Review of Essential Algebra Concepts and Skills for Calculus

(Prepared for Fall 2006 Algebra/Calculus Review Workshop for engineering majors who have a gap of one year or more in their study of mathematics)

"The best way to learn is to do; the worst way to teach is to talk." (Paul R. Halmos)

I. Equation Types and Methods of Solution

Methods of solution of solving algebraic equations rely on **fundamental axioms of equality**:

- **Addition property of equality**: If $a = b$ then $a + c = b + c$ for all real numbers c.
- **Multiplication property of equality**: If $a = b$ then $ac = bc$ for all real numbers c.
- **Commutative properties of addition and multiplication**:
  - $a + b = b + a$
  - $ab = ba$
- **Associative properties of addition and multiplication**:
  - $(a + b) + c = a + (b + c)$
  - $(ab)c = a(bc)$
- **Distributive property of multiplication with respect to addition**:
  $a(b + c) = ab + ac$

Linear Equations

The simplest type of equation is a *linear* equation. The method for solving such equations after collecting like terms is to isolate the desired variable. At each step of the solution process, the resulting "new" form of the equation is an *equivalent equation.* That is, the new form of the equation has the same solution set as the original equation. The remaining expression or value of the variable is the solution to the equation.

Example 1 Solve for *x*: $3(2x - 5) = 4x + 3$

Expanding and collecting like terms, we obtain

$$\begin{aligned} 3(2x - 5) &= 4x + 3 && \text{Use the distributive property} \\ 6x - 15 &= 4x + 3 && \text{to obtain an equivalent equation} \\ 2x &= 18 \\ x &= 9 \end{aligned}$$

Thus, 9 is the value of the variable *x* that satisfies (or solves) the original equation. The set of all solutions to the original equation is $\{9\}$.

To check, we could use substation into the original equation:

3(2(9) - 5) = 4(9) + 3
3(18 – 5) = 36 + 3
3(13) = 39
39 = 39 (correct)

It should be emphasized that **it is good practice to check your solution**(s) to an equation.

In general, we expect a linear equation in one variable to have one *unique* solution.

However, there are two other possibilities.

Example 2 (No solution) Solve for *x*: $3(2x-5)-2x=4x+3$

We have

$$\begin{aligned} 3(2x-5)-2x &= 4x+3 \\ 6x-15-2x &= 4x+3 \\ 4x-15 &= 4x+3 \qquad \text{("Houston, we have a problem...")} \\ -15 &= 3 \end{aligned}$$

The discerning student recognizes that there is a problem at the third line. The last line produces a *false* statement, namely, -15 is not equal to 3. The absence of a variable in the last line means that the solution *does not depend on x*. This means that **there is *no solution* to the original equation**. That is to say, there is no replacement value for *x* that will result in a *true* statement of the equation.

The next example should come as no surprise.

Example 3 (Infinite number of solutions) Solve $3(2x-5)-2x=4x-15$

In this case, we find

$$\begin{aligned} 3(2x-5)-2x &= 4x-15 \\ 6x-15-2x &= 4x-15 \\ 4x-15 &= 4x-15 \qquad \text{(Identical sides)} \\ 0 &= 0 \end{aligned}$$

In this case, the resulting statement is *true* but does not depend on *x* so the original equation is true for *all* values of *x*. Therefore, the solution set contains *all real numbers*. Any value for *x* in the original equation will result in a true statement.

This type of equation is known as an *identity*. You may be familiar with other identities like $\sqrt{x^2}=|x|$ or the many trigonometric identities that result from Pythagorean and geometric relationships between trigonometric functions like $\sin^2 x+\cos^2 x=1$, $\sin(2x)=2\sin(x)\cos(x)$ or $\sin^2(x)=\dfrac{1-\cos(2x)}{2}$.

Nonlinear Algebraic Equations

Virtually all other algebraic *nonlinear* equations require a different method of solution.

- Higher degree polynomial equations require transforming one side equal to zero, factoring (if possible) and using the **Zero Factor Property**: $ab = 0$ if and only if either $a = 0$ or $b = 0$.
- Rational equations require obtaining a single rational term and setting the numerator (a *polynomial*) equal to zero and solving.
- Radical equations require eliminating the radical(s) and then solving the resulting polynomial equation.
- Logarithmic and exponential equations require using the properties of logarithms and exponentials to rewrite the equation in a polynomial form and then solving.

Since this is a review of algebra in preparation for calculus, we will consider the common types of nonlinear equations encountered in calculus courses in their full complexity.

Example 4 A common type of equation to solve is

$$3x^3 - 9x^2 - 30x = 0$$

Factoring, we have

$$\begin{aligned} 0 &= 3x^3 - 9x^2 - 30x \\ &= 3x(x-5)(x+2) \end{aligned}$$

which has solutions $x = -2, 0, 5$.

Example 5 Not all equations factor so easily. Consider

$$x^3 + x^2 - 4x + 2 = 0\ .$$

While it is not obvious if this factors and the quadratic formula is of no use (yet), it is easy to see that $x = 1$ is a solution of this equation so x -1 is a factor of the corresponding polynomial. Long division allows us to factor:

$$\begin{array}{r} x^2 + 2x - 2 \\ x-1 \overline{\big) x^3 + x^2 - 4x + 2} \\ \underline{x^3 - x^2 + 0x + 0} \\ 2x^2 - 4x - 2 \\ \underline{2x^2 - 2x + 0} \\ -2x + 2 \\ \underline{-2x + 2} \\ 0 \end{array}$$

Thus, we have

$$\begin{aligned} 0 &= x^3 + x^2 - 4x + 2 \\ &= (x-1)(x^2 + 2x - 2) \end{aligned}$$

Since the quadratic factor does not easily factor, we use the quadratic formula to solve

$$x^2 + 2x - 2 = 0$$

giving

$$\begin{aligned} x &= \frac{-2 \pm \sqrt{4+8}}{2} \\ &= -1 \pm \sqrt{3} \end{aligned}$$

So the solutions set is $\{1, -1-\sqrt{3}, -1+\sqrt{3}\}$.

Example 6 Solve $\dfrac{2x\sqrt{x} - \dfrac{x^2+1}{\sqrt{x}}}{x} = 0$.

Clearly, the denominator x cannot be zero. The square root function requires that $x > 0$. Multiplying both sides by x gives

$$\frac{x}{1} \cdot \frac{2x\sqrt{x} - \dfrac{(x^2+1)}{\sqrt{x}}}{x} = 0 \cdot x$$

$$2x\sqrt{x} - \frac{(x^2+1)}{\sqrt{x}} = 0 \qquad \text{Now multiply both sides by } \sqrt{x}$$

$$2x^2 - (x^2+1) = 0$$

$$x^2 - 1 = 0$$

$$(x+1)(x-1) = 0$$

so either $x = -1$ or $x = 1$. Due to previously mentioned restrictions, the only solution is $x = 1$.

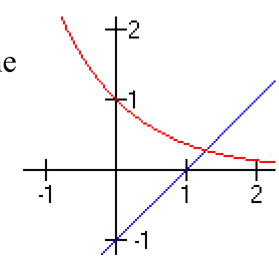
Example 7 Solve $\dfrac{2x(e^x+1)^2 - 2x^2e^x(e^x+1)}{(e^x+1)^4} = 0$.

Factoring the numerator, we obtain

$$\begin{aligned} 0 &= \frac{2x(e^x+1)(e^x+1-xe^x)}{(e^x+1)^4} \\ &= \frac{2x(e^x+1-xe^x)}{(e^x+1)^3} \end{aligned}$$

Again, only the numerator can be zero for this statement to be true. Thus, we must have either $2x = 0$ or $e^x + 1 - xe^x = 0$.

The second case is not easily solved by standard techniques. However, we can graph the function $y = e^x + 1 - xe^x$ and examine its x-intercepts. The only one is approximately $x = 1.279$



Algebraic Inequalities

Another common algebraic construct used extensively in calculus is solving *inequalities*. These are fundamentally different than equations yet the solutions have a strong relationship.

Example 8 A common type of nonlinear inequality is

$$3x^3 - 9x^2 - 30x \le 0$$

Now we must *interpret* the meaning of this inequality. This product of 3 factors is *negative* or equal to zero. The equality case has solutions $x = -2$, 0, and 5.
These values break up the number line (x-axis) into four intervals:

$$(-\infty,-2),\ (-2,0),\ (0,5), \text{ and } (5,\infty).$$

Beginning with the factored form of the inequality

$$3x(x+2)(x-5) \le 0$$

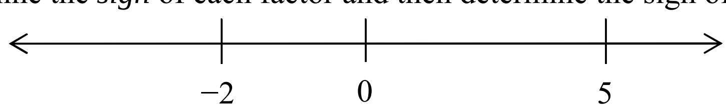
we use these factors and observe that the product of three factors is *negative* when

- only one factor is negative: $3x$, $x + 2$, or $x - 5$
- all three factors are negative

The solution to the original inequality will include the solutions $x = -2$, 0, and 5 and some of the intervals.

We may **select one value from each interval** and test that value to see if it satisfies the inequality:

We determine the *sign* of each factor and then determine the sign of the product:



$x+2$	- - - - - - - - - - - 0 +
$3x$	- - - - - - - - - - - - - - - - - - - 0 + + + + + + + + + + + + + +
$x-5$	- 0 + + + +
$3x(x+2)(x-5)$	- - - - - - - - - - - 0 + + + + + + 0 - - - - - - - - - - - 0 + + + + (Sign of product)

Thus, we can see that the shaded regions indicate where the product of the three factors is *negative*. Therefore, the solution is the set $(-\infty,2]\cup[0,5]$. **This is why we can test a single point in each interval as in Method 1. The sign of a factor changes at the corresponding zero for that factor.**

Example 8 Solve $\dfrac{2x - \dfrac{x^2+1}{x}}{x} > 0$.

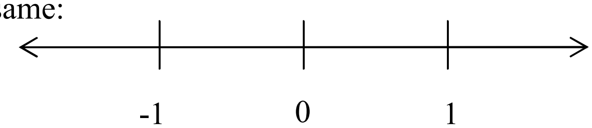
Clearly, x cannot be zero. Factoring out $\dfrac{1}{x}$ in the numerator gives

$$\frac{\frac{1}{x}\left[2x^2 - \left(x^2+1\right)\right]}{x} > 0$$

$$\frac{x^2-1}{x^2} > 0$$

$$\frac{(x+1)(x-1)}{x^2} > 0$$

In this case, using the reasoning above reveals that the factor x^2 does not change sign at its zero. That is because x^2 is nonnegative. As a result, the *other* factors determine where the inequality is true. Thus, the inequality is true wherever the sign of $x - 1$ and $x + 1$ are the same:

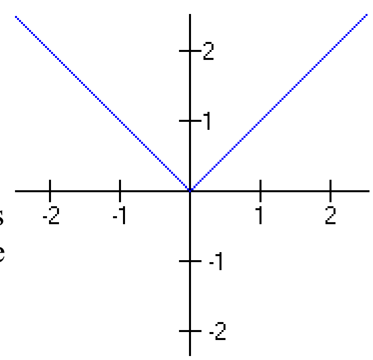


$x + 1$	- - - - - - - 0 + + + + + + + + + + + + + + + + + +	
$x - 1$	- 0 + + + + + +	
$x^2 - 1$	+ + + + + 0 - - - - - - - - - - - - - - - - 0 + + + + +	(Sign of product)

Therefore, the solution is $(-\infty,-1)\cup(1,\infty)$.

Absolute Value Geometrically, the absolute value of a number, $|x|$, represents the *distance* from the number to the origin (zero). Algebraically, we describe this as a *piecewise function*:

$$|x| = \begin{cases} x & x \ge 0 \\ -x & x < 0 \end{cases}$$



The graph of the absolute value function is shown at the right. In tems of solving equations and inequalities involving this function, we note that *linear* absolute value expressions satisfy

- $|x| \ge 0$ for all values of x
- Equations such as $|x| = 5$ have two solutions, $x = -5$ and 5
- Inequalities such as $|x| \le -5$ have no solution
- Inequalities such as $|x| \ge -5$ are true for all values of x
- Inequalities such as $|x| \le 5$ have a single interval solution set: $[-5,5]$
- Inequalities such as $|x| \ge 5$ have a two interval solution set: $(-\infty,-5]\cup[5,\infty)$

Example 9 Solve the equation $|5x-7|=22$.

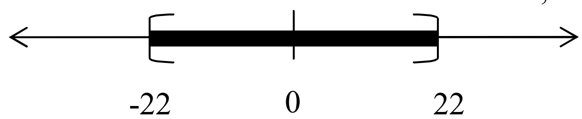
We have two cases to consider:

$$5x-7=22 \qquad \text{or} \qquad 5x-7=-22$$

In the first case we obtain $x = 29/5$. In the second case, we obtain $x = -3$.

Example 10 Solve the inequality $|5x-7|<22$.

Interpreting this statement in terms of *distance from zero*, we see that the expression $5x - 7$ is less than 22 units from zero. That is,



This can be written as a single inequality

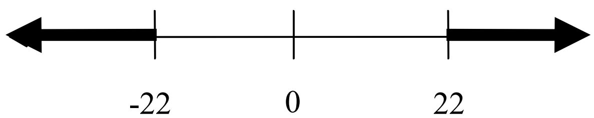
$$-22 < 5x-7 < 22$$

Solving this compound inequality gives

$$-3 < x < \frac{29}{5} \qquad \text{or in interval form} \qquad \left(-3, \frac{29}{5}\right).$$

Example 11 Solve the inequality $|5x-7| \ge 22$.

Interpreting this statement in terms of distance from zero, we see that the expression $5x - 7$ is 22 units or ***more*** units from zero.



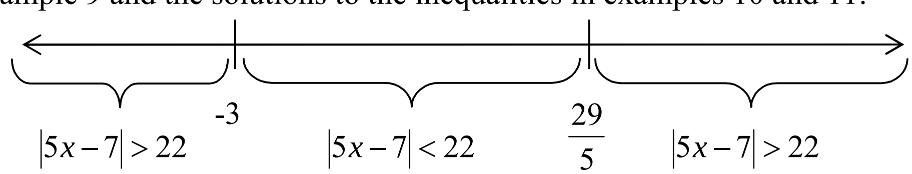
This must be written as two inequalities:

$$5x-7 \ge 22 \qquad \text{or} \qquad 5x-7 \le -22$$

Solving this compound inequality gives

$$x \le -3 \ \text{ or } \ x \ge \frac{29}{5} \qquad \text{or in interval form} \qquad (-\infty,-3) \cup \left(\frac{29}{5}, \infty\right).$$

Once again, note the relationship between the solutions to the corresponding equation from example 9 and the solutions to the inequalities in examples 10 and 11:



Practice Problems Solve each of the following.

1. $2x^3 = 2x^2 + 12x$

2. $\dfrac{3x^2 + 6x}{\sqrt{x+2}} = \sqrt{x+2}$

3. $\dfrac{2x^2 + 5x - 3}{2x - 1} = x + 3$

4. $\sqrt{3x+4} = x - 2$

5. $\dfrac{2x(x+1) - \dfrac{(x+1)^2}{x}}{\sqrt{x-1}} = 0$

6. $x^4 < 5x^3 - 6x^2$

7. $\dfrac{2x(x+1) - \dfrac{(x+1)^2}{x}}{\sqrt{x-1}} \geq 0$

8. $|15x + 8| \leq 3$

9. $|15x + 8| > 3$

10. $3(x-5)(x+1) + 10 = 2(x-1)(x+2) + x^2 - 14x - 1$

II. Functions and Graphs of Functions

Fundamentally, the **graph** of an equation $y = f(x)$ is the set of all points (x,y) whose coordinates satisfy the equation.

Lines: Slope-Intercept Form $y = mx + b$ Point-Slope Form $y - y_1 = m(x - x_1)$

Lines are used extensively in calculus and as such they play a fundamental role in the study of functions.

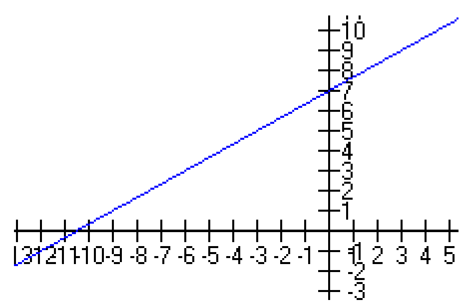
Example 1 Find the equation of the line containing the point $(-3,5)$ perpendicular to the line $3x + 2y = 10$. Graph the line and determine algebraically whether the point $(-4,5)$ is on the line.

Solution To determine the equation of a line, you need to know the slope and a point on the line (which we have). To find the slope, we must determine the slope if the given line and use the *opposite of its reciprocal*. We were given $3x + 2y = 10$, so solving for y yields $y = -\frac{3}{2}x + 10$. The slope of this line is $m = -\frac{3}{2}$ so the slope of the line we seek is $m = \frac{2}{3}$. Using the point-slope formula, we may substitute $m = \frac{2}{3}$, $x = -3$, and $y = 5$ into the formula:

$$5 = \frac{2}{3}(-3) + b$$

$$5 = -2 + b$$

$$b = 7 \quad \text{which is the } y-\text{intercept}$$



So the equation of the line is $y = \frac{2}{3}x + 7$ and its graph is shown at right. For the point $(-4,5)$, we have

$$5 = \frac{2}{3}(-4) + 7$$

$$5 = -\frac{8}{3} + 7$$

$$5 \neq \frac{13}{3}$$

So the point is $(-4,5)$ is *not* on the line.

Slope as rate of change The slope of a line or "steepness" is ratio of the change in y (rise) to the change in x (run). The negative sign indicates that the line *decreases* from left to right. That is to say, the values of the y-coordinate decreases as the x-coordinate increases.

Example 2 A New York city taxi service charges an initial fee of \$2.00 and then \$0.20 for every 1/5 mile traveled. Determine the function representing the cost (fare) for a taxi ride of *x* miles and use it to find the cost of a 34 mile taxi ride.

Solution Since we are asked to find the cost as a function of miles traveled, the rate per mile is 5(\$0.20 per 1/5mile) = \$1.00 per mile. This **rate of change** is the slope for this linear function. The initial fee of \$2.00 is the *y*-intercept. Thus, we have

$y = 1x + 2$ or in function notation $c(x) = x + 2$

Therefore, a 34 mile taxi ride in New York city costs

$$\begin{aligned} c(34) &= 34 + 2 \\ &= \$47 \end{aligned}$$

Polynomial Functions

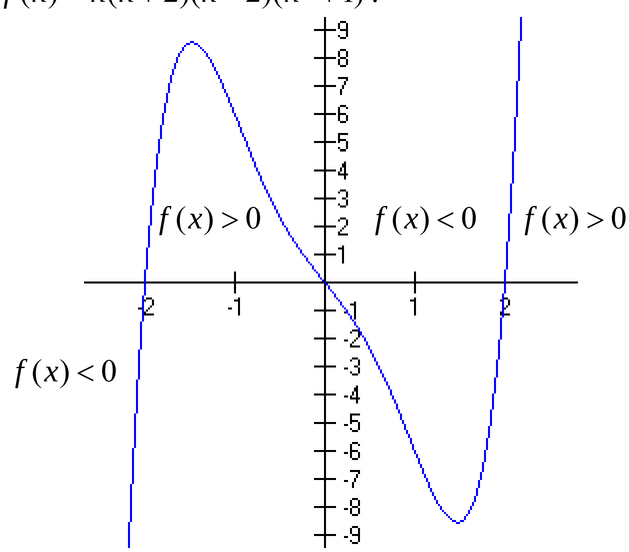
A polynomial function of degree *n* is a finite sum of nonnegative integer powers of *x*:

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_2 x^2 + a_1 x + a_0$$

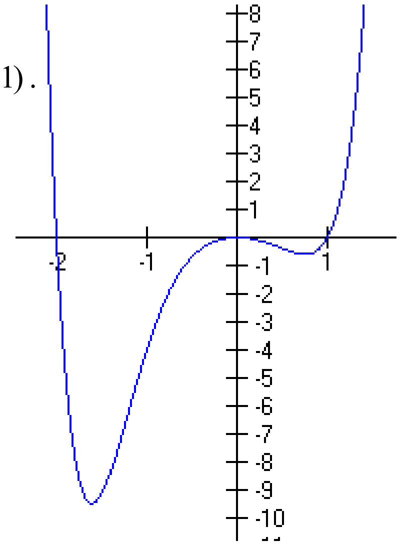
Key Characteristics of Graphs of Polynomial Functions

- The domain is the set of all real numbers
- There are at most *n* *x*-intercepts
- The ends of the graph point in the same direction if *n* is **even**
- The ends of the graph point in opposite directions if *n* is **odd**

Example 3 Describe the graph of $f(x) = x(x+2)(x-2)(x^2+1)$.

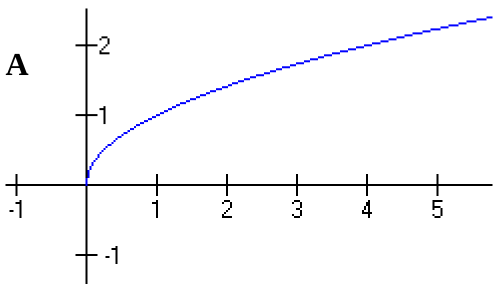


Example 4 Describe the graph of $f(x) = x^2(x+2)(x-1)(x^2+1)$.



Identify the graphs of $y = x^2$, $y = x^3$, $y = \sqrt{x}$ and $y = |x|$

A



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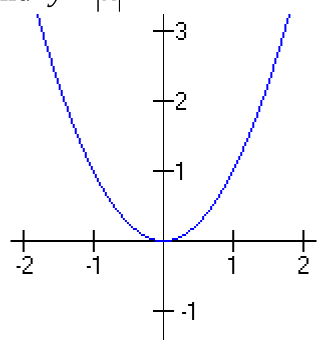

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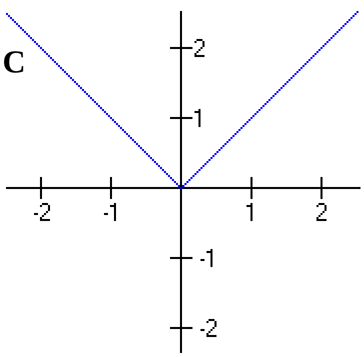

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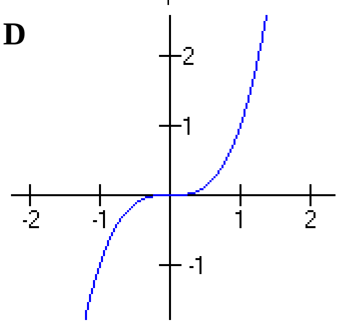


Transformation of Graphs

Let c be a positive number. Then

- The graph of $y = f(x) + c$ is a **vertical shift** of the graph of $y = f(x)$ c units ______
- The graph of $y = f(x) - c$ is a **vertical shift** of the graph of $y = f(x)$ c units ______
- The graph of $y = f(x+c)$ is a **horizontal shift** of the graph of $y = f(x)$ c units ______
- The graph of $y = f(x-c)$ is a **horizontal shift** of the graph of $y = f(x)$ c units ______
- The graph of $y = -f(x)$ is a **reflection** of the graph of $y = f(x)$ across the ________
- The graph of $y = f(-x)$ is a **reflection** of the graph of $y = f(x)$ across the ________

Example 5 Sketch the graph of each of the following.

(a) $y = -(x+3)^2$ (b) $g(x) = \sqrt{x-2} - 1$ (c) $y = |x+3| + 5$ (d) $f(x) = -x^3 + 2$

Stretches and Shrinks

- If $|c| > 1$, then the graph of $y = cf(x)$ is a *stretch* of the graph of $y = f(x)$.
- If $0 < |c| < 1$, then the graph of $y = cf(x)$ is a shrink of the graph of $y = f(x)$.

Example 6 Sketch the graph of each of the following.

(a) $y = -\frac{1}{2}|x+3|$ (b) $g(x) = 3\sqrt{x-2}$ (c) $p(x) = -3x^2$

Practice Problems

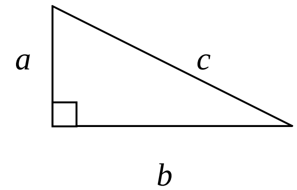
Identify the *x*-intercepts for each of the following functions.

1. $y = (x+2)^2 - 25$ 2. $f(x) = 3x^4 - 15x^3 + 18x^2$ 3. $y = \sqrt{x} - 7$

4. Find a parabola whose vertex is (2,-3) opening down passing through the point (0,-11).

5. Determine a polynomial of minimum degree having zeros x = -2, 0, 3 passing through the point (1,6).

III Pythagorean Theorem, Distance, and Circles



Pythagorean Theorem: In a right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse.

$$a^2 + b^2 = c^2$$

Quadratic Formula: The solutions to the equation $ax^2 + bx + c = 0$ are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Completing the Square: A useful technique when working with quadratic expressions is completing the square. This is based on the form of the square of a binomial:

$$(x+a)^2 = x^2 + 2ax + a^2$$

Note the factor of two in the linear coefficient. For *any* monic quadratic expression (the leading coefficient is 1), we can *complete the square* in the following manner:

$$\begin{aligned} x^2 + bx + c &= x^2 + bx + \left(\frac{b}{2}\right)^2 - \left(\frac{b}{2}\right)^2 + c \\ &= \underbrace{\left(x + \frac{b}{2}\right)^2}_{\text{Completed Perfect Square}} + c - \left(\frac{b}{2}\right)^2 \end{aligned}$$

Adding and subtracting the same expression does not change the value

Example 1 Complete the square: $x^2 - 10x + 17$. We have

$$\begin{aligned} x^2 - 10x + 17 &= x^2 - 10x + (5^2 - 5^2) + 17 \\ &= \underbrace{x^2 - 10x + 25} - 25 + 17 \\ &= (x-5)^2 - 8 \end{aligned}$$

Example 2 Solve by completing the square. $3x^2 + 24x = 12$

We first divide through by 3 to obtain a leading coefficient of 1:

$$3x^2 + 24x = 12 \quad \Rightarrow \quad x^2 - 8x = 4$$

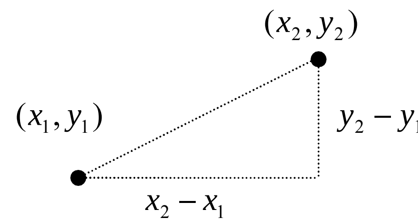
We now complete the square:

$$\begin{aligned} &x^2 - 8x - 5 = 0 \\ &x^2 - 8x + (4^2 - 4^2) - 5 = 0 \\ &(x-4)^2 - 21 = 0 \\ &(x-4)^2 = 21 \\ &x - 4 = \pm\sqrt{21} \quad \text{so} \quad x = 4 \pm \sqrt{21} \end{aligned}$$

Distance Formula

From the Pythagorean Theorem, we obtain the **distance between two points** (x_1, y_1) and (x_2, y_2):

$$D = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



Equation of a Circle

The equation of a circle of radius *r* centered at the point (*h*, *k*) is given by

$$(x-h)^2 + (y-k)^2 = r^2$$

Example 3 Complete the square to determine the center and radius of the circle given by

$$x^2 - 6x + y^2 + 12y = 11$$

$$x^2 - 6x + (-3)^2 + y^2 + 12y + 6^2 = 11 + (-3)^2 + 6^2$$

$$(x-3)^2 + (y+6)^2 = 56$$

Therefore, the center of the circle is (3, −6) and the radius is $2\sqrt{14}$.

Practice Problems

1. Find the equation of the circle of radius 5 in the first quadrant tangent to the *x*-axis at the point (9,0).

2. Find the equation of the circle with a diameter having endpoints (1,-2) and (5,6).

IV Rational Functions and the Difference Quotient

A rational function is a *quotient*. It is obtained by dividing two non-constant polynomials, say, $p(x)$ and $q(x)$:

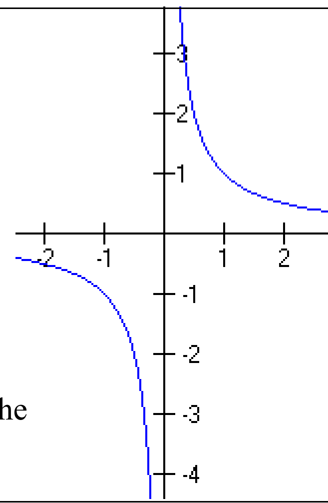
$$f(x) = \frac{p(x)}{q(x)} \qquad q(x) \neq 0$$

We make the following observations about rational functions:

- **The zeros** (x-intercepts) **of the numerator** $p(x)$ are the zeros (x-intercepts) of the rational function $f(x)$.
- **The zeros of the denominator** $q(x)$ are where the rational function $f(x)$ is **undefined**. These x values break the graph into unconnected sections referred to as *branches*.

The primary example of a rational function is $y = \frac{1}{x}$ whose graph is shown at right. Notice that

- the function is undefined at $x = 0$
- the "ends" of the graph approach the x-axis



Rational functions are not easy to graph accurately by hand because they require carefully plotted points. However, we can easily identify key characteristics of such graphs based only upon the numerator $p(x)$ and the denominator $q(x)$.

End Behavior of Rational Functions

Let m be the degree of $p(x)$ and n be the degree of $q(x)$.

- **If $m - n > 1$**, the graph has end behavior like a polynomial of degree $m - n$.
- **If $m = n + 1$**, the graph has end behavior like (is asymptotic to) the diagonal line obtained by long division (ignoring the rational remainder)
- **If $m = n$**, the graph has a horizontal asymptote $y = k$ where k is the ratio of the leading coefficients of $p(x)$ and $q(x)$
- **If $m < n$**, then the ends of the graph are asymptotic to the x-axis (the line $y = 0$)

As with the factors of a polynomial, we will see the same behavior is exhibited by the factors of rational functions.

Example 1 Describe the graph of $f(x) = \frac{x^2 - 5x - 14}{x^3 - 4x^2}$ in terms of x-intercepts, location above or below the x-axis, and end behavior.

Example 2 Determine a rational function $f(x)$ with x-intercepts at $x = -2$ and 4 having vertical asymptotes at $x = 0$ and 7. Identify each x value across which the graph changes sign. **What determines whether there is a change in sign?**

Example 3 Describe the graph of $f(x) = \dfrac{x^2 - x - 12}{x^2 + x - 6}$ in terms of x-intercepts, location above or below the x-axis, and end behavior. **What happens if the numerator and denominator have the same linear factor?**

Example 4 Use long division to show that the rational function $g(x) = \dfrac{3x^2 - 9x - 30}{2x - 2}$ has a diagonal (slant) asymptote of $y = \dfrac{3}{2}x - 3$. **What happens to the value of the remainder term as |*x*| becomes large (unbounded)?**

Example 5 Describe the graph of $h(x) = \dfrac{x^2 - 4}{x^2 + 1}$ in terms of symmetry, x-intercepts, location above the x-axis, and end behavior. Sketch the graph.

Practice Problems

Describe each graph in terms of *x*-intercepts, location above or below the *x*-axis, and end behavior. If the graph has a horizontal or slant asymptote, determine the equation of the line (asymptote).

1. $f(x) = \dfrac{x}{x^2 + 2x - 8}$

2. $g(x) = \dfrac{3x^2 + 3x - 18}{x^2 + 2x - 8}$

3. $h(x) = \dfrac{x^2 + 2x - 8}{2x}$

Partial Fractions

Combining fractions through addition or subtraction involves obtaining a common denominator and forming a single fraction. There are occasions in which we actually want to break up a single fraction into "partial" fractions whose denominator involves a single "prime" factor.

As a simple example, add the fractions $\dfrac{3}{x-4}+\dfrac{1}{x+2}$.

We have

$$\begin{aligned}\frac{3}{x-4}+\frac{1}{x+2} &= \frac{3}{x-4}\cdot\frac{x+2}{x+2}+\frac{1}{x+2}\cdot\frac{x-4}{x-4} \\ &= \frac{(3x+6)+(x-4)}{(x+2)(x-4)} \\ &= \frac{4x+2}{(x+2)(x-4)}\end{aligned}$$

We observe that this means that we could "decompose" the fraction $\dfrac{4x+2}{(x+2)(x-4)}$ as the sum of fractions $\dfrac{3}{x-4}+\dfrac{1}{x+2}$. But how would we reverse this process in general?

Partial Fraction Decomposition

Consider the rational expression $\dfrac{x-1}{x^2+3x+2}$. (Note that the degree of the numerator is *less than* the degree of the denominator so the rational expression is in "lowest terms".)

Factoring reveals that

$$\frac{x-1}{x^2+3x+2}=\frac{x-1}{(x+1)(x+2)}$$

Because the numerator has degree less than the denominator, there must exist constants A and B with

$$\frac{x-1}{(x+1)(x+2)}=\frac{A}{x+1}+\frac{B}{x+2}$$

Obtaining a common denominator, we have

$$\begin{aligned}\frac{x-1}{(x+1)(x+2)} &= \frac{A}{x+1}+\frac{B}{x+2} \\ &= \frac{A(x+2)}{(x+1)(x+2)}+\frac{B(x+1)}{(x+1)(x+2)} \\ &= \frac{A(x+2)+B(x+1)}{(x+1)(x+2)}\end{aligned}$$

Equating the numerators on each side of the equal sign, we have

$$x-1 = A(x+2)+B(x+1)$$

We may now expand to determine the coefficients *A* and *B*:

$$\begin{aligned} x-1 &= Ax+Bx+2A+B \\ &= (A+B)x+(2A+B) \end{aligned}$$

Equating like terms, we obtain

$$\begin{aligned} A+B &= 1 \\ 2A+B &= -1 \end{aligned}$$

Eliminating *B*, we get

$$A=-2$$

which allows us to determine the value $B = 3$. Therefore, we may write

$$\frac{x-1}{(x+1)(x+2)} = \frac{-2}{x+1}+\frac{3}{x+2}$$

<u>Practice Problem</u>

1. Decompose $\dfrac{3x+2}{x^2-x-12}$ into fractions with linear denominators.

The Difference Quotient

The concept of slope ("steepness" or rate of change) applies to more than straight lines. It applies to any curve or process (population, water volume, pollution, elimination of drugs from the body, etc.). As a result, we frequently consider the *average rate of change* in a quantity over a small interval.

Let $f(x)$ denote a function that represents a curve or variable process. The average change of f on the interval $[x, x+h]$ is given by the quotient

$$\frac{f(x+h)-f(x)}{(x+h)-x}=\frac{f(x+h)-f(x)}{h}$$

Examples Set up and simplify the different quotient (average rate of change) for the following functions on a general interval $[x, x+h]$.

(a) $f(x)=\sqrt{x}$

We have

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{\sqrt{x+h}-\sqrt{x}}{h}\\&=\frac{\sqrt{x+h}-\sqrt{x}}{h}\cdot\frac{\sqrt{x+h}+\sqrt{x}}{\sqrt{x+h}+\sqrt{x}}\\&=\frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\frac{h}{h\left(\sqrt{x+h}+\sqrt{x}\right)}\\&=\frac{1}{\sqrt{x+h}+\sqrt{x}}\end{aligned}$$

(b) $g(x)=\dfrac{1}{x}$

In this case, we get

$$\begin{aligned}\frac{g(x+h)-g(x)}{h}&=\frac{\frac{1}{x+h}-\frac{1}{x}}{h}\\&=\frac{\frac{1}{x+h}\cdot\frac{x}{x}-\frac{1}{x}\cdot\frac{x+h}{x+h}}{h}\\&=\frac{x-(x+h)}{x(x+h)}\cdot\frac{1}{h}\\&=\frac{-h}{x(x+h)h}\\&=-\frac{1}{x(x+h)}\end{aligned}$$

Practice Problems

Set up and simplify the different quotient (average rate of change) for the following functions on a general interval $[x, x+h]$.

1. $f(x)=x^2+2x+1$

2. $g(x)=\dfrac{2}{x+3}$

V Exponents, Exponential and Logarithmic Functions

Order of Operations

The operations of arithmetic are performed, in order, from left to right:

1. Parentheses or grouping symbols
2. Exponents
3. Multiplication and Division
4. Addition and Subtraction

The Properties of Exponents

Exponents are a short hand for multiplication. When multiplying a common base a raised to various powers, the following properties hold for all **integers** r and s:

1. $a^r a^s = a^{r+s}$
2. $\dfrac{a^r}{a^s} = a^{r-s} \quad (a \neq 0)$
3. $\left(a^r\right)^s = a^{rs}$
4. $a^0 = 1 \quad (a \neq 0)$
5. $a^{-r} = \dfrac{1}{a^r} \quad (a \neq 0)$

For rational exponents, we have

6. $a^{m/n} = \sqrt[n]{a^m} = \left(\sqrt[n]{a}\right)^m$

Furthermore, we say that $\sqrt[n]{a^m}$ is in reduced form if $m < n$.

Examples Write each of the following expressions in simplest (reduced) form without negative exponents.

(a) $\dfrac{9a^3b^{-2}(2c^3)^4}{36ab^{-3}c^5}$

(b) $\sqrt[3]{24x^4y^5z^6}$

(c) $\dfrac{5^{3/2}r^{1/3}}{\sqrt{5}r^{-2/5}}$

<u>Exponential Functions</u>

The question arises of, "What do we mean by $3^{\sqrt{2}}$ or 2^{π}?"

The reason is that we want to consider functions of the form $f(x) = a^x$ **but is this function defined for all real numbers?**

Before we answer that question, let's see if we can motivate the reasonability of such values.

Consider $\sqrt{3}^{\sqrt{2}}$. We have an irrational number raised to an irrational power. Is this a rational number or not? If not, what rational number is it? If it is irrational, then consider

$$\left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}}$$

Evaluate this number using the properties of exponents.

$$\begin{aligned}\left(\sqrt{3}^{\sqrt{2}}\right)^{\sqrt{2}} &= \sqrt{3}^{\sqrt{2}\cdot\sqrt{2}} \\ &= \left(\sqrt{3}\right)^2 \\ &= 3\end{aligned}$$

Somewhere, somehow, in this process we have managed to raise an irrational number to an irrational power and obtain a rational number!

In the case of 2^{π}, we could use the following sequence of approximations using 2 raised to a *rational* power:

$$2^3, 2^{3.1}, 2^{3.14}, 2^{3.141}, 2^{3.1415}, 2^{3.14159}, 2^{3.141592}, 2^{3.1415926}, \ldots$$

This sequence approaches a single value that is represented by 2^{π}. We cannot produce the actual value in decimal form but we can estimate it to any required accuracy. In this manner, we can *define* the value of any positive real base raised to any real power.

Exponential Functions

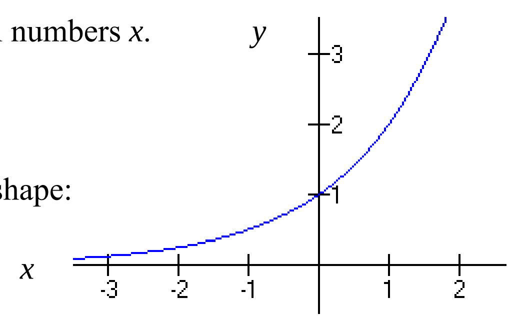
Using the previous ideas, we may define an exponential function to be

$$f(x) = a^x \quad (a > 0).$$

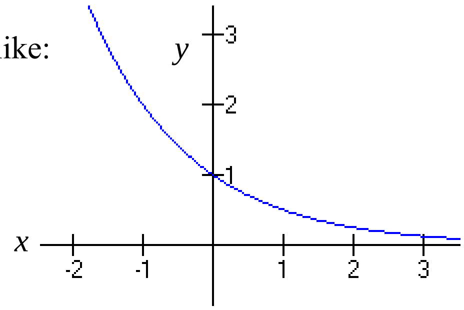
It is immediate that for $a > 0$, $a^x > 0$ for all real numbers x.

Graph of an Exponential function

If $a > 1$, the graph of $y = a^x$ has the following shape:



On the other hand, if $0 < a < 1$, the graph looks like:



This is due to the fact that $a^{-1} = \dfrac{1}{a}$

Examples Graph the following exponential functions together by plotting points.

(a) $y = 2^x$ and $y = \left(\dfrac{1}{2}\right)^x$ (b) $y = 4^x$ and $y = \left(\dfrac{1}{4}\right)^x$

Modeling with exponential functions and solving equations

Population and radioactive materials exhibit exponential behavior. In general, this model is

$$A = A_0 a^t$$

where A_0 is the initial population and a is a growth factor occurring every t units of time.

Example 1 Suppose a certain colony of bacteria grows exponentially so that it doubles every 6 hours. If the initial population is 50 bacteria, find the population after 30 hours.

We have

$$A = 50(2^t)$$

since $A_0 = 50$ bacteria and $a = 2$. Thus, $A(30) = 50(2^{30/6}) = 1600$

Example 2: Using the natural exponential function, e^{kt}, we have

$$A = 50e^{kt}.$$

To determine the value of k needed for this model, we use that fact that $A(6) = 100$ since the population doubles every 6 hours.

$$100 = 50e^{6k} \quad \Rightarrow \quad 2 = e^{6k} \quad \Rightarrow \quad \ln 2 = 6k \quad \Rightarrow k = \frac{\ln 2}{6}$$

so the population of bacteria is modeled by $A(t) = 50e^{t \ln 2/6}$.