cos\theta_1\sin\theta_2+\sin\theta_1\cos\theta_2\end{aligned}$$

Multiple angle formulas for the cosine and sine can be found by taking real and imaginary parts of the following identity (which is known as de Moivre's formula):

$$\begin{aligned}\cos(n\theta)+i\sin(n\theta) =&e^{in\theta}\\ =&(e^{i\theta})^n\\ =&(\cos\theta+i\sin\theta)^n\end{aligned}$$

For example, taking $n=2$ we get the double angle formulas

$$\begin{aligned}\cos(2\theta) =&\mathrm{Re}((\cos\theta+i\sin\theta)^2)\\ =&\mathrm{Re}((\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta))\\ =&\cos^2\theta-\sin^2\theta\end{aligned}$$

and

$$\begin{aligned}\sin(2\theta) =&\mathrm{Im}((\cos\theta+i\sin\theta)^2)\\ =&\mathrm{Im}((\cos\theta+i\sin\theta)(\cos\theta+i\sin\theta))\\ =&2\sin\theta\cos\theta\end{aligned}$$

4.2 Derivatives of trigonometric functions

Writing the cosine and sine as the real and imaginary parts of $e^{i\theta}$, one can easily compute their derivatives from the derivative of the exponential. One has

$$\begin{aligned}\frac{d}{d\theta}\cos\theta =& \frac{d}{d\theta}\mathrm{Re}(e^{i\theta})\\ =& \frac{d}{d\theta}(\frac{1}{2}(e^{i\theta}+e^{-i\theta}))\\ =& \frac{i}{2}(e^{i\theta}-e^{-i\theta})\\ =& -\sin\theta\end{aligned}$$

and

$$\begin{aligned}\frac{d}{d\theta}\sin\theta =& \frac{d}{d\theta}\mathrm{Im}(e^{i\theta})\\ =& \frac{d}{d\theta}(\frac{1}{2i}(e^{i\theta}-e^{-i\theta}))\\ =& \frac{1}{2}(e^{i\theta}+e^{-i\theta})\\ =& \cos\theta\end{aligned}$$

4.3 Integrals of exponential and trigonometric functions

Three different types of integrals involving trigonmetric functions that can be straightforwardly evaluated using Euler's formula and the properties of exponentials are:

- Integrals of the form

  $$\int e^{ax}\cos(bx)dx \quad \text{or} \quad \int e^{ax}\sin(bx)dx$$

  are typically done in calculus textbooks using a trick involving two integrations by parts. They can be more straightforwardly evaluated by using Euler's formula to rewrite them as integrals of complex exponentials, for instance

$$\begin{aligned}
\int e^{ax}\cos(bx)dx =& \mathrm{Re}(\int e^{ax}e^{ibx}dx) \\
=& \mathrm{Re}(\int e^{(a+ib)x}dx) \\
=& \mathrm{Re}(\frac{1}{a+ib}e^{(a+ib)x}) + C \\
=& \mathrm{Re}(\frac{a-ib}{a^2+b^2}e^{ax}e^{ibx}) + C \\
=& \mathrm{Re}(\frac{a-ib}{a^2+b^2}e^{ax}(\cos(bx) + i\sin(bx))) + C \\
=& \frac{1}{a^2+b^2}e^{ax}(a\cos(bx) + b\sin(bx)) + C
\end{aligned}$$

- Integrals of the form

$$\int \cos(ax)\cos(bx)dx, \quad \int \cos(ax)\sin(bx)dx \quad \text{or} \int \sin(ax)\sin(bx)dx$$

are usually done by using the addition formulas for the cosine and sine functions. They could equally well be be done using exponentials, for instance (assuming $a \neq b$)

$$\begin{aligned}
\int \cos(ax)\cos(bx)dx =& \int \frac{1}{2}(e^{iax} + e^{-iax})\frac{1}{2}(e^{ibx} + e^{-ibx})dx \\
=& \frac{1}{4}\int (e^{i(a+b)x} + e^{i(a-b)x} + e^{-i(a-b)x} + e^{-i(a+b)x})dx \\
=& \frac{1}{2}\int (\cos((a+b)x) + \cos((a-b)x))dx \\
=& \frac{1}{2}(\frac{1}{a+b}\sin((a+b)x) + \frac{1}{a-b}\sin((a-b)x)) + C
\end{aligned}$$

- When a and b are integers m, n, and one integrates over an interval of size 2π (for instance $[-\pi, \pi]$), the above integrals give very simple results. This is due to the fact that

$$\int_{-\pi}^{+\pi} e^{imx}e^{inx}dx = \int_{-\pi}^{+\pi} e^{i(m+n)x}dx = \begin{cases} 0 \text{ if } m \neq n \\ 2\pi \text{ if } m = -n \end{cases}$$

One can show this by integrating the exponential, or more simply by noticing that the real and imaginary parts of the answer will, for $m \neq -n$, be given by integrating a cosine and sine over $m+n$ periods, This gives zero since the area under the curves is the same above and below the x-axis. For $m = -n$, the integrand is just 1, so the integral is the length of the interval of integration.

- Integrals of the form

$$\int \cos^m x\ dx, \quad \int \cos^m x \sin^n x\ dx \ \text{ or} \int \sin^m x\ dx$$

are performed in calculus textbooks by a combination of use of the substitution $u = \sin(x)$ or $u = \cos(x)$, of the identity $\cos^2 x + \sin^2 x = 1$ to turn even powers of the cosine into even powers of the sine (and vice-versa), as well as the double angle formulas for the cosine and sine. Such methods are often the simplest ones, but one can also do such integrals by expressing them in terms of exponentials. For example

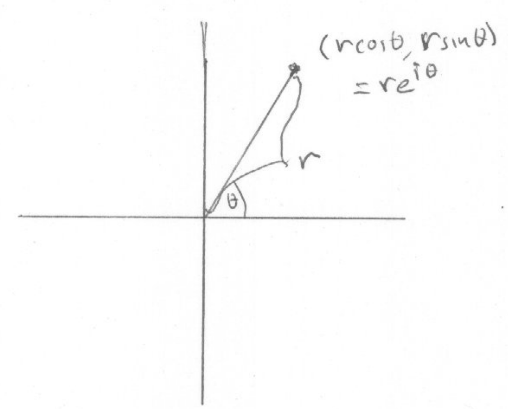
$$\begin{aligned}\int \cos^3 x\ dx =& \int (\frac{1}{2}(e^{ix} + e^{-ix}))^3 dx \\ =& \frac{1}{8}\int (e^{3ix} + 3e^{ix} + 3e^{-ix} + e^{-i3x})dx \\ =& \frac{1}{4}\int (\cos(3x) + 3\cos(x))dx \\ =& \frac{1}{12}\sin(3x) + \frac{3}{4}\sin(x) + C\end{aligned}$$

Note that this technique will typically give answers in a different form than the technique used in the book, giving not powers of the cosine or the sine, but something equivalent related to these by multiple-angle formulas.

4.4 Polar coordinates

Instead of Cartesian coordinates x and y, one can parametrize points in the plane by polar coordinates r (the distance from the origin) and θ (the angle with the positive x axis. A point with polar coordinates (r, θ) has (x, y) coordinates

$$x = r\cos\theta, \quad y = r\sin\theta$$



The point with polar coordinates (r, θ) has a simple complex coordinate, which, using Euler's formula, will be given by

$$z = x + iy = r\cos\theta + ir\sin\theta = re^{i\theta}$$

5 Problems

1. Compute
$$\int e^{ax}\sin(bx)dx$$
for arbitrary real constants a and b.

2. Compute
$$\int \cos(ax)\sin(bx)dx$$
for arbitrary real constants a and b.

3. Compute $\int \cos^3 x\ dx$ using Euler's formula, and show that the result is the same as what one gets by the textbook method (using a substitution $u = \sin x$).