



**SCHOOL OF ENGINEERING & BUILT  
ENVIRONMENT**

**Mathematics**

**Trigonometry**

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- 2. Measurement of Angles – Radian Measure
- 3. The Trigonometric Ratios: Sine, Cosine and Tangent
- 4. Three Other Trig Ratios
- 5. Some Exact Trig Values
- 6. Important Trig Identities
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- 8. Trigonometric Equations
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## Trigonometry

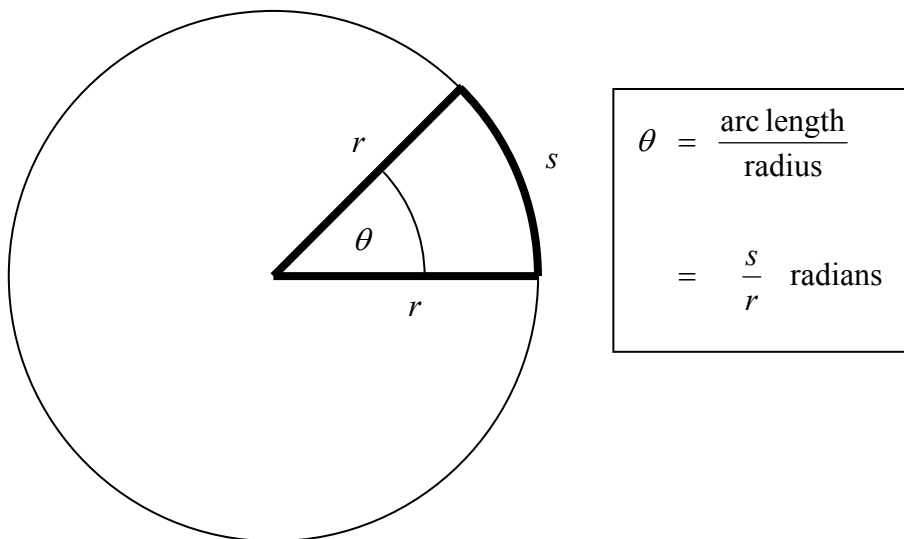
### 1) Introduction

Essentially, trigonometry is the study of triangles and the interrelationships between angles and lengths of lines. This has given rise to many applications in areas such as navigation, surveying and civil engineering. However trigonometry evolved and linked-in with “functional” mathematics and the modelling of periodic phenomena such as mechanical vibrations and electrical oscillations, which we shall look at towards the end of this section.

### 2) Measurement of Angles – Radian Measure

You should be familiar with the use of degrees for measuring angles. However degrees are not always convenient when we want to use trigonometric concepts in mathematical modelling and calculations involving calculus. The breaking down of a full circle into  $360^\circ$  is too arbitrary; we need something that is more natural, mathematically speaking. This is provided by **radian measure**.

Basic definition:



From this definition, the full circle angle in radians is obtained by setting  $s = 2\pi r$ , the circle's full circumference:

$$\text{Full circle angle} = \frac{2\pi r}{r} = 2\pi \text{ radians.}$$

**360° corresponds to 2π radians**

Using proportion we can determine conversion factors:

$$1^\circ \text{ corresponds to } \frac{2\pi}{360} = \frac{\pi}{180} \text{ radians ;}$$

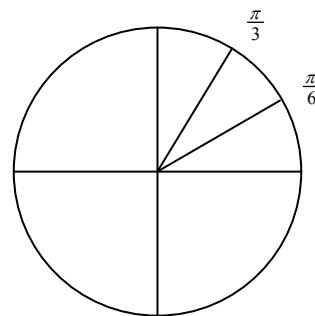
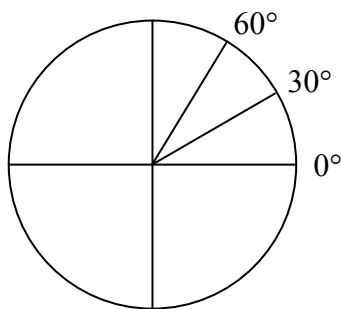
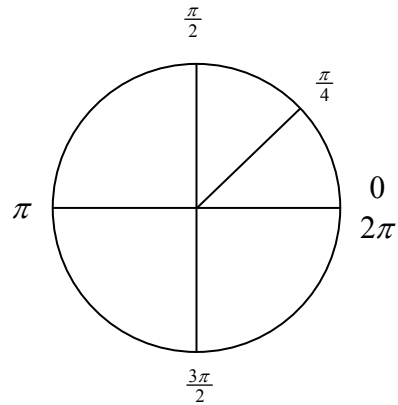
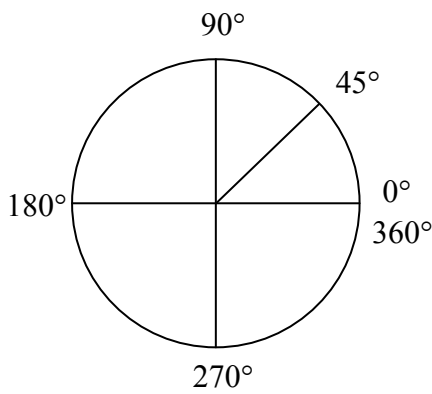
$$\theta^\circ \text{ corresponds to } \frac{\pi}{180} \theta \text{ radians .}$$

It follows that

$$\theta \text{ rad corresponds to } \left( \frac{180}{\pi} \theta \right)^\circ .$$

### Examples

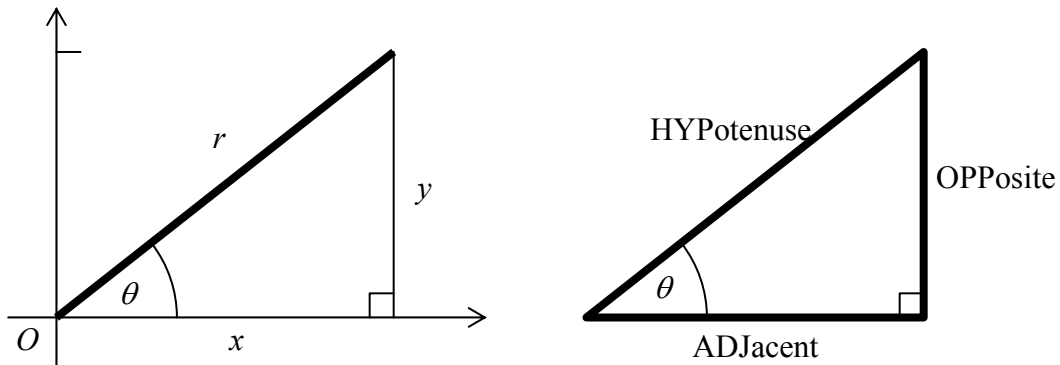
- (1) In the right-hand circles below, note the corresponding radian values as fractions or multiples of  $\pi$ .



- (2)  $35^\circ = 35 \times \frac{\pi}{180} \text{ radians} = 0.6109 \text{ radians}$
- (3)  $1 \text{ radian} = \left(1 \times \frac{180}{\pi}\right) \text{ degrees} = 57.30^\circ$
- (4)  $2.3 \text{ radians} = \left(2.3 \times \frac{180}{\pi}\right) \text{ degrees} = 131.78^\circ$

### 3) The Trigonometric Ratios: Sine, Cosine and Tangent

We can depict an angle on a Cartesian (i.e.  $Oxy$ ) axes system as being swept out by a radial arm of length  $r$  in an **anti-clockwise** direction from the positive  $x$ -axis (see below, left). This we can do for any angle, positive or negative; for a negative angle we move clockwise from  $0^\circ$ . If we are dealing with an acute angle (i.e. between  $0^\circ$  and  $90^\circ$ ) then it is sometimes useful to regard it as embedded in a right-angled triangle (see below, right).



The trig ratios are defined as follows:

Ratio	Abbreviation	Definition (any angle)	Definition ( $0^\circ < \theta < 90^\circ$ )
cosine	$\cos \theta$	$\frac{x}{r}$	$\frac{\text{ADJ}}{\text{HYP}}$
sine	$\sin \theta$	$\frac{y}{r}$	$\frac{\text{OPP}}{\text{HYP}}$
tangent	$\tan \theta$	$\frac{y}{x}$	$\frac{\text{OPP}}{\text{ADJ}}$

As the radial arm sweeps around the axes system and changes quadrant, the values of  $x$  and  $y$  change sign (+ / -). This means that the trig ratios are positive or negative in different quadrants. If we regard  $r$  as always positive, then we see that

		$90^\circ$		
2nd quadrant	<b>sin +ve</b>		<b>sin +ve</b>	1st quadrant
	<b>cos -ve</b>		<b>cos +ve</b>	
	<b>tan -ve</b>		<b>tan +ve</b>	
$180^\circ$	<b>sin -ve</b>		<b>sin -ve</b>	$0^\circ$
3rd quadrant	<b>cos -ve</b>		<b>cos +ve</b>	4th quadrant
	<b>tan +ve</b>		<b>tan -ve</b>	
		$270^\circ$		

This information will prove extremely important later.

#### 4) Three Other Trig Ratios

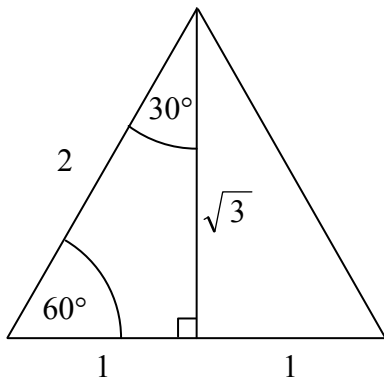
Secant:  $\sec \theta = \frac{1}{\cos \theta}$

Cosecant:  $\operatorname{cosec} \theta = \frac{1}{\sin \theta}$  (sometimes denoted by “csc”)

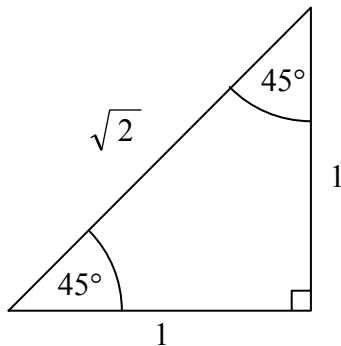
Cotangent:  $\cot \theta = \frac{1}{\tan \theta}$

#### 5) Some Exact Trig Values

Determining trig values is easy with a calculator. Just make sure your calculator is set to the appropriate mode (degrees or radians). Except for a few cases, calculators will give approximate values (albeit highly accurate ones). However, for some angles we can be exact:



$\cos 60^\circ = \frac{1}{2}$	$\cos 30^\circ = \frac{\sqrt{3}}{2}$
$\sin 60^\circ = \frac{\sqrt{3}}{2}$	$\sin 30^\circ = \frac{1}{2}$
$\tan 60^\circ = \frac{\sqrt{3}}{1}$	$\tan 30^\circ = \frac{1}{\sqrt{3}}$



$\cos 45^\circ = \frac{1}{\sqrt{2}}$
$\sin 45^\circ = \frac{1}{\sqrt{2}}$
$\tan 45^\circ = 1$

## 6) Important Trig Identities

There are lots of equations relating the trig ratios; below are the most important ones. Others can be found in the module formula sheets.

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin^2 A + \cos^2 A = 1 \quad [\text{Note how we write powers of the trig ratios}]$$

$$\cos(-A) = \cos(A)$$

$$\sin(-A) = -\sin(A)$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B$$

$$\sin 2A = 2 \sin A \cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

## 7) Trig Functions and Inverse Trig Functions

If we write

$$y = \sin x$$

$$y = \cos x$$

$$y = \tan x$$

[Note: Trig value from angle.]

we have created mathematical functions from the trig ratios. Later on in this section we shall look at the graphs of the sine and cosine functions and use modified forms of the functions to model periodic behaviour.

Each of these trig functions has a corresponding inverse function (note the notation used):

$$x = \sin^{-1} y$$

$$x = \cos^{-1} y$$

$$x = \tan^{-1} y$$

[Note: Angle from trig value.]

As stated earlier, to evaluate a trig value from a given angle we can use a calculator, making sure it is set to the appropriate mode (degrees or radians). This is a straightforward process. However to do the reverse, that is, determine an angle corresponding to a given trig value, is a little bit more complicated. We can still use a calculator, but the calculator will only give one angle when, in fact, there are an infinite number of possible angles. We need a systematic way of making sure that we don't miss any angles that may be relevant.

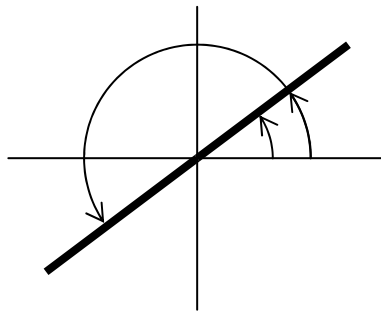
## Examples

(5) Determine  $x$  when  $\tan x = 0.75355$ .

### Step 1

It is tempting to go straight to a calculator and calculate  $\tan^{-1}(0.75355)$ . We will do that, but **not** right away! First of all, we remember that there is not just a single answer, but multiple answers. We must establish in which quadrants those multiple answers lie. To do this we use the quadrant diagram at the bottom of p3.

In this example we are dealing with a tangent value which is positive. The quadrant diagram tells us that a tangent is positive in the 1st and 3rd quadrants. We draw a little sketch showing an angle in the 1st quadrant and a corresponding angle reflected through the origin into the 3rd quadrant:



These show the quadrants where the **angles of inversion** lie.

### Step 2

Now invert 0.75355 by calculator:

$$\begin{aligned} x &= \tan^{-1}(0.75355) \\ &= 37.0^\circ \end{aligned}$$

This is clearly the 1st quadrant angle.

### Step 3

With the aid of the quadrant diagram, compute the 3rd quadrant angle:

$$\begin{aligned} x &= 37.0^\circ + 180^\circ \\ &= 217.0^\circ \end{aligned}$$

### Step 4

To each of these angles of inversion, we can add or subtract any number of complete rotations of  $360^\circ$  and still obtain angles with the same trig value. So all possible angles of inversion are covered by the expressions:

$$\left. \begin{aligned} x &= 37.0^\circ + n \cdot 360^\circ \\ x &= 217.0^\circ + n \cdot 360^\circ \end{aligned} \right\} \text{ (where } n \text{ is an integer).}$$

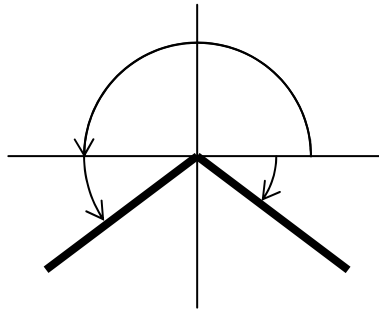


**Example**

(6) Determine  $x$  when  $\sin x = -0.43775$ .

**Step 1**

This time we are dealing with a sine value which is negative. The quadrant diagram tells us that a sine is negative in the 3rd and 4th quadrants. We draw a sketch showing an angle in the 3rd quadrant and a corresponding angle reflected across the  $y$ -axis into the 4th quadrant:



These show the quadrants where the **angles of inversion** lie.

**Step 2**

Now invert  $-0.43775$  by calculator:

$$\begin{aligned} x &= \sin^{-1}(-0.43775) \\ &= -26.0^\circ \end{aligned}$$

This is clearly the 4th quadrant angle.

**Step 3**

With the aid of the quadrant diagram, compute the 3rd quadrant angle:

$$\begin{aligned} x &= 180^\circ + 26.0^\circ \\ &= 206.0^\circ \end{aligned}$$

**Step 4**

To each of these angles of inversion, we include the arbitrary number of complete rotations of  $360^\circ$ . The angles of inversion are given by the expressions:

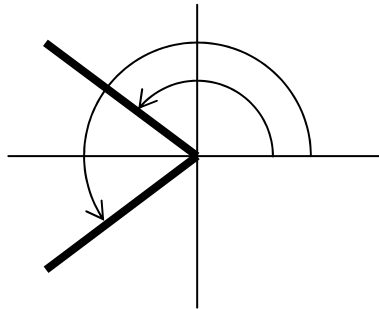
$$\left. \begin{aligned} x &= -26.0^\circ + n.360^\circ \\ x &= 206.0^\circ + n.360^\circ \end{aligned} \right\} \text{ (where } n \text{ is an integer).}$$

**Example**

(7) Determine  $x$  when  $\cos x = -0.34215$ .

**Step 1**

This time we are dealing with a cosine value which is negative. The quadrant diagram tells us that a cosine is negative in the 2nd and 3rd quadrants. Draw the sketch showing an angle in the 2nd quadrant and a corresponding angle reflected across the  $x$ -axis into the 3rd quadrant:



These show the quadrants where the **angles of inversion** lie.

**Step 2**

Now invert  $-0.34215$  by calculator:

$$\begin{aligned} x &= \cos^{-1}(-0.34215) \\ &= 110.0^\circ \end{aligned}$$

This is clearly the 2nd quadrant angle.

**Step 3**

With the aid of the quadrant diagram, compute the 3rd quadrant angle:

$$x = -110^\circ \quad \text{or} \quad x = -110^\circ + 360^\circ = 250^\circ$$

**Step 4**

Including the arbitrary number of complete rotations of  $360^\circ$  we obtain:

$$\left. \begin{aligned} x &= 110.0^\circ + n.360^\circ \\ x &= -110.0^\circ + n.360^\circ \end{aligned} \right\} \quad \text{or} \quad \left. \begin{aligned} x &= 110.0^\circ + n.360^\circ \\ x &= 250.0^\circ + n.360^\circ \end{aligned} \right\} .$$

**WARNING: DO NOT CUT CORNERS WHEN INVERTING TRIG FUNCTIONS. FOLLOW ALL THE STEPS.**

We can work in a similar fashion with the calculator set to radian mode. However you may find it easier to work in degrees and convert to radians at the end. Either way, show that the solutions to Example (5) – (7) in radians are:

$$(5) \quad \left. \begin{array}{l} x = 0.6458 + 2n\pi \\ x = 3.7874 + 2n\pi \end{array} \right\} \text{ (where } n \text{ is an integer)}$$

$$(6) \quad \left. \begin{array}{l} x = -0.4531 + 2n\pi \\ x = 3.5947 + 2n\pi \end{array} \right\} \text{ (where } n \text{ is an integer)}$$

$$(7) \quad \left. \begin{array}{l} x = 1.9200 + 2n\pi \\ x = -1.9200 + 2n\pi \end{array} \right\} \text{ or } \left. \begin{array}{l} x = 1.9200 + 2n\pi \\ x = 4.3632 + 2n\pi \end{array} \right\} \text{ (where } n \text{ is an integer).}$$

**Please note the following special cases where angles of inversion do not lie in quadrants, but between quadrants:**

$$\sin x = 0 \rightarrow \left. \begin{array}{l} x = 0^\circ + n.360^\circ \\ x = 180^\circ + n.360^\circ \end{array} \right\} \rightarrow x = n.180^\circ$$

$$\sin x = +1 \rightarrow x = 90^\circ + n.360^\circ$$

$$\sin x = -1 \rightarrow x = 270^\circ + n.360^\circ$$

$$\cos x = 0 \rightarrow \left. \begin{array}{l} x = 90^\circ + n.360^\circ \\ x = 270^\circ + n.360^\circ \end{array} \right\}$$

$$\cos x = +1 \rightarrow x = 0^\circ + n.360^\circ$$

$$\cos x = -1 \rightarrow x = 180^\circ + n.360^\circ$$

$$\tan x = 0 \rightarrow \left. \begin{array}{l} x = 0^\circ + n.360^\circ \\ x = 180^\circ + n.360^\circ \end{array} \right\} \rightarrow x = n.180^\circ$$

## 8) Trigonometric Equations

As its name suggests, a trig equation is any equation that contains a trig function. Examples (5) – (7) illustrate the simplest forms of trig equation and how to solve them. For more complicated trig equations, we have to work a little harder. In all cases we try to manipulate the equation into one (or more) basic forms that can be inverted. We then do the inversion(s) as described earlier. After that, a little more algebra may be required to complete the solution.

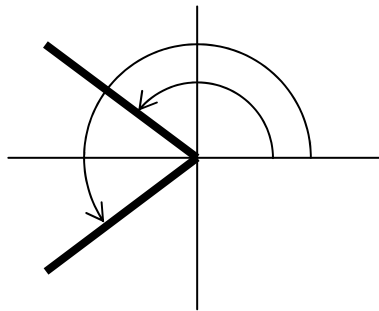
### Examples

(8) Determine the solution of  $2 \cos(4x) + 1 = 0$  for  $0^\circ \leq x \leq 360^\circ$ .

First, manipulate the equation into the form “trig function equals a number”:

$$\cos(4x) = -\frac{1}{2} .$$

Next, prepare for the inversion by identifying the relevant quadrants:



Note that this diagram only relates to the angles of inversion and **not** the angles of the final solution.

Now we can invert by calculator:  $\cos^{-1}(-\frac{1}{2}) = 120^\circ$  .

This is a 2nd quadrant angle; the corresponding 3rd quadrant angle is  $-120^\circ$  or  $240^\circ$ . Given the range of interest specified in the question, we shall work with the positive angle. This means that

$$\left. \begin{aligned} 4x &= 120^\circ + n.360^\circ \\ 4x &= 240^\circ + n.360^\circ \end{aligned} \right\} .$$

Solving for  $x$  gives:

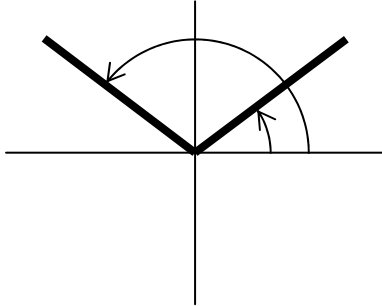
$$\left. \begin{aligned} x &= 30^\circ + n.90^\circ \\ x &= 60^\circ + n.90^\circ \end{aligned} \right\} .$$

This constitutes the **general** solution. We now select the particular solutions in the range  $0^\circ \leq x \leq 360^\circ$  :

$$\underline{\underline{x = 30^\circ, 60^\circ, 120^\circ, 150^\circ, 210^\circ, 240^\circ, 300^\circ, 330^\circ}}$$

(9) Determine the general solution of  $\sin(2t + 1) = 0.6$ , where  $t$  is in radians.

This equation is already in the form “trig function equals a number” so we can proceed with the inversion:



$$\sin^{-1}(0.6) = 0.6435$$

This is a 1st quadrant angle; the corresponding 2nd quadrant angle is  $\pi - 0.6435 = 2.4981$ , so

$$\left. \begin{aligned} 2t + 1 &= 0.6435 + 2n\pi \\ 2t + 1 &= 2.4981 + 2n\pi \end{aligned} \right\} .$$

Solve for  $t$ ; subtract 1 and divide by 2 :

$$\left. \begin{aligned} 2t &= -0.3565 + 2n\pi \\ 2t &= 1.4981 + 2n\pi \end{aligned} \right\}$$

$$\left. \begin{aligned} t &= -0.1782 + n\pi \\ t &= 0.7490 + n\pi \end{aligned} \right\}$$


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(10) Solve  $4 \sin^2 \theta - 3 = 0$  for  $0^\circ \leq \theta \leq 360^\circ$ .

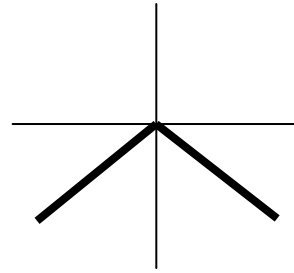
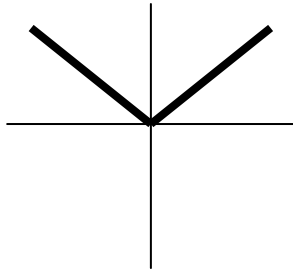
Rearrange equation to give

$$\sin^2 \theta = 0.75 .$$

Here we have to be very careful that we don't miss any solutions. When we take the square root we must remember to consider both the positive and negative roots:

$$\sin \theta = +0.8660254$$

$$\sin \theta = -0.8660254$$



$$\left. \begin{aligned} \theta &= 60^\circ + n.360^\circ \\ \theta &= 120^\circ + n.360^\circ \end{aligned} \right\}$$

$$\left. \begin{aligned} \theta &= -60^\circ + n.360^\circ \\ \theta &= 240^\circ + n.360^\circ \end{aligned} \right\}$$

Solution:

$$\underline{\underline{\theta = 60^\circ, 120^\circ, 240^\circ, 300^\circ}}$$

Sometimes we may need the help of trig identities to convert a trig equation into an invertible form.

Recall the following identities:

$$\sin^2 A + \cos^2 A = 1$$

$$\frac{\sin A}{\cos A} = \tan A$$

$$\sin(A + B) = \sin A \cos B + \cos A \sin B$$

$$\sin(A - B) = \sin A \cos B - \cos A \sin B$$

$$\cos(A + B) = \cos A \cos B - \sin A \sin B$$

$$\cos(A - B) = \cos A \cos B + \sin A \sin B .$$

We shall use versions of these in the following examples:

**Example**

(11) Solve  $3 \sin x + 4 \cos x = 2$ ,  $0 \leq x \leq 2\pi$ , by converting the LHS of the equation to the form  $R \sin(x + \alpha)$ .

First, use trig IDs to determine and solve equations for  $R$  and  $\alpha$ :

$$\begin{aligned} 3 \sin x + 4 \cos x &= R \sin(x + \alpha) \\ &= R [\sin x \cos \alpha + \cos x \sin \alpha] \\ &= R \cos \alpha \sin x + R \sin \alpha \cos x \end{aligned}$$

Comparing sides gives

$$\left. \begin{aligned} R \cos \alpha &= 3 \\ R \sin \alpha &= 4 \end{aligned} \right\} \begin{array}{l} [1] \\ [2] \end{array}$$

Squaring and adding [ 1 ] and [ 2 ] gives

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 3^2 + 4^2$$

$$R^2 [\cos^2 \alpha + \sin^2 \alpha] = 25$$

$$R^2 = 25$$

$$R = 5 .$$

[ 2 ]  $\div$  [ 1 ] gives

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{4}{3}$$

$$\tan \alpha = \frac{4}{3}$$

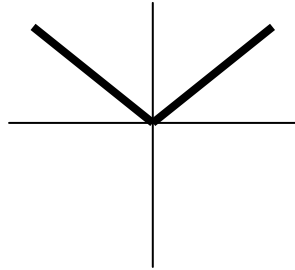
$$\alpha = 0.9273 .$$

The original equation can therefore be written as

$$5 \sin(x + 0.9273) = 2 .$$

We can now solve this by trig inversion:

$$\sin(x + 0.9273) = 0.4$$



$$\left. \begin{aligned} x + 0.9273 &= 0.4115 + 2n\pi && \text{(1st quadrant)} \\ x + 0.9273 &= 2.7301 + 2n\pi && \text{(2nd quadrant)} \end{aligned} \right\}$$

$$\left. \begin{aligned} x &= -0.5158 + 2n\pi \\ x &= 1.8028 + 2n\pi \end{aligned} \right\}.$$

Selecting the values of  $x$  between  $0$  and  $2\pi$  gives the required solution:

$$\underline{\underline{x = 1.8028, 5.7674}}$$

(12) Solve  $5 \cos(2x) - 12 \sin(2x) = -6.5$ ,  $0 \leq x \leq 2\pi$ , by converting the LHS of the equation to the form  $R \cos(2x + \alpha)$ .

Similar to before, use trig IDs to determine  $R$  and  $\alpha$ :

$$\begin{aligned} 5 \cos(2x) - 12 \sin(2x) &= R \cos(2x + \alpha) \\ &= R [\cos(2x) \cos \alpha - \sin(2x) \sin \alpha] \\ &= R \cos \alpha \cos(2x) - R \sin \alpha \sin(2x) \end{aligned}$$

Comparing sides gives

$$\left. \begin{aligned} R \cos \alpha &= 5 \\ R \sin \alpha &= 12 \end{aligned} \right\} \begin{array}{l} [1] \\ [2] \end{array}$$



Squaring and adding [ 1 ] and [ 2 ] gives

$$R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 5^2 + 12^2$$

$$R^2 [\cos^2 \alpha + \sin^2 \alpha] = 169$$

$$R^2 = 169$$

$$R = 13 .$$

[ 2 ]  $\div$  [ 1 ] gives

$$\frac{R \sin \alpha}{R \cos \alpha} = \frac{12}{5}$$

$$\tan \alpha = \frac{12}{5}$$

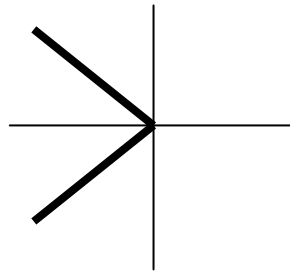
$$\alpha = 1.1760 .$$

The original equation can therefore be written as

$$13 \cos(2x + 1.1760) = -6.5 .$$

Solving this by trig inversion:

$$\cos(2x + 1.1760) = -0.5$$



$$\left. \begin{aligned} 2x + 1.1760 &= 2.0944 + 2n\pi && \text{(2nd quadrant)} \\ 2x + 1.1760 &= 4.1888 + 2n\pi && \text{(3rd quadrant)} \end{aligned} \right\}$$

$$\left. \begin{aligned} 2x &= 0.9184 + 2n\pi \\ 2x &= 3.0128 + 2n\pi \end{aligned} \right\}$$

$$\left. \begin{aligned} x &= 0.4592 + n\pi \\ x &= 1.5064 + n\pi \end{aligned} \right\}$$

Selecting the values of  $x$  between 0 and  $2\pi$  gives the required solution:

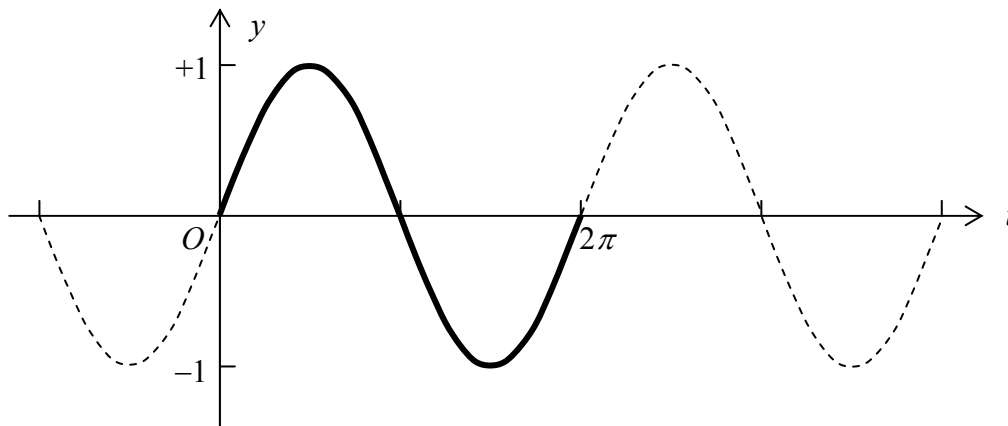
$$\underline{\underline{x = 0.4592 , 1.5064 , 3.6008 , 4.6478 .}}$$

## 9) The Periodic Nature of the Trigonometric Functions

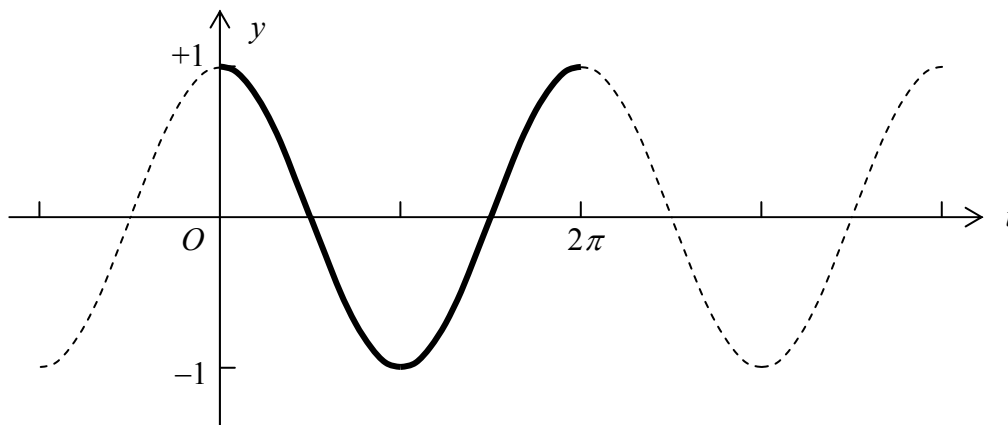
In this section we shall look at the graphs of the basic trig functions and how to modify these basic functions in order to model periodic phenomena such as mechanical vibrations and electrical oscillations.

Since many applications deal with time dependence, we shall use  $t$  as the independent variable.

**The graph of  $y = \sin t$ :**



**The graph of  $y = \cos t$ :**



We won't consider the graph of  $y = \tan t$  here.

Both the sine and cosine functions display **simple harmonic motion** and oscillate between  $+1$  and  $-1$ ; we say they have an **amplitude** equal to 1.

The graphs exhibit periodic behaviour, that is, their shape is repetitive with increasing  $t$ . We define the following terms:

**Cycle:** A portion of the function graph that is repeated. One cycle starting at  $t = 0$  is drawn solid in each of the graphs on p16.

**Period:** The time taken for one cycle to elapse.

**Frequency:** The number of cycles per unit of time.

For the basic sine and cosine functions,  $y = \sin t$  and  $y = \cos t$ :

$$\text{Amplitude} = 1$$

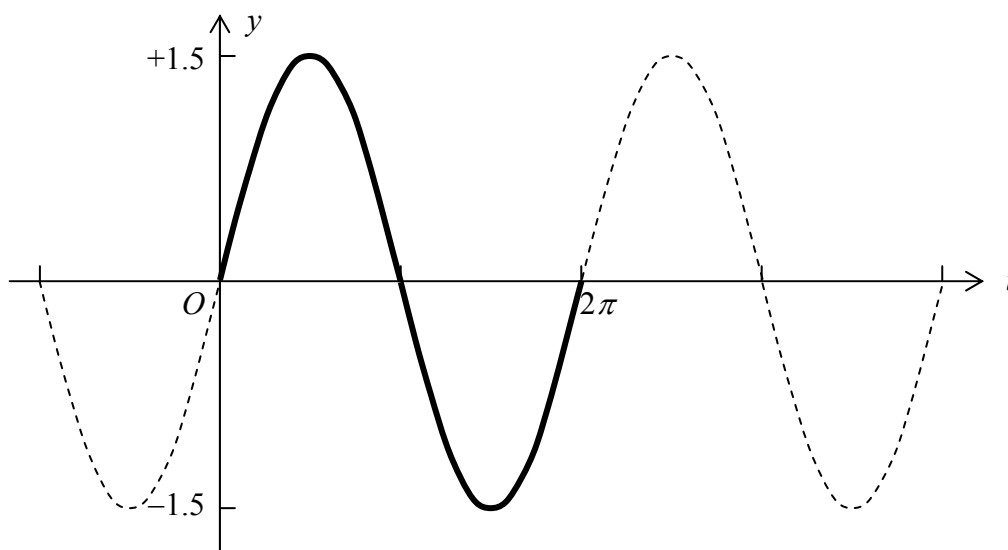
$$\text{Period} = 2\pi$$

$$\text{Frequency} = \frac{1}{2\pi}.$$

By introducing parameters into the trig functions that affect their amplitude, period and frequency, we can make them more flexible with respect to applications. Consider the following:

$$y = A \sin t \text{ and } y = A \cos t:$$

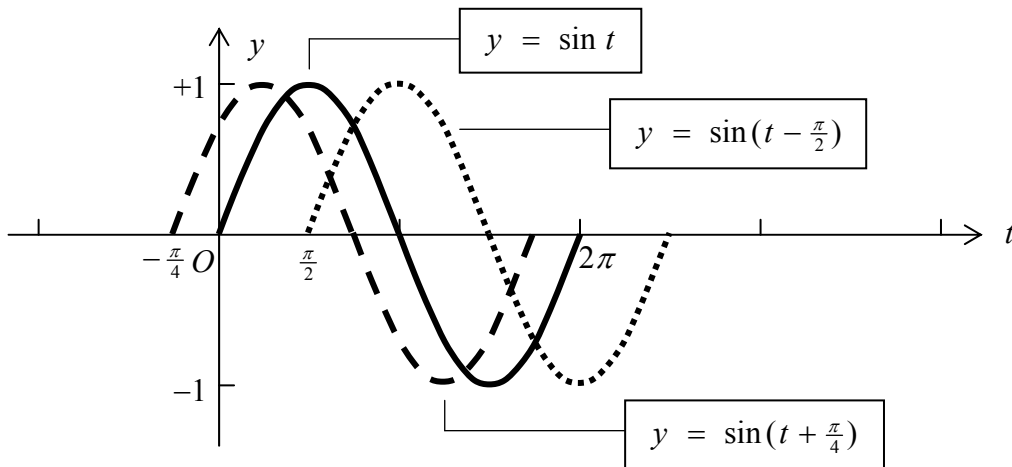
Multiplying the basic trig functions by a constant only affect the amplitude. For example, the graph of  $y = 1.5 \sin t$  looks like



Normally we try and ensure that the constant  $A$  is positive.

Now compare the graphs of

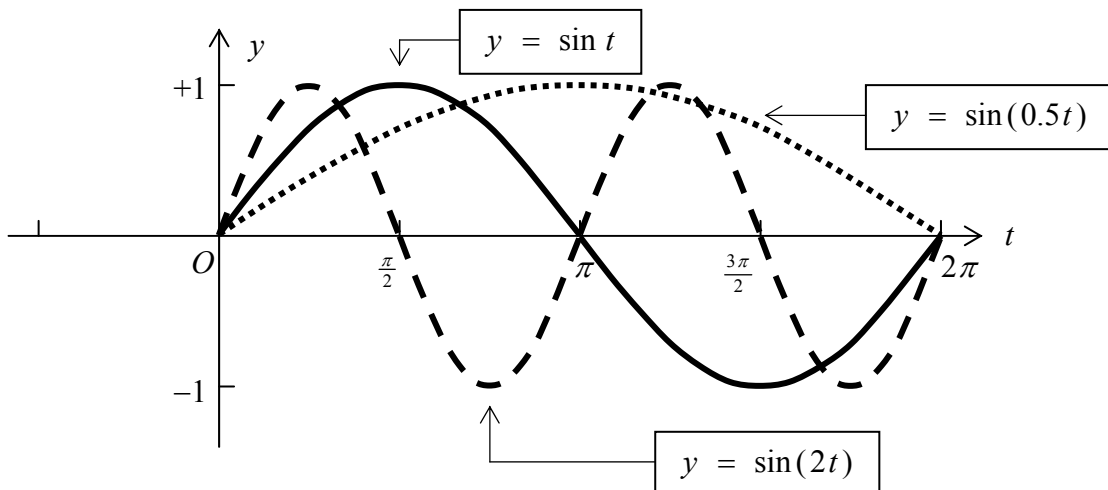
$$y = \sin t \quad y = \sin\left(t + \frac{\pi}{4}\right) \quad \text{and} \quad y = \sin\left(t - \frac{\pi}{2}\right) :$$



Adding a constant to the  $t$  inside the trig function results in a shift of the basic graph to the left or the right depending on whether the constant is positive or negative. We say that the constant results in a **phase shift**.

Next, compare the graphs of

$$y = \sin t \quad y = \sin(2t) \quad \text{and} \quad y = \sin(0.5t) :$$



Multiplying the  $t$  inside the trig function by a constant stretches or compresses the basic wave-form horizontally, i.e. it alters the period and the frequency. In general, we obtain the following results:

$y = \sin(Bt)$	Period = $\frac{2\pi}{B}$	Frequency = $\frac{B}{2\pi}$
----------------	---------------------------	------------------------------

By putting all these effects together we can construct a “generalised” sine function:

$$y = A \sin(Bt + C),$$

where

$A$  is the amplitude

$B$  determines the period and the frequency

$C$  effects a phase shift .

**Note:**  $C$  is called the **phase angle** and does not, on its own, give the amount or direction of shift. This is actually given by:

$$\text{Amount of shift} = -\frac{C}{B},$$

where a negative value for the amount of shift indicates a shift to the left and a positive value, a shift to the right.

All the effects described above also hold for the “generalised” cosine function:

$$y = A \cos(Bt + C) .$$

**Examples / Over the page . . .**

(13) Sketch the graph of  $y = 3 \sin(2t + \frac{\pi}{3})$ .

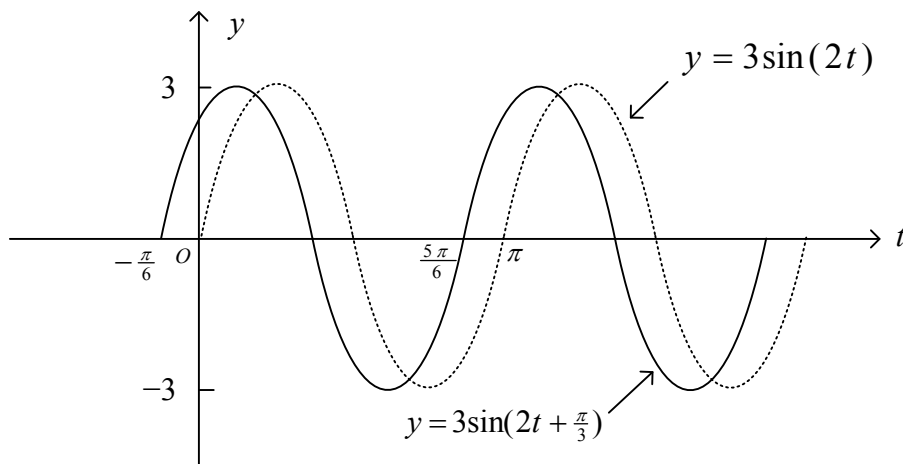
$$\text{Amplitude} = 3$$

$$\text{Period} = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$$

$$\text{Amount of shift} = -\frac{C}{B} = -\frac{\pi}{6}$$

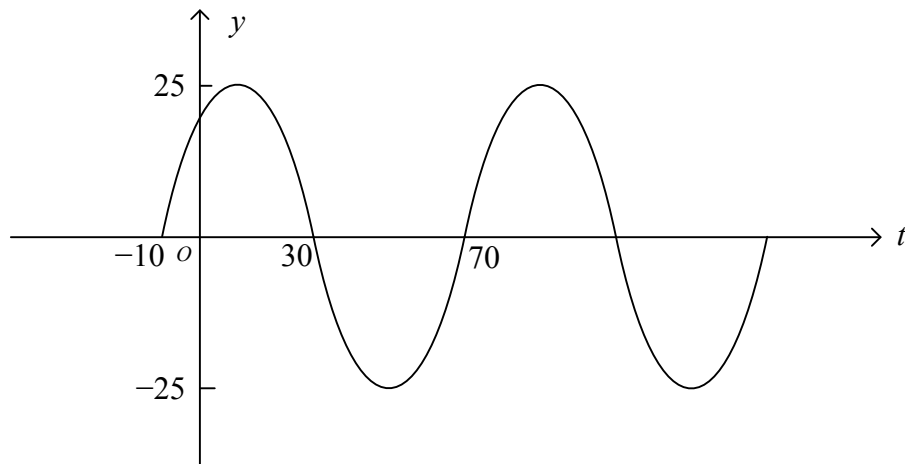
Graph  $y = 3 \sin(2t)$ ; easy, since it passes through the origin.

Then graph  $y = 3 \sin(2t + \frac{\pi}{3})$  by shifting graph of  $y = 3 \sin(2t)$  to the left by  $\frac{\pi}{6}$  units.



(14) / Over the page ...

(14) Determine an expression that describes the graph:



Use  $y = A \sin(Bt + C)$ .

Read from Graph: Amplitude = 25  $\rightarrow A = 25$

Period = 80  $\rightarrow B = \frac{2\pi}{P} = \frac{2\pi}{80} = \frac{\pi}{40}$

Amount of shift = -10

$$-\frac{C}{B} = -10$$

$$C = 10B = \frac{10\pi}{40} = \frac{\pi}{4}$$

Substitute  $A$ ,  $B$  and  $C$  into  $y = A \sin(Bt + C)$  to give

$$\underline{\underline{y = 25 \sin\left(\frac{\pi}{40}t + \frac{\pi}{4}\right)}}$$





**Tutorial Exercises****Measurement of Angles / Evaluation of Trig Functions**

(1) Convert the following degree values to radian measure. Express answers to 4 decimal places:

- |                   |                    |                     |
|-------------------|--------------------|---------------------|
| (i) $15^\circ$    | (ii) $30^\circ$    | (iii) $45^\circ$    |
| (iv) $60^\circ$   | (v) $75^\circ$     | (vi) $90^\circ$     |
| (vii) $180^\circ$ | (viii) $270^\circ$ | (ix) $360^\circ$    |
| (x) $450^\circ$   | (xi) $540^\circ$   | (xii) $720^\circ$ . |

(2) Repeat Question (1), but express the answers exactly as fractions or multiples of  $\pi$ .

**Example**

$$15^\circ \rightarrow 15 \times \frac{\pi}{180} = \frac{15}{180} \pi = \frac{1}{12} \pi \text{ radians}$$

(3) Convert the following radian measures to degrees:

- |                      |                       |                       |
|----------------------|-----------------------|-----------------------|
| (i) $\frac{\pi}{8}$  | (ii) $\frac{\pi}{6}$  | (iii) $\frac{\pi}{4}$ |
| (iv) $\frac{\pi}{3}$ | (v) $\frac{5\pi}{12}$ | (vi) $\frac{5\pi}{3}$ |
| (vii) 1.0            | (viii) 6.2            | (ix) 150 .            |

(4) Use your calculator to determine the following trig values:

- |  |                         |                         |
|--|-------------------------|-------------------------|
| (i) $\sin(62^\circ)$                   | (ii) $\cos(24^\circ)$   | (iii) $\tan(36^\circ)$  |
| (iv) $\sin(1.5)$                       | (v) $\cos(2.3)$         | (vi) $\tan(4.2)$        |
| (vii) $\operatorname{cosec}(37^\circ)$ | (viii) $\sec(65^\circ)$ | (ix) $\cot(49^\circ)$ . |

**Trig Inversions and Equations**

(5) Determine the general solutions (in degrees) of the following trig equations:

(i)  $\sin x = 0.6$

(ii)  $\sin x = 0.25$

(iii)  $\sin x = -0.15$

(iv)  $\sin x = -0.75$

(v)  $\sin x = 0$

(vi)  $\sin x = 1$

(vii)  $\sin x = -1$

(viii)  $\cos x = 0.4$

(ix)  $\cos x = 0.8$

(x)  $\cos x = -0.35$

(xi)  $\cos x = -0.65$

(xii)  $\cos x = 0$

(xiii)  $\cos x = 1$

(xiv)  $\cos x = -1$

(xv)  $\tan x = 0.7$

(xvi)  $\tan x = 10.3$

(xvii)  $\tan x = -2.6$

(xviii)  $\tan x = -0.3$

(xix)  $\tan x = 0$

(xx)  $\tan x = 1$

(6) Repeat parts (i), (viii) and (xv) of Question (5) working in radian measure.

(7) For each of the following equations, determine the general solution and the particular solutions lying in the range  $0^\circ \leq x \leq 360^\circ$  :

(i)  $\sin(2x) = 0.68$

(ii)  $2 \cos(3x) + 1 = 0$

(iii)  $\cos(4x - 20^\circ) = 0.4$

(iv)  $\sin(0.5x + 40^\circ) = 0$

(v)  $4 \sin(2x) - 3 = 0$

(vi)  $\cos^2(x) = 0.04$

(vii)  $6 \sin^2(x) + \sin(x) - 2 = 0$  [Hint: Let  $\sin(x) = X$ ]

(viii)  $4 \cos^2(x) - 7 \cos(x) - 2 = 0$  [Hint: Let  $\cos(x) = X$ ]

- (8) Solve the following equations by first expressing the LHSs in the form  $R \sin(kx + \alpha)$  or  $R \sin(kx - \alpha)$  (whichever is most appropriate):

(i)  $\sqrt{3} \sin x + \cos x = 2$

(ii)  $\sin(4x) - \sqrt{3} \cos(4x) = \sqrt{2}$  .

- (9) Solve the following equations by first expressing the LHSs in the form  $R \cos(kx + \alpha)$  or  $R \cos(kx - \alpha)$  (whichever is most appropriate):

(i)  $\sqrt{3} \cos x - \sin x = 1$

(ii)  $5 \cos(2x) + 12 \sin(2x) = 6.5$  .

### Periodic Functions

- (10) For each of the following trig functions, determine (a) the amplitude, (b) the period, (c) the phase shift. Hence sketch the graph of the function over two cycles.

(i)  $y = 5 \sin(3t)$

(ii)  $y = 5 \sin(3t - 45^\circ)$

(iii)  $y = 10 \sin(4t)$

(iv)  $y = 10 \sin(4t + 20^\circ)$

(v)  $y = 8 \cos(2t)$

(vi)  $y = 8 \cos(2t - 40^\circ)$

(vii)  $y = 0.5 \cos(5t)$

(viii)  $y = 0.5 \cos(5t + 60^\circ)$

- (11) For each of the following trig functions, determine (a) the amplitude, (b) the period, (c) the frequency, (d) the phase shift. Hence sketch the graph of the function over two cycles.

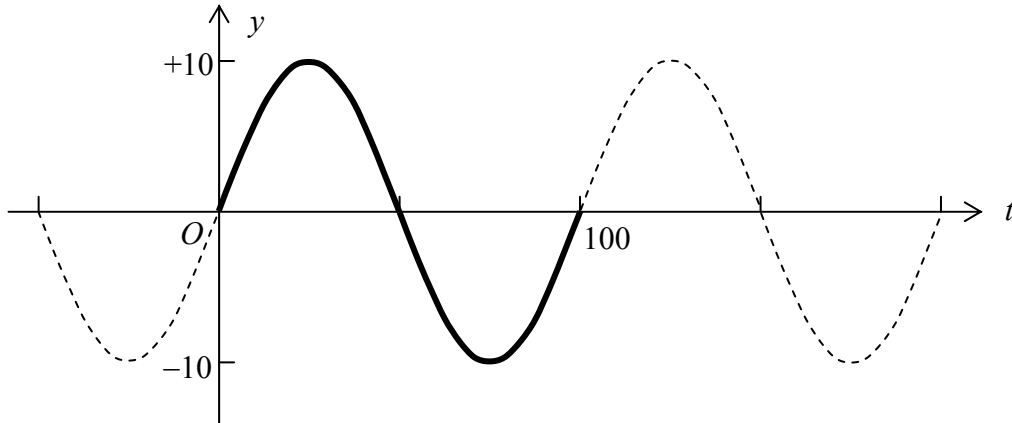
(i)  $y = 2 \sin(3t - \frac{\pi}{6})$

(ii)  $y = 0.75 \cos(2t + \frac{\pi}{3})$

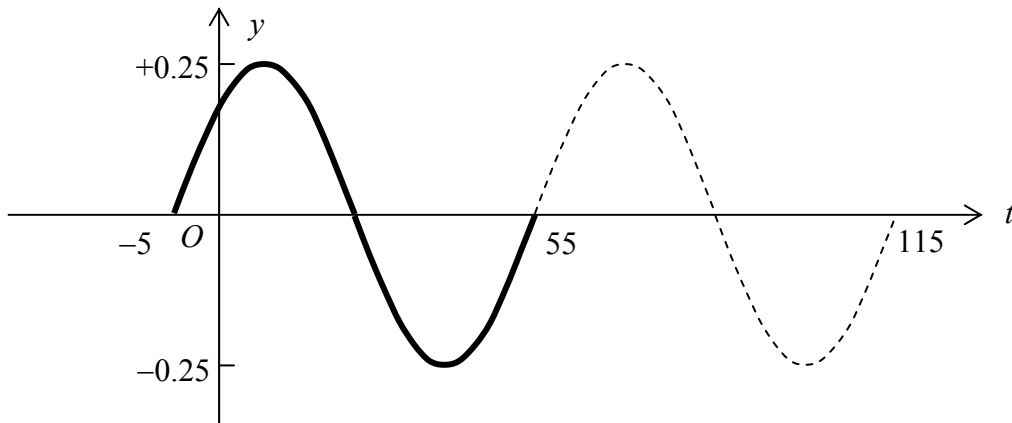
(12) For each of the following graphs, determine an expression of the form

$$y = A \sin(Bt + C).$$

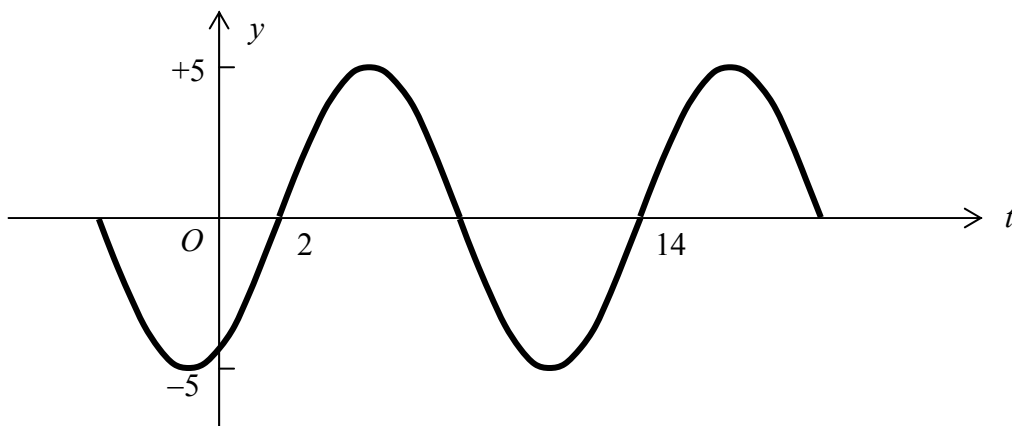
(i)



(ii)



(iii)



(13) Repeat Question (12) but instead use  $y = A \cos(Bt + C)$

**Answers**

- |     |       |        |        |        |       |         |
|-----|-------|--------|--------|--------|-------|---------|
| (1) | (i)   | 0.2618 | (ii)   | 0.5236 | (iii) | 0.7854  |
|     | (iv)  | 1.0472 | (v)    | 1.3090 | (vi)  | 1.5708  |
|     | (vii) | 3.1416 | (viii) | 4.7124 | (ix)  | 6.2832  |
|     | (x)   | 7.8540 | (xi)   | 9.4248 | (xii) | 12.5664 |

- |     |       |                  |        |                   |       |                 |
|-----|-------|------------------|--------|-------------------|-------|-----------------|
| (2) | (i)   | $\frac{\pi}{12}$ | (ii)   | $\frac{\pi}{6}$   | (iii) | $\frac{\pi}{4}$ |
|     | (iv)  | $\frac{\pi}{3}$  | (v)    | $\frac{5\pi}{12}$ | (vi)  | $\frac{\pi}{2}$ |
|     | (vii) | $\pi$            | (viii) | $\frac{3\pi}{2}$  | (ix)  | $2\pi$          |
|     | (x)   | $\frac{5\pi}{2}$ | (xi)   | $3\pi$            | (xii) | $4\pi$          |

- |     |       |                |        |                |       |                |
|-----|-------|----------------|--------|----------------|-------|----------------|
| (3) | (i)   | $22.5^\circ$   | (ii)   | $30^\circ$     | (iii) | $45^\circ$     |
|     | (iv)  | $60^\circ$     | (v)    | $75^\circ$     | (vi)  | $300^\circ$    |
|     | (vii) | $57.296^\circ$ | (viii) | $355.23^\circ$ | (ix)  | $8594.4^\circ$ |

- |     |       |        |        |         |       |        |
|-----|-------|--------|--------|---------|-------|--------|
| (4) | (i)   | 0.8829 | (ii)   | 0.9135  | (iii) | 0.7265 |
|     | (iv)  | 0.9975 | (v)    | -0.6663 | (vi)  | 1.7777 |
|     | (vii) | 1.6616 | (viii) | 2.3662  | (ix)  | 0.8693 |

(5) (i)	$\left. \begin{array}{l} 36.87^\circ + n.360^\circ \\ 143.13^\circ + n.360^\circ \end{array} \right\}$	(ii)	$\left. \begin{array}{l} 14.48^\circ + n.360^\circ \\ 165.52^\circ + n.360^\circ \end{array} \right\}$
(iii)	$\left. \begin{array}{l} -8.63^\circ + n.360^\circ \\ 188.63^\circ + n.360^\circ \end{array} \right\}$	(iv)	$\left. \begin{array}{l} -48.59^\circ + n.360^\circ \\ 228.59^\circ + n.360^\circ \end{array} \right\}$
(v)	$\left. \begin{array}{l} 0^\circ + n.360^\circ \\ 180^\circ + n.360^\circ \end{array} \right\} = n.180^\circ$	(vi)	$90^\circ + n.360^\circ$
(vii)	$270^\circ + n.360^\circ$	(viii)	$\left. \begin{array}{l} 66.42^\circ + n.360^\circ \\ -66.42^\circ + n.360^\circ \end{array} \right\}$
(ix)	$\left. \begin{array}{l} 36.87^\circ + n.360^\circ \\ -36.87^\circ + n.360^\circ \end{array} \right\}$	(x)	$\left. \begin{array}{l} 110.49^\circ + n.360^\circ \\ 249.51^\circ + n.360^\circ \end{array} \right\}$
(xi)	$\left. \begin{array}{l} 130.54^\circ + n.360^\circ \\ 229.46^\circ + n.360^\circ \end{array} \right\}$	(xii)	$\left. \begin{array}{l} 90^\circ + n.360^\circ \\ 270^\circ + n.360^\circ \end{array} \right\}$
(xiii)	$n.360^\circ$	(xiv)	$180^\circ + n.360^\circ$
(xv)	$\left. \begin{array}{l} 34.99^\circ + n.360^\circ \\ 214.99^\circ + n.360^\circ \end{array} \right\}$	(xvi)	$\left. \begin{array}{l} 84.45^\circ + n.360^\circ \\ 264.45^\circ + n.360^\circ \end{array} \right\}$
(xvii)	$\left. \begin{array}{l} -68.96^\circ + n.360^\circ \\ 111.04^\circ + n.360^\circ \end{array} \right\}$	(xviii)	$\left. \begin{array}{l} -16.70^\circ + n.360^\circ \\ 163.30^\circ + n.360^\circ \end{array} \right\}$
(xix)	$\left. \begin{array}{l} 0^\circ + n.360^\circ \\ 180^\circ + n.360^\circ \end{array} \right\} = n.180^\circ$	(xx)	$\left. \begin{array}{l} 45^\circ + n.360^\circ \\ 225^\circ + n.360^\circ \end{array} \right\}$

(6) (i)	$\left. \begin{array}{l} 0.6435 + 2n\pi \\ 2.4981 + 2n\pi \end{array} \right\}$
(viii)	$\left. \begin{array}{l} 1.1593 + 2n\pi \\ -1.1593 + 2n\pi \end{array} \right\}$
(xv)	$\left. \begin{array}{l} 0.6107 + 2n\pi \\ 3.7523 + 2n\pi \end{array} \right\}$

(7) Answers in range  $0^\circ \leq x \leq 360^\circ$  :

(i)  $21.42^\circ, 68.58^\circ, 201.42^\circ, 248.58^\circ$

(ii)  $40^\circ, 80^\circ, 160^\circ, 200^\circ, 280^\circ, 320^\circ$

(iii)  $21.61^\circ, 78.39^\circ, 111.61^\circ, 168.39^\circ, 201.61^\circ, 258.39^\circ, 291.61^\circ, 348.39^\circ$

(iv)  $280^\circ$

(v)  $24.30^\circ, 65.70^\circ, 204.30^\circ, 245.70^\circ$

(vi)  $78.46^\circ, 101.54^\circ, 258.46^\circ, 281.54^\circ$

(vii)  $30^\circ, 150^\circ, 221.81^\circ, 318.19^\circ$

(viii)  $104.48^\circ, 255.52^\circ$

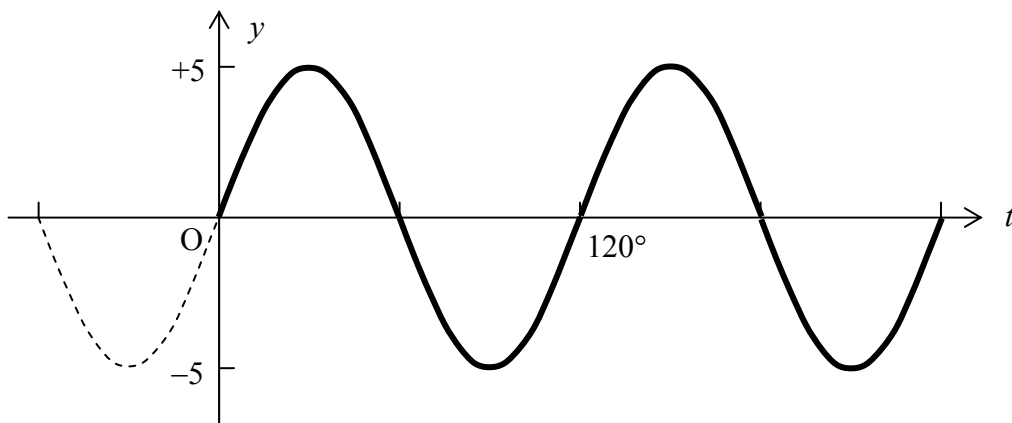
(8) (i)  $2 \sin\left(x + \frac{\pi}{6}\right)$   
 $x = \frac{\pi}{3} + 2n\pi$

(ii)  $2 \sin\left(4x - \frac{\pi}{3}\right)$   
 $x = \frac{7\pi}{48} + \frac{n\pi}{2}$  ,  $x = \frac{13\pi}{48} + \frac{n\pi}{2}$

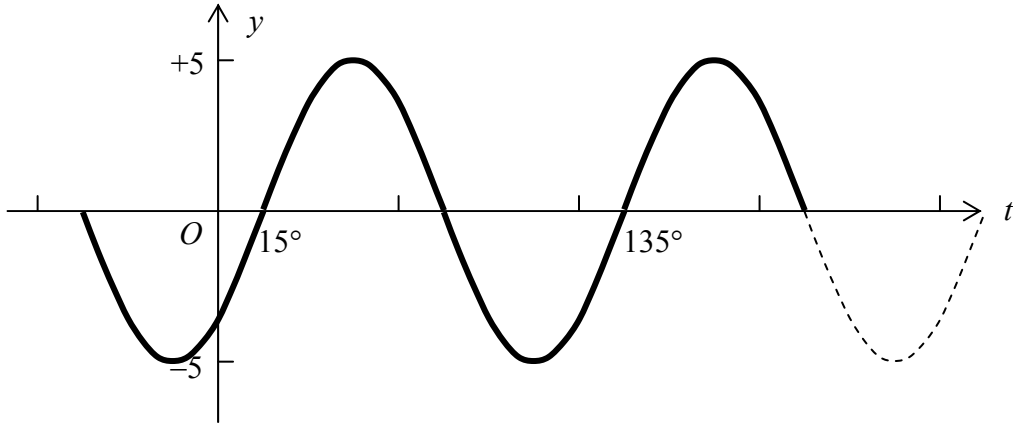
(9) (i)  $2 \cos\left(x + \frac{\pi}{6}\right)$   
 $x = \frac{\pi}{6} + 2n\pi$  ,  $x = -\frac{\pi}{2} + 2n\pi$

(ii)  $13 \cos(2x - 1.176)$   
 $x = 0.064 + n\pi$  ,  $x = 1.112 + n\pi$

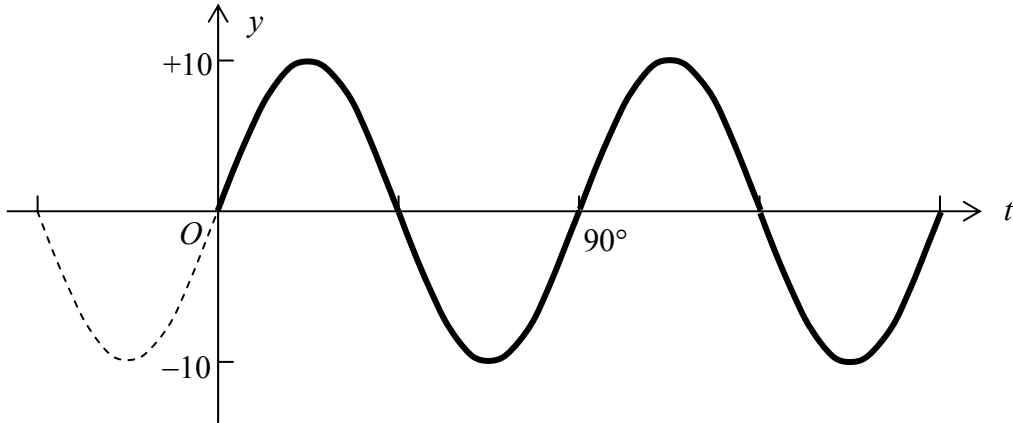
(10) (i) Amplitude = 5 , Period =  $120^\circ$  , Phase shift =  $0^\circ$  (relative to basic sine)



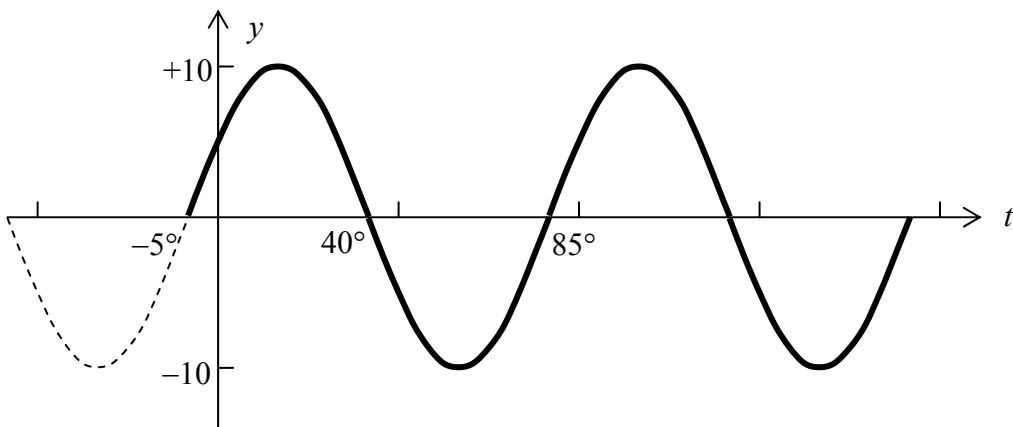
(ii) Amplitude = 5 , Period =  $120^\circ$  , Phase shift =  $15^\circ$  (relative to basic sine)



(iii) Amplitude = 10 , Period =  $90^\circ$  , Phase shift =  $0^\circ$  (relative to basic sine)

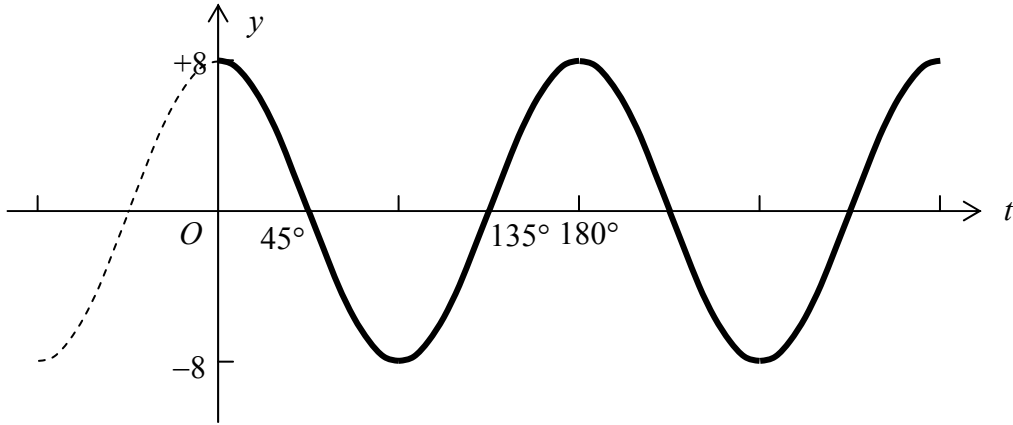


(iv) Amplitude = 10 , Period =  $90^\circ$  , Phase shift =  $-5^\circ$  (relative to basic sine)

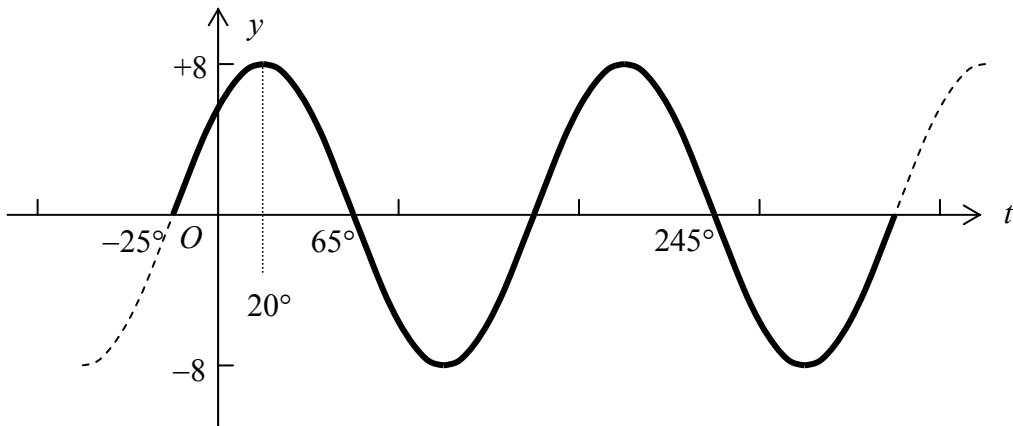




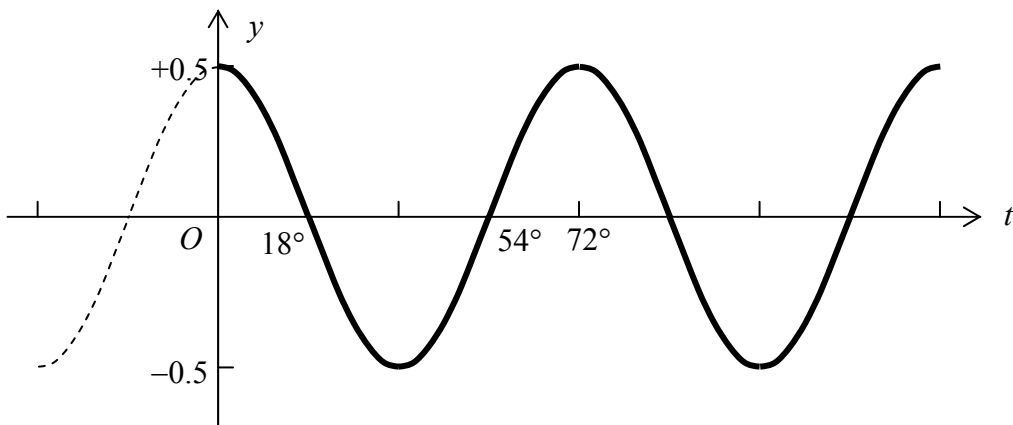
(v) Amplitude = 8 , Period =  $180^\circ$  , Phase shift =  $0^\circ$  (relative to basic cosine)



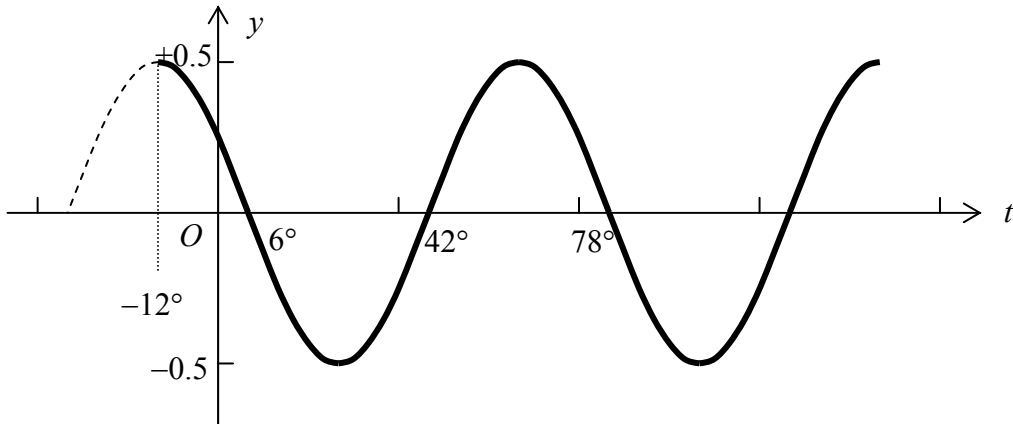
(vi) Amplitude = 8 , Period =  $180^\circ$  , Phase shift =  $20^\circ$  (relative to basic cosine)



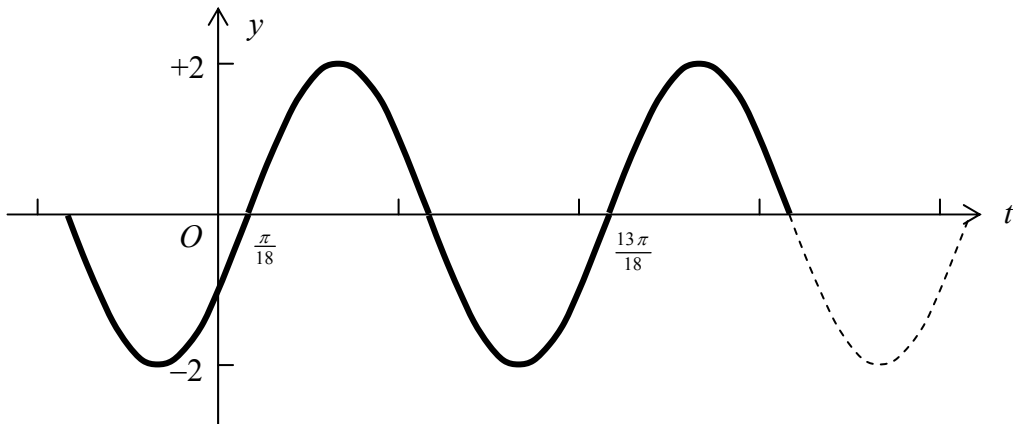
(vii) Amplitude = 0.5 , Period =  $72^\circ$  , Phase shift =  $0^\circ$  (relative to basic cosine)



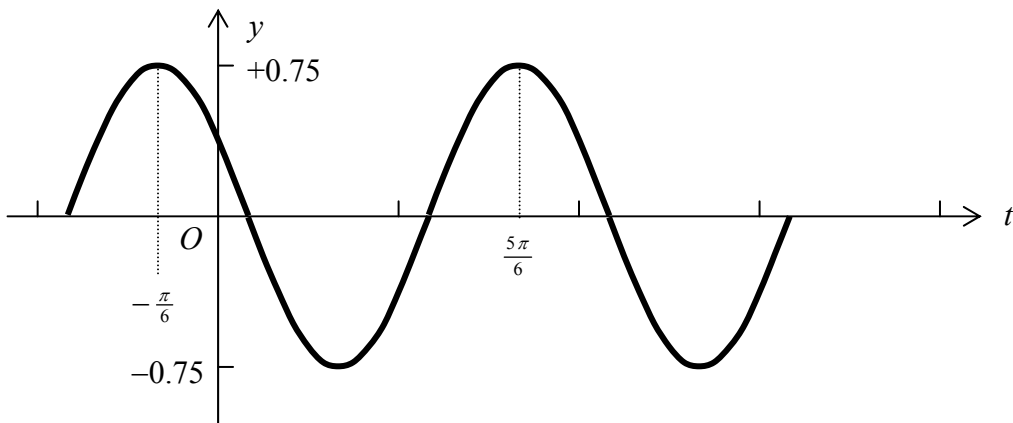
(viii) Amplitude = 0.5 , Period =  $72^\circ$  , Phase shift =  $-12^\circ$  (relative to basic cosine)



(11) (i) Amplitude = 2 , Period =  $\frac{2\pi}{3}$  , Frequency =  $\frac{3}{2\pi}$  , Phase shift =  $\frac{\pi}{18}$  (relative to basic sine)



(ii) Amplitude = 0.75 , Period =  $\pi$  , Frequency =  $\frac{1}{\pi}$  , Phase shift =  $-\frac{\pi}{6}$  (relative to basic cosine)



$$(12) \text{ (i) } y = 10 \sin\left(\frac{\pi}{50}t\right)$$

$$\text{(ii) } y = 0.25 \sin\left(\frac{\pi}{30}t + \frac{\pi}{6}\right)$$

$$\text{(iii) } y = 5 \sin\left(\frac{\pi}{6}t - \frac{\pi}{3}\right)$$

$$(13) \text{ (i) } y = 10 \cos\left(\frac{\pi}{50}t - \frac{\pi}{2}\right)$$

$$\text{(ii) } y = 0.25 \cos\left(\frac{\pi}{30}t - \frac{\pi}{3}\right)$$

$$\text{(iii) } y = 5 \cos\left(\frac{\pi}{6}t - \frac{5\pi}{6}\right)$$