Accelerated Inclusive Mathematics Project

YuMi Deadly Maths

AIM Module O4

Year C, Term 3

Operations:

Arithmetic and Algebra Principles

Prepared by the YuMi Deadly Centre
Queensland University of Technology
Kelvin Grove, Queensland, 4059

ACKNOWLEDGEMENT

The YuMi Deadly Centre acknowledges the traditional owners and custodians of the lands in which the mathematics ideas for this resource were developed, refined and presented in professional development sessions.

YUMI DEADLY CENTRE

The YuMi Deadly Centre is a research centre within the Faculty of Education at the Queensland University of Technology which is dedicated to enhancing the learning of Indigenous and non-Indigenous children, young people and adults to improve their opportunities for further education, training and employment, and to equip them for lifelong learning.

"YuMi" is a Torres Strait Islander Creole word meaning "you and me" but is used here with permission from the Torres Strait Islanders' Regional Education Council to mean working together as a community for the betterment of education for all. "Deadly" is an Aboriginal word used widely across Australia to mean smart in terms of being the best one can be in learning and life.

YuMi Deadly Centre's motif was developed by Blacklines to depict learning, empowerment, and growth within country/community. The three key elements are the individual (represented by the inner seed), the community (represented by the leaf), and the journey/pathway of learning (represented by the curved line which winds around and up through the leaf). As such, the motif illustrates the YuMi Deadly Centre's vision: *Growing community through education*.

The YuMi Deadly Centre (YDC) can be contacted at ydc@qut.edu.au. Its website is http://ydc.qut.edu.au.

DEVELOPMENT OF THE AIM MODULES

The Accelerated Inclusive Mathematics (AIM) modules were originally developed from 2010 to 2013 as part of the Accelerated Indigenous Mathematics project funded under the Commonwealth Government's *Closing the Gap: Expansion of Intensive Literacy and Numeracy* program for Indigenous students. The project aimed to assist secondary schools with beginning junior-secondary Indigenous students who were at Year 2/3 level in mathematics by developing a junior-secondary mathematics program that accelerates the students' learning to enable access to mathematics subjects in the senior-secondary years and therefore enhance employment and life chances. The project developed three years of modules (Years A to C) that are vertical sequences of learning to take students from their ability level to their age level in mathematics. The YuMi Deadly Centre acknowledges the role of the Department of Education, Employment and Workplace Relations in the development of the AIM project and these modules.

Contents

Module Overview

This is the fourth Operations module of five – it follows on from Module O1 *Addition and Subtraction*, Module O2 *Multiplication and Division*, and Module O3 *Common and Decimal Fraction Operations*, and leads into Module O5 *Financial Mathematics* – see **Appendix B**. However, Modules N4 *Percent, Rate and Ratio* and N5 *Directed Number, Indices and Systems* also cover operations in part. The module is also based on Module A1 *Equivalence and Equations*.

This module, Module O4 *Arithmetic and Algebra Principles*, covers the area of operation principles, or what are sometimes called **number sense** principles. Along with concepts and strategies, principles provide the knowledge that underlies operations in both arithmetic and algebra. Thus, extending arithmetic principles to algebra principles is a major mathematics outcome for Years 7 to 9. This module is also a major part of the development of mathematics because it outlines the structure that is common to arithmetic and algebra. It focuses on some very important big ideas.

As well as the principles, this module also covers one of the major operation skills still very useful in the modern age, **computational estimation**. This is a topic where expertise is strongly based on principles.

Thus this module is important in that it completes the work started in O1 and O2 by filling in the sections that were missed out in these two modules, and extends this new information to algebra. It has been placed into Year C in the AIM scope and sequence, but its work is crucial across all year levels.

Finally, this module, O4 *Arithmetic and Algebra Principles*, is important for its relationship with Module A4 *Algebraic Computation*. To some extent A4 and O4 form one module. Module O4 ends with the arithmetic principles being shown to work in algebra. Module A4 uses these principles for algebraic computation. As has been stated before in earlier modules, algebraic proficiency grows out of using the processes of arithmetic (i.e. the principles of arithmetic), not the computational algorithms that give answers in arithmetic.

Background information for teaching arithmetic and algebra principles

This section looks at the two sides of operations and then discusses connections and big ideas.

Two foci of operations

As we have said across the Operations modules, there are two components for operations. The first of these deals with **meaning or operating** – what does addition and subtraction mean (what are their concepts), where are they used in their world, and what are their properties. The second deals with **computation or calculating** – what is the answer to the operation, what ways can we work it out, and how accurate does this have to be. Interestingly, problem solving is based on meaning, not computation, as is understanding algebraic sentences with addition and/or subtraction. Obviously algorithms are computation along with basic facts and estimation; however, as well, most of the operations work is based on strategies and properties of the operations (which this module will call principles). Finally, the four operations act on the world in two ways – on individual objects, and on measures. The major components of operations are given below, followed by a description of how they interact.

Meanings

1. **Concepts**. These are meanings that define the operation in terms of the everyday world; they also cover the different models in which the meanings can be encapsulated (for addition and subtraction, there are two models – set, and number line or length); they relate story, action, drawing, language and symbols (these are called representations). As concepts, they are big ideas.

1. **Principles**. These are properties that hold no matter what the numbers are (i.e. they hold for whole numbers, decimals, fractions, and so on); for example, the commutative law, that first + second always equals second + first. As properties for all numbers and algebra, these are big ideas.

2. **Problem-solving strategies**. These are general rules of thumb that direct towards the answer; for example, the major ones in this module are "identify given and wanted", "act it out", "make a drawing", "restate the problem", "solve a simpler problem", "check the answer", and "learn from the solution". The strategies will also include Polya's plan of attack (See, Plan, Do, Check).These are also big ideas.

3. **Word problems**. Being able to interpret problems given in words and determine the operation to use, and construct problems from equations.

4. **Extension to algebra**. Being able to repeat concepts for when there are variables (i.e. relate algebra sentences to actions and stories).

Computation

1. **Basic facts and basic fact strategies**. Basic facts are automated answers to addition and subtraction facts (one-digit numbers) from 0+0 to 9+9, from 18−9 to 0−0, and for multiples of 10 (e.g. 400 + 500, 13000 − 8000), while strategies are ways to find answers to addition and subtraction facts (albeit slowly); the major ones in this module are "count on", "near doubles", "near tens", "turnarounds", "families", and "think addition".

2. **Algorithms and computation strategies**. Algorithms are ways of calculating answers to addition and subtraction algorithms (more than one digit numbers – mental, written and calculator), while strategies are ways of finding answers to addition and subtraction algorithms. The major strategies are "separation", "sequencing" including "additive-subtraction sequencing", and "compensation" – these three strategies are big ideas.

3. **Estimation and estimation strategies**. Estimation is calculating approximate answers to large-number computations and uses principles to be accurate, while strategies are ways to find approximate answers to large-number computations. The major strategies are "front end", "rounding", "straddling", and "getting closer".

Connections and big ideas

Connections

The major connections between operations and the other topics are to topics that use number and/or operations as the basis of their mathematics (i.e. number, algebra, measurement, statistics and probability). Major connections are as follows.

1. **Operations and number**. This is an obvious connection as operations need numbers to act on. In particular, the strategies for computation relate to the numeration concepts, that is: (a) separation strategy relies on a place-value understanding of 2- to 4-digit numeration; and (b) sequencing and compensation strategies rely on a rank understanding of numeration.

2. **Operations and algebra**. Again, this is an obvious relationship as algebra is generalisation of arithmetic activities. In particular, x + 3 relates to an example like 5 + 3. The difference is that 5 is an actual number while x is a variable.

3. **Operations and measurement**. Measurement involves a lot of operations particularly with respect to formulae (e.g. perimeter, area).

4. **Operations and Statistics and probability**. Both of these involve operations (e.g. in calculating mean and chance).

As well as between operations and other topics, there are connections between topics within the two foci of addition and subtraction and multiplication and division; and between topics within operations. The major connections within operations are as follows:

(a) *addition and subtraction* – inverses of each other;

(b) *multiplication* – one meaning of multiplication is repeated addition;

(c) *subtraction with division* – one meaning of division is repeated subtraction;

(d) *concepts and problem solving* – the meanings of the operations are the basis of solving problems as they determine which operations relate to which situations; and

(e) *calculation and estimation* – estimation requires calculation but they also have strategies in common (the calculation strategies help to estimate).

Big ideas

The meanings of operations are big ideas as are the major strategies for computation. These have been covered above. As well as these, the properties of operations are big ideas and the major of these are listed below. These big ideas apply in algebra as well as arithmetic. They are divided into global/teaching (big ideas that apply beyond operations) and principles or properties of operations (big ideas that apply for number and algebra).

Global/teaching big ideas

1. **Symbols tell stories**. The symbols of mathematics enable the world to be described succinctly and in a generalised way (e.g. 2 + 3 = 5 means caught 2 fish and then caught another 3 fish, or bought a $2 chocolate and $3 drink, or joined a 2 m length of wood to a 3 m length … and so on).

2. **Relationship vs change**. Mathematics has three components – objects, relationships between objects, and changes/transformations between objects. All relationships can be perceived as changes and vice versa. This is particularly applicable to operations; 2 plus 3 can be perceived as relationship 2 + 3 = 5 or change $2 \xrightarrow{+3} 5$.

3. **Interpretation vs construction**. Things can either be interpreted (e.g. what operation for this problem, what properties for this shape) or constructed (write a problem for 2 + 3 = 5; construct a shape of 4 sides with 2 sides parallel).

4. **Accuracy vs exactness**. Problems can be solved accurately (e.g. find 5275 + 3873 to the nearest 100 – rounding and estimation) or exactly (e.g. 5275 + 3873 = 9148 – basic facts and algorithms).

5. **Part-part-total/whole**. Two parts make a total or whole, and a total or whole can be separated to form two parts – this is the basis of numbers and operations (e.g. fraction is part-whole, ratio is part to part; addition is knowing parts, wanting total).

Field properties for operations

1. **Closure.** Numbers and an operation always give another number (e.g. 2.17 + 4.34 = 6.51 – for any numbers a and b, a + b = c which is another number; and 2.17 × 4.3 = 9.331 – for any numbers a and b, a × b = c, where c is another number).

2. **Identity**. 0 and 1 do not change things (+/− and ×/÷ respectively). Adding/subtracting zero leaves numbers unchanged (e.g. 9 +/− 0 = 9, where 0 can equal +1−1, +6−3−3, +11−14+3, and so on). Anything multiplied by 1 = itself (e.g. for any a, a × 1 = 1 × a = a). Anything multiplied by 0 = 0.

3. **Inverse**. A change that undoes another change. Addition is undone by subtraction and vice versa (e.g. + 5 − 5 = − 5 + 5 = 0, so 2 + 5 = 7 means 7 – 5 = 2). Multiplication's inverse is division and vice versa (e.g. × 5 ÷ 5 = ÷ 5 × 5 = 1, so 2 × 5 = 10 means 10 ÷ 5 = 2). This principle holds for fractions and indices (e.g. for fractions, the inverse of $\frac{2}{3}$ is the reciprocal $\frac{3}{2}$ (or 1 over $\frac{2}{3}$) because $\frac{2}{3} \times \frac{3}{2} = \frac{3}{2} \times \frac{2}{3} = 1$; for indices, the inverse of 6^3 is 6^{-3} and vice versa because 3 + −3 = 0 and $6^3 \times 6^{-3} = 6^{(3+-3)} = 6^0 = 1$).

4. **Commutativity**. Order does not matter for addition but does for subtraction (e.g. 3 + 4 = 4 + 3, but 7 – 5 ≠ 5 – 7). Order does not matter for multiplication but does for division (e.g. 12 × 4 = 4 × 12 but 12 ÷ 4 ≠ 4 ÷ 12; for any a, b and c, (a × b) × c = a × (b × c)). Also known as turnarounds.

5. **Associativity**. What is done first does not matter for addition and multiplication but does matter for subtraction and division (e.g. (8 + 4) + 2 = 8 + (4 + 2), and (8 × 4) × 2 = 32 × 2 = 64 and 8 × (4 × 2) = 8 × 8 = 64 but (8 ÷ 4) ÷ 2 ≠ 8 ÷ (4 ÷ 2).

6. **Distributivity**. Multiplication and division are distributed across addition and subtraction and act on everything (e.g. 3 × (4 + 5) = (3 × 4) + (3 × 5); (21 – 12) ÷ 3 = (21 ÷ 3) – (12 ÷ 3)). Distributivity does hold for all operations (e.g. 7 × (8 – 3) = (7 × 8) – (7 × 3), (56 + 21) ÷ 7 = (56 ÷ 7) + (21 ÷ 7) and (56 – 21) ÷ 7 = (56 ÷ 7) – (21 ÷ 7)).

Extension of field properties

1. **Compensation**. Ensuring that a change is compensated for so the answer remains the same – related to inverse (e.g. 5 + 5 = 7 + 3; 48 + 25 = 50 + 23; 61 – 29 = 62 − 30).

2. **Equivalence.** Two expressions are equivalent if they relate by adding or subtracting 0 and multiplying or dividing by 1; also related to inverse (e.g. 48 + 25 = 48 + 2 + 25 – 2 = 73; 50 + 23 = 73; $\frac{2}{3} = \frac{2}{3} \times \frac{2}{2} = \frac{4}{6}$).

3. **Inverse relation**. The higher the second number in subtraction and division, the smaller the result (e.g. 12 ÷ 2 = 6 > 12 ÷ 3 = 4; $\frac{1}{2} > \frac{1}{3}$). For division, the more you divide by, the less you have (e.g. 24 ÷ 8 is less than 24 ÷ 6). This principle does not apply to addition or multiplication.

4. **Triadic relationships.** When three things are related there are three problem types where each of the parts are the unknowns. For example, 2 + 3 = 5 can have a problem for: ? + 3 = 5, 2 + ? = 5, 2 + 3 = ?. This principle holds for all four operations: a + b = c (? + b = c, a + ? = c, a + b = ?); a – b = c (? – b = c, a – ? = c, a – b = ?,); a × b = c (? × b = c, a × ? = c, a × b = ?); a ÷ b = c (? ÷ b = c, a ÷ ? = c, a ÷ b = ?).

Equals properties

1. **Reflexivity principle**. Anything equals itself (e.g. 2 = 2).

2. **Symmetry principle**. Equals can be turned around (e.g. if 2 + 3 = 5 then 5 = 2 + 3).

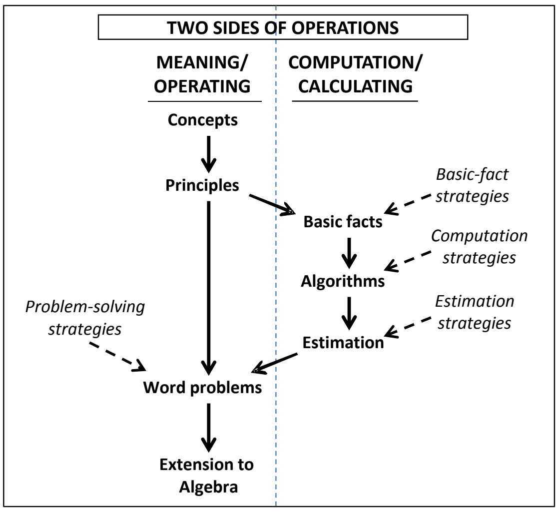
3. **Transitivity principle**. Equals continues across equal relationships; the first in a sequence, equals the last (e.g. if 2 + 3 = 5 and 5 = 6 − 1 then 2 + 3 = 6 − 1).

4. **Balance principle**. Equations stay true if whatever is done to one side is done to the other (e.g. 2 + 3 = 5 means that 2 + 3 + 5 = 5 + 5, that is, 2 + 8 = 10).

Sequencing for arithmetic and algebra principles

This section looks at sequencing for operations and then for this particular module.

Sequencing in operations



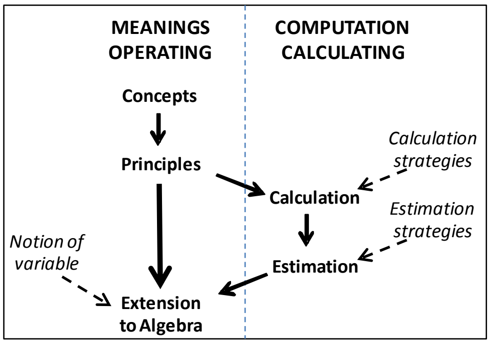
The diagram on the right shows how the various components of meanings and computation interact. There are two columns. Concepts, principles, word problems, problem-solving strategies, and algebra are in the meaning side while basic facts, algorithms and estimation, along with their strategies, are on the computation side.

In Modules O1 and O2, we focused on concepts, word problems and extension to algebra on the meaning side and on basic facts and algorithms on the computation side. This means that, so far in the AIM modules, we have not looked at the two missing sections, principles and estimation, although there had to be some reference to principles to do the computation side in O1 and O2. Also, the extension to algebra was only done for concepts in O1 and O2; in this module it will be done for principles as well.

This extension to algebra will mean be restricted to the principles. The computational methods of substitution, simplification, expansion and factorisation will be left to Module A4 *Algebraic Computation*.

Sequencing in this module

The **sequence for this module** is based on the interaction of the ideas in the figure on the right. It covers, in sequence, concepts and principles to calculation and estimation, and extension to algebra.



The principles are all **big ideas**; they consist of:

(a) *number-size principles* – relationships such as increasing a number increases the answer for addition and multiplication, while increasing the second number decreases the answer for subtraction and division;

(b) *operation properties or field principles* – identity (0 and 1), inverse (+ and −, × and ÷), commutativity (e.g. 3+5=5+3), associativity (e.g. 3+4+5 = 7+5 and 3+9) and distributivity (e.g. 3×(4+5) = 3×4 + 3×5); and

(c) *properties of equals and order* (equivalence and order class principles) – reflexivity (A=A but A is not greater/smaller than A), symmetry (A+B=C means C=A+B, A+B<C means C>A+B), and transitivity (A=B, B=C means A=C, and same for order).

The module has sections and units as follows.

Overview: Background information, sequencing, and relation to Australian Curriculum

Unit 1: Number-size, equals and order principles

Unit 2: Field principles and their teaching

Unit 3: Application to estimation

Unit 4: Extending principles to algebra

Test item types: Test items associated with the four units above which can be used for pre- and post-tests

Appendix A: RAMR cycle components and description

Appendix B: AIM scope and sequence showing all modules by year level and term.

The module provides teaching information in sequence and relates these to the **RAMR** cycle (see **Appendix A**) and where possible activities are given following the RAMR headings. YDM believes that activities should be related to the real world of the students, as far as possible, and that teaching should be active, involving the students acting out situations. Because the ideas in this module are often new for teachers as well as students, the sections in each unit have **two parts**: one **providing information**, and the second **discussing methods of teaching** the ideas using the RAMR model.

Note: Some of the units also use the length and mass models used in Module A1 *Equivalence and Equations*. Use the ideas from that module to help with the teaching of this module.

Relation to Australian Curriculum: Mathematics

AIM O4 meets the Australian Curriculum: Mathematics (Foundation to Year 10)					
Unit 1: Number-size, equals and order principles Unit 2: Field principles and their teaching Unit 3: Application to estimation Unit 4: Extending principles to algebra					
Content Description	**Year**	**O4 Unit**			
		1	**2**	**3**	**4**
Explore and describe number patterns resulting from performing multiplication (ACMNA081)	4	✓			
Use equivalent number sentences involving addition and subtraction to find unknown quantities (ACMNA083)		✓			
Use estimation and rounding to check the reasonableness of answers to calculations (ACMNA099)	5			✓	
Describe, continue and create patterns with fractions, decimals and whole numbers resulting from addition and subtraction (ACMNA107)		✓			
Use equivalent number sentences involving multiplication and division to find unknown quantities (ACMNA121)		✓			
Continue and create sequences involving whole numbers, fractions and decimals. Describe the rule used to create the sequence (ACMNA133)	6	✓			
Explore the use of brackets and order of operations to write number sentences (ACMNA134)			✓		
Apply the associative, commutative and distributive laws to aid mental and written computation(ACMNA151)	7		✓		
Create algebraic expressions and evaluate them by substituting a given value for each variable (ACMNA176)					✓
Extend and apply the laws and properties of arithmetic to algebraic terms and expressions (ACMNA177)		✓	✓		
Carry out the four operations with rational numbers and integers, using efficient mental and written strategies and appropriate digital technologies (ACMNA183)	8	✓			
Extend and apply the distributive law to the expansion of algebraic expressions (ACMNA190)					✓
Factorise algebraic expressions by identifying numerical factors (ACMNA191)					✓
Simplify algebraic expressions involving the four operations (ACMNA192)					✓

Unit 1: Number-Size, Equals and Order Principles

This unit looks at the properties of the four operations and of equals that hold regardless of the numbers being added and subtracted. It begins by looking at the number-size principles for addition and subtraction, the number-size principles for multiplication and division, and concludes by looking at the equals and order principles. Teaching the principles will be discussed in separate sections.

1.1 Number-size principles for addition/subtraction

The number-size principles are properties of addition and subtraction that hold for all numbers (whole numbers, decimal numbers, common fractions, measures, and so on). We can work the principles out by looking at two simple examples: 7 + 4 = 11 and 7 – 4 = 3, and the activities below. For the activities, a number and an upward arrow means that the number is increasing; a number and a downward arrow means that the number is decreasing; and a number and an equals sign means that the number stays the same.

Explore what happens to addition and subtraction examples as numbers change. Each line will have three numbers, two with changes and one without. As you explore, your job is to provide through arrows or an equals sign whether the number that has nothing beside it goes up [↑], goes down [↓] or stays the same [=].

Activity 1: Addition

Consider the sum 7 + 4 = 11. Two of the 7, 4 and 11 can change or not change and you have to work out what will happen to the number without a sign on it. The symbols to use are ↑ goes up/increases; ↓ goes down/decreases; and = stays the same.

1. Complete the signs for the number left free (put sign above underlining):

(a)	7 ↑	4 =	11____	(b)	7 ↓	4 =	11____	(c)	7 =	4 ↑	11____
(d)	7 =	4 ↓	11____	(e)	7____	4 =	11 ↑	(f)	7____	4 =	11 ↓
(g)	7 =	4____	11↑	(h)	7 =	4____	11 ↓				

2. Complete the signs for the number left free (put sign above underlining):

(a)	7 ↑	4____	11 =	(b)	7 ↓	4____	11 =	(c)	7____	4 ↑	11 =
(d)	7____	4 ↓	11 =								

Activity 2: Subtraction

Consider the sum 7 – 4 = 3. Two of the 7, 4 and 3 can change or not change and you have to work out what will happen to the number without a sign on it. The signs to use are ↑ goes up/increases; ↓ goes down/decreases; and = stays the same.

1. Complete the signs for the number left free (put sign above underlining):

(a)	7 ↑	4 =	3____	(b)	7 ↓	4 =	3____	(c)	7 =	4 ↑	3____
(d)	7 =	4 ↓	3____	(e)	7____	4 =	3 ↑	(f)	7____	4 =	3 ↓
(g)	7 =	4____	3↑	(h)	7 =	4____	3 ↓				

2. Complete the signs for the number left free (put sign above underlining):

(a)	7 ↑	4____	3 =	(b)	7 ↓	4____	3 =	(c)	7____	4 ↑	3 =
(d)	7____	4 ↓	3 =								

Activity 3: Comparing addition and subtraction

Consider what is the same and different about the addition and subtraction results.

Activities 1, 2 and 3 enable number-size principles for addition and subtraction to be identified. They also enable differences between addition and subtraction to be identified that are important (i.e. that the second number acts differently for subtraction than it does for addition, and what keeps the answer the same is different in addition than what it is in subtraction). These principles and differences are summarised below (*** shows differences).

	CHANGE	ADDITION	SUBTRACTION
1.	First number increases/decreases, other number stays the same	Answer increases/decreases	Answer increases/decreases
2.	Second number increases/decreases, other number stays the same	Answer increases/decreases	Answer decreases/increases***
3.	First number increases/decreases, answer stays the same	Second number decreases/ increases	Second number increases/ decreases ***
4.	Second number increases/decreases, answer stays the same	First number decreases/ increases	First number increases/ decreases ***
5.	Answer increases/decreases, second number stays the same	First number increases/ decreases	First number increases/ decreases
6.	Answer increases/decreases, first number stays the same	Second number increases/ decreases	Second number decreases/ increases ***

7. Addition and subtraction act differently for change in second number (subtraction change is called ***inverse relation principle***) – this applies at 2 and 6 above. 1 and 5 are **direct relation**.
8. Addition and subtraction act differently to keep answer the same (called ***compensation principle***) – this applies at 3 and 4 above.

1.2 Number-size principles for multiplication/division

The number-size principles are properties of multiplication and division that hold for all numbers (whole numbers, decimal numbers, common fractions, measures, and so on). We can work the principles out by looking at two simple examples: 6 × 3 = 18 and 8 ÷ 4 = 2 and the activities below. For these examples, a number and an upward arrow means that the number (in the algorithm) is increasing; a number and a downward arrow means that the number is decreasing; and a number and an equals sign means that the number stays the same. Explore what happens to multiplication and division examples as numbers change. Nothing next to a number means that the activity requires determining whether that number ↑, ↓ or = as a result of the changes given.

For example, "6 ↑" means that the 6 in 6×3=18 is increasing, while "3 =" means that the 3 stays the same. The "18 " means that you have to find what will happen to 18 in this situation. Thus, the activity requires the student to determine whether 18 is ↑, ↓ or = when 6 is ↑ and 3 is =. The way to do this is to think of different scenarios and check what happens. If the 6 increases to 9, for example, then 9×3=27, which means that 18 increases ↑.

Activity 1: Multiplication

Consider the sum 6×3=18. Two of the 6, 3 and 18 can change or not change and you have to work out what will happen to the number without a sign on it. The symbols to use are ↑ goes up/increases; ↓ goes down/decreases; and = stays the same.

1. Complete the signs for the number left free:

(a)	6 ↑	3 =	18____	(b)	6 ↓	3 =	18____	(c)	6 =	3 ↑	18
(d)	6 =	3 ↓	18____	(e)	6____	3 =	18 ↑	(f)	6____	3 =	18 ↓
(g)	6 =	3____	18↑	(h)	6 =	3____	18 ↓				

2. Complete the signs for the number left free:

(a)	6 ↑	3____	18 =	(b)	6 ↓	3____	18 =	(c)	6____	3 ↑	18 =
(d)	6____	3 ↓	18 =								

Activity 2: Division

Consider the sum 8÷4=2. Two of the 8, 4 and 2 can change or not change and you have to work out what will happen to the number without a sign on it. The signs to use are ↑ goes up/increases; ↓ goes down/decreases; and = stays the same.

1. Complete the signs for the number left free:

(a)	8 ↑	4 =	2____	(b)	8 ↓	4 =	2____	(c)	8 =	4 ↑	2____
(d)	8 =	4 ↓	2____	(e)	8____	4 =	2 ↑	(f)	8____	4 =	2 ↓
(g)	8 =	4____	2↑	(h)	8 =	4____	2 ↓				

2. Complete the signs for the number left free:

(a)	8 ↑	4____	2 =	(b)	8 ↓	4____	2 =	(c)	8____	4 ↑	2 =
(d)	8____	4 ↓	2 =								

Activity 3: Comparing multiplication and division, and comparing all operations

Consider what is the same and different about multiplication and division results.

Consider what is the same and different about all operations.

Activities 1, 2 and 3 enable number-size principles for multiplication and division to be identified. They also enable differences between multiplication and division to be identified that are important (i.e. that the second number acts differently for division than it does for multiplication, and what keeps the answer the same is different in multiplication than what it is in division). These principles and differences are summarised below (*** shows differences).

	CHANGE	MULTIPLICATION	DIVISION
1.	First number increases/decreases, second number stays the same	Answer increases/decreases	Answer increases/decreases
2.	Second number increases/ decreases, first number stays the same	Answer increases/decreases	Answer decreases/increases ***
3.	First number increases/decreases, answer stays the same	Second number decreases/ increases	Second number increases/ decreases ***
4.	Second increases/decreases, answer stays the same	First number decreases/ increases	First number increases/ decreases ***
5.	Answer increases/decreases, second number stays the same	First number increases/ decreases	First number increases/ decreases
6.	Answer increases/decreases, first number stays the same	Second number increases/ decreases	Second number decreases/ increases ***

7. Multiplication and division act differently for change in second number (the division change is called the **inverse relation principle**) – this applies for 2 and 6 above. 1 and 5 are **direct relation**.
8. Multiplication and division act differently to keep answer the same (called the **compensation principle**) – this applies for 3 and 4 above.
9. With regard to all the operations, there are similarities between addition and multiplication and between subtraction and division, and differences between addition/multiplication and subtraction/division, particularly with regard to the inverse relation and compensation principles.

1.3 Teaching the number-size principles (RAMR cycle)

The main teaching approach is to keep following the RAMR cycle – reality → abstraction → mathematics → reflection as follows.

Reality

Check that students have the prerequisite knowledge that **addition is joining**, and **subtraction is separating or taking away**. If students do not grasp this, or the principles are unfamiliar, then these need covering first. The properties of addition and subtraction are based on the concepts of addition and subtraction and these hold regardless of the numbers or variables being added and subtracted.

Also check that students have the prerequisite knowledge that **multiplication is combining**, and **division is partitioning**. These also need covering first. Like addition and subtraction, the concepts and properties of multiplication and division hold regardless of the numbers or variables being multiplied and divided.

Stories should be used as a basis for students to relate to the sums. Try to think of reality/kinaesthetic activities from the world of the students for the principles – a good one for addition is a relay race for 2 people: *Two runners, if both run 5km, then total is 10km; if the first runner runs 6km and the second runs as normal, then total is 11 km*. Obviously if one runner runs further, the second has to run less if the total distance stays the same. Get the students outside in pairs with batons – set up walking short course relay races – have the students change how far they walk. Act out/model what is happening with counters and on number lines.

The relay race can also be modified for multiplication – 3 runners each doing 4 km. More runners mean going further as does each runner running further. If 12 km is the distance then 2 runners each run 6 km and 6 runners each run 2 km.

Abstraction

Each of the scenarios can be abstracted by acting it out, manipulating materials and creating in the mind.

Scenario 1: 7+4=11. Encourage students to discover the following. In this example 7 and 4 are the parts that make up the total of 11. If the scenario/reality changes and the 7 increases and the 4 remains the same, what happens to the total? The total increases. If the reality changes and the 4 increases but the 7 stays the same, the total will also increase. If the reality changes and the 11 increases, we know that one or both of the parts must have increased, OR, another part was added. The results when the parts or total decrease should also be examined.

Scenario 2: 7–3=4. In this example 3 and 4 are the parts that make up the total of 7. If the scenario/reality changes and the 3 increases and the total (7) remains the same, what happens to the other part? It decreases. If the reality changes and the 4 increases but the 7 stays the same, the other part will also decrease. If the reality changes and the 7 increases, we know that one or both of the parts (3 and/or 4) must have increased, OR, another part was added. The results when the parts or total decrease should also be examined.

Do the same as above for scenarios 6×3=18 and 8÷4=2. Relate multiplication and division to the array and area models.

Mathematics

One powerful way to teach the principles is to run this as an investigation – let students simply investigate the operations for properties and record all they find. The activities in 1.1 and 1.2 are a way to do this.

After this, students should be able to recognise and state the actions and resultant changes. For example:

- In an addition sum, if one of the parts increases and the other part stays the same, the total will increase. In multiplication, if the number of groups increases, the total will also increase.

- In a subtraction sum, if the part being taken away increases and the total stays the same then the other part will decrease. Similarly, in division, if the number of groups increases, there is less in each group if the total stays the same.

It is crucial that students record their findings in some way – and that the teacher helps the students to create findings that are useful.

Reflection

Get students to find real-world situations where properties hold (particularly compensation and inverse relation, but also direct relation). For example, students could be given a sum, 7 + 3 = 10 and the change 7 + 5 = 12 and they make a story for the original sum and then one for the change.

Get students to think of situations where more sharing gives you less and more earning gives you more.

1.4 Equals and order principles and their teaching (RAMR cycle)

The equals principles are the properties that always hold for equals and the order properties are the properties that always hold for order – there are three of them and they apply differently to equals than to order as follows.

Equals principles	Order principles
1. *Reflexive* – any number a is equal to itself, that is, a=a	1. *Non-reflexive* – any number a is never greater or less than itself, that is, a is not < a or > a
2. *Symmetric* – for all numbers a and b, if a=b then b=a (order of an equation can be reversed or "turned around"); this is important for equations as it means that 3 + 4 = 7 and 7 = 3 + 4, 3 + 4 = 8 − 1 and 8 − 1 = 3 + 4 are all allowed and are all correct and true.	2. *Antisymmetric* – for all numbers a and b, if a < b then b > a, and if a > b then b < a; this is also important because it means that if 3 + 4 > 5 then 5 < 3 + 4.
3. *Transitive* – for all numbers a, b and c, a=b and b=c means a=c; this is also important for equations because it means we can say, for example, 4 + 8 – 3 = 12 – 3 = 9 and so 4 + 8 – 3 = 9.	3. *Transitive* – for all numbers a, b and c, a>b and b>c means a>c; and a<b and b<c means a<c.

The implication of these principles is that equals must be understood in terms of **"same value as"**, not as the position to write the answer. This makes the idea of balance as the best way to think of equals. For example, if 7+2 = ?+4, then ? must be 5 not 9 because we are looking for a number which when added to 4 gives the same value as 7+2, not the place to write the answer to 7+2.

Years of doing "sums" usually means that most students do not see equals as same value as, they see it as where you write the answer, and most students believe that 7+2=9 but believe 9=7+2 is wrong and not allowed.

The way to teach them the equals and order principles is to use the RAMR cycle and a model for representing equals and operations that can be real world, physical, virtual and pictorial. The best model for these principles is the mass model – to think of equals as two sides being balanced. However, the length model is also good.

Reality

Think of real-world situations that could be used for equals, greater than and less than – money, height, mass, length? There are two important models to use.

1. **Balance or mass model**. Here equals is represented with groceries on a balanced beam balance. The principles are easily shown physically with a balance. So have the students experience the principles with beam balances and groceries. Direct the students to state out loud the equation as you move your hand from their left-hand side of the balance to their right-hand side – “equals” is said as the hand goes past the balance point (centre) of the balance (*Note:* it can be useful to stick “=” on the centre of the balance).
2. **Length model**. Here equals and order are represented by same length and different length respectively. Again it begins by strips of paper, moves onto lines (the double number line) and finally to images in the mind.

The order principles can also be taught with the balance (mass) and length (strips of paper) models. (*Note:* for the balance, the heavier item pushes down more, so higher is lighter).

It is always best to start with (a) *human body* (e.g. hang plastic bags on arms and become a beam balance and walk different distances); and, as stated before, (b) in *unnumbered* situations (e.g. compare groceries, use unmeasured strips of paper) before using numbers.

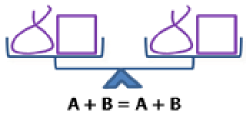
The use of balances begins as a real balance in early activities and then abstracts to a mathematical balance drawing in middle activities and finally to imagining of a balance in later activities. The drawing and the imagining balance can allow any operation.

Abstraction

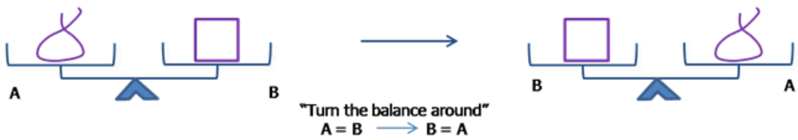
Equals principles

Balance or mass model. Undertake unnumbered activities before numbered. Direct the students to record the balanced groceries as informal equations, e.g. “salt equals soap plus pasta”. Discuss any generalities they find; encourage them to see the generalities below.

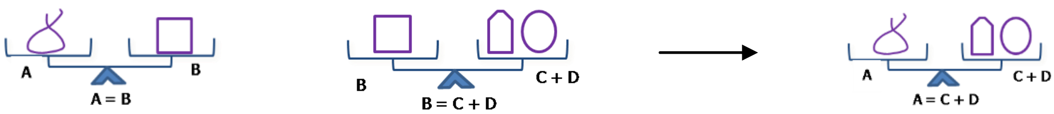
1. **Reflexivity**. This is fairly obvious (that two things that are the same will be equal) but it needs to be made explicit with equations. Students can investigate if same things always balance.



2. **Symmetry**. This can be seen by turning the balance 180°.



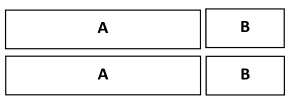
3. **Transitivity**. This requires the students comparing three things in three different ways to show A=B and B=C means A=C.



Length model. The equivalence properties are easily represented as in the three examples below. Once again, effective strategies involve the students saying out loud the equalities, writing informal equations, and discussing generalities.

1. **Reflexivity**.

   A + B = A + B

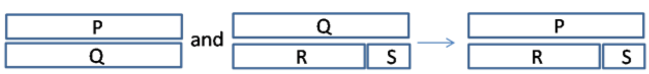


2. **Symmetry**.

   C = A + B → A + B = C

3. **Transitivity**

   P = Q and Q = R + S → P = R + S



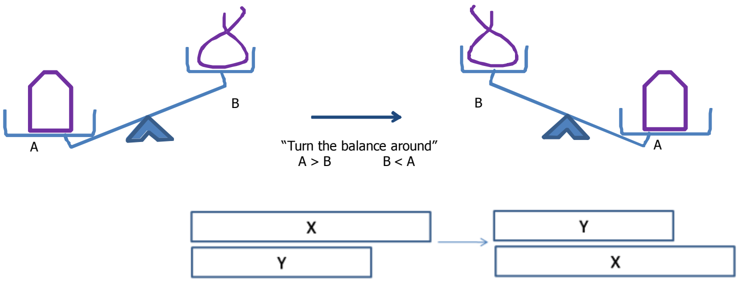
Order principles

Once again, effective strategies involve the students saying out loud the equalities, writing informal equations, and discussing generalities. Drawings now show how both balances with weights and paper strips can give students experiences with the order principles on which to base discussion.

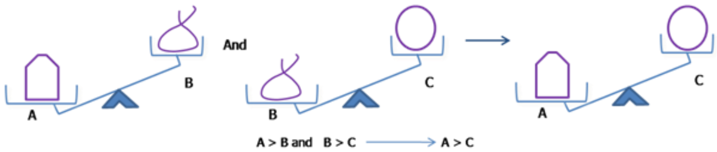
1. **Antisymmetry**. As for equivalence and symmetry, this can be seen by turning the balance 180° or changing the order of the strips.

A > B → B < A

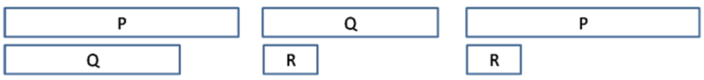
X > Y → Y < X



2. **Transitivity**. Once again the students compare three things in three different ways to show A=B and B=C means A=C.



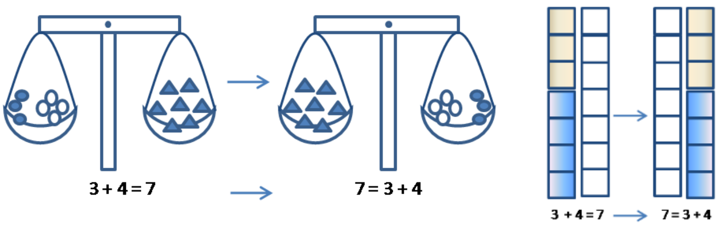
P > Q and Q > R → P > R



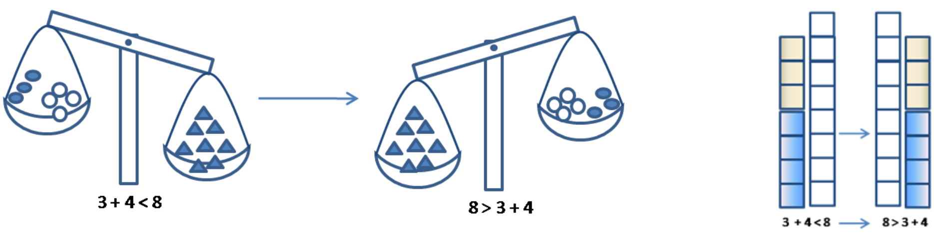
Mathematics

Practise the ideas above. Use the common weights and unifix cubes to introduce numbered justification for the principles, e.g. balance beams with numbers of weights (e.g. coat-hanger balances with red, blue and green baked bean cans) and lines of blocks. Undertake these activities so that students can record and experience equals and order as a basis for discussion and recognition of generalisations.

1. **Symmetry**. This is important as it shows that an equation can be reversed, e.g. 3+4=7 and 7=3+4 are the same. Symmetry is difficult for students used to "sums" to grasp so return to stories. Discuss that the following equations are allowed and what they mean, and discuss how to relate equations to stories and vice versa: 2 × 3 = 6, 6 = 2 × 3; 8 ÷ 4 = 2, 2 = 8 ÷ 4; 2 × 2 = 8 ÷ 2 and 8 ÷ 2 = 2 × 2. Stress the need to have understandings that cover stories with two equal components as well as stories with answers (e.g. There are six families that each have two cats, this makes the same number of cats as four families that each have three cats). This shows that 3 × 4 = 4 × 3 = 12. Get students to write their own same-value stories.



2. **Antisymmetry**. This extends symmetry in equivalence to order showing that when inequations are reversed the order changes from greater than to less than and vice versa.

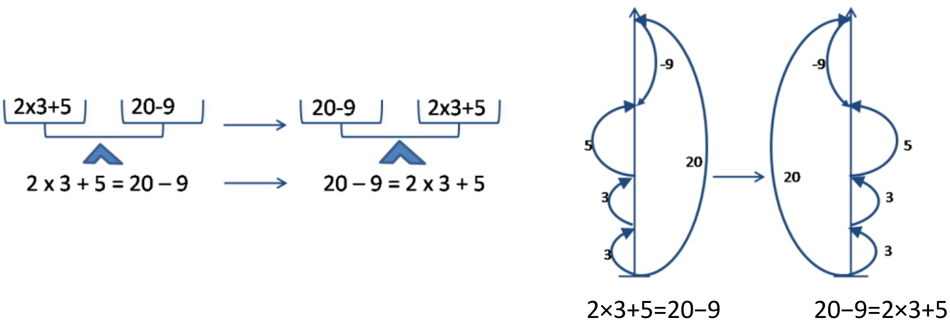


3. **Transitivity**. This is showing that if one thing is equal to/less than/greater than a second thing and this second thing is equal to/less than/greater than a third, then the first is equal to/less than/greater than the third.

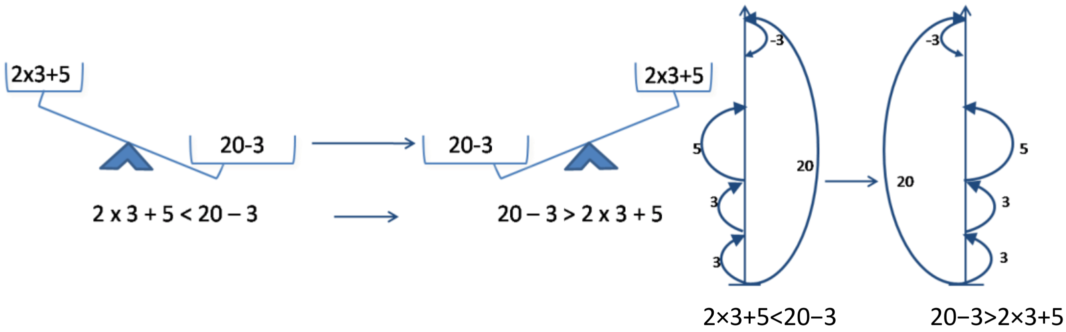
Reflection

There are two things to be done here. **The first** is to ensure that students can **take the understandings of equals and order to their world** – this should be pretty straightforward as most people understand that if John is taller than Sue, and Sue is taller than Jane, then John is taller than Jane, and Jane is shorter than John. **The second** is that we begin to **look at the principles in more abstract situations** using pictures of balances ("mathematical balances") and jumps along number lines (here vertical number lines).

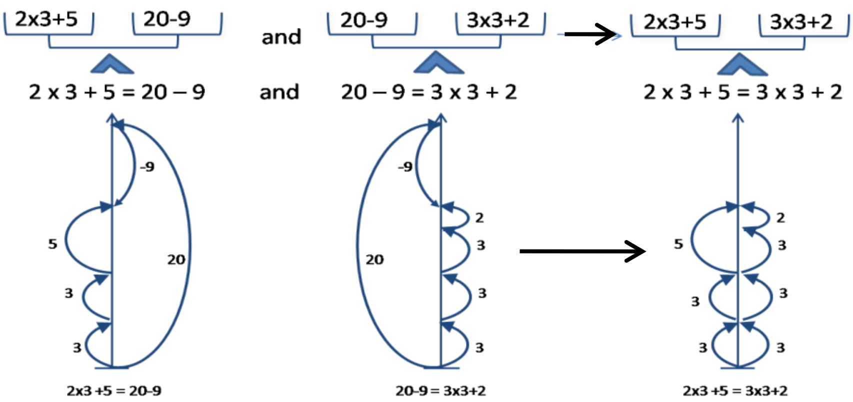
1. **Symmetry**.



2. **Antisymmetry**.



3. **Transitivity**.



Unit 2: Field Principles and Their Teaching

The number-size and equals/order principles are the first to be developed as they relate to equations (and even to "sums"). It is important to know that "equals" (=) means "same value as" and to understand larger and smaller from counting and very early operations. The number-size principles can be developed from exploring one-digit operations and, in fact, come from the meanings of the operations. Young children learn early that getting extra (adding) gives more, that taking a lot away (subtraction) leaves one with a little, that many bags of lollies (multiplication) is better than a few (if the bags contain the same), and that more sharing (division) means less for everyone.

After the two sets of principles above have been acquired, then the most important principles, the field principles, can begin to be taught. These principles hold for arithmetic and algebra, and right across P–12 and into university.

2.1 Field principles and their extensions

The **operation principles** or the **field principles** are properties of numbers and operations. Although there are four operations in school mathematics, all the field principles only work for addition and multiplication. Thus, addition and multiplication are the only real operations in mathematics – subtraction occurs when you add the additive inverse (5–2=5+(−2)) and division occurs when you multiply by the multiplicative inverse (12÷4=12×(1/4)).

The **relationship** principles are as follows (*** denotes principles which do not hold for subtraction or division).

1. **Closure** – numbers and operations (addition and multiplication) always give another number, that is, a+b is still a number and a×b is still a number for any numbers a and b. Examples are 2.17 + 4.3 = 6.47 which is still a number and 2.17 × 4.3 = 9.331 which is still a number. ***
2. **Identity** – there exists identity numbers, 0 for addition and 1 for multiplication, that do not change the number when added (e.g. a+0 = 0+a = a for any a) or when multiplied (e.g. a×1 = 1×a = a for any a).
3. **Inverse** – for any number a, there exists an additive inverse number −a which when added to a gives the identity 0, that is, a+(−a) = (−a)+a = 0, and a multiplicative inverse number a^{-1} or 1/a which when multiplied by a gives the identity 1, that is $a \times a^{-1} = a^{-1} \times a = 1$. Examples are the additive inverse of 7 is −7 and the multiplicative inverse of 5 is 5^{-1} or 1/5. This is because 7+(−7) = (−7)+7 = 0 and 7×1/7 = 1/7×7 = 1. (Please note that the additive identity 0 does not have an inverse in multiplication.)
4. **Associative** – numbers can be associated/added or multiplied in any way and this gives the same answer, that is, (a+b)+c=a+(b+c) and (a×b)×c=a×(b×c) for any a, b and c. Examples are 3+5+2 = 8+2 = 3+7 = 10, and 6×5×2 = 30×2 = 6×10 = 60. ***
5. **Commutative** – numbers can change order without changing the answer, that is, a+b = b+a and a×b = b×a for any a and b. Examples are 7+4 = 4+7 = 11 and 3×4 = 4×3 = 12. ***
6. **Distributive** – multiplication distributes across all additions, that is, a × (b + c) = (a × b) + (a × c) for any a, b and c. Example is 5 × (4+3) = (5×4) + (4×3).

These six field properties of operations have also given rise to a collection of principles that are based on the six but extend them. These **extension principles** are as follows.

1. **Compensation** – ensuring that a change is compensated so answer remains the same – related to inverse. Examples are: 5+5 = 7+3 (+2, −2); 47+25 = 50+22 (+3, −3); and 61−29 = 62−30 (+1, +1).
2. **Equivalence** – two expressions are equivalent if they relate by +0 and ×1; also related to inverse and similar to compensation. Examples are 48+25 = 48+25+0 = 48+25+2−2 = 50+23 = 73; 2/3 = 2/3×1 = 2/3×2/2 = 4/6.

3. **Inverse relation** – the higher the second number the smaller the result (and vice versa) for subtraction and division. Examples are 8−5=3 < 8−2 = 6 yet 5>2; 12÷2 = 6 > 12÷3 = 4 yet 2<3; 1/2 > 1/3 yet 2<3.

4. **Backtracking** – using inverse to reverse (backtrack) and solve problems. Example is $2y+3 = 11$. This means that y is the unknown and that y has been ×2 and +3 to total 11. Backtracking means that we −3 and ÷2. This means that $2y = 11−3 = 8$, and $y = 8÷2 = 4$.

2.2 Teaching the field principles with the RAMR model (3 RAMR cycles)

The way to teach the field principles is the same way as for the number-size principles: the **RAMR cycle** (reality → abstraction → mathematics → reflection) with attention to set and number-line models, and kinaesthetic activities. A powerful way to assist recall of these principles is to use patterns. This is best done with **calculator patterns**. The students are encouraged to see the patterns in the calculator activities that enable them to do the non-calculator activities without a calculator.

Below we give some ideas for teaching with the RAMR model. It is not complete – there is not the space to show in detail how to teach each principle, so **diagrams will show how models may be used** to teach them. In some cases, interesting and effective methods will be highlighted.

RAMR cycle 1: Identity and inverse principle

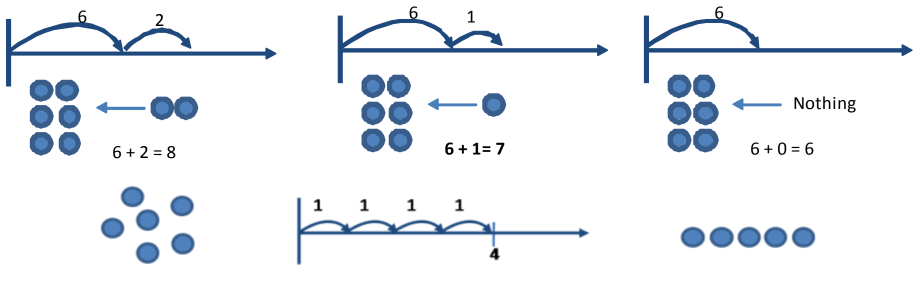
Reality

Discuss actions that leave things unchanged – do nothing, rotate 360 degrees, drive in a circle. Discuss actions that change a previous action back to no change – e.g. driving out of the garage, backing back into the garage, coming to school, going home from school, and so on.

Now look for identity and inverse situations involving addition and multiplication – e.g. getting no money for your Birthday (+0), making no profit (getting back what you put in (×1), borrowing $5 and returning $5 (+5−5=0), and getting $5 from each of your 4 friends and sharing $20 amongst your four friends (×4÷4=1).

Abstraction

For identity, show that +0 leaves numbers unchanged through sets and number lines. Show that ×1 leaves numbers unchanged through sets and number lines. That is, show that adding 0 and multiplying by 1 do not change anything. For addition, the best idea is to add 2, then add 1 and finally add 0. For multiplication, we have 1 group, row or jump of the number or a number of groups, rows and jumps of 1 (both of which equal the number).

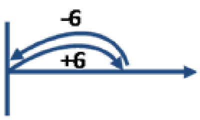


1 group of 6 (1×6=6) 4 jumps of 1 (4×1=4) 1 row of 5 (1×5=5)

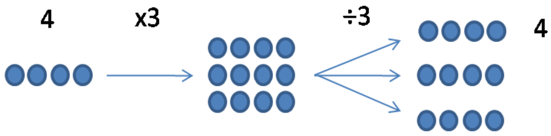
Note: This may be difficult for some students to understand and you may have to use patterns, for example:

5 + 3 = 8, 5 + 2 = 7, 5 + 1 = 6, 5 + 0 = ??.
7 × 4 = 28, 7 × 3 = 21, 7 × 2 = 14, 7 × 1 = 7, 7 × 0 = ??.

For inverse, show that addition and subtraction, and multiplication and division are inverses. Act out situations like: *Share 12 amongst 3, what do we get?* [4]. *Make 3 groups of this, what do we get?* [12]. Join and separate, rejoin and reseparate; make groups and share out groups, remake and reshare. Highlight that one action is the opposite and undoes the other (e.g. 7+2 = 9 and 9−2 = 7).



+6−6



1 row of 4×3 makes 12; 12 shared among 3 makes 4 each

Mathematics

Record the above activities with symbols. Validate with a calculator for large numbers. Revise with calculator patterns and, if required, do the patterns with materials and recordings.

Reflection

Go back into reality and find places where things are left unchanged – any of these ×1 or +0. Look also for where one operation undoes another operation.

Apply the identity principle and the inverse principle to: (a) introduce the compensation strategy for bridging 10 basic facts (e.g. 8+5 = 8+2+5−2 = 10+3 = 13) and algorithms; (b) introduce equivalent fractions (e.g. 2/3 × 1 = 2/3 × 2/2 = 4/6); (c) develop the *think addition* and *think multiplication* basic fact strategies for subtraction and division basic facts; (d) develop the idea of checking algorithms by doing the opposite or inverse calculation (e.g. adding to check subtraction); and (d) to introduce additive subtraction (e.g. 63−27 is 27+?=63, the ? is +3, +30, and +3 = 36) which is particularly useful for time.

RAMR cycle 2: Commutative principle ("turnarounds")

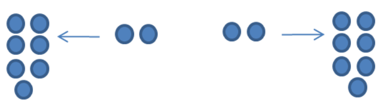
Reality

Look for things in the world where order does not matter; for example, two busloads – does it matter who gets off first? Does it matter if you have 4 bags with 3 chocolates in each or 3 bags with 4 chocolates in each bag?

Abstraction

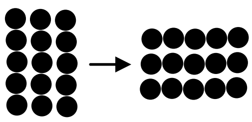
Show that order does not matter for addition and multiplication. This is achieved by showing two numbers added gives the same regardless of the order, and (highly recommended) turn an array by 90 degrees to show, e.g. that 3 rows of 4 is the same as 4 rows of 3.

For addition, show with objects (set model) that, for example, 2 joining 7 is the same as 7 joining 2. Show with number line that, for example, a hop of 7 followed by a hop of 2 is the same as 2 followed by 7 (see below). Validate this with calculator for any two numbers (e.g. 456 + 327 = 327 + 456). Explore whether this is true for subtraction.



7 joined by 2   2 joined by 7

Show with objects that 3 groups of 5 is the same as 5 groups of 3. Show with number line that 5 hops of 3 is the same as 3 hops of 5. **Show by rotating array 90 degrees** that 5 rows of 3 is 3 rows of 5 (as on right). Validate this with calculator for any two numbers (e.g. 456 × 327 = 327 × 456). Explore whether this is true for division.



Try to come up with general rules. Shut eyes and imagine two groups being joined – does it matter whether left moves to right or right moves to left?

Mathematics

Do all these activities with materials and record with symbols. Then use calculator patterns – this approach allows students to recognise the pattern by repeating the process. Calculators may be used to begin with to find the pattern but students should be continually looking for a pattern and then applying it when they can and moving to not requiring the calculator as soon as they are able. The final stage is formulating a generalised rule for each of the principles. This method can be done using a worksheet that students can work on individually or in groups. If students are not adept with this and cannot grasp the generalisations, step back to using set/number line models and activities with materials.

Example of a calculator patterns worksheet

Do these with calculators:

1. 2.52 + 3.8 =________; 3.8 + 2.52 =_________;
2. 34.9 + 7.86 =________; 7.86 + 34.9 =_______; and so on.

What pattern have you found? ____

Do these without calculators:

1. 46.4 + 3.92 = 50.32 ; 3.92 + 46.4 =_______;
2. 8.56 + 7.84 = 16.4 ; 7.84 + 8.56 =_______; and so on.

How could we write this as a general rule? _____________________________

Reflection

Look at where this can help us – adding/multiplying numbers by reversing order? Does it work for subtraction/division? Does it work for variables – where you can have any number?

Apply the principal to basic facts to reduce the number of facts to be learnt – also enables changes in algorithms (can add tens first or ones first – answer is the same).

RAMR cycle 3: Associative and distributive principles

Reality

Find something in the world where you can add/multiply with numbers in any order and from any position first, and it does not matter. Look at situations where you have lots of combined things; for example, we buy 4 snack packs each having 2 pieces of chicken, 3 wicked wings, a roll, and a small bag of chips – how many rolls?, how many pieces of chicken?, and so on.

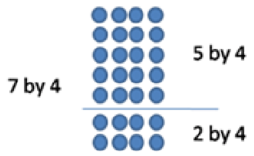
Abstraction

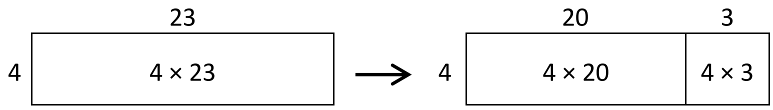
For associativity, show with objects and number lines that, for any three numbers, say 3, 5 and 6, it does not matter whether 3 and 5 or 5 and 6 are joined or hopped first, (3 + 5) + 6 always equals 3 + (5 + 6). Show with objects and number lines that, for any three numbers, say 3, 5 and 6, it does not matter whether 3 and 5 or 5 and 6 are joined or hopped first, (3 × 5) × 6 always equals 3 × (5 × 6). Imagine this in your mind.

It appears straightforward for addition (joining 3 sets so which sets join first is irrelevant – see below), but not so easy for multiplication because you need (3 groups of 4) groups of 5 objects to equal 3 groups of (4 groups of 5). This is more easily seen with calculation (and a calculator) than from the models.

$$3 + 4 + 5 = (3 + 4) + 5 = 3 + (4 + 5)$$

For distributivity, it is best done by breaking arrays, or the more abstract rectangles, into two parts.





Mathematics

Do all these activities with materials and record with symbols as you go. Validate principle with calculator for large numbers. Revise with calculator patterns and, if required, do the patterns with materials and recordings.

Reflection

Discuss whether this works for subtraction/division. What does this mean?

Apply the two principles to multiplication basic facts (e.g. $7 \times 8 = 5 \times 8 + 2 \times 8 = 40 + 16 = 56$) and algorithms. This principle is the basis of expansion and factorisation in algebra (see Unit 4 and also Module A4).

2.3 Teaching the extension field principles (RAMR cycle)

The method for teaching principles by the RAMR cycle used in section 2.2 has certain commonalities that may help in teaching other principles not covered. They are:

1. The **reality** is done by using examples of basic, obvious addition and multiplication contexts that the students will relate to.

2. Where possible, the **abstraction** phase should be done body → hand → mind using the set, number line and array models and ending with symbols and imagining in the mind.

3. The **mathematics** part of the cycle is to record using appropriate symbols – then look for patterns for the rule.

4. The **reflection** is the extension of the rule beyond whole numbers to decimals, fractions and letters. The rule should be generalised to an algebraic form. The other two operations subtraction and division should be explored.

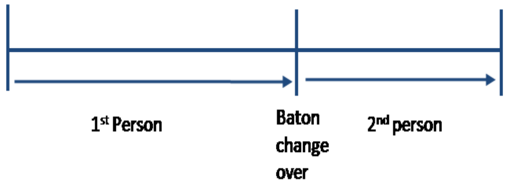
This can be used again for the extension field properties. Two examples are below.

RAMR cycle 4: Compensation and inverse relation principles

Reality

Look for things where if someone does more, another person does less.

This is one area where using kinaesthetic or whole body activity is useful. Students have difficulty looking at materials and pictures and seeing that, if increasing the 8 in $8 + 5 = 13$, it is necessary to decrease the 5 to keep 13 as the answer. However, one effective way to overcome this difficulty is to consider addition as a relay race in which one member does more than their share, and to act this out. Get students to form into pairs, mark out a relay walk (as on right) and a baton change and direct the pairs to walk the relay.



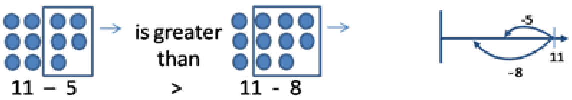
Discuss what would happen if the first person walked further (as on right) – what happens to the second person? Students can see that the second person has to walk less by the amount the first person walked more.

Abstraction

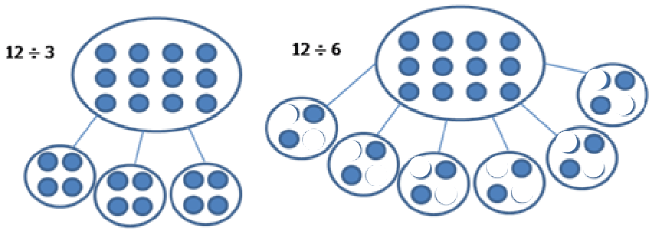
This is where we show that any two numbers have the same addition or multiplication if a change in one number is undone in the other number by use of inverse. It is difficult to show with models. In fact, it is sometimes more easily seen by discussion; for example, *Look at 2+3=5, what happens if 2 goes to 4, what if we want the sum to remain as 5?* However, models can be used as below.

This method of teaching appears to make it easier for students to understand the compensation property.

For the inverse relation principal, subtracting more and dividing by more decrease the computations, as can be seen in the examples below.



The more that is taken away, the less that is left.



"More shares, less each share"

Mathematics

Practice compensation and inverse relation, making sure everything is covered. For example, it should be noted that although the field/operation principles apply to addition and multiplication, there is also compensation for subtraction and division. However, it can be confusing for students because compensation for addition or multiplication is the inverse of the first change but compensation for subtraction or division is the same as the first change. For example, 8+5 = 10+3 because, for addition, 8 increased by 2 means that 5 should be decreased by 2; but 8–5 = 10−7 because, for subtraction, 8 increased by 2 means that 5 should also be increased by 2. Similarly, 12×4 = 6×8 because, for multiplication, 12 divided by 2 means that 4 should be multiplied by 2; but 12÷4 = 6÷2 because, for division, 12 divided by 2 means that 4 should also be divided by 2.

To avoid confusion either teach compensation for addition and multiplication, and subtraction and division separately, or place compensation under the inverse principle. Since addition and subtraction, and multiplication and division, are inverses, then it is reasonable that they would do the opposite with regard to the compensation principle. Since addition and multiplication compensation require the opposite change, it is reasonable that subtraction and division compensation require the opposite of opposite which is the same change.

This can also lead to the inverse relation principle – that sometimes more gives less.

Reflection

Let the students take their new knowledge and see if it makes sense in the world.

Apply the principles to basic facts, algorithms and estimation – compensation is a major computational strategy and inverse relation important for the getting closer strategy in estimation (see section 3.1). Extend compensation to equivalence of expressions.

Unit 3: Application to Estimation

Estimation is a method of finding approximate answers for large-number additions and subtractions. With a calculator for accurate answers where required, estimation is adequate for students' computation in terms of YDM. It is based on four strategies and the principles from Unit 2. In this unit, we will look at its strategies and its teaching, and show some practice activities.

3.1 Estimation strategies

There are four strategies for teaching estimation. They are described below. Like basic facts and algorithms, strategies are the most effective ways to teach anything because, in most cases, the strategies, once learnt, are useful in other areas.

1. **Front-end**. This strategy requires that only the highest place-value (PV) position(s) are considered in addition and subtraction to give a simple estimate that, sometimes, is not that accurate. However, it can be a good start. It is called front-end because one way to teach it is to put a card over all the sum but the "front-end" of the highest PV position(s). The strategy uses multiple-of-ten basic facts. Examples are:

   4 678 + 3 402 is found by 4 000 + 3 000 = 7 000.

   52 901 – 36 287 is found by 50 000 – 30 000 = 20 000.

   4 678 × 34 is found by 4 000 × 30 = 120 000.

   42 901 ÷ 362 is found by 42 000 ÷ 300 = 140.

2. **Rounding**. This strategy rounds the highest PV position(s) to the nearest of these PV positions and then adds and subtracts these numbers. It is a more accurate method than front end in most cases (but not all). The strategy uses multiple-of-ten basic facts. Examples are:

   4 678 + 3 402 is found by rounding to 5 000 + 3 000 = 8 000.

   52 901 – 34 287 is found by rounding to 50 000 – 30 000 = 20 000.

   4 678 × 34 is found by rounding to 5 000 × 30 = 150 000.

   42 901 ÷ 362 is found by rounding to 44 000 ÷ 400 = 110.

3. **Straddling**. This strategy determines, by rounding up and down, approximate answers that are less and more than the example. The strategy uses multiple-of-ten basic facts and the number-size principles (particularly the inverse relation principle). Examples are:

   4 678 + 3 402 is straddled by rounding up, 5 000 + 4 000 = 9 000, and rounding down, 4 000 + 3 000 = 7 000. So the addition is between 9 000 and 7 000. Looking at the numbers, they are both about halfway in between, so estimate is 8 000.

   52 901 – 34 287 is straddled by rounding up the first number and down the second, 60 000 – 30 000 = 30 000, and down the first and up the second, 50 000 – 40 000 = 10 000 (this is because of the inverse relation, i.e. bigger second number is a smaller answer). So the sum is between 30 000 and 10 000. Looking at the numbers, they are both nearer the 10 000 below, so estimate is 20 000.

   4 678 × 34 is straddled by rounding up, 5 000 × 40 = 200 000, and rounding down, 4 000 × 30 = 120 000. So the multiplication is between 200 000 and 120 000. Looking at the numbers, they seem to be about halfway in between, so estimate is 160 000.

42 901 ÷ 362 is straddled by rounding up the first number and down the second (but making the round up divisible by 3), 45 000 ÷ 300 = 150, and down the first and up the second (making the round divisible by 4), 40 000 ÷ 400 = 100. This is because of inverse relation for division, i.e. bigger second number means a smaller answer. So the estimation is between 150 and 100. Looking at the numbers, 42 901 changes by a similar amount in the round up and round down, but 362 changes more in the round down, so the higher straddle is less correct, so the estimate should be closer to the lower straddle, say 120.

4. **Getting closer**. This strategy follows one of the above and uses the number-size principles and multiple-of-ten basic facts to get a more accurate estimate. Examples are:

   The front-end strategy's estimate for 4 678 + 3 402 is 7 000 and is obviously too low as both numbers reduced, so a better estimate is about 1 100 more, or 8 100.

   The rounding estimate for 52 901 – 34 287 is 20 000 and is slightly high as the second number is rounded down an extra 1 500 (more should be subtracted) and so a more accurate estimate is 1 500 lower, or 18 500.

   The front-end strategy's estimate for 4 678 × 34 is 120 000 and is obviously low as both numbers are reduced, so a better estimate is about 160 000

   The rounding estimate for 42 901 ÷ 362 is 110. The rounding is 44 000 ÷ 400. Both numbers have increased, but the 362 has increased more (proportionately). So the answer is too low. A better estimate is 120.

3.2 Teaching estimation by strategies and principles (RAMR cycle)

Reality

Look at where in the world we need estimation – money, time, distance, and so on. The following are some money situations:

A. *You have to buy 3 groceries for $8.45, $5.60 and $4.25. Is $20 enough? If it is, estimate how much change.*

B. *You need $350. You get casual work for $17.56 per hour. Estimate how many hours you need to work.*

C. *The Council is paying $436 564 and $287 647 for building repairs. Estimate about how much this is in total. Will the $750 000 allocation cover it?*

D. *47 men were employed and their annual pay was $38 628 each. About how much will this cost?*

E. *The payout of $73 484 had to be shared amongst 67 people. Estimate how much each person got.*

Abstraction

Introduce the two-step process: (1) use one of the strategies *front end*, *rounding* and *straddling* to get a starting estimate; and (2) use *getting closer* to become more accurate.

Step 1. In turn, teach each of strategies *front end*, *rounding* and *straddling* separately. Check accuracy with calculator. Discuss with students the effectiveness of each strategy – its advantages and disadvantages. Make sure students see examples where the strategies are not that effective.

Provide a variety. Discuss which strategy works best for which calculations – front end needs numbers that are not too much bigger than the left place value, rounding is better but needs to be in terms of "nice numbers" for division (e.g. 3 768 ÷ 94 is best rounded to 3 600 ÷ 90), and straddling is good for multiplications.

Step 2. Use number-size principles to help with *getting closer* once you have done the first estimate. For the examples A to E:

A. Front end will give \$8 + \$5 + \$4 which is \$17 but this is inaccurate because all the front ends are less than the real amounts – we have to add at least another \$1. This can be seen in the comparison below.

Comparing	Original	\$8.45 + \$5.60 + \$4.25
	Estimate	\$8 + \$5 + \$4 = \$17

B. Rounding \$350 ÷ 17.56 is best done as 360 ÷ 18 = 20 and would be reasonably accurate as both numbers have been increased.

C. Rounding (\$400 000 + \$300 000 = \$700 000) would be better than front end (\$400 000 + \$200 000 = \$600 000) and could be reasonably accurate as one is rounded down and one is rounded up. If we were subtracting \$436 564 minus \$287 647 then front end would be better as rounding (\$400 000 – \$300 000 = \$100 000) is too low because the first number is rounded down and the second number is rounded up (direct and inverse relation principle). The "too-lowness" of front end for addition can be seen below:

Comparing	Original	\$436 564 + \$287 647
	Estimate	\$400 000 + \$200 000 = \$600 000

D. Straddling is good here to estimate 47 × \$38 628. It is less than 50 × 40 000 = 2 000 000 and more than 40 × 30 000 = 1 200 000. Because the numbers are closer to 50 and 40 000, the estimate would be closer to the high estimate, say 1 800 000.

Comparing	Estimate	50 × \$40 000 = \$2 000 000
	Original	47 × \$38 628
	Estimate	40 × \$30 000 = \$1 200 000

E. This one is hard to do in "nice numbers" – ordinary rounding of 70 000 ÷ 70 = 1000 would be best but it is too low as the first number is decreased (direct relation) and the second number is increased (inverse relation). Again this can be seen in the comparison below:

Comparing	Original	\$73 484 ÷ 67
	Estimate	\$70 000 ÷ 70 = 1000

Mathematics

Practise estimation. Use the activities/games in section 3.5. Always check with a calculator how close you are – and try to understand why you were inaccurate so you don't make the same error again.

Keep putting the Step 1 estimate beside the original exercise when using the Step 2 strategy *getting closer* to see which direction to go in getting a more accurate estimate. For example E above, we have:

Comparing	Original	\$73 484 ÷ 67
	Estimate	\$70 000 ÷ 70 = 1000

We can see that the estimate has less money and more people sharing so it is too low (direct and inverse relation principle for division). Trying a few examples and comparing with the exact answer from a calculator will give experience in how much to increase the estimate.

Reflection

Ask students to find situations in their world in which they can use estimation – make a poster "When guessing is good".

3.3 Estimation with fractions

Strategies. No matter the number, the estimation strategies of rounding, straddling, and getting closer still apply. However, front-end is not applicable.

Equivalence. The above strategies rely on being able to easily order fractions – so equivalence will be needed for unlike-denominator fractions. For example, $10\ ^2/_3 \times {}^5/_7 = {}^{32}/_3 \times {}^5/_7$ is easy to round to ${}^{32}/_3 \times {}^6/_8 = {}^{32}/_8 \times {}^6/_3 = 4 \times 2 = 8$. Is this too low or too high? Equivalent fractions show that ${}^6/_8 = {}^{42}/_{56}$ and ${}^5/_7 = {}^{40}/_{56}$, so ${}^6/_8$ is larger than ${}^5/_7$, so we are too high.

Benchmarking. For rounding, we need to know whether fractions are less than or greater than half. For example $7^3/_7$ is rounded to 7 for the nearest whole number because ${}^1/_2$ is $3^1/_2$ out of 7 whereas ${}^3/_7$ is 3 out of 7, therefore ${}^3/_7$ is smaller than ${}^1/_2$.

3.4 Estimating percent, rate and ratio calculations

For percent, rate and ratio, the important calculations are multiplication and division. As the following will show, this means that we need (a) good estimation of multiplication and division; and (b) an understanding of what percents, rates, and ratios mean in terms of the numbers they multiplicatively compare.

1. **Estimate first**. Always try to estimate the answer before calculating. For example, 52 × 1.64 is larger than 52 × 1 = 52 and smaller than 52 × 2 = 104. As 1.64 is a little over halfway between 1 and 2, then the estimate is a little over halfway between 52 and 104 (around about 80–85). As another example, 52 ÷ 1.64 would be a little under halfway between 52 ÷ 1 = 52 and 52 ÷ 2 = 26 (under because we are dividing by more than halfway between 1 and 2). (*Note:* This is using the straddling estimation strategy.)

2. **Inverse relation**. As some of the numbers can be decimals or fractions that are less than 1, students need to know that multiplying by a number less than 1 reduces the answer while dividing by a number less than 1 increases the answer. For example, 30% is 0.3 and 30% of $70 is 70 × 0.3 and this will be less than 70. Similarly 120 ÷ 0.3 will be larger than 120 (and more than double as 0.3 is less than half).

3. **Relate estimation to what percents, rates and ratios are**. For example, a profit of 40% will make a number larger, while a loss of 25% will make a number smaller. Similarly, red:yellow in ratio 4:7 means yellow is always nearly double red, and red is always just above half of yellow. Finally a rate of $1.64/L means the number of dollars is more than the number of litres and the number of litres is less than the number of dollars, while a rate of $0.45/kg means the number of dollars is less than the number of kg and the number of kg is more than the number of dollars.

4. **Realise difference between additive and multiplicative comparison**. Number multiplication is different and **faster acting** than addition. Eight being four more than four is the same as 21 being four more than 17, but eight being twice four means it requires 34 (instead of 21) to be twice 17.

3.5 Practising estimation using games

For expertise in estimation, it is necessary to practise because accurate estimation combines basic facts, multiple-of-ten facts, number-size principles, place value and rounding in situations where decisions have to be made on which of these prerequisites to use – a lot to bring together at the same time as making decisions. Thus familiarity through practice is needed to reduce cognitive load. There are a lot of games available to practise estimation – many use calculators.

Here are a few of them.

1. **Guestimate** – an estimation guessing game

2. **Tic Off** – a tic-tac-toe or noughts and crossed estimation game

3. **Target** – a game for estimating divisions

4. **Estimation Clouds** – an estimation game where you have to pick numbers to give required answers

5. **Estimation Hex** – using estimation to win a strategy game.

Estimation games

1. Guestimate!

This game practises the three strategies – it is suitable for one or many players/teams. It requires the teacher to make up a set of 9 estimation examples – 3 for front-end, 3 for rounding and 3 for straddling; finally, all 9 are made more accurate through getting closer.

1. Each player/team receives a game sheet.

2. Players/teams complete A, B and C using strategies then move to Getting Closer for a better solution.

3. Players/teams score 1 point if their answer has the same number of digits as the correct answer (use a calculator for this); 1 point if it has the same first digit; and 1 point if it is within 10% (use a calculator for this also).

A. FRONT-END

1. ____________________
2. ____________________
3. ____________________

B. ROUNDING

4. ____________________
5. ____________________
6. ____________________

C. STRADDLING

7. ____________________
8. ____________________
9. ____________________

GETTING CLOSER

1. ____________
2. ____________
3. ____________
4. ____________
5. ____________
6. ____________
7. ____________
8. ____________
9. ____________

2. Tic Off!

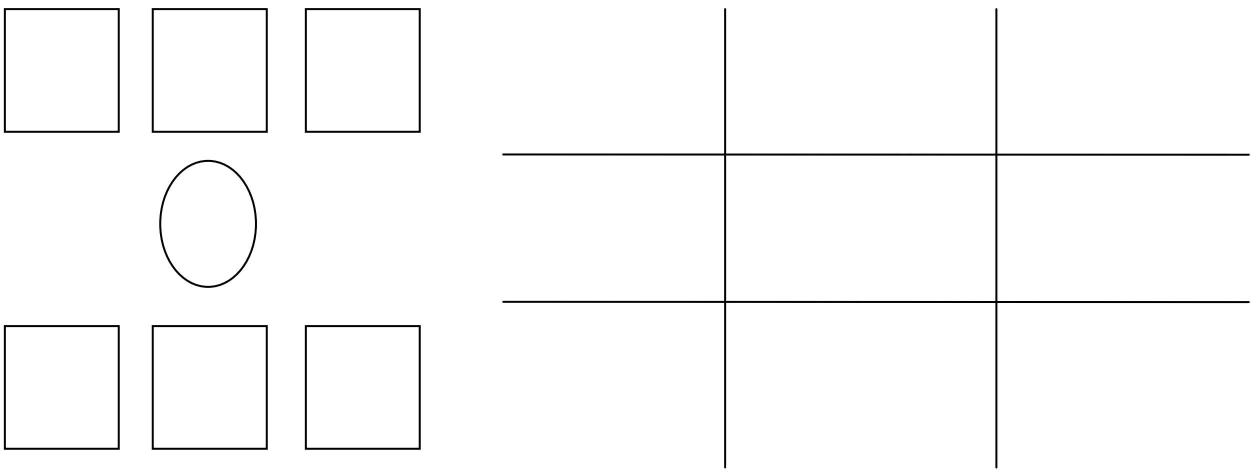
This game is suitable for two players/teams.

For addition and subtraction, this game requires the teacher to make up 6 numbers to be added or subtracted. Then + or – are put in the circle and the 6 numbers randomly in the boxes. This gives 9 additions or subtractions; they are calculated and put in the tic-tac-toe figure.

For multiplication and division it is the same – the teacher makes up 6 numbers to be multiplied (or divided – but this is harder as have to exactly divide) and puts them randomly in the boxes while × or ÷ is put in the circle. This gives 9 multiplications or divisions; they are calculated and put in the tic-tac-toe figure.

Two students can play on one sheet with a calculator.

1. Each player/team chooses a colour for their counters. Players/teams take turns in selecting two numbers – one from the top and one from the bottom.
2. Players/teams use a calculator to operate on the two numbers. Players/teams cover the answer on tic-tac-toe board below with their colour counter.
3. First player/team with three-in-a-row, wins.



3. Target

This game is suitable for one to many players/teams. Materials – calculators, table worksheet.

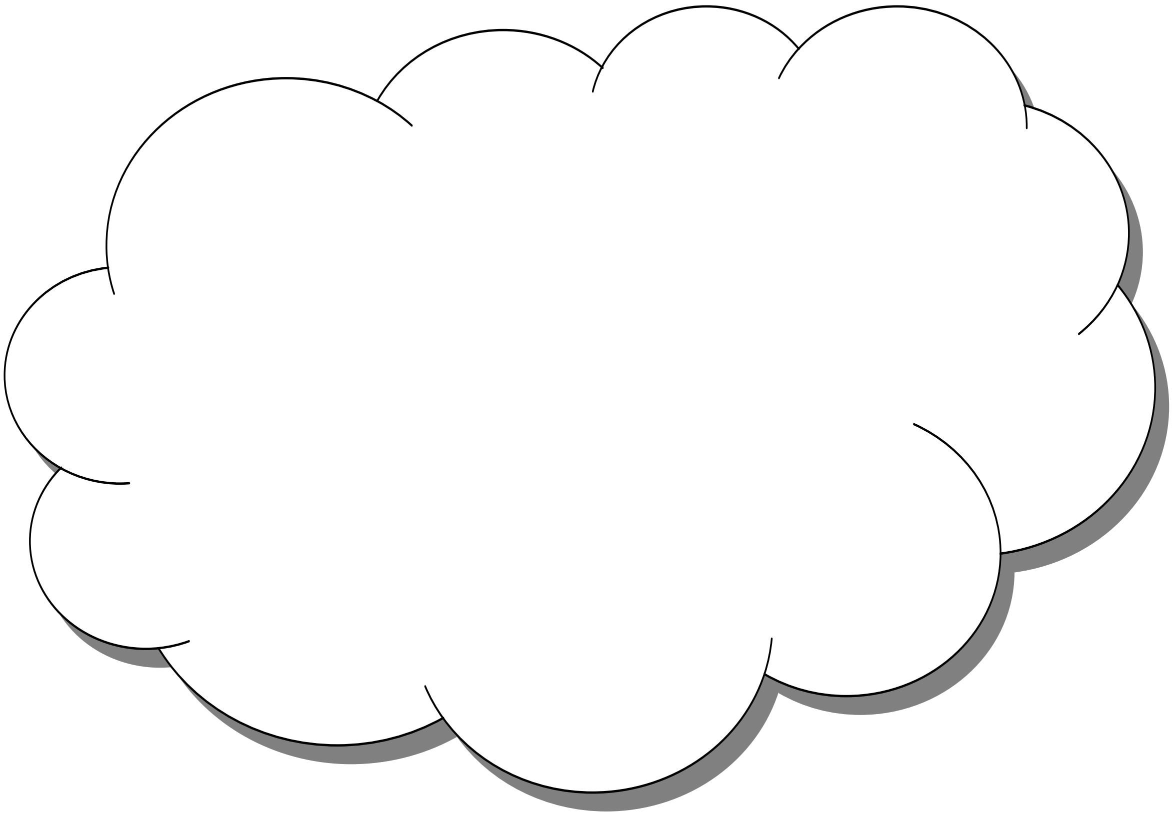
1. Players/teams enter the starting number in calculator and press ×. Players/teams guess what number multiplied by the starting number gives the target and type their guess then = into calculator.
2. Players/teams use whether their guess was too high or too low to make a better guess. Players record guesses in the "too high" or "too low" columns on chart below. Score is the number of guesses to get to the answer.
3. Player/team with least total guesses wins.

Start	Target	Too high	Too low	Correct guess	No. of guesses
37	1702				
41	14 227				

4. Estimation Clouds

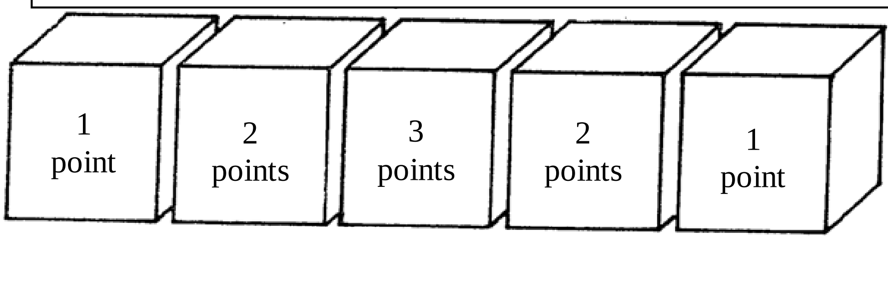
This game is suitable for two players/teams. Materials – calculators, cloud area, game board (boxes).

1. An operation is chosen, then numbers are chosen to suit the operation and so that the biggest multiplication is just under a multiple of 10. These numbers are randomly placed in the cloud. For example, the operation chosen could be multiplication and the cloud could be filled with two-digit numbers. In this case, the multiplication of any two of them ranges from 0 to 10 000.
2. The boxes below are numbered to match the multiple of 10. For the example of 0 to 10 000, the boxes are marked as below.



3. Players/teams take turns in selecting any two numbers from the cloud. Circle them. (They can be used only once.)
4. Using a calculator, players/teams multiply the two numbers, find the box for their answer, and keep track of their points. (The boxes are divided between 0 and 10 000, for this example, and given a point value.)
5. The winner is the player/team with the greater number of points after all numbers have been used.

0	2 000	4 000	6 000	8 000	10 000

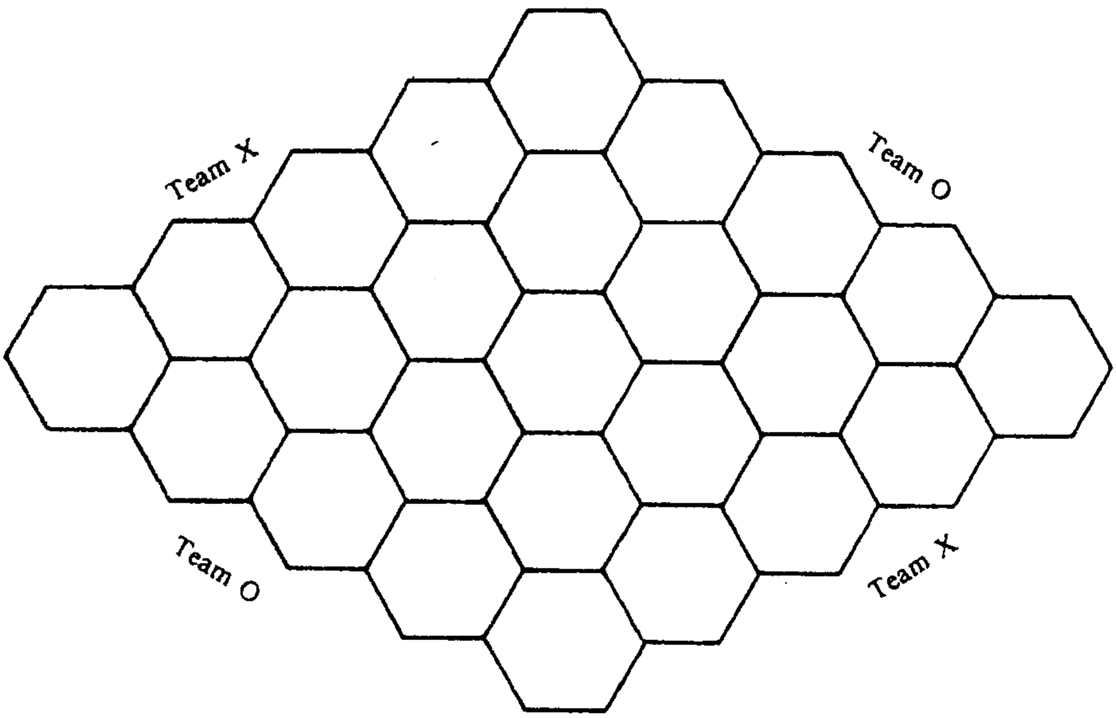


5. Estimation Hex

This game is suitable for two players/teams. Materials – calculators, game board (hexagons with answers in each space), and a collection of exercises in a box.

1. Each player/team chooses a colour for their counters. Players/teams take turns in selecting an exercise in a list – all answers on the Hex board have an exercise in the list but there can be extra exercises.
2. Using a calculator, players/teams calculate the exercise selected.
3. Players/teams look for answer on the game board, and cover it with their colour counter if it is there.
4. The first player/team to get a path across the game board wins.

COLLECTION OF EXERCISES



Unit 4: Extending Principles to Algebra

In the later years, the principles need to be reinforced for number and then applied to variables. This should be seen as an extension of number work and the basis of algebra work. This unit looks at how the arithmetic principles can be extended to various algebraic skills (simplification, expansion and numerical factorisation). This is done by using patterns of arithmetic, modelling variable and number and using the distributive principle.

Thus, in this unit, we extend the principles to algebra and introduce simplification and expansion within algebra by patterns from arithmetic. We use the knowledge about principles in arithmetic to extend the principles to algebra based on:

(a) variable being seen as any number and anything that is true for any number being seen as true in regard to variable; and

(b) using patterns to see that particular results lead to results for any number.

The unit is in two parts: the extension work, and then reinforcement work by checking with calculators.

In Module A4 *Algebraic Computation*, we will look at: (a) simplification by modelling variables and numbers with cups and counters; (b) expansion by extending the distributive law from arithmetic to algebra using the area model; and (c) reversing the process to introduce factorisation.

We focus on addition and multiplication in this unit, but the above ideas can also hold for division and subtraction. Subtraction situations can be undertaken similar to addition. However, division is more difficult and may require being approached in two ways:

1. Division is actually the inverse of multiplication. Expansion with division may need factorisation before the division can be completed. For example, $\frac{6a+9b}{3}$ cannot be immediately factorised. The numerator must first be factorised as follows, $6a + 9b$ factorises to $3(2a + 3b)$. Then the division can be completed so that $\frac{6a+9b}{3} = \frac{3(2a+3b)}{3} = 2a + 3b$. This approach requires division examples to be explored after students are comfortable with multiplication and addition/subtraction.

2. Division can also be taught as multiplication by reciprocal. In this way the multiplication ideas can be extended simply to cover division. For example, $\frac{6a+9b}{3} = {}^1/_3 \times (6a + 9b) = {}^1/_3 \times 6a + {}^1/_3 \times 9b = 2a + 3b$.

Note: This unit provides ideas for teaching but not in the form of RAMR cycles.

4.1 Extending principles using patterns

An effective way to show that the principles apply to algebra is to build the principles from arithmetic as follows (for two examples – identity and distributive). The method starts by looking at arithmetic examples, replaces the numbers with "any number", and finishes with a letter representing a variable. The examples of identity and distributive are provided. (*Note:* A more structured way to do this is given for the distributive law in section 4.2).

Identity

7 + 0 = 7	8 × 1 = 8
23 + 0 = 23	47 × 1 = 47

... and so on until students can see that it leads to

any number + 0 = any number	any number × 1 = any number
$y + 0 = y$	$y \times 1 = y$

Distributive

3 tens + 4 tens = 7 tens (3 + 4 = 7)	3 twos and 3 fives = 3 sevens (two + five = seven)
3 eights + 4 eights = 7 eights ...	3 elevens + 3 nines = 3 twenties (eleven + nine = twenty)

... and so on until students can see it leads to

3 any no. + 4 same no. = 7 same no.	3 any no. + 3 any other no. = 3 (any no. + same other no.)
3a + 4a = 7a	3a + 3b = 3 (a + b)

continuing in the same manner can lead to these other algebra findings:

$4x + 4y = 4\ (x + y)$
$12x + 12y = 12(x + y)$ ...
thus, $px + py = p(x + y)$ for any number p

The associative property also can be shown to hold for multiplication in the same way:	3 lots of 4 eights = 12 eights (12 = 3 × 4) 8 lots of 6 twelves = 48 twelves 4 lots of 5 anythings = 20 anythings, ... p lots of q anythings = ($p \times q$) anythings $p \times (qa)$ = $(p \times q)a$

4.2 Reinforcing principles in arithmetic using patterns

An effective way to reinforce the principles for numbers is to give students calculators and encourage them to explore properties (e.g. *Is the first number plus the second always equal to the second plus the first? Is a number multiplied by 1 always equal to itself?*). There are two steps to this process as the examples for the distributive principle and the symmetry principle show.

Step 1. Provide students with examples to check with the calculator. For the example of the **distributive principle**, this would mean activities like below.

Are these the same?	11 × 13 + 11 × 24 = ________; 11 × (13 + 24) = ___________
Check with a calculator!	23 × 27 + 23 × 56 = ________; 23 × (27 + 56) = ___________
	34 × 78 – 34 × 23 = ________; 34 × (78 − 23) = ___________
	and so on

For the example of the **symmetry principle** , students could be given examples to check as below. Examples should involve all operations.

Are these both correct/true? Yes/No	245 × 23 < 67 231 _______; 67 231 > 245 × 23 _______
Use your calculator!	and so on

Step 2. Give students examples to solve that require using the principle being checked. For the example of the **distributive principle**, this would mean activities like below. Students should be allowed to check by adding or subtracting the numbers in the brackets.

Calculate these without adding or	54 × (76 + 28) = ___________
subtracting the numbers in the brackets.	186 × (259 + 543) = ___________
You can use a calculator!	74 × (168 − 89) = ___________
	and so on

For the example of the **symmetry principle**, students could be given examples as below. Once again, the calculator should be used to check.

Use your calculator to determine the order in the first	49 × 53 ____ 2733; 2733 ____ 49 × 53
activity (<, > or =). Do the second without calculator!	54 × 58 ____ 29 × 109; 29 × 109 ____ 54 × 58
	and so on

For Step 2 in other field principles, students could be asked to calculate as follows.

(a) Commutative – calculate 345 + 672 another way, that is, without entering [345], [+] and [672] on their calculator in that order (answer – reverse the order).

(b) Associative – calculate 158 + 436 + 277 another way, that is without entering [158], [+], [436], [+], [277] in that order (answer – add the last two numbers first).

Note: The patterning activities in section 4.1 could be seen as abstraction, while these calculator-checking activities could be seen as mathematics in a RAMR cycle.

Test Item Types

This section presents instructions and the test item types for the subtests associated with the units. These will form the bases of the pre-test and post-test for this module.

Instructions

Selecting the items and administering the pre-post tests

This section provides an item bank of test item types, constructed around the units in the module. From this bank, items should be selected for the pre-test and post-test; these selected items need to suit the students and may need to be modified, particularly to make post-test items different to pre-test items. The purpose of the tests is to measure students' performance before and after the module is taught. The questions should be selected so that the level of difficulty progresses from easier items to more difficult items. In some modules this will follow the order of the units and subtests, and in other modules it will not, depending on the sequencing across the module and within units. The pre-test items need to allow for students' existing knowledge to be shown but without continual failure, and the post-test items need to cover all the sections in a manner that maximises students' effort to show what they can do.

In administering the pre-test, the students should be told that the test is not related to grades, but is to find out what they know before the topic is taught. They should be told that they are not expected to know the work as they have not been taught it. They should show what they know and, if they cannot do a question, they should skip it, or put "not known" beside questions. They will be taught the work in the next few weeks and will then be able to show what they know. Stress to students that **any pre-test is a series of questions to find out what they know** before the knowledge is taught. They should do their best but the important questions come at the end of the module. For the **post-test**, the students should be told that **this is their opportunity to show how they have improved**.

For all tests, **teachers should continually check to see how the students are going.** Items in later subtests, or more difficult items within a particular subtest, should not be attempted if previous similar items in earlier subtests show strong weaknesses. Students should be allowed to skip that part of the test, or the test should be finished. Students can be marked zero for these parts.

Information on the arithmetic and algebra principles item types

The arithmetic and algebra principles item types have been divided into four subtests, one for each of the units in this module. Units 1 and 2 introduce the major principles that hold across arithmetic and algebra – number-size, equals and order principles, and field principles (closure, identity, inverse, commutativity, associativity, distributivity). Unit 3 applies these principles to estimation and Unit 4 extends them to algebra. Therefore, although the four units cover different topics, they are reasonably in sequence.

Consequently, the pre-test can begin with Subtest 1 items and move on to the other subtests in sequence, and the post-test should cover all subtest item types. It is important that sufficient content is included in the pre-test to ensure that: (a) teaching begins where students are at; (b) what is missed out is because the students cannot answer the questions; and (c) the pre-test provides both achievement level and diagnostic information. It is also important that sufficient content is included in the post-tests to ensure that: (a) what is not included is because students can do this; (b) what is included will give the level of achievement at the end of the module; and (c) legitimate comparisons can be made between pre- and post-tests in terms of effect. Finally, always read the questions to the students and explain any contextual information as long as this does not direct to the answers.

Subtest item types

Subtest 1 (Unit 1: Number-size, equals and order principles)

1. 8↑ means 8 increases

   6↓ means 6 decreases

   11↔ means 11 stays the same

   Which of the following are correct? Tick the correct ones.

   (a) 6↑ + 5↔ = 11↑ _____

   (b) 8↔ − 3↑ = 5↓ _____

   (c) 8↑ − 3↓ = 5↔ _____

   (d) 7↔ + 4↓ = 11↔ _____

   (e) 3↑ × 8↔ = 24↔ _____

   (f) 12↔ ÷ 4↑ = 3↔ _____

   (g) 7↑ × 6↔ = 42↓ _____

   (h) 32↓ ÷ 8↑ = 4↓ _____

2. Tick the equations that are correct.

   (a) 7 + 8 = 15 _____

   (b) 3 × 8 = 24 − 2 _____

   (c) 36 = 9 × 4 _____

   (d) 27 ÷ 3 = 9 _____

   (e) 5 + 3 = 8 + 2 _____

   (f) 16 – 2 = 7 × 2 _____

3. With regard to the **number-size** principles:

   (a) What is the same about + and ×? ____

   (b) What is different between × and ÷? ______________________________________

   (c) What is the same about – and ÷? ____

4. With regard to the **equals** principles:

   (a) What is transitivity? ____

   (b) What is symmetry? ____

   (c) How is order different to equals with regard to symmetry? ______________________

   ____

Subtest 2 (Unit 2: Field principles and their teaching)

1. What are the following (give an example if you can)?

   (a) Inverse: ____

   ____

   (b) Commutative principle: ____

   ____

   (c) Distributive principle: ____

   ____

2. How are identity and inverse related? ____

   ____

3. What are the following (give an example if you can)?

   (a) Compensation principle: ____

   ____

   (b) Inverse relation principle: ____

   ____

   (c) Equivalence principle: ____

   ____

4. Which of the following are correct? Tick the correct ones (do not calculate).

   (a) 38 + 23 = 40 + 21 _____

   (b) (21 + 35) ÷ 7 = (21 ÷ 7) + 35 _____

   (c) 82 – 27 = 80 + 29 _____

   (d) 67 × 8 = 60 × 8 + 7 × 8 _____

   (e) 4387 ÷ 7 < 4387 ÷ 9 _____

   (f) 4378 – 2491 < 4178 – 2291 _____

   (g) 45 × 63 > 45 × 7 × 9 _____

   (h) 18 × (536–38) = 18 × 536 – 18 × 38 _____

Subtest 3 (Unit 3: Application to estimation)

1. Consider 3 687 + 4 502. Do not calculate it.

   (a) What is the **front-end** strategy for estimation? ________________________________

   ____

   (b) Apply the front-end strategy to the example to get an estimate. Show your working.

   (c) Is the estimate too low or too high? _____________________

   (d) What is the "**getting closer**" strategy? ______________________________________

   ____

   (e) Use "getting closer" to make a better estimate. Show your working.

2. Consider 58 621 ÷ 78. Do not calculate it.

   (a) What is the **rounding** strategy for estimation? ________________________________

   ____

   (b) Apply the rounding strategy to the example to get an estimate. Show your working.

   (c) Is the estimate too low or too high? _____________________

   (d) Use "getting closer" to make a better estimate. Show your working.

3. Consider 37 × 86. Do not calculate it.

(a) What is the **straddling** strategy for estimation? ________________________________

____

(b) Apply the straddling strategy to the example to get an estimate. Show your working.

(c) Now use "getting closer" to make a better estimate. Show your working.

Subtest 4 (Unit 4: Extending principles to algebra)

1. Use patterns to extend the following from arithmetic to algebra. Show all working and give an algebra example for each principle. An example has been done for you.

Example

Additive inverse:
2 + − 2 = 0
5 + − 5 = 0
21 + − 21 = 0
any number + − same number = 0
x + − x = 0

(a) Identity:

(b) Commutative principle:

2. What principle(s) is/are the following based on? Show how they work for arithmetic and translate to algebra. Show all the steps.

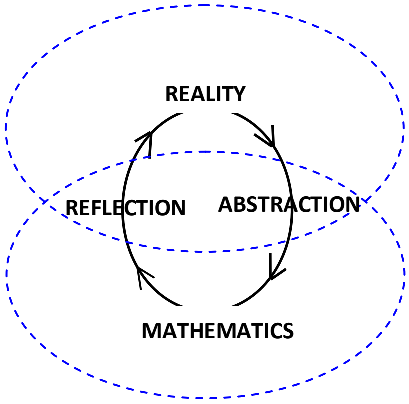
(a) $4y + 2y = 6y$

(b) $a \times \frac{1}{a} = 1$

(c) $(a \times b) \times c = c \times (a \times b)$

(d) $3x \times 2 = 6x$

Appendix A: RAMR Cycle



AIM advocates using the four components in the figure on right, reality–abstraction–mathematics–reflection (RAMR), as a cycle for planning and teaching mathematics. RAMR proposes: (a) working from reality and local culture (prior experience and everyday kinaesthetic activities); (b) abstracting mathematics ideas from everyday instances to mathematical forms through an active pedagogy (kinaesthetic, physical, virtual, pictorial, language and symbolic representations, i.e. body → hand → mind); (c) consolidating the new ideas as mathematics through practice and connections; and (d) reflecting these ideas back to reality through a focus on applications, problem solving, flexibility, reversing and generalising (see figure on right).

The innovative aspect of RAMR is that the right half develops the mathematics idea while the left half reconnects it to the world and extends it. For example, whole-number place value built around the **pattern of threes** where hundreds-tens-ones applies to ones, thousands, millions, and so on, can be easily extended to metrics by considering the ones to be millimetres, the thousands to be metres and the millions to be kilometres.

Planning the teaching of mathematics is based around the RAMR cycle if it is deconstructed into components that are applied to a mathematical idea. By breaking instruction down into the four parts and taking account of the pedagogical approaches described above, the cycle can lead to a structured instructional sequence for teaching the idea. The table below briefly outlines how this can be done. Prerequisite mathematical ideas are considered in the Reality and Mathematics components of the cycle, while extensions and follow-up ideas are considered in the Reflection component.

REALITY

- **Local knowledge**: Identify local student cultural-environmental knowledge and interests that can be used to introduce the idea.
- **Prior experience**: Ensure existing knowledge and experience prerequisite to the idea is known.
- **Kinaesthetic**: Construct kinaesthetic activities, based on local context, that introduce the idea.

ABSTRACTION

- **Representation**: Develop a sequence of representational activities (physical to virtual to pictorial materials to language to symbols) that develop meaning for the mathematical idea.
- **Body-hand-mind**: Develop two-way connections between reality, representational activities, and mental models through body → hand → mind activities.
- **Creativity**: Allow opportunities to create own representations, including language and symbols.

MATHEMATICS

- **Language/symbols**: Enable students to appropriate and understand the formal language and symbols for the mathematical idea.
- **Practice**: Facilitate students' practice to become familiar with all aspects of the idea.
- **Connections**: Construct activities to connect the idea to other mathematical ideas.

REFLECTION

- **Validation**: Facilitate reflection of the new idea in terms of reality to enable students to validate and justify their new knowledge.
- **Applications/problems**: Set problems that apply the idea back to reality.
- **Extension**: Organise activities so that students can extend the idea (use reflective strategies – *flexibility*, *reversing*, *generalising*, and *changing parameters*).

Appendix B: AIM Scope and Sequence

Yr	Term 1	Term 2	Term 3	Term 4
A	**N1: Whole Number Numeration** Early grouping, big ideas for H-T-O; pattern of threes; extension to large numbers and number system **N2: Decimal Number Numeration** Fraction to decimal; whole number to decimal; big ideas for decimals; tenths, hundredths and thousandths; extension to decimal number system	**O1: Addition and Subtraction for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M1: Basic Measurement (Length, Mass and Capacity)** Attribute; direct and indirect comparison; non-standard units; standard units; applications	**O2: Multiplication and Division for Whole Numbers** Concepts; strategies; basic facts; computation; problem solving; extension to algebra **M2: Relationship Measurement (Perimeter, Area and Volume)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae	**G1: Shape (3D, 2D, Line and Angle)** 3D and 2D shapes; lines, angles, diagonals, rigidity and properties; Pythagoras; teaching approaches **SP1: Tables and Graphs** Different tables and charts; bar, line, circle, stem and leaf, and scatter graphs; use and construction
B	**M3: Extension Measurement (Time, Money, Angle and Temperature)** Attribute; direct and indirect comparison; non-standard units; standard units; applications and formulae **N3: Common Fractions** Concepts and models of common fractions; mixed numbers; equivalent fractions; relationship to percent, ratio and probability	**G2: Euclidean Transformations (Flips, Slides and Turns)** Line-rotation symmetry; flip-slides-turns; tessellations; dissections; congruence; properties and relationships **O3: Common and Decimal Fraction Operations** Addition, subtraction, multiplication and division of common and decimal fractions; models, concepts and computation	**A1: Equivalence and Equations** Definition of equals; equivalence principles; equations; balance rule; solutions for unknowns; changing subject **N4: Percent, Rate and Ratio** Concepts and models for percent, rate and ratio; proportion; applications, models and problems	**SP2: Probability** Definition and language; listing outcomes; likely outcomes; desired outcomes; calculating (fractions); experiments; relation to inference **G3: Coordinates and Graphing** Polar and Cartesian coordinates; line graphs; slope and y-intercept; distance and midpoints; graphical solutions; nonlinear graphs
C	**A2: Patterns and Linear Relationships** Repeating and growing patterns; position rules, visual and table methods; application to linear and nonlinear relations and graphs **N5: Directed Number, Indices and Systems** Concept and operations for negative numbers; concept, patterns and operations for indices; scientific notation and number systems	**A3: Change and Functions** Function machine; input-output tables; arrowmath notation, inverse and backtracking; solutions for unknowns; model for applications to percent, rate and ratio **G4: Projective and Topology** Visualisation; divergent and affine projections; perspective; similarity and trigonometry; topology and networks	**O4: Arithmetic and Algebra Principles** Number-size, field and equivalence principles for arithmetic; application to estimation; extension to algebra; simplification, expansion and factorisation **SP3: Statistical Inference** Gathering and analysing data; mean, mode, median, range and deviation; box and whisker graphs; large data sets, investigations and inferences	**A4: Algebraic Computation** Arithmetic to algebra computation; modelling-solving for unknowns; simultaneous equations, quadratics **O5: Financial Mathematics** Applications of percent, rate and ratio to money; simple and compound interest; best buys; budgeting and planning activities

Key: N = Number; O = Operations; M = Measurement; G = Geometry; SP = Statistics and Probability; A = Algebra.

Accelerated Inclusive Mathematics Project

YuMiDeadly
Growing community through education

Faculty of Education
School of Curriculum
S Block, Room 404
Victoria Park Road
KELVIN GROVE QLD 4059

CRICOS No. 00213J

Phone: +61 7 3138 0035
Fax: +61 7 3138 3985
Email: ydc@qut.edu.au
Website: ydc.qut.edu.au