CHAPTER 1

GRAPH THEORY

1 Graphs and Graph Models

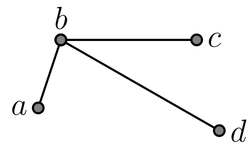
Definition 1.1
A graph (رسم) $G = (V, E)$ *is a structure consisting of a set* V *of vertices* (رؤُوس) *(also called nodes), and a set* E *of edges* (أضلَاع)*, which are lines joining vertices. Each edge has either one or two vertices associated with it, called its endpoints. An edge is said to connect its endpoints. If the edge* e *links the vertex* a *to the vertex* b*, we write* $e = \{a, b\}$*.*
The order of a graph $G = (V, E)$ *is the cardinality of its vertex set, and the size of a graph is the cardinality of its edge set.*

There is several type of graphs, (undirected, directed, simple, multigraph,...) have different formal definitions, depending on what kinds of edges are allowed.

Definition 1.2

1. *A simple graph* (رسم بسيط) G *is a graph that has no loops* (عروَات)*, (that is no edge* $\{a, b\}$ *with* $a = b$*) and no parallel edges between any pair of vertices.*
2. *A multigraph* G *is a graph that has no loop and at least two parallel edges between some pair of vertices.*

1.1 Simple Undirected Graph (رسم بسيط غير موجه)

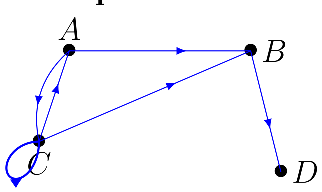


Only undirected edges, at most one edge between any pair of distinct nodes, and no loops.

1.2 Directed Graph (Digraph) (with loops)

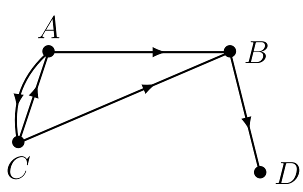
Definition 1.3
A directed graph (digraph) , $G = (V, E)$, consists of a non-empty set, V, of vertices (or nodes), and a set $E \subset V \times V$ of directed edges (or ordered pairs). Each directed edge $(a, b) \in E$ has a start (tail) vertex a, and a end (head) vertex b.
a is called the initial vertex (اَلرَأس اَلإِبتِدَائِي) and b is the terminal vertex (اَلرَأس اَلنِهَائِي).
Nőte: a directed graph $G = (V, E)$ is simply a set V together with a binary relation E on V.

Example 1 :



Only directed edges, at most one directed edge from any node to any node, and loops are allowed.

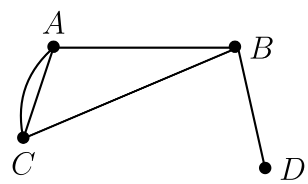
1.3 Simple Directed Graph



Only directed edges, at most one directed edge from any node to any other node, and no loops allowed.

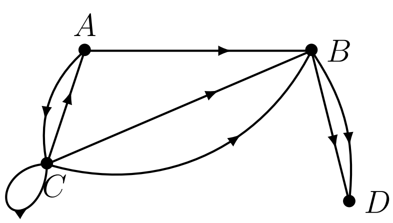
1.4 Undirected Multigraph

Definition 1.4
A (simple,undirected) multigraph, $G = (V, E)$, consists of a non-empty set V of vertices (or nodes), and a set $E \subset [V]^2$ of (undirected) edges, but no loops.



Only undirected edges, may contain multiple edges between a pair of nodes, but no loops.

1.5 Directed Multigraph



Only directed edges, may contain multiple edges from one node to another, the loops are allowed.

1.6 Graph Terminology

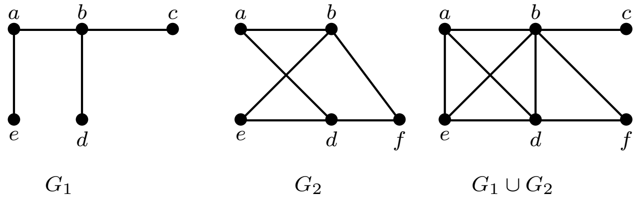
Graph Terminology

	Type	Edges	Multi-Edges	Loops
1	(Simple undirected) graph	Undirected	No	No
2	(Undirected) multigraph	Undirected	Yes	No
3	(Undirected) pseudograph	Undirected	Yes	Yes
4	Directed graph	Directed	No	Yes
5	Simple directed graph	Directed	No	No
6	Directed multigraph	Directed	Yes	Yes
8	Mixed graph	Both	Yes	Yes

1.7 New Graphs

Definition 1.5
The union of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the simple graph with vertex set $V_1 \cup V_2$ and edge set $E_1 \cup E_2$. The union of G_1 and G_2 is denoted by $G_1 \cup G_2$.

Example 2 :



2 Degree and neighborhood of a vertex

Remark 2.1
The set of vertices V of a graph G may be infinite. A graph with an infinite vertex set or an infinite number of edges is called an infinite graph, and in comparison, a graph with a finite vertex set and a finite edge set is called a finite graph.
In this course we will consider only finite graphs.

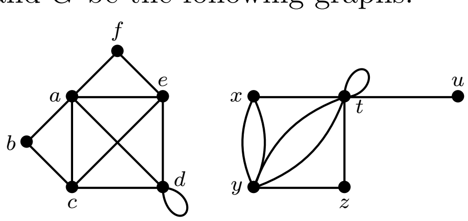
Definition 2.2
Two vertices a, b in a graph G are called adjacent (متجَاورة) in G if $\{a, b\}$ is an edge of G. If $e = \{a, b\}$ is an edge of G, then e is called incident with the vertices a and b or e connects a and b.

2.1 Degree and neighborhood of a vertex

Definition 2.3
The degree of a vertex a in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex. The degree of the vertex a is denoted by $\deg(a)$.

Definition 2.4
The neighborhood (neighbor set) of a vertex a in an undirected graph, denoted $N(a)$ is the set of vertices adjacent to a.

Example 3 :
Let F and G be the following graphs:



The degrees of the vertices in the graphs F and G are respectively: $\deg(a) = 5$, $\deg(b) = 2$, $\deg(c) = 4$, $\deg(d) = 5$, $\deg(e) = 4$, $\deg(f) = 2$.
$\deg(x) = 3$, $\deg(y) = 5$, $\deg(z) = 2$, $\deg(t) = 7$, $\deg(u) = 1$.

$N(a) = \{b, c, d, e, f\}$, $N(b) = \{a, c\}$, $N(c) = \{a, b, d, e\}$. $N(d) = \{a, c, e\}$, $N(e) = \{a, c, d, f\}$, $N(f) = \{a, e\}$.
$N(x) = \{y, z, t\}$, $N(y) = \{x, z, t\}$, $N(z) = \{x, y, t\}$, $N(t) = \{x, y, z, t, u\}$, $N(u) = \{t\}$.

Definition 2.5
For any graph G,we define

$$\delta(G) = \min\{\deg v;\ v \in V(G)\}$$

and

$$\Delta(G) = \max\{\deg v;\ v \in V(G)\}.$$

If all the points of G have the same degree r, then $\delta(G) = \Delta(G) = r$ and in this case G is called a regular graph of degree r.
A regular graph of degree 3 is called a cubic graph.

2.2 Handshaking Theorem

Theorem 2.6 (Handshaking Lemma)
If $G = (V, E)$ is a undirected graph with m edges, then:

$$2m = \sum_{a \in V} \deg(a).$$

Proof
Each edge contributes twice to the degree count of all vertices. Hence, both the left-hand and right-hand sides of this equation equal twice the number of edges.

Corollary 2.7
Every cubic graph has an even number of points.

Proof
Let G be a cubic graph with p points, then $\sum_{v \in V} \deg(v) = 3p$ which is even by Handshaking Theorem. Hence p is even.

Corollary 2.8
An undirected graph has an even number of vertices of odd degree.

Proof
Let V_1 be the vertices of even degree and V_2 be the vertices of odd degree in graph $G = (V, E)$ with m edges. Then

$$2m = \sum_{a \in V_1} \deg(a) + \sum_{a \in V_2} \deg(a).$$

$$\sum_{a \in V_1} \deg(a) \text{ must be even since } \deg(a) \text{ is even for each } a \in V_1.$$

$\sum_{a \in V_2} \deg(a)$ must be even because $2m$ and $\sum_{a \in V_1} \deg(a)$ are even.

Example 4 :
Every graph has with at least two vertices contains two vertices of equal degree.

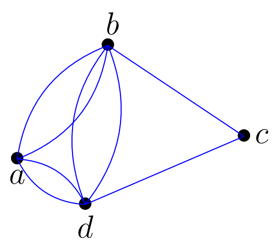
Suppose that the all n vertices have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be $\{0, 1, 2, \ldots, n-2, n-1\}$.

But that's not possible, because the vertex with degree $n-1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

Example 5 :
If a graph has 7 vertices and each vertices have degree 6. The nombre of edges in the graph is 21. $(6 \times 7 = 42 = 2m = 2 \times 21)$.

Example 6 :
There is a graph with four vertices a, b, c, and d with $\deg(a) = 4$, $\deg(b) = 5 = \deg(d)$, and $\deg(c) = 2$.

The sum of the degrees is $4 + 5 + 2 + 5 = 16$. Since the sum is even, there might be such a graph with $\frac{16}{2} = 8$ edges.

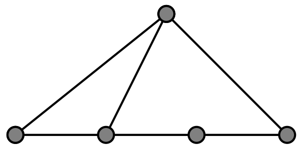


Example 7 :
A graph with 4 vertices of degrees $1, 2, 3$, and 3 does not exist because $1+2+3+3 = 9$ (The Handshake Theorem.)
Also there is not a such graph because, there is an odd number of vertices of odd degree.

Example 8 :
For each of the following sequences, find out if there is any graph of order 5 such that the degrees of its vertices are given by that sequence. If so, give an example.

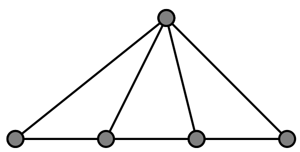
1. $3, 3, 2, 2, 2$
2. $4, 4, 3, 2, 1$.
3. $4, 3, 3, 2, 2$.
4. $3, 3, 3, 2, 2$.
5. $3, 3, 3, 3, 2$.
6. $5, 3, 2, 2, 2$.

1. $3, 3, 2, 2, 2$



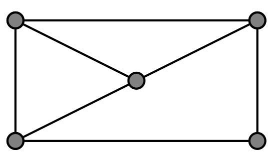
2. $4, 4, 3, 2, 1$. It does not exist. (One vertice v_1 which has degree 4, then there is one edge between v_1 and the others vertices. Also there is an other vertice v_2 which has degree 4, then there is one edge between v_2 and the others vertices. Then the minimum of degree is 2 and not 1).

3. $4, 3, 3, 2, 2$.



4. It does not exist. (The number of vertives with odd edges is odd).

5. $3, 3, 3, 3, 2$.

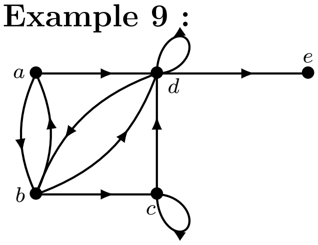


6. $5, 3, 2, 2, 2$. It does not exist. (The order is 5 and one vertive has degree 5).

2.3 Directed Graphs

Definition 2.9
The in-degree of a vertex a, denoted $\deg^-(a)$, is the number of edges directed into a. The out-degree of a, denoted $\deg^+(a)$, is the number of edges directed out of a. Note that a loop at a vertex contributes 1 to both in-degree and out-degree.

Example 9 :



In the graph we have: $\deg^-(a) = 1$, $\deg^+(a) = 2$, $\deg^-(b) = 2$, $\deg^+(b) = 3$, $\deg^-(c) = 2$, $\deg^+(c) = 2$, $\deg^-(d) = 4$, $\deg^+(d) = 3$, $\deg^-(e) = 1$, $\deg^+(e) = 0$.

Theorem 2.10
Let $G = (V, E)$ be a directed graph. Then:

$$|E| = \sum_{v \in V} \deg^-(v) = \sum_{v \in V} \deg^+(v).$$

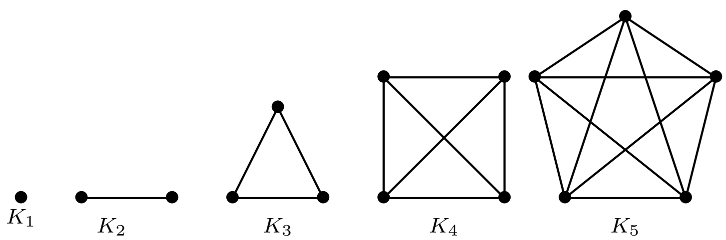
Proof The first sum counts the number of outgoing edges over all vertices and the second sum counts the number of incoming edges over all vertices. Both sums must be $|E|$.

3 Special Types of Graphs

Definition 3.1
A null graph (or totally disconnected graph) is one whose edge set is empty. (A null graph is just a collection of points.)

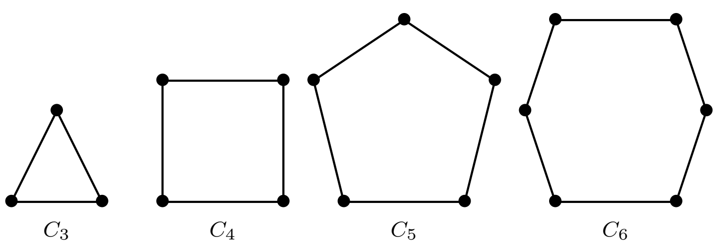
3.1 Complete Graphs

A complete graph on n vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



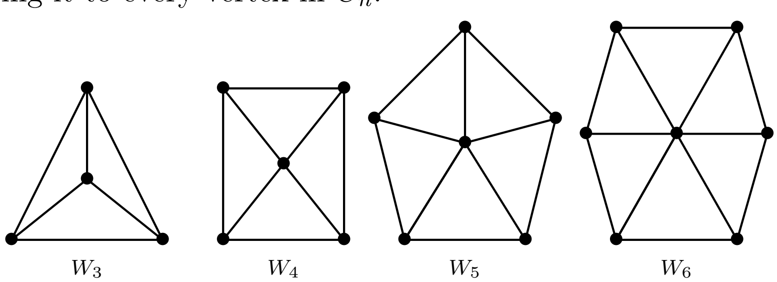
3.2 Cycles

A cycle for $n \geq 3$ consists of n vertices v_1, v_2,..., v_n, and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$,..., $\{v_{n-1}, v_n\}$, $\{v_n, v_1\}$.



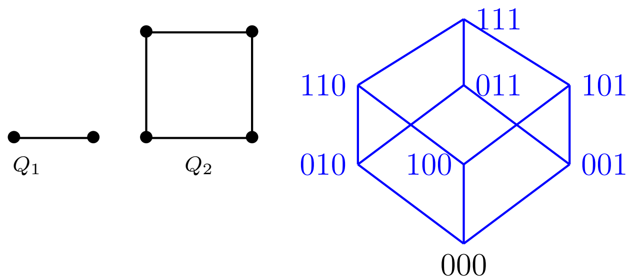
3.3 The Wheel Graph

The wheel graph W_n ($n \geq 3$) is obtained from C_n by adding a vertex a inside C_n and connecting it to every vertex in C_n.



3.4 $n-$Cubes

An $n-$dimensional hypercube, or $n-$cube, is a graph with 2^n vertices representing all bit strings of length n, where there is an edge between two vertices if and only if they differ in exactly one bit position.



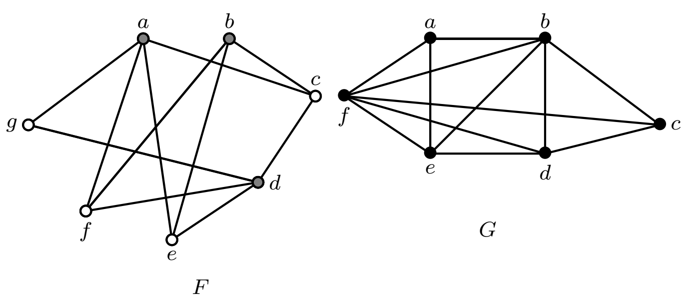
4 Bipartite graphs

4.1 Bipartite Graphs

Definition 4.1
A bipartite graph is an (undirected) graph $G = (V, E)$ whose vertices can be partitioned into two disjoint sets (V_1, V_2), with $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = V$, such that for every edge $e \in E$, $e = \{a, b\}$ such that $a \in V_1$ and $b \in V_2$.

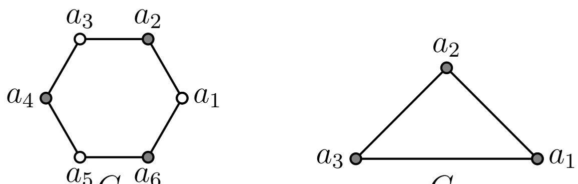
In other words, every edge connects a vertex in V_1 with a vertex in V_2. Equivalently, a graph is bipartite if and only if it is possible to color each vertex red or blue such that no two adjacent vertices have the same color.

Definition 4.2
An equivalent definition of a bipartite graph is one where it is possible to color the vertices either red or blue so that no two adjacent vertices are the same color.



F is bipartite. $V_1 = \{a, b, d\}$, $V_2 = \{c, e, f, g\}$.
In G if we color a red, then its neighbors f and b must be blue. But f and b are adjacent. G is not bipartite

Example 10 :



C_6 C_3

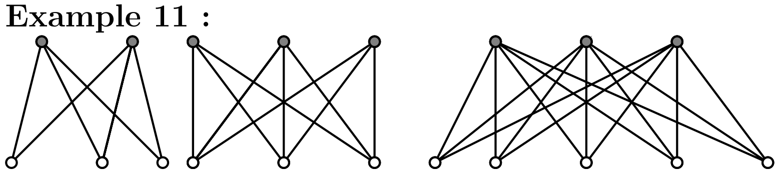
C_6 is bipartite. Partition the vertex set of C_6 into $V_1 = \{a_1, a_3, a_5\}$ and $V_2 = \{a_2, a_4, a_6\}$.
If we partition vertices of C_3 into two nonempty sets, one set must contains two vertices. But every vertex is connected to every other. So, the two vertices in the same partition are connected. Hence, C_3 is not bipartite.

Theorem 4.3
Let G be a graph of n vertices. Then G is bipartite if and only if it contains no cycles of odd length.

4.2 Complete Bipartite Graphs

Definition 4.4
A complete bipartite graph is a graph that has its vertex set partitioned into two subsets V_1 of size m and V_2 of size n such that there is an edge from every vertex in V_1 to every vertex in V_2.

Example 11 :



$K_{2,3}$ $K_{3,3}$ $K_{3,5}$

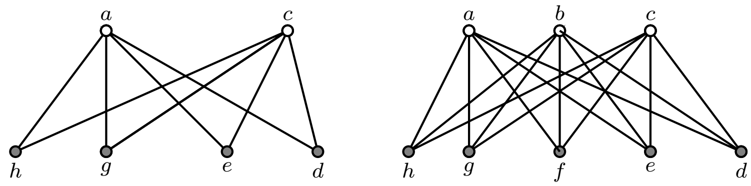
5 Subgraphs

5.1 Subgraphs

Definition 5.1
A subgraph of a graph $G = (V, E)$ is a graph (W, F), where $W \subset V$ and $F \subset E$.
A subgraph F of G is a proper subgraph of G if $F \neq G$.

5.2 Induced Subgraphs

Definition 5.2
Let $G = (V, E)$ be a graph. The subgraph induced by a subset W of the vertex set V is the graph $H = (W, F)$, whose edge set F contains an edge in E if and only if both endpoints are in W.



$F = K_{2,4}$ $K_{3,5}$

$K_{2,4}$ is the subgraph of $K_{3,5}$ induced by $W = \{a, c, e, g, h\}$.

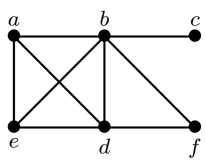
6 Representing Graphs and Graph Isomorphism

6.1 Representing Graphs: Adjacency Lists

Definition 6.1

An adjacency list represents a graph (with no multiple edges) by specifying the vertices that are adjacent to each vertex.

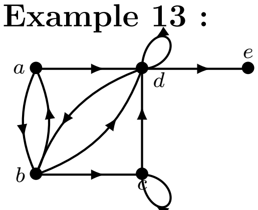
Example 12 :



G

An adjacency list for a simply graph	
Vertex	Adjacent vertices
a	b, d, e
b	a, c, e, d, f
c	b
d	a, b, e, f
e	a, b, d
f	b, d

Example 13 :

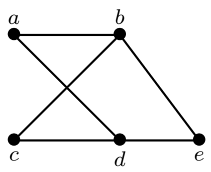


An adjacency list for a directed graph	
Initial vertex	Terminal vertices
a	b, d
b	a, c, d
c	c, d
d	b, d, e
e	

6.2 Representation of Graphs: Adjacency Matrices

Definition 6.2
Let $G = (V, E)$ be a simple graph where $|V| = n$. If $a_1, a_2, \ldots, a_n$ are the vertices of G. The adjacency matrix, A, of G, with respect to this listing of vertices, is the $n \times n$ matrix with its $(i, j)^{\text{th}}$ entry is 1 if a_i and a_j are adjacent, and 0 if they are not adjacent. ($A = (a_{i,j})$, with $a_{i,j} = 1$ if $\{a_i, a_j\} \in E$ and $a_{i,j} = 0$ if $\{a_i, a_j\} \notin E$.)

Example 14 :

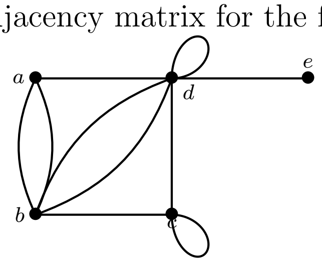


G

The adjacency matrix is $\begin{pmatrix} 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

The adjacency matrix of an undirected graph is symmetric: Also, since there are no loops, each diagonal entry is zero:

Example 15 :
The adjacency matrix for the following pseudograph is:



$$\begin{pmatrix} 0 & 2 & 0 & 1 & 0 \\ 2 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 2 & 1 & 2 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

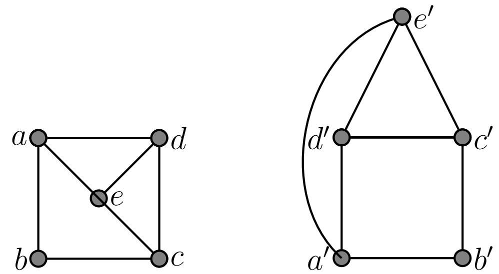
6.3 Isomorphism of Graphs

Definition 6.3
Two (undirected) graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are called isomorphic if there is a bijection, $f\colon V_1 \longrightarrow V_+2$, with the property that for all vertices $a, b \in V_1$

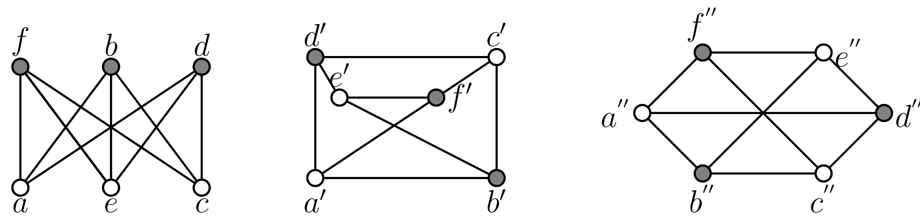
$$\{a, b\} \in E_1 \iff \{f(a), f(b)\} \in E_2.$$

Such a function f is called an isomorphism.

The following graphs are isomorphic.



The following graphs are isomorphic.



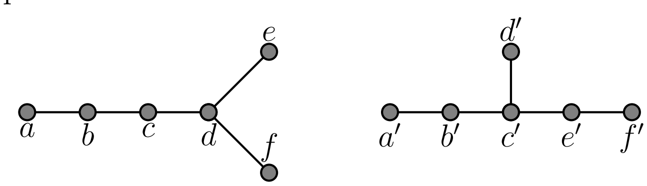
Theorem 6.4
Let f be an isomorphism of the graph $G_1 = (V_1, E_1)$ to the graph $G_2 = (V_2, E_2)$. Let $v \in V_1$. Then $\deg(v) = \deg(f(v)$. i.e., isomorphism preserves the degree of vertices.

Proof A point $u \in V_1$ is adjacent to v in G_1 if and only if $f(u)$ is adjacent to $f(v)$ in G_2. Also f is bijection. Hence the number of points in V_1 which are adjacent to v is equal to the number of points in V_2 which are adjacent to $f(v)$. Hence $\deg(v) = \deg(f(v))$.

Remarks 6.5

1. *Two isomorphic graphs have the same number of points and the same number of edges.*
2. *Two isomorphic graphs have equal number of points with a given degree. However these conditions are not sufficient to ensure that two graphs are isomorphic.*

Example 16 :
Consider the two graphs given in figure below. Under any isomorphism d must correspond to c', a, e, f must correspond to a', d', f' in some order. The remaining two points b, c are adjacent whereas b', e' are not adjacent. Hence there does not exist an isomorphism.



7 Connectedness in undirected graphs

7.1 Paths (in undirected graphs)

Informally, a path is a sequence of edges connecting vertices.

Definition 7.1

1. *For an undirected graph $G = (V, E)$, an integer $n \geq 0$, and vertices $a, b \in V$, a path of length n from a to b in G is a sequence: $x_0, e_1, x_1, e_2, \ldots, x_{n-1}, e_n, x_n$ of interleaved vertices $x_j \in V$ and edges $e_i \in E$, such that $x_0 = a$ and $x_n = b$, and such that $e_i = \{x_{i-1}, x_i\} \in E$ for all $i \in \{1, \ldots, n\}$.*
   Such path starts at a and ends at b.
   The trivial path from v to v consists of the single vertex v.

Definition 7.2

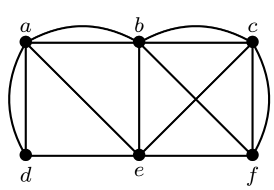
2. *A path of length $n \geq 1$ is called a circuit (or cycle) if $n \geq 1$ and the path starts and ends at the same vertex, i.e., $a = b$.*

3. *A path or circuit is called simple if it does not contain the same edge more than once.*

Remarks 7.3

1. *When $G = (V, E)$ is a simple undirected graph a path $x_0, e_1, \ldots, e_n, x_n$ is determined uniquely by the sequence of vertices $x_0, x_1, \ldots, x_n$. So, for simple undirected graphs we can denote a path by its sequence of vertices $x_0, x_1, \ldots, x_n$.*

2. *Don't confuse a simple undirected graph with a simple path. There can be a simple path in a non-simple graph, and a non-simple path in a simple graph.*



1. d, a, b, c, f is a simple path of length 4.

2. d, e, c, b, a, d is a simple circuit of length 5.

3. d, a, b, c, f, b, a, e is a path, but it is not a simple path, because the edge $\{a, b\}$ occurs twice in it.

4. c, e, a, d, e, f is a simple path, but it is not a tidy path, because vertex e occurs twice in it.

7.2 Paths in directed graphs

Definition 7.4

1. *For a directed graph $G = (V, E)$, an integer $n \geq 0$, and vertices $a, b \in V$, a path of length n from a to b in G is a sequence of vertices and edges $x_0, e_1, x_1, e_2, \ldots, x_n, e_n$, such that $x_0 = a$ and $x_n = b$, and such that $e_i = (x_{i-1}, x_i) \in E$ for all $i \in \{1, \ldots, n\}$.*

2. *When there are no multi-edges in the directed graph G, the path can be denoted (uniquely) by its vertex sequence $x_0, x_1, \ldots, x_n$.*

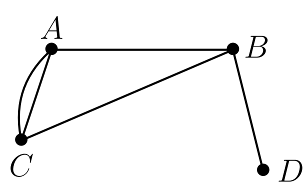
3. *A path of length $n \geq 1$ is called a circuit (or cycle) if the path starts and ends at the same vertex, i.e., $a = b$.*

Definition 7.5

4. *A path or circuit is called simple if it does not contain the same edge more than once. (And we call it tidy if it does not contain the same vertex more than once, except possibly the first and last in case $a = b$ and the path is a circuit (cycle).)*

7.3 Connectedness in undirected graphs

Definition 7.6
An undirected graph $G = (V, E)$ is called connected, if there is a path between every pair of distinct vertices. It is called disconnected otherwise.



This graph is connected

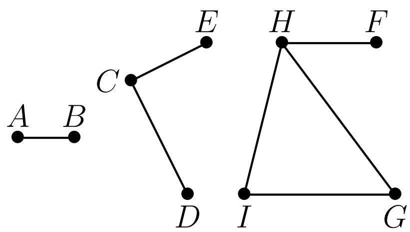
Theorem 7.7
A graph G is connected if and only if for any partition of V into subsets V_1 and V_2 there is an edge joining a vertex of V_1 to a vertex of V_2.

Theorem 7.8
There is always a simple, and tidy, path between any pair of vertices a, b of a connected undirected graph G.

Proof By definition of connectedness, for every pair of vertices a, b, there must exist a shortest path $x_0, e_1, x_1, \ldots, e_n, x_n$ in G such that $x_0 = a$ and $x_n = b$. Suppose this path is not tidy, and $n \geq 1$. (If $n = 0$, the Proposition is trivial.) Then $x_j = x_k$ for some $0 \leq j < k \leq n$. But then $x_0, e_1, x_1, \ldots, x_j, e_{k+1}, x_{k+1}, \ldots, e_n, x_n$ is a shorter path from a to b, contradicting the assumption that the original path was shortest.

7.4 Connected Components of Undirected Graphs

Definition 7.9
A connected component $H = (V', E')$ of a graph $G = (V, E)$ is a maximal connected subgraph of G, meaning H is connected and $V' \subset V$ and $E' \subset E$, but H is not a proper subgraph of a larger connected subgraph R of G.

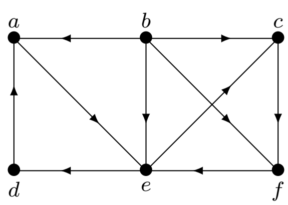


This graph, $G = (V, E)$, has 3 connected components. (It is thus a disconnected graph.)

7.5 Connectedness in Directed Graphs

Definition 7.10

1. *A directed graph $G = (V, E)$ is called strongly connected, if for every pair of vertices a and b in V, there is a (directed) path from a to b, and a directed path from b to a.*
2. *($G = (V, E)$ is weakly connected if there is a path between every pair of vertices in V in the underlying undirected graph (meaning when we ignore the direction of edges in E.) A strongly connected component of a directed graph G, is a maximal strongly connected subgraph H of G which is not contained in a larger strongly connected subgraph of G.*



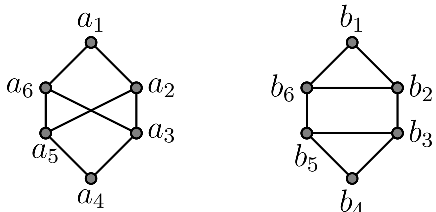
This digraph, G, is not strongly connected, because, for example, there is no directed path from e to b.
One strongly connected component of G is $H = (V_1, E_1)$, where $V_1 = \{a, c, d, e, f\}$ and $E_1 = \{(a, e), (e, c), (c, f), (f, e), (e, d), (d, a)\}$.

8 Paths and Isomorphism

8.1 Paths and Isomorphism

There are several ways that paths and circuits can help determine whether two graphs are isomorphic. For example, the existence of a simple circuit of a particular length is a useful invariant that can be used to show that two graphs are not isomorphic. In addition, paths can be used to construct mappings that may be isomorphisms. As we mentioned, a useful isomorphic invariant for simple graphs is the existence of a simple circuit of length k, where k is a positive integer greater than 2.

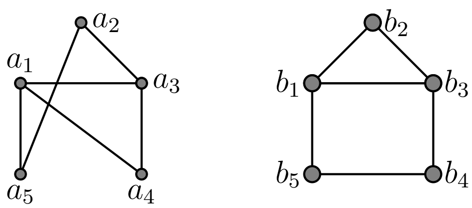
Let G and H be the following graphs.



G H

Both G and H have six vertices and eight edges. Each has 4 vertices of degree 3, and two vertices of degree 2. So, the three invariants number of vertices, number of edges, and degrees of vertices all agree for the two graphs. However, H has a simple circuit of length 3, namely, b_1, b_2, b_6, b_1, whereas G has no simple circuit of length 3. Then G and H are not isomorphic.

Example 17 :
Let G and H be the following graphs.

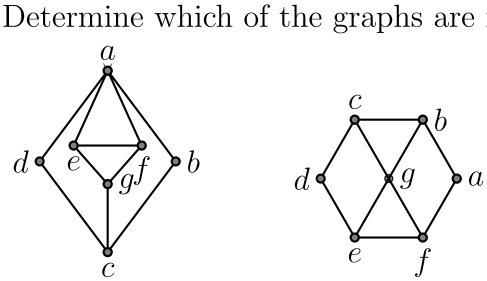


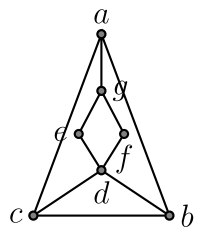
G H

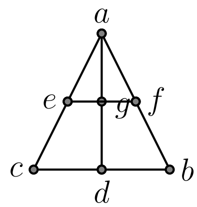
Both G and H have 5 vertices and 6 edges, both have 2 vertices of degree 3 and 3 vertices of degree 2, and both have a simple circuit of length 3, a simple circuit of length 4, and a simple circuit of length 5.

Because all these isomorphic invariants agree, G and H may be isomorphic. To find a possible isomorphism, we can follow paths that go through all vertices so that the corresponding vertices in the two graphs have the same degree. For example, the paths a_1, a_4, a_3, a_2, a_5 in G and b_3, b_2, b_1, b_5, b_4 in H both go through every vertex in the graph, start at a vertex of degree 3, go through vertices of degrees 2, three, and two, respectively, and end at a vertex of degree 2. By following these paths through the graphs, we define the mapping f with $f(a_1) = b_3$, $f(a_4) = b_2$, $f(a_3) = b_1$, $f(a_2) = b_5$, and $f(a_5) = b_4$.

Determine which of the graphs are isomorphic.







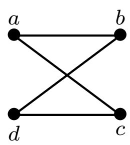
9 Counting Paths Between Vertices

9.1 Counting Paths Between Vertices

The number of paths between two vertices in a graph can be determined using its adjacency matrix.

Theorem 9.1
Let G be a graph with adjacency matrix A with respect to the ordering $b_1, b_2, \ldots, b_n$ of the vertices of the graph (with directed or undirected edges, with multiple edges and loops allowed). The number of different paths of length r from b_i to b_j, where r is a positive integer, equals the $(i,j)^{th}$ entry of A^r.

Example 18 :
How many paths of length four are there from a to d in the simple graph G



The adjacency matrix of G (ordering the vertices as a, b, c, d) is

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}.$$

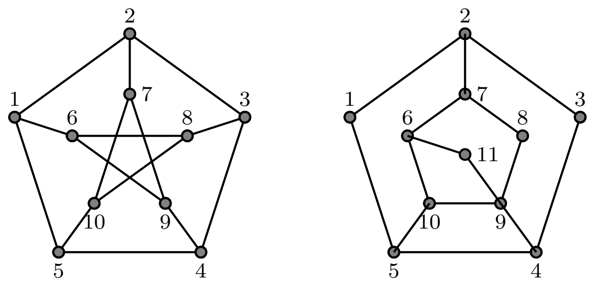
Hence, the number of paths of length 4 from a to d is the $(1,4)^{th}$ entry of A^4.

Because $A = \begin{pmatrix} 8 & 0 & 0 & 8 \\ 0 & 8 & 8 & 0 \\ 0 & 8 & 8 & 0 \\ 8 & 0 & 0 & 8 \end{pmatrix}$.

There are exactly eight paths of length four from a to d. By inspection of the graph, we see that a, b, a, b, d; a, b, a, c, d; a, b, d, b, d; a, b, d, c, d; a, c, a, b, d; a, c, a, c, d; a, c, d, b, d; and a, c, d, c, d are the eight paths of length four from a to d.

10 Exercises

Exercise 1 :
In each of the following graphs, find paths of length 9 and 11, and cycles of length $5, 6, 8$ and 9, if possible.



Solution of Exercise 1:
G_1: Path of length 9: 1 2 3 4 5 10 7 9 6 8. There are no paths of length 11 because G_1 has order 10.
Cycles: 1 2 3 4 5 1, 1 2 3 8 10 5 1, 1 6 8 10 7 9 4 5 1, 1 2 3 4 9 7 10 8 6 1.
G_2: 1 2 3 4 5 10 6 7 8 9. There are no paths of length 11 because G_2 has order 11.
Cycles: 1 2 3 4 5 1, 5 10 6 11 9 4 5, 2 3 4 5 10 9 8 7 2, 5 1 2 3 4 9 11 6 10 5.

Exercise 2 :

Solution of Exercise 2:

Exercise 3 :
A graph is self-complementary if it is isomorphic to its complement.

1. How many edges does a self-complementary graph of order n have?
2. Prove that if n is the order of a self-complementary graph, then n is congruent with 0 or with 1 modulo 4.
3. Check that for $n = 4k$ with $k \geq 1$, the following construction yields a self-complementary graph of order n, let us take $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where each V_i contains k vertices, the vertices of V_1 and V_2 induce complete graphs, also, we have all edges between V_1 and V_3, between V_3 and V_4, and between V_4 and V_2.
4. How could we modify the previous construction to build a self-complementary graph of order $4k + 1$?

Solution of Exercise 3:

Exercise 4 :

Solution of Exercise 4:

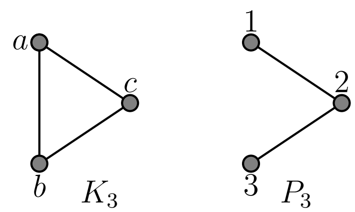
Exercise 5 :
Consider a graph $G = (V, E)$ of order n and size m. Let v be a vertex and e an edge of G. Give the order and the size of $\bar{G} = G^c$, $G - v$ and $G - e$.

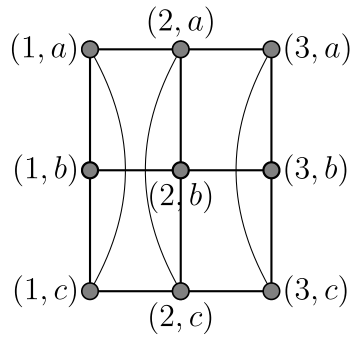
Solution of Exercise 5:
The order of $\bar{G} = G^c$ is the order of G. The size of $\bar{G} = G^c$ is $\dfrac{n(n+1)}{2} - m$.
The order of $G - v$ is $n - 1$. The size of $G - v$ is $m - \deg(v)$.
The order of $G - e$ is n. The size of $G - e$ is $m - 1$.

Exercise 6 :
Give the set of edges and a drawing of the graphs $K_3 \cup P_3$ and $K_3 \times P_3$, assuming that the sets of vertices of K_3 and P_3 are disjoint.

Solution of Exercise 6:

1. $K_3 \cup P_3$, $E = \{ab, ac, bc, 12, 23\}$





2. $K_3 \times P_3$

   $E = \{(1,a)(1,b); (1,a)(1,c); (1,a)(2,a); (1,b)(1,c); (1,b)(2,b); (1,c)(2,c); (2,a)(2,b);$
   $(2,a)(2,c); (2,a)(3,a); (3,a)(3,b); (3,a)(3,c); (3,b)(3,c)\}$

Exercise 7 :
Consider the graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$. Give the order, the degree of the vertices and the size of $G_1 \times G_2$ in terms of those of G_1 and G_2.

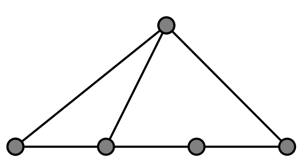
Solution of Exercise 7:
The order of $G_1 \times G_2$ is $|V_1|\ |V_2|$, $\deg_{G_1 \times G_2}(v) = \deg_{G_1}(v) + \deg_{G_2}(v)$ and size $|V_1|\ |E_2| + |V_2|\ |E_1|$.

Exercise 8 :
For each of the following sequences, find out if there is any graph of order 5 such that the degrees of its vertices are given by that sequence. If so, give an example.

1. $3, 3, 2, 2, 2$
2. $4, 4, 3, 2, 1.$
3. $4, 3, 3, 2, 2.$
4. $3, 3, 3, 2, 2.$
5. $3, 3, 3, 3, 2.$
6. $5, 3, 2, 2, 2.$

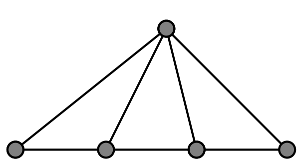
Solution of Exercise 8:

1. $3, 3, 2, 2, 2$



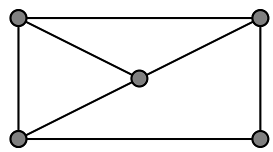
2. $4, 4, 3, 2, 1$. It does not exist. (One vertice v_1 which has degree 4, then there is one edge between v_1 and the others vertices. Also there is an other vertice v_2 which has degree 4, then there is one edge between v_2 and the others vertices. Then the minimum of degree is 2 and not 1).

3. $4, 3, 3, 2, 2.$



4. It does not exist. (The number of vertives with odd edges is odd).

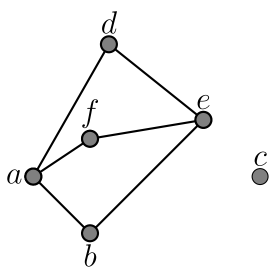
5. 3, 3, 3, 3, 2.



6. 5, 3, 2, 2, 2. It does not exist. (The order is 5 and one vertive has degree 5).

Exercise 9 :
Let $V = \{a, b, c, d, e, f\}$, $E = \{ab, af, ad, be, de, ef\}$ and $G = (V, E)$. Determine all the subgraphs of G of order 4 and size 4.

Solution of Exercise 9:



$(\{a, b, e, d\}, \{ab, be, ed, da\})$
$(\{a, b, e, f\}, \{ab, be, ef, fa\})$
$(\{a, d, e, f\}, \{ad, de, ef, fa\})$

Exercise 10 :
Prove that if a graph is regular of odd degree, then it has even order.

Solution of Exercise 10:
If the graph is of order n and regular of $2p + 1$ degree, then $(2p + 1)$ must be evn, then n is even.

Exercise 11 :
Let G be a bipartite graph of order n and regular of degree $d \geq 1$. Which is the size of G? Could it be that the order of G is odd?

Solution of Exercise 11:
The size of G is $2d$. The order of G is $2d$.

Exercise 12 :
Prove that the size of a bipartite graph of order n is at most $\dfrac{n^2}{4}$.

Solution of Exercise 12:
If $n = p + q$ is the order of G and p is the order of V_1 and q is the order of V_2. The size of G is $p(n - p)$. The minimum of $p(n - p)$ is reached for $p = \frac{n}{2}$. Then the size of G is at most $\dfrac{n^2}{4}$.

Exercise 13 :

Let $G = (V, E)$ and $H = (W, B)$ be two graphs. Prove that G and H are isomorphic if, and only if, $\bar{G}$ and $\bar{H}$ are isomorphic.

Solution of Exercise 13:

Exercise 14 :
Let G be a graph with order 9 so that the degree of each vertex is either 5 or 6. Prove that there are either at least 5 vertices of degree 6 or at least 6 vertices of degree 5.

Solution of Exercise 14:

Exercise 15 :
Let G be a (p, q) graph all of whose points have degree k or $k+1$. If G has $m > 0$ points of degree k, show that $m = p(k+1) - 2q$. (A graph with p points and q lines is called a (p, q) graph).

Solution of Exercise 15:
Since G has m points of degree k, the remaining $p - m$ points have degree $k+1$. Hence $\sum\limits_{v \in V} \deg(v) = mk + (p-m)(k+1) = 2q$. Then $m = p(k+1) - 2q$.

Exercise 16 :
Show that in any group of two or more people, there are always two with exactly the same number of friends inside the group.

Solution of Exercise 16:
We construct a graph G by taking the group of people as the set of points and joining two of them if they are friends, then $\deg(v)$ is equal to number of friends of v and hence we need only to prove that at least two points of G have the same degree. Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. Clearly $0 \leq \deg(v_i) \leq p - 1$ for each i. Suppose no two points of G have the same degree. Then the degrees of $v_1, v_2, \ldots, v_p$. are the integers $0, 1, 2, \ldots, p - 1$ in some order. However a point of degree $p - 1$ is joined to every other point of G and hence no point can have degree zero which is a contradiction. Hence there exist two points of G with equal degree.

Exercise 17 :
Prove that $\delta \leq 2q/p \leq \Delta$.

Solution of Exercise 17:
Let $V(G) = \{v_1, v_2, \ldots, v_p\}$. We have $\delta \leq \deg(v_i) \leq \Delta$ for all i. Hence $p\delta \leq \sum_{i=1}^{p} \deg(v_i) \leq p\Delta$. Then $p\delta \leq 2q \leq p\Delta$ (by Handshaking Theorem). We deduce that $\delta \leq 2q/p \leq \Delta$.

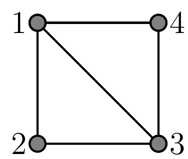
Exercise 18 :
Let G be a $k-$regular bibgraph with bipartion (V_1, V_2) and $k > 0$. Prove that $|V_1| = |V_2|$.

Solution of Exercise 18:
Since every line of G has one end in V_1 and other end in V_2 it follows that $\sum\limits_{v \in V_1} \deg(v) = \sum\limits_{v \in V_2} \deg(v) = q$. Also $\deg(v) = k$ for all $v \in V = V_1 \cup V_2$. Hence $\sum\limits_{v \in V_1} \deg(v) k |V_1|$ and

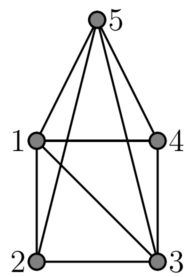
$\sum_{v \in V_2} \deg(v) = k|V_2|$ so that $k|V_1| = k|V_2|$. Since $k > 0$, we have $|V_1| = |V_2|$.

Exercise 19 :
Let $V = \{1, 2, 3, \ldots, n\}$. Let $X = \{\{i, j\};\ i, j \in V$ and are relatievly prime$\}$. The resulting graph (V, X) is denoted by G_n. Draw G_4 and G_5.

Solution of Exercise 19:



W_4



W_5

Exercise 20 :
Let G be a graph with minimum degree $p > 1$. Prove that G contains a cycle of length at least $p + 1$.

Solution of Exercise 20:
Let $v_1, \ldots, v_k$ be a maximal path in G, $k \geq m + 1$ and the path has length at least m.
The neighbor of v_1 that is furthest along the path must be v_j with $j \geq m + 1$. Then $v_1, \ldots, v_j, v_1$ is a cycle of length at least $m + 1$.

Exercise 21 :
Show that every graph on at least two vertices contains two vertices of equal degree.

Solution of Exercise 21:
Suppose that the n vertices all have different degrees, and look at the set of degrees. Since the degree of a vertex is at most $n-1$, the set of degrees must be $\{0, 1, 2, \ldots, n-2, n-1\}$.

But that's not possible, because the vertex with degree $n - 1$ would have to be adjacent to all other vertices, whereas the one with degree 0 is not adjacent to any vertex.

Exercise 22 :
• The null graph of order n, denoted by N_n, is the graph of order n and size 0. The graph N_1 is called the trivial graph.
• The complete graph of order n, denoted by K_n, is the graph of order n that has all possible edges. We observe that K_1 is a trivial graph too.
• The path graph of order n, denoted by $P_n = (V, E)$, is the graph that has as a set of edges $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$.
• The cycle graph of order $n \geq 3$, denoted by $C_n = (V, E)$, is the graph that has as a set of edges $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$.
• The wheel graph of order $n \geq 4$, denoted by $W_n = (V, E)$, is the graph that has as a set of edges $E = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_1, x_nx_1\} \cup \{x_nx_1, x_nx_2, \ldots, x_nx_{n-1}\}$

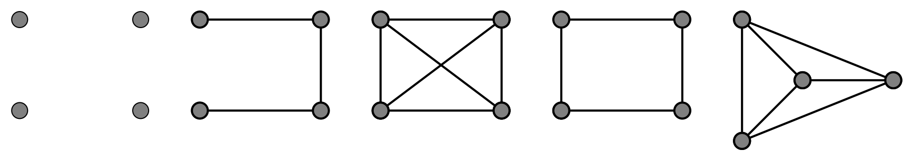
•

For each of the graphs N_n, K_n, P_n, C_n and W_n, give:

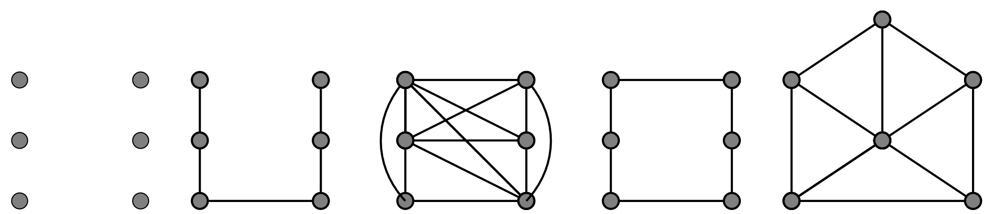
1. a drawing for $n = 4$ and $n = 6$,
2. the adjacency matrix for $n = 5$,
3. the order, the size, the maximum degree and the minimum degree in terms of n.

Solution of Exercise 22:

1.



N_4 T_4 K_4 C_4 W_4



N_6 T_6 K_6 C_6 W_6

2. $M_{N_5} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$ $M_{K_5} = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

$M_{P_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$ $M_{C_5} = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$ $M_{W_5} = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

3. For a $n \geq 3$
$N_n = (V, E)$, $|V| = n$, —E— $= 0, \delta(N_n) = 0, \Delta(N_n) = 0$.
$K_n = (V, E)$, $|V| = n$, $|E| = C_n^2$, $\delta(K_n) = n - 1$, $\Delta(K_n) = n - 1$.
$P_n = (V, E)$, $|V| = n$, $|E| = n - 1$, $\delta(P_n) = 1$, $\Delta(P_n) = 2$.
$C_n = (V, E)$, $|V| = n$, $|E| = n$, $\delta(C_n) = 2$, $\Delta(C_n) = 2$.
$W_n = (V, E)$, $|V| = n$, $|E| = 2n - 2$, $\delta(W_n) = 3$, $\Delta(W_n) = n - 1$.

Exercise 23 :

1. Is C_n a subgraph of K_n?
2. For what values of n and m is Kn, n a subgraph of K_m?

3. For what n is C_n a subgraph of $K_{n,n}$?

Solution of Exercise 23:

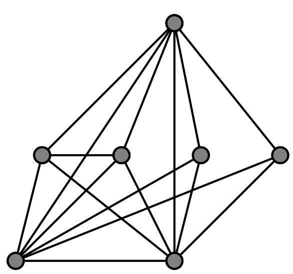
1. Yes! (by definition of subgraph , or just simply by the fact that K_n has all the possible edges a graph on n vertices can have.)
2. We must have $m = |V(K_m)|, |V(K_{n,n})| = 2n$. On the other hand, by a similar reasoning as part (1), we get that the statement holds for all m, n with $m \geq 2n$.
3. First, note that a bipartite graph cannot have any cycle of odd length, so n cannot be odd. For even n, one can check that $K_{n,n}$ has a cycle of length n.

Exercise 24 :
Given a graph G with vertex set $V = \{v_1, \ldots, v_n\}$ we define the degree sequence of G to be the list $\deg(v_1), \ldots, \deg(v_n)$ of degrees in decreasing order. For each of the following lists, give an example of a graph with such a degree sequence or prove that no such graph exists:

1. $3, 3, 2, 2, 2, 1$
2. $6, 6, 6, 4, 4, 3, 3$
3. $6, 6, 6, 4, 4, 2, 2$
4. $6, 6, 6, 6, 5, 4, 2, 1$

Solution of Exercise 24:

1. There is no such graph, since the number of odd-degree vertices in a graph is always even.
2. Consider the following graph:



3. No, since otherwise we have 3 vertices of degree 6 which are adjacent to all other vertices of the graph, so each vertex in the graph must be of degree at least 3.
4. No! Note that each vertex of the degree 6 is adjacent to all but one other vertices. In particular, each such vertex is adjacent to at least one of v_1 and v_2 (where $\deg(v_1) = 1$ and $\deg(v_2) = 2$). But that would mean at least four edges touching v_1 or v_2, contradicting $\deg(v_1) + \deg(v_2) = 3 < 4$.

Exercise 25 :
Construct two graphs that have the same degree sequence but are not isomorphic.

Solution of Exercise 25:
Let F be of a cycle on 6 vertices, and let G be the union of two disjoint cycles on 3 vertices each. In both graphs each vertex has degree 2, but the graphs are not isomorphic, since one is connected and the other is not.

Exercise 26 :
A graph is $k-$regular if every vertex has degree k. Describe all $1-$regular graphs and all $2-$regular graphs.

Solution of Exercise 26:
A $1-$regular graph is just a disjoint union of edges. A $2-$regular graph is a disjoint union of cycles.

Exercise 27 :

Solution of Exercise 27:

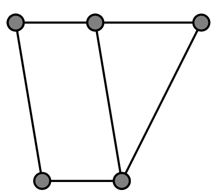
Exercise 28 :
Draw an example graph for each of these.

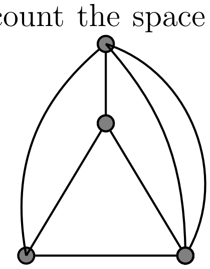
1. A planar graph has 5 vertices and 3 faces. How many edges does it have?

2. A planar graph has 7 edges and 5 faces. How many vertices does it have?

Solution of Exercise 28:
We use Euler's formula: $V + F = E + 2$.

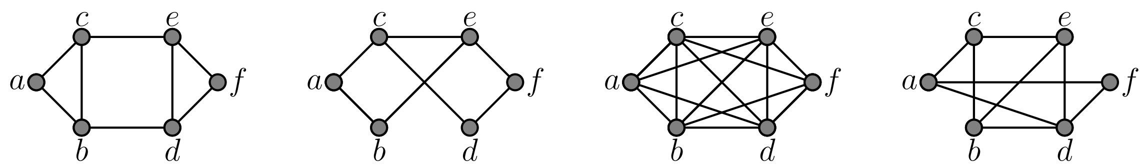
1. There are $E = V + F - 2 = 6$ edges. Here's an example: (Note that the outer face is also counted!)

2. There should be $V = E - F + 2 = 4$ vertices. However, this is not possible without creating duplicate edges. With duplicate edges, it is possible, and the formula gives the correct answer if we count the space between two duplicate edges also as a face. Here's an example: 

   Note, however, that this is not a graph, but a multigraph.

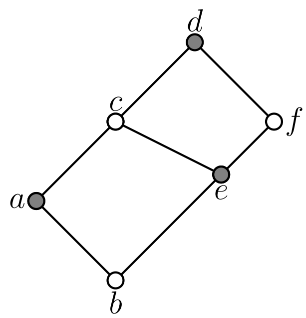
Exercise 29 :
Answer for each of these graphs: Is it planar? Is it bipartite?



Solution of Exercise 29:

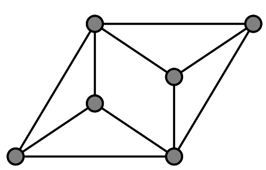
1. This graph is planar, since there are no edge crossings in its drawing. It is not bipartite, since it has a cycle of odd length (a, b, c).

2. This graph is planar: we can flip part of the graph to obtain a planar graph as follows. It is also bipartite, since we can colour all vertices with two colours.



3. This graph is not planar: it has 6 vertices and 14 edges, and by Euler's formula a planar graph with 6 vertices can have at most $3V - 6 = 12$ edges. It is also not bipartite, since it contains triangles.

4. This graph is planer, as can be seen in this drawing of the same graph. It is not bipartite, since it contains triangles.



Exercise 30 :
Draw diagrams to represent each of the graphs whose adjacency matrix is given below. Write down the degree of each vertex, and state whether the graph is (a) simple; (b) regular.

1. $$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

2. $\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$

3. $\begin{pmatrix} 1 & 2 & 0 & 2 & 1 \\ 2 & 1 & 2 & 0 & 1 \\ 0 & 2 & 1 & 2 & 1 \\ 2 & 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$

Exercise 31 :
Decide whether there exists a graph with four vertices of degrees $1, 2, 3$, and 3.

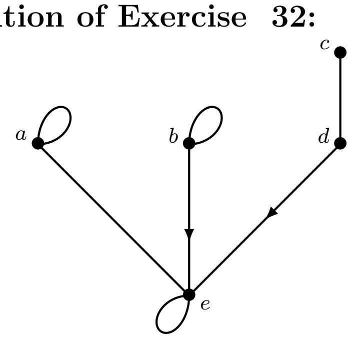
Solution of Exercise 30:
The Handshake Theorem states that the number of edges of the graph is $8 = \dfrac{0+2+2+3+9}{2}$.

Exercise 32 :
Decide whether there exists a graph with four vertices of degrees $1, 2, 3$, and 3.

Solution of Exercise 31:
The graph does not exist because $1+2+3+3=9$ by Handshake Theorem.
Also there is not such a graph because, there is an even number of vertices of odd degree in any graph, hence there cannot be 3.

Exercise 33 :
Decide whether there exists a graph with 5 vertices of degree $1, 2, 3, 3$, and 5, respectively.

Solution of Exercise 32:

Yes. 

Exercise 34 :
Is there a simple graph G with four vertices of degrees $1, 2, 3$, and 4?

Solution of Exercise 33:
Such a graph does not exist. A vertex v with $\deg(v) = 4$ needs to be connected to 4 distinct vertices, since a simple graph is not allowed to have loops or parallel edges.

Exercise 35 :
Simple graph with six edges and all vertices of degree 3.

Solution of Exercise 34:
For having all vertices of degree 3, the graph should have 4 vertices with two diagonals.

Exercise 36 :
Is there a simple graph, each of whose vertices has even degree?

Solution of Exercise 35:
Yes. Consider a graph that forms a geometric figure, e.g., a triangle. This is a simple circuit and each vertex has degree 2.

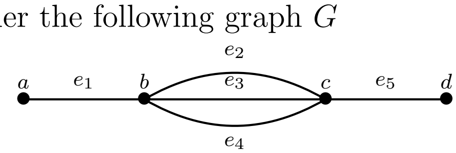
Exercise 37 :
Recall that K_n denotes a complete graph on n vertices, that is, a simple graph with n vertices and exactly 1 edge between each pair of distinct vertices. Show that for all integers $n \geq 1$, the number of edges of K_n is $\dfrac{n(n-1)}{2}$.

Solution of Exercise 36:
The statement can be proved by induction, since K_{n+1} can be obtained starting from K_n and by adding a vertex and connecting it to the other n vertices. K_1 has 1 vertex and 0 edges $= \frac{1.0}{2}$.

Assume that K_n has $\dfrac{n(n-1)}{2}$ edges. K_{n+1} is obtained by K_n adding an $(n+1)^{\text{th}}$ vertex, and connecting it with all the other n vertices through n distinct edges. Therefore K_{n+1} has $n + \dfrac{n(n-1)}{2}$ edges, that is $\dfrac{n(n+1)}{2}$.

Alternatively, use the Handshake Theorem: 2 times number of edges of $G = \deg(G) = \sum_{i=1}^{n} \deg(v_i)$. Since, by definition, v_i has $(n-1)$ edges (1 for each of the other $(n-1)$ vertices), then, for each $i = 1 \ldots n$, $\deg(v_i) = (n-1)$. Therefore, 2 times the number of edges of $G = n(n-1)$, that is, the number of edges of $G = \frac{n(n-11)}{2}$.

Exercise 38 :
Consider the following graph G



1. How many paths are there from v_1 to v_4?

2. How many trails are there from v_1 to v_4?

3. How many walks are there from v_1 to v_4?

Solution of Exercise 37:
Remember what follows:

1. A walk from a vertex v to a vertex w is a finite alternating sequence of adjacent vertices and edges of G.
2. A trail from a vertex v to a vertex w is a walk from v to w that does not contain a repeated edge.
3. A path from a vertex v to a vertex w is a trail from v to w that does not contain a repeated vertex.

1. G has 3 paths,
2. G has $3 + 3!$ trails,
3. G has infinitely many walks.