9. Integration Techniques

We begin this chapter by reviewing all those results which we already know, and perhaps a few we have yet to assimilate. In particular, I add the hyperbolic functions to our required lexicon of functions. If you don't know what the hyperbolic sine or cosine is then you should look it up in the text book or perhaps read Chapter 2 of my notes.

The principle aim of this chapter is to complete your knowledge of basic integration techniques. The methods presented here are foundational to other schemes. Moreover, the algebraic insights implicit within our discussions have use far beyond integration. To be honest, this chapter is about building mathematical character.

Improper integration involves either bounds which diverge or integrands which diverge. In either case the integral is to be understood in terms of definite integral with a varying bound. If the integrand diverges at some point in the integration region then we have to take the limit of definite integrals that approach that point. On the other hand, if we write that the upper integration bound is infinity then that is meant to indicate we take the limit of definite integrals with ever increasing upper bounds. L'Hopital's Rule is sometimes needed to determine the behavior of the limits that arise from improper integrations.

We conclude this chapter with an introductory discussion of numerical integration techniques. The midpoint, trapezoid and Simpson's Rule are contrasted. Some basic ideas about error bounds are also discussed.

9.1. BASIC INTEGRALS

I assume that you know the integrals (1-14) given below:

$$1.)\ \int x^n\ dx = \frac{x^{n+1}}{n+1} + c\ \ , n \neq -1$$

$$2.)\ \int \frac{dx}{x} = \ln|x| + c$$

$$3.)\ \int e^x\ dx = e^x + c$$

$$4.)\ \int a^x\ dx = \frac{1}{\ln(a)} a^x + c\ \ a > 0$$

$$5.)\ \int \cos(x)\ dx = \sin(x) + c$$

$$6.)\ \int \sin(x)\ dx = -\cos(x) + c$$

$$7.)\ \int \sec^2(x)\ dx = \tan(x) + c$$

$$8.)\ \int \sec(x)\tan(x)\ dx = \sec(x) + c$$

$$9.)\ \int \csc^2(x)\ dx = -\cot(x) + c$$

$$10.)\ \int \csc(x)\cot(x)\ dx = \csc(x) + c$$

$$11.)\ \int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1}(x) + c$$

$$12.)\ \int \frac{1}{1+x^2}\ dx = \tan^{-1}(x) + c$$

$$13.)\ \int \cosh(x)\ dx = \sinh(x) + c$$

$$14.)\ \int \sinh(x)\ dx = \cosh(x) + c$$

The integrals that follow are also basic, but I don't assume you have them memorized. If there was a question on a test concerning these then I would give a hint.

$$15.)\ \int \operatorname{sech}^2(x)\ dx = \tanh(x) + c$$

$$16.)\ \int \frac{1}{\sqrt{x^2-1}}\ dx = \cosh^{-1}(x) + c$$

$$17.)\ \int \frac{1}{\sqrt{x^2+1}}\ dx = \sinh^{-1}(x) + c$$

$$18.)\ \int \frac{1}{1-x^2}\ dx = \tanh^{-1}(x) + c$$

9.2. EXPLICIT SUBSTITUTIONS

An explicit substitution is often called a u-substitution. The basic idea here is that it may be possible to recast a given non-basic integral in the initial variable (usually x) as a basic integral in the substituted variable (usually u). In other words, the goal is to get back to one of the known integrals listed in the last section.

Example 9.2.1: Consider $\int xe^{x^2}dx$. Choose $u = x^2$ which produces $du = 2xdx \implies xdx = \frac{du}{2}$. Thus the given integral simplifies as follows:

$$\int xe^{x^2}\ dx = \int e^u \frac{du}{2} = \frac{1}{2}e^u + c = \boxed{\frac{1}{2}e^{x^2} + c}$$

Example 9.2.2: Consider $\int \cos(x)\cos(\sin(x))\ dx$. Choose $u = \sin(x)$ which yields $du = \cos(x)dx$. Thus the given integral simplifies as follows:

$$\int \cos(x)\cos(\sin(x))\ dx = \int \cos(u)du = \sin(u) + c = \boxed{\sin(\sin(x)) + c}$$

Example 9.2.3: Consider $\int x\sqrt{x+3}\ dx$. Choose $u = x + 3$ so that $x = 3 - u$ and $du = dx$. Thus the integral simplifies as follows:

$$\begin{aligned} \int x\sqrt{x+3}\ dx &= \int (3-u)\sqrt{u}\ du \\ &= \int \left(3\sqrt{u} - u^{\frac{3}{2}}\ \right)du \\ &= 2u^{\frac{3}{2}} - \frac{2}{5}u^{\frac{5}{2}} + c \\ &= \boxed{2(x+3)^{\frac{3}{2}} - \frac{2}{5}(x+3)^{\frac{5}{2}} + c} \end{aligned}$$

Example 9.2.4: Consider $\int \sec(x)\ dx$. Choose $u = \sec(x) + \tan(x)$ so that $\sec(x)dx = \frac{du}{u}$ (this is not really obvious, you should check this is true):

$$\int \sec(x)\ dx = \int \frac{du}{u} = \ln|u| + c = \boxed{\ln|\sec(x) + \tan(x)| + c}$$

This integral is worth remembering, we'll see it again. The examples given here are not meant to be comprehensive. You might need to go review my calculus I notes if you are rusty on u-substitution.

Definite Integrals involving u-substitution:

There are two ways to do these. You should understand both methods.

iii. Find the antiderivative via u-substitution and then use the FTC to evaluate in terms of the given upper and lower bounds in x. (see E18 below)

iv. Do the u-substitution and change the bounds all at once, this means you will use the FTC and evaluate the upper and lower bounds in u. (see E17 below)

The notation is not decorative, it is necessary and important to use correct notation. If the measure in your definite integral is "du" then you had best have bounds which refer to the value of "u".

Method ii. illustrated:

E17)

$$\int_{4\pi^2}^{9\pi^2} \frac{\sin\sqrt{x}}{\sqrt{x}}\,dx = \int_{2\pi}^{3\pi} \frac{\sin(u)}{\sqrt{x}}\,2\sqrt{x}\,du$$

$$u = u(x) = \sqrt{x}$$

$$\frac{du}{dx} = \frac{1}{2\sqrt{x}} \quad \therefore \quad dx = 2\sqrt{x}\,dx$$

$$u(4) = \sqrt{4\pi^2} = 2\pi$$

$$u(9) = \sqrt{9\pi^2} = 3\pi$$

$$= \int_{2\pi}^{3\pi} 2\sin(u)\,du$$

$$= -2\cos(u)\Big|_{2\pi}^{3\pi}$$

$$= -2\cos(3\pi) + 2\cos(2\pi)$$

$$= \boxed{4}$$

Method i. illustrated:

E18)

$$\int_0^{\pi/4} \tan^3\theta\,d\theta = \left[\ln|\cos\theta| + \frac{1}{2\cos^2\theta}\right|_0^{\pi/4} = \left(\ln\left|\frac{\sqrt{2}}{2}\right| + 1\right) - \left(\ln(1) + \frac{1}{2}\right)$$

$$= \boxed{\ln(\sqrt{2}/2) + \frac{1}{2}}$$

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using work below.

$$\int \tan^3\theta\,d\theta = \int \frac{\sin^3\theta}{\cos^3\theta}\,d\theta$$

$$= \int \frac{1-\cos^2\theta}{\cos^3\theta}\sin\theta\,d\theta$$

$$= \int \frac{1-u^2}{u^3}(-du)$$

$$u = \cos\theta$$

$$\frac{du}{d\theta} = -\sin\theta \Rightarrow -du = \sin\theta\,d\theta$$

$$= \int\left(\frac{1}{u} - \frac{1}{u^3}\right)du$$

$$= \ln|u| + \frac{1}{2u^2} + C$$

$$= \ln|\cos\theta| + \frac{1}{2\cos^2\theta} + C$$

9.3. TRIGONOMETRY ALL GROWN UP

It is time we settled what is known and unknown about trigonometry. If you had Calculus I with me then you have already heard many of these things but for the most part I did not require you remember them. That time is over. I expect you to absorb the material in this section one way or another. I do expect you remember all the trigonometric identities I present in this section, more than that, I expect you learn how to derive similar identities if the need arises.

Logically there are a variety of routes to remember this material. I choose the route that allows us to derive as much as possible through essentially algebraic arguments. I call this the *imaginary exponential technique.* There is nothing really "imaginary" about this, its unfortunate terminology since complex numbers are just as "real" as real numbers. Both real and imaginary numbers are well-defined mathematical objects. There are rules and equations which govern them. Both provide a language which is used to describe a plethora of physical systems (although, I would argue, that is not necessary for them to be sensible mathematical objects)

9.3.1: What is a complex number?

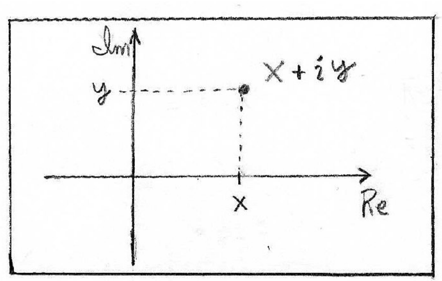
Complex numbers are pairs of real numbers that enjoy a certain rather beautiful multiplication; $(a,b)*(c,d) = (ac-bd, ad+bc)$. This is usually denoted

$$\boxed{(a+ib)(c+id) = ac + iad + ibc + i^2bd = ac - bd + i(ad+bc)}$$

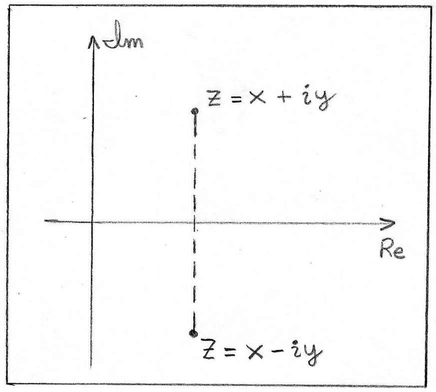
Where the observation $i^2 = (0,1)*(0,1) = (-1,0) = -1$. We typically introduce this by saying that $i = \sqrt{-1}$ but this is the same as saying $i^2 = -1$. Complex numbers can be added, subtracted, multiplied and divided just the same as real numbers. Complex number have a real and imaginary part,

$$\boxed{Re(a,b) = Re(a+ib) = a \qquad Im(a,b) = Im(a+ib) = b}$$

In general if $z \in \mathbb{C}$ then $z = Re(z) + iIm(z)$. It should be emphasized that $Re(z), Im(z) \in \mathbb{R}$ so there is a natural correspondence between complex numbers and the Cartesian Plane $\mathbb{R}^2$; I use this correspondence when I write $(x,y) = x + iy$. This plane is called the complex plane. The x-axis is called the real-axis, the y-axis is called the imaginary-axis.



Every complex number $z = x + iy$ has a complex-conjugate $z^* = x - iy$. In the complex plane the mapping $z \to z^*$ is a reflection across the x-axis.



Recall that any polynomial with real-coefficients can be completely factored over the complex numbers. For example, we usually say that $x^2 + 1$ is an irreducible quadratic. This is true with respect to real numbers, however if we use complex numbers to assist with the factorization then we can factor $x^2 + 1 = (x - i)(x + i)$. Generally, a quadratic polynomial $ax^2 + bx + c$ with $b^2 - 4ac < 0$ is called ***irreducible*** because we cannot factor it over the real numbers. Notice that the quadratic formula still makes sense in this case it just gives complex solutions. We can pull an $i = \sqrt{-1}$ out of the square root; $\sqrt{b^2 - 4ac} = \sqrt{-1(4ac - b^2)} = i\sqrt{4ac - b^2}$ where the quantity $\sqrt{4ac - b^2} \in \mathbb{R}$ since $4ac - b^2 > 0$. If $ax^2 + bx^2 + c = 0$ then it can be shown,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-b \pm i\sqrt{4ac - b^2}}{2a} = \alpha \pm i\beta$$

Where I have defined $Re(x) = \alpha = -\frac{b}{2a}$ and $Im(x) = \beta = \frac{\sqrt{4ac - b^2}}{2a}$. The quadratic polynomial factors as follows in this case:

$$x^2 + bx + c = a\big(x - (\alpha + i\beta)\big)\big(x - (\alpha + i\beta)\big)$$

The roots $\alpha + i\beta$ and $\alpha - i\beta$ form a ***conjugate pair***. Any polynomial with real coefficients can be completely factored with the help of complex numbers. When an irreducible quadratic appears in the factorization it gives rise to a pair of linear factors whose roots form a conjugate pair. There is much more to say about complex numbers but this little subsection will more than suffice for the purposes of this course.

9.3.2: What is the complex exponential function?

We define $exp : \mathbb{C} \to \mathbb{C}$ by the following formula:

$$\boxed{exp(z) = exp(Re(z) + iIm(z)) \equiv e^{Re(z)}\big(\cos(Im(z)) + i\sin(Im(z))\big)}$$

We can show that this definition yields the following desirable properties:

$$\begin{aligned}
&1.)\ exp(z+w) = exp(z)exp(w)\\
&2.)\ e^{Re(z)} = Re(exp(z))\\
&3.)\ exp(iIm(z)) = \cos(Im(z)) + i\sin(Im(z))\\
&4.)\ exp(0) = 1\\
&5.)\ exp(-z) = \frac{1}{exp(z)}
\end{aligned}$$

Here $e^{Re(z)}$ denotes the plain-old real exponential function we discussed at length in previous chapters. Essentially, the second condition says that the complex exponential function must reproduce the real exponential function when the input is a complex number with zero imaginary part. Condition 3.) is called Euler's Identity.

Let me show you the proof of 1.). Suppose that $z = x + iy$ and $w = a + ib$ where $x, y, a, b \in \mathbb{R}$. Observe:

$$\begin{aligned}
exp(z+w) &= exp(x + iy + a + ib)\\
&= exp\big(x + a + i(y+b)\big)\\
&= e^{x+a}\big(\cos(y+b) + i\sin(y+b)\big), \quad \text{used the definition here}\\
&= e^x e^a\Bigg(\cos(y)\cos(b) - \sin(y)\sin(b) + i\big(\sin(y)\cos(b) + \cos(y)\sin(b)\big)\Bigg)
\end{aligned}$$

In the last step we used the adding angles formulas for sine and cosine. These can be derived geometrically. They follow from the law of cosines. On the other hand observe:

$$\begin{aligned}
exp(z)exp(w) &= exp(x+iy)exp(a+ib)\\
&= e^x\big(\cos(y) + i\sin(y)\big)e^a\big(\cos(b) + i\sin(b)\big)\\
&= e^x e^a\Bigg(\cos(y)\cos(b) - \sin(y)\sin(b) + i\big(\sin(y)\cos(b) + \cos(y)\sin(b)\big)\Bigg)
\end{aligned}$$

Comparing the equations above we verify that $exp(z+w) = exp(z)exp(w)$. I will use the notation $exp(z) = e^z$ from this point onward.

9.3.3: Deconstructing sine and cosine

We can calculate that $e^{\pm i\theta} = \cos(\pm\theta) + i\sin(\pm\theta) = \cos(\theta) \pm i\sin(\theta)$ because cosine is even and sine is odd. If we add and subtract $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ and $e^{-i\theta} = \cos(\theta) - i\sin(\theta)$ then we obtain formulas for sine and cosine in terms of the imaginary exponentials $e^{i\theta}, e^{-i\theta}$ as follows:

$$\boxed{\cos(\theta) = \frac{1}{2}\big(e^{i\theta} + e^{-i\theta}\big) \qquad \sin(\theta) = \frac{1}{2i}\big(e^{i\theta} - e^{-i\theta}\big)}$$

Now we can derive pretty much any trigonometric identity you run across.

Example 9.3.3.1: Notice this gives us a way to calculate $\int \cos^2(x)\ dx$

$$\begin{aligned}
\cos^2(x) &= \frac{1}{2}\left(e^{ix} + e^{-ix}\right)\frac{1}{2}\left(e^{ix} + e^{-ix}\right) \\
&= \frac{1}{4}\left((e^{ix})^2 + 2e^{ix}e^{-ix} + (e^{-ix})^2\right) \\
&= \frac{1}{4}\left(e^{2ix} + 2 + e^{-2ix}\right) \\
&= \frac{1}{2}\left[1 + \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right)\right] \\
&= \frac{1}{2}\left(1 + \cos(2x)\right).
\end{aligned}$$

Example 9.3.3.2: Notice this gives us a way to calculate $\int \sin^2(x)\ dx$

$$\begin{aligned}
\sin^2(x) &= \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)\frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \\
&= \frac{1}{4i^2}\left((e^{ix})^2 - 2e^{ix}e^{-ix} + (e^{-ix})^2\right) \\
&= \frac{1}{-4}\left(e^{2ix} - 2 + e^{-2ix}\right) \\
&= \frac{1}{2}\left[1 - \frac{1}{2}\left(e^{2ix} + e^{-2ix}\right)\right] \\
&= \frac{1}{2}\left(1 - \cos(2x)\right).
\end{aligned}$$

Notice that $\sin^2(x) + \cos^2(x) = \frac{1}{2}(1 - \cos(2x)) + \frac{1}{2}(1 + \cos(2x)) = 1$. Another way to look at this is that if you have either of these identities then you could use the pythagorean identity to obtain the other one:

$$\sin^2(x) = 1 - \cos^2(x) = 1 - \frac{1}{2}(1 + \cos(2x)) = \frac{1}{2}(1 - \cos(2x)).$$

Let's see another application of the imaginary exponentials.

Example 9.3.3.3: The identity below makes the integration of $\cos(x)\sin(4x)$ easy.

$$
\begin{aligned}
\cos(x)\sin(4x) &= \frac{1}{2}\left(e^{ix}+e^{-ix}\right)\frac{1}{2i}\left(e^{4ix}-e^{-4ix}\right)\\
&= \frac{1}{4i}\left((e^{ix}e^{4ix}-e^{ix}e^{-4ix}+e^{-ix}e^{4ix}-e^{-ix}e^{-4ix}\right)\\
&= \frac{1}{4i}\left(e^{5ix}-e^{-5ix}+e^{ix}-e^{-ix}\right)\\
&= \frac{1}{2}\frac{1}{2i}\left(e^{5ix}-e^{-5ix}\right)+\frac{1}{2}\frac{1}{2i}\left(e^{ix}-e^{-ix}\right)\\
&= \frac{1}{2}\sin(5x)+\frac{1}{2}\sin(x).
\end{aligned}
$$

Example 9.3.3.4: The identity below makes the integration of $\cos(ax)\cos(bx)$ easy.

$$
\begin{aligned}
\cos(ax)\cos(bx) &= \frac{1}{2}\left(e^{aix}+e^{-aix}\right)\frac{1}{2i}\left(e^{bix}-e^{-bix}\right)\\
&= \frac{1}{4}\left((e^{aix}e^{bix}+e^{aix}e^{-bix}+e^{-aix}e^{bix}+e^{-aix}e^{-bix}\right)\\
&= \frac{1}{4}\left(e^{(a+b)ix}+e^{-(a+b)ix}+e^{(a-b)ix}+e^{-(a-b)ix}\right)\\
&= \frac{1}{2}\cos\big((a+b)x\big)+\frac{1}{2}\cos\big((a-b)x\big).
\end{aligned}
$$

I hope you can see the idea here. Perhaps you do not yet appreciate why we need to find trigonometric identities for integration, but I'll fix that in a quiz sometime soon... insert maniacal laughter here…

9.3.4: Identities that we can derive (or memorize)

The identities that follow are either geometrically motivated or can be derived via the methods advertised at length in the last section.

$$
\begin{aligned}
&1.)\ \cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)\\
&2.)\ \sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)\\
&3.)\ \sin(x+\frac{\pi}{2}) = \cos(x)\\
&4.)\ \cos(x-\frac{\pi}{2}) = \sin(x)\\
&5.)\ \cos^2(x) = \frac{1}{2}\Big(1+\cos(2x)\Big)\\
&6.)\ \sin^2(x) = \frac{1}{2}\Big(1-\cos(2x)\Big)\\
&7.)\ \sin(2x) = 2\sin(x)\cos(x)\\
&8.)\ \cos(2x) = \cos^2(x) - \sin^2(x)\\
&9.)\ \cos(A)\cos(B) = \frac{1}{2}\cos(A-B) + \frac{1}{2}\cos(A+B)\\
&10.)\ \sin(A)\sin(B) = \frac{1}{2}\cos(A-B) - \frac{1}{2}\cos(A+B)\\
&11.)\ \sin(A)\cos(B) = \frac{1}{2}\sin(A-B) + \frac{1}{2}\sin(A+B)
\end{aligned}
$$

I do expect you can derive or recall all of the identities above. As we will see in the next subsection these identities will help us integrate many otherwise intractable integrals.

9.3.5: Integrating powers and products of sine or cosine

Let's begin with the easy cases:

Example 9.3.5.1: (all of these use $u = \cos(x)$)

$$\begin{aligned}
\int \sin^3(x)\,dx &= \int \sin^2(x)\sin(x)\,dx \\
&= \int (1-\cos^2(x))\sin(x)\,dx \\
&= \int (u^2-1)du \\
&= \boxed{\frac{1}{3}\cos^3(x) - \cos(x) + c}
\end{aligned}$$

$$\begin{aligned}
\int \sin^5(x)\,dx &= \int \sin^4(x)\sin(x)\,dx \\
&= \int (1-\cos^2(x))^2\sin(x)\,dx \\
&= -\int (1-2u^2+u^4)du \\
&= \boxed{-\cos(x) + \frac{2}{3}\cos^3(x) - \frac{1}{5}\cos^5(x) + c}
\end{aligned}$$

$$\begin{aligned}
\int \sin^7(x)\,dx &= \int \sin^6(x)\sin(x)\,dx \\
&= \int (1-\cos^2(x))^3\sin(x)\,dx \\
&= -\int (1-3u^2+3u^4-u^6)du \\
&= \boxed{-\cos(x) + \frac{3}{4}\cos^3(x) - \frac{3}{5}\cos^5(x) + \frac{1}{7}\cos^7(x) + c}
\end{aligned}$$

It should be obvious how to calculate $\int \sin^9(x)\,dx$ in view of the calculations above. Moreover, I hope you can see how to calculate $\int \cos^3(x)\,dx,\ \int \cos^5(x)\,dx$ and so forth. For odd powers of cosine a substitution of $u = \sin(x)$ will prove useful.

Example 9.3.5.2: I'll use $\cos^2(x) = \frac{1}{2}\big(1+\cos(2x)\big)$ in what follows:

$$\begin{aligned}\int \cos^2(x)\,dx &= \int \frac{1}{2}\big(1+\cos(2x)\big)\,dx \\ &= \frac{1}{2}\int \big(1+\cos(2x)\,\big)dx \\ &= \boxed{\frac{x}{2}+\frac{1}{4}\sin(2x)+c}\end{aligned}$$

The integral of $\cos^4(x)$ follows a similar pattern,

$$\begin{aligned}\int \cos^4(x)\,dx &= \int \frac{1}{2}\big(1+\cos(2x)\big)\frac{1}{2}\big(1+\cos(2x)\big)\,dx \\ &= \frac{1}{4}\int \big(1+2\cos(2x)+\cos^2(2x)\,\big)dx \\ &= \frac{1}{4}\int \left[1+2\cos(2x)+\frac{1}{2}\big(1+\cos(4x)\big)\right]dx \\ &= \int \left[\frac{3}{8}+\frac{1}{2}\cos(2x)+\frac{1}{4}\cos(4x)\right]dx \\ &= \boxed{\frac{3x}{8}+\frac{1}{4}\sin(2x)+\frac{1}{16}\sin(4x)+c}\end{aligned}$$

If you understand this example then $\int \cos^6(x)\,dx$ shouldn't be much more trouble. Integrals of $\sin^2(x), \sin^4(x), \sin^6(x)$ follow from very similar calculations.

9.3.6: Powers of tangent and secant

Essentially the calculations in this subsection follow from the identity $1+\tan^2(x) = \sec^2(x)$ as well as the derivatives $\frac{d(\sec(x))}{dx} = \sec(x)\tan(x)$ and $\frac{d(\tan(x))}{dx} = \sec^2(x)$ which suggest substitutions of $u = \sec(x)$ or $u = \tan(x)$ when the opportunity presents itself.

Example 9.3.6.1:

$$\int \tan^2(x)\,dx = \int \big(\,\sec^2(x)-1\,\big)dx = \boxed{\tan(x)-x+c}$$

Example 9.3.6.2:

$$\begin{aligned}
\int \tan^4(x)\,dx &= \int \big(\sec^2(x) - 1\big)^2 dx \\
&= \int \big(\sec^4(x) - 2\sec^2(x) + 1\big) dx \\
&= \int \big((1+\tan^2(x))\sec^2(x) - 2\sec^2(x) + 1\big) dx \\
&= x - 2\tan(x) + \int (1+u^2)du, \quad \text{where } u = \tan(x) \text{ so } du = \sec^2(x)dx \\
&= \boxed{x - \tan(x) + \frac{1}{3}\tan^3(x) + c}
\end{aligned}$$

We stumbled across the integral of $\sec^4(x)$ in the calculation above. You may recall we learned the integral of $\sec(x)$ in Ex. 9.2.4. The integrals of odd powers of secant are sometimes quite challenging. For example, $\sec^3(x)$ is not easy with the techniques we currently have discussed. On the other hand it should be clear enough that even powers and even products of tangent or secant can be tackled easily enough by calculations similar to those contained in the preceding example.

9.4. TRIGONOMETRIC SUBSTITUTIONS

In principle these are really nothing more than an implicit u-substitution. If you keep the the identities $\sin^2(\theta) + \cos^2(\theta) = 1$ and $\tan^2(\theta) + 1 = \csc^2(\theta)$ the technique of trig-substitution is very natural. The principle aim of trig-substitution is to remove square roots, however the method is useful for a wider class of examples. Let us proceed by example, we'll summarize the cases after we've played a little.

Examples 9.4.1 and 9.4.2: (can you see the typo in E1? Its missing a 2 somewhere)

$\boxed{E1}$

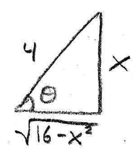
$$\int \sqrt{16 - x^2}\, dx = \int (4\cos\theta)(4\cos\theta\, d\theta) \leftarrow \boxed{\begin{array}{c} x = 4\sin\theta \\ dx = 4\cos\theta\, d\theta \\ \sqrt{16 - x^2} = \sqrt{16(1 - \sin^2\theta)} = 4\cos\theta \end{array}}$$

$$= 16\int \cos^2\theta\, d\theta$$

$$= \frac{16}{2}\int (1 + \cos(2\theta))\, d\theta$$

$$= 8\left(\theta + \tfrac{1}{2}\sin(2\theta)\right) + C$$

$$= \boxed{8\sin^{-1}\left(\frac{x}{4}\right) + 4\sin\left(\sin^{-1}\left(\frac{x}{4}\right)\right) + C}$$



$\boxed{E2}$

$$\int \frac{\sqrt{9 - x^2}}{x^2}\, dx = 9\int \frac{\cos\theta\,\cos\theta\, d\theta}{9\sin^2\theta} \leftarrow \boxed{\begin{array}{l} x = 3\sin\theta \\ dx = 3\cos\theta\, d\theta \\ \sqrt{9 - x^2} = \sqrt{9 - 9\sin^2\theta} = 3\cos\theta \end{array}}$$

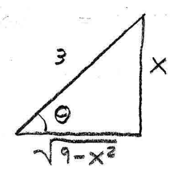
$$= \int \cot^2\theta\, d\theta$$

$$= \int (\csc^2\theta - 1)\, d\theta$$

$$= -\cot\theta - \theta + C$$

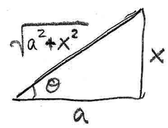
$$= \frac{-3\cos\theta}{3\sin\theta} - \theta + C$$

$$= \boxed{\frac{-\sqrt{9 - x^2}}{x} - \sin^{-1}\left(\frac{x}{3}\right) + C}$$

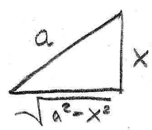


I like to draw the triangle to illustrate the substitution. It also helps reverse the subsitutition. For example, given $x = 3\sin(\theta)$ it might not have been immediately obvious that $\frac{-3\cos(\theta)}{3\sin(\theta)} = -\cot(\theta) = -\frac{\sqrt{9-x^2}}{x^2}$. That fact should be clear from the triangle. In other cases the triangle cannot remove the ugliness, like in E1 where $\theta = \sin^{-1}(\frac{x}{4})$. I know of no particular way to clean that up.

Three Patterns we can reduce via Trigonometric Subsitution:

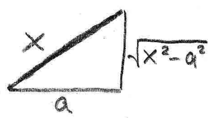


$$x = a\tan\theta$$
$$\sqrt{a^2+x^2} = \sqrt{a^2+a^2\tan^2\theta} = a\sqrt{\sec^2\theta} = a|\sec\theta|$$
$$a^2+x^2 = a^2\sec^2\theta$$



$$x = a\sin\theta$$
$$\sqrt{a^2-x^2} = \sqrt{a^2(1-\sin^2\theta)} = a\sqrt{\cos^2\theta} = a|\cos\theta|$$
$$a^2-x^2 = \cos^2\theta$$

$x = a\cos\theta$ also works.



$$x = a\sec\theta$$
$$\sqrt{x^2-a^2} = \sqrt{a^2(\sec^2\theta-1)} = a\sqrt{\tan^2\theta} = a|\tan\theta|$$
$$x^2-a^2 = a^2\tan^2\theta$$

You can use could also use $x = a\cos(\theta)$ in middle case. In each substiution the square-root is eliminated. We trade an integral with a square root for a new integral of some trigonometric function. We know how to integrate a large variety of trigonometric functions so this is a good bargain for most examples. There are more advanced trigonometric substitutions, but we will focus on just these three basic cases.

Examples 9.4.3

E3 Evaluate $\int \frac{dx}{\sqrt{4+x^2}}$. We'll try the 1st. subst.

$x = 2\tan\theta \implies 4+x^2 = 4+4\tan^2\theta = 4(1+\tan^2\theta) = 4\sec^2\theta$;

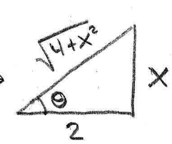
$dx = 2\sec^2\theta\, d\theta$ & $4+x^2 = 4\sec^2\theta$ hence,

$$\int \frac{dx}{\sqrt{4+x^2}} = \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}}$$
$$= \int \sec(\theta)\, d\theta$$
$$= \int \frac{du}{u}$$
$$= \ln|u| + C$$
$$= \ln|\sec\theta + \tan\theta| + C$$
$$= \boxed{\ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C}$$

let $u = \sec\theta + \tan\theta$
$du = (\sec\theta\tan\theta + \sec^2\theta)d\theta$
$du = \sec\theta(\sec\theta + \tan\theta)d\theta$
$\frac{du}{u} = \sec\theta\, d\theta$

Using figure below,

this Δ corresponds to the subst. $x = 2\tan\theta$



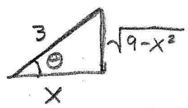
$\sec\theta = \frac{1}{\cos\theta} = \frac{hyp}{adj.} = \frac{\sqrt{4+x^2}}{2}$

$\tan\theta = \frac{opp}{ADJ.} = \frac{x}{2}$

Examples 9.4.4

$\boxed{E4}$ $\displaystyle\int \frac{x^3\,dx}{\sqrt{9-x^2}}$ for $-3 < x < 3$. (109)

Could use
$x = 3\sin\theta$
try it for yourself.



$x = 3\cos\theta$ & $dx = -3\sin\theta\, d\theta$
$9 - x^2 = 9 - 9\cos^2\theta = 9\sin^2\theta$

$$\int \frac{x^3\,dx}{\sqrt{9-x^2}} = \int \frac{(3\cos\theta)^3(-3\sin\theta\, d\theta)}{\sqrt{9\sin^2\theta}}$$

since $\dfrac{\sin\theta}{|\sin\theta|} = 1$ for $0 < \theta < 90°$

$$= -\int 27\cos^3\theta\, d\theta$$

$$= -27\int (1-\sin^2\theta)\cos\theta\, d\theta$$

$$= 27\int (u^2 - 1)\, du$$

$u = \sin\theta$
$du = \cos\theta\, d\theta$

$$= 27\left[\frac{u^3}{3} - u\right] + C$$

$$= 9\sin^3\theta - 27\sin\theta + C$$

note $\sin\theta = \dfrac{\sqrt{9-x^2}}{3}$

$$= 9\left(\frac{\sqrt{9-x^2}}{3}\right)^3 - 27\left(\frac{\sqrt{9-x^2}}{3}\right) + C$$

$$= \boxed{-9\sqrt{9-x^2} + \frac{1}{3}(9-x^2)^{3/2} + C}$$

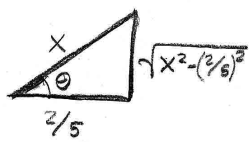
Subtle Remark: I used $0 < \theta < 90° \Rightarrow |\sin\theta| = \sin\theta > 0$ implicitly when I simplified $\sqrt{\sin^2\theta} = \sin\theta$. In principal you might find $\sqrt{\sin^2\theta} = -\sin\theta$, but that is not the case here, because $\sin\theta > 0$ and squareroots are by convention positive. Notice that $0 < \theta < 90°$ follows from $-3 < x = 3\sin\theta < 3 \Rightarrow -1 < \sin\theta < 1$ $\Rightarrow 0 < \theta < 90°$. From the beginning I knew that $|x| < 3$ because otherwise the integrand is not real-valued.

Examples 9.4.5

$\boxed{E5}$ $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}}$ for $x > 2/5$

look's like $\sqrt{x^2-a^2}$ but not quite yet, do algebra on it.

$$\sqrt{25x^2-4} = \sqrt{25(x^2-4/25)}$$
$$= 5\sqrt{x^2-(2/5)^2} \quad \text{suggests we use } x = \tfrac{2}{5}\sec\theta$$



$$x = \tfrac{2}{5}\sec(\theta)$$
$$dx = \tfrac{2}{5}\sec(\theta)\tan\theta\, d\theta =$$
$$x^2-(2/5)^2 = (2/5)^2[\sec^2\theta - 1] = \left(\tfrac{2}{5}\tan\theta\right)^2$$

$$\int \frac{dx}{\sqrt{25x^2-4}} = \frac{1}{5}\int \frac{dx}{\sqrt{x^2-2/5}}$$
$$= \frac{1}{5}\int \frac{\frac{2}{5}\sec(\theta)\tan(\theta)\,d\theta}{\sqrt{\left(\frac{2}{5}\tan\theta\right)^2}}$$
$$= \frac{1}{5}\int \sec\theta\, d\theta$$
$$= \frac{1}{5}\ln|\sec\theta + \tan\theta| + C \quad \text{using } \boxed{E3}$$
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{x^2-(2/5)^2}}{2/5}\right| + C$$
$$= \boxed{\frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C}$$

I should remind you that last semester when I calculated the area of the circle I had to use a trigonometric substitution. You might go back and look at Example 7.3.6 of page 166 in my notes. Perhaps the integration I did there will seem easy now.

9.5. PRODUCT RULE FOR INTEGRALS

Students often seem to invent new, but wrong, methods of integration on tests. For example, a very popular mistake goes as follows: $\int fg\,dx = \int f\,dx \int g\,dx$ **[incorrect !!!]** This is almost never true. However, there is a way to deal with products in integrals. This method is known as *Integration By Parts (IBP):* what follows is the method and its proof,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \Rightarrow u\frac{dv}{dx} = \frac{d}{dx}(uv) - v\frac{du}{dx}$$

Now integrate,

$$\int u\frac{dV}{dx}\,dx = \int \frac{d}{dx}(uv)\,dx - \int V\frac{du}{dx}\,dx$$

$$\therefore \boxed{\int U\,dV = uv - \int v\,du}$$

In practice, what makes this difficult is seeing how to choose what should be the "dv" and consequently what the "u" ought to be. This often requires some trial and error before you arrive at a profitable path forward.

Example 9.5.1(I draw the box to the side to organize my thoughts)

E1

$$\int xe^x dx = \int U\,dV = xe^x - \int e^x dx = \boxed{xe^x - e^x + C}$$

$u = x$	$dV = e^x dx$
$du = dx$	$V = e^x$

Example 9.5.2

E2

$$\int x\cos(x)\,dx = x\cos(x) - \int \sin(x)\,dx = \boxed{x\cos(x) + \cos(x) + C}$$

$u = x$	$dV = \cos(x)dx$
$du = dx$	$V = \sin(x)$

Examples E1 and E2 show that we can remove x by a proper application of IBP.

Example 9.5.3

E3

$$\int \ln(x)\,dx = x\ln(x) - \int x\frac{dx}{x} = x\ln(x) - x + C = \boxed{x(\ln(x) - 1) + C}$$

$u = \ln(x)$	$dV = dx$
$du = \frac{dx}{x}$	$V = x$

Example 9.5.4

$\boxed{E4}$ $$\int \tan^{-1}(x)\,dx = x\tan^{-1}(x) - \int \frac{x}{1+x^2}\,dx$$ $\leftarrow$ $u = \tan^{-1}(x)$, $dv = dx$; $du = \frac{1}{1+x^2}dx$, $v = x$

$$= x\tan^{-1}(x) - \frac{1}{2}\int \frac{dw}{w}$$ $\leftarrow$ $w = 1+x^2$; $dw = 2x\,dx$

$$= x\tan^{-1}(x) - \frac{1}{2}\ln|w| + c$$

$$= \boxed{x\tan^{-1}(x) - \frac{1}{2}\ln(1+x^2) + C}$$

notice $1+x^2 > 0$ so we can drop the absolute value bars.

By now you may be wondering, is there any good way to anticipate what to choose for u in setting up IBP? The answer is, **L.I.A.T.E.** this works pretty good for most examples.

Logarithmic function
Inverse function
Algebraic function
Trigonometric function
Exponential function

• listed in order of choosing u.

For many examples (if it appears to be IBP) we can choose u by simply going down this list and making u whatever we see first. Lets see how this worked

$\boxed{E1}$ $\int xe^x dx$ has x and e^x which are algebraic and exponential functions respectively.

L
I $\Rightarrow$ no log or inverse
A $\leftarrow$ 1^{st} our example has one of $\Rightarrow$ $\underline{u = x}$
T
E

$\boxed{E3}$ $\int \ln(x)\,dx$ has a logarithm $\Rightarrow$ L
I
A
T
E
$\Rightarrow$ chose $u = \ln(x)$

$\boxed{E4}$ $\int \tan^{-1}(x)dx$ has inverse function (& no log. fnct.) $\Rightarrow$ L
I
A
T
E
$\Rightarrow$ $u = \tan^{-1}(x)$.

A better answer, is to practice and then practice some more. Just as with u-substitution integration technique requires both skill of calculation and creative insight.

Example 9.5.5

E5

$$\int x^3 e^x\,dx = x^3e^x - 3\int x^2e^x\,dx \quad \leftarrow \begin{array}{|c|c|} u = x^3 & dV = e^x dx \\ \hline du = 3x^2dx & V = e^x \end{array}$$

$$= x^3e^x - 3\left[x^2e^x - 2\int xe^x dx\right] \quad \leftarrow \begin{array}{|c|c|} u = x^2 & dV = e^x dx \\ \hline du = 2xdx & V = e^x \end{array}$$

$$= x^3e^x - 3\left[x^2e^x - 2\left(xe^x - \int e^x dx\right)\right] \quad \leftarrow \begin{array}{|c|c|} u = x & dV = e^x dx \\ \hline du = dx & V = e^x \end{array}$$

$$= x^3e^x - 3\left[x^2e^x - 2xe^x + 2e^x\right] + C$$

$$= x^3e^x - 3x^2e^x + 6xe^x - 6e^x + C$$

$$= \boxed{e^x\left(x^3 - 3x^2 + 6x - 6\right) + C}$$

I hope you can see how to do $\int x^4 e^x\,dx$ and so forth, it's just a matter of patience and persistence. People tend to find the next example a little unsettling upon first exposure.

Example 9.5.6

E6

$$\int e^x\cos(x)\,dx = e^x\cos(x) - \int \sin(x)e^x dx \quad \leftarrow \begin{array}{|c|c|} u = e^x & dV = \cos(x)\,dx \\ \hline du = e^x dx & V = \sin(x) \end{array}$$

$$= e^x\cos(x) - \left[\sin(x)e^x + \int e^x\cos(x)dx\right] \quad \leftarrow \begin{array}{|c|c|} u = e^x & dV = \sin(x) \\ \hline du = e^x & V = -\cos(x) \end{array}$$

$$= e^x(\cos(x) - \sin(x)) - \int e^x\cos(x)\,dx$$

: we've come full-circle back to where we began.

$$\Rightarrow 2\int e^x\cos(x)dx = e^x(\cos(x) - \sin(x))$$

$$\Rightarrow \boxed{\int e^x\cos(x)\,dx = \tfrac{1}{2}e^x(\cos(x) - \sin(x)) + C}$$

Remark: There is another way to calculate this if we allow ourselves the luxury of some complex variable calculus. Technically speaking, I probably should lay a little more ground work before showing you the calculation below, but you'll forgive me I think,

$$\begin{aligned}
\int e^x \cos(x)dx &= \int e^x \frac{1}{2}\left(e^{ix} + e^{-ix}\right)dx \\
&= \frac{1}{2}\int \left(e^{(1+i)x} + e^{(1-i)x} \right)dx \\
&= \frac{1}{2}\left(\frac{1}{1+i}e^{(1+i)x} + \frac{1}{1-i}e^{(1-i)x} \right) + c \\
&= \frac{1}{2}\left(\frac{1-i}{1-i^2}e^{(1+i)x} + \frac{1+i}{1-i^2}e^{(1-i)x} \right) + c \\
&= \frac{1}{2}\left(\frac{1}{2}e^{(1+i)x} + \frac{1}{2}e^{(1-i)x} \right) + \frac{1}{2}\left(\frac{-i}{2}e^{(1+i)x} + \frac{i}{2}e^{(1-i)x} \right) + c \\
&= \frac{1}{2}\left(\frac{1}{2}\left(e^{ix} + e^{-ix}\right) + \frac{1}{2i}\left(e^{ix} - e^{-ix}\right) \right)e^x + c \\
&= \boxed{\frac{1}{2}(\cos(x) + \sin(x))e^x + c.}
\end{aligned}$$

I made some major assumptions in the calculation above, I assumed that the integral of the complex exponential works according to the same calculus pattern as the real exponential. This is in fact true, and I will justify this in part in a later section. In your complex variables course you will learn the complete background to justify such integrations. In fact, the integration I just completed is not that exotic, no deep theorem of complex variables is required because the integrand is *analytic everywhere, it has no poles in the complex plane.*

Example 9.5.7

E7

$$\int \sin^{-1}(x)\,dx = \sin^{-1}(x) - \int \frac{x}{\sqrt{1-x^2}}dx \qquad \begin{array}{c|c} u = \sin^{-1}(x) & dv = dx \\ \hline du = \frac{dx}{\sqrt{1-x^2}} & v = x \end{array}$$

$$= \sin^{-1}(x) + \frac{1}{2}\int \frac{du}{\sqrt{u}} \qquad \begin{array}{l} u = 1-x^2 \\ du = -2x\,dx \;\therefore\; \frac{du}{2} = -x\,dx \end{array}$$

$$= \sin^{-1}(x) + \sqrt{u} + C$$

$$= \boxed{\sin^{-1}(x) + \sqrt{1-x^2} + C}$$

Example 9.5.8

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E8

$$I = \int \cos(\ln(x))\,dx = x\cos(\ln(x)) + \int \sin(\ln(x))\,dx \qquad \begin{array}{c|c} u = \cos(\ln(x)) & dV = dx \\ \hline du = \frac{-\sin(\ln(x))}{x}dx & V = x \end{array}$$

$$= x\cos(\ln(x)) + \left[x\sin(\ln(x)) - \int \cos(\ln(x))dx\right] \qquad \begin{array}{c|c} u = \sin(\ln(x)) & dV = dx \\ \hline du = \frac{\cos(\ln(x))}{x}dx & V = x \end{array}$$

$$= x\left(\cos(\ln(x)) + \sin(\ln(x))\right) - I$$

$$\therefore \boxed{I = \frac{1}{2}x\left(\cos(\ln(x)) + \sin(\ln(x))\right) + C}$$

Example 9.5.9

Some people prefer these arguments over those which are based on trigonometric identities. Personally, I prefer the arguments we made in 9.3.5 since they are a bit more direct. Also, I see a need for students to learn more complex variables early on because: 1.) it's not really that hard, 2.) they're really neat, 3.) there is a silly but undeniable bias against complex numbers simply because of their label "complex" , this bias must be confronted since it may cause the student to ignore or dismiss the best solution to many problems,

E9

$$\int \cos^n(x)\,dx = \int \cos^{n-1}(x)\cos(x)\,dx$$

$u = \cos^{n-1}(x)$
$du = (n-1)\cos^{n-2}(x)(-\sin(x))$
$dv = \cos(x)dx$
$v = \sin(x)$

$$= \cos^{n-1}(x)\sin(x) - \int \sin(x)(n-1)\cos^{n-2}(x)(-\sin(x))$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int \sin^2(x)\cos^{n-2}(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\int (1-\cos^2(x))\cos^{n-2}(x)\,dx$$

$$= \cos^{n-1}(x)\sin(x) + (n-1)\left(\int \cos^{n-2}(x)dx + \int \cos^n(x)\,dx\right)$$

$$\therefore \boxed{\int \cos^n(x)\,dx = \frac{1}{n}\cos^{n-1}(x)\sin(x) + \frac{n-1}{n}\int \cos^{n-2}(x)\,dx}$$

Solving for the integral algebraically

Let $n=2$ then we find:

$$\int \cos^2(x)dx = \frac{1}{2}\cos(x)\sin(x) + \frac{1}{2}\int dx = \boxed{\frac{1}{2}\left(\cos(x)\sin(x) + x\right) + C}$$

Let $n=3$ then:

$$\int \cos^3(x)\,dx = \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\int \cos(x)\,dx$$

$$= \frac{1}{3}\cos^2(x)\sin(x) + \frac{2}{3}\sin(x) + C$$

$$= \boxed{\frac{1}{3}\sin(x)\left(\cos^2(x) + 1\right) + C}$$

this is problem 34 a & b pg. 398

<u>Remark</u>: this is the other popular method to compute integrals of powers of cosine, there is also a similar "recurrence" relation for $\sin\theta$.

(Actually this is problem 44a-b on page 494 of the version of Stewart we are using at the moment)

9.6. INTEGRATING RATIONAL FUNCTIONS

We can integrate any polynomial, it's easy, just use linearity and the power rule. What about rational functions? Is there some method to integrate an arbitrary rational function? It is just the quotient of two polynomials, how bad can it be? Pretty bad actually, pretty bad. However, the difficulty is not insurmountable.

The key is realizing we can undo the algebraic maneuver of making a common denominator. If I have a product XY in the denominator then it can be split into a sum of a term with denominator X and another term with denominator Y. I call this idea ***reverse common "denominatoring"***, but ok that's not really word.

Example 9.6.0

$\boxed{E0}$ How to integrate $\frac{x+5}{x^2+5x+6}$? Upto now none of the previous techniques seem to help with this one. So notice that we can break up this fraction into two fractions, following from the factorization of $x^2+5x+6 = (x+2)(x+3)$,

$$\frac{x+5}{(x+2)(x+3)} = \frac{A}{x+3} + \frac{B}{x+2}$$

this guess is verified by algebra below

We can figure out the now unknown A & B as follows, first multiply by denominator to obtain,

$$(x+2)(x+3)\frac{x+5}{(x+2)(x+3)} = (x+2)(x+3)\left[\frac{A}{x+3} + \frac{B}{x+2}\right]$$

Which gives the (simple to solve) eq$^{\underline{n}}$,

$$x+5 = (x+2)A + (x+3)B$$

So plug in the roots $x=-2$ then $x=-3$ to get

$$-2+5 = 3 = \cancel{(-2+2)}^{0}A + (-2+3)B = B \quad \therefore \boxed{B=3}$$
$$-3+5 = 2 = (-3+2)A + \cancel{(-3+3)}^{0}B = -A \quad \therefore \boxed{A=-2}$$

Thus we have by the algebra above,

$$\frac{x+5}{(x+2)(x+3)} = \frac{-2}{x+3} + \frac{3}{x+2}$$

Now we can integrate the RHS of the above, remember how?

$$\int \frac{x+5}{(x+2)(x+3)}\,dx = -2\int \frac{1}{x+3}\,dx + 3\int \frac{1}{x+2}\,dx$$
$$= -2\int \frac{du}{u} + 3\int \frac{dw}{w} \qquad \begin{array}{l} u = x+3 \\ w = x+2 \end{array}$$
$$= \boxed{-2\ln|x+3| + 3\ln|x+2| + C}$$

Integrals that motivate the algebra for partial fractions

You will learn over the course of many examples that to do partial fractions means we rewrite a given rational function as a sum of other rational functions. It is not terribly surprising this alone is possible, what is perhaps surprising is that there are certain types of rational functions which integrate nicely. I call these special rational functions the "**basic rational functions"**. A basic rational function is one which cannot be further reduced into a sum of other basic rational functions, loosely speaking.

We can integrate the basic rational functions: (you'll prove these in one of you homeworks, it's a combination of u-substitution and trig-subst.)

Known Integrals of Basic Rational Functions:

1.) Let $a_n, a_{n-1}, \ldots a_1, a_0 \in \mathbb{R}$, then $P(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0$ is a polynomial. It can be shown

$$\boxed{\int P(x)\,dx = \frac{a_n}{n+1}x^{n+1} + \cdots + \frac{a_2}{3}x^3 + \frac{a_1}{2}x^2 + a_0 x + C}$$

2.) Let $a \in \mathbb{R}$ then $R(x) = \frac{1}{x+a}$ is a basic rational function. It is the reciprocal of a linear factor. It can be shown

$$\boxed{\int \frac{1}{x+a}\,dx = \ln|x+a| + C}$$

3.) Let $a \in \mathbb{R}$ and $n \in \mathbb{N}$ such that $n \geq 2$ then $R(x) = \frac{1}{(x+a)^n}$ is a basic rational function. It is the reciprocal of a repeated linear factor.

$$\boxed{\int \frac{1}{(x+a)^n}\,dx = \frac{-1}{n(x+a)^n} + C}$$

4.) Let $b, c \in \mathbb{R}$ such that $b^2 - 4c < 0$ then $R(x) = \frac{1}{x^2+bx+c}$ is a basic rational function. It is the reciprocal of an irreducible quadratic. Let $\alpha = -b/2$ and $\beta = \sqrt{c - b^2/4}$ then the integral is given as follows:

$$\boxed{\int \frac{1}{x^2+bx+c}\,dx = \frac{1}{\beta}\tan^{-1}\left[\frac{1}{\beta}(x-\alpha)\right] + C}$$

5.) Let $b, c \in \mathbb{R}$ such that $b^2 - 4c < 0$ then $R(x) = \frac{x}{x^2+bx+c}$ is a basic rational function. It is x over an irreducible quadratic. Let $\alpha = -b/2$ and $\beta = \sqrt{c - b^2/4}$ then the integral is given as follows:

$$\boxed{\int \frac{x}{x^2+bx+c}\,dx = \frac{1}{2}\ln(x^2+bx+c) - \frac{b}{2\beta}\tan^{-1}\left[\frac{1}{\beta}(x-\alpha)\right] + C}$$

Cases 4.) and 5.) are considerably simplified in the case $\alpha = 0$. For example,

$$\int \frac{dx}{x^2+4} = \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C \qquad \int \frac{x\,dx}{x^2+4} = \frac{1}{2}\ln\left(x^2+4\right) + C$$

Many examples have this simplified form. It wouldn't be unreasonable to ask you to integrate $\frac{1}{(x^2+4)^2}$, this is a basic rational function. It is not possible to break it down further. The substitution $x = 2\tan(\theta)$ will make that integration work out nicely. Once you understand that example then you (by "you" I actually just mean me in this case) can do the general case for a repeated quadratic in the denominator:

$$\int \frac{1}{(x^2+bx+c)^2}\,dx = \int \frac{1}{[(x+b/2)^2+\alpha^2]^2}\,dx$$

$$= \int \frac{1}{[\alpha^2((\frac{1}{\alpha}(x+b/2))^2+1)]^2}\,dx$$

$$= \int \frac{1}{(\alpha^2)^2\,[(\frac{1}{\alpha}(x+b/2))^2+1]^2}\,dx$$

$$= \int \frac{1}{\alpha^4\,[u^2+1]^2}\,\alpha\,du \qquad : \qquad u = \frac{1}{\alpha}(x+b/2), \quad du = \frac{1}{\alpha}dx \ \therefore dx = \alpha\,du$$

$$= \frac{1}{\alpha^3}\int\frac{du}{(1+u^2)^2}$$

Let's consider this integral seperately for a moment,

$$\int\frac{du}{(1+u^2)^2} = \int\frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta \quad \leftarrow \boxed{\begin{array}{l} u = \tan\theta \\ du = \sec^2\theta\,d\theta \\ 1+u^2 = 1+\tan^2\theta = \sec^2\theta \end{array}}$$

$$= \int\frac{1}{\sec^2\theta}\,d\theta$$

$$= \int \cos^2\theta\,d\theta$$

$$= \int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{1}{2}\int d\theta + \frac{1}{2}\int\cos(v)\,\frac{dv}{2} \qquad \begin{array}{l} v = 2\theta \\ dv = 2\,d\theta \end{array}$$

$$= \frac{1}{2}\theta + \frac{1}{4}\sin(2\theta) + C$$

$$= \frac{1}{2}\tan^{-1}(u) + \frac{1}{4}\sin(2\tan^{-1}(u)) + C$$

Hence returning to our original pursuit,

$$\int\frac{1}{(x^2+bx+c)^2}\,dx = \frac{1}{\alpha^3}\left\{\frac{1}{2}\tan^{-1}(u) + \frac{1}{4}\sin(2\tan^{-1}(u))\right\} + C$$

$$= \boxed{\left(c-\frac{b^2}{4}\right)^{-\frac{3}{2}}\left\{\frac{1}{2}\tan^{-1}\left(\frac{1}{\sqrt{c-b^2/4}}(x+b/2)\right) + \frac{1}{4}\sin\left(2\tan^{-1}\left(\frac{1}{\sqrt{c-b^2/4}}(x+b/2)\right)\right)\right\} + C}$$

(if you take a close look you'll see I'm using "α" in the place β resided previously oops)

It is not too hard to see that irreducible quadratics raised to higher powers will also be manageable by an appropriate trig-subst. There is also an iterative formula known to tackle the integral of $f(x) = (x^2 + bx + c)^{-n}$ for $n \in \mathbb{N}$ such that $n \geq 1$. If you present and derive that iterative formula I will award 4 bonus points if you do a good job. Ok, enough about all that, by in large we will only need to know 1-5.) and even then most often we have $\alpha = 0$ in our examples.

<u>Example 9.6.1(I'd call the given function an improper rational function)</u>

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PARTIAL FRACTIONS:

Goal: given some rational function $f(x) = \frac{P(x)}{Q(x)}$ integrate it.

Outline: Long ÷, Partial Fractal Decompose it, integrate the "Basic" rational functions that result.

$\boxed{E1}$ $f(x) = \dfrac{x^4 + x^3 + 2x^2 + 3x - 2}{x^2 - 3x + 2}$ ← need to do long ÷

$$
\begin{array}{r l}
 & x^2 + 4x + 12 \\
x^2 - 3x + 2 \,\big) & x^4 + x^3 + 2x^2 + 3x - 2 \\
 & -(x^4 - 3x^3 + 2x^2) \\
 & 4x^3 + 0x^2 + 3x - 2 \\
 & -(4x^3 - 12x^2 + 8x) \\
 & 12x^2 - 5 - 2 \\
 & -(12x^2 - 36x + 24) \\
 & 0 \quad \boxed{31x - 26} \leftarrow \text{remainder}
\end{array}
$$

$$\therefore \quad f(x) = \underbrace{x^2 + 4x + 12}_{\text{can integrate this no problem}} + \underbrace{\frac{31x - 26}{x^2 - 3x + 2}}_{\text{we'll use partial fractions to break it up.}}$$

$$\frac{31x - 26}{(x-2)(x-1)} = \frac{A}{x-2} + \frac{B}{x-1}$$ ← Partial Fractal Decomp. for distinct linear factors.

$\therefore \quad 31x - 26 = A(x-1) + B(x-2)$: Multiplying by $(x-2)(x-1)$

$\underline{x=1}$ $5 = -B$

$\underline{x=2}$ $36 = A$

$$\boxed{f(x) = x^2 + 4x + 12 + \frac{36}{x-2} - \frac{5}{x-1}}$$

$$\boxed{\int f(x)\,dx = \frac{x^3}{3} + 2x^2 + 12x + 36\ln|x-2| - 5\ln|x-1| + C}$$

Long division, thought you'd gotten away from it didn't you. There is a small typo in E1 can you find it?

Example 9.6.2

$\boxed{E2}$ $f(x) = \dfrac{x^2+2x+3}{x^3+x} = \dfrac{x^2+2x+3}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

$x^2+2x+3 = A(x^2+1) + (Bx+C)(x)$

$\underline{x=0|}$ $\boxed{3 = A}$

$\underline{x=1|}$ $6 = 2A + B + C \Rightarrow 6 = 6 + B + C \Rightarrow \underline{B = -C}$

$\underline{x=2|}$ $11 = 5A + 4B + 2C \Rightarrow 11 = 15 - 4C + 2C \Rightarrow 2C = 4 \Rightarrow C = 2$

$\therefore A = 3, \; B = -2, \; C = 2$

$$\int f(x)\,dx = \int \frac{3}{x}dx + \int \frac{2}{x^2+1}dx - \int \frac{2x}{x^2+1}dx$$

$$= 3\ln|x| + 2\tan^{-1}(x) - \int \frac{du}{u} \quad ; \leftarrow \boxed{\begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}}$$

$$= \boxed{3\ln|x| + 2\tan^{-1}(x) - \ln|x^2+1| + C}$$

Example 9.6.3

$\boxed{E3}$ $f(x) = \dfrac{2x^2-3}{(x+1)(x^2-1)} = \dfrac{2x^2-3}{(x+1)(x+1)(x-1)} = \dfrac{A}{(x+1)} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x-1}$

$2x^2-3 = A(x+1)(x-1) + B(x-1) + C(x+1)^2$

$\underline{x=-1|}$ $-1 = -2B \;\therefore\; \boxed{B = 1/2}$

$\underline{x=1|}$ $-1 = 4C \;\therefore\; \boxed{C = -1/4}$

$\underline{x=0|}$ $-3 = -A - B + C = -A - 3/4 \Rightarrow -12 = -4A - 3$

$\Rightarrow -9 = -4A$

$\Rightarrow \boxed{A = 9/4}$

$$\int f(x)\,dx = \frac{9}{4}\int \frac{dx}{x+1} - \frac{1}{4}\int \frac{dx}{x-1} + \frac{1}{2}\int \frac{dx}{(x+1)^2}$$

$$= \frac{9}{4}\int \frac{du}{u} - \frac{1}{4}\int \frac{dw}{w} + \frac{1}{2}\int \frac{du}{u^2} \quad \leftarrow \boxed{\begin{array}{ll} u = x+1 & du = dx \\ w = x-1 & dw = dx \end{array}}$$

$$= \frac{9}{4}\ln|u| - \frac{1}{4}\ln|w| - \frac{1}{2}\frac{1}{u} + C$$

$$= \boxed{\frac{9}{4}\ln|x+1| - \frac{1}{4}\ln|x-1| - \frac{1}{2}\frac{1}{(x+1)} + C}$$

Remark: you are probably curious, how do I know how to break it up into pieces? Why is there A,B for one problem then A,B,C for another? What is the algorithm? The examples that follow probably have enough variety for you to see the algorithm.

Example 9.6.4 and 9.6.5

E4) $$f(x) = \frac{1}{(x+1)(x-2)^2(x^2+1)(x^2+2)^2(x^2+3)^3}$$

$$= \frac{A}{(x+1)} + \frac{B}{(x-2)} + \frac{C}{(x-2)^2} + \frac{Dx+E}{x^2+1}$$

$$+ \frac{Fx+G}{x^2+2} + \frac{Hx+I}{(x^2+2)^2} + \frac{Jx+K}{x^2+3} + \frac{Lx+M}{(x^2+3)^2} + \frac{Nx+O}{(x^2+3)^3}$$

I'm not going to find $A, B, \ldots, N, O$ for you but it should be clear how you would go about it.

E5) $$f(x) = \frac{x^2+2x-7}{(x-3)^5}$$

$$= \frac{A}{(x-3)} + \frac{B}{(x-3)^2} + \frac{C}{(x-3)^3} + \frac{D}{(x-4)^4} + \frac{E}{(x-5)^5}$$

Example 9.6.6

E6) $$f(x) = \frac{x^2+3}{(x-2)^2} = \cancel{\frac{A}{(x-2)} + \frac{B}{(x-2)^2}}$$ ← NO must have deg(numerator) < deg(denom.)

First we need to do long %.

$$\begin{array}{r|l} & \quad 1 \\ x^2-4x+4 & \; x^2 + 0\cdot x + 3 \\ & \underline{-(x^2 - 4x + 4)} \\ & \quad 4x - 1 \end{array} \quad \Rightarrow \quad \frac{x^2+3}{(x-2)^2} = 1 + \frac{4x-1}{(x-2)^2}$$

Now we can do partial fractions on $\frac{4x-1}{(x-2)^2} = \frac{A}{x-2} + \frac{B}{(x-2)^2}$

$$4x - 1 = A(x-2) + B$$

$x=0$) $-1 = -2A + B$

$x=2$) $\boxed{7 = B} \Rightarrow 2A = 7 + 1 \therefore \boxed{A = 4}$

$$\int f(x)\,dx = \int \left(1 + \frac{4}{x-2} + \frac{7}{(x-2)^2}\right) dx$$

$$= x + 4\int \frac{1}{u}\,du + 7\int \frac{1}{u^2}\,du \quad \leftarrow \boxed{\begin{array}{l} u = x-2 \\ du = dx \end{array}}$$

$$= x + 4\ln|u| - 7\frac{1}{u} + C$$

$$= \boxed{x + 4\ln|x-2| - \frac{7}{x-2} + C}$$

I hope you see the idea now. There are additional examples in the suggested homework solutions posted on the course website. Let me sketch the general pattern now that we have a good sampling of examples:

The question is how to integrate *any* rational function $f(x) = \frac{P(x)}{Q(x)}$. This reduces by polynomial long division to the problem of integrating

$$f(x) = S(x) + \frac{R(x)}{Q(x)}$$

Where $R(x)$ is the remainder so $\deg(R(x)) < \deg(Q(x))$. So how to integrate $R(x)/Q(x)$ in general? (Note $S(x)$ is easy! it's a polynomial we can integrate those, no problem). Then from algebra we know that any POLYNOMIAL $f(x)$ can be factored to:

$$f(x) = (\text{linear factors})(\text{irred. quad. factors})$$

Hence $R(x)$ & $Q(x)$ can be factored likewise, (irred. means irreducible)

$$\frac{R(x)}{Q(x)} = \frac{(\text{linear factors})(\text{irred. quad. factors})}{(\text{linear factors})(\text{irred. quad factors})}$$

$$= \underbrace{\frac{A}{x-r_1} + \frac{B}{(x-r_1)^2} + \cdots + \frac{Cx+D}{x^2+bx+c} + \frac{Ex+F}{(x^2+bx+c)^2} + \cdots}_{(*)}$$

$(*)$ which and how many of these "Basic Rational" terms depends on the details of $R(x)$ & $Q(x)$.

Personally, I think this belongs at the beginning of the section, but my students tell me examples first then the general story, you can thank them for this section being less than logically ordered. In any event, it should be clear we can integrate any rational function via these methods. In practice it would be wise to use Mathematica or a TI-89 for complicated examples. I often use my TI-89 to check my partial fractions decomposition for silly errors. If you are going to buy a calculator the TI-89 is hard to beat, I probably shouldn't tell you, but it can do most everything we learn this semester. Of course the same is true for Mathematica, but that is just inconvenient enough to keep you doing your homework.

<u>In-class Exercise 9.6.7:</u> integrate the function below.

$f(x) = \frac{1}{x^3+x}$

9.7. IMPROPER INTEGRATION

Don't worry. The Liberty Way will not be violated in this section. The "improper" here refers to one of two possibilities:

1.) The integrand in $\int_a^b f(x)\,dx$ has a vertical asymptote at some $c \in [a,b]$.
2.) The integral is something like $\int_a^\infty f(x)\,dx$.

In both cases the integral is defined in terms of a natural limiting process. In both cases the integrals may converge or diverge depending on the details of the limiting process. We will need to recall the various tricks and common sense observations we made about limits in calculus I. On occasion L'Hopital's Rule may be necessary.

Begin with case 2.).

Def$^{\underline{n}}$/ Assuming the limits below exist,

a) $\displaystyle\int_a^\infty f(x)\,dx \equiv \lim_{t\to\infty}\int_a^t f(x)\,dx$

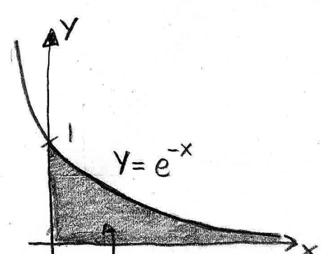
b) $\displaystyle\int_{-\infty}^b f(x)\,dx \equiv \lim_{t\to-\infty}\int_t^b f(x)\,dx$

c.) $\displaystyle\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$ (usually use $a=0$)

These are all convergent integrals, when these limits exist. When the limits d.n.e we say the the integrals are divergent.

Example 9.7.1

E1
$$\int_0^\infty e^{-x}\,dx \equiv \lim_{t\to\infty}\int_0^t e^{-x}\,dx = \lim_{t\to\infty}\left(-e^{-x}\Big|_0^t\right) = \lim_{t\to\infty}\left(-e^{-t}+1\right) = \boxed{1}$$



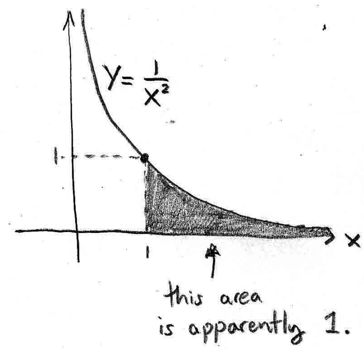
integral calculates the pictured area.

I suppose it may be surprising that we can have a shape with an infinite length on one side yet in total a finite area. The non-intuitive feature that makes this possible is that the height of the object gets very small very quickly so the net area does not blow up. It is not enough that the height goes to zero, we'll see that soon.

Example 9.7.2

$\boxed{E2}$

$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx$$
$$= \lim_{t\to\infty} \left(\frac{-1}{x}\Big|_1^t \right)$$
$$= \lim_{t\to\infty} \left(\frac{-1}{t}\,^{\to 0} + 1 \right)$$
$$= \boxed{1}$$



Example 9.7.3

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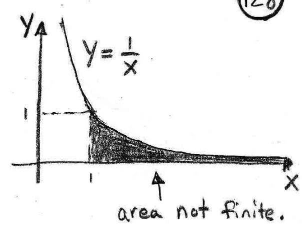
$\boxed{E3}$

$$\int_1^\infty \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx$$
$$= \lim_{t\to\infty} \left(\ln(x)\Big|_1^t \right)$$
$$= \lim_{t\to\infty} \left(\ln(t) \right)$$
$$= \infty \quad (\textit{divergent})$$

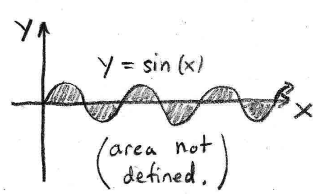


Now here is the example I was referring to after E1. The integrand $\frac{1}{x}$ goes to zero as $x \to \infty$ yet the total area under the curve diverges to infinity. What does that mean? It means that as we take larger and larger upper bounds the area under the curve from one up to those bounds keeps changing, it never settles down to just one value. Instead with each higher bound we'd find the area gets larger and larger without any end in sight. In E1 and E2 we found the limits converged to one, geometrically this means that if we actually calculate those integrals for very large upper bounds we would find the values got very close to one. The next example has a different kind of divergence.

Example 9.7.4

$\boxed{E4}$

$$\int_{-\infty}^0 \sin(x)\,dx = \lim_{t\to-\infty} \int_t^0 \sin(x)\,dx$$
$$= \lim_{t\to-\infty} \left(-\cos(x)\Big|_t^0 \right)$$
$$= \lim_{t\to-\infty} \left(-1 + \cos(t) \right)$$
$$= d.n.e \quad (\textit{divergent} \text{ because } \cos(t) \text{ "oscilates at } \infty\text{"})$$



Example 9.7.5

$\boxed{E5}$ $$\int_0^{\infty} \frac{2x}{1+x^4}\,dx = \lim_{t\to\infty}\left(\int_0^{t} \frac{2x}{1+x^4}\,dx\right)$$

$$= \lim_{t\to\infty}\left(\int_0^{t^2} \frac{du}{1+u^2}\right) \qquad \begin{array}{ll} u = x^2 & u(0) = 0 \\ du = 2x\,dx & u(t) = t^2 \end{array}$$

$$= \lim_{t\to\infty}\left(\tan^{-1}(u)\Big|_0^{t^2}\right)$$

$$= \lim_{t\to\infty}\left(\tan^{-1}(t^2) - \cancel{\tan^{-1}(0)}^{\,0}\right)$$

$$= \boxed{\pi/2}$$

Example 9.7.6

Sometimes we need to use some integration technique before applying the appropriate limiting process. Here I do a u-substitution before applying the definition of improper integration for the given limits.

$\boxed{E6}$

$$\int x^2 e^{-x^3}\,dx = \int \frac{-1}{3} e^{u}\,du \qquad \leftarrow \boxed{\begin{array}{l} u = -x^3 \\ du = -3x^2\,dx \end{array}}$$

$$= \frac{-1}{3} e^{u} + C$$

$$= \frac{-1}{3} e^{-x^3} + C$$

$$\int_{-\infty}^{\infty} x^2 e^{-x^3}\,dx = \lim_{t_1\to-\infty} \int_{t_1}^{0} x^2 e^{-x^3}\,dx + \lim_{t_2\to\infty} \int_0^{t_2} x^2 e^{-x^3}\,dx$$

$$= \lim_{t_1\to-\infty}\left(\frac{-1}{3} e^{-x^3}\Big|_{t_1}^{0}\right) + \lim_{t_2\to\infty}\left(\frac{-1}{3} e^{-x^3}\Big|_0^{t_2}\right)$$

$$= \lim_{t_1\to-\infty}\left(\frac{-1}{3} + \underbrace{\frac{e^{-t_1^3}}{3}}_{\infty}\right) + \lim_{t_2\to\infty}\left(\underbrace{\frac{-1}{3}e^{-t_2^3}}_{0} + \frac{1}{3}\right)$$

$$= \text{divergent}$$

Example 9.7.7

$$\boxed{E7}\quad \int \frac{\ln(x)}{x^3}\,dx = \frac{-\ln(x)}{2x^2} - \int \frac{-1}{2x^2}\frac{dx}{x}$$

$u = \ln(x)$	$dV = dx/x^3$
$du = \frac{dx}{x}$	$V = \frac{-1}{2x^2}$

$$= \frac{-1}{2x^2}\ln(x) + \frac{1}{2}\frac{-1}{2x^2} + C$$

$$= \frac{-1}{2x^2}\left(\ln(x) + \frac{1}{2}\right) + C$$

$$\int_1^\infty \frac{\ln(x)}{x^3}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln(x)\,dx}{x^3}$$

$$= \lim_{t\to\infty}\left(\frac{-1}{2x^2}\left(\ln(x)+\frac{1}{2}\right)\Big|_1^t\right)$$

$$= \lim_{t\to\infty}\left(\frac{\ln(t)+1/2}{-2t^2} + \frac{\ln(1)+1/2}{2}\right)$$

$$\overset{\mathcal{L}}{\underset{(\frac{\infty}{\infty})}{=}} \lim_{t\to\infty}\left(\frac{1/t}{-4t}\right) + \frac{1}{4}$$ ← pulled this out before doing L-Hopital's Rule on the 1st term.

$$= \lim_{t\to\infty}\underbrace{\left(\frac{-1}{4t^2}\right)}_{0} + \frac{1}{4}$$

$$= \boxed{1/4}$$

In E6 and E7 I decided to complete the integration then apply the limiting process to the appropriate integral. There is also notation to do this all at once. Care must be taken to change bounds in the other notation, I avoid the issue by just completing the antiderivative separately in E6 and E7.

Lets continue on to case 2.).

Def^n/ Provide the limits below exist (are real numbers)

a.) $\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx \quad (b \notin dom(f))$

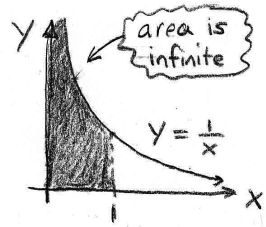
b.) $\int_a^b f(x)\,dx = \lim_{t\to a^+} \int_t^b f(x)\,dx \quad (a \notin dom(f))$

c.) $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (c \notin dom(f))$

If the limit in a, b or c d.n.e we say the integrals diverge.

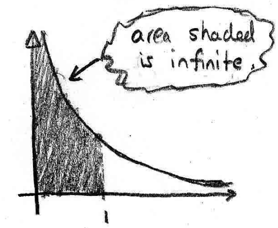
Example 9.7.8

E8
$$\int_0^1 \frac{1}{x}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x}\,dx$$
$$= \lim_{t\to 0^+} \left(\ln(x) \Big|_t^1 \right)$$
$$= \lim_{t\to 0^+} \left(\underbrace{\ln(1)}_{0} - \underbrace{\ln(t)}_{\infty} \right) = \boxed{\infty} \quad \underline{\text{divergent.}}$$



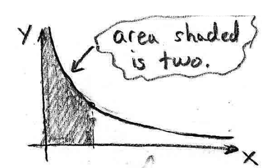
Example 9.7.9

E9
$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$
$$= \lim_{t\to 0^+} \frac{-1}{x}\Big|_t^1$$
$$= \lim_{t\to 0^+} \left(-1 + \underbrace{\frac{1}{t}}_{\infty} \right) = \boxed{\infty} \quad \underline{\text{divergent}}$$



Example 9.7.10

E10
$$\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{\sqrt{x}}\,dx$$
$$= \lim_{t\to 0^+} \left(2\sqrt{x} \Big|_t^1 \right)$$
$$= \lim_{t\to 0^+} \left(2 - 2\sqrt{t} \right) = \boxed{2} \quad \underline{\text{convergent}}$$



Example 9.7.11

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E11
$$\int_0^{\pi/4} \csc^2(x)\,dx = \lim_{t\to 0^+} \int_t^{\pi/4} \csc^2(x)\,dx$$
$$= \lim_{t\to 0^+} \left[-\cot(x) \Big|_t^{\pi/4} \right]$$
$$= \lim_{t\to 0^+} \left[-\underbrace{\cot(\pi/4)}_{1} + \underbrace{\cot(t)}_{\infty} \right] \quad \underline{\text{diverges.}}$$

Example 9.7.12
It is possible for the answer to be negative in examples like this one. The integral calculates the signed-area which can be negative in certain cases.

$\boxed{E12}$

$$\int_1^9 \frac{1}{\sqrt[3]{x-9}}\,dx = \lim_{t\to 9^-} \int_1^t \frac{1}{\sqrt[3]{x-9}}\,dx$$

$$= \lim_{t\to 9^-} \left(\frac{3}{2}(x-9)^{2/3} \Big|_1^t \right)$$

$$= \lim_{t\to 9^-} \left(\frac{3}{2}\underbrace{(t-9)}_{\to 0}{}^{2/3} - \frac{3}{2}(-8)^{2/3} \right)$$

$$= \boxed{-6}$$

Example 9.7.13
Our final example combines cases 1.) and 2.)

$\boxed{E13}$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}}$$

BOTH TYPES OF IMPROPREITY HERE.
(V.A. at $x=2$)

$$\int \frac{dx}{x\sqrt{x^2-4}} = \sec^{-1}\left(\tfrac{1}{2}x\right) + C$$

follows from the trig-subst. $x = 2\sec\theta$. it takes a little work, try it.

$$\int_2^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \int_t^3 \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+} \left(\sec^{-1}\left(\tfrac{3}{2}\right) - \sec^{-1}\left(\tfrac{t}{2}\right) \right)$$

$$= \boxed{\sec^{-1}(3/2) - \underbrace{\sec^{-1}(1)}_{0}}$$

$$\int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to\infty} \left(\sec^{-1}\left(\tfrac{1}{2}x\right) \Big|_3^t \right)$$

$$= \lim_{t\to\infty} \left(\sec^{-1}\left(\tfrac{t}{2}\right) - \sec^{-1}(3/2) \right)$$

$$= \boxed{\pi/2 - \sec^{-1}(3/2)}$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \cancel{\sec^{-1}(3/2)} + \frac{\pi}{2} - \cancel{\sec^{-1}(3/2)} = \boxed{\frac{\pi}{2}}$$

In-Class Exercise 9.7.14: Calculate $\displaystyle\int_0^\pi \tan^2(x)\,dx.$

In-Class Exercise 9.7.15: Calculate $\displaystyle\int_\infty^\infty \frac{1}{x}\,dx.$

9.8. APPROXIMATE INTEGRATION

Numerical Integration: An Introduction:

Questions: ① If we use L_n, R_n, M_n then how close is our aprox. to the real answer $\int_a^b f(x)dx \equiv I$

② Are there any other approximation shemes?

③ How do the different approx's compare (usually)

Let's begin with question ②

Trapezoidal Rule with n-trapezoids: $T_n \cong \int_a^b f(x)\,dx$

$$\boxed{T_n = \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)\right] \qquad \Delta x = \frac{b-a}{n} \quad \text{and} \quad x_i = a + i\Delta x}$$

"Proof"

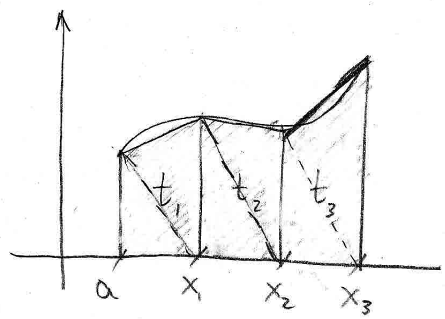

Illustration of T_3

We can geometrically see 6 triangles to get area by summing them.

$$t_1 = \frac{1}{2}f(x_0)\Delta x + \frac{1}{2}f(x_1)\Delta x$$

$$t_2 = \frac{1}{2}f(x_1)\Delta x + \frac{1}{2}f(x_2)\Delta x$$

$$t_3 = \frac{1}{2}f(x_2)\Delta x + \frac{1}{2}f(x_3)\Delta x$$

$$T_3 = t_1 + t_2 + t_3 = \frac{1}{2}\Delta x\left(f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right) \quad //$$

Remark: It's not hard to see that $T_n = \frac{1}{2}(L_n + R_n)$

> **Def^n/** Let A_n be some approx. to $I = \int_a^b f(x)dx$
> We define the absolute error in A_n to be
> $$I - A_n \equiv E_{A_n}$$

Some limitations and comparisons :

1. As n gets large the accuracy increases for each approx., modulo the limiting accuracy of the computer.
2. L_n and R_n have errors with opposite signs, and are halfed when we double n.
3. T_n & M_n are better approx. then L_n or R_n
4. Errors in T_n and M_n are opposite in sign and are quarted when we double n
5. $E_{M_n} \approx \frac{-1}{2} E_{T_n}$

Besides these rules of thumb we can add some more concrete limiting comments,

Th$^{\underline{m}}$(3) Error Bounds. Suppose $|f''(x)| \leq K$ for $a \leq x \leq b$. If E_T and E_M are errors in trap. and midpt. rule then

$$|E_T| \leq \frac{K(b-a)^3}{12n^2} \qquad |E_M| \leq \frac{K(b-a)^3}{24n^2}$$

I expect you to be able to apply these once given them, if there is some question on the test I would supply the the relevant inequality. It's upto you to use them,

Example: Put an upper bound on E_{T_n} where T_n is used to approximate $\int_5^{11} e^x\,dx$. Notice $f(x) = e^x \Rightarrow f''(x) = e^x$ which is increasing since $f'''(x) > 0$ for $5 \leq x \leq 11$ thus $f''(5) \leq f''(x) \leq f''(11)$ $\Rightarrow f''(11)$ is an upper bound, specifically $f''(11) = e^{11} = K \geq |f''(x)|$

$$|E_T| \leq \frac{e^{11}(11-5)^3}{12n^2}$$

- Crucial to using the Th$^{\underline{m}}$(3) is finding the K.

For the most part this section closely mirrors Stewart's text. I also feel no need to justify these wild claims about the error bounds. Proof by example is not proof. A good graduate course in numerical methods would derive these results. Error bounds are important since they tell us the worst case scenario when we replace the true integral with a much easier to calculate finite sum. However, math aside, beware the machine epsilon seeks to devour and corrupt. Just because the mathematics says the error should be smaller than a particular value does not mean the machine we are using is capable of the precision we assume. And, I haven't even bothered to factor in the robot holocaust. That said, I don't plan on doing Simpson's rule without a computer.

Simpson's Rule

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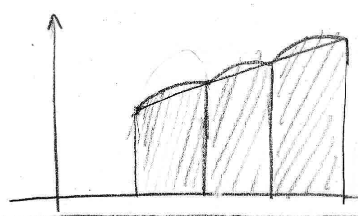
Simpson's Rule approximates the area by integrating the area under a parabolic approx. of the function



S_3 for a linear function

Simpson's Rule

$$S_n = \frac{\Delta x}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)\right]$$

Where n must be even and $\Delta x = \frac{b-a}{n}$.

Proof/ See book.

Remark: Notice pattern $1, 4, 2, 4, 2, \ldots, 2, 4, 1$

How accurate is simpson's rule? Calculators base their numeric int. on it,

Th^{m} (4) Error Bound for Simpson's Rule: Let $|f''''(x)| \leq K$ for $a \leq x \leq b$ and let E_S be error in simpson's rule S_n

$$|E_{S_n}| \leq \frac{K(b-a)^5}{180n^4}$$

Remark: If we double n then the error is $\frac{1}{16}$ of what it was.

Example: If we approximate $\int_2^4 \sin\theta \, d\theta$ with S_n then find $|E_{S_n}| \leq$?
Notice that $f(\theta) = \sin(\theta) \Rightarrow f''''(\theta) = \sin(\theta)$ which has the property $|\sin(\theta)| \leq 1 \Rightarrow |f''''(\theta)| \leq 1$. Thus

$$|E_{S_n}| \leq \frac{1 \cdot (4-2)^5}{180n^4}$$

If we demand accuracy within $0.01 \Rightarrow |E_{S_n}| \approx 0.01$ thus

$$0.01 = \frac{32}{180n^4} \Rightarrow n = \left(\frac{32}{(0.01)180}\right)^{1/4} = 2.05 \Rightarrow \boxed{S_3 \text{ will certainly do the job.}}$$

Perhaps this section has left you a little dazed. Why are we doing this anyway? Didn't the methods we used earlier in this chapter give better, nice exact results? The answer is that yes those methods do give exact results but there are many problems which simply do not admit a nice closed-form answer in terms of elementary functions. For example,

$$1.)\ \int \frac{\sin(x)}{x} dx \qquad \text{arises in study of optics}$$

$$2.)\ \int e^{-x^2} \quad dx \qquad \text{the bell curve from probability}$$

$$3.)\ \int \sqrt{a^2\cos^2(t) + b^2\cos^2(t)} \qquad \text{ellipitical function}$$

In a few special cases 3.) does permit an solution in terms of elementary functions. It can be shown that the integral converges for piecewise continuous functions. However, there is no guarantee that the antiderivative for the given integrand has a nice formula. We will find approximations to the antiderivative in a later chapter using power series arguments. This section has dealt only with the less challenging question of how to approximate a definite integral.

Some definite integrals require we take a brute-force like approach. That is in essence what this section is. When we can't find the antiderivative for the integrand, or when there is no formula given for the integrand, in such cases we have no alternative but to use brute force. This discussion in this section is just quibbling over which type of brute we want to be. In practice, the thing most people do is to use Mathematica or the Wolfram Integrator (it's online) to calculate definite integrals which defy closed form solutions. If you could look inside Mathematica odds are you'd find something like Simpson's Rule being used to find the answer.

Finally, there is a nice summary section in Stewart giving some grand advice about how to integrate. It might help you gather your thoughts. Try reading section 8.5. I don't plan to formally cover integration tables. Who needs a table when you can create the table? And if you can't do the integral then Mathematica beats the table 99.9% of the time. I would strongly caution over using Mathematica, you need to suffer when doing the homework from this chapter. The burning sensation in your brain may be needed for you to level-up mathematically speaking.

As usual there are some additional examples in the suggested homework solutions.

***Remark:** we are skipping Chapter 9 for now. I'd like to do some of the sections in Chapter 9 from a "parametric viewpoint". That's hard if we haven't yet discussed what the "parametric viewpoint" is. Stewart takes a purely Cartesian viewpoint so he avoids the problem. I'd like to do it right the first time around so we'll wait on Chapter 9.*