Be An Actuary A Career Without Boundaries

# Mathematics of Risk

## *Introduction*

There are many mechanisms that individuals and organizations use to protect themselves against the risk of financial loss.  Government organizations and public and private companies provide various forms of protection, including insurance contracts, such as homeowners, auto, health and life insurance; pension plans; social insurance programs such as Medicare, Medicaid and Social Security; and other programs to protect against a wide variety of risks.

The process of managing risk is highly mathematical and quantitative.  The insurance, pension and social insurance industry employs certified professionals called actuaries with the specific skills required to address risk management.  These skills include advanced analytical and mathematical expertise, problem solving abilities, and general business acumen.

The attached case study and accompanying problems are intended for advanced high school math students.  They illustrate the types of mathematics commonly used in these types of risk management programs.  These problems are representative of the type of work that actuaries do.

## *Case Study #1 – Personal Auto Insurance Pricing*

### Mathematical Concepts Illustrated

The following mathematical topics are illustrated in this case study:

- Random variables
- Probability
- Expected value

### Background

Auto Insurers Inc. (AII) sells 6-month auto insurance policies.  In exchange for the payment of a single premium at the beginning of the 6-month period, the insurance policy covers the cost of repairing any damage to the insured vehicle during that period. For ease in performing the calculations, we will make some simplifying assumptions:

1. We assume that damage is incurred in \$5,000 increments, up to \$20,000 (for a totaled vehicle).

2. We assume that all policyholders buy their policy on the same day.

3. We assume that all policyholders are charged the same premium.

The company's pricing actuary needs to determine how much to charge each person (called the insured) for the policy. The amount charged is called the premium. To determine this, the actuary needs to make some sort of prediction about how much will be paid out in claims against the policy during the 6-month period. Different companies may use different formulas to make this determination. The pricing actuary decides to use the following formula:

Premium = expected claims amount for each driver

= expected number of accidents * expected amount paid on each accident.

In the insurance industry, the expected number of accidents is known as the <u>frequency</u>, while the amount incurred for each accident is called the <u>severity</u>. Thus, our formula can be expressed as:

Expected claims amount = expected frequency * expected severity.

## Probability and Expected Value

Frequency and severity are each <u>random variables</u>. Random variables are variables that can take on random values, based on an underlying <u>probability distribution</u>. A probability distribution describes the probability associated with each potential value of a random variable. We can use a die as an example of a random variable. A die has 6 sides. A normal die will land showing any one of the 6 sides, each with equal probability of 1/6. Thus, on any roll, there is a 1/6 chance that a 1 will appear, a 1/6 chance that a 2 will appear, etc. The notation that we use to denote the probability that a 1 will appear is $\Pr(X = 1) = 1/6$, read as, "The probability that the random variable X will take on a value of 1 is equal to 1/6." The probability distribution for a die would look like:

| Side Showing | Probability |
|---|---|
| 1 | 1/6 |
| 2 | 1/6 |
| 3 | 1/6 |
| 4 | 1/6 |
| 5 | 1/6 |
| 6 | 1/6 |

Note that the probabilities in the right-hand column sum to 100%. One rule of probability distribution is that the probabilities must sum to 100% over all potential outcomes.

Probability distributions can be used to provide several different types of information. For example, the probability of a random variable Y taking on value A *or* value B is the sum of the two probabilities, Pr(Y = A) + Pr(Y = B). The probability that a random variable Y takes on value A on one outcome and B on the next is Pr(Y = A) * Pr(Y = B). The probability that a random variable Y takes on a value other than A is 1 – Pr(Y = A). As noted above, 1 (or 100%) is the total of all probabilities over the entire set of outcomes, so this is equivalent to taking the entire set of outcomes and subtracting the probability of the one event we want to exclude.

**Check your understanding:**

What is the probability of rolling a 1 or a 2?
Pr(Die = 1) + Pr(Die = 2) = 1/6 + 1/6 = 2/6.

What is the probability of rolling a 1 followed by a 2 on the next roll? 1/6 * 1/6 = 1/36.
Pr(Die = 1) * Pr(Die = 2) = 1/6 * 1/6 = 1/36.

What is the probability of rolling anything other than a 1 or a 2?
1 – Pr(Die = 1) – Pr (Die = 2) = 1 – 1/6 – 1/6 = 4/6.

If we know the probability of each value occurring, then we can calculate the expected value of the die. The expected value of random variable X is denoted E(X) and is calculated as:

$$E(X) = \Sigma Pr(X) * X$$

Thus, it is the sum of each value multiplied by the probability of that value occurring. Note that the expected value does not have to be equal to any of the possible outcomes of the event.

**Check your understanding:**

What is the expected value of the die?
E(X) = 1/6 * 1 + 1/6 * 2 + 1/6 * 3 + 1/6 * 4 + 1/6 * 5 + 1/6 * 6 = 3.5.
Note that 3.5 is not a possible outcome of rolling the die.

## Real-World Application

As shown above, to calculate the premium each insured must pay, the pricing actuary needs to calculate the expected number of claims each driver will have over the 6 month period and the expected amount of each of those claims. Using the mathematical notation introduced above, we can restate the pricing formula as:

$$\text{Premium} = E(\text{\# of claims}) * E(\text{amount of claim})$$

Now let's assume that the pricing actuary has the following distribution for the number of claims for a driver in the 6-month period covered by the policy, also known as the frequency:

| Number of Claims | Probability |
|---|---|
| 0 | 50% |
| 1 | 25% |
| 2 | 15% |
| 3 | 10% |

In this situation, based on the given probability distribution, the probability that someone has more than 3 accidents in the 6-month period is 0%. With this distribution, the expected value of the number of claims can be calculated as:

E(# of claims) = Pr(# of claims = 0) * 0 + Pr(# of claims = 1) * 1 + Pr(# of claims = 2) * 2 + Pr(# of claims = 3) * 3 = 50% * 0 + 25% * 1 + 15% * 2 + 10% * 3 = 0.85.

Next, assume the pricing actuary has the following distribution for the severity, or the cost for each claim:

| Cost of Claim | Probability |
|---|---|
| $5,000 | 10% |
| $10,000 | 40% |
| $15,000 | 45% |
| $20,000 | 5% |

As you can see from this probability distribution, it is much more likely for the claim to cost $10,000 or $15,000 than either the low amount ($5,000) or the high amount ($20,000). We can calculate the expected value of this random variable as:

E(amount of claim) = Pr(amount of claim = $5,000) * $5,000 + Pr(amount of claim = $10,000) * $10,000 + Pr(amount of claim = $15,000) * $15,000 + Pr(amount of claim = $20,000) * $20,000 = 10% * $5,000 + 40% * $10,000 + 45% * $15,000 + 5% * $20,000 = $12,250.

Note that the expected value is expressed in the same format (in dollars) as the various outcomes of the random variable.

Now, we can calculate the premium using our original formula as:

Premium = E(# of claims) * E(amount of claim)

= 0.85 * $12,250 = $10,412.50.

## Additional Problems

1. In the next year, the pricing actuary does another study and finds that the probabilities of the number of accidents and the amount per accident are as follows:

| Number of Claims | Probability |
|---|---|
| 0 | 40% |
| 1 | 35% |

| | |
|---|---|
| 2 | 10% |
| 3 | 10% |
| 4 | 5% |

| Cost of Claim | Probability |
|---|---|
| $5,000 | 20% |
| $10,000 | 50% |
| $15,000 | 20% |
| $20,000 | 10% |

Recalculate the premium the company should charge using the formula provided in the text above.

2. Note that the expected value is only an expectation, and that actual results may differ from the expected value. Say that the company sells two policies, charging the premium found in question 1 for each. Assume that the two drivers have a total of 3 accidents, with the amount of claims shown below for each accident.

| Accident | Amount |
|---|---|
| # 1 | $5,000 |
| # 2 | $15,000 |
| # 3 | $10,000 |

The company's profit or loss is equal to the amount of premium they take in minus the claims they pay out. What profit or loss does the company make on these two drivers?

3. The company would like to build a level of conservatism into their premium formula. They want to charge a premium such that 90% of the premium will cover the expected claims, while the other 10% will kept by the company as profit if actual results turn out as expected. What premium should the company charge each driver?

4. Upon further review of accident statistics and individual driver's records, the company determines that its insured drivers can be divided into two categories, simply identified as "good drivers" and "bad drivers." The distribution of accidents for the good drivers is:

| Number of Claims | Probability |
|---|---|
| 0 | 70% |
| 1 | 20% |
| 2 | 10% |

These drivers also experience a lower dollar amount per claim. The distribution of claims for these drivers is:

| Cost of Claim | Probability |
|---|---|
| $5,000 | 80% |
| $10,000 | 15% |
| $20,000 | 5% |

The bad drivers have the same accident and claim distribution as in problem #1 above. Ignore the profit component introduced in #3 above. If the company wants to charge each good driver a premium equal to his or her expected claims, excluding any additional amount for profit, what amount should the company charge a good driver? If 40% of the company's insured drivers are classified as good drivers and the other 60% are classified as bad drivers, what premium amount should the company charge if it wishes to charge all drivers, both good and bad, the same amount, excluding any additional amount for profit?

5. Similar to auto insurance, home owners insurance also has the same possibility of multiple claims on a single premium with varying amounts. With home owners insurance, the dollar amount of a claim can be much higher than on an auto insurance policy. For simplicity and a die and spinner can be used to represent the frequency and severity of home owners insurance claims; the number rolled on the die represents the number of claims and the spinner represents the claim amount.

| Die | Probability |
|---|---|
| # 1 | 1/6 |
| # 2 | 1/6 |
| # 3 | 1/6 |
| # 4 | 1/6 |
| # 5 | 1/6 |
| # 6 | 1/6 |

| Spinner | Probability |
|---|---|
| $0 | .40 |
| $500 | .35 |
| $1,500 | .15 |
| $5,000 | .05 |
| $10,000 | .045 |
| $300,000 | .005 |

What is the premium amount needed to cover the expected claims amount?

## *Glossary of Terms*

Expected value – The sum of the probability of each possible outcome multiplied by the value of that outcome.

Frequency – The measurement of the number of occurrences of a specific event over a period of time.

Probability distribution – The distribution of probabilities associated with the various values that a random variable can take.

Random variable – A random variable is a variable that can take on different, random values.

Severity – The measurement of the impact of an event.