

# POL502: Differential and Integral Calculus

Kosuke Imai  
Department of Politics, Princeton University

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We have come a long way and finally are about to study calculus. Many of you might have taken some courses in the past where you learned a number of formulas to calculate the derivatives and integrals of certain functions. The purpose of this course, however, is not to memorize these formulas mindlessly. Rather, our goals are to understand the mathematical concepts underlying such formulas and to develop a solid understanding of calculus. This should not be too challenging given that we are now armed with the knowledge of sequential and functional limits.

## 1 Derivatives

First, we start with the familiar definition of a derivative.

**Definition 1** *Let  $f : X \mapsto \mathbf{R}$  be a function and  $c \in X$  be an accumulation point of  $X$ . Then, the derivative is defined as*

$$f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}.$$

*We say  $f$  is differentiable at  $c$  if this limit exists. If such a limit exists at all  $c \in X$ , then we say  $f$  is differentiable (on  $X$ ).*

Before moving on, it is important to develop a geometric understanding of this definition. That is, the derivative represents a rate of change of the function. Also, make sure that you can apply the definition above to the following examples.

**Example 1** *Are the following functions differentiable at  $c \in \mathbf{R}$ ?*

1.  $f : \mathbf{R} \mapsto \mathbf{R}$  where  $f(x) = 2x + 1$ .
2.  $f : \mathbf{R} \mapsto \mathbf{R}$  where  $f(x) = x^n$  with  $n \in \mathbf{N}$ .
3.  $f : \mathbf{R} \mapsto \mathbf{R}$  where  $f(x) = |x|$ .

The last example suggests a connection between differentiability and continuity. Indeed, if a function is differentiable at a certain point, then it is also continuous at that point.

**Theorem 1 (Differentiability and Continuity)** *Let  $f : X \mapsto \mathbf{R}$  be a differentiable function at  $c \in X$ . Then,  $f$  is continuous at  $c$ .*

We note that although a function must be continuous if it is differentiable, its derivative might not be continuous. That is, the derivative of a derivative, called the second derivative, may not exist.

A function whose derivative is differentiable is said to be twice differentiable. The same principle applies to the third, fourth,  $\dots$ , derivatives.

There is important connection between derivatives and sequential limits. The next theorem immediately follows from Theorem 1 of the previous chapter.

**Theorem 2 (Derivatives and Sequential Limits)** *Let  $f : X \mapsto \mathbf{R}$  be a function and  $c$  be an accumulation point of  $X$ . Then,  $f$  is differentiable at  $c$  if and only if for every sequence  $\{x_n\}_{n=1}^{\infty}$  of  $X$  with  $x_n \neq c$  for all  $n \in \mathbf{N}$  and  $\lim_{n \rightarrow \infty} x_n = c$ , the sequence  $\left\{ \frac{f(x_n) - f(c)}{x_n - c} \right\}_{n=1}^{\infty}$  converges to  $f'(c)$ .*

Now, we prove the algebraic rules concerning derivatives

**Theorem 3 (Algebraic Operations of Derivatives)** *Let  $f, g : X \mapsto \mathbf{R}$  be differentiable at  $c \in X$  and  $X \subset \mathbf{R}$ . Then,*

1.  $(kf)'(c) = kf'(c)$  for all  $k \in \mathbf{R}$ ,
2.  $(f + g)'(c) = f'(c) + g'(c)$ ,
3.  $(fg)'(c) = f'(c)g(c) + f(c)g'(c)$  (Product Rule),
4.  $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}$  for  $g(c) \neq 0$  (Quotient Rule).

**Example 2** *Find the derivative of the following functions,  $f : X \mapsto \mathbf{R}$  where  $X \subset \mathbf{R}$ .*

1.  $f(x) = (9x^2 - 2)(3x + 1)$ .
2.  $f(x) = \frac{x^2 + 3}{x}$ .

Another well-known fact is the so called Chain rule.

**Theorem 4 (Chain Rule)** *Let  $f : X \mapsto \mathbf{R}$  and  $g : Y \mapsto \mathbf{R}$  with  $f(X) \subset Y$  so that  $g \circ f$  is defined. If  $f$  is differentiable at  $c \in X$  and  $g$  is differentiable at  $f(c) \in Y$ , then  $g \circ f$  is differentiable at  $c$  with  $(g \circ f)'(c) = g'(f(c))f'(c)$ .*

**Example 3** *Find the derivative of the following functions,  $f : X \mapsto \mathbf{R}$  where  $X \subset \mathbf{R}$ .*

1.  $f(x) = (3x^2 - 13)^3$ .
2.  $f(x) = \sqrt{1 + 3x^2}$ .

Finally, we consider some important special functions.

**Definition 2 (Exponential and Logarithmic Functions)** *The exponential function  $f : \mathbf{R} \mapsto (0, \infty)$  is defined as  $f(x) = e^x$  where  $e = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n = 2.71828\dots$ . The inverse of the exponential function,  $f^{-1} : (0, \infty) \mapsto \mathbf{R}$  is called the (natural) logarithmic function and is denoted by  $\log x$ .*

That is, if  $y = e^x$ , then  $x = \log y$ . The important properties of exponential and logarithmic functions are:  $e^x e^y = e^{x+y}$ ,  $(e^x)^y = e^{xy}$ ,  $\log x^y = y \log x$ ,  $\log x + \log y = \log xy$ , and  $\log x - \log y = \log \frac{x}{y}$ . You may occasionally encounter a log function whose base is different from  $e$ . For example,  $g : (0, \infty) \mapsto \mathbf{R}$  with  $g(x) = \log_{10} x$  is called the common logarithmic function. That is, if  $y = 10^x$  then  $x = \log_{10} y$ . Now, we derive the derivative of the natural logarithmic function.

**Theorem 5 (Derivative of Logarithmic Function)** *Consider  $f : (0, \infty) \mapsto \mathbf{R}$  with  $f(x) = \log(x)$ . Then,  $f'(c) = \frac{1}{c}$  for all  $c \in (0, \infty)$ .*

## 2 The Mean Value Theorem and Its Applications

Derivatives are often used to solve the optimization problems of functions where the goal is to find a point where an *objective* function attains its maximum or minimum. We first define the concept of *local* (or *relative*) extremum.

**Definition 3 (Local Extremum)** Let  $f : X \mapsto \mathbf{R}$  be a function with  $X \subset \mathbf{R}$ . A point  $c \in X$  is a local maximum (minimum) if there exists a neighborhood  $Q$  of  $c$  such that if  $f(x) \leq f(c)$  ( $f(x) \geq f(c)$ ) for any  $x \in Q \cap X$ .

Sometimes, one use the word *global* (or *absolute*) extremum to distinguish extremum from local extremum. Obviously, a local extremum is not guaranteed to be a global extremum because the former is an extremum only in a neighborhood. The next theorem shows that  $f'(c) = 0$  is a necessary condition for a point  $c$  being an extremum.

**Theorem 6 (Interior Extremum Theorem)** Let  $f : X \mapsto \mathbf{R}$  be differentiable on  $(a, b) \subset \mathbf{R}$  with  $a < b$ . If  $f$  attains a maximum or minimum value at some point  $c \in (a, b)$ , then  $f'(c) = 0$ .

Note that as we see below,  $f'(c) = 0$  is NOT a sufficient condition for an extremum, local or global. Now, recall the Extremum Value Theorem from the previous chapter which states that continuous functions on compact sets always attain maximum and minimum values. By combining this with Theorem 6, we have the following result.

**Theorem 7 (Rolle's Theorem)** Let  $f : [a, b] \mapsto \mathbf{R}$  be a function that is continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $a < b$ . If  $f(a) = f(b)$ , there exists a point  $c \in (a, b)$  such that  $f'(c) = 0$ .

It turns out that this is a special case of an even more general result, which is the key theorem of this section! After proving it, we also collect important implications of the Mean Value Theorem.

**Theorem 8 (Mean Value Theorem)** Let  $f : [a, b] \mapsto \mathbf{R}$  be a function that is continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $a < b$ . Then, there exists a point  $c \in (a, b)$  such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

**Theorem 9 (Corollaries of the Mean Value Theorem)** Let  $f : [a, b] \mapsto \mathbf{R}$  be a function that is continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $a < b$ .

1. If  $f'(c) \neq 0$  for all  $c \in (a, b)$ , then  $f$  is one-to-one.
2. If  $f'(c) = 0$  for all  $c \in (a, b)$ , then  $f$  is constant.
3. If  $f'(c) > 0$  for all  $c \in (a, b)$ , then  $f$  is strictly increasing.
4. If  $f'(c) < 0$  for all  $c \in (a, b)$ , then  $f$  is strictly decreasing.

Recall that the derivative of a function represents a rate of change of the function. A positive (negative) value of the derivative indicates that the function is increasing (decreasing) in the neighborhood. A larger positive (negative) value of the derivative implies a faster rate of increase (decrease). When the derivative is zero, the function is at a stationary point.

Before we turn to the optimization, we prove the following useful result.

**Theorem 10 (Inverse Function Theorem)** Suppose  $f : [a, b] \mapsto \mathbf{R}$  is continuous and differentiable with  $f'(x) \neq 0$  for all  $x \in [a, b]$ . Then,  $f$  is one-to-one, and  $f^{-1}$  is continuous and differentiable on  $f([a, b])$ . Furthermore, for all  $x \in [a, b]$ ,

$$(f^{-1})'(f(x)) = \frac{1}{f'(x)}.$$

Here are some examples.

**Example 4** Let  $0 < a < b$ . Find the derivative of  $f^{-1}$  for the following functions.

1.  $f : [a, b] \mapsto \mathbf{R}$  where  $f(x) = x^2$ .
2.  $f : [a, b] \mapsto \mathbf{R}$  where  $f(x) = \log x$ .

We can now characterize stationary points (i.e., the points where the derivative is zero).

**Theorem 11 (Characterization of Stationary Points)** Let  $f : X \mapsto \mathbf{R}$  be a differentiable function and  $f'(c) = 0$  with  $c \in X$ .

1.  $c$  is a local maximum,  $f'(c)$  changes its sign from positive to negative as it moves from the immediate left of the point  $c$  to its immediate right.
2.  $c$  is a local minimum,  $f'(c)$  changes its sign from negative to positive.
3.  $c$  is an inflection point,  $f'(c)$  does not change the sign.

**Example 5** Find the stationary points of the following functions and determine whether they are a local minimum, a local maximum, or an inflection point.

1.  $f : \mathbf{R} \mapsto \mathbf{R}$  where  $f(x) = x^3 - 12x^2 + 36x + 8$ .
2.  $f : \mathbf{R} \mapsto \mathbf{R}$  where  $f(x) = x^3 + 1$ .

In order to figure out more about the shape of a function, we need the derivative of a derivative, called the *second* (or *second-order*) *derivative* (as opposed to the *first* or *first-order* derivative). We often denote the second derivative of  $f : X \mapsto \mathbf{R}$  at  $c \in X$  by  $f''(c)$ . Note that in order for the second derivative to exist, the first derivative has to be differentiable. Theorem 2 suggests that the second derivative represents a rate of change of the slope of a function. This allows us to investigate the following characteristics of functions.

**Definition 4** Let  $f : [a, b] \mapsto \mathbf{R}$  be a function.

1.  $f$  is concave on  $[x_0, x_1] \in X$  if  $f(\lambda x_0 + (1 - \lambda)x_1) \geq \lambda f(x_0) + (1 - \lambda)f(x_1)$  for all  $x_0, x_1 \in [a, b]$  and all  $\lambda \in [0, 1]$ .
2.  $f$  is convex on  $[x_0, x_1] \in X$  if  $f(\lambda x_0 + (1 - \lambda)x_1) \leq \lambda f(x_0) + (1 - \lambda)f(x_1)$  for all  $x_0, x_1 \in [a, b]$  and all  $\lambda \in [0, 1]$ .

**Theorem 12 (Concavity and Convexity)** Let  $f : [a, b] \mapsto \mathbf{R}$  be a twice differentiable function.

1.  $f$  is concave on  $[a, b]$  if  $f''(c) \leq 0$  for all  $c \in [a, b]$ .
2.  $f$  is convex on  $[a, b]$  if  $f''(c) \geq 0$  for all  $c \in [a, b]$ .

Now, we can use the second derivative to find a local minimum and maximum.

**Theorem 13 (First-Order and Second-Order Conditions)** Let  $f : [a, b] \mapsto \mathbf{R}$  be a twice differentiable function and  $f'(c) = 0$  for  $c \in [a, b]$ . Then,

1.  $f(c)$  is a local maximum if  $f''(c) < 0$ .
2.  $f(c)$  is a local minimum if  $f''(c) > 0$ .

Let's apply this theorem to Example 2.

Another important application of the Mean Value theorem (especially for statistics) is that one can approximate differentiable functions using the derivatives.

**Theorem 14 (Taylor's Theorem)** Let  $f : [a, b] \mapsto \mathbf{R}$  be  $n$  times differentiable on  $[a, b]$  with  $a < b$ , and its  $n$ th derivative  $f^{(n)}$  is also continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Let  $x_0 \in [a, b]$ . Then, for each  $x \in [a, b]$  with  $x \neq x_0$  there exists  $c$  between  $x$  and  $x_0$  such that

$$f(x) = f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}$$

The second term of the right hand side equation is called the Taylor series, and the last term is called Lagrange remainder.

**Example 6** Apply the Taylor's theorem to the following functions.

1. Expand  $f(x) = \frac{1}{1+x}$  around the point  $x_0 = 2$  with  $n = 2$ .
2. Expand  $f(x) = e^x$  around the point  $x_0 = 0$ .

Recall from the previous chapter that when taking a limit of a quotient of functions, we can write  $\lim_{x \rightarrow c} \frac{f}{g}(x) = \frac{\lim_{x \rightarrow c} f(x)}{\lim_{x \rightarrow c} g(x)}$  provided that the denominator is not zero. But, what if both the denominator and numerator go to zero?

**Theorem 15 (L'Hospital's Rule)** Let  $f, g : [a, b] \mapsto \mathbf{R}$  be continuous on  $[a, b]$  and differentiable on  $(a, b)$  with  $a < b$ . If  $g'(x) \neq 0$  for all  $x \in (a, b)$ ,  $f(c) = g(c) = 0$ , and  $\frac{f'}{g'}$  has a limit at  $c$ , then  $\frac{f}{g}$  has a limit at  $c$  and

$$\lim_{x \rightarrow c} \frac{f}{g}(x) = \lim_{x \rightarrow c} \frac{f'}{g'}(x).$$

**Example 7** Compute  $\lim_{x \rightarrow 2} \frac{f}{g}(x)$  where  $f, g : [2, 3] \mapsto \mathbf{R}$  with  $f(x) = \sqrt{x} - \sqrt{2} + \sqrt{x-2}$  and  $g(x) = \sqrt{x^2 - 4}$ .

Although what we have learned so far can be extended to the functions of multiple variables, at this point we are not yet prepared to study the optimization problems of multivariate functions. We shall come back to this topic later in the course.

### 3 The Riemann Integral

In this section, we study the integral calculus from a particular perspective, i.e., that of the Riemann integral. Riemann's idea was to use the notion of "area under the curve" for the definition of integration. This is not the only way to define integrals and has its own limitations (e.g., the Lebesgue integral is commonly used in modern probability theory). Nevertheless, the Riemann integral gives a useful starting point for studying integral calculus.

**Definition 5** Let  $f : [a, b] \mapsto \mathbf{R}$  be a bounded function.

1. A partition  $P$  of  $[a, b]$  is any finite set  $P = \{x_0, x_1, x_2, \dots, x_n\}$  with  $a = x_0 < x_1 < x_2 < \dots < x_n = b$ .
2. The lower sum of  $f$  with respect to a partition  $P$  is

$$L(f, P) = \sum_{k=1}^n m_k(x_k - x_{k-1}) \quad \text{where} \quad m_k = \inf\{f(x) : x \in [x_{k-1}, x_k]\}.$$

3. The upper sum of  $f$  with respect to a partition  $P$  is

$$U(f, P) = \sum_{k=1}^n M_k(x_k - x_{k-1}) \quad \text{where} \quad M_k = \sup\{f(x) : x \in [x_{k-1}, x_k]\}.$$

Note that a function must be bounded in order for "area under the curve" to be defined. It is clear from this definition that  $U(f, P) \geq L(f, P)$ . Next, we prove some other important properties.

**Theorem 16 (Upper/Lower Sums)** Let  $P$  and  $Q$  partitions  $[a, b]$  and  $f : [a, b] \mapsto \mathbf{R}$  be a bounded function.

1. If  $P \subset Q$  (we call  $Q$  as a refinement of  $P$ ), then  $L(f, P) \leq L(f, Q)$  and  $U(f, P) \geq U(f, Q)$ .
2.  $L(f, P) \leq U(f, Q)$ .

A geometric understanding of these concepts is important. As the partitions get finer, the upper sums and lower sums get closer to each other. With this intuition at hand, we are ready to provide the definition of the Riemann integral.

**Definition 6** Let  $\mathcal{P}$  be the set of all possible partitions of  $[a, b]$  and  $f : [a, b] \mapsto \mathbf{R}$  be a bounded function.

1. The upper integral of  $f$  is  $U(f) = \inf\{U(f, P) : P \in \mathcal{P}\}$ .
2. The lower integral of  $f$  is  $L(f) = \sup\{L(f, P) : P \in \mathcal{P}\}$ .

The next theorem follows immediately from the previous theorem.

**Theorem 17 (Upper/Lower Integrals)** Let  $f : [a, b] \mapsto \mathbf{R}$  be a bounded function. Then,  $L(f) \leq U(f)$ .

**Definition 7** Let  $f : [a, b] \mapsto \mathbf{R}$  be a bounded function.  $f$  is said to be Riemann-integrable if

$$\int_a^b f(x) dx = U(f) = L(f).$$

That is, in order for a (bounded) function to be integrable, the inequality in Theorem 17 must become equality. Now, we relate this definition with the concepts of the upper and lower sums.

**Theorem 18 (Riemann Criteria)** *A bounded function  $f : [a, b] \mapsto \mathbf{R}$  is Riemann-integrable on  $[a, b]$  if and only if for any  $\epsilon > 0$  there exists a partition  $P$  of  $[a, b]$  such that  $U(f, P) - L(f, P) < \epsilon$ .*

Let's look at some example to understand the concepts introduced so far.

**Example 8** *Are the following functions Riemann-integrable?*

1.  $f : [0, 1] \mapsto \mathbf{R}$  with  $f(x) = x^2$ .
2.  $f : [0, 1] \mapsto \mathbf{R}$  with  $f(x) = 0$  if  $x$  is rational and  $f(x) = 1$  if  $x$  is irrational.

The first function is both monotone and continuous while the second is neither. We can show more generally that a monotone function is Riemann-integrable and so is a continuous function. This gives a locally large class of integrable functions.

**Theorem 19 (Monotonicity and Integrability)** *If  $f : [a, b] \mapsto \mathbf{R}$  is monotone, then it is Riemann-integrable on  $[a, b]$ .*

**Theorem 20 (Continuity and Integrability)** *If  $f : [a, b] \mapsto \mathbf{R}$  is continuous on  $[a, b]$ , then it is Riemann-integrable on  $[a, b]$ .*

Finally, we prove the familiar algebraic properties about the integral. To prove the algebraic operations of integrals, we use the fact that a function  $f : [a, b] \mapsto \mathbf{R}$  is integrable on  $[a, b]$  if and only if there exists a sequence of partitions  $\{P_n\}_{n=1}^{\infty}$  such that  $\lim_{n \rightarrow \infty} [U(f, P_n) - L(f, P_n)] = 0$  (you should be able to prove this using Theorem 18 without too much difficulty).

**Theorem 21 (Algebraic Operations of Integrals)** *Let  $f, g : [a, b] \mapsto \mathbf{R}$  be bounded functions and be Riemann-integrable on  $[a, b]$ .*

1.  $\int_a^b c f(x) dx = c \int_a^b f(x) dx$  for any  $c \in \mathbf{R}$ .
2.  $\int_a^b f(x) + g(x) dx = \int_a^b f(x) dx + \int_a^b g(x) dx$ .

**Theorem 22 (Order of Integrals)** *Let  $f, g : [a, b] \mapsto \mathbf{R}$  be bounded functions and be Riemann-integrable on  $[a, b]$ .*

1. If  $m \leq f(x) \leq M$  for all  $x \in [a, b]$ , then  $m(b - a) \leq \int_a^b f dx \leq M(b - a)$ .
2. If  $f(x) \leq g(x)$  for all  $x \in [a, b]$ , then  $\int_a^b f(x) dx \leq \int_a^b g(x) dx$ .
3.  $|f|$  is Riemann-integrable on  $[a, b]$ , and  $|\int_a^b f(x) dx| \leq \int_a^b |f(x)| dx$ .

**Theorem 23 (Partitions and Integrals)** *Let  $f : [a, b] \mapsto \mathbf{R}$  be a bounded function and  $c \in (a, b)$ . Then,  $f$  is Riemann-integrable on  $[a, b]$  if and only if  $f$  is Riemann-integrable on  $[a, c]$  and  $[c, b]$ . And, we have*

$$\int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx.$$

## 4 The Fundamental Theorem of Calculus and Its Applications

Finally, we establish the connection between the derivative and the integral. To prove the following theorem, we make use of the Mean Value theorem. To illustrate this idea, first consider Example 8 (1). Now, use another function  $g(x) = \frac{x^3}{3}$  to find the integral.

**Theorem 24 (The Fundamental Theorem of Calculus)** *Let  $f : [a, b] \mapsto \mathbf{R}$  be a differentiable function on  $[a, b]$  and  $f'$  be Riemann-integrable on  $[a, b]$ . Then,*

$$\int_a^b f'(x) dx = f(b) - f(a).$$

The above is called the *definite* integral. If the region of integration is not specified, the integral is called *indefinite* and is defined as  $\int f'(x)dx = f(x) + C$  where  $C$  is any constant. Now, we can use this theorem to calculate the integral of some functions.

**Example 9** *Find the following integrals.*

1.  $\int_1^2 (4x^2 + \sqrt{x} - \frac{3}{x}) dx$
2.  $\int (x^3 + 3x^2 + 1)^3(x^2 + 2x) dx$

The previous theorem shows us how to compute the integral of a function. The next theorem gives more theoretical results. First, the integral of a function is continuous. This contrasts with the fact that the derivative of a function may not be continuous. Secondly, every continuous function is the derivative of its integral. This second property is very important, and it is the reason why an integral is sometimes called an “anti-derivative.”

**Theorem 25 (Continuity and Integrals)** *Let  $f : [a, b] \mapsto \mathbf{R}$  be Riemann-integrable on  $[a, b]$  and  $F(c) = \int_a^c f(x) dx$  for all  $c \in [a, b]$ . Then,*

1.  $F$  is continuous on  $[a, b]$ .
2. If  $f$  is continuous at  $c \in [a, b]$ , then  $F$  is differentiable at  $c$  and  $F'(c) = f(c)$ .

We now present the two useful techniques of integration: Integration by Substitution and Integration by Parts. The next theorem follows immediately from the application of the Chain Rule and Theorem 25.

**Theorem 26 (Integration by Substitution)** *Let  $f : [a, b] \mapsto \mathbf{R}$  be differentiable on  $[a, b]$  with  $f'$  being continuous and  $f([a, b]) = [f(a), f(b)]$ . If  $g : [f(a), f(b)] \mapsto \mathbf{R}$  is a continuous function, then*

$$\int_a^b (g \circ f(x))f'(x) dx = \int_{f(a)}^{f(b)} g(t) dt.$$

Indeed, this amounts to applying the Chain rule backwards. Note that integration by substitution can be applied to indefinite integral exactly in the same way. Let's try some examples.

**Example 10** *Compute the following integrals.*

1.  $\int 2x\sqrt{x^2 + 1} dx$ .
2.  $\int_0^1 8e^{2x+3} dx$ .



The idea of integration by substitution allows one to do the “change-of-variables” within integrals. To see this, recall that the derivative of  $f(x)$  can also be written as  $\frac{d}{dx}f(x)$ . Now, suppose you want to change the variable from  $x$  to  $z$ , which is an one-to-one function of  $x$ . Let  $x = f(z)$  (or equivalently  $z = f^{-1}(x)$ ). Then,  $dx = f'(z) dz$ . This relationship gives you  $\int g(x) dx = \int g(f(z))f'(z) dz$ .

The next theorem immediately follows from the application of the Product Rule.

**Theorem 27 (Integration by Parts)** *Let  $f, g : [a, b] \mapsto \mathbf{R}$  be differentiable on  $[a, b]$  with  $f', g'$  being Riemann-integrable on  $[a, b]$ . Then,  $f'g$  and  $g'f$  are also Riemann-integrable on  $[a, b]$ , and*

$$\int_a^b f'(x)g(x) dx = [f(b)g(b) - f(a)g(a)] - \int_a^b g'(x)f(x) dx.$$

For indefinite integrals, we have

$$\int f'(x)g(x) dx = f(x)g(x) - \int g'(x)f(x) dx.$$

One may write this theorem for indefinite integrals as  $\int u dv = uv - \int v du$  by setting  $u = g(x)$  and  $v = f(x)$  and noting  $f'(x) = \frac{d}{dx}f(x)$  and  $g'(x) = \frac{d}{dx}g(x)$ . Let's apply this technique to the following examples.

**Example 11** *Compute the following integrals.*

1.  $\int x^2 e^x dx$ .
2.  $\int x \log x dx$ .

Before ending this chapter, we present some interesting results concerning the integral. First, we derive an alternative expression for Lagrange remainder in Taylor's theorem.

**Theorem 28 (Alternative Form of Taylor's Theorem)** *Let  $f : [a, b] \mapsto \mathbf{R}$  be  $n$  times differentiable on  $[a, b]$  with  $a < b$ , and its  $n$ th derivative  $f^{(n)}$  is also continuous on  $[a, b]$  and differentiable on  $(a, b)$ . Let  $x_0 \in [a, b]$ . Then, for each  $x \in [a, b]$  with  $x \neq x_0$*

$$f(x) = f(x_0) + \sum_{k=1}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k + \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t) (x - t)^n dt.$$

Next, we prove an analogue of the Mean-Value Theorem for integrals.

**Theorem 29 (Mean-Value Theorem for Integrals)** *Suppose  $f : [a, b] \mapsto \mathbf{R}$  is continuous on  $[a, b]$ , and  $g : [a, b] \mapsto \mathbf{R}$  is Riemann-integrable on  $[a, b]$  with  $g(x) \geq 0$  for all  $x \in [a, b]$ . Then, there exists  $c \in [a, b]$  such that*

$$\int_a^b f(x)g(x) dx = f(c) \int_a^b g(x) dx.$$

By setting  $g(x) = 1$ , we obtain the result that  $\int_a^b f(x) dx = f(c)(b - a)$ , which gives a nice geometric interpretation of this mean value theorem. Using this theorem, one can also easily prove the equivalence between the two alternative expressions for the Lagrange remainder in the Taylor's theorem.

This ends our study of differential and integral calculus. Unfortunately, we are not yet ready to study calculus that involves more than one variable. We shall come back to that topic after studying linear algebra.