River Hydraulics

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ABSTRACT

This initial lecture establishes the fundamental principles of open-channel flow. Mass and momentum budgets are derived and then exploited to obtain a few relations central to hydraulics. The equations governing nonlinear waves are treated by the method of characteristics, which is then applied to solve the dam-breaking problem.

1.1 Introduction

Rivers are, in first approximation, nearly one-dimensional flows driven by gravity down a slope and resisted by friction. While this may seem simple from a physical perspective, nonlinearities in the dynamics can make the mathematical description quite intresting. In particular, we shall see that multiple equilibrium states are possible and that the velocity and water depth at a particular point are dependent on boundary conditions that may be upstream at times and downstream at other times. In other words, the physical condition at a certain location can either be the downstream evolution of some upstream condition or alternatively be controlled by an obstacle lying further downstream, such as a retaining dam. The situation is not unlike road traffic, in which car density and speed may alternatively be a free stream emerging from a congested area or form a retrograde bottleneck at the approach of a traffic light or construction zone.

In a first step, we shall establish the equations governing the water velocity and water depth as functions of the downstream distance and time, with particular attention paid to the case of a rectangular channel bed. Then, we shall consider a series of particular cases of interest: steady and unsteady flow, frictionless and frictional flows.

1.2 Equations of Motion

River flow is actually three-dimensional because the velocity depends not only on downstream distance but also on depth and lateral position (Figure 1-1). This is so because

Figure 1-1. The Penobscot River in Maine, USA. [Photo ©2001 Ian Adams]

friction against the bottom and banks causes the velocity to decrease from a maximum at the surface near the middle of the stream to zero along the bottom and sides. In addition, centrifugal effects in river bends generate secondary circulations that render the velocity a full three-dimensional vector.

Because we wish to emphasize here the manner by which the flow varies in the downstream direction, we will neglect cross-stream velocity components as well as cross-stream variations of the downstream component, by considering the speed u as the water velocity averaged across the stream and a function of only the downstream distance x and time t. Because the flow in a river almost never reverses, the fact that we take x directed downstream implies that u is a positive quantity.

With a free surface exposed to the atmosphere, the water depth in a river can, too, vary in space and time. This implicates a second flow variable, namely the water depth, which we denote h and take as function of x and t, like the velocity. And like u, h must be positive everywhere. The existence of two dependent variables, $u(x,t)$ and $h(x,t)$, calls for two governing equations. Naturally, these are statements of mass conservation and momentum budget.

To establish the pair of governing equations, consider a slice of river as depicted in Figure 1-2. Geometric quantities are: A the cross-sectional area occupied by the water, P the *wetted perimeter* (shortest underwater distance from bank to opposite bank following the curved bottom), $S = \sin\theta$ the bottom slope, and h the water depth at the deepest point. The cross-sectional area A and wetted perimeter P are each a function of the water depth h, because as h rises A and P increase in a way that depends on the shape of the channel bed.

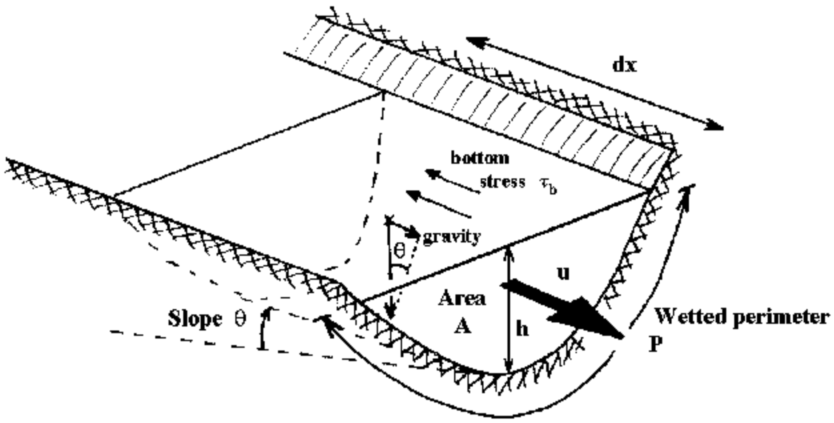

Figure 1-2. A slice of length dx along a river for the formulation of mass conservation and momentum budget. The notation is: velocity averaged across the stream u, water depth h, cross-sectional area of the stream A, wetted perimeter P, and bottom slope $S = \sin\theta$.

For example, a channel bed with rectangular cross-section of width W (Figure 1-3) yields $A = Wh$ and $P = W + 2h$.

Mass conservation

Conservation of mass is relatively straightforward. We simply need to state that the accumulation over time of mass $\rho A dx$ inside the slice of length dx is caused by a possible difference between the amount of mass $\rho A u$ that enters per time at position x and the amount that leaves per time at position $x + dx$. For a short time interval dt, this mass budget is:

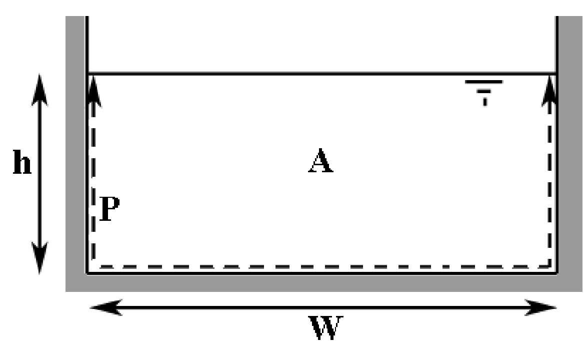

Figure 1-3. A channel bed with rectangular cross-section. In this simplest of cases, $A = Wh$ and $P = h + W + h = W + 2h$.

$$\begin{aligned} \rho A dx|_{\text{at } t+dt} &= \rho A dx|_{\text{at } t} \\ &+ \rho A u|_{\text{at } x} - \rho A u|_{\text{at } x+dx} \end{aligned}$$

which, in the limit of dt and dx going to zero, becomes:

$$\frac{\partial}{\partial t}(\rho A) + \frac{\partial}{\partial x}(\rho A u) = 0,$$

or, because ρ is a constant,

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0. \tag{1.1}$$

This equation is attributed to Leonardo da Vinci (1452–1519), although he did not write it in terms of derivatives.

Since the manner in which the cross-sectional area A increases with the water depth h is known from the shape of the channel bed, the preceding equation actually governs the temporal evolution of the water depth h. It requires the knowledge of the velocity u, a second equation is necessary. This will be fulfilled once we have established the momentum budget.

In the meantime, it is instructive to write the mass-conservation equation in the case of a rectangular cross-section of constant width. With $A = Wh$, Equation (1.1) reduces to:

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(hu) = 0. \tag{1.2}$$

Momentum budget

The momentum budget is an expression of Newton's second law, namely that the sum of unbalanced forces creates a change in momentum. For our slice of river, we write that the time rate of change of momentum is the momentum flux entering from the rear, minus the momentum flux exiting downstream, plus the sum of accelerating forces (acting in the direction of the flow), and minus the sum of the decelerating forces (acting against the flow). Symbolically:

$$\begin{aligned} \frac{d}{dt}\text{[Momentum inside the slice]} &= \text{Momentum flux entering at } x \\ &- \text{Momentum flux exiting at } x + dx \\ &+ \text{Pressure force in the rear} \\ &- \text{Pressure force ahead} \\ &+ \text{Downslope gravitational force} \\ &- \text{Frictional force along the bottom} \end{aligned}$$

The momentum is the mass times the velocity, that is $(\rho dV)u = \rho A u dx$, whereas the momentum flux is the mass flux times the velocity, that is $(\rho A u)u = \rho A u^2$. The pressure force F_p at each end of the slice is obtained from the inegration of the depth-dependent pressure over the cross-section:

$$\text{Pressure force} \;=\; F_p \;=\; \int\!\!\int p dA \;=\; \int_0^h p(z)w(z)dz,$$

in which $p(z)$ and $w(z)$ are, respectively, the pressure and channel width at level z, with z varying from zero at the bottom-most point to h at the surface. Under the assumption of a hydrostatic balance, the pressure increases linearly with depth according to

$$p(z) \;=\; \rho g(h - z),$$

discounting the atmospheric pressure which acts all around and has no net effect on the flow. The pressure force is thus equal to:

$$F_p \;=\; \int_0^h \rho g(h-z)w(z)dz,$$

and is a function of how filled the channel is. In other words, it is a function of depth h. Taking the h derivative (which will be needed later), we have:

$$\begin{aligned} \frac{dF_h}{dh} &= [\rho g(z-h)w(z)]_{z=h} \;+\; \int_0^h \rho g w(z)dz \\ &= \rho g \int_0^h w(z)dz \;=\; \rho g A. \end{aligned} \tag{1.3}$$

The gravitational force is the weight of the water slice projected along the x–direction, which is mg times the sine of the slope angle θ:

$$\text{Gravitational force} \;=\; [(\rho dV)g]\sin\theta \;=\; \rho g A S dx. \tag{1.4}$$

Finally, the frictional force is the bottom stress τ_b multiplied by the wetted surface area:

$$\text{Frictional force} \;=\; \tau_b P dx.$$

River flows are typically in a state of turbulence, and, within a certain level of approximation, the bottom stress is proportional to the square of the velocity. If we introduce a drag coefficient C_D, we may write:

$$\text{Bottom stress} \;=\; \tau_b \;=\; C_D \rho u^2, \tag{1.5}$$

which evokes a Reynolds stress ($\tau = -\rho\overline{u'w'}$, with the turbulent fluctuations u' and w' each proportional to the average velocity u). The frictional force exerted on the slice of water is expressed as

$$\text{Frictional force} \;=\; \tau_b P dx \;=\; C_D \rho P u^2 dx. \tag{1.6}$$

Values for the drag coefficient in rivers vary between 0.003 and 0.01, but there is no universal value for a given channel bed because C_D varies with the Reynolds number of the flow as well as with the shape and roughness of the channel bed. For the sake of mathematical simplicity, however, we do not enter into those details here and shall assume, quite audaciously, that C_D is a constant.

We are now in a position to write the momentum budget:

$$\begin{aligned} \frac{[\rho A u dx|_{\text{at } t+dt} - \rho A u dx|_{\text{at } t}]}{dt} &= \rho A u^2|_{\text{at } x} - \rho A u^2|_{\text{at } x+dx} \\ &+ F_p|_{\text{at } x} - F_p|_{\text{at } x+dx} \\ &+ \rho g A S dx \\ &- C_D \rho P u^2 dx, \end{aligned}$$

or, in differential form,

$$\frac{\partial}{\partial t}(\rho A u) + \frac{\partial}{\partial x}(\rho A u^2) = - \frac{\partial F_p}{\partial x} + \rho g A S - C_D \rho P u^2.$$

Using the mass-conservation equation (1.1), we can reduce the left-hand side of this equation. Then, thanks to (1.3), the gradient of the pressure force becomes

$$\frac{\partial F_p}{\partial x} = \frac{dF_p}{dh}\frac{\partial h}{\partial x} = \rho g A \frac{\partial h}{\partial x} .$$

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A division by ρA finally yields:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - g \frac{\partial h}{\partial x} + gS - C_D \frac{u^2}{R} . \tag{1.7}$$

In this equation the ratio of the cross-sectional area A over the wetted perimeter P, which has the dimension of a length, was defined as

$$R = \frac{A}{P} . \tag{1.8}$$

This is called the *hydraulic radius.* Because most rivers are much wider than they are deep, the wetted perimeter is not much more than the width ($P \simeq W$), and the hydraulic radius is approximately the average depth $\bar{h}$, which itself is not very different from the center depth if the channel has a broad flat bottom, as is often the case with natural streams:

$$R \simeq \frac{A}{W} = \bar{h} \simeq h. \tag{1.9}$$

The average depth $\bar{h}$ is exactly equal to the maximum depth h for a rectangular cross-section.

In Equation (1.7), the quantity R is a function of the water depth h. The momentum equation, therefore, establishes a new relation between the velocity u and depth h, which together with mass conservation (1.1) forms a closed set of two equations for two unknowns.

Because each equation contains a first-order derivative in time and also one in space, the system is of second order in both time and space. Two initial conditions and two boundary conditions are thus required to specify fully the problem. The initial conditions are naturally the spatial distribution of $h_o(x)$ and $u_o(x)$ at some original time, but it is far less clear what the boundary conditions ought to be and where they should be applied. As we shall see, imposing an upstream value of h and an upstream value of u does not necessarily work.

For a wide channel with broad flat bottom or with a rectangular cross-section, R may be replaced by h, and the momentum equation reduces to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} + gS - C_D \frac{u^2}{h} . \tag{1.10}$$

Credit for the pair of governing equations (1.1)–(1.7) is due to Adhémar Jean Claude Barré de Saint-Venant (1797–1886), a French civil engineer who spent a significant part of his career working for the country's Bridges and Highways Department. Occasionally but not often, the pair of equations are called the Saint-Venant equations. Even in its simplified form, the set (1.2)–(1.10) for a wide rectangular channel is highly nonlinear. So, non-unique solutions and other surprises may occur.

1.3 Uniform Frictional Flow

Our first particular case is that of a steady and uniform flow down a constant slope. With the temporal and spatial derivatives set to zero, Equation (1.1) reduces to $0 = 0$ while the momentum budget (1.7) reduces to:

$$C_D \frac{u^2}{R} = gS,$$

which simply states that the downslope force of gravity is resisted entirely by bottom friction. This is similar to a parachute in action, in which the downward force of gravity, which is constant, is balanced by the upward force of air drag, which is proportional to the square of the velocity. The equation can be readily solved for the velocity:

$$u = \sqrt{\frac{gRS}{C_D}} . \tag{1.11}$$

This is known as the Chézy formula, in honor of Antoine Léonard de Chézy (1718-1798), a French engineer who designed canals for supplying water to the city of Paris. The formula in the preceding form dates back to 1776 and has been the subject of countless improvements over the years. We shall not dwell into these and consider instead the basic formula (1.11).

For a wide channel with broad flat bottom, the hydraulic radius R is nearly the water depth h, and (1.11) reduces to:

$$u = \sqrt{\frac{ghS}{C_D}} , \tag{1.12}$$

Thus, the river velocity increases as the square root of the water depth. What sets a value to each water depth and velocity is the river volumetric flow rate, called the *discharge* and noted Q. With

$$Q = Au \tag{1.13}$$

Equations (1.11) and (1.13) for a two-by-two system of equations for h and u.

The solution in the particular case of a wide rectangular channel ($A = Wh$ and $R \simeq h$) is:

$$h = \left(\frac{C_D Q^2}{gSW^2}\right)^{\frac{1}{3}} \tag{1.14}$$

$$u = \left(\frac{gSQ}{C_D W}\right)^{\frac{1}{3}}. \tag{1.15}$$

Because the water depth h varies like $Q^{2/3}$ whereas the velocity u varies as $Q^{1/3}$, we deduce that an increase in discharge generates a larger increase in depth than in velocity. The interesting result, however, is that the two quantities are intimately related to each other. It is presumed that this is the reason why Roman engineers of antiquity were successful at conveying clean water by aquaducts and removing waste water by sewers. Indeed, Romans did not have a notion of time on the scale of the second and minute, only on the scale of hours and days by following the motion of the sun, and therefore the concept of a velocity was foreign to them. They only had at their disposal the water depth, which they could measure with a stick. So, all their calculations were solely based on the water depth, but the fact that h and u are tightly related to each other allowed them to obtain inaccurate but practical estimates for the design of their water lines.

When the flow is not uniform but gradually varying, because the slope is not constant or there are other elements that activate the derivatives in (1.1) and (1.7), the value of h given by (1.14) is not necessarily the water depth realized by the stream but nonetheless serves as a useful reference against which the actual water depth may be compared. In this case, it is called the *normal depth* and is denoted by h_n:

$$h_n = \left(\frac{C_D Q^2}{gSW^2}\right)^{\frac{1}{3}}. \tag{1.16}$$

As we shall see later, the cases $h < h_n$ (flow is too thin and fast) and $h > h_n$ (the flow is too thick and slow) exhibit different dynamical properties.

1.4 Steady Frictionless Flow

As a second particular case, let us now consider a steady and frictionless flow. With no time derivative and $C_D = 0$, the governing equations (1.1) and (1.7) become

$$\frac{d}{dx}(Au) = 0 \tag{1.17}$$

$$u\,\frac{du}{dx} = -g\,\frac{dh}{dx} + gS. \tag{1.18}$$

The first equation is readily integrated over x to yield that the volumetric flowrate (discharge)

$$Q = Au \tag{1.19}$$

is constant along the stream. This is evident because there is no water lost or gained along the way, and there is no possibility of local accumulation in steady state.

The second equation, too, may be integrated over x. To do so, we express the bottom slope as minus the derivative of the bottom elevation $b(x)$ above a reference datum (such as sea level):

$$S = - \frac{db}{dx} , \tag{1.20}$$

which permits to gather all terms of (1.18) in a single x–derivative:

$$\frac{d}{dx} \left(\frac{u^2}{2} + gh + gb \right) = 0$$

Therefore, the expression

$$B = \frac{u^2}{2} + gh + gb \tag{1.21}$$

is conserved along the flow. This result is known as the Bernoulli principle in honor of Daniel Bernoulli (1700–1782), the most famous member of an illustrious Swizz family counting a number of great mathematicians and physicists. Note that Daniel Bernoulli preceded Saint Venant and therefore established this flow property from first principles.

The essence of the Bernoulli principle is conservation of energy, for it states that the sum of the kinetic energy $u^2/2$ and the potential energy $g(b+h)$, each on a per-mass basis, is conserved. Note, however, that there is an element of surprise here. Indeed, while we expect energy to be conserved for the whole system in the absence of friction, it could not have been anticipated that energy conservation would actually hold true for every individual particle.

For the moment, we only note this result, leaving for a later lecture the rich results that can be deduced from it.

1.5 Surface Waves

Our third particular case is that of frictionless, unsteady flow ($C_D = 0$, $\partial/\partial t \neq 0$) in the absence of a bottom slope ($S = 0$). The governing equations (1.1) and (1.7) now reduce to:

$$\frac{\partial A}{\partial t} + \frac{\partial}{\partial x}(Au) = 0$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = - g \frac{\partial h}{\partial x} ,$$

which for a rectangular channel further simplify to:

$$\frac{\partial h}{\partial t} + u \frac{\partial h}{\partial x} = -h \frac{\partial u}{\partial x} \tag{1.22}$$
$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = -g \frac{\partial h}{\partial x} . \tag{1.23}$$

The multiple quadratic terms render these equations strongly nonlinear, but some near symmetry between the two permits a transformation in some form that more easily leads to a solution. For this let us substitute for the water depth h the quantity c defined as

$$c \;=\; \sqrt{gh} \quad \longrightarrow \quad h \;=\; \frac{c^2}{g}\,. \tag{1.24}$$

This new quantity c has the dimension of a speed, just like the remaining variable u. With this notation, the equations become:

$$\frac{\partial c}{\partial t} \;+\; u\,\frac{\partial c}{\partial x} \;=\; -\,\frac{c}{2}\,\frac{\partial u}{\partial x} \tag{1.25}$$

$$\frac{\partial u}{\partial t} \;+\; u\,\frac{\partial u}{\partial x} \;=\; -\,2c\,\frac{\partial c}{\partial x}\,. \tag{1.26}$$

which have quadratic nonlinearities in all terms that do not include the temporal derivative.

Adding twice (1.25) to (1.26) yields

$$\begin{aligned}\left(\frac{\partial}{\partial t} \;+\; u\,\frac{\partial}{\partial x}\right)(u+2c) \;&=\; -\,c\,\frac{\partial u}{\partial x} \;-\; 2c\,\frac{\partial c}{\partial x} \\ &=\; -\,c\,\frac{\partial}{\partial x}(u+2c),\end{aligned}$$

or, after gathering all terms on the left:

$$\left[\frac{\partial}{\partial t} \;+\; (u+c)\,\frac{\partial}{\partial x}\right](u+2c) \;=\; 0. \tag{1.27}$$

Likewise, if we now subtract twice (1.25) from (1.26), we get

$$\begin{aligned}\left(\frac{\partial}{\partial t} \;+\; u\,\frac{\partial}{\partial x}\right)(u-2c) \;&=\; +\,c\,\frac{\partial u}{\partial x} \;-\; 2c\,\frac{\partial c}{\partial x} \\ &=\; +\,c\,\frac{\partial}{\partial x}(u-2c),\end{aligned}$$

which in turn can be made more compact:

$$\left[\frac{\partial}{\partial t} \;+\; (u-c)\,\frac{\partial}{\partial x}\right](u-2c) \;=\; 0. \tag{1.28}$$

We recognize here a hyperbolic problem, which may be solved by the method of characteristics. Indeed, Equations (1.27)–(1.28) state respectively that the quantity $(u+2c)$ remains unchanged when traveling at the speed $(u+c)$ and that $(u-2c)$ remains unchanged when traveling at the speed $(u-c)$. The quantities $(u \pm 2c)$ are the Riemann invariants of the problem.

From this, we can conclude that c acts as a wave speed that either adds to the flow velocity, $(u+c)$, or subtracts from it, $(u-c)$. Thus, there is a single kind of wave, which is able to propagate information both with and against the flow. Since $c = \sqrt{gh}$ implicates

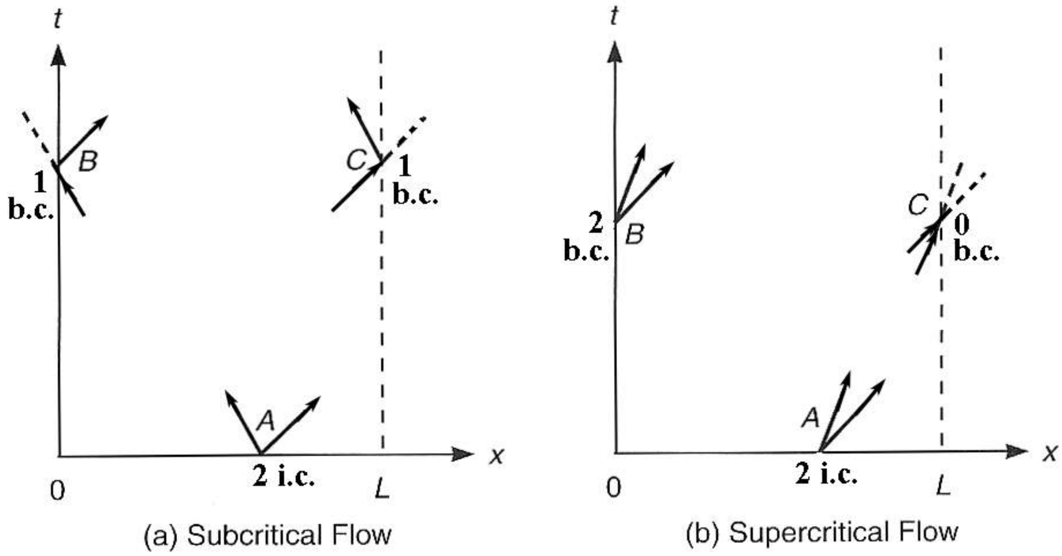

Figure 1-4. The manner in which the characteristics intersect the domain boundaries specifies where the boundary conditions need to be applied. The numbers indicate how many initial conditions (i.c.) and boundary conditions (b.c.) are required. [Adapted from Sturm, 2001]

gravity, this is naturally a gravity wave. [Important note: Although the equations are nonlinear, the present representation of the gravity wave does not include the full set of possible nonlinearities. Indeed, the assumption of a vertically uniform velocity and the use of the hydrostatic balance for the determination of pressure does not permit the investigation of wave breaking.]

It is noteworthy to remark that the wave speed c is not necessarily larger than the water velocity u. Therefore the expression $(u - c)$ may be either positive or negative. This leads to the definition of the so-called Froude number:

$$Fr = \frac{u}{c} = \frac{u}{\sqrt{gh}}. \tag{1.29}$$

which is credited to William Froude (1810–1870), a British naval architect who found this ratio to be important in the estimation of wave drag on ship propulsion.

For weak flow, defined as when u is less than c ($Fr < 1$) and called *subcritical flow*, $(u - c)$ is negative whereas $(u + c)$ is positive. Thus, waves travel in opposite directions, communicating information both downstream and upstream. The two boundary conditions must then be such that one lies upstream and the other downstream (Figure 1-4a). For example, the water depth h can be set at both ends of the domain. A more common situation, however, is setting the volumetric flow Q upstream and the water depth h downstream.

In the reverse case, when the flow is fast, defined as when u surpasses c ($Fr > 1$) and called *supercritical flow*, both $(u \pm c)$ are positive, and information can only be propagated

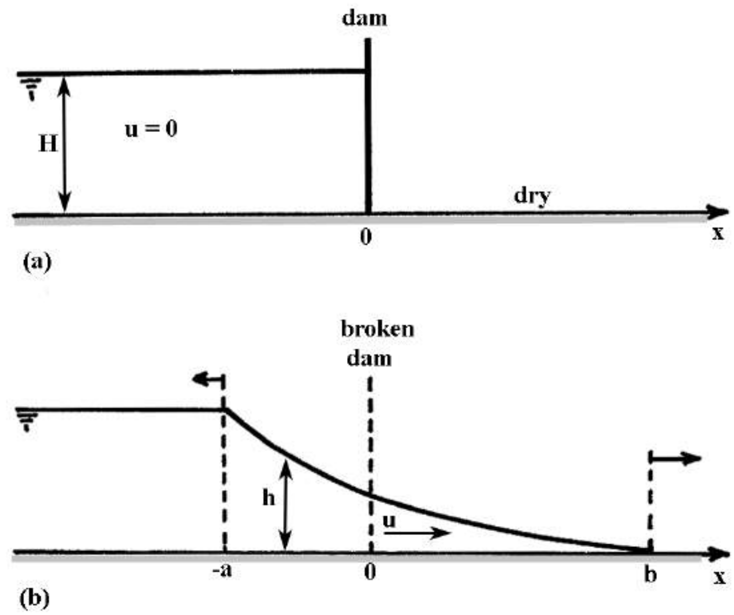

Figure 1-5. The dam-breaking problem: (a) initial condition and (b) some time later.

downstream. In this case both boundary conditions must be applied upstream (Figure 1-4b). Most often this done by imposing both h and u (or, equivalently h and Q) at the entrance of the domain.

Because both u and c depend on the solution of the problem, one is not sure ahead of time where to impose the boundary conditions. Furthermore, it is very possible that the flow is simultaneously subcritical in some regions and supercritical in others. And, the size of these regions is also susceptible to change over time. Needless to say, the situation may be quite complex. The numerical solution of the equations in general cases is not a completely solved problem. For a textbook treatment of the problem, the reader is referred to Sturm (2001).

1.6 The Dam-Breaking Problem

An interesting application of the preceding dynamics is the dam-breaking problem. For this, we imagine an initial condition consisting in water piled to a depth H behind a dam situated at $x = 0$ and dry land on the other side. The bottom is horizontal everywhere and there is no initial velocity (Figure 1-5a). Thus, the initial conditions are

$$h = H, \; u = 0 \text{ for } x < 0, \quad \text{and} \quad h = 0, \; u = 0 \text{ for } x > 0. \tag{1.30}$$

The dam suddenly fails at $t = 0$, and for the sake of simplicity, it is imagined that it is completely removed instantaneously. The problem is to describe the flooding on the

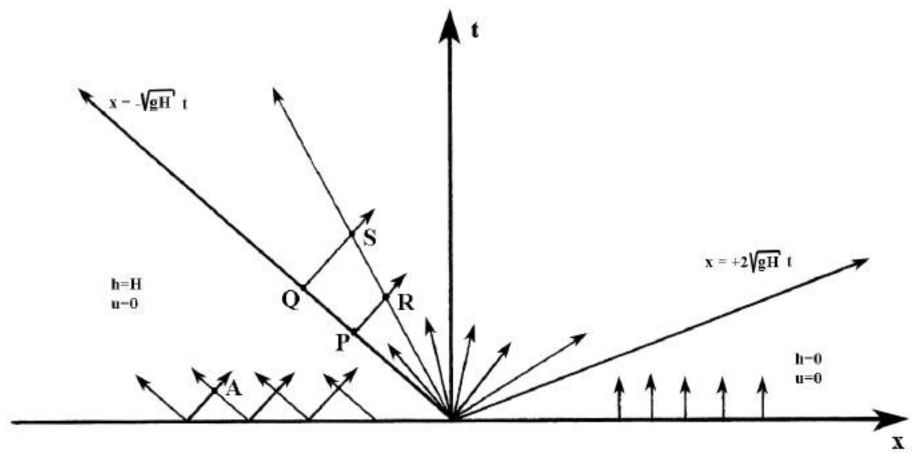

Figure 1-6. Characteristics configuration for the dam-breaking problem.

formerly dry side and the water depletion on the backside. This can be done analytically by the method of characteristics.

The first thing to notice is that finite water depth and finite water speed can only generate finite values of $(u \pm c)$. Therefore, the signals propagating upstream and downstream do so at finite speeds and, at some finite time t, the situation consists in three regions, a far upstream region $[x < -a(t)]$ and a far downstream region $[x > +b(t)]$, each still untouched and bracketing the zone $[-a(t) < x < +b(t)]$ where change is taking place (Figure 1-5b).

In the far upstream region, a point **A** at some early time (see Figure 1-6) is crossed by a pair of characteristics that carry the Riemann invariants $(u \pm 2c)$, which are equal to $\pm 2\sqrt{gH}$. It follows that $u + 2c = +2\sqrt{gH}$ and $u - 2c = -2\sqrt{gH}$. The solution is obviously still $u = 0$ and $c = \sqrt{gH}$. Progressing through this region, we conclude that $u = 0$ and $c = \sqrt{gH}$ throughout the region, as expected. The limiting characteristics is the one that begins where the dam was and travels backwards, namely $dx/dt = -c = -\sqrt{gH}$ or $x = -t\sqrt{gH}$. Along that characteristic, u and c remain constant at 0 and $\sqrt{gH}$, respectively, setting a initial/boundary condition on the upstream side of the changing region.

Since the complementary quantity $(u + 2c)$ is carried unchanged along characteristics such as **P**-**R** and **Q**–**S** emanating from this line, it follows that

$$u + 2c \;=\; 2\sqrt{gH}$$

throughout the region in which motion occurs, and Equation (1.27) is trivially satisfied.

Let us now take two points, **P** and **Q**, along the aforementioned characteristic and follow their respective progression along the other family of characteristics to the points **R** and **S**, respectively, as shown in Figure 1-6. Along the **P**–**Q** and **R**–**S** paths, the quantity $(u + 2c)$ is conserved and thus

$$u_R \;+\; 2c_R \;\;=\;\; u_P \;+\; 2c_P \;=\; 2\sqrt{gH}$$

$$u_S + 2c_S = u_Q + 2c_Q = 2\sqrt{gH}$$

which yields

$$c_R = \sqrt{gH} = \frac{u_R}{2}$$
$$c_S = \sqrt{gH} = \frac{u_S}{2}.$$

If points **R** and **S** lie on the same crossing characteristic, they must also share the same value of $(u - 2c)$ value and therefore

$$u_S - 2c_S = u_R - 2c_R,$$

from which follows that

$$u_S - 2\left(\sqrt{gH} - \frac{u_S}{2}\right) = u_R - 2\left(\sqrt{gH} - \frac{u_R}{2}\right)$$

and therefore

$$u_S = u_R \quad \text{and} \quad c_S = c_R \quad \text{and} \quad u_S - c_S = u_R - c_R.$$

From this follows that the crossing characteristic has a constant slope and is therefore a straight line. Since point **R** lies at an arbitrary distance from point **P**, we can further generalize by stating that all $(u - c)$ characteristics in the region of movement are straight lines, with

$$\frac{dx}{dt} = u - c = \text{a constant}.$$

Time integration provides $x(t) = (u-c)t+x_o$. Since we know that $u+2c = 2\sqrt{gH}$ everywhere in the region, the characteristic slope $u - c$ must be equal to $2\sqrt{gH} - 3c$. But, the initial value of h is undetermined at $x = 0$, and there are many $c = \sqrt{gh}$ possible values with which to begin characteristics. There thus exists a family of characteristics fanning outward from the same point $(x = 0, t = 0)$, each with its own slope $2\sqrt{gH} - 3c$, ranging between $-\sqrt{gH}$ (for $h = H$) to $+2\sqrt{gH}$ (for $h = 0$). This is also the only family of characteristics, for those that emanate from $x > 0$ all begin with $u = c = 0$ and do not propagate. It follows that the zone of motion consists entirely in the set of straight characteristics that emanate from $x = 0$ with the aforementioned range of slopes. The leading edges (Figure 1-5) are thus $a(t) = \sqrt{gH}t$ and $b(t) = 2\sqrt{gH}t$.

For this set of characteristics starting from the origin $(x_o = 0)$, the progression of x with time is given by $x = (2\sqrt{gH} - 3c)t$. Inverting this equation to solve for c yields

$$c = \frac{2}{3}\sqrt{gH} - \frac{x}{3t} \tag{1.31}$$

which is the solution of the problem in terms of c. Switching back to the water depth h, we have

$$h = \frac{c^2}{g} = \frac{4H}{9} - \frac{4x\sqrt{gH}}{9gt} + \frac{x^2}{9gt^2}, \tag{1.32}$$

which describes a upward-facing parabola starting from $h = 0$ at the leading edge $x = 2\sqrt{gH}t$, passing through $h = 4H/9$, a fixed value, at $x = 0$, where the dam used to be, and ending with $h = H$ at the backward end $x = -\sqrt{gH}t$ (see Figure 1-5).

The velocity is obtained from the $u + 2c = 2\sqrt{gH}$ constraint:

$$u \;=\; \frac{2}{3}\,\sqrt{gH} \;+\; \frac{2x}{3t}\;, \tag{1.33}$$

which varies from zero at the backward end to $2\sqrt{gH}$ at the leading edge. At the location of the broken dam ($x = 0$), the velocity u and wave speed c are both equal to $(2/3)\sqrt{gH}$. At this point, therefore, the flow is critical, separating a subcritical region upstream (where the water depth is thicker and the flow slower) from a supercritical region downstream (where the water depth is thinner and the flow faster).

The discharge per unit width of channel is given by

$$q \;=\; hu \;=\; \frac{8H\sqrt{gH}}{27} \;-\; \frac{2x^2\sqrt{gH}}{9gt^2} \;+\; \frac{2x^3}{27gt^3}\;, \tag{1.34}$$

which varies from zero at $x = -a$ back to zero at $x = b$. It must thus reach a maximum in the region of motion. Taking the x-derivative and setting it to zero, we note that there are two extrema, the first at $x = 0$ (at the former location of the dam) and $x = b$ (the forward edge). Since $q(x = b) = 0$, the maximum is the value at $x = 0$, namely

$$q_{\max} \;=\; q(0) \;=\; \frac{8}{27}H\sqrt{gH}\;, \tag{1.35}$$

which is invariant over time.

The whole scenario bears a strong analogy with road traffic. The accumulation of non-moving vehicles at a red light is analogous to the high water at rest, and the light turning green is similar to the breaking of the dam. The first few cars speeding ahead from the pack behave like the flooding waters at the leading of the released water, while the gradual setting in motion of the line of vehicles is a backward decompression wave analogous to the upstream wave that sets ever more water in motion.

REFERENCES

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