MATH 36000: Real Analysis I Lecture Notes

Created by: Dr. Amanda Harsy

July 20, 2020

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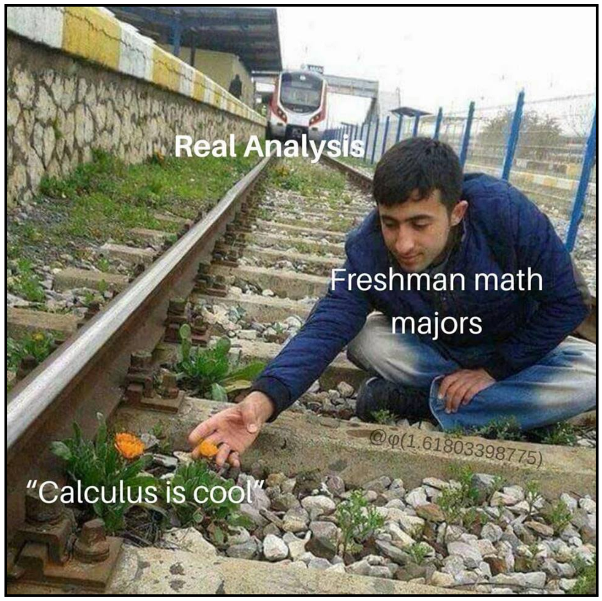


Thanks for taking Real Analysis I with me! Real Analysis is one of my favorite courses to teach. In fact, it was my favorite mathematics course I took as an undergraduate. You may be wondering, "What exactly is Real Analysis?"

Analysis is one of the principle areas in mathematics. It provides the theoretical underpinnings of the calculus you know and love. In your calculus courses, you gained an intuition about limits, continuity, differentiability, and integration. Real Analysis is the formalization of everything we learned in Calculus. This enables you to make use of the examples and intuition from your calculus courses which may help you with your proofs. Throughout the course, we will be formally proving and exploring the inner workings of the Real Number Line (hence the name *Real* Analysis). But Real Analysis is more than just proving calculus, and I think Dr. Carol Schumacher of Kenyan College describes it extremely well by when she calls *Analysis* the "Mathematics of Closeness." At its core, this is what Real Analysis is above. When you think about the derivatives and integration, remember we talk about taking small changes, Δx whether it be a $\frac{\Delta y}{\Delta x}$ or a partition for our Riemann Sums. Our job in Real Analysis is to understand how to formally describe closeness and the process of getting "closer and closer" (limits).

This course starts with very abstract concepts and gets more concrete as the semester goes on. Believe me, the hardest part of the class is at the beginning! We start by talking about bounds of real numbers which allows us to prove that there is in fact a unique limit we want to reach. We then explore sequences which we will use to get as close as we can to these numbers/bounds. Next we discuss closeness in a function setting along with continuity. We need continuity later for our integration and special derivative theorems. We then we revisit and use sequences and functions to discuss rate of change (derivatives) and optimization. We end with Riemann Sums and the beautiful Fundamental Theorem of Calculus.

I hope you will enjoy this semester and learn a lot! Please make use of my office hours and plan to work hard in this class. My classes have a high work load (as all math classes usually do!), so make sure you **stay on top of your assignments and get help early**. Remember you can also email me questions if you can't make my office hours or make an appointment outside of office hours for help. When I am at Lewis, I usually keep the door open and feel free to pop in at any time. If I have something especially pressing, I may ask you to come back at a different time, but in general, I am usually available. I have worked hard to create this course packet for you, but it is still a work in progress. Please be understanding of the typos I have not caught, and politely bring them to my attention so I can fix them for the next time I teach this course. I look forward to meeting you and guiding you through the wonderful course that is Real Analysis.

Cheers,
Dr. H
Acknowledgments: No math teacher is who she is without a little help. I would like to thank my own undergraduate professors from Taylor University: Dr. Ken Constantine, Dr. Matt Delong, and Dr. Jeremy Case for their wonderful example and ideas for structuring excellent learning environments. I also want to thank Dr. Annalisa Crannell, Dr. Tom Clark, Dr. Alyssa Hoofnagel, Dr. Alden Gassert, Dr. Francis Su, Dr. Brian Katz, and Dr. Christian Millichap for sharing some of their resources from their own courses. *And finally, I would like to thank you and all the other students for making this job worthwhile and for all the suggestions and encouragement you have given me over the years to improve.*

Important Norms for the Class:

1. I will embrace challenges because they help me learn.
2. I will not be afraid of making mistakes and taking risks because they provide learning opportunities.
3. I will be respectful of the diversity in the room.
4. I will be a mindful contributor and work as a team during classroom activities.
5. I will have a positive attitude about this class because my attitude is something I can control.
6. I will be appreciative of the effort others put forth during this semester.
7. I understand that assessment opportunities give me a chance to demonstrate my growth and learning.
8. I will minimize distractions during class.
9. I will help to create an inclusive learning community.

2 Syllabus Crib Notes

The full syllabus is posted in Blackboard. Here are some highlights from the syllabus:

2.1 Office Hours

Please come to my office hours! Helping you with the material is the best part of my job! Normally I have 5 weekly office hours which I hold, but due to us being remote, I will be only holding 3 standing drop-in remote hours. I encourage you to instead make appointments for me to meet with you at a time that works for both of us! My office is in **AS-124-A**, but this semester I will hold my office hours in a Bb collaborate classroom. I will have a link for each of these posted in Bb. Remember if none of these times work, send me an email and we can schedule another time to meet. I can also answer questions through email! This semester my office hours will be:

Mondays: 3:00-4:00
Thursdays: 12:30-1:30
Fridays: 2:00-3:00
Or By Appointment!

Note: Sometimes I have meetings or class that goes right up to my office hours, so if I am not there, please wait a few minutes. Also sometimes I have unexpected meetings that get scheduled during my office hours. If this happens, I will do my best to let you know as soon as possible and I usually hold replacement office hours.

Help: Don't wait to get help. Visit me during my office hours, use the discussion forum in Blackboard, go to the Math Study Tables, find a study partner, get a tutor!

2.2 Grades

Category	Percentage
Definition Quizzes	10
Homework/ LaTeXPortfolio	25
Mastery Exams	45
Take-Home/Oral Exams	15
Productive Engagement/ ICE Group Work	5

Dr. Harsy reserves the right to change the percentages of these portions. The necessary, but not sufficient conditions to earn each grade is as follows:

Grade	Requirements
A:	an A average in Mastery Exams and at least a B average in HW and Oral Exams
B:	a B average in Mastery Exams, and at least a C average in HW and Oral Exams
C:	a C average in Mastery Exams, and at least a C average in HW and Oral Exams

We may revisit these conditions throughout the semester and see where we are at. Dr. Harsy reserves the right to adapt these terms, but only in your favor.

Final Exam: We will not have a formal final exam. Instead Finals week will be a final Testing Week (see description above under Master-Based Testing).

Homework: Almost every week, I will collect a homework assignment. I will post these homework assignments on Blackboard. You may work with others on the homework, but it must be your own work. If I catch you copying homework, you will get a 0. Please see the academic honesty section in the posted syllabus in Bb.

Productive Engagement: In order to achieve the maximum points for this portion of your grade you must actively present, facilitate, and participate in class activities, present problems and proofs from homework and ICE sheets, and participate in class discussions.

1. *Weekly Problem Sets:* These problems will be posted on blackboard and will be due almost every Tuesday. These will be graded traditionally. While you may work with people, you must do your own work. If I suspect there is academic dishonesty, you may be given a 0 on the assignment.
2. *Weekly LaTeXProblems:* One proof from your weekly homework problems (see above) will need to be done in a LaTeXPortfolio. I will give you a chance to revise this problem and work on your proof writing skills. It will also give you a chance to learn LaTeX. These should be written in a LaTeXtemplate which will be provided. You will share your portfolio with me through `sharelatex.com`.
3. *ICE group work:* Sometimes I will assign problems from ICE sheets or class notes which will be due the following class and will be graded on completeness and count towards your Productive Engagement portion of your grade. Whatever is not completed in class, you will need to be complete as homework before the next class. You should work with your group and may be asked to present your work as a learning opportunity.

Quizzes: Between each Mastery Exam, I will give 1-2 short basic definition quizzes. These will cover any section learned before the quiz day and since the last definition quiz. Definitions are vital in mathematics and it is important that we understand them. I may add a pop (unscheduled) quiz.

2.2.1 Exams

Take-Home/Oral Exams: Three times during the semester, you will be given a 2-4 proofs which will assess your creativity and ability to understand and write slightly longer proofs. You will not turn in these problems, but will schedule a time with me during the "oral exam" period in which I will pick one of the proofs for you to present to me in my office. These are low pressure and will give you experience presenting and explaining a proof in front of an audience. You will not need to write down every detail, but I may ask you to justify a step. You cannot retest an oral exam. **You may not use the Internet or other people, but can use your textbook and course notes for these oral exams.**

In-Class Exams: This course will use a testing method called, "Mastery-Based Testing." There will be three (3) paper-and-pencil, in-class Mastery Exams given periodically throughout the semester. In mastery-based testing, students receive credit only when they display "mastery", but they receive multiple attempts to do so. The primary source of extra attempts comes from the fact that test questions appear on every subsequent test. In this Real Analysis course, Exam 1 will have approximately 7 questions. Exam 2 will have approximately 13 questions -a remixed version of the five from Test 1 and five new questions. Exam 3 will have 16 questions- a remixed version of the questions from Exam 2 and a few new questions. The Final Exam will have no new questions and will be conducted as a retesting week (see below). It contains only remixed versions of questions from Exam 3.

Retesting Weeks: We will also have a total of 3 re-testing weeks (including the final retesting week). During these weeks, students can use doodle to sign up to retest concepts during (extended) office hours or math study tables. Students are allowed to test any concept, but cannot retest that concept the rest of the week. So for example, a student can test concepts 2,3 and 5 on Monday and concept 6 on Tuesday, but would not be able to test concept 5 again.

Exam #	Date
Exam 1	10/9
Oral Exam 1	10/16
Testing Week 1:	10/19-10/23
Exam 2	11/6
Oral Exam 2	11/13
Testing Week 2	11/16-11/20
Exam 3	12/4
Oral Exam 3	12/18
Final Testing Week	12/14-12/18

Mastery-Based Testing: This course will use a testing method called, "Mastery-Based Testing." There will be four (4) paper-and-pencil, in-class Mastery Exams given periodically throughout the semester. In mastery-based testing, students receive credit only when they display "mastery", but they receive multiple attempts to do so. The primary source of extra attempts comes from the fact that test questions appear on every subsequent test. In this Calculus III course, Test 1 will have 5 questions. Test 2 will have 10 questions -a remixed version of the five from Test 1 and five new questions. Test 3 will have 15 questions- a remixed version of the ten from Test 2 and five new questions. Test 4 will have 18 questions-a remixed version of the fifteen from Test 3 and three new questions. We will also have four testing weeks. During these weeks, students can use doodle to sign up to retest concepts during (extended) office hours. Students are allowed to test any concept, but cannot retest that concept the rest of the week. So for example, a student can test concepts 2,3 and 5 on Monday and concept 6 on Tuesday, but would not be able to test concept 5 again.

Grading of Mastery-Based Tests: The objectives of this course can be broken down into 16 main concepts/problems. For each sort of problem on the exam, I identify three levels of performance: master level, journeyman level, and apprentice level. I will record how well the student does on each problem (an M for master level, a J for journeyman level, a 0 for apprentice level) on each exam. After the Final testing week, I will make a record of the highest level of performance the student has made on each sort of problem or project and use this record to determine the student's total exam grade. Each of the first 8 concepts/questions students master will count 9% points towards their exam grade. After that, each concept/question will be worth 3.5% towards your exam grade. So for example, if you master 10 of the 15 concepts your grade will be a 79%.

This particular way of arriving at the course grade is unusual. It has some advantages. Each of you will get several chances to display mastery of almost all the problems. Once you have displayed mastery of a problem there is no need to do problems like it on later exams. So it can certainly happen that if you do well on the midterms you might only have to do one or two problems on the Final. (A few students may not even have to take the final.) On the other hand, because earlier weak performances are not averaged in, students who come into the Final on shaky ground can still manage to get a respectable grade for the course. This method of grading also stresses working out the problems in a completely correct way, since accumulating a lot of journeyman level performances only results in a journeyman level performance. So it pays to do one problem carefully and correctly as opposed to trying to get four problems partially correctly. Finally, this method of grading allows you to see easily which parts of the course you are doing well with, and which parts deserve more attention. The primary disadvantage of this grading scheme is that it is complicated. At any time, if you are uncertain about how you are doing in the class I would be more than glad to clarify the matter.

2.3 Expectations

This is a college level Math class and is much different than one taught in high school. We cover a lot of (very different) material in a very limited class time. You cannot expect to be able to pass this class if you do not spend several hours every day reading the sections and working problems outside of class. Paying attention and taking notes only during class time will not be enough. After the problems are worked, find a common thread, idea, or technique.

Technology Policy: Please do not have your phones out unless it is for a class activity. If

I see a phone out during a quiz or exam, you will receive an F on that quiz. You may use calculators in this course.

Academic Integrity: Scholastic integrity lies at the heart of Lewis University. Plagiarism, collusion and other forms of cheating or scholastic dishonesty are incompatible with the principles of the University. This includes using "tutoring"' sites for homework, quizzes, and exams. Students engaging in such activities are subject to loss of credit and expulsion from the University. Cases involving academic dishonesty are initially considered and determined at the instructor level. If the student is not satisfied with the instructors explanation, the student may appeal at the department/program level. Appeal of the department /program decision must be made to the Dean of the college/school. The Dean reviews the appeal and makes the final decision in all cases except those in which suspension or expulsion is recommended, and in these cases the Provost makes the final decision.

Make-Ups: There will be no make-ups for any assignments. If you are late or miss class, your assignment will not be accepted and there will be no make-up offered, except in extenuating and unpredictable circumstances. If you will miss class for a justifiable & unavoidable reason, you can contact me **before** you miss class & it is possible you can have a make-up. If you do not contact me & explain your absence, you will not be allowed a make-up.

Dr. Harsy's web page: For information on undergraduate research opportunities, about the Lewis Math Major, or about the process to get a Dr. Harsy letter of recommendation, please visit my website: http://www.cs.lewisu.edu/~harsyram.

Blackboard: Check the Blackboard website regularly (at least twice a week) during the semester for important information, announcements, and resources. It is also where you will find the course discussion board. Also, check your Lewis email account every day. I will use email as my primary method of communication outside of office hours.

The full syllabus and schedule is subject to change and the most updated versions are posted in the Blackboard.

Real Analysis I Schedule Fall 2020

The table below outlines the topics to be covered each day, the quiz and homework due dates. The daily homework problems from each section will be posted on my website. **This schedule is subject to change.**

Monday	Wednesday	Friday	Tentative Weekly Topics
8/31	9/2	9/4	Prelims, Ordered Fields
9/7	9/9 **HW 1 Due**	9/11	Completeness Axiom, inequalities
9/14 **Definition Quiz 1**	9/16 **HW 2 Due**	9/18	Suprema, Infima Archimedean Property,
9/21	9/23 **HW 3 Due**	9/25	Sequences
9/28 **Definition Quiz 2**	9/30 **HW 4 Due**	10/2	Sequence Theorems,
10/5	10/7	10/9 **Mastery Exam 1** *Start Oral Exam 1 Take-home*	Sequences Theorems and Properties, Subsequences
10/12 .	10/14	10/16 **Oral Exam 1 Due 4pm Fri**	Cauchy Sequences Functional Limits
10/19 **Definition Quiz 3** *Testing Week 1*	10/21 **HW 5 Due** *Testing Week 1*	10/23 *Testing Week 1*	Continuity
10/26	10/28 **HW 6 Due**	10/30	Continuity Theorems
11/2 **Definition Quiz 4**	11/4	11/6 **Mastery Exam 2** *Start Oral Exam 2 Take-home*	Differentiation
11/9	11/11	11/13 **Oral Exam 2 Due 4pm**	Differentiation Theorems
11/16 **Definition Quiz 5** *Testing Week 2*	11/18 **HW 7 Due** *Testing Week 2*	11/20 *Testing Week 2*	Taylor Approximation Riemann Sums *Testing Week 2 with new topics*
11/23 **Thanksgiving Break** **No Class**	11/25 **Thanksgiving Break** **No Class**	11/27 **Thanksgiving Break** **No Class**	
11/30 **Definition Quiz 6**	12/2 **HW 8 Due**	12/4 **Mastery Exam 3** *Start Oral Exam 3 Take-home*	Riemann Integrals (if time)
12/7	12/9 **HW 9 and HW 10 Due** **Make up Quiz**	12/11	Riemann Integrals (if time)
12/14 *Testing Week 3*	12/16 *Testing Week 3*	12/18 *Testing Week 3* **Oral Exam 3 Due 2pm**	

Tentative Topics for Exams:

Exam 1: Completeness Axioms, Sups, Infs,

Exam 2: Sequences, Continuity,

Exam 3: Differentiation, Integration (if time)

Assessment and Mapping of Student Learning Objectives:

Baccalaureate Characteristics:

BC 1. The baccalaureate graduate of Lewis University will read, write, speak, calculate, and use technology at a demonstrated level of proficiency.

Measurable Student Learning Outcome:

Advocate for a cause or idea, presenting facts and arguments, in an organized and accurate manner using some form of technology. Include qualitative and quantitative reasoning.

BC6: The baccalaureate graduate of Lewis University will think critically and creatively.

Measurable Student Learning Outcome:

Employ critical and creative thinking skills *by articulating or crafting an argument's major assertions and assumptions and evaluating its supporting evidence, using both qualitative and quantitative analysis.*

Course Student Learning Objectives	Baccalaureate Characteristics	Demonstrated by
1. Prove basic set theoretic statements.	1- mastered 6- mastered	Homework, Exams (in-class and oral/take-home exams)
2. Prove various statements by induction.	1 -reinforced	Homework, Exams (in-class)
3. Construct examples of sequences with specific properties ….	1- mastered	Homework, Quizzes, Exams (in-class)
4. Calculate the limit superior, limit inferior, and the limit of a sequence.	1- mastered	Homework, Quizzes, Exams (in-class)
5. Define the limit of a function at a value, a limit of a sequence, and the Cauchy Criterion.	1- mastered	Homework, Quizzes, Exams (in-class and oral/take-home exams)
6. Prove various theorems about limits of sequences and functions.	1- mastered 6- mastered	Homework, Exams (in-class and oral/take-home exams)
7. Define continuity of a function and uniform continuity of a function.	1- mastered	Homework, Quizzes, Exams (in-class)
8. Prove various theorems about continuous functions and emphasize the proofs' development.	1- mastered 6- mastered	Homework, Exams (in-class and oral/take-home exams)
9. Define the derivative of a function.	1- mastered	Homework, Quizzes, Exams (in-class and oral/take-home exams)
10. Prove various theorems about the derivatives of functions.	1- mastered 6- mastered	Homework, Exams (in-class and oral/take-home exams)
11. State and use theorems, lemmas, and axioms from Real Analysis …	1- mastered	Homework, Quizzes, Exams (in-class and oral/take-home exams)
12. Comprehend rigorous arguments developing the theory underpinning real analysis. …	1- mastered 6- mastered	Homework, In-class group assignments, Exams (in-class and oral/take-home exams)
13. Demonstrate an understanding of limits and how they are used in sequences, continuity, differentiation, and integration.	1- mastered	Homework, Exams (in-class and oral/take-home exams)

3 Mathematical and Proof Writing

3.1 Example of a Basic Proof Rubric

Typical Basic Proof Rubric	
Delivery & Organization	2 pts
Language	2 pts
Logic	2 pts
Notation	2 pts
Central Idea	2 pts

This rubric will often be the basis for some of the more basic proofs you will write in your homework.

3.1.1 Delivery & Organization

[1] A mathematician must make sure her proof is concise and easy to understand. This involves using connecting statements and making sure the flow of the proof is easy to understand. Delivery of a proof also requires organization. In some proofs, each step must fit in exactly one place to be correct, but in most there are choices. This component skill involves making careful choices about the order in time and location on the board of the ideas. At a developing level, the presenter may end up repeating ideas because the salient information is not collected for use in moving forward. At an acquired level, the presenter makes the organization of the information consistently visible. At a masterful level, the presenter has taken control of the organization (perhaps differently than the way the challenge was posed or the solution discovered) in a way that adds perspective to the presentation in addition to the solution.

3.1.2 Language

Mathematics requires a very precise use of language. At a developing level, the presenter is close enough with language use for the listener interpret the communication but struggles with things like the pronunciation of the terms, the verb choices, precise use of the quantifiers ("there exists" and "for all"), and unambiguous language. At an acquired level, the presenter makes only very few errors from the developing level and usually corrects them immediately. At the masterful level, the presenter finds a way to say exactly what is needed, clearly and concisely, to get to the heart of the proof.

3.1.3 Proof

Any mathematical proof should make use of correct and formal logic. Every statement should logically follow from a rule of inference. Be careful when you are writing your proof that you understand what statement is the asking you to prove. The eloquent proof-writer should correctly identify the antecedent and the consequent. Beware of assuming what you are wanting to prove.

[1]Special Thanks to Dr. Francis Su and Dr. Brian Katz for sharing their input for this section

3.1.4 Notation

Mathematics require the correct and appropriate use of notation. There should be a balance between overuse and underuse of notation to help make the proof readable. Define all of your notation. The first time you use a symbol, state explicitly what that symbol means (even if the symbol previously appeared in the statement of a problem, theorem, or definition). All mathematics should be written in complete sentences. The rules of spelling, punctuation, and grammar apply to mathematics as well. Open any mathematics text and youll see that this is true. Equations, even displayed ones, have punctuation that help you see where it fits in the context of a larger sentence. Consider this piece of writing:

$$(x-2)^2+(x-1)^2=5^2 \qquad 5^2=25$$
$$(x-2)^2=x^2-4x+4+x^2-2x+1=25$$
$$2x^2-6x-20$$
$$2(x+2)(x-5) \quad x=-2,5 \quad x>0 \quad x=5$$

Can you figure out what the writer is doing? Whats being assumed? Whats being proved? Where does one thought end and another begin? Whats the relationship between these phrases? Some phrases are dangling, and others, as statements, are not even true. The reader should not have to figure out what the writer was thinking! Now consider the work of another writer who has attempted the same problem:

Problem. Find a point on the line $y=x$ which is distance 5 from the point $(2,1)$ and such that $x>0$.

Solution. We wish to solve $(x-2)^2+(x-1)^2=5^2$, an equation obtained from the distance formula in the plane. Some algebra turns this equation into:

$$2x^2-6x-20=0.$$

Factoring the left side, we obtain

$$2(x+2)(x-5)=0,$$

whose solutions are evidently $x=-2$ and $x=5$. Since we assumed $x>0$, we have $(5,5)$ as the desired point on the line $y=x$.

Here, the writer has clearly stated the problem and described her path to a solution. She has set an invitational tone, and every thought is expressed in a complete sentence. Now it is clear that $x>0$ is a condition, not a result. Notice the punctuation in equations: one ended with a period because her thought was complete, the other ended with a comma because she wanted to continue the thought. Since she assumed her audience could do algebra, she didnt bore them with trivial algebraic manipulation, which would only obscure the thread of her arguments. But she did show the most interesting parts: the resulting polynomial and its factoring. And she made sure she answered the original question.

3.1.5 Central Idea:

There is usually an idea or definition that is vital for proving or disproving that statement. This section of the rubric allows Dr. Harsy to look for this central idea/concept vital for the proof.

3.2 More Advanced Proof Rubric

The next page is an example of a proof rubric I will use to grade more involved proofs from your homework and take-home exams. Sometimes this rubric will be scaled (that is, it won't be out of 24 points. For example, the oral exam includes the hints section of the rubric.)

Advanced Proof Rubric

Score:		**Excellent (6 pts)**	**Acceptable (4pts)**	**Repairable (2pts)**	**Unacceptable (0 pts)**
Reasoning	**Hypothesis and Structure (direct, induction, contradiction, etc.)**	Structure and hypothesis are appropriate and correctly applied.	Correct, but with needless complication or minor flaw(s), hypothesis use implied	Inappropriate or seriously flawed structure (requires many corrections), hypothesis incorrectly used	No identifiable or incomplete proof structure; proves different result, hypothesis ignored
	Definitions, Theorems, Terminology	Each used or referenced correctly The proof uses accurate and appropriate mathematical notation and terminology. Symbolic notation is used where it clearly simplifies the discourse, and avoided when English will better serve the reader.	Minor terminology or definition error or burdensome notation. Notation and terminology are correctly used, but there may be instances where the discourse would benefit from either more or less use of symbols versus English. The proof accurately invokes all needed definitions, though they may appear other than precisely where needed.	Terminology confused, distracting notation, Some relevant definitions or theorems are missing or misstated, but the proof is otherwise understandable.. Most, but not all, the notation and terminology is used accurately. Errors are identifiable and correctable by a reader of experience similar to the author.	Terminology incorrect or notation indecipherable, complete misuse of definition of theorem. Notation and/or terminology is frequently misused. The writer may use personal rather than standard notation.
	Logic and Reasoning	A clear, complete, concise, cohesive and properly ordered chain of deductive steps leads from the hypothesis to the conclusion.	Correct and reasonably well organized, all variables defined. The chain of deductive steps is complete and correctly ordered.	Contains flaws or omissions, at least one undefined variable. One or more intermediate deductive steps are missing or unclear, but the correctness of the proof is not compromised.	At least one fatal flaw(e.g., assumed what is being proved or omits key step) or The stated chain of deductions does not lead to the stated conclusion.
		Excellent (3 pts)	**Acceptable (2pts)**	**Repairable (1pt)**	**Unacceptable (0 pts)**
Presentation	**Style and Clarity**	The proof is well-organized and succinct, easy to read, without inclusion of irrelevant definitions or theorems. The proof moves seamlessly between symbolic notation and standard English.	Unnecessary detail(s) or awkwardness but start and conclusion clearly marked, The author generally avoids digressions, but may repeat some ideas in an unnecessary way.	The proof is well-organized but includes extraneous steps, definitions, theorems, or unnecessary repetition. Disjointed, arrows to additions, scratch outs	Totally confused (e.g. scattered about the page or nonsensical statements)
	Grammar	Grammar and spelling correct; equations are part of the sentence structure	Structure apparent, but with sentence fragments;	Spelling and grammar (or lack of) distracting	Difficult to decipher
	Hints (oral exam only)	No hints needed	Needs 1 hint and is able to continue from there	Needs more than one hint or a huge hint, but is able to continue afterwards	Many hints and prompts needed & was not able to continue with proof without prompting

4 Review and Preliminaries

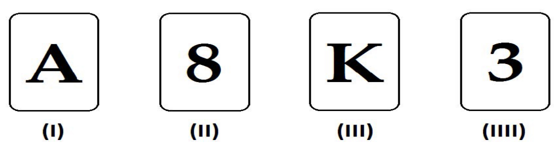
Remember whatever is not completed in class, will need to be completed as homework before the next class. You should work with your group and may be asked to present your work as a learning opportunity.

1. Name as many different methods of proof as you can.

2. You are given a deck of four cards, each of which has a **letter** on one side and a **digit** on the other side. Your job is to verify that the deck obeys the following rule:

   Rule: If a card has a vowel on one side, then it must have an even digit on the other side.

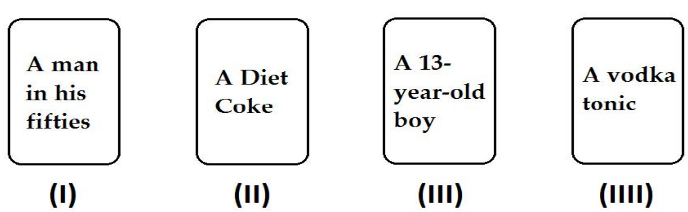
   Decide which of the following cards you would need to turn over in order to verify that the deck of cards obeys the rule.



3. You are a bouncer at a bar. You are responsible for enforcing the following rule:

   Rule: If a patron at the bar is drinking an alcoholic beverage, he/she must be at least 21 years of age.

   There are four patrons in the bar. Some patrons are visible so that you can see how old they are, but their drinks are hidden. Other patrons are invisible, though you can see what they are drinking. Decide which patrons you would need to investigate in order to be sure the rule is being followed.



4. Which of these sets contain $\sqrt{2}$?

   (a) $A = \{x \in \mathbb{Q} : x^2 < 3\}$

   (b) $B = \{x \in \mathbb{R} : x^2 < 3\}$

   (c) $C = A \cup B$

   (d) $D = A \cap B$

   (e) $E = A^c$

5. For the statement $p \implies q$, what is the contrapositive? What is the converse? Which is equivalent to $p \implies q$?

6. Rewrite the following statement by using quantifiers $(\exists, \forall, \notin, ...)$: "The sum of any two rational numbers is rational."

7. Write the negation of "$\forall x \in \mathbb{Z} \implies (x + y = 0) \cap (xy < 0)$."

8. Induction Practice:

   a) For $n = 1, 2, 3, 4$, compute $\displaystyle\sum_{k=1}^{n}(2k)$.

   b) Make a guess for the value of this sum for arbitrary n.

c) Use the PMI to prove that your conjecture. Recall the **Principle of Mathematical Induction:** Suppose P_n is a sequence of statements depending on a natural number $n = 1, 2, ...$ If we show that

(a) **Base Case:**

(b) **Inductive Step:**

Then we may conclude that all the statements P_n are true for all $n = 1, 2, ...$

9. Let $P_1, P_2, P_3, ...$ be a sequence of statements. Suppose the following two fact are known:

- The statement P_2 is true.
- If P_n is true, then P_{n+3} is true.

Which of the statements $P_1, P_2, P_3, P_4, P_5, P_6, P_7, P_8, P_9$, and P_{10} must be true?

5 Ordered Fields

5.1 Fields -but I thought this was Real Analysis!

Field Axioms: Let $\mathbb{R}$ be a set and assume the existence of two binary operations on $\mathbb{R}$ called $+$ & $\cdot$. If $\mathbb{R}$ is a field and $a, b \in \mathbb{R}$, we assume that our operations $+$ & $\cdot$ are:

- closed: if $a, b \in \mathbb{R}$, then $a + b \in \mathbb{R}$ and $a \cdot b \in \mathbb{R}$.
- commutative: $a + b = b + a$ and $a \cdot b = b \cdot a$.
- associate: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
- have identities: additive: $a + 0 = a$ and multiplicative: $a \cdot 1 = a$
- have inverses: $a + (-a) = 0$ and $a \cdot \frac{1}{a} = 1$
- $\cdot$ distributes over $+$: $a \cdot (b + c) = ab + ac$

Example 5.1. *Which of the following are fields:* $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}^c$*?*

We can introduce "order," denoted by __________, to a field by requiring the following *axioms*:

1. **Trichotomy Law:** $\forall x, y \in \mathbb{R}$, exactly one of the following relations holds:
2. $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $y < z$ then
3. $\forall x, y, z \in \mathbb{R}$, if $x < y$ then $x + z <$
4. $\forall x, y, z \in \mathbb{R}$, if $x < y$ and $z > 0$ then $xz <$

A field that satisfies the "order axioms" is called an **ordered field**. What are some examples of ordered fields?

Theorem 5.1. *Let $x, y, z \in \mathbb{R}$, then the following are true:*

1. *If $x + z = y + z$, then $x = y$*
2. $x \cdot 0 = 0$
3. $-0 = 0$
4. $(-1) \cdot x = -x$
5. $xy = 0 \iff x = 0$ *or* $y = 0$
6. *If $x < y$ and $z < 0$, then $xz > yz$*
7. $x < y \iff -y < -x$

Pf. See Text, but let's prove one way of #7.

Theorem 5.2. *Let $x, y \in \mathbb{R}$ such that $x \leq y + \epsilon$ for every $\epsilon > 0$. Then $x \leq y$.*

State the contrapositive of this Theorem:

Which statement is easier to prove?

Proof:

5.2 Inequalities

Definition 5.1. *Recall, the **absolute value** of a real number x, is*

$$|x| = \begin{cases} x & if \\ -x & if \end{cases}$$

Property 5.1. *for any x,* __________ $\leq x \leq$ __________

Definition 5.2. *The distance between a and b is $dist(a, b) =$* __________

Theorem 5.3. *Inequality Theorems: For $x, y \in \mathbb{R}$ and $a \geq 0$, then*

- $|x| \geq 0$
- $|x| \leq a \iff -a \leq x \leq a$
- $|xy| = |x| \cdot |y|$

Proof of $|x| \geq 0$:

Proof. Case 1: $x \geq 0$
By definition, $|x| = x \geq 0$.

Case 2: $x < 0$
By definition, $|x| = -x \geq 0$.

Therefore $|x| \geq 0$ ■

Proof of $|x| \leq a \iff -a \leq x \leq a$:

See text for proof of $|xy| = |x| \cdot |y|$
How many cases should we check?

Property 5.2. *For any a and positive r. TFAE:*

1. $|x - a| < r$
2.
3. *x belongs to the open interval:*

Pf. Omitted

Theorem 5.4. *The Triangle Inequality: For every pair of real numbers a and b,*

Proof:

Corollary 5.1. *Useful variation of Triangle Inequality:* $||a| - |b|| \leq |a - b|$

Pf. On your HW.

5.3 ICE 1: Ordered Fields

When you see a statement listed as [T/F] , I expect you to provide a proof (if you decide the statement is true) or a counter-example (if you decide the statement is false).

1. Archer is trying to identify the antecedent (the p) and consequent (the q) of the statement, "A cat can work here only if the cat has a college degree." He starts his direct proof with "Assume a cat has a college degree." Is he using the correct antecedent?

2. Let $b < 0$. Prove if $|x - b| < \frac{|b|}{2}$, then $x < \frac{b}{2}$.

3. Eva and Archer are trying to determine the validity of the following *similar* statements.

   (a) [T/F] For every pair of real numbers a and b, $|a - b| \leq |a| - |b|$.

(b) [T/F] Given a, $b \in \mathbb{R}$, we have $a = b$ only if for every $\epsilon > 0$, it follows that $|a - b| < \epsilon$.

(c) [T/F] Given a, $b \in \mathbb{R}$, we have $a = b$ if for some $\epsilon > 0$, it follows that $|a - b| < \epsilon$.

(d) [T/F] Given a, $b \in \mathbb{R}$, we have $a = b$ only if for some $\epsilon > 0$, it follows that $|a - b| < \epsilon$.

6 The Axiom of Completeness

Optional Reading:

- Feeling like you are a little lost with some notation and remembering your introduction to proofs? I suggest reading our suggested texts, *Understanding Analysis* by Abbott pages 1-13 and Pugh's *Real Mathematical Analysis* Section 1: Preliminaries (pages 1-10) both which review topics from previous math courses.
- Want to preview what we will be talking about next? Either read the section in our text about cuts or read *Understanding Analysis* pages 13-17. This has reviews about sets, unions, induction, functions, and more.
- Our textbook has an interesting section on "cuts" which are used to justify and prove the field axioms we stated in the previous section. Basically we like the $\mathbb{R}$ because it is complete (has no gaps). Sets like $\mathbb{Q}$ are not complete which causes problems. The real numbers fill the gaps that $\mathbb{Q}$ has. Pugh, the author of *Real Mathematical Analysis* uses *Dedekind cuts* -a way to imagine cutting the real number line in parts. We will talk about this on the next page!

6.1 Cuts

As we will learn in our next section, some fields, like $\mathbb{Q}$ are not ideal to work in because they are incomplete. That is, $\mathbb{Q}$ has gaps if we were to graph this set as a subset of $\mathbb{R}$. Pugh defines $\mathbb{R}$ as a *cut* in $\mathbb{Q}$.

Definition 6.1. *A* <u>*cut*</u> *in* $\mathbb{Q}$ *is a pair of subsets* $A, B \subseteq \mathbb{Q}$ *such that*
1) $A \sqcup B = \mathbb{Q}$ *(note this means the disjoin union so* $A \cap B = \emptyset$ *and* $A \neq \emptyset$ *and* $B \neq \emptyset$*)*
2) If $a \in A$ *and* $b \in B$*, then* $a < b$
3) A *contains no largest element.*

We denote a cut as $x = A|B$. So $\mathbb{R}$ is the class of all real numbers $x = A|B$.

Example 6.1. $A|B = \{r \in \mathbb{Q} : r < 2\}|\{r \in \mathbb{Q} : r \geq 2\}$

Example 6.2. $A|B = \{r \in \mathbb{Q} : r \leq 0 \text{ or } r^2 < 2\}|\{r \in \mathbb{Q} : r > 0 \text{ and } r^2 \geq 2\}$

Example 6.3. *Is* $A|B = \{r \in \mathbb{Q} : r \leq 3\}|\{r \in \mathbb{Q} : r > 3\}$ *a cut of* $\mathbb{Q}$*?*

Example 6.4. *Create your own cut of* $\mathbb{Q}$*. Justify it satisfies the criteria for this*

6.2 Suprema, Infima, and Bounds

This course is called **REAL** analysis because we will be working with the field, $\mathbb{R}$. Unless otherwise noted, you can assume variables are real variables. We're working our way up to the following, which is the fundamental difference between $\mathbb{Q}$ and $\mathbb{R}$ which is the property of *completeness*.

Definition 6.2. *Let $A \subset \mathbb{R}$, and $s \in \mathbb{R}$, then s is an upper bound for A if*

Definition 6.3. *Let $A \subset \mathbb{R}$, and $s \in \mathbb{R}$, then s is the LEAST upper bound or supremum of A if*

1.

2.

Definition 6.4. *Let $A \subset \mathbb{R}$, M is the greatest or maximum element of A if*

Definition 6.5. *Let $A \subset \mathbb{R}$, A is bounded above if*

Draw the Picture:

Example 6.5. *a) List 3 upper bounds in $\mathbb{R}$ of the interval $(-9, 9]$*

b)What is the least upper bound (or sup) of $(-9, 9]$?

c) Is it the greatest/maximum element?

Definition 6.6. *Let $A \subset \mathbb{R}$, and $l \in \mathbb{R}$, then l is a lower bound for A if $l \leq a$ $\forall a \in A$*

Definition 6.7. *Let $A \subset \mathbb{R}$, and $l \in \mathbb{R}$, then l is the GREATEST lower bound or infimum of S if*

1. *l is a lower bound AND*
2. *If $\tilde{l} > l$, then $\exists a \in A$ s.t. $a < \tilde{l}$.*

Definition 6.8. *Let $A \subset \mathbb{R}$, then m is the least or minimum element of A if $m \in A$ and $m \leq a$ $\forall a \in A$.*
(That is, m is the greatest lower bound in A.)

Definition 6.9. *Let $A \subset \mathbb{R}$, we say A is bounded below if there exists a lower bound.*

Definition 6.10. *Let $A \subset \mathbb{R}$, we say A is bounded if S has both an upper and lower bound.*

Example 6.6. *a) List 3 lower bounds in* $\mathbb{R}$ *of the interval* $(-9, 9]$

b)What is the greatest lower bound (or inf) of $(-9, 9]$*?*

c) Is it the least/minimum element?

Example 6.7. *What is the* $\sup(S)$ *for rational set* $S = \{3, 3.1, 3.14, 3.141, 3.1415, ...\}$ *?*

6.3 Least Upper Bound Property of $\mathbb{R}$

Axiom 6.1. ***The Axiom of Completeness [AoC]:*** *Every non-empty subset of the real numbers that is bounded above has at least one least upper bound.*

Considering $\mathbb{R}$ to be constructed by Dedekin cuts, our textbook actually proves that $\mathbb{R}$ satisfies the Axiom of Completeness and has the following theorem:

Theorem 6.1. $\mathbb{R}$ *is complete in that it satisfies the Axiom of Completeness or the "Least Upper Bound Property."*

Pf. See Text.

Corollary 6.1. *Every nonempty subset of* $\mathbb{R}$ *which is bounded below has an* __________

Proof. Suppose A is a nonempty subset of $\mathbb{R}$ that is bounded below. Then consider the set $-A := \{-a : a \in A\}$. $-A$ is a nonempty set that is bounded above since A is bounded below. By our Axiom of Completeness, $-A$ has a supremum say, s. Thus $-s$ is the infimum of A. ■

$\mathbb{Q}$ is not all of $\mathbb{R}$, as we will show next. This proof doesn't need the lemma below from your introduction of proofs course, but I like using it to prove the existence of irrational numbers.

Lemma 6.1. ***Euclid's Lemma:*** *If p is a prime and* $p|ab$ *then either* $p|a$ *or* $p|b$*.*

Property 6.1. *There is no rational number whose square is prime.*

Proof. We will prove this by a proof by contradiction. Assume if p is a prime, then $\sqrt{p}$ is rational. Then, by the definition of a rational number, $\sqrt{p} = \dfrac{a}{b}$ where $a, b \in \mathbb{Z}$ and are relatively prime. If $\sqrt{p} = \dfrac{a}{b} \implies p = \dfrac{a^2}{b^2} \implies (b^2)(p) = a^2$. So $p|a^2$ which by Euclid's Lemma, implies $p|a$. Since $p|a$, $\exists k \in \mathbb{Z}$ such that $a = pk$. Then since $p = \dfrac{a^2}{b^2}$, $p = \dfrac{(kp)^2}{b^2} \implies pb^2 = p^2k^2$. Thus $pk^2 = b^2$ and $p|b$. But this is a contradiction since according to Euclid's Lemma, if $p|ab$ then $p|a$ or $p|b$. Therefore, if p is prime, then $\sqrt{p}$ is irrational. ■

Theorem 6.2. *Given a nonempty set $A \subseteq \mathbb{R}$ with upper bound s, then $s = \sup(A)$ if and only if for every $\epsilon > 0$, there exists $a \in A$ such that $s - \epsilon < a$.*

6.4 ICE 2: The Completeness Axiom

1. For each set below, determine any min, maxs, sup, or inf and determine whether it is bounded above or below.

a) $[1, 2]$

b) $\{2, 7\}$

c) $\{r \in \mathbb{Q} : r^2 < 4\}$

d) $\{n + \frac{(-1)^n}{n} : n \in \mathbb{N}\}$

e) $[0, 1] \cup [2, 4]$

f) $\{0\}$

g) $\{x \in \mathbb{R} : x < 0\}$

h) $\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime }\}$

i) $\bigcap_{n=1}^{\infty} (1 - \frac{1}{n}, 1 + \frac{1}{n})$

2. If $A = \{\frac{n}{n+1} \mid n \in \mathbb{N}\}$, then which if any of these numbers are an upper bound for A: $\frac{1}{2}$, 1, 5?

3. When does $\max(A) \neq \sup(A)$?

4. [T/F] An upper bound for $A \subset \mathbb{R}$ is necessarily an element of A.

5. [T/F] A least upper bound for $A \subset \mathbb{R}$ is necessarily an element of A.

6. [T/F] A set $A \subset \mathbb{R}$ has at least one maximum.

7. [T/F] A set $A \subset \mathbb{R}$ has at most one maximum.

7 Density of Rationals

7.1 The Archimedean Property

Property 7.1. ***The Archimedean Property*** *The set of natural numbers is unbounded above in* $\mathbb{R}$.

** Idea: If x is quite small and y is quite large, some multiple of x will exceed y. **

The following 3 properties are equivalent to the Archimedean Property:

1. $\forall x \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $n > x$
2. $\forall x > 0$ and $y \in \mathbb{R}$, $\exists n \in \mathbb{N}$ s.t. $nx > y$
3. $\forall x > 0$, $\exists n \in \mathbb{N}$ s.t. $\frac{1}{n} < x$

Rephrased Archimedean Property: Given any positive number $x \in \mathbb{R}$, there are integers n and m so that $0 < 1/n < x < m$.

Proof:

Definition 7.1. *A set* $S \subset \mathbb{R}$ *is dense in* $\mathbb{R}$ *if every interval* (a, b) *contains an element of* S.

In other words, $\exists s \in S$ such that $s \in (a, b) \iff \exists s \in S$ such that $a < s < b$.

Theorem 7.1. ***Density of*** $\mathbb{Q}$ ***Theorem:*** *The set of rational numbers are "dense" in* $\mathbb{R}$.

In other words: for any real $a < b$,

"Heuristic" Proof of Density of $\mathbb{Q}$:

To show: $\forall a, b \in \mathbb{R}$,

In other words, we want to show: For some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$,

This is true if $\exists$ m and n such that

If "an" and "bn" differ by more than 1, then

Now $bn - an =$

By the Archimedean Property, $\exists n$ large enough such that

Reworking the inequality, we see that $\exists n$ large enough such that

Thus we have at least one integer m between __________ and __________.

Corollary 7.1. ***Density of*** $\mathbb{Q}^c$ ***Theorem:*** *The set of irrational numbers are "dense" in* $\mathbb{R}$.

Pf. Similar to Density of $\mathbb{Q}$ Theorem.

Next we will show that there is a real number, x, that satisfies $x^2 = 2$. Up until now, we have just shown that $\sqrt{p}$ is not rational, that is such a number x that satisfies $x^2 = p$ is not in $\mathbb{Q}$. Thus we want to show the existence of irrational numbers in $\mathbb{R}$ by proving there exists an $x \in \mathbb{R}$ such that $x^2 = 2$. (That is, we are proving this for $p = 2$.) We will do this by completing the next few exercises.

Example 7.1. *Prove that given the set* $T = \{t \in \mathbb{R} : t^2 < 2\}$, *the number* $\alpha = \sup(T)$ *exists.*

Example 7.2. *Prove that it is impossible for the number* $\alpha = \sup(T) = \sup\{t \in \mathbb{R} : t^2 < 2\}$ *to satisfy* $\alpha^2 < 2$.

Example 7.3. *Prove it is impossible for the number* $\alpha = \sup(T) = \sup\{t \in \mathbb{R} : t^2 < 2\}$ *to satisfy* $\alpha^2 > 2$. *Note this follows the method above only we now use* $(\alpha - \frac{1}{n})^2 = \alpha^2 - \frac{2\alpha}{n} + \frac{1}{n^2} > \alpha^2 - \frac{2\alpha}{n}$.

Proof. Suppose $\alpha^2 > 2$. Our goal is to find an element in T that is smaller than α, but whose square is larger than T.
Consider the element slightly less than α, $\alpha - \dfrac{1}{n}$ for $n \in \mathbb{N}$.
Then $(\alpha - \dfrac{1}{n})^2 = \alpha^2 - \dfrac{2\alpha}{n} + \dfrac{1}{n^2} > \alpha^2 - \dfrac{2\alpha}{n}$.
Because $\alpha^2 > 2$, $\alpha^2 - 2 > 0 \implies \dfrac{\alpha^2 - 2}{2\alpha} > 0$. [Note: Recall $\alpha = \sup(T)$, so $\alpha > 0$.] By the Archimedean Property, $\exists n \in \mathbb{N}$ such that $\dfrac{1}{n} < \dfrac{\alpha^2 - 2}{2\alpha} \implies \alpha^2 - 2 > \dfrac{2\alpha}{n} \implies$ $\alpha^2 - \dfrac{2\alpha}{n} > 2 \implies (\alpha - \dfrac{1}{n})^2 > \alpha^2 - \dfrac{2\alpha}{n} > 2$. Thus, $(\alpha - \dfrac{1}{n})^2 > 2$ which means $\alpha - \dfrac{1}{n}$ is an upper bound for T. But this is a contradiction because $\alpha = \sup(T)$ and thus must be an upper bound of T. Therefore, α^2 is not greater than 2.

■

Property 7.2. $\sqrt{2} \in \mathbb{R}$.

From the previous exercises, we have shown that $\alpha^2 = 2$; that is, $\sqrt{2}$ exists in $\mathbb{R}$.

8 Sequences

When you ask most calculus students what it means for a sequence to have a limit, they say that the sequence gets "closer and closer" to some number. But that's not a very good working definition. For example, the sequence

$$(\frac{\sin(n^2)}{n^2}) = (.84, -.18, .04, -.01, -.005, -.028, -.02, .01, -.01, -.005, .008, -.003, \dots)$$

converges to the limit 0, but sometimes the terms of the sequence get closer to 0 than the previous terms, and sometimes the terms get further from 0
(like jumping from $-.005$ to $-.028$).

By the same token, the sequence

$$(\frac{n+1}{n}) = (2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \dots)$$

gets "closer and closer" to 1 (its limit), but it also gets closer and closer to 0, or to -37, for that matter. To be true mathematicians, we're going to need a more precise set of definitions.

8.1 Introduction to Sequences

Definition 8.1. *A* ***sequence*** *is a countable list of things (possibly repeated). We write* $(a_1, a_2, a_3, \dots) = (a_n)$*. Note the shape of the brackets: "$\{\}$" is for sets, and "()" or "$<>$" are for sequences.*

Note: Some texts define a sequence a function from $\mathbb{N}$ to $\mathbb{R}$. Why does this definition make sense?

Definition 8.2. *A sequence* (a_n) ***converges*** *to a number* a *if for all* $\epsilon > 0$*, there exists an* $N \in \mathbb{N}$ *so that whenever* $n > N$*, we have* $|a_n - a| < \epsilon$*.*
In this case, we write $a = \lim_{n\to\infty} a_n$ *or* $a_n \to a$*.*

Note that a sequence converges to a finite value.

Unpacking this definition

A sequence (a_n) *converges* to a number a if for all $\epsilon > 0$, there exists an $N \in \mathbb{N}$ so that whenever $n > N$, we have $|a_n - a| < \epsilon$.

- The conditional "if for every $\epsilon > 0 \ldots$" describes the measure of "closeness." Any good proof of convergence will therefore begin with some variant of the simple sentence, "Let $\epsilon > 0$."

  Basically we are picking ϵ to be an ________________ ________________ of any real positive number.

- "…there exists an $N \in \mathbb{N} \ldots$" is a description of the term after which the sequence is always ϵ-close to the limit.

  Your job in the proof will be to name that N. Usually this requires doing a lot of work *before* you start writing the proof.

- "… whenever $n > N$ … " means the next part of your proof will say something like "Assume $n > N$."

- "… so that $|a_n - a| < \epsilon$" means at this point you need to demonstrate the terms of the sequence actually do fall ϵ-close to a.

The second bullet is worth emphasizing and repeating. Before you even start writing the proof, it will be your job to do some work to determine what an appropriate value of N is. When you are doing your scratch work, you will usually be starting with the inequality $|a_n - a| < \epsilon$ and work your way backwards to determine an appropriate value for N, usually describing N in terms of ϵ.

Example 8.1. *Let's prove that* $(\dfrac{n+1}{n})$ *converges to* 1.

First, scratch work: Assume $|\frac{n+1}{n} - 1| < \epsilon$. Rearrange this inequality to get an inequality for n in terms of ϵ.

Now, more scratch work. What does this mean we should choose for a value N?

Finally, we put it all together by filling in the holes in the proof below.

Proof. Let "______________", and let $N \in \mathbb{N}$ such that "______________". Suppose "______________". Then

$$\begin{aligned} |\tfrac{n+1}{n} - 1| &= |\underline{\hspace{3cm}}| && \text{(simplify this algebraically)} \\ &< \underline{\hspace{3cm}} && \text{(convert from } n \text{ to } N\text{)} \\ &< \epsilon && \text{(use your choice of } N \text{ to draw this conclusion)} \end{aligned}$$

Therefore, if $n > N$, we have $|\frac{n+1}{n} - 1| < \epsilon$, as desired. ■

Example 8.2. *Let's try (but fail) to prove that* $(\frac{n+1}{n})$ *converges to* 0*.*

Just like in Ex 8.1, try to write up your scratch work. Assume $|\frac{n+1}{n} - 0| < \epsilon$. Do some basic algebra to rewrite $|\frac{n+1}{n} - 0|$.

Can we choose $N \in \mathbb{N}$ so that $|\frac{n+1}{n} - 0|$ can be made arbitrarily small for $n > N$? If not, what problem do we run into?

Is $|\frac{n+1}{n} - 0|$ always going to be larger than a particular positive number, no matter what positive integer n might be?

Prove $(\frac{n+1}{n})$ does NOT converges to 0 :

Example 8.3. *Prove* $\lim_{n\to\infty} \frac{3n+1}{5n-3} = \frac{3}{5}$.

Definition 8.3. *If a sequence does not converge to a finite limit, we say the sequence is* ***divergent.***

Definition 8.4. ***Definition for the Nonexistence of a Limit:***
We say $\lim_{n\to\infty} a_n \neq L$ *if*

Definition 8.5. ***Definition of a Sequence with an Infinite limit:***
Give a sequence a_n*,* $\lim_{n\to\infty} a_n = \infty$ *means for any* $M > 0$*, we can find an integer* N *such that if* $n > N$*, then*

Example 8.4. *Prove that* $(\sqrt{n}) \to \infty$*.*

Pf.

8.1.1 ICE 3: Intro to Sequences

1. Prove $(\frac{1}{n^3} + \frac{3}{n} + 7) \to 7$.

2. Prove $\lim_{n\to\infty} \frac{3n+1}{n+3} = 3$

3. Prove $\dfrac{n^4 + 8n}{n^2 + 2} \to \infty$

4. Eva and Archer are preparing for their next Real Analysis class. They are wondering if the following statements are true or false. You do not need to provide any proofs, but give a conjecture about the truth of the following statements:

   (a) [T/F] Limits of sequences are unique.

   (b) [T/F] If $a_n \to a$ and $b_n \to b$, then $(a_n + b_n) \to (a + b)$.

   (c) [T/F] If $(a_n + b_n) \to (a + b)$, then $a_n \to a$ and $b_n \to b$.

   (d) [T/F] Every bounded sequence is convergent. (A sequence (a_n) is *bounded* if there exists a real number $M > 0$ such that $|a_n| < M$ for all $n \in \mathbb{N}$.)

   (e) [T/F] Every convergent sequence is bounded.

5. Consider the sequence of garden-variety functions (f_n) given

$$f_n(x) = \begin{cases} n & \frac{1}{n} \leq x \leq \frac{2}{n} \\ 0 & otherwise \end{cases}$$

Graph the first few functions. Does this sequence seem to converge? If so, to what function?

8.2 More Advanced Sequences

Theorem 8.1. *Let b_n, a_n be sequences of Real numbers and let $b \in \mathbb{R}$. If $a_n \to 0$ and for some $k > 0$ and for some $m \in \mathbb{N}$, we have $|b_n - b| \leq k|a_n|$, $\forall n \geq m$, then $\lim_{n\to\infty} b_n = b$.*

Proof. Proof: Let $\epsilon > 0$ and $k > 0$. Suppose a_n, b_n are sequences and suppose $a_n \to 0$. Since $a_n \to 0$, $\exists$ __________ $\in$ __________ such that if __________ $>$ __________ then
$|a_n - 0| <$ __________. We also know there exists an m such that for $n > m$, $|b_n - b| \leq k|a_n|$. So we want to choose an N such that both of the previous statements are true so we let $N =$ max{__________, __________} Then for $n > N$, we have that $n >$ __________ and $n >$ __________ and therefore $|b_n - b| \leq k|a_n| <$ ____________________
Thus $b_n \to b$. ∎

Property 8.1. *Limits of sequences are unique.*

Proof:

Example 8.5. *Prove* $\lim_{n\to\infty} \frac{n^2+2n}{n^3-5} = 0$

Property 8.2. *If* $a_n > 0\ \forall n \in \mathbb{N}$, *then* $\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} (\frac{1}{a_n}) = 0$.

8.2.1 ICE 4: Sequences Continued

1. Prove $\lim_{n\to\infty} \frac{3n+1}{n-3} = 3$. (How is the proof different from $\lim_{n\to\infty} \frac{3n+1}{n+3} = 3$?)

2. Prove $2^n \to \infty$

3. Eva and Archer remember from their Calculus I class that $\lim_{x\to 0} \dfrac{\sin x}{x} = 0$. Now that they are hot shots in Real Analysis, they want to rigorously prove that $\frac{\sin(n^2)}{n^2} \to 0$. As they are working on the proof, Archer is concerned saying "Doesn't $\sin(n^2)$ change as n approaches 0?" Eva says that we don't even have to worry about this since we can just use a super nice property about $\sin x$... Help Eva and Archer complete their proof.

Note: Want to read more about sequences? I suggest reading our suggested text, *Understanding Analysis* by Abbott pages 35-42 which reiterates some of our sequence terms and also introduces "ϵ-neighborhoods" which is helpful in more general metric spaces.

4. Complete the proof of Proposition 8.2, "If $a_n > 0$ $\forall n \in \mathbb{N}$, then $\lim_{n\to\infty} a_n = \infty \iff \lim_{n\to\infty} (\frac{1}{a_n}) = 0$.." (Prove whichever way we we didn't complete in class.)

8.3 Sequence Theorems

We continue our discussion about sequences by proving some statements about general sequences!

Definition 8.6. *A sequence (a_n) is* **convergent** *if there exists some real number a so that (a_n) converges to a.*

Definition 8.7. *A sequence (a_n) is* **bounded** *if there exists a finite real number $M > 0$ such that $|a_n| < M$ for all $n \in \mathbb{N}$.*

Example 8.6. *Let $a_1 = 1$ and $a_{n+1} = (\frac{n}{n+1})(a_n)^2$.*

a) Find $a_2, a_3,$ & a_4.

Determine the $\inf, \sup$*, max, or min, if it exists for* $\{a_n : n \in \mathbb{N}\}$

b) Is a_n bounded?

c) If so, use induction to prove a_n is bounded above.

Theorem 8.2. ***The Squeeze Theorem:*** *Let $b_n \leq a_n \leq c_n$ $\forall n \in \mathbb{N}$. And suppose b_n and c_n both converge to L. Then $a_n \to$ __________.*

Pf.

Theorem 8.3. *Every convergent sequence is bounded.*

Pf.

Theorem 8.4. *If $(a_n) \to a$ and $c \in \mathbb{R}$, then $(c \cdot a_n) \to c \cdot a$.*

Pf. See #5 on Ice 8.3.1

Theorem 8.5. *If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n + b_n) \to a + b$.*

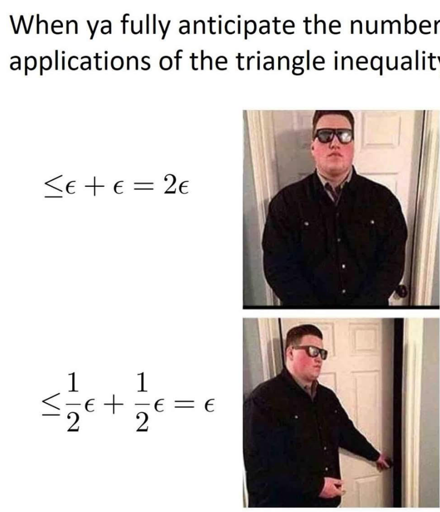
Pf.

Theorem 8.6. *If* $(a_n) \to a$ *and* $(b_n) \to b$*, then* $(a_n \cdot b_n) \to a \cdot b$.

Pf.

Theorem 8.7. *If* $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$*, with* $b \neq 0$*, then* $(a_n) \to a$ *and* $(b_n) \to b$.

Pf. Omitted.



[2]

[2]Meme from https://awwmemes.com/i/when-ya-fully-anticipate-the-number-of-applications-of-the-47be2f2

8.3.1 Ice 4: Sequence Theorems

For these problems, first go through and determine if the statements are true or false. Then prove the true statements. We may have proved some of these statements in class. If so, state the theorem or property.

1. [T/F] A convergent sequence of negative numbers has a negative limit.

2. [T/F] A convergent sequence of rational numbers has a rational limit.

3. [T/F] Every bounded sequence is convergent.

4. [T/F] Every convergent sequence is bounded.

5. [T/F] If $(a_n) \to a$ and $c \in \mathbb{R}$, then $(c \cdot a_n) \to c \cdot a$.

6. [T/F] If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n + b_n) \to a + b$.

7. [T/F] If $(a_n + b_n) \to a + b$, then $(a_n) \to a$ and $(b_n) \to b$.

8. [T/F] If $(a_n) \to a$ and $(b_n) \to b$, then $(a_n \cdot b_n) \to a \cdot b$.

9. [T/F] If $(a_n \cdot b_n) \to a \cdot b$, then $(a_n) \to a$ and $(b_n) \to b$.

10. [T/F] If $(a_n) \to a$ and $(b_n) \to b$, then $\dfrac{a_n}{b_n} \to \dfrac{a}{b}$, provided $b \neq 0$.

Draw a sequence of cats. Is it convergent?

8.4 Monotonic Sequences

Definition 8.8. *A sequence (a_n) is* **increasing** *if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and* **decreasing** *if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is* **monotone** *if it is either increasing or decreasing. We say (a_n) is strictly increasing if $a_n < a_{n+1}$ and a_n is strictly decreasing if $a_n > a_{n+1}$.*

Example 8.7. *Determine whether the following sequences are monotonic:*

a) $\dfrac{1}{n^2}$

b) n^3

c) $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$

d) $\dfrac{(-1)^n}{n^2}$

Theorem 8.8. ***Monotone Convergence Theorem:*** *Every Bounded & Monotone Sequence is Convergent.*

Theorem Restated: A monotone sequence converges $\iff$ it is bounded.

In fact,

1) If a_n is non-decreasing/ monotonically increasing and bounded, $a_n \to$ __________

2) If a_n is non-increasing/ monotonically decreasing and bounded, $a_n \to$ __________

We will prove this statement, but just do the case where a_n is bounded and non-increasing $\to$ convergent.)

Proof. Pf. Let a_n be a bounded and monotonically decreasing sequence. If a_n is bounded then $A = \{a_n\}$ is bounded in $\mathbb{R}$ so by the Axiom of Completeness, A has an infimum. Let L=inf A. Now we will show $a_n \to L$. Let $\epsilon > 0$. Since $L = \inf A$, $L + \epsilon$ is not a ____________________. So, $\exists\ a_N \in A$ s.t. $a_N < L + \epsilon$. Because a_n is monotonically decreasing $\forall\ n > N$, $a_n \leq a_N < L + \epsilon$. So $a_n < L + \epsilon$. Furthermore, $L - \epsilon$ is a lower bound since $L - \epsilon < L$. So $L - \epsilon < a_n$ $\forall n > N$. Therefore, $L - \epsilon < a_n < L + \epsilon \to |a_n - L| < \epsilon$. ■

Theorem 8.9. ***Unbounded Monotone Sequence Theorem:*** *An unbounded monotonically increasing sequence diverges to* __________ *and an unbounded monotonically decreasing sequences diverges to* __________.

Proof. Pf. (unbounded monotonically increasing case)

Let a_n be an unbounded and monotonically increasing sequence. [To show $a_n > M$] Let $M > 0$. Since a_n is unbounded and increasing $A = \{a_n\}$ is unbounded above. Thus $\exists N \in \mathbb{N}$ s.t. $a_N > M$ because a_n is monotonically increasing. So $\forall n > N$, since $a_n > a_N$, $a_n > M \implies a_n \to \infty$. ■

Notice: From our theorems, a monotone sequence either converges to a finite number, diverges to $+\infty$, or diverges to $-\infty$.

8.4.1 Ice 5: Monotonic Sequences

1. Consider the sequence defined by $a_1 = 1$ and $a_{n+1} = \sqrt{1 + a_n}$
   a) Determine a_2, a_3, & a_4

   b) Is a_n bounded? Is a_n monotonic?

   c) Show a_n is bounded above by 2.

   d) Show that a_n is monotonically increasing.

   e) What does this tell us about a_n?

   f) Determine a such that $a_n \to a$.
   Key Technique: If a_n converges to a, then $\lim_{n\to\infty} a_{n+1} =$ __________

2. [T/F] Every monotone sequence is convergent.

3. [T/F] Every convergent sequence is monotone.

4. [T/F] If $(a_n) \to a$ and $a_n \geq 0$ for all n, then $a \geq 0$. [Hint: you can prove this by contradiction.]

5. [T/F] If $(a_n) \to a$ and $(b_n) \to b$ with $a_n \geq b_n$ for all n, then $a \geq b$.
(Hint: use previous result!)

8.5 Subsequences

Definition 8.9. *Let (a_n) be a sequence of real numbers and let $n_1 < n_2 < n_3 < \cdots$ be an increasing sequence of natural numbers. Then the sequence*

$$a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, \ldots$$

is called a **subsequence** *of a_n and is denoted by (a_{n_j}), where $j \in \mathbb{N}$ indexes the subsequence.*

Example 8.8. *Consider the sequence $a_n = \{-1, \frac{1}{2}, -3, \frac{1}{4}, -5, \frac{1}{6}, -7, ...\}$*

a) What are some subsequences of a_n

b) What is the subsequence we get from the subsequence selection sequence $n_k = \{2, 3, 5, 7, 9, ...\}$?

Property 8.3. *For a sequence a_n, $n_k \geq k$ for all k. (The index of the subsequence is greater than or equal to index of sequence.*

Theorem 8.10. *Suppose $\lim\limits_{n\to\infty} a_n = a$, then* ***every*** *subsequence of a_n converges to a*

Proof. Suppose $a_n \to a$ and let a_{n_j} be an arbitrary subsequence of a_n. Let $\epsilon > 0$. Since $a_n \to a$, $\exists N \in \mathbb{N}$ such that for $j > N$, $|a_j - a| < \epsilon$. By Property 8.3, the index of a sequence's subsequence is greater than or equal to index of sequence, $n_j > j > N$, $|a_{n_j} - a| < \epsilon$, $a_{n_j} \to a$. ■

Corollary 8.1. *If there are two subsequences of a_n which does not converge to the same value, then a_n is not a convergent sequence.*

Examples:

Theorem 8.11. *Every sequence has a* ____________________ *subsequence!*

Pf. Will prove in HW.

Theorem 8.12. ***Bolzano-Weierstrass Theorem:*** *Every bounded sequence has a ____________________ subsequence!*

Pf.

Definition 8.10. *A* ***subsequential limit*** *of a sequence is any "value" __________ or __________ to which some subsequence converges.*

Question: How many subsequential limits can a sequence have?

Definition 8.11. *Let S be the set of subsequential limits of s_n.*
That is $S =$

The ***limit superior*** *of a bounded sequence s_n is $\limsup_{n\to\infty} s_n = \sup S$*

The ***limit inferior*** *of a bounded sequence s_n is $\liminf_{n\to\infty} s_n = \inf S$*

Property 8.4. *If s_n is in $\mathbb{R}$ then $s_n \to s \iff$*

Example 8.9. *Find the set S of subsequential limits of the following sequences:*

$a_n = \{0, 1, 2, 0, 1, 3, 0, 1, 4, ...\}$

$c_n = (-n)^n$

$b_n = \frac{(-1)^n}{n}$

$d_n = (-1)^n(2 - \frac{1}{n})$

Definition 8.12. *A set $S \subset \mathbb{R}$ is said to be* ***sequentially compact*** __ *if every sequence in S has a subsequence that*

Example 8.10. *Is $(0, 5]$ sequentially compact?*

Example 8.11. *Is $[5, \infty)$ sequentially compact?*

Theorem 8.13. ***Sequential Compactness Theorem:*** *Assume $a < b$, then $[a, b]$ is sequentially compact. That is every sequence in $[a, b]$ has a subsequence that converges to a point in* __________.

Proof: Note this is just...

Property 8.5. *A closed and bounded subset of $\mathbb{R}$ is sequentially compact.*

Proof: Follows Sequential Compactness Theorem.

Example 8.12. *Prove there exists bounded intervals in $\mathbb{R}$ that are not sequentially compact.*

Example 8.13. *Prove there exists closed intervals in $\mathbb{R}$ that are not sequentially compact.*

8.5.1 Ice 6: Subsequences

1. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 2, 5, 13, \dots)$.

2. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 2, 1, 5, 3, 13, 8, \dots)$.

3. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 1, 2, 3, 5, 8, \dots)$.

4. [T/F] One subsequence of $(1, 1, 2, 3, 5, 8, 13, 21, 34, \dots)$ is $(1, 1, 2, 2, 5, 5, 13, 13, \dots)$.

5. [T/F] If some subsequence of (x_n) converges to x, then (x_n) converges to x as well.

6. [T/F] If every subsequence of (x_n) converges to x, then (x_n) converges to x as well.

7. [T/F] Every monotone sequence of real numbers contains a convergent subsequence.

8. Use subsequences to prove that $a_n = (-1)^n$ does not converge.

9. Create a sequence that has 3 subsequential limits.

8.6 Cauchy Sequences

Definition 8.13. *A sequence (a_n) is called a Cauchy (pronounced COE-shee) sequence if for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $n, m \geq N$, it follows that $|a_m - a_n| < \epsilon$.*

How is the definition for a Cauchy sequence different from the definition for a convergent sequence? A *convergent* sequence's terms get arbitrarily close to __________________ in the long run.

A *cauchy* sequence's terms are mutually and arbitrarily close to __________________ in the long run.

Theorem 8.14. *If a sequence is convergent, then it is Cauchy.*

Careful: The converse is not generally true, but it is true for sequences of real numbers!

Proof of Theorem:

Theorem 8.15. *If a sequence of real numbers is a Cauchy Sequence, then it must converge to a finite value.*

Pf. See Text.

Idea

Thus...

Theorem 8.16. *A real sequence a_n is convergent if and only if a_n is* __________.

Why is this theorem so useful?

Consider the infinite decimal: 4.1020030001000020000030000001...

Does this "equal" a real number?

If so, what kind of number?

In either case, what is the value of that number?

Suppose we wrote the number as a sequence: $\{4.1, 4.10, 4.102, 4.1020, 4.10200, 4.102003, ...\}$

Does this sequence converge?

If so, what is the value of its limit?

What is the trouble?

8.6.1 Ice 7: Cauchy Sequences

1. [T/F] Every Cauchy sequence in $\mathbb{R}$ is bounded.

2. [T/F] Every bounded sequence in $\mathbb{R}$ is Cauchy.

3. [T/F] Every Cauchy sequence in $\mathbb{R}$ is monotone.

4. [T/F] Every monotone sequence in $\mathbb{R}$ is Cauchy.

5. [T/F] If a sequence (a_n) of real numbers is Cauchy, then (a_n) has a convergent subsequence.

6. Consider the sequence $a_n = \{-1, \frac{1}{2}, -3, \frac{1}{4}, -5, \frac{1}{6}, -7, ...\}$

   a) Is a_n bounded?

   b) Is a_n monotone?

   c) Does a_n converge?

   d) Is a_n Cauchy?

   e) What is the $\sup a_n$?

   f) What is the $\inf a_n$?

   g) Does a_n have a max?

   h) Does a_n have a min?

   i) $\limsup\limits_{n\to\infty} a_n =$

   j) $\liminf\limits_{n\to\infty} a_n =$

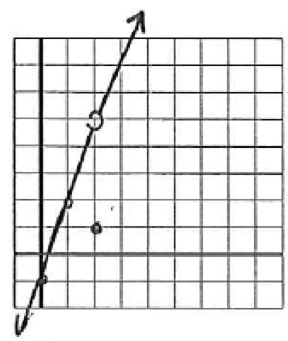
7. Show that the sequence defined by $a_1 = 1$ and $a_{n+1} = 3 - \dfrac{1}{a_n}$ is convergent by showing that it is monotonic and bounded. What is the limit?

9 Limits of Functions

As you may recall from previous calculus classes, Calculus is based on the concept of a limit. But up until now, we have been using intuitive phrases like, "$f(x)$ gets closer and closer to," "as x is sufficiently close to," or "$f(x)$ gets arbitrarily large." All of these phrases give us intuition, but are vague. We need to transform this intuitive idea of a limit into rigorous and precise mathematical statements.

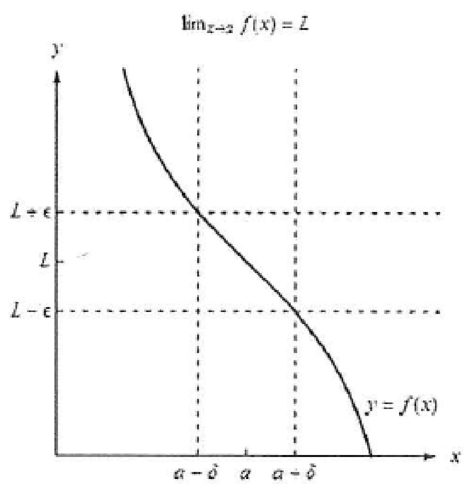
Example 9.1. *Consider the function below*

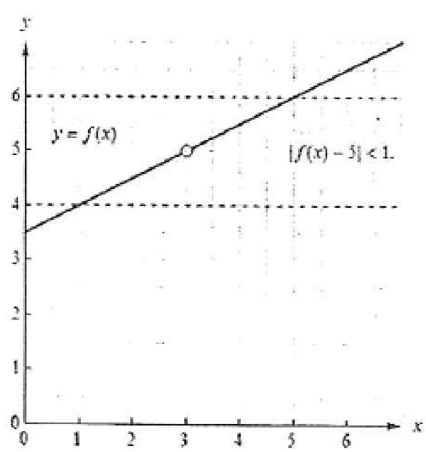
$$f(x) = \begin{cases} 3x - 1, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

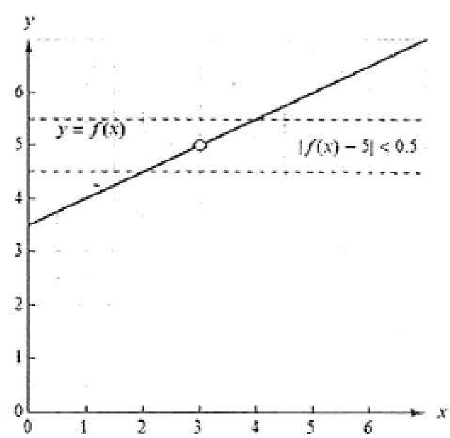


We want to show that the limit of $f(x)$ as x approaches 2 is L= __________
That means we want to show that $f(x)$ is close to __________ when x is close to __________.

That is, we want __________, the distance between $f(x)$ and our limit, to be __________ when the distance between x and __________, __________ is small.







This really is just a game. Suppose your classmate, Reid, gives you a small number, say, __________. He says he wants you to get $f(x)$ within __________ of our supposed limit. Your goal is to find another small number, denoted, __________ such that when our x is within __________ distance of 2, __________ $<$ __________. That is we want to find __________ such that when __________ __________, then __________ __________.

Let's find our δ:

Well, wasn't that just splendid. Now suppose Reid responds, "Well what about if I choose __________ instead!" We just do it again.

Unfortunately, as fun as this is, we are beginning to get a bit fed up. Before Reid gives you another small number, you want to be ready for him. Now you want to find a rule that will allow you to quickly pick your δ given any small challenging number Reid gives you. Suppose Reid gives you any small number. Let's call this number "epsilon" __________. Given $\epsilon > 0$, want to find δ such that $|f(x) - 5| < \epsilon$ whenever $|x - 2| < \delta$
Let's find our δ:

Now we have shown that $f(x)$ gets **arbitrarily** close to 5 when x is close enough to 2. Using our game/work as a model we now get our precise definition of a limit.

9.1 The Precise Definition of the Limit of a Function

Let $f(x)$ be defined on some open interval containing a (except possibly a itself). We say the limit of $f(x)$ as x approaches a is L, written if for any number $\epsilon > 0$ there is a corresponding number $\delta > 0$ such that if

Definition 9.1. ***<u>The Precise Definition of the Limit of a Function:</u>***
$\lim_{x \to a} f(x) = L$ *if*

One thing students often miss, is that we control ________ NOT ________!
"Real" Life Illustration:
Suppose we are a factory that makes the raw materials to make a bolt. We give the metal to Bolts 'R Us and they make it into a bolt. Bolts 'R Us tells us that they need their bolts to be within a certain error. This means we have make sure the error we have in cutting the metal is within an appropriate amount so that it will also be within the error after Bolts 'R Us makes it into a bolt. Bolts 'R Us is giving us an ϵ value and we have make sure our error, the δ value is small enough so that once Bolts 'R Us takes our metal and makes it into a bolt (like a function), the bolt is still within the error Bolts R' Us needs.

Can you think of a metaphor, or simile, or story that describes what we are doing when we are finding the limit of a function using our precise definition? This will be part of your next homework assignment.

General Format for a δ, ϵ Proof
Prove $\lim_{x \to a} f(x) = L$

Pf. Let $\epsilon > 0$

Let $\delta =$

If $|x - a| < \delta$

Then

$\vdots$
$|f(x) - L| < \epsilon$

Example 9.2. *Prove* $\lim_{x \to -1} (3x + 5) = 2$ *using the* ϵ, δ *definition of a limit.*

Example 9.3. *Prove* $\lim_{x \to 2}(x^2 + 2x - 7) = 1$ *using the* ϵ, δ *definition of a limit.*

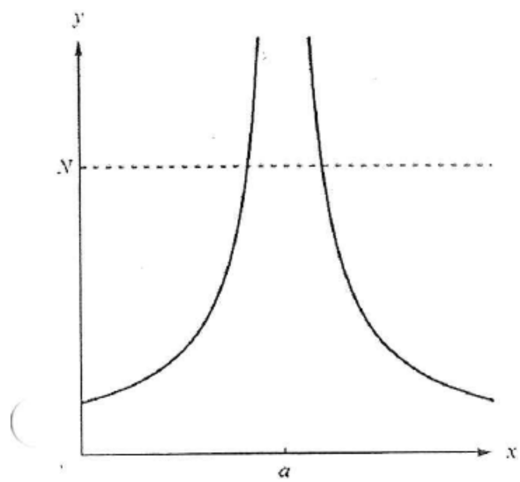
9.2 One-sided and Infinite Limits

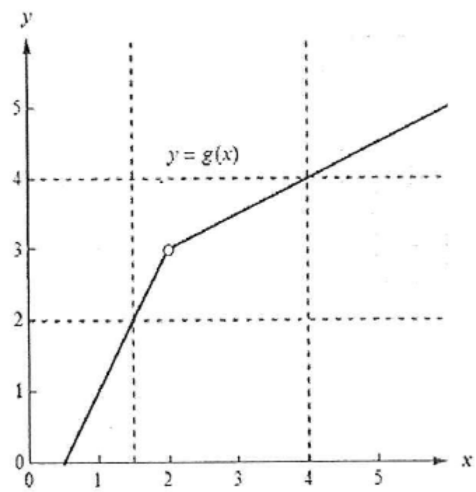
Definition 9.2. ***The Precise Definition of the Left-Hand Limit of a Function:***
$\lim\limits_{x\to a^-} f(x) = L$ *if for all $\epsilon > 0$, there is a $\delta > 0$ such that*

Definition 9.3. ***The Precise Definition of the Right-Hand Limit of a Function:***
$\lim\limits_{x\to a^+} f(x) = L$ *if for all $\epsilon > 0$, there is a $\delta > 0$ such that*

Definition 9.4. ***The Precise Definition of the Infinite Limit of a Function:***
We say $\lim\limits_{x\to a} f(x) = \infty$ if for every $M > 0$, there is a $\delta > 0$ such that

We say $\lim\limits_{x\to a} f(x) = -\infty$ if for every **negative** number N ($N < 0$), there is a $\delta > 0$ such that





Example 9.4. *Prove that* $\lim\limits_{x\to 3} \dfrac{1}{(x-3)^2} = \infty$.

9.3 Proving a limit does not exist

Definition 9.5. ***Definition for the Nonexistence of a Limit:***
Assume f is defined for all values of x near a (except possibly at a). We say $\lim_{x\to a} f(x) \neq L$ if for some $\epsilon > 0$, there is NO value of $\delta > 0$ satisfying $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

Idea:

Example 9.5. *Prove $\lim_{x\to 0} f(x)$ does not exist for $f(x) = \begin{cases} 0, & x \in Q \\ 1, & x \notin Q \end{cases}$*
Fact: Remember between every rational number, there is an irrational number!

9.4 Ice 8: Limits of Functions

1. Use Calculus 1 methods to determine what $\lim_{x \to 3} \frac{x^2 - 5x + 6}{x - 3}$ is. Then prove it is true using the precise definition of a limit (aka use δ, ϵ).

2. Prove $\lim_{x \to -2} (\frac{1}{x}) = -\frac{1}{2}$ using the precise definition of a limit (aka use δ, ϵ).

3. Prove using a ϵ, δ proof that $\lim_{x \to 0^+} (\frac{1}{x} + 1) = \infty$

4. What is the $\lim_{x \to 2} \dfrac{x^3 - 8}{x - 2}$? Prove it using our formal definition of a limit.

10 Continuity

We are now ready to define what it means for a function to be continuous. Intuitively what does it mean?

Idea: As the independent variable x approaches a point ______ in the domain of f, then the images $f(x)$ approach ______.

Let's make this idea more precise...

Definition 10.1. *Let $f : D \to \mathbb{R}$, and let $a \in D$. We say that f* ***is continuous at a*** *if $\forall \epsilon > 0$ there exists a $\delta > 0$ such that whenever $0 < |x - a| < \delta$ (and $x \in D$), we have $|f(x) - f(a)| < \epsilon$.*

In other words, $\lim_{x \to a} f(x) = f(a)$.
Can we rephrase this definition of a functional limit in terms of sequences?

Theorem 10.1. *Continuity Theorem: Given a function $f : D \to \mathbb{R}$ and a limit point a of D. TFAE (The following are equivalent):*

- *f is continuous at a:*
  That is $\lim_{x \to a} f(x) = f(a)$
- *For every sequence $(x_n) \subset D$ s.t. $(x_n) \to a$, it follows that $(f(x_n)) \to f(a)$.*
  In other words,

We will prove this on the next page.

The following is a useful corollary to the Continuity Theorem:

Corollary 10.1. *Divergence Criterion for Functional Limits Let f be a function defined on A, and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim_{n \to \infty} x_n = \lim_{n \to \infty} y_n = c, \text{ but } \lim_{n \to \infty} f(x_n) \neq \lim_{n \to \infty} f(y_n),$$

then the functional limit $\lim_{x \to c} f(x)$ does not exist.

To prove Theorem 10.1 is equivalent to our definition:

Pf. *To show the sequence definition implies the limit definition:*
Assume not. Suppose that for an arbitrary sequence, x_n which converges to a that $f(x_n) \to f(a)$, but that $\lim_{x \to a} f(x) \neq f(a)$. This means $\exists \epsilon > 0$ such that if $|x - a| < \delta$, then $|f(x) - f(a)| > \epsilon$. But if $x_n \to a$, $\exists N \in \mathbb{N}$ such that for $n > N$, $|x_n - a| < \delta$ but $|f(x_n) - f(a)| > \epsilon$ which is a contradiction. Therefore $\lim_{x \to a} f(x) = f(a)$.

To show the limit definition implies the sequence definition:

Definition 10.2. *Global Continuity: We say f is a continuous function if f is continuous at every point in its domain.*

To use the sequence definition to prove continuity, like with δ, ϵ proofs, we need structure!

Strategy for using the Sequence Definition of Continuity:
To prove Continuous at a:
1) Let x_n be any arbitrary sequence in the domain of f such that $x_n \to a$.
2) Use our tools from sequences to show $f(x_n) \to f(a)$

To prove **DIS**continuous at a:
1) Pick a sequence x_n in the domain of f s.t $x_n \to a$
2) Use our tools from sequences to show $\lim_{n \to \infty} f(x_n)$ does not exist or does not equal $f(a)$

Strategy for using the δ, ϵ Definition of Continuity:
To prove Continuous at a:

1) Prove $\lim_{x \to a} f(x) = f(a)$:
That is, show $\forall \epsilon > 0, \exists \delta > 0.$ s.t. if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.
To prove **DIS**continuous at a:
1) Prove $\lim_{x \to a} f(x) \neq f(a)$.

Theorem 10.2. *If g is continuous at x_0 and f is continuous at $g(x_0)$, then $f \circ g$ is continuous at x_0.*

Pf. See Text

Example 10.1. *Prove that $f(x) = 2x^2 - 1$ is continuous.*
a) Prove by using the sequence definition.

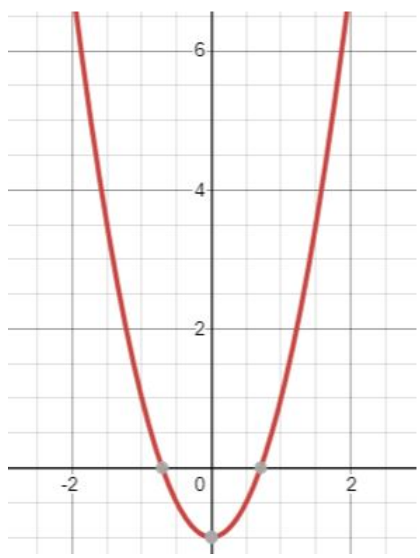


Figure 1: $f(x) = 2x^2 - 1$

b) Prove using our Continuity Theorem (δ, ϵ proof).

Example 10.2. *Let domain of* $(f) = (-\infty, 0) \cup (0, \infty)$ *and define* f *as stated below. Is* f *continuous? If not, where is* f *discontinuous?*

$$f(x) = \begin{cases} 1, & x > 0 \\ -1, & x < 0 \end{cases}$$

Example 10.3. *Consider* $g(x) = \begin{cases} x^2 \sin(\frac{1}{x}), & x \neq 0 \\ 0, & x = 0 \end{cases}$

Is $g(x)$ *continuous at 0? Prove it? Is* g *a continuous function?*

Example 10.4. *Consider* $f(x) = \begin{cases} 0, & x \in Q \\ 1, & x \notin Q \end{cases}$
Prove $f(x)$ *is not continuous at 0. [We proved this in Ex 9.5, let's prove this using our sequence definition.]*

The following is another useful corollary to the Continuity Theorem:

Corollary 10.2. *Corollary (Algebraic Limit Theorem for FLs) Let f and g be functions defined on a domain $A \subset \mathbb{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then,*

1. $\lim_{x \to c} kf(x) = kL$, *for all* $k \in \mathbb{R}$,
2. $\lim_{x \to c} [f(x) + g(x)] = L + M$,
3. $\lim_{x \to c} [f(x)g(x)] = LM$, *and*
4. $\lim_{x \to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, *provided* $M \neq 0$.

10.1 Ice 9: Continuity of Functions

1. Is it possible for a function to be discontinuous at all points in its domain? If so, explicitly state a function with this property.

2. Is it possible for a function to be continuous at just one point in its domain? If so, explicitly state a function with this property.

3. Is it possible for a function to be discontinuous at just one point in its domain? If so, explicitly state a function with this property.

4. Sketch the modified Dirichlet function h given below. Where, if anywhere is the function continuous?
$h(x) = \begin{cases} x & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$

5. Sketch the Thomae function given below. Where, if anywhere is the function continuous?

$$t(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/n & \text{if } x = m/n \in \mathbb{Q}\backslash\{0\} \text{is in lowest terms with } n > 0 \\ 0 & \text{if } x \notin \mathbb{Q}. \end{cases}$$

6. Let f and g be continuous functions defined on a domain $D \subset \mathbb{R}$.

   (a) Prove for constant k, $kf(x)$ is a continuous function. Prove it 2 different ways.

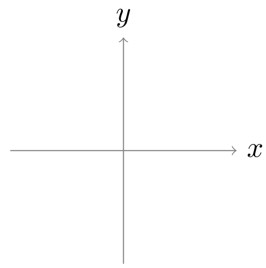
(b) Prove $(f \cdot g)(x)$ is a continuous function.

10.2 Uniform Continuity

When we deal with continuous functions, we have been talking about f being continuous at a given arbitrary point x_0. And typically our δ depends on ________ and ________.

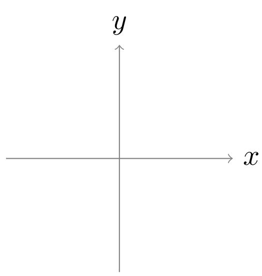
Example 10.5. *Let f be defined below. Graph it.*

$$f(x) = \begin{cases} 1 + 2x, & x > 0 \\ -1 + 2x, & x < 0 \end{cases}$$



Notice for any $\epsilon > 0$, and any x_0 in the domain of f, you can show that $\delta = \min\{\frac{\epsilon}{2}, |x_0|\}$. Loosely speaking, δ has to get really small for $x_0's$ near 0 (the ________ neighborhood.)

Example 10.6. *Let $g(x) = \dfrac{1}{1+|x|}$ $\forall x$. Graph it.*



Notice for any $\epsilon > 0$, and any x_0 in the domain of g, you can show that $\delta = \epsilon$ will work. Loosely speaking, the "nastiest point" in g's domain is $x = 0$ and $\delta = \epsilon$ works there... hence everywhere else too.

Idea for "Uniform Continuity": Some continuous functions are *tame* enough that, once ϵ is specified, the same δ will work for all ________.

Definition 10.3. *δ, ϵ Definition of Uniform Cont. Let $f : D \to \mathbb{R}$, we say f is* ***uniformly continuous on*** *D if*

Note the difference from regular continuity:
f is **continuous** at a given point a if...

Definition 10.4. ***Sequence Definition of Uniform Continuity:*** *$f : D \to \mathbb{R}$ is uniformly continuous provided that whenever x_n and y_n are sequences in D s.t.*
if $\lim_{n\to\infty}[x_n - y_n] = 0$, then $\lim_{n\to\infty}[f(x_n) - f(y_n)] = 0$

Again, the essence of uniform continuity is that the same δ works __________ __________ __________!

Also uniform continuity is a __________ property of the function f and its domain D; it is NOT a local property of f alone.

Analogy:

- Saying "f is continuous on D is like saying "for all destinations in Illinois and for all vehicles, there exists an amount of gasoline which will get you there in that vehicle."
- Saying "f is **uniform** continuous on D is like saying "for all destinations in Illinois, there exists an amount of gasoline which will get you there no matter what vehicle you drive."

These are two very different statements. The latter hints that a "nastiest case (vehicle) scenario" might be crucial. Watch for a "nastiest case" mindset in what follows.

From Examples 10.5 and 10.6 above, which function is uniform continuous? f or g?

Example 10.7. *Prove $f(x)$ is not uniformly continuous*

$$f(x) = \begin{cases} 1 + 2x, & x > 0 \\ -1 + 2x, & x < 0 \end{cases}$$

Example 10.8. *Prove* $g(x) = \dfrac{1}{1+|x|}$ *is uniformly continuous.*

10.2.1 Uniform Continuity Theorems

Theorem 10.3. *If a function is continuous on* $[a, b]$ *(a closed and bounded interval), then* f *is* __________ *continuous on* $[a, b]$*.*

Idea:

Note: if f is continuous on $[a, b] \cup [c, d]$, we get the same result.

Proof omitted. See text.

Property 10.1. *If* f *is uniform continuous, then* f *is* __________*.*

Is the converse true?

Theorem 10.4. ***Cauchy Criteria for Uniform Continuity Theorem*** *If* f *is uniformly continuous on* D*, and* s_n *is a Cauchy sequence in* D*, then* $f(s_n)$ *is a Cauchy sequence.*

Pf. Next page.

Corollary 10.3. *If* f *is a function on* D *and there exists a Cauchy sequence,* s_n*, in* D *such that* $f(s_n)$ *is not a Cauchy sequence, then* f *is not uniformly continuous.*

Proof of Thm 10.4, the **Cauchy Criteria for Uniform Continuity Theorem**: “If f is uniformly continuous on D, and s_n is a Cauchy sequence in D, then $f(s_n)$ is a Cauchy sequence.”

Theorem 10.5. *$f : (a, b) \to \mathbb{R}$ is uniformly continuous on (a, b) if and only if it can be extended to a function $\bar{f}$ that is continuous on $[a, b]$.*

Pf: Omitted. See Text (too much topology needed.)

Example 10.9. *Let $f(x) = \dfrac{1}{x^2}$. Prove that f is uniformly continuous on $(1, 3)$.*

Example 10.10. *Let* $f(x) = \dfrac{1}{x^2}$. *Prove* f *not uniformly continuous on* $(0, 1]$ *(even though it is continuous on* $(0, 1]$*).*

General Comments from Dr. H:

- When you are given a specific function and you think that function is continuous or uniformly continuous,
  - using the $\delta - \epsilon$ definition (structured proof) is often nice when proving the function is continuous **at a given point**.
  - using the sequence definition (structured proof) is nice when proving global continuity and uniform continuity
- When given an arbitrary function, and you think the function is continuous or uniformly continuous,
  - using the sequence definition is often easier.
- When given a specific function which you think is NOT continuous (or NOT uniformly continuous)
  - using the sequence definition, theorems, and properties is often easier.
- When given an arbitrary function which you think is NOT continuous (or NOT uniformly continuous)
  - using the sequence definition, theorems, and properties is often easier.

10.2.2 ICE 10: Uniform Continuity

1. Which of the following functions are uniform continuous on the specified set? Justify your answers. Use any theorems you wish for justification.

   a) $f(x) = \dfrac{e^x}{x}$ on $[3, 5]$

   b) $k(x) = \dfrac{e^x}{x}$ on $(0, 2)$

   c) $g(x) = mx + b$ on $\mathbb{R}$

d) $h(x) = \dfrac{x^2 - 4}{x - 2}$ on $(3, 4)$

e) $h(x) = \dfrac{x^2 - 4}{x - 2}$ on $(2, 4)$

2. $[T/F]$ and justify: Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is uniformly continuous.

3. $[T/F]$ and justify: Every continuous function $f:(0,1]\to\mathbb{R}$ is uniformly continuous.

4. $[T/F]$ and justify: Every continuous function $f:[0,1]\to\mathbb{R}$ is uniformly continuous.

5. $[T/F]$ and justify: Every uniformly continuous function $f:D\to\mathbb{R}$ is continuous.

6. Prove if $f : D \to \mathbb{R}$ & $g : D \to \mathbb{R}$ are uniformly continuous & bounded, then fg is uniformly continuous. Hint: write $f(x)g(x) - f(y)g(y) = f(x)[g(x) - g(y)] + g(y)[f(x) - f(y)]$

10.3 Continuity Theorems

The next few questions look at a "theorem" you used in Calculus many times without realizing you were taking advantage of deep, mysterious properties of the Real numbers. That is, you'd assume a function had a max or a min, and you'd look for it by taking derivatives and checking endpoints. Is that theorem actually true, or were there things we were hiding from you? Does the max always exist?

1. [T/F] If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f attains a maximum value. That is, there exists $x_0 \in \mathbb{R}$ such that $f(x) \leq f(x_0)$ for all $x \in \mathbb{R}$.

2. [T/F] Let $A \subset \mathbb{R}$ be an open interval (set). If $f : A \to \mathbb{R}$ is continuous, then f attains a maximum value on A. That is, there exists $x_0 \in A$ such that $f(x) \leq f(x_0)$ for all $x \in A$.

3. [T/F] Let $B \subset \mathbb{R}$ be a bounded interval (set). If $f : B \to \mathbb{R}$ is continuous, then f attains a maximum value on B.

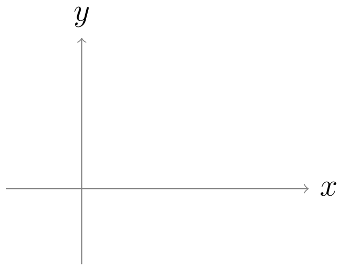
4. [T/F] Let $C \subset \mathbb{R}$ be a closed interval (set). If $f : C \to \mathbb{R}$ is continuous, then f attains a maximum value on C.

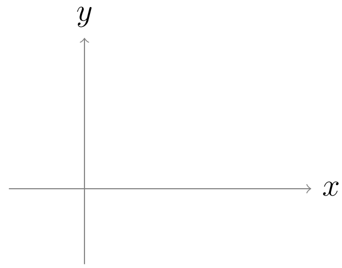
5. [T/F] Let $K \subset \mathbb{R}$ be a closed and bounded interval (compact set). If $f : K \to \mathbb{R}$ is continuous, then f attains a maximum value on K.

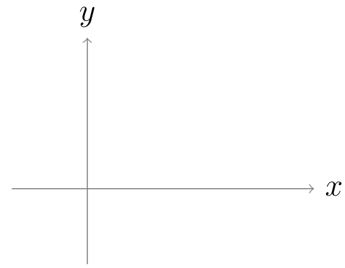
10.3.1 Extreme Value Theorem

Theorem 10.6. ***Extreme Value Theorem:*** *Let $f : D \to \mathbb{R}$ be a ________ function and let D be ________ and ________ (so for example think $[a, b]$). then..*

a) f is ________ on ________
b) $\exists \hat{x} \in D$ s.t. $f(\hat{x}) =$ ________ and $\exists \tilde{x} \in D$ s.t. $f(\tilde{x}) =$
Where M is the ________ value of f on D and m is the ________ value of f on D.







EVT Restated:

Corollary 10.4. *If $f : [a, b] \to \mathbb{R}$ is a continuous function , then f is bounded on $[a, b]$ and attains a maximum and minimum value on $[a, b]$.*

Why do we need closed?

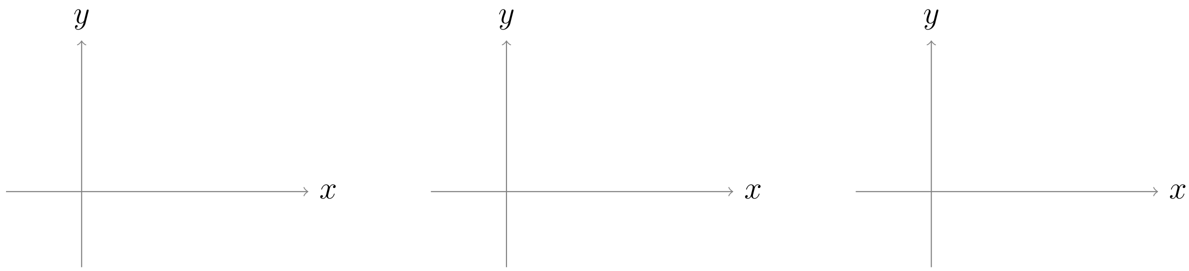
Why do we need bounded?

Proof. Pf. [T.S. f is bounded:] Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Assume f is not bounded on $[a, b]$, this means for every $\tilde{n} \in \mathbb{N}$, $\exists x_n \in [a, b]$ s.t. $|f(x_n)| >$ ________. Thus we have created a sequence $x_n \in [a, b]$ s.t. $f(x_n) \to$ ________. But $x_n \in [a, b]$ which means ________ $< x_n <$ ________ so x is ________. So by the ________ ________ Theorem, $\exists$ a ________ subsequence, say $x_{n_k} \to L$. Since $[a, b]$ is sequentially compact, $L \in [a, b]$ which means L is bounded and thus is finite. Since f is continuous, $f(x_n) \to$ ________ $< \infty$ This is a contraction since from earlier we found that $f(x_n) \to$ ________. Thus, f is bounded.

[T.S. f has a maximum (note this proof needs to appeal to compactness which we discuss slightly in our reading quiz, but we won't cover until Real Analysis II...):] Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Since $[a, b]$ is bounded and compact, by The Axiom of Completeness, $[a, b]$ achieves both a max and min value and furthermore by Theorem 5.3.2 in your book, $f([a, b])$ compact and thus $\exists M \in f([a, b])$ such that $\sup_{x \in [a,b]} f(x) = M < \infty$. Recall during our Sup Ice Sheet Exercise 6.2, we proved that if $M = \sup(A)$, then $\forall \epsilon > 0, \exists a \in A$ s.t. $a > M - \epsilon$. Using this fact, note that for each $n \in \mathbb{N}$ $\exists y_n \in [a, b]$ s.t. $M - \frac{1}{n} < f(y_n) < M$. Since $\lim\limits_{n\to\infty}(M - \dfrac{1}{n}) = \lim\limits_{n\to\infty} f(y_n) = \lim\limits_{n\to\infty} M$, by the __________ Theorem, $\lim\limits_{n\to\infty} f(y_n) = M$. Since, y_n is convergent, it is bounded, thus by the ____________________ Theorem, y_n has a convergent subsequence, say $y_{n_k} \to y_0$. Since f is continuous, then $f(y_{n_k}) \to f(y_0)$ and therefore $f(y_n) \to f(y_0) = M$. Therefore, $\exists y_0$ s.t. $f(y_0)$ is the maximum of f. ■

10.3.2 Intermediate Value Theorem

Theorem 10.7. ***Intermediate Value Theorem:*** *Suppose f is continuous on $[a, b]$ (a closed and bounded domain), then for any y s.t. $f(a) < y < f(b)$, $\exists \hat{x} \in (a, b)$ s.t* __________.



Pf. Omitted (see text -in part because it uses topology, we don't have time to go over.)

Example 10.11. *Prove $x2^x = 1$ for some $x \in (0, 1)$.*

Definition 10.5. *x_0 is a fixed point if* __________

Theorem 10.8. ***Fixed Point Theorem:*** *Let $f : [a, b] \to [a, b]$ be continuous, then there exists at least one "fixed point" on $[a, b]$.*

Pf.

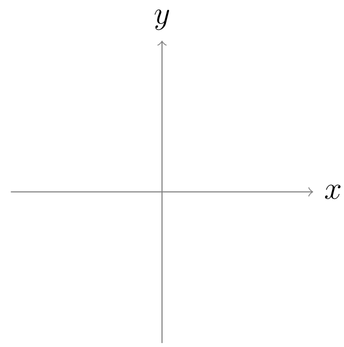
10.3.3 ICE 11: Continuity Theorems

1. If $f : [-1, 1] \to \mathbb{R}$ is continuous and $f(-1) > -1$ and $f(1) < 1$, show that f has a fixed point.

2. Suppose $h : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ are continuous and $h(a) \leq g(a)$ and $h(b) \geq g(b)$. Show that there exists an $x \in [a, b]$ such that $h(x) = g(x)$

3. Let f be defined below. Graph it. Is it continuous on its domain?

$$f(x) = \begin{cases} 1 + 2x, & x > 0 \\ -1 + 2x, & x < 0 \end{cases}$$



4. Is $f(x) = \frac{1}{x^2}$ continuous on $(0, 1]$?

5. Suppose f is continuous on $[0, 2]$ and $f(0) = f(2)$. Prove there exist $x, y \in [0, 2]$ s.t. $|y - x| = 1$ and $f(x) = f(y)$. Hint: Consider $g(x) = f(x + 1) - f(x)$ on $[0, 1]$.

11 Differentiation

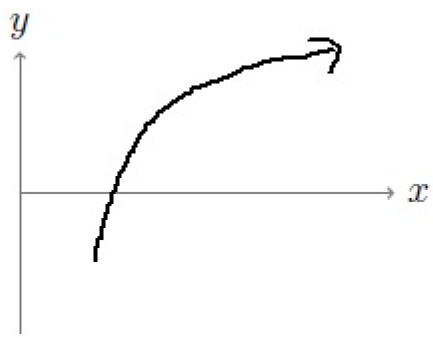
11.1 The Derivative

Idea: The derivative of a function f at a point $x = a$ is the slope of its local linear approximation. If the function is not locally linear there, we'll say that it has no derivative at that point.

Definition 11.1. *The **derivative of a function** f **at the point** $x = a$ equals the number $f'(a)=$ (Provided the limit ________.)*

Recall, the slope of the line tangent to $f(x)$ at $x = a$ is ________

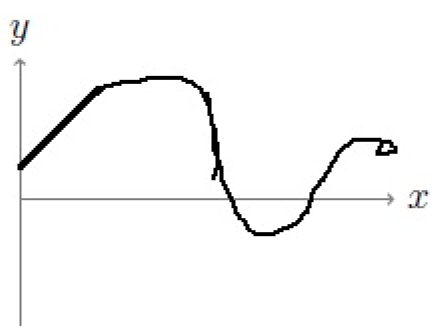
The equation of the tangent line of $f(x)$ at $x = a$ is



Under what circumstances is a function NOT differentiable at a point?

-
-
-

Definition 11.2. *The **derivative function** of f is given by $f'(x)=$ (Provided the limit ________.)*



Example 11.1. *If* $f(x) = C$ *then* $f'(x) =$ __________.

Since $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h) - f(x)}{h} = \lim\limits_{h\to 0} \dfrac{C - C}{h} = \lim\limits_{h\to 0} \dfrac{0}{h} =$ __________

Example 11.2. *If* $f(x) = x$ *then* $f'(x) =$ __________.

Since $f'(x_0) = \lim\limits_{x\to x_0} \dfrac{f(x) - f(x_0)}{x - x_0} = \lim\limits_{x\to x_0} \dfrac{x - x_0}{x - x_0} = \lim\limits_{x\to x_0} 1 =$ __________

Example 11.3. *If* $f(x) = x^2$ *then* $f'(a) =$ __________.

Since $f'(a) = \lim\limits_{x\to a} \dfrac{f(x) - f(a)}{x - a} = \lim\limits_{x\to a} \dfrac{x^2 - a^2}{x - a} = \lim\limits_{x\to a}$

Example 11.4. *If* $f(x) = \sqrt{x}$ *for* $x > 0$ *then* $f'(x) =$ __________.

Property 11.1. *If* $f(x) = x^n$ *for* $n \in \mathbb{N}$, *then* $f'(x) =$ __________

Proof: Recall the difference of powers formula:
$x^n - x_0^n = (x - x_0)(x^{n-1} + x^{n-2}x_0 + ... + xx_0^{n-2} + x_0^{n-1})$

Theorem 11.1. *If I is an interval containing the point c, f is differentiable at c iff for every sequence in $x_n \in I - \{c\}$ that converges to c, the following sequence converges:* $\dfrac{f(x_n) - f(c)}{x_n - c}$

Furthermore, if f is differentiable at c, then the sequence above converges to $f'(c)$.

Proof: This is just applying the sequential criterion for limits. This theorem is often useful when trying to show a given function is *NOT* differentiable.

Example 11.5. *True or false: If a function is continuous, it is differentiable.*

Example 11.6. *Let $f(x)$ be defined below. To see if f is differentiable at $x = 1$, we note that $f(1) =$ __________ and we examine the one-sided limits for the derivative at $x = 1$.*

$$f(x) = \begin{cases} 2x^3 + 2, & x \geq 1 \\ 3x^2 + 1, & x < 1 \end{cases}$$

Theorem 11.2. *If $f : I \to \mathbb{R}$ is differentiable at $a \in I$, then f is **continuous** at a.*

Idea for proof:

Proof of Thm 11.2:

Proof. Suppose f is differentiable at a. So by definition, $f'(a) = \lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}$ exists. Consider the equation of the tangent line at $x = a$:

$$f(x) = f'(a)(x-a) + f(a).$$

$\lim_{x\to a} f(x) = \lim_{x\to a}(f'(a)(x-a) + f(a)) = f'(a)(a-a) + f(a) = f'(a)\cdot(0) + f(a) = f(a)$.
$\therefore \lim_{x\to a} f(x) = f(a)$ ■

Theorem 11.3. ***The Chain Rule:*** *Let I and J be intervals in $\mathbb{R}$ and $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where the range of f is a subset of J. If f is differentiable at a and g is differentiable at $f(a)$, then $g \circ f$ is differentiable at a and $(g\circ f)'(a) =$*

Proof. Suppose I and J be intervals in $\mathbb{R}$ and $f : I \to \mathbb{R}$ and $g : J \to \mathbb{R}$, where the range of f is a subset of J. Suppose f is differentiable at a and g is differentiable at $f(a)$. Notice that the difference quotient,

$$\frac{g(f(x)) - g(f(a))}{x-a} = \frac{g(f(x)) - g(f(a))}{f(x)-f(a)} \cdot \frac{f(x)-f(a)}{x-a}.$$

[Note the key algebra manipulation is to multiple by a factor of $\dfrac{f(x)-f(a)}{f(x)-f(a)}$.]

At first it may seem like we can just take the limit of this quotient as $x \to a$. But there is the possibility that $(f(x) - f(a))$ may be 0 even when $(x - a) \neq 0$. To help with this we consider this limit:

$$\lim_{y\to f(a)} \frac{g(y) - g(f(a))}{y - f(a)} = g'(f(a))$$

Note this is equal to $g'(f(a))$ because g is differentiable at $f(a)$. Thus we can now define a new function $H : J \to \mathbb{R}$ by:

$$H(y) = \begin{cases} \dfrac{g(y) - g(f(a))}{y - f(a)} & y \neq f(a) \\ g'(f(a)) & y = f(a) \end{cases}$$

Notice H is continuous at $f(a)$. Because f is differentiable at a, by Thm 11.2, f is continuous at a. Thus by Thm 10.2, $H \circ f$ is continuous at a. Thus

$$\lim_{x\to a} H(f(x)) = H(f(a)) = g'(f(a)).$$

There for by the definition of H, $g(a) - g(f(a)) = H(y)(y - f(a))$ for all $y \in J$. Thus if $x \in I$ then $f(x) \in J$ so

$$g(f(x)) - g(f(a)) = H(f(x))(f(x) - f(a))$$

And for $x \in I$ such that $x \neq a$, we have $\dfrac{g(f(x)) - g(f(a))}{x - a} = H(f(x))\dfrac{f(x) - f(a)}{x - a}$. Thus

$$(g \circ f)'(a) = \lim_{x \to a} \frac{g(f(x)) - g(f(a))}{x - a} = \lim_{x \to a} H(f(x))\frac{f(x) - f(a)}{x - a} = \lim_{x \to a} H(f(x)) \cdot \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

$= g'(f(a)) \cdot f'(a)$. ■

Property 11.2. ***Derivative Rules:*** *Let f and g be differentiable, then...*

1. $(cf)' =$ __________
2. $(f \pm g)' =$ __________
3. $(fg)' =$ __________
4. $(\frac{1}{g})' =$ __________
5. $(\frac{f}{g})' =$ __________

Proof of 3:

Proof of 4: On HW

Property 11.3. *For $n \in \mathbb{Z}$, define the set D to be $\mathbb{R}$ if $n \geq 0$ and to be $\{x \in \mathbb{R} : x \neq 0\}$ then the function $f : D :\to \mathbb{R}$ defined as $f(x) = x^n$ is differentiable with derivative $=nx^{n-1}$.*

Proof: We can use property 4 from above!

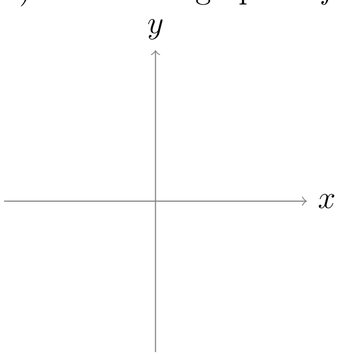
11.2 Ice 12: Derivatives

1. True or false and justify: If a function is continuous at x_0, f is differentiable at x_0.

2. True or false and justify: If a function is differentiable at x_0, then f is continuous at x_0.

3. True or false and justify: If $f^2 : \mathbb{R} \to \mathbb{R}$ is differentiable, then $f : \mathbb{R} \to \mathbb{R}$ is differentiable.

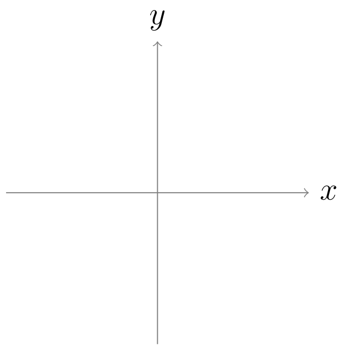
4. Let $f(x) = x^2$ for $x \geq 0$ and $f(x) = 0$ for $x < 0$.
   a) Sketch the graph of f



   b) Use the definition of the derivative to show that f is differentiable at $x = 0$.

   c) Use a theorem and part b to prove f is continuous at $x = 0$.

d) Calculate f' on $\mathbb{R}$ and sketch its graph.



e) Prove that f' is continuous at $x = 0$.

f) Prove f' is not differentiable at $x = 0$.

5. Determine where $f(x) = |x^2 - 1|$ is differentiable. Then find the equation of $f'(x)$

6. Go back to Question 4 and prove f' is continuous on $\mathbb{R}$ and differentiable for $x \in \mathbb{R} - \{0\}$

11.3 Derivative Theorems

The next few theorems will allow us to work up to proving the Mean Value Theorem, which is a very useful theorem.

11.3.1 Fermat's Theorem

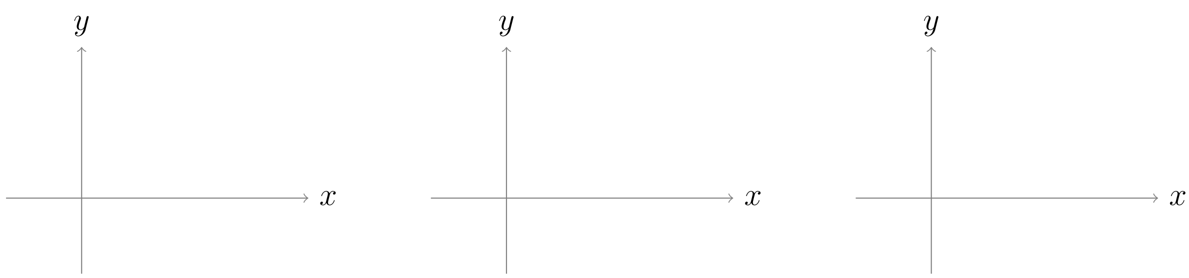
Recall, in Calculus I, if we wanted to find a max or min of a function, we needed to check points where:

1.

2.

3.

Theorem 11.4. ***Fermat's Theorem:*** *If f is differentiable on (a, b) and if f assumes its max or min at a point $c \in (a, b)$, then* ______



Main technique of Proof:

Proof of Fermat's Theorm: We will just prove this for c is a max. In the case where f has a minimum at c, just use the same method by apply the results to the function "$-f$".

Proof. Let f be differentiable on (a, b) and assume f has a maximum value at $c \in (a, b)$. Thus, $f(c) \geq f(x)$ $\forall x \in (a, b)$. Let x_n be a sequence such that $a < x_n < c$. (For example we could use the sequence $x_n = (c - \frac{1}{n})$ for large enough n.) Because f is differentiable and because

$$f(c) \geq f(x), \forall x \in (a, b) \implies f(x_n) - f(c) \leq 0$$

and because

$$x_n < c \implies x_n - c < 0,$$

then $f'(c) = \lim\limits_{n\to\infty} \dfrac{f(x_n) - f(c)}{x_n - c} \geq 0$. Let $\tilde{x_n}$ be a sequence such that $c < x_n < b$, then because $f(\tilde{x_n}) - f(c) \leq 0$ and $\tilde{x_n} - c > 0$,

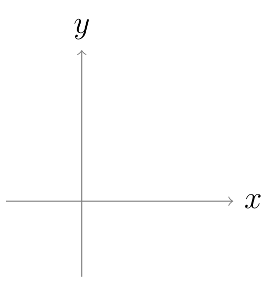
$$f'(c) = \lim_{n\to\infty} \frac{f(\tilde{x_n}) - f(c)}{\tilde{x_n} - c} \leq 0$$

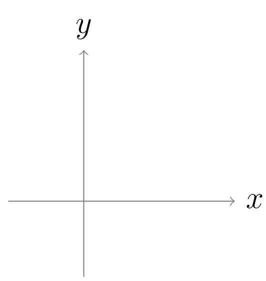
Thus $0 \leq f'(c) \leq 0 \implies f'(c) = 0$. ■

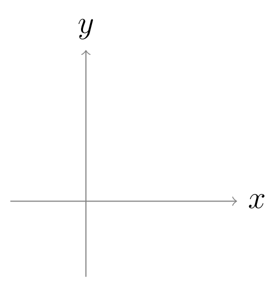
11.3.2 Rolle's Theorem

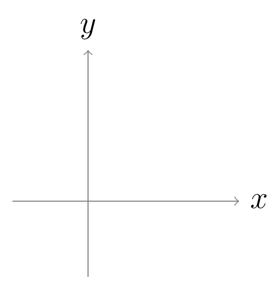
Theorem 11.5. ***Rolle's Theorem:*** *Suppose f is*
1) continuous on $[a, b]$,
2) differentiable on (a, b), and
3) $f(a) = f(b)$
Then there is at least one number $c \in (a, b)$ such that $f'(c) =$

Examples:









Key for the Proof:

Proof of Rolle's Theorem:

Proof. Let f be a function that is continuous on $[a, b]$, differentiable on (a, b), and $f(a) = f(b)$. Since f is continuous on $[a, b]$, by the EVT, $\exists x_1, x_2 \in (a, b)$ such that $f(x_1)$ and $f(x_2)$ are the maximum value and minimum of f on (a, b) respectively. Thus $f(x_2) \leq f(x) \leq f(x_1)$ for every $x \in (a, b)$. We now have two cases:

Case 1: If $f(x_1) = f(x_2)$:
Then $f(x) = f(x_2)\ \forall x \in (a, b) \implies f$ is a constant function so $f'(x) = 0\ \forall x \in (a, b)$.

Case 2: If $f(x_1) > f(x_2)$:
Then by Fermat's Theorem, $f'(x_1) = 0 = f(x_2)$. So Rolle's Theorem is satisfied since we can pick $c = x_1$ or $c = x_2$. ∎

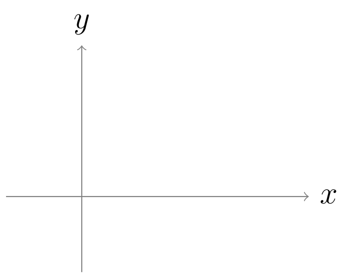
Example 11.7. *Show $f(x) = 2x + \cos(x)$ has exactly one real root.*

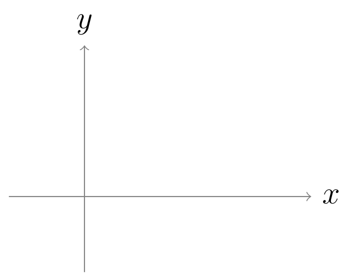
11.3.3 Mean Value Theorem

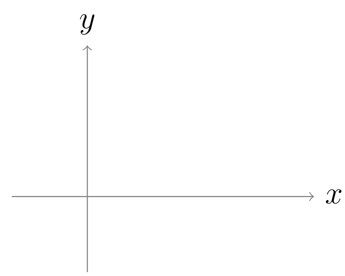
Theorem 11.6. ***Mean Value Theorem (MVT):*** *Suppose f is*
1) continuous on $[a, b]$ and
2) differentiable on (a, b)
Then $\exists c \in (a, b)$ such that $f'(c) =$

Equivalently, we can write: $f(b) - f(a) =$

Examples:







Idea for Proof:

Proof of Mean Value Theorem:

Proof. Let f be continuous on $[a,b]$ and differentiable on (a,b). Let $g(x)$ be the secant line between points $(a,f(a))$ and $(b,f(b))$. So

$$g(x) = \frac{f(b)-f(a)}{b-a}(x-a)+f(a).$$

Let $h(x) = f(x) - g(x)$ which is continuous on $[a,b]$ and differentiable on (a,b) because f and g both are. Thus,

$$h(a) = f(a) - g(a) = f(a) - \frac{f(a)-f(a)}{b-a}(a-a) - f(a) = f(a) - f(a) = 0.$$

and

$$h(b) = f(a) - g(a) = f(b) - \frac{f(b)-f(a)}{b-a}(b-a) - f(a) = f(b) - f(b) + f(a) - f(a) = 0.$$

Thus $h(a) = 0 = h(b)$ so by Rolle's Theorem, $\exists c \in (a,b)$ such that $h'(c) = 0 \implies f'(c) - g'(c) = 0 \implies f'(c) = g'(c) = \dfrac{f(b)-f(a)}{b-a}$. ■

Example 11.8. *Use the MVT to derive Bernoulli's Inequality for $x > 0$: If $x > 0$, then $(1+x)^n \geq 1 + nx \; \forall n \in \mathbb{N}$.*

Theorem 11.7. *Let f be differentiable on an interval I. Then*
a) If $f'(x) > 0\ \forall x \in I$, then f is strictly increasing on I. That is, if $x > y$, $f(x) > f(y)$.
b) If $f'(x) < 0\ \forall x \in I$, then f is strictly decreasing on I. That is, if $x > y$, $f(x) < f(y)$.

Proof Idea: Just use MVT: For any $c \in [x, y]$ (or $[y, x]$), $f'(c) = \dfrac{f(y) - f(x)}{y - x}$.

11.4 ICE 13: Consequences of the Mean Value Theorem

1. The amazing math kitties, Eva and Archer, are working on proving the following corollaries from the Mean Value Theorem. Help them finish their proofs

   Corollary 11.1. ***Zero Derivative Implies Constant Function:***
   If f is continuous and $f'(c) = 0$ at all points in (a, b), then f is a constant function on (a, b).

   The Kitties have decided to use a proof by contradiction. They assume f is continuous function whose derivative is equal to 0 for all points in (a, b), but which is not a constant function. Since f is not a constant, Eva says we have at least two distinct points $x_1, x_2 \in [a, b]$ such that $a \leq x_1 < x_2 \leq b$ and that $f(x_1) \neq f(x_2)$. Help them use the MVT to finish their proof.

   Corollary 11.2. ***Functions with Equal Derivatives Differ by a Constant:***
   If $f'(x) = g'(x)$ for all points in (a, b),then $\exists$ a constant C such that $f(x) - g(x) = C$.

   Eva says this proof isn't too bad, we should just apply the previous corollary to a new function, $h(x) = f(x) - g(x)$. Help them finish proving the corollary.

2. Eva says that we can also use the MVT to estimate the value of a function at a point. Use the MVT to prove the following inequalities:

a) $6\frac{2}{7} < \sqrt{40} < 6\frac{1}{3}$
Hint: Apply the MVT to the function $f(x) = \sqrt{x}$ on $[36, 40]$. Why do you suppose we picked the number 36 as our lower bound?

b) $e^x > 1 + x$ for $x > 0$
Hint: Consider the function $f(t) = e^t$ on the interval $[0, x]$.

11.5 Taylor's Theorem

In you Calculus class, you learned about sequences and series. These topics were mainly used in order to provide theoretical background for Power Series. In fact, we discussed how you can use sums of polynomials (power series) to approximate complex functions. This is especially helpful because polynomials are easy to differentiate, integrate, and graph - especially for computers. In fact, we found that we could create a power series called a Taylor Series using higher derivatives of functions. Recall a Taylor Series for $f(x)$ is given by $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)(a)}}{n!}(x-a)^n$. Of course in practice, we usually don't want to use an infinite series to model a function, so instead we __________________ and use something called a __________ __________________ instead.

In this section, we will show how the theory behind these Taylor Polynomials actually comes from the Mean Value Theorem. But first let's explore higher derivatives.

11.5.1 Higher Derivatives

If $f(x)$ is differentiable on (a,b), then we can also explore its derivative $f'(x)$ as a function on (a,b). If $f'(x)$ is differentiable at $x_0 \in (a,b)$, then we say f is *twice differentiable at* x_0 and we call the derivative of $f'(x)$, the *second derivative* of f at x_0 and we denote it by $f''(x_0)$ or $f^{(2)}(x_0)$. We can continue in this fashion and consider the n^{th} derivatives of f, $f^{(n)}$.

We say f is n-times differentiable if $f^{(n)}$ exists. Furthermore, if $f^{(n)}$ exists, then the $(n-1)$-st derivative $f^{(n-1)}$ also exists.

Example 11.9. *Let $f(x) = |x^3|$ for $x \in \mathbb{R}$.*
a) Write $f(x)$ as a piecewise function and determine a formula for $f'(x)$, if it exists.

b) Does $f''(x)$ exists anywhere? What about $f'''(x)$?

Theorem 11.8. *Taylor's Theorem*[3] *Let $n \in \mathbb{N}$ and suppose f is an $(n+1)$-times differentiable function on (a,b) and continuous on $[a,b]$. Fix $x_0 \in [a,b]$. Then for each $x \in [a,b]$ such that $x \neq x_0$, $\exists c$ between x and x_0 such that*

$$f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \ldots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$$

Note: Sometimes we denote $\dfrac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ as $R_n(x)$.

Proof:

[3] Note if we let $x = b, x_0 = a$, and $n = 0$, Taylor's Theorem is just the MVT.

Example 11.10. *Remember the whole purpose of these "Taylor Polynomials" are to approximate functions near a given point, x_0. Suppose we were to approximate a given function, $f(x)$ as a power series. Then we would have $f(x) = \sum_{n=0}^{\infty} c_n(x-x_0)^n$. Find the values of the coefficients $c_0, c_1, c_2, ...$*

a) Find c_0 by plugging in $x = x_0$ into $f(x)$.

b) Find c_1 by plugging in $x = x_0$ into $f'(x)$

c) Find c_2 by plugging in $x = x_0$ into $f''(x)$.

d) Find c_3 by plugging in $x = x_0$ into $f'''(x)$.

e)Find c_4 by plugging in $x = x_0$ into $f^{(4)}(x)$.

f) Can you see the pattern to find c_n ?

g) Thus, $f(x) = \sum_{n=0}^{\infty}$

This infinite series is known as the **Taylor Series** for $f(x)$ at the point x_0.

12 The Riemann Integral

We will now discuss the definition and basic properties of the Riemann Integral for real-valued functions of one variable.

There are 3 types of integrals and we will study one of them in this course.

1. Darboux Integrals or __________ Integrals
2. Riemann-Stieltjies Integrals
3. Lesbesgue Integrals

Very Rough Idea: An integral is a sum of __________ many terms, each __________ small.

Applications: Anything that can be formulated as a __________

- area
- volume
- arc length
- probability
- work

Ingredients:

- a function
- an interval

Notation:

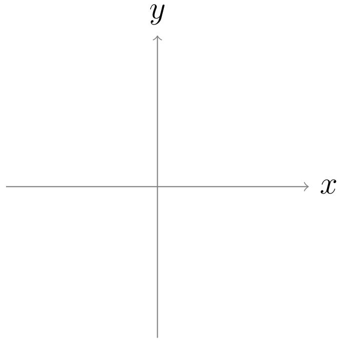
Remember An integral equals a __________!

12.1 Darboux Sums

Definition 12.1. *The* ***Maximum of a function*** f ***on a set*** S, $M(f, S) =$

Definition 12.2. ***The minimum of a function*** f ***on a set*** S, $m(f, S) =$

Example 12.1. *Consider* $f(x) = x^2$ *on* $[0, b]$*. Draw the picture:*

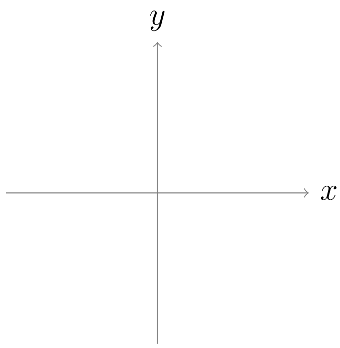


What is $M(f, [0, b])$?

What is $m(f, [a, 1])$?

Example 12.2. *Consider $f(x)$ on $[0,1]$ where* $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.

Draw the picture:



What is $M(f,[0,1])$?

What is $M(f,[0,\frac{1}{\pi}])$?

What is $m(f,[\frac{1}{2},\frac{2}{\pi}])$?

Definition 12.3. *Let $[a,b] \subset \mathbb{R}$, a* ***partition*** *of $[a,b]$ is a finite set of points $\{x_0, x_1, x_2, ..., x_n\}$ in $[a,b]$ such that $a = x_0 < x_1 < x_2 < x_3 < ... < x_{n-1} < x_n = b$.*

We can have many different partitions, if we have two partitions P and Q of $[a,b]$ and $P \subseteq Q$, Q is called a **refinement** of P.

Example: Consider $[0,1]$

Example 12.3. *Consider the interval $[0,b]$, we can create a partition of $[0,b]$ by assigning $x_k = \dfrac{(b-0)k}{n} = \dfrac{(b)k}{n}$ for $k = 0, ..., n$. Then our partition, P=*

This is called a “**regular partition**” because the partitioning points are evenly spaced. The distance between successive partitioning points is $x_k - x_{k-1} =$

Example 12.4. *Consider the interval* $[0, 1]$*, we can create a partition of* $[0, 1]$ *by assigning* $x_k = \dfrac{k^2}{n^2}$ *for* $k = 0, ..., n$*. Then our partition,* P=

Is this a "regular partition"?
$x_k - x_{k-1} =$

Definition 12.4. *Given a bounded function,* f *defined on the interval* $[a, b]$*, let* P *be a partition of* $[a, b]$*. Then for each* $i = 1, ...n$*, we define*

$\Delta x_i =$

$M_i(f) =$

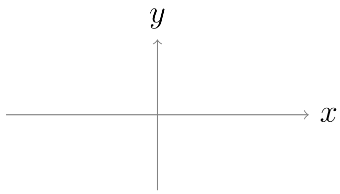
$m_i(f) =$

Sometimes we abbreviate these with M_i and m_i

Definition 12.5. ***The Upper Darboux Sum of*** f ***with respect to partition P*** *is* $U(f, P) =$

Definition 12.6. ***The Lower Darboux Sum of*** f ***with respect to partition P*** *is* $L(f, P) =$

Note: If f is bounded on $[a,b]$, this means $\exists m$ and M such that $m \leq f \leq M$ for all $x \in [a,b]$. So for any partition, P, we actually have:

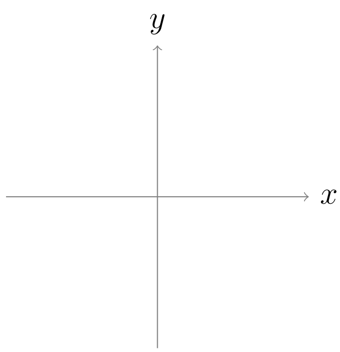


Theorem 12.1. *Let f be bounded on $[a,b]$. If P and Q are partitions of $[a,b]$ and Q is a refinement of$P$, we have:*

Proof: Omitted.

Key Idea: Refining the partition makes the lower sums larger and the upper sums smaller. Ultimately a function is integrable if we can refine the partition enough to make the upper and lower sums arbitrarily close (see last theorem of these notes).

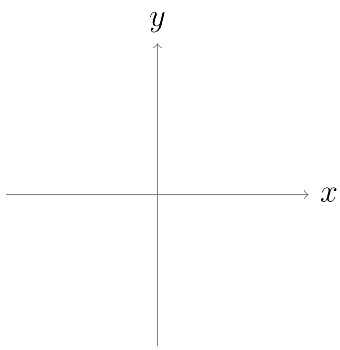
Example 12.5. *Consider $f(x) = x^2$ on $[0,b]$. Compute the following for $n = 3$ and use the partition from example 3 and draw the picture:*



$L(f,P) =$

$U(f,P) =$

Example 12.6. *Consider $f(x)$ on $[0,1]$ where* $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.
Compute the following for $n = 4$ and use the partition from example 12.4 and draw the picture:



$L(f, P) =$

$U(f, P) =$

Note: For $f > 0$, $U(f, P) - L(f, P) =$ (circumscribed area)- (inscribed area)

Helpful Sums:

- $\sum_{i=1}^{n} c = cn$
- $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$
- $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{i=1}^{n} i^3 = [\frac{n(n+1)}{2}]^2$

Example 12.7. *Consider $f(x) = x^2$ on $[0, b]$. Compute the following for a general n:*

$U(f, P) =$

$L(f, P) =$

Example 12.8. *Consider $f(x)$ on $[0,1]$ where $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.*
Compute the following for a general n:

$L(f, P) =$

$U(f, P) =$

12.2 Darboux Integrals (Riemann Integrals)

Definition 12.7. *The* ***Upper Darboux Integral*** *is* $U(f) =$

The ***Lower Darboux Integral*** *is* $L(f) =$

What is the relationship between $L(f, P)$ and $L(f)$?

What is the relationship between $U(f, P)$ and $U(f)$?

Lemma 12.1. *For any partitions P and Q, $L(f, P) \leq U(f, Q)$.*

Pf.

Theorem 12.2. *Let f be bounded on $[a, b]$. Then $L(f) \leq U(f)$.*

Pf.

Definition 12.8. *f is **Darboux-integrable or Riemann-integrable** on $[a, b]$ if* _________ = _________.

*If it is, we denote the common value by $\int_b^a f$ and this is called the **Riemann or Darboux integral or integral of f on** $[a, b]$.*

Example 12.9. *Consider $f(x) = x^2$ on $[0, b]$. Compute the following :*

$U(f) =$ $L(f) =$

$\int_0^b f =$

Example 12.10. *Consider $f(x)$ on $[0, 1]$ where $f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$.*
Compute the following for a general n:

$U(f) =$

$L(f) =$

$\int_0^1 f =$

Theorem 12.3. *f is Darboux-integrable $\iff \forall \epsilon > 0$, $\exists$ a partition P of $[a, b]$ s.t. $U(f, P) - L(f, P) < \epsilon$.*

Pf.

12.3 Ice Darboux Sums

Example 12.11. *Let $g(x) = 1$ for rational x and $f(x) = 0$ for irrational x.*
a) Calculate the upper and lower Darboux integrals for g on the interval $[0, 2]$.
b) Is g integrable on $[0, 2]$?

Example 12.12. *Use the tools we have learned so far about Darboux sums to find the value of* $\int_0^1 [x+1]dx$

Example 12.13. *Use the tools we have learned so far about Darboux sums to show that* $\int_a^b x dx = \dfrac{b^2 - a^2}{2}$

Example 12.14. *Let $g(x) = x^2$ for rational x and $g(x) = 0$ for irrational x.*
a) Calculate the upper and lower Darboux integrals for g on the interval $[0, b]$.
b) Is g integrable on $[0, b]$?

Property 12.1. *Let f be a bounded function on $[a,b]$. Suppose there exists sequences U_n and L_n of upper and lower Darboux sums for f such that $\lim_{n\to\infty}(U_n - L_n) = 0$. Show f is integrable and $\int_a^b f = \lim_{n\to\infty} U_n = \lim_{n\to\infty} L_n$.*

Proof:

12.4 Properties of Integrals

Review: The key tool to showing functions are integrable is showing for each $\epsilon > 0$, $\exists$ a partition of $[a, b]$ s.t.

Theorem 12.4. *If f is monotonic on $[a, b]$, then f is integrable on $[a, b]$*

Pf.

Theorem 12.5. *If f is continuous on $[a, b]$, then f is integrable on $[a, b]$*

Pf.

Theorem 12.6. *If f & g are integrable on $[a, b]$ and $c \in \mathbb{R}$, then*
a) cf is integrable and $\int cf = c \int f$
b) $f + g$ is integrable and $\int (f + g) = \int f + \int g$

Theorem 12.7. *If f & g are integrable on $[a, b]$ and $f(x) \leq g(x)$ throughout $[a, b]$, then $\int_a^b f \leq \int_a^b g$.*

Pf.

Theorem 12.8. *If f is integrable on $[a, b]$, then $|f|$ is integrable on $[a, b]$ and $|\int_a^b f| \leq \int_a^b |f|$.*

Pf.

Theorem 12.9. *If f is integrable on $[a, b]$ and $a < c < b$, then f is integrable on $[a, c]$ and $[c, b]$ and $\int_a^c f + \int_c^b f = \int_a^b f$.*

Pf. Omitted. See text.

Definition 12.9. *A function is* ***piecewise monotonic function*** *if there exists a partition on $[a, b]$, $P = \{a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b\}$ such that f is monotonic on each $[x_{i-1}, x_i]$.*

Definition 12.10. *A function is* ***piecewise continuous function*** *if if there exists a partition on $[a, b]$, $P = \{a = x_0 < x_1 < x_2 < ... < x_{n-1} < x_n = b\}$ such that f is continuous on each $[x_{i-1}, x_i]$.*

Theorem 12.10. *Any function that is either piecewise monotonic or piecewise continuous on $[a, b]$ is integrable there.*

Pf.

13 The Fundamental Theorem of Calculus

Theorem 13.1. ***The Fundamental Theorem of Calculus Part 1:*** *Let f be integrable on $[a, b]$. For each $x \in [a, b]$, let $F(x) = \int_a^x f(t)dt$.*
Then F is uniformly continuous on $[a, b]$ and if f is continuous at each $c \in [a, b]$, then F is differentiable at c and $F'(c) = f(c)$

Pf.

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Example 13.1. *Find the derivative of* $\int_0^x \sin(t^2 - 3t)\sqrt{t}dt$ *for* $x \geq 0$.

Corollary 13.1. *Let* f *be continuous on* $[a, b]$ *an let* g *be differentiable on*$[c, d]$, *where* $g([c, d]) \subseteq [a, b]$. *let* $F(x) = \displaystyle\int_a^{g(x)} f$ $\forall x \in [c, d]$. *Then* F *is differentiable on* $[c, d]$ *and* $F'(x) =$

Pf.

Theorem 13.2. ***The Fundamental Theorem of Calculus Part 2:*** *If* f *is differentiable on* $[a, b]$ *and* f' *is integrable on* $[a, b]$, *then* $\int_a^b f' =$

Pf. Below.

Theorem 13.3. *Integration by Parts: Suppose* f *and* g *are differentiable on* $[a, b]$ *and that* f' *and* g' *are integrable on* $[a, b]$, *then* $\int_a^b fg' =$

Pf in hw

Theorem 13.4. *Integration Using Substitution/Change of Variables: Let* u *be a differentiable function on an open interval* J *such that* u' *is continuous and let* I *be an open interval such that* $u(x) \in I$ $\forall x \in J$. *If* f *is continuous on* I, *then* $f \circ u$ *is continuous on* J *and* $\int_a^b f \circ u(x)u'(x)dx =$
for all $a, b \in J$.

Pf. In Ice

13.1 ICE Fundamental Theorem of Calculus

1. $[T/F]$ If $f : [a, b] \to \mathbb{R}$ is integrable and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.

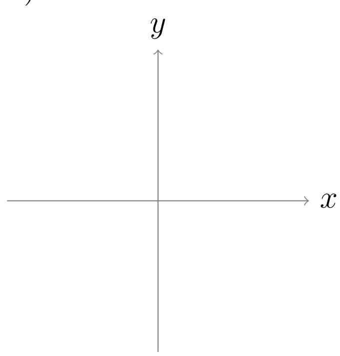
2. $[T/F]$ If $f : [a, b] \to \mathbb{R}$ is integrable, then $f : [a, b] \to \mathbb{R}$ is continuous.

3. $[T/F]$ If $f : [a, b] \to \mathbb{R}$ is integrable and $f \geq 0\ \forall x \in [a, b]$, then $\int_a^b f \geq 0$

4. $[T/F]$ Let $F(x) = \int_0^x xe^{t^2} dt$ for $x \in [0, 1]$, Find $F''(x)$ for $x \in (0, 1)$. Careful, $F'(x) \neq xe^{x^2}$

5. Let $f(t) = t$ for $0 \leq t \leq 2$ and $f(t) = 3$ for $2 < t \leq 4$.
a) Find an explicit expression for $F(x) = \int_0^x f(t)dt$ as a function of x on $[0, 4]$.

b) Sketch F and determine where F is differentiable.



c) Find a formula for $F'(x)$ wherever F is differentiable.

6. Prove the change of variable theorem. Let $u(x)$ be differentiable on (a,b) and $u'(x)$ be integrable on $[c,d]$. Suppose f is continuous on the range of $u(x)$ (For simplicity, assume range of u is $[a,b]$, the $\int_a^b f \circ u(x)u'(x)dx = \int_{u(a)}^{u(b)} f(u)du$. Step 1: Show $f \circ u$ is continuous. Step 2: Fix $c \in [a,b]$ and let $F(u) = \int_c^u f(t)dt$. Then determine $F'(u)$ equal for all $u \in (a,b)$. Step 3: Let $g = F \circ u$ and find is $g'(x)$.Step 4: Then $\int_a^b f \circ u(x)u'(x)dx =$...

7. Let $f(x) = xsgn(\sin(\frac{1}{x}))$ for $x \neq 0$ and $f(0) = 0$.

a) Show that f is not piecewise continuous on $[-1, 1]$

Key Technique: Assume it is piecewise continuous which means it is only discontinuous at finitely many points, say "n points." The find a contradiction by finding an "n +1" point of discontinuity.

b) Show that f is not piecewise monotonic on $[-1, 1]$

c) Actually f is integrable on $[-1, 1]$

14 Finale

14.1 Goals for our Math Majors

Many of you are finishing up your time as math majors. Some of the goals from our mathematics program include:

1. Model the essential components of a problem mathematically so that it can be solved through mathematical means.

2. Solve problems through the appropriate technology.

3. Write clear and concise formal proofs of mathematical theorems, and evaluate the validity of proofs written by others.

4. Solve mathematics problems as a member of a group and in front of a group.

5. Describe and leverage the interdependency of different areas of mathematics, the connections between mathematics and other disciplines, and the historical context for the development of mathematical ideas.

6. Develop critical and creative thinking skills.

14.2 Problem Solving

What are some of the problem solving techniques you use to solve problems?

14.3 Proof Writing Skills

What are some things you have learned about proof writing during your time so far at Lewis?

14.4 Calculus

Most of Real Analysis is proving some of the properties and theorems we use in Calculus. Let's review some of the main exciting uses of Calculus:

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A Review Materials and Mastery Concepts

The following pages are practice problems and main concepts you should master on each exam.

These problems are meant to help you practice for the exam and are often harder than what you will see on the exam. You should also look over all ICE sheets, Homework problems, Quizzes, and Lecture Notes. Know basic theorems and definitions. **Note this is not an exhaustive list.**

Disclaimer: The following lists are topics that you should be familiar with, and these are problems that you should be able to solve. This list may not be complete. You are responsible for everything that we have covered thus far in this course.

I will post the solutions to these practice exams on our blackboard site. If you find any mistakes with my solutions, please let me know right away and feel free to email at any hour.

Good Luck,
Dr. H

Math 36000 – **Review Exam 1**

Review all examples from Notes and Ice sheets and homework. Know basic theorems and definitions. **Note this is not an exhaustive list.**

Concepts on Exam 1:

Concept 1: Basic Definitions and Theorems: Sups, infs, and sequences
Concept 2: Basic Examples of sups, infs, and sequences
Concept 3: Basic Proof for convergent sequence using the precise definition
Concept 4: More complex sequence proof using the precise definition
Concept 5: Basic Proof of known theorem or easy to prove fact about an arbitrary sequence.
Concept 6: Induction Proof usually about sequences

Definitions:

- $|x|$
- upper bound
- LEAST upper bound or supremum (sup)
- lower bound
- GREATEST upper bound or infimum (inf)
- Maximum
- Minimum
- Dense Set
- Limit of a Sequence, $\lim_{n\to\infty} a_n = L$
- non-existence of a limit, $\lim_{n\to\infty} a_n \neq L$
- sequence diverging to infinity, $\lim_{n\to\infty} a_n = \infty$

Theorems:

- The Archimedean Property
- The Completeness Axiom
- The Triangle Inequality
- Squeeze Theorem for Sequences
- Limits, sups, infs, maxs, mins are all unique if they exist.
- Every Convergent Sequence is Bounded

Example Problems:

(Also go over Ice and HW problems!)

Expect some T/F questions where you must provide an explicit counterexample (and justify your counterexample) if you think the statement is false. If a statement is true, be prepared to give a brief description of why it is true or reference the relevant theorems. These will be similar to the T/F questions we do in our classroom discussion activities. Some of these T/F questions will also test whether or not you know key definitions and theorems.

Example 1: [T/F] If x is an irrational number, then there exists an irrational number y such that $x + y$ is a rational number.

Correct Answer: True. If x is irrational, then so is $-x$. Then $x + (-x) = 0$, and 0 is a rational number.

Example 2: According to the Axiom of Completeness, every nonempty set of positive real numbers has a supremum.

Correct Answer: False. The Axiom of Completeness states that every nonempty *bounded* subset of $\mathbb{R}$ has a supremum. For an explicit counterexample, consider $\mathbb{N} \subset \mathbb{R}$. $\mathbb{N}$ is certainly a nonempty subset of the positive real numbers. However, $\mathbb{N}$ is not bounded above, and so, this set does not have a supremum.

Expect a number of questions that ask for an explicit example to be given with certain properties. Also, expect to briefly justify that your example meets the desired criteria!

Example 3: Give an example of a sequence that is bounded and converges, but is NOT monotone.

Correct Answer: Let $(a_n) = (\frac{(-1)^n}{n})$. This sequence is bounded since $|a_n| \leq 1$, for all $n \in \mathbb{N}$. Also, this sequence converges to 0. However, this sequence is not monotone since it alternates between positive and negative terms.

Expect a few proof-writing question on the exam. You will be asked to prove that a particular sequence converges to a particular value.

Example 4: Prove that $\lim\limits_{n\to\infty} \dfrac{2n+1}{5n+4} = \dfrac{2}{5}$, $\lim\limits_{n\to\infty} \dfrac{2n+1}{5n^2+4} = 0$, $\lim\limits_{n\to\infty} \dfrac{n^3}{n-1} = \infty$, $\lim\limits_{n\to\infty} \dfrac{\sin(n^2)}{n^2} = 0$, $\lim\limits_{n\to\infty} \dfrac{2n+1}{n^2-4} = 0$

Correct Answer: Hopefully you have some similar examples done in your homework and notes :)

Other example problems:

- Give an example (if it exists) of a sequence that...

  a) is bounded but not monotonic
  b) bounded but not convergent
  c) monotonic but not convergent
  d) Convergent but not monotonic
  e) Monotonic but not bounded
  f) Convergent but not bounded

- Prove if $a_1 > 1$ and $a_{n+1} = 1 + \sqrt{a_n - 1}$ for $n \in \mathbb{N}$, then a_n is bounded below and a_n is monotonic.

- Suppose $c \in \mathbb{R}$ such that $|c| < 1$. Prove $\lim\limits_{n\to\infty} c^n = 0$.

- Prove if $a_n \to \infty$ then $\frac{1}{a_n} \to 0$.

- Prove if $a_n \to a < \infty$ and $b_n \to b < \infty$ then $a_n + b_n \to a + b$.

- Prove $a_n = 3 + \frac{1}{n} \to 3$ using the definition of a convergent sequence.

- Prove $a_n = \frac{2n^2+1}{3n^2} \to \frac{2}{3}$ using the definition of a convergent sequence.

- Be able to prove any of the sup/inf examples from HW and notes

Some Big Ideas:

Preliminaries

Theorem (Triangle Inequality)**.** *The absolute value function, defined by*

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0. \end{cases}$$

satisfies

1. $|ab| = |a||b|$, *and*
2. $|a + b| \leq |a| + |b|$

for all a and b in $\mathbb{R}$.

Theorem (Reverse Triangle Inequality)**.** *The absolute value function also satisfies*

1. $|a - b| \geq |a| - |b|$

for all a and b in $\mathbb{R}$.

Theorem. *Two real numbers a and b are equal if and only if for every real number $\epsilon > 0$ it follows that $|a - b| < \epsilon$.*

Definition 1. *The function $f : A \to B$ is one-to-one (1-1) on injective if $a_1 \neq a_2$ in A implies that $f(a_1) \neq f(a_2)$.*

Definition 2. *The function $f : A \to B$ is onto or surjective if for every $b \in B$, there is some $a \in A$ with $f(a) = b$.*

The Axiom of Completeness

Axiom 1 (Axiom of Completeness)**.** *Every nonempty set of real numbers that is bounded above has a least upper bound (supremum). (We proved in class that every set of real numbers bounded below has a greatest lower bound (infimum) as well. You may recall the definitions of bounded above, bounded below, least upper bound, and greatest lower bound from page 15.)*

Lemma 1. *Let $s \in \mathbb{R}$ be an upper bound for a set $A \subseteq \mathbb{R}$. Then, $s = \sup A$ if and only if, for every choice of $\epsilon > 0$, there exists an element $a \in A$ satisfying $s - \epsilon < a$.*

Consequences of Completeness

Theorem ((Nested Interval Property))**.** *For each $n \in \mathbb{N}$, assume we are given a closed interval $I_n = [a_n, b_n] = \{x \in \mathbb{R} : a_n \leq x \leq b_n\}$. Assume also that each I_n contains I_{n+1}. Then, the resulting nested sequence of closed intervals*
$I_1 \supseteq I_2 \supseteq I_3 \supseteq I_4 \supseteq \ldots$
has a nonempty intersection; that is $\bigcap_{n=1}^{\infty} I_n \neq \emptyset$.

Theorem ((Archimedean Property))**.**

1. *Given any number $x \in \mathbb{R}$, there exists an $n \in \mathbb{N}$ satisfying $n > x$.*
2. *Given any real number $y > 0$, there exists an $n \in \mathbb{N}$ satisfying $1/n < y$.*

Theorem ((Density of $\mathbb{Q}$ in $\mathbb{R}$))**.** *For every two real numbers a and b with $a < b$, there exists a rational number r satisfying $a < r < b$.*

Corollary 1. *Given any two real numbers $a < b$, there exists an irrational number t satisfying $a < t < b$.*

Theorem. *There exists a real number α satisfying $\alpha^2 = 2$.*

Theorem.

1. *If $a, b \in \mathbb{Q}$, then $ab \in \mathbb{Q}$ and $a + b \in \mathbb{Q}$.*
2. *If $a \in \mathbb{Q} \backslash \{0\}$ and t is irrational, then $a + t$ and at are irrational.*

Limit of a Sequence

Definition 3. *A* ***sequence*** *is a function whose domain is $\mathbb{N}$.*

Definition 4 (Convergence of a Sequence)**.** *A sequence (a_n)* ***converges*** *to a real number a if, for every positive number ϵ, there exists an $N \in \mathbb{N}$ such that whenever $n \geq N$ it follows that $|a_n - a| < \epsilon$.*

Theorem (Uniqueness of Limits)**.** *The limit of a sequence, when it exists, must be unique.*

Definition 5 (Divergence of a Sequence)**.** *A sequence that does not converge is said to* ***diverge****.*

§Algebraic and Order Limit Theorems

Definition 6. *A sequence (x_n) is* ***bounded*** *if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.*

Theorem. *Every convergent sequence is bounded.*

Theorem (Algebraic Limit Theorem).
Let $\lim_{n\to\infty} a_n = a$ *and* $\lim_{n\to\infty} b_n = b$. *Then,*

1. $\lim_{n\to\infty} (ca_n) = ca$, *for all* $c \in \mathbb{R}$;
2. $\lim_{n\to\infty} (a_n + b_n) = a + b$;
3. $\lim_{n\to\infty} (a_n b_n) = ab$;
4. $\lim_{n\to\infty} (a_n/b_n) = a/b$, *provided* $b \neq 0$.

Theorem (Order Limit Theorem). *Assume* $\lim a_n = a$ *and* $\lim b_n = b$.

1. *If* $a_n \geq 0$ *for all* $n \in \mathbb{N}$, *then* $a \geq 0$.
2. *If* $a_n \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $a \leq b$.
3. *If there exists* $c \in \mathbb{R}$ *for which* $c \leq b_n$ *for all* $n \in \mathbb{N}$, *then* $c \leq b$. *Similarly, if* $a_n \leq c$ *for all* $n \in \mathbb{N}$, *then* $a \leq c$.

Theorem (Squeeze Theorem). *If* $x_n \leq y_n \leq z_n$ *for all* $n \in \mathbb{N}$, *and if* $\lim x_n = l = \lim z_n$, *then* $\lim y_n = l$ *as well.*

MATH 36000 – Review Exam 2

Review all examples from Notes and Ice sheets and homework. Know basic theorems and definitions. **Note this is not an exhaustive list.**

Concepts on Exam 2:

Concept 1: Basic Definitions and Theorems: Sups, Infs, and Sequences
Concept 2: Basic Examples of sups, infs, and sequences
Concept 3: Basic Proof about Sups or Infs
Concept 4: Convergent/divergent sequence proof using the precise definition
Concept 5: Basic Proof of known theorem or easy to prove fact about an arbitrary sequence.
Concept 6: Induction Proof usually about sequences
Concept 7: Basic Definitions and Theorems about sequence properties
Concept 8: Basic Examples of sequences and subsequences
Concept 9: Sequence Example (determine sups, infs, max, bounded, monotone,limsups, etc)
Concept 10: Delta -Epsilon Limit Proof
Concept 11: Proofs showing a function is continuous or not continuous
Concept 12: Uniform Continuity

Concepts 1-9 can be done during retesting week 1.

New Definitions:

- Limit of a Sequence, also non-existence, infinite
- Bounded sequence
- Closed Set
- Cauchy Sequence
- Monotonic Sequence
- Monotonically increasing Sequence
- Monotonically decreasing Sequence
- non-decreasing sequence
- non-increasing sequence
- Subsequence
- subsequential limit
- Limsup or limit superior

- Liminf or limit inferior
- sequentially compact set
- Monotone Convergent Theorem
- Definition of $\lim_{x \to a} f(x) = L$
- Definition of $\lim_{x \to a^+} f(x) = L$
- Definition of $\lim_{x \to a^-} f(x) = L$
- Definition of $\lim_{x \to a} f(x) = \infty$
- Definition of $\lim_{x \to a} f(x) = -\infty$
- Sequence Definition of Continuity
- δ, ϵ definition of Continuity
- δ, ϵ definition of Uniform Continuity
- Sequence definition of Uniform Continuity

Example problems:

(Also go over Ice and HW problems!)
Expect some T/F questions where you must provide an explicit counterexample (and justify your counterexample) if you think the statement is false. If a statement is true, be prepared to give a brief description of why it is true or reference the relevant theorems. These will be similar to the T/F questions we do in our classroom discussion activities. Some of these T/F questions will also test whether or not you know key definitions and theorems.

1. Give an example (if it exists) of a sequence that...

   a) is bounded but not monotonic
   b) bounded but not convergent
   c) monotonic but not convergent
   d) Convergent but not monotonic
   e) Monotonic but not bounded
   f) Convergent but not bounded

2. Prove if a_n monotonically increasing an unbounded then $\lim_{n \to \infty} a_n = \infty$

3. Prove if a_n is cauchy, a_n is bounded.

4. Prove if a_n is convergent, a_n is cauchy.
5. Prove if $a_1 > 1$ and $a_{n+1} = 1 + \sqrt{a_n - 1}$ for $n \in \mathbb{N}$, then a_n is bounded below and a_n is monotonic.
6. Given the sequence $a_n = 2 + (-1)^n$, answer the following questions:
   a) Is it bounded?
   b) Is it monotone?
   c) Does it converge?
   d) Is it Cauchy?
   e) Find the Supremum and Infimum, if it exists for the sequence.
   f) Find the maximum or minimum, if it exists for the sequence.
   g) Are there any monotonic subsequences? If so, list **2**.
   h) Are there any convergent subsequences? If so, list **2**.
   i) List the subsequential limits.
   j) Find the limsup(a_n) and liminf(a_n).
7. Prove $\lim_{x \to 1}(x^2 - 3x + 2) = 0$ using the delta epsilon definition of a limit.
8. Prove that $f(x) = x^m$ for $m \in \mathbb{N}$ is continuous on $\mathbb{R}$.
9. Let f be continuous on (a, b). Show that if $f(r) = 0 \forall r \in (a, b)$ with $r \in \mathbb{Q}$, then $f(x) = 0 \forall x \in (a, b)$
10. Prove that a linear function, $f(x) = a + bx$ is continuous for all x.
11. Prove $g(x) = \sin(\frac{1}{x})$ is discontinuous at $x = 0$.
12. Prove the $sgn(x)$ is discontinuous at 1 point. $sgn(x) = -1$ for $x < 0$ and $sgn(x) = 1$ for $x > 0$ and $sgn(0) = 0$.
13. Which of the following are uniformly continuous:
    a) $3x^5 - \cos(3x) + 5$ on $[0, 7]$
    b) $f(x) = x^3$ on $[0, 1]$ on $(0, 1]$
    c) $f(x) = \frac{1}{x^7}$ on $(0, 1]$
    d) $f(x) = \frac{5x}{2x-1}$ on $[1, \infty)$
    e) $f(x) = \sqrt{x}$ on $[0, \infty)$.
14. If you can, give an example of a continuous function that is not uniformly continuous.
15. If you can, give an example of a uniform continuous function that is not continuous.
16. Given an example of a uniform continuous function on an open interval.
17. Give an example a function that is continuous only at 1 point.
18. Give an example of a function that is continuous everywhere but 1 point.

19. Prove if $f : A \to B$ and $g : B \to A$ are uniformly continuous, then $f + g$ is uniformly continuous.

20. Give an example of two real functions f and g that are uniformly continuous, but where fg is not uniformly continuous.

Some Big Ideas:

§Algebraic and Order Limit Theorems

Definition 1. *A sequence (x_n) is* ***bounded*** *if there exists a number $M > 0$ such that $|x_n| \leq M$ for all $n \in \mathbb{N}$.*

Theorem. *Every convergent sequence is bounded.*

Theorem (Algebraic Limit Theorem)**.**
Let $\lim_{n\to\infty} a_n = a$ and $\lim_{n\to\infty} b_n = b$. Then,

1. $\lim_{n\to\infty} (ca_n) = ca$, *for all* $c \in \mathbb{R}$;
2. $\lim_{n\to\infty} (a_n + b_n) = a + b$;
3. $\lim_{n\to\infty} (a_n b_n) = ab$;
4. $\lim_{n\to\infty} (a_n/b_n) = a/b$, *provided* $b \neq 0$.

Theorem (Order Limit Theorem)**.** *Assume $\lim a_n = a$ and $\lim b_n = b$.*

1. *If $a_n \geq 0$ for all $n \in \mathbb{N}$, then $a \geq 0$.*
2. *If $a_n \leq b_n$ for all $n \in \mathbb{N}$, then $a \leq b$.*
3. *If there exists $c \in \mathbb{R}$ for which $c \leq b_n$ for all $n \in \mathbb{N}$, then $c \leq b$. Similarly, if $a_n \leq c$ for all $n \in \mathbb{N}$, then $a \leq c$.*

Theorem (Squeeze Theorem)**.** *If $x_n \leq y_n \leq z_n$ for all $n \in \mathbb{N}$, and if $\lim x_n = l = \lim z_n$, then $\lim y_n = l$ as well.*

§Monotone Convergence Theorem and Infinite Series

Definition 2. *A sequence (a_n) is* ***increasing*** *if $a_n \leq a_{n+1}$ for all $n \in \mathbb{N}$ and* ***decreasing*** *if $a_n \geq a_{n+1}$ for all $n \in \mathbb{N}$. A sequence is* ***monotone*** *if it is either increasing or decreasing.*

Theorem (Monotone Convergence Theorem)**.** *If a sequence is monotone and bounded, then it converges.*

§Subsequences and the Bolzano-Weierstrass Theorem

Definition 3. *Let (a_n) be a sequence of real numbers, and let $n_1 < n_2 < n_3 < n_4 < n_5 < \ldots$ be an increasing sequence of natural numbers. Then the sequence*

$$(a_{n_1}, a_{n_2}, a_{n_3}, a_{n_4}, a_{n_5}, \ldots)$$

is called a ***subsequence*** *of (a_n) and is denoted by (a_{n_k}), where $k \in \mathbb{N}$ indexes the subsequence.*

Theorem. *Subsequences of a convergent sequence converge to the same limit as the original sequence.*

Theorem (Bolzano-Weierstrass Theorem)**.** *Every bounded sequence of real numbers contains a convergence subsequence.*

§The Cauchy Criterion

Definition 4 (2.6.1)**.** *A sequence (a_n) is called a* ***Cauchy sequence*** *if, for every $\epsilon > 0$, there exists an $N \in \mathbb{N}$ such that whenever $m, n \geq \mathbb{N}$ it follows that $|a_n - a_m| < \epsilon$.*

Theorem (Cauchy Criterion)**.** *A sequence of real numbers converges if and only if it is a Cauchy sequence.*

Examples of Discontinuous Functions

Example. *Dirichlet's Function* $g(x) = \begin{cases} 1 \ \textit{if } x \in \mathbb{Q} \\ 0 \ \textit{if } x \notin \mathbb{Q} \end{cases}$

Example. *Modified Dirichlet's Function* $g(x) = \begin{cases} x \ \textit{if } x \in \mathbb{Q} \\ 0 \ \textit{if } x \notin \mathbb{Q} \end{cases}$

Example. *Thomae's Function* $t(x) = \begin{cases} 1 & \textit{if } x = 0 \\ 1/n & \textit{if } x = m/n \in \mathbb{Q}\backslash\{0\} \textit{ is in lowest terms with } n > 0 \\ 0 & \textit{if } x \notin \mathbb{Q} \end{cases}.$

Limits of Functions

Definition 5 ((Functional Limit))**.** *Let $f : A \to \mathbb{R}$, and let c be a limit point of the domain $A \subseteq \mathbb{R}$. We say that $\lim_{x\to c} f(x) = L$ provided that, for all $\epsilon > 0$, there exists $\delta > 0$ such that whenever $0 < |x - c| < \delta$ (and $x \in A$) it follows that $|f(x) - L| < \epsilon$.*

Definition 6. *Given a limit point c of a set A and a function $f : A \to \mathbb{R}$ we write* ***the right hand limit***

$$\lim_{x \to c^+} f(x) = L$$

if for all ϵ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - c < \delta$. The ***left hand limit***

$$\lim_{x \to c^-} f(x) = L$$

is similarly defined.

Theorem. *Given $f : A \to \mathbb{R}$ and a limit point c of A, $\lim_{x \to c} f(x) = L$ if and only if*

$$\lim_{x \to c^+} f(x) = L \quad \text{and} \quad \lim_{x \to c^-} f(x) = L.$$

Theorem ((Sequential Criterion for Functional Limits))**.** *Given a function $f : A \to \mathbb{R}$ and a limit point c of A, the following two statements are equivalent:*

1. $\lim\limits_{x \to c} f(x) = L$
2. *For all sequences $(x_n) \subseteq A$ satisfying $x_n \neq c$ and $(x_n) \to c$, it follows that $f(x_n) \to L$.*

Corollary 1 ((Algebraic Limit Theorems for Functional Limits))**.** *Let f and g be functions defined on a domain $A \subseteq \mathbb{R}$, and assume $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$ for some limit point c of A. Then,*

1. $\lim\limits_{x \to c} kf(x) = kL$ *for all* $k \in \mathbb{R}$,
2. $\lim\limits_{x \to c} [f(x) + g(x)] = L + M$,
3. $\lim\limits_{x \to c} [f(x)g(x)] = LM$,
4. $\lim\limits_{x \to c} [f(x)/g(x)] = L/M$, *provided* $M \neq 0$.

Corollary 2 ((Divergence Criterion for Functional Limits))**.** *Let f be a function defined on A, and let c be a limit point of A. If there exist two sequences (x_n) and (y_n) in A with $x_n \neq c$ and $y_n \neq c$ and*

$$\lim x_n = c = \lim yn \qquad \text{but} \qquad \lim f(x_n) \neq \lim f(y_n),$$

then the functional limit $\lim\limits_{x \to c} f(x)$ does not exist.

Continuous Functions

Definition 7 ((Continuity -delta epsilon))**.** *A function $f : A \to \mathbb{R}$ is **continuous at a point** $c \in A$ if, for all $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $|x - c| < \delta$ (and $x \in A$) if follows that $|f(x) - f(c)| < \epsilon$. If f is continuous at every point in the domain A, the we say that f is **continuous on** A.*

Theorem ((Characterizations of Continuity))**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$. The function f is continuous at c if and only if any one of the following three conditions is met:*

1. *For all $\epsilon > 0$, there exists a $\delta > 0$ such that $|x - c| < \delta$ (and $x \in A$) implies $|f(x) - f(c)| < \epsilon$;*
2. *If $(x_n) \to c$ (with $x_n \in A$), then $f(x_n) \to f(c)$.*

If c is a limit point of A, then the above conditions are also equivalent to

3. *$\lim_{x \to c} f(x) = f(c)$.*

Corollary 3 ((Criterion for Discontinuity))**.** *Let $f : A \to \mathbb{R}$, and let $c \in A$ be a limit point of A. If there exists a sequence $(x_n) \subseteq A$ where $(x_n) \to c$ but $f(x_n)$ does not converge to $f(c)$, then f is not continuous at c.*

Theorem ((Algebraic Continuity Theorem))**.** *Assume $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ are continuous at a point $c \in A$. Then,*

1. *$kf(x)$ is continuous at c for all $k \in \mathbb{R}$.*
2. *$f(x) + g(x)$ is continuous at c.*
3. *$f(x) \cdot g(x)$ is continuous at c.*
4. *$f(x)/g(x)$ is continuous at c, provided the quotient is defined.*

Theorem ((Composition of Continuous Functions))**.** *Given $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$, assume that the range $f(A) = \{f(x) : x \in A\} \subseteq B$, so that the composition $g \circ f(x) = g(f(x))$ is defined on A. If f is continuous at $c \in A$ and if g is continuous at $f(c) \in B$, then $g \circ f$ is continuous at c.*

Definition 8 ((Uniform Continuity))**.** *A function $f : A \to \mathbb{R}$ is **uniformly continuous on** A if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in A$, $|x - y| < \delta$ implies $|f(x) - f(y)| < \epsilon$.*

Theorem ((Sequential Criterion for Absence of Uniform Continuity))**.** *A function $f : A \to \mathbb{R}$ fails to be uniformly continuous on A if and only if there exists a particular $\epsilon_0 > 0$ and two sequences (x_n) and (y_n) in A satisfying*

$$|x_n - y_n| \to 0 \quad \textit{but} \quad |f(x_n) - f(y_n)| \geq \epsilon_0.$$

Example. *The function $h(x) = \sin(1/x)$ is continuous on $(0,1)$ but not uniformly continuous.*

Theorem ((Uniform Continuity of Closed on Bounded (Compact) Sets))**.** *A function that is continuous on a closed and bounded (compact) set K is uniformly continuous on K. (Note K often is in the form of $[a,b]$)*

Theorem ((Cauchy Check for Uniform Continuity))**.** *If f is uniformly continuous on D, and s_n is a Cauchy sequence in D, then $f(s_n)$ is a Cauchy sequence.*

MATH 36000 – **Review Exam 3**

Review all examples from Notes and Ice sheets and homework. Know basic theorems and definitions. **Note this is not an exhaustive list.**

Concepts on Exam 3:

Concept 1: Basic Definitions and Theorems: Sups, Infs, and sequences
Concept 2: Basic Examples of sups, infs, and sequences
Concept 3: Basic Proof about Sups or Infs
Concept 4: Convergent/divergent sequence proof using the precise definition
Concept 5: Basic Proof of known theorem or easy to prove fact about an arbitrary sequence.
Concept 6: Induction Proof usually about sequences
Concept 7: Basic Definitions and Theorems about sequence properties
Concept 8: Basic Examples of sequences and subsequences
Concept 9: Sequence Example (determine sups, infs, max, bounded, monotone,limsups, etc)
Concept 10: Delta -Epsilon Limit Proof
Concept 11: Proofs showing a function is continuous or not continuous
Concept 12: Uniform Continuity
Concept 13: Basic Definitions from Continuity and Differentiation
Concept 14: Basic Examples from Continuity and Differentiation
Concept 15: Basic Proofs using the definition of the derivative to show a function is differentiable or not differentiable
Concept 16: Basic Proofs using the definition of the derivative to prove a basic theorem for arbitrary function.

Definitions/Theorems:

- δ, ϵ definition of Uniform Continuity
- Sequence definition of Uniform Continuity
- Sequence Definition of Continuity
- δ, ϵ definition of Continuity
- If a function is continuous on $[a, b]$ (a closed and bounded interval), then f is.... on $[a, b]$
- If f is uniformly continuous on D, and s_n is a Cauchy sequence in D, then $f(s_n)$ is
- Definition of global continuity
- Composition, sum,difference, product, absolute value, etc theorems for continuous functions
- Extreme Value Theorem
- Intermediate Value Theorem
- Fixed Point

- derivative at x=a
- derivative function
- Sequence definition for the derivative of a function
- Fermat's Theorem
- Rolles Theorem
- Mean Value Theorem
- Consequences of MVT Theorems

Example problems:

(Also go over Ice and HW problems!)

Expect some T/F questions where you must provide an explicit counterexample (and justify your counterexample) if you think the statement is false. If a statement is true, be prepared to give a brief description of why it is true or reference the relevant theorems. These will be similar to the T/F questions we do in our classroom discussion activities. Some of these T/F questions will also test whether or not you know key definitions and theorems.

1. Let f, g be continuous real functions on $[a, b]$ and suppose $f(a) < g(a)$ and $f(b) > g(b)$. Prove there must exist a point $c \in (a, b)$ where $f(c) = g(c)$.

2. Prove $x3^x = 1$ for some $x \in (0, 1)$.

3. Show that $f(x) = \sin(\frac{1}{x})$ has the IVT property on $\mathbb{R}$ even though it is not continuous at 0.

   Derivative section:

4. Be able to prove
   - $(cf)' = cf'(x)$
   - $(f \pm g)' = f' \pm g'$
   - $(fg)' = f'g + fg'$
   - $(\dfrac{1}{g})' = \dfrac{-g'}{g^2}$
   - $(\dfrac{f}{g})' = \dfrac{gf' - fg'}{g^2}$

5. Examples (if they exist) of functions that are continuous but not differentiable

6. Examples (if they exist) of functions that are differentiable but not continuous

7. Examples (if they exist) of functions such that $f^2 : \mathbb{R} \to \mathbb{R}$ is differentiable, but $f : \mathbb{R} \to \mathbb{R}$ is not differentiable.

8. Examples (if they exist) of functions such that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, but $f^2 : \mathbb{R} \to \mathbb{R}$ is not differentiable.
9. Be able to find derivatives using the definition of the derivative of piecewise functions
10. Determine where a function is NOT differentiable
11. Have examples of continuous functions that are not differentiable (more than 1).
12. Use induction to prove $f'(x) = nx^{n-1}$ for $f(x) = x^n$ and $n \in \mathbb{N}$.
13. Give an example of two functions not differing by a constant such that $f'(x) = g'(x) = f(x)$ for all $x \neq 0$.
14. Draw pictures for MVT and Rolle's Theorem
15. Find 2 functions that satisfy **exactly 2** of the three conditions we need satisfied for Rolle's Theorem, but for which the conclusion of Rolle's theorem does not follow. That is, there is no point c in (a,b) such that $f'(c) = 0$.

Some Big Ideas:

Continuous Functions

Theorem ((Preservation of Compact Sets))**.** *Let $f : A \to \mathbb{R}$ be continuous on A. If $K \subset A$ is compact, then $f(K)$ is compact as well.*

Theorem ((Extreme Value Theorem))**.** *If $f : K \to \mathbb{R}$ is continuous on a compact set $K \subset \mathbb{R}$, then f attains a maximum and a minimum value. In other words, there exists $x_0, x_1 \in K$ such that $f(x_0) \leq f(x) \leq f(x_1)$ for all $x \in K$.*

The Intermediate Value Theorem

Theorem ((Intermediate Value Theorem))**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous. If L is a real number satisfying $f(a) < L < f(b)$ or $f(a) > L > f(b)$, then there exists a point $c \in (a, b)$ where $f(c) = L$.*

Derivatives and the Intermediate Value Property

Definition (Differentiability)**.** *Let $g : A \to \mathbb{R}$ be a function defined on an interval A. Given $c \in A$, the **derivative of** g **at** c is defined by*

$$g'(c) = \lim_{x \to c} \frac{g(x) - g(c)}{x - c},$$

*provided this limit exists. In this case we say g is **differentiable at** c. If g' exists for all points $c \in A$, we say that g **is differentiable on A**.*

Theorem 1 (Theorem)**.** *If $g : A \to \mathbb{R}$ is differentiable at a point $c \in A$, then g is continuous at c as well.*

Theorem 2 (Algebraic Differentiability Theorem)**.** *Let f and g be defined on an interval A, and assume both are differentiable at some point $c \in A$. Then,*

(i) $(f+g)'(c) = f'(c) + g'(c)$.

(ii) $(kf)'(c) = kf'(c)$, *for all* $k \in \mathbb{R}$.

(iii) $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$, *and*

(iv) $(f/g)'(c) = \frac{g(c)f'(c) - f(c)g'(c)}{[g(c)]^2}$, *provided that* $g'(c) \neq 0$.

Theorem 3 (Fermat's Theorem)**.** *Let f be differentiable on an open interval (a, b). If f attains a maximum value at some point $c \in (a, b)$ (i.e., $f(c) \geq f(x)$ for all $x \in (a, b)$), then $f'(c) = 0$. The same is true if $f(c)$ is a minimum value.*

Mean Value Theorem

Theorem 4 (Rolle's Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $f(a) = f(b)$, then there exists a point $c \in (a, b)$ where $f'(c) = 0$.*

Theorem 5 (Mean Value Theorem)**.** *If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists $c \in (a, b)$ where*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Corollary. *If $g : A \to \mathbb{R}$ is differentiable on an interval A and satisfies $g'(x) = 0$ for all $x \in A$, then $g(x) = k$ for some constant $k \in \mathbb{R}$.*

Corollary. *If f and g are differentiable functions on an interval A and satisfy $f'(x) = g'(x)$ for all $x \in A$, then $f(x) = g(x) + k$ for some constant $k \in \mathbb{R}$.*

B Supplemental Topics

B.1 Review and Preliminaries

Here is a reading guide to Pugh's *Real Mathematical Analysis* Section 1: Preliminaries (pages 1-10).

1. Is $\emptyset \in \mathbb{Q}$? Explain why or why not.

2. Is $0 \in \emptyset$? Explain why or why not.

3. Explain what is wrong with the following notations: $\mathbb{N} \in \mathbb{Q}$ and $1 \subset \mathbb{N}$.

4. Write in words what the following set means: $A = \{x \in \mathbb{Q} : x^2 < 3\}$.
   Dr. Harsy notation comment: I sometimes use "$|$" instead of ":" so in our class $A := \{x \in \mathbb{Q} : x^2 < 3\} = \{x \in \mathbb{Q} | x^2 < 3\}$

5. Draw a picture representing what $A \Delta B$ represents for arbitrary sets A and B.

6. How do you show two sets are equal?

7. What are the 3 properties of an equivalence relation?

8. How do we denote equivalence classes of a subset?

9. of the list of guidelines for writing a rigorous mathematical proof, what is the step that you often omit or do not think to do?

10. Our textbook says to try to avoid reading $\forall$ as "for all" (Dr. H will try to break this habit too...). What do they say to use instead?

11. Explain why even though "Every integer is less than some prime number." is equivalent to "For each integer n there is a prime number p which is greater than n," the latter is better to use when converting to mathematical symbolic notation.

12. Give an example which shows that the following statement is false: $(\forall n \in \mathbb{N})(\exists p \in P)$ such that $(\forall m \in \mathbb{N})(nm < p)$.

13. Dr. Harsy sometimes uses $\sim$ for negation. What symbol does our textbook use?

14. Review DeMorgan's Law (note $\cap$ means "&"): $\neg(A \cap B) \equiv$

15. $a \implies b \equiv$ __________ $\equiv$ __________

16. What is the first piece of advice given at the end of this section?

17. Out of the reading you did, what did you find most interesting?

18. Out of the reading, what did you find to be the "muddiest point"?

B.2 Density

Read our textbook, *Real Mathematical Analysis* pages 18-21 (stop at the ϵ principle). The first two pages gives a preview for our next section and the last part gives a a further discussion of the density of $\mathbb{Q}$ and $\mathbb{Q}^c$ in $\mathbb{R}$ and the Archimedean Principle. The focus of the quiz starts at the bottom of page 19.

There are questions on the back of this page.

1. Our textbook proves the density of rational and irrational numbers (Theorem 7) a different way than we do/did in class. They start with an interval (a, b) and think of a and b as cuts. What else do they use in their proof? Did the proof make sense? What parts (if any) were confusing to you?

2. The book lists 3 facts which come from Theorem 7. What is another way to state fact (a) "There is o smallest rational number in $(0, 1)$" using a term from our completeness axiom notes?

3. For what field does the Archimedean property fail? Describe this field.

4. What was the most interesting part of your reading?

5. What was the muddiest or most confusing part of your reading?

B.3 A Taste of Topology

Over the next two weeks, we will be introducing some topology idea which are often alluded to or discussed in Analysis courses. Most of these topics will be covered more in depth in Real Analysis II or a Topology course.

Quiz 3: Please read our textbook, *Real Mathematical Analysis* pages 58-70 and focus on pages 58-59 and 65-70. There are questions on both sides of this page.

B.4 Metric Spaces

Definition B.1. ***Metric Space:*** *We say a nonempty set X with associated function $d : X \times X \to \mathbb{R}$ is a* ***metric space*** *if the following conditions are satisfied:*

1. $\forall x, y \in X$, $d(x, y) \geq 0$
2. $\forall x, y \in X$, $d(x, y) = 0$ if and only if $x = y$
3. $\forall x, y \in X$, $d(x, y) = d(y, x)$
4. $\forall x, y, z \in X$, $d(x, y) + d(y, z) \geq d(x, z)$

If all of the conditions are satisfied, d is called a **distance function** or **metric**. The elements of a metric space are called **elements.**

1. What is the main metric for $\mathbb{R}^n$?

   $d(\mathbf{x}, \mathbf{y}) =$

2. Given a set $X = \{a, b, c\}$, what is $d(a, b)$, $d(b, c)$ and $d(c, c)$ if d is the **discrete metric**?

This quiz continues on the next **two pages**.

B.5 Open and Closed Sets

The definitions we'll encounter in this section are fundamental to two important areas of mathematics: analysis and topology. For reasons you might discover later, topologists love open sets most of all. Analysts love closed sets most of all.

Definition B.2. *Given a subset of S of a metric space, X, a point $p \in X$ is a **limit point** or **limit** of S if there exits a sequence $p_n \in S$ such that p_n converges to p.*

Definition B.3. *A subset of S of a metric space, X, is said to be **closed** in X if it contains all of its limit points.*

To help you understand the definition of open subset from our text, I have the following two definitions:

Definition B.4. *An **open ball** with center $\boldsymbol{p}$ and radius $r > 0$ in metric space X, is given by $B(p, r) = \{x \in X | d(p, x) < r\}$*

Definition B.5. *A subset of S of a metric space, X, is said to be* **open** *in X if and only if for every point $p \in S$, there is some $r > 0$ so that $B(p, r) \subset S$. That is, any $q \in S$ such that $d(p, q) < r \implies q \in S$.*

3. Is the set $(0, 1]$ closed?

4. Basically a set, S, is open if for any point $p \in S$, you can draw a little "ball" around it so that the ball stays in S Draw a picture that demonstrates the definition of an open set.

5. If the set E is open, what can we say about E^c?

6. Why are sets like doors?

7. Why are sets not like doors?

8. What is a **clopen** set?

9. Give an example of a clopen set.

10. The set (a, ∞) is open in $\mathbb{R}$, what other *intervals* in $\mathbb{R}$ are open?

11. Given an example of a closed set in $\mathbb{R}$.

12. What was the most interesting part of your reading?

13. What was the muddiest or most confusing part of your reading?

Quiz 4: Please read our textbook, *Real Mathematical Analysis* pages 75-84 (you can also read pages 71-84 all the way through for some more topology information). There are questions on both sides of this page. The main thing I want you to get out of the Product Metric section is that we can define metrics on products of spaces. But first we need to define what a product space is... (pg 22):

Definition B.6. *Given sets A and b, the Cartesian Product of A and B is the set* $A \times B := \{(a, b) : a \in A, b \in B\}$.

Example B.1. $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ *is a Cartesian product of* $\mathbb{R}$ *and* $\mathbb{R}$.
$p = (1, 4)$ *is an example of an element of* $\mathbb{R} \times \mathbb{R} = \mathbb{R}^2$.

Page 75 gives examples of possible ways to calculate distances for product spaces. For example, if $\mathbb{R}$ is given the Euclidean metric ($d(a, b) = |a - b|$), for $p = (1, 3)$, $q = (-1, 7)$ $\in \mathbb{R} \times \mathbb{R}$,

$$d_{sum}(p, q) = d_{sum}((1, 3), (-1, 7)) = d(1, 3) + d(5, 7) = |3 - 1| + |7 - (-1)| = 2 + 8 = 10$$

B.6 Complete Metric Spaces

Here are some examples to keep in mind during this section.

- $\mathbb{R}$ is nice because it "has no gaps."
- $\mathbb{Q}$ is not so nice because it *does* "have gaps."
- $(0, 1]$ is not so nice because it doesn't contain all of its limit points. That is sequences like $a_n = \frac{1}{n}$ can "leak out" in the limit.

Definition B.7. *A metric space, X, is called* ***complete*** *if and only if every Cauchy sequence in X converges to an element in X.*

1. Give and example of a metric space that is complete.

2. Give and example of a metric space that is **not** complete.

3. If S is a closed subset of a complete metric space X, what property does S have?

4. Is $[a, b] \subset \mathbb{R}$ complete?

B.7 Compact Metric Spaces

Definition B.8. *A subspace S of metric space X, is* ***sequentially compact*** *if every sequence in S has a subsequence that converges to a point in S.*

5. If I have a compact set. What two properties does it have?

6. True or false: If a set is closed and bounded, it is compact.

7. True or false: $[a, b] \subset \mathbb{R}$ is compact.

8. The Bolzano-Weierstrass Theorem we learned in class for $\mathbb{R}$ extends to $\mathbb{R}^n$. That is every bounded sequence in $\mathbb{R}^n$ has a ____________________ subsequence.

9. In general, a set that is closed and bounded is not necessarily compact, but what does the **Heine-Borel Theorem** tell us?

   We actually have a **generalization of our Extreme Value Theorem**:
   If $f : X \to Y$ in continuous and X is compact, then $f(X)$ is compact.
   That is, $f(X)$ is bounded which means it assumes maximum and minimum values.

10. What was the most interesting part of your reading?

11. What was the muddiest or most confusing part of your reading?