Electrical Circuit Analysis

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- we first review general graphs
  - incidence matrix
  - flows
  - potentials

- electrical circuit analysis uses very similar ideas, with some small differences
  - reduced incidence matrix
  - (electrical) currents
  - (electrical) potentials

- we focus on resistor circuits, but same ideas apply to more general circuits with other devices

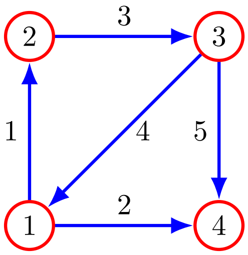
Outline

General graphs, flows, and potentials

Analysis of electrical circuits

Graph

- graph with n nodes, labeled $1, \ldots, n$
- m directed edges, labeled $1, \ldots, m$
- in example below, edge 4 goes from node 3 to node 1
- we say edge 4 is incident to nodes 3 and 1



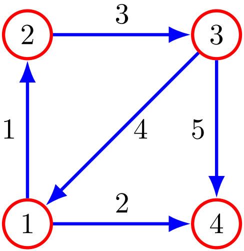
Incidence matrix of a graph

- $n \times m$ incidence matrix A defined as

$$A_{ij} = \begin{cases} 1 & \text{edge } j \text{ points to node } i \\ -1 & \text{edge } j \text{ points from node } i \\ 0 & \text{otherwise.} \end{cases}$$

- each column is associated with an edge, and has one $+1$ and one -1 entry
- row i is associated with node i, and can have zero or multiple $+1$ and -1 entries

Example



$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

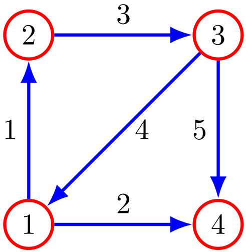
Flows

- m-vector f denotes a *flow*, f_i is the flow along edge i
- $f_i > 0$ means flow is in the edge direction
- $f_i < 0$ means flow is in the direction opposite the edge
- n-vector Af gives the total net flow into the nodes
- $(Af)_i$ is the total net flow into node i
- $Af = 0$ means the flow is conserved; f is a *circulation*

Potentials

- n-vector p denotes a *potential* at each node
- p_i is the potential at node i
- m-vector $A^T p$ gives the potential differences across the m edges
- $(A^T p)_j$ is the potential difference across edge j
- potential difference is the incoming node potential minus outgoing node potential

Example



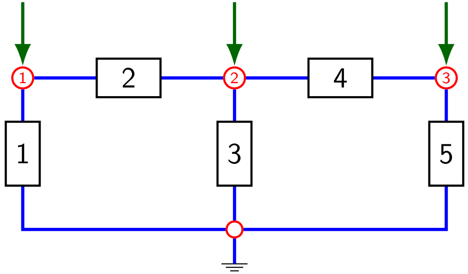
$$A = \begin{bmatrix} -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

- for flow f, $(Af)_3 = f_3 - f_4 - f_5$ (net flow into node 3)
- for potential p, $(A^T p)_2 = p_4 - p_1$ (potential difference across edge 2)

Outline

Circuit nomenclature

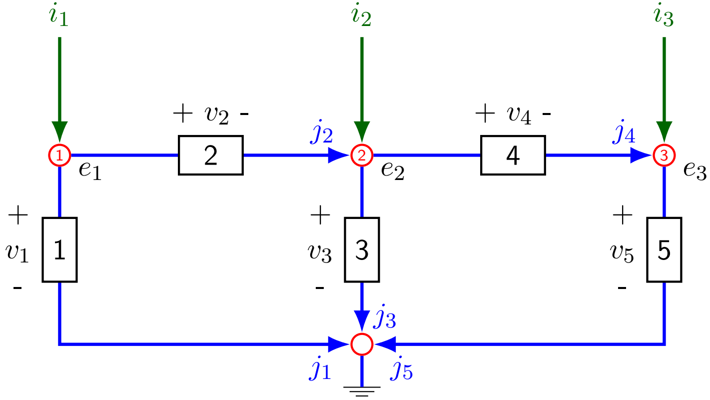
- (resistor) circuit consists of
  - n nodes, drawn as dots, plus a special *ground node*
  - b edges or branches, each containing a *resistor*, drawn as a box
  - n external sources, injecting current into the nodes
- often described using *circuit schematic diagram*
- simple example with $n = 3$, $b = 5$



Circuit variables

- we index nodes by $k = 1, \ldots, n$, branches by $l = 1, \ldots, b$ (can think of ground node as node $n + 1$)
- j_l is the electrical current in branch l (in A, Amperes)
- v_l is the voltage across branch l (in V, Volts) (measured from outgoing node to incoming node)
- electrical potential (relative to ground) at node k is e_k (in V)
- node k has external current injected, denoted i_k (in A)
- we'll work with b-vectors j and v, and n-vectors e and i

Circuit schematic diagram



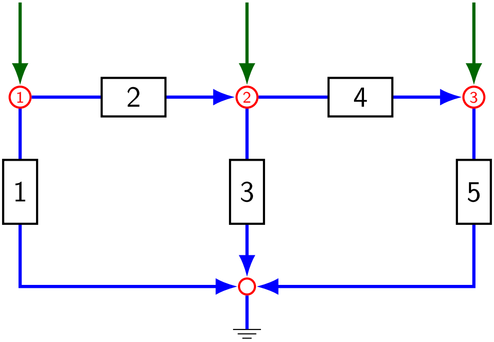
Reduced incidence matrix

- circuit analysis uses $n \times b$ *reduced incidence matrix*

$$A_{kl} = \begin{cases} +1 & \text{branch } l \text{ goes into node } k \\ -1 & \text{branch } l \text{ goes out of node } k \\ 0 & \text{otherwise} \end{cases}$$

- same as incidence matrix of the circuit graph, with last row (associated with ground node) removed
- columns of A have two entries (one $+1$ and one -1) for branches between nodes
- columns of A have one entry ($+1$ or -1) for branches that go to or from ground node

Example



$$A = \begin{bmatrix} -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

Kirchhoff's circuit laws

Kirchhoff's current law (KCL):

- current is conserved at each node, *i.e.*, $(Aj)_k + i_k = 0$
- in matrix notation: $Aj + i = 0$

Kirchhoff's voltage law (KVL):

- branch l voltage is the potential difference across it (using circuit convention, outgoing minus incoming potential): $(A^T e)_l + v_l = 0$
- in matrix notation: $A^T e + v = 0$

Ohm's law

- each branch contains a resistor
- characterized by *Ohm's law*, $v_l = R_l j_l$
- $R_l > 0$ is the *resistance* (in Ohms, denoted Ω) of branch l
- in matrix notation: $v = Rj$, with $R = \mathbf{diag}(R_1, \ldots, R_b)$

Circuit equations

- circuit quantities are b-vectors v and j, n-vector e
- circuit equations are

$$\begin{array}{ll} Aj + i = 0 & \text{(KCL)} \\ A^T e + v = 0 & \text{(KVL)} \\ v = Rj & \text{(Ohm's law)} \end{array}$$

- combine KVL and Ohm's law to get $A^T e + Rj = 0$, and express as

$$\begin{bmatrix} R & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} j \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ -i \end{bmatrix}$$

- a system of $b + n$ linear equations with $b + n$ variables

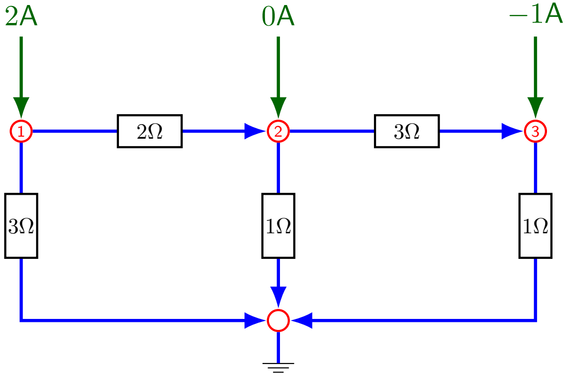
Solution of circuit equations

- assuming matrix is invertible,

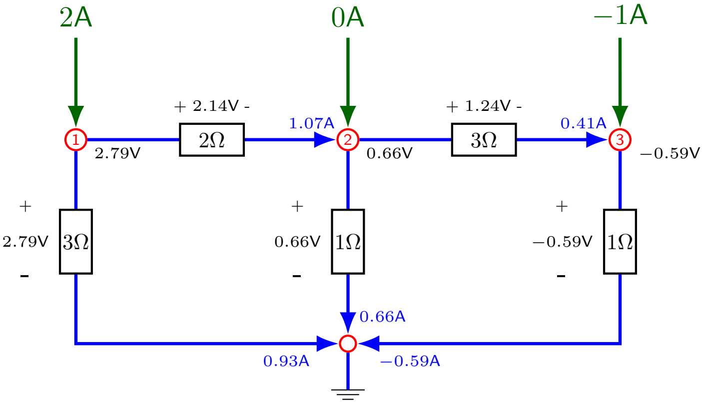
$$\left[\begin{array}{c} j \\ e \end{array}\right] = \left[\begin{array}{cc} R & A^T \\ A & 0 \end{array}\right]^{-1} \left[\begin{array}{c} 0 \\ -i \end{array}\right]$$

  (and $v = Rj$)
- so v, j, and e are all linear functions of i

Example



Example — solution



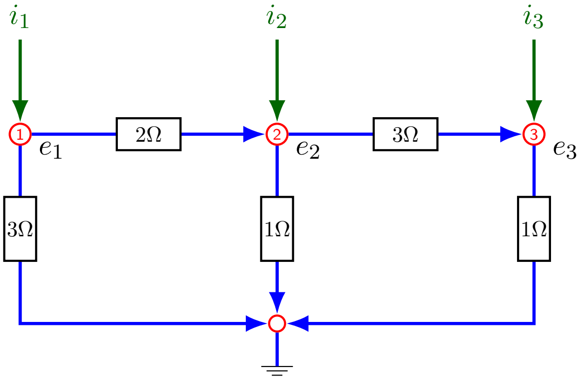
Resistance matrix

- e is a linear function of i so it has form $e = \mathcal{R}i$
- $n \times n$ matrix $\mathcal{R}$ is called the *resistance matrix*
- $\mathcal{R}$ maps injected currents to resulting node potentials
- $\mathcal{R}_{ij}$ is the potential at node i when 1A is injected into node j
- don't confuse with R, which maps branch currents to branch voltages (and is diagonal)

Resistance matrix

- we can find $\mathcal{R}$ from circuit equations
- from $A^T e + Rj = 0$ we get $j = -R^{-1} A^T e$
- from $Aj + i = 0$ we get $-AR^{-1} A^T e = -i$
- so $e = \left(AR^{-1} A^T\right)^{-1} i$
- so $\mathcal{R} = \left(AR^{-1} A^T\right)^{-1}$

Example



$$\mathcal{R} = \begin{bmatrix} 1.448 & 0.414 & 0.103 \\ 0.414 & 0.690 & 0.172 \\ 0.103 & 0.172 & 0.793 \end{bmatrix}$$

Conservation of power

- power dissipated in branch l is $j_l v_l$; total power is $P^{\text{diss}} = j^T v$
- power entering circuit via external current at node k is $i_k e_k$; total is $P^{\text{ext}} = i^T e$
- *conservation of power*: $P^{\text{diss}} = P^{\text{ext}}$
- *i.e.*, the total power dissipated in the circuit branches is the total power entering the nodes via the external currents
- to see this:

$$j^T v = -j^T(A^T e) = -(Aj)^T e = i^T e$$

  (using $Aj + i = 0$, $A^T e + v = 0$)
- does not depend on branch resistances ...

Maxwell's minimum energy principle

- the circuit equations are

$$\begin{bmatrix} R & A^T \\ A & 0 \end{bmatrix} \begin{bmatrix} j \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ -i \end{bmatrix}$$

- these are also the KKT optimality conditions for the problem

$$\begin{array}{ll} \text{minimize} & (1/2) j^T R j \\ \text{subject to} & Aj + i = 0 \end{array}$$

  with variable j
- $j^T R j = j^T v$, the total power dissipated in the circuit

Maxwell's minimum energy principle

Maxwell concluded that

- *branch currents minimize the dissipated power, subject to satisfying KCL*
- the optimal Lagrange multipliers are the node potentials