Aerodynamics C - Exam January 2005 Problems and Solutions

1 Engine Intake of an Aircraft

1. Consider the air flow through the engine-intake of an aircraft flying at Mach 4 at an altitude of $10000m$ ($p = 2.65 \cdot 10^4 N/m^2$, $T = 223K$). The intake has a protruding wedge which deflects the flow by $30°$. The flow is further decelerated to the subsonic regime with a normal shock wave that follows the first shock.
   (a) Draw a schematic of the intake geometry, including flow streamlines and shock waves.
   (b) Determine the value of the pressure and the temperature in the subsonic flow region where it is assumed that $M = 0.2$.
2. Determine the value of the specific heat ratio γ for Helium (He), Nitrogen (N_2) and Carbon Dioxide (CO_2) at ambient conditions. (Hint: $\gamma = (n+2)/n$ where n is the number of degrees of freedom of the molecular motion).

1 Solution

1. Let's gather some data. We know that $M = 4$, $p = 2.65 \cdot 10^4 Pa$, $T = 223K$ and $\theta = 30°$. We immediately see that M and θ are given. Of course we can get β from that, using the β–θ–M graph. It follows that $\beta = 45°$. Now let's look at the problem.
   (a) Let's draw the intake. You can see it in figure 1. The wedge causes an oblique shock wave, with wave angle $\beta = 45°$. This causes the flow (see the thick stream line) to be deflected by an angle $\theta = 30°$. After this, it passes through a normal shock wave, causing the flow to turn subsonic. After the normal shock wave, the subsonic flow simply goes further into the engine.

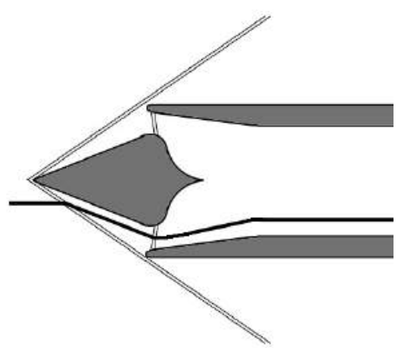

Figure 1: A schematic of the intake.

   (b) Now we need to get the pressure and temperature in the subsonic flow region after it has slowed down to $M = 0.2$. Let's call the situation before the oblique shock wave 1, the situation between the oblique shock wave and the normal shock wave 2, the situation just after the normal shock wave 3, and the situation after which the flow has slowed down to $M = 0.2$ situation 4. We thus know that $M_4 = 0.2$ and we need to get p_4 and T_4.

We know that $M_1 = 4$. The component normal to the oblique shock wave is $M_{n,1} = M_1 \sin\beta = 4\sin 45^\circ = 2.82$. This is approximately between 2.8 and 2.85. So when we use table B, we could take average values between the 2.8-row and the 2.85-row. We then get

$$p_2/p_1 = 9.15 \qquad \Rightarrow \qquad p_2 = 9.15 \cdot 2.65 \cdot 10^4 = 2.42 \cdot 10^5 Pa, \tag{1.1}$$

$$T_2/T_1 = 2.48 \qquad \Rightarrow \qquad T_2 = 223 \cdot 2.48 = 553K, \tag{1.2}$$

$$M_{n,2} = 0.485 \qquad \Rightarrow \qquad M_2 = \frac{M_{n,2}}{\sin(\beta - \theta)} = \frac{0.485}{\sin 15^\circ} = 1.87. \tag{1.3}$$

Now we know enough of situation 2. Let's turn to situation 3. We now have a normal shock wave with incoming Mach number $M_2 = 1.87$. From table B we can find that

$$p_3/p_2 = 3.96 \qquad \Rightarrow \qquad p_3 = 9.5 \cdot 10^5 Pa, \tag{1.4}$$

$$T_3/T_2 = 1.59 \qquad \Rightarrow \qquad T_3 = 879.3K. \tag{1.5}$$

We also see that $M_3 = 0.60$. Between situations 3 and 4 the flow simply slows down isentropically. Using the isentropic flow relations we find that

$$p_4 = p_3 \left(\frac{2 + (\gamma - 1) M_3^2}{2 + (\gamma - 1) M_4^2}\right)^{\frac{\gamma}{\gamma - 1}} = 9.5 \cdot 10^5 \left(\frac{2 + (1.4 - 1)\, 0.6^2}{2 + (1.4 - 1)\, 0.2^2}\right)^{\frac{1.4}{1.4-1}} = 1.18 \cdot 10^6 Pa. \tag{1.6}$$

$$T_4 = T_3 \frac{2 + (\gamma - 1) M_3^2}{2 + (\gamma - 1) M_4^2} = 879.3 \frac{2 + (1.4 - 1)\, 0.6^2}{2 + (1.4 - 1)\, 0.2^2} = 935K. \tag{1.7}$$

And that's all there is to it.

2. On to the next question. First let's examine the "degrees of freedom" thing a bit more closely. What is it? Well, the amount of degrees of freedom can be interpreted as the amount of ways in which a molecule can move in a meaningful way. Let's examine the Helium molecule. This is just one symmetric atom. It can move in three directions. However, rotating it isn't meaningful, as you won't see a difference. So for Helium $n = 3$ and thus $\gamma = (3 + 2)/3 = 1.67$.

   Now let's look at Nitrogen. A Nitrogen molecule consists of two Nitrogen atoms. Let's suppose these two atoms lie on the x-axis. The molecule can (just like Helium) move in three directions. However, it can also rotate about the y-axis and the z-axis. But, if we rotate it about the x-axis, nothing meaningful happens, since the molecule is rotationally symmetric about the x-axis. (If you don't see this, just take two soccer balls and stick a long pole through both of them. The pole is now your x-axis. Rotate the pole about its axis, and the orientation of the balls won't really change. Well, assuming they're still round and filled with air, which might be troubling, considering you just put a huge pole through them. Naturally molecules don't have this problem.) So, we see that a Nitrogen molecule can move in 5 meaningful ways. We therefore have $n = 5$ and $\gamma = (5+2)/5 = 1.4$.

   Finally, there's CO_2. This molecule is a bit wedged-shaped. So it isn't rotationally symmetric about any axis. If we rotate it about any axis, we always have a "meaningful" rotation. We can rotate it about 3 independent axes. We can also still move it in three directions. Therefore $n = 6$ and $\gamma = (6 + 2)/6 = 1.33$.

2 Designing a Supersonic Wind Tunnel

1. We need to design a supersonic wind tunnel reproducing at $M = 2.8$ free-stream flow. The cross section of the test section area is $A_{test} = 1m^2$. The air in the reservoir is at ambient temperature $T_0 = 280K$.

   (a) Determine the nozzle throat cross section $A_{t,1}$.

(b) Determine the total pressure in the reservoir p_0 and the mass flow $\dot{m}$ through the nozzle in the hypothesis that the nozzle discharges the air directly in the ambient ($p_{amb} = 1atm$, $T_{amb} = 280K$) at supersonic conditions and without any shock.

(c) A supersonic diffuser is placed behind the test section. Now the flow is decelerated through a normal shock wave at $M_{shock} = 2.8$. Determine the diffuser throat minimum cross section $A_{t,2}$, the minimum total pressure p_0 in the reservoir and the mass flow through the wind tunnel.

2. Consider air at a pressure of $0.3atm$. Calculate the values of the isothermal compressibility τ_T and isentropic compressibility τ_S expressing them in SI units.

2 Solution

1. Let's solve this question!

(a) The first subquestion (finding $A_{t,1}$) is, in fact, quite easy. We know that at the throat $M = 1$. At the test section $M = 2.8$ and $A_{test} = 1$. Looking up $A_{test}/A_{t,1} = A/A^*$ at $M = 2.8$ (table A) gives $A_{test}/A_{t,1} = 3.5$. It follows that $A_{t,1} = 0.286m^2$.

(b) Now we will determine the total pressure p_0 and the mass flow $\dot{m}$. At $M = 2.8$ we find that

$$p_0/p = 27.14 \qquad \Rightarrow \qquad p_0 = 27.14 \cdot 1 \cdot 1.01325 \cdot 10^5 = 2.75 \cdot 10^6 Pa, \tag{2.1}$$

$$T_0/T = 2.568 \qquad \Rightarrow \qquad T = 280/2.568 = 109K. \tag{2.2}$$

You may be wondering what we need the temperature for. We use it to find the mass flow. We will first derive an expression for that. The mass flow $\dot{m}$ equals

$$\dot{m} = \rho V A = \frac{p}{RT}\left(\sqrt{\gamma RT}M\right)A = pA\sqrt{\frac{\gamma}{RT}}M. \tag{2.3}$$

Examining this equation at the test section gives

$$\dot{m} = 1 \cdot 1.01325 \cdot 10^5 \cdot 1 \cdot \sqrt{\frac{1.4}{287 \cdot 109}} \cdot 2.8 = 1898kg/s. \tag{2.4}$$

(c) The next question is a bit more difficult. We define four situations. Situation 1 is the situation in the reservoir. Situation 2 is the situation just before the shock wave. Situation 3 is the situation just after the shock wave. Situation 4 is the situation where the flow has slowed down in the air behind the wind tunnel.

First let's examine the shock wave. This shock wave has an incoming Mach number of $M_2 = 2.8$. From this we find the ratio of total pressures $p_{0,3}/p_{0,2} = 0.3895$. We now know that the ratio of throat areas is

$$\frac{A_{t,1}}{A_{t,2}} = \frac{p_{0,3}}{p_{0,2}} = 0.3895 \qquad \Rightarrow \qquad A_{t,2} = \frac{0.286}{0.3895} = 0.734m^2. \tag{2.5}$$

We also know that the total pressure stays constant (as long as there are no shock waves), so $p_{0,1} = p_{0,2}$ and $p_{0,3} = p_{0,4}$. This means that

$$\frac{p_{0,4}}{p_{0,1}} = \frac{p_{0,3}}{p_{0,2}} = 0.3895 \qquad \Rightarrow \qquad p_{0,1} = \frac{p_{0,4}}{0.3895} = \frac{1 \cdot 1.01325 \cdot 10^5}{0.3895} = 2.60 \cdot 10^5 Pa. \tag{2.6}$$

Now we only need to find the mass flow. We can find that

$$p_2 = \frac{p_1}{27.14} = \frac{p_{0,1}}{27.14} = \frac{2.60 \cdot 10^5}{27.14} = 9.59 \cdot 10^3 Pa. \tag{2.7}$$

No other properties have changed in situation 2. So we can use situation 2 to find the mass flow, just like we previously did. We then get

$$\dot{m} = 9.59 \cdot 10^3 \cdot 1 \cdot \sqrt{\frac{1.4}{287 \cdot 109}} \cdot 2.8 = 179.6 kg/s. \tag{2.8}$$

So we see that by adding a diffuser, we reduce the necessary reservoir pressure by about a factor 10. The same goes for the mass flow. Therefore it's wise to add diffusers to wind tunnels.

2. The equation with which the compressibility can be found is

$$\tau = -\frac{1}{v}\frac{dv}{dp}. \tag{2.9}$$

First let's look at the isothermal case. In this case T is constant. Differentiating the equation of state $pv = RT$ then gives

$$p\,dv + v\,dp = 0 \qquad \Rightarrow \qquad -\frac{1}{v}\frac{dv}{dp} = \frac{1}{p}. \tag{2.10}$$

There we've got the first one already! Filling in values (in SI units) gives

$$\tau_T = \frac{1}{p} = \frac{1}{0.3 \cdot 1.01325 \cdot 10^5} = 3.29 \cdot 10^{-5} Pa^{-1}. \tag{2.11}$$

Now let's try to find τ_s. Now we know that the entropy stays constant. From the definition of entropy follows

$$0 = ds = de + p\,dv = dh - v\,dp = c_p\,dT - v\,dp. \tag{2.12}$$

The part dT now is a bit annoying. Let's get rid of that. We once more differentiate the equation of state $pv = RT$ to get

$$\frac{p\,dv + v\,dp}{R} = dT. \tag{2.13}$$

Inserting this in the previous equation and working everything out will give

$$\tau_s = -\frac{1}{v}\frac{dv}{dp} = \left(1 - \frac{R}{c_p}\right)\frac{1}{p} = \left(1 - \frac{287}{1004}\right)\frac{1}{0.3 \cdot 1.01325 \cdot 10^5} = 2.35 \cdot 10^{-5} Pa^{-1}. \tag{2.14}$$

3 Analyzing a Supersonic Airfoil

1. A semi convex airfoil is immersed in a uniform supersonic flow at Mach $M_\infty = 2.5$ with an incidence angle $\alpha = 3$ degrees, as is shown in figure 2.

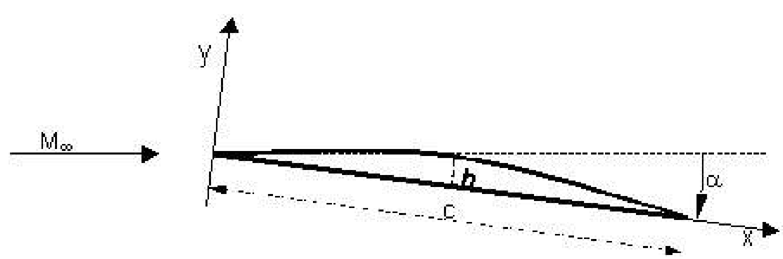

Figure 2: The airfoil corresponding to this question.

The upper surface has the following equation with respect to the body system of reference:

$$y = -\frac{4hx^2}{c^2} + \frac{4hx}{c}, \tag{3.1}$$

where $h/c = 0.02$. The lower surface is flat. Determine:

(a) The pressure coefficient distribution $C_p(x)$ on the bottom and top side of the airfoil.
(b) The airfoil lift coefficient C_L.
(c) The airfoil drag coefficient C_d.

2. Give the definition of the critical Mach number M_{cr} for an airfoil.

3 Solution

1. The airfoil is quite thin, and it has a low angle of attack. So our theory is applicable. How nice...

   We know that, both for the upper and lower side of the wing, we have

$$C_p = \frac{2\theta}{\sqrt{M_\infty^2 - 1}}. \tag{3.2}$$

   For the lower side of the airfoil we simply have $\theta = \alpha = 3°$. We need to fill in θ in radians though, so $\theta = 3 \cdot \pi/180 = 0.0524$. For the upper side, we have

$$\theta \approx \tan\theta = \frac{dy}{dx} - \alpha = \frac{4h}{c} - \frac{8hx}{c^2} - \alpha. \tag{3.3}$$

   Inserting θ in the expression for C_p will give the pressure coefficient for both the upper and the lower side.

   Let's find the lift and drag coefficients. The normal force coefficient c_n is

$$c_n = \frac{1}{c}\int_0^c (C_{p,l} - C_{p,u})\, dx = \frac{2}{c}\frac{1}{\sqrt{M_\infty^2 - 1}}\int_0^c \left(2\alpha - \frac{4h}{c} + \frac{8hx}{c^2}\right) dx. \tag{3.4}$$

   This is then equal to

$$c_n = \frac{4\alpha}{\sqrt{M_\infty^2 - 1}} = \frac{4 \cdot 0.0524}{\sqrt{2.5^2 - 1}} = 0.0914. \tag{3.5}$$

   The tangential force coefficient c_a now is

$$c_a = \frac{1}{c}\int_{LE}^{TE} (C_{p,u} - C_{p,l})\, dy = \frac{1}{c}\int_0^c \left(C_{p,u}\left(\frac{dy}{dx}\right)_u - C_{p,l}\left(\frac{dy}{dx}\right)_l\right) dx. \tag{3.6}$$

   Note that $(dy/dx)_l = 0$. We can now work the above equation further out to

$$c_a = \frac{2}{c}\frac{1}{\sqrt{M_\infty^2 - 1}}\int_0^c \left(\frac{4h}{c} - \frac{8hx}{c^2} - \alpha\right)\left(\frac{4h}{c} - \frac{8hx}{c^2}\right) dx. \tag{3.7}$$

   Solving this integral will eventually give

$$c_a = \frac{32}{3\sqrt{M_\infty^2 - 1}}\frac{h^2}{c^2} = \frac{32}{3\sqrt{2.5^2 - 1}} 0.02^2 = 0.00186. \tag{3.8}$$

   The lift and drag coefficient can now be found using

$$c_l = c_n - c_a\alpha = 0.0914 - 0.00186 \cdot 3 \cdot \pi/180 = 0.0913, \tag{3.9}$$

$$c_d = c_a + c_n\alpha = 0.00186 + 0.0914 \cdot 3 \cdot \pi/180 = 0.00665. \tag{3.10}$$

2. The critical Mach number is the free-stream Mach number such that sonic conditions $M = 1$ are achieved for the first time at one point on the airfoil surface.