Differential Equations I
MATB44H3F

Version September 15, 2011-1949

Contents

Chapter 1

Introduction

1.1 Preliminaries

Definition (Differential equation)
A *differential equation* (DE) is an equation involving a function and its derivatives.

Differential equations are called *partial differential equations* (PDE) or *ordinary differential equations* (ODE) according to whether or not they contain partial derivatives. The *order* of a differential equation is the highest order derivative occurring. A *solution* (or *particular solution*) of a differential equation of order n consists of a function defined and n times differentiable on a domain D having the property that the functional equation obtained by substituting the function and its n derivatives into the differential equation holds for every point in D.

Example 1.1. An example of a differential equation of order 4, 2, and 1 is given respectively by

$$\left(\frac{dy}{dx}\right)^3 + \frac{d^4y}{dx^4} + y = 2\sin(x) + \cos^3(x),$$
$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0,$$
$$yy' = 1.$$

*

Example 1.2. The function $y = \sin(x)$ is a solution of

$$\left(\frac{dy}{dx}\right)^3 + \frac{d^4y}{dx^4} + y = 2\sin(x) + \cos^3(x)$$

on domain $\mathbb{R}$; the function $z = e^x \cos(y)$ is a solution of

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

on domain $\mathbb{R}^2$; the function $y = 2\sqrt{x}$ is a solution of

$$yy' = 2$$

on domain $(0, \infty)$. $*$

Although it is possible for a DE to have a unique solution, e.g., $y = 0$ is the solution to $(y')^2 + y^2 = 0$, or no solution at all, e.g., $(y')^2 + y^2 = -1$ has no solution, most DE's have infinitely many solutions.

Example 1.3. The function $y = \sqrt{4x + C}$ on domain $(-C/4, \infty)$ is a solution of $yy' = 2$ for any constant C. $*$

Note that different solutions can have different domains. The set of all solutions to a DE is call its *general solution*.

1.2 Sample Application of Differential Equations

A typical application of differential equations proceeds along these lines:

Real World Situation

↓

Mathematical Model

↓

Solution of Mathematical Model

↓

Interpretation of Solution

Sometimes in attempting to solve a DE, we might perform an irreversible step. This might introduce extra solutions. If we can get a short list which contains all solutions, we can then test out each one and throw out the invalid ones. The ultimate test is this: **does it satisfy the equation?**

Here is a sample application of differential equations.

Example 1.4. The half-life of radium is 1600 years, i.e., it takes 1600 years for half of any quantity to decay. If a sample initially contains 50 g, how long will it be until it contains 45 g? $*$

Solution. Let $x(t)$ be the amount of radium present at time t in years. The rate at which the sample decays is proportional to the size of the sample at that time. Therefore we know that $dx/dt = kx$. This differential equation is our mathematical model. Using techniques we will study in this course (see §3.2, Chapter 3), we will discover that the general solution of this equation is given by the equation $x = Ae^{kt}$, for some constant A. We are told that $x = 50$ when $t = 0$ and so substituting gives $A = 50$. Thus $x = 50e^{kt}$. Solving for t gives $t = \ln(x/50)/k$. With $x(1600) = 25$, we have $25 = 50e^{1600k}$. Therefore,

$$1600k = \ln\left(\frac{1}{2}\right) = -\ln(2),$$

giving us $k = -\ln(2)/1600$. When $x = 45$, we have

$$\begin{aligned} t = \frac{\ln(x/50)}{k} &= \frac{\ln(45/50)}{-\ln(2)/1600} = -1600 \cdot \frac{\ln(8/10)}{\ln(2)} = 1600 \cdot \frac{\ln(10/8)}{\ln(2)} \\ &\approx 1600 \cdot \frac{0.105}{0.693} \approx 1600 \times 0.152 \approx 243.2. \end{aligned}$$

Therefore, it will be approximately 243.2 years until the sample contains 45 g of radium. ♢

Additional conditions required of the solution ($x(0) = 50$ in the above example) are called *boundary conditions* and a differential equation together with boundary conditions is called a *boundary-value problem* (BVP). Boundary conditions come in many forms. For example, $y(6) = y(22)$; $y'(7) = 3y(0)$; $y(9) = 5$ are all examples of boundary conditions. Boundary-value problems, like the one in the example, where the boundary condition consists of specifying the value of the solution at some point are also called *initial-value problems* (IVP).

Example 1.5. An analogy from algebra is the equation

$$y = \sqrt{y} + 2. \tag{1.1}$$

To solve for y, we proceed as

$$\begin{aligned} y - 2 &= \sqrt{y}, \\ (y-2)^2 &= y, \quad \text{(irreversible step)} \\ y^2 - 4y + 4 &= y, \\ y^2 - 5y + 4 &= 0, \\ (y-1)(y-4) &= 0. \end{aligned}$$

Thus, the set $y \in \{1, 4\}$ contains all the solutions. We quickly see that $y = 4$ satisfies Equation (1.1) because

$$4 = \sqrt{4} + 2 \Longrightarrow 4 = 2 + 2 \Longrightarrow 4 = 4,$$

while $y = 1$ does not because

$$1 = \sqrt{1} + 2 \Longrightarrow 1 = 3.$$

So we accept $y = 4$ and reject $y = 1$. $*$

Chapter 2

First Order Ordinary Differential Equations

The complexity of solving DE's increases with the order. We begin with first order DE's.

2.1 Separable Equations

A first order ODE has the form $F(x, y, y') = 0$. In theory, at least, the methods of algebra can be used to write it in the form* $y' = G(x, y)$. If $G(x, y)$ can be factored to give $G(x, y) = M(x)\,N(y)$,then the equation is called *separable.* To solve the separable equation $y' = M(x)\,N(y)$, we rewrite it in the form $f(y)y' = g(x)$. Integrating both sides gives

$$\int f(y)y'\,dx = \int g(x)\,dx,$$
$$\int f(y)\,dy = \int f(y)\frac{dy}{dx}\,dx.$$

Example 2.1. Solve $2xy + 6x + \left(x^2 - 4\right) y' = 0$. $\ast$

*We use the notation $dy/dx = G(x, y)$ and $dy = G(x, y)\,dx$ interchangeably.

Solution. Rearranging, we have

$$\begin{aligned}
\left(x^2-4\right) y' &= -2xy - 6x, \\
&= -2xy - 6x, \\
\frac{y'}{y+3} &= -\frac{2x}{x^2-4}, \quad x \neq \pm 2 \\
\ln(|y+3|) &= -\ln\left(\left|x^2-4\right|\right) + C, \\
\ln(|y+3|) + \ln\left(\left|x^2-4\right|\right) &= C,
\end{aligned}$$

where C is an arbitrary constant. Then

$$\begin{aligned}
\left|(y+3)\left(x^2-4\right)\right| &= A, \\
(y+3)\left(x^2-4\right) &= A, \\
y+3 &= \frac{A}{x^2-4},
\end{aligned}$$

where A is a constant (equal to $\pm e^C$) and $x \neq \pm 2$. Also $y = -3$ is a solution (corresponding to $A = 0$) and the domain for that solution is $\mathbb{R}$. ◊

Example 2.2. Solve the IVP $\sin(x)\,dx + y\,dy = 0$, where $y(0) = 1$. *

Solution. Note: $\sin(x)\,dx + y\,dy = 0$ is an alternate notation meaning the same as $\sin(x) + y\,dy/dx = 0$.

We have

$$\begin{aligned}
y\,dy &= -\sin(x)\,dx, \\
\int y\,dy &= \int -\sin(x)\,dx, \\
\frac{y^2}{2} &= \cos(x) + C_1, \\
y &= \sqrt{2\cos(x) + C_2},
\end{aligned}$$

where C_1 is an arbitrary constant and $C_2 = 2C_1$. Considering $y(0) = 1$, we have

$$1 = \sqrt{2 + C_2} \Longrightarrow 1 = 2 + C_2 \Longrightarrow C_2 = -1.$$

Therefore, $y = \sqrt{2\cos(x) - 1}$ on the domain $(-\pi/3, \pi/3)$, since we need $\cos(x) \geq 1/2$ and $\cos(\pm\pi/3) = 1/2$.

An alternate method to solving the problem is

$$\begin{aligned}
y\,dy &= -\sin(x)\,dx, \\
\int_1^y y\,dy &= \int_0^x -\sin(x)\,dx, \\
\frac{y^2}{2} - \frac{1^2}{2} &= \cos(x) - \cos(0), \\
\frac{y^2}{2} - \frac{1}{2} &= \cos(x) - 1, \\
\frac{y^2}{2} &= \cos(x) - \frac{1}{2}, \\
y &= \sqrt{2\cos(x) - 1},
\end{aligned}$$

giving us the same result as with the first method. ♢

Example 2.3. Solve $y^4 y' + y' + x^2 + 1 = 0$. $*$

Solution. We have

$$\begin{aligned}
\left(y^4 + 1\right) y' &= -x^2 - 1, \\
\frac{y^5}{5} + y &= -\frac{x^3}{3} - x + C,
\end{aligned}$$

where C is an arbitrary constant. This is an implicit solution which we cannot easily solve explicitly for y in terms of x. ♢

2.2 Exact Differential Equations

Using algebra, any first order equation can be written in the form $F(x, y)\,dx + G(x, y)\,dy = 0$ for some functions $F(x, y)$, $G(x, y)$.

Definition

An expression of the form $F(x, y)\,dx + G(x, y)\,dy$ is called a *(first-order) differential form*. A differentical form $F(x, y)\,dx + G(x, y)\,dy$ is called *exact* if there exists a function $g(x, y)$ such that $dg = F\,dx + G\,dy$.

If $\omega = F\,dx + G\,dy$ is an exact differential form, then $\omega = 0$ is called an *exact differential equation*. Its solution is $g = C$, where $\omega = dg$.

Recall the following useful theorem from `MATB42`:

Theorem 2.4
If F and G are functions that are continuously differentiable throughout a simply connected region, then $F\,dx + G\,dy$ is exact if and only if $\partial G/\partial x = \partial F/\partial y$.

Proof. Proof is given in `MATB42`. □

Example 2.5. Consider $\left(3x^2y^2 + x^2\right)dx + \left(2x^3y + y^2\right)dy = 0$. Let

$$\omega = \underbrace{\left(3x^2y^2 + x^2\right)}_{F}dx + \underbrace{\left(2x^3y + y^2\right)}_{G}dy$$

Then note that

$$\frac{\partial G}{\partial x} = 6x^2y = \frac{\partial F}{\partial y}.$$

By THEOREM 2.4, $\omega = dg$ for some g. To find g, we know that

$$\frac{\partial g}{\partial x} = 3x^2y^2 + x^2, \tag{2.1a}$$

$$\frac{\partial g}{\partial y} = 2x^3y + y^2. \tag{2.1b}$$

Integrating Equation (2.1a) with respect to x gives us

$$g = x^3y^2 + \frac{x^3}{3} + h(y). \tag{2.2}$$

So differentiating that with respect to y gives us

$$\begin{aligned}
\overbrace{\frac{\partial g}{\partial y}}^{\text{Eq. (2.1b)}} &= 2x^3y + \frac{dh}{dy}, \\
2x^3y + y^2 &= 2x^3y + \frac{dh}{dy}, \\
\frac{dh}{dy} &= y^2, \\
h(y) &= \frac{y^3}{3} + C
\end{aligned}$$

for some arbitrary constant C. Therefore, Equation (2.2) becomes

$$g = x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} + C.$$

Note that according to our differential equation, we have

$$d\left(x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} + C\right) = 0 \text{ which implies } x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} + C = C'$$

for some arbitrary constant C'. Letting $D = C' - C$, which is still an arbitrary constant, the solution is

$$x^3y^2 + \frac{x^3}{3} + \frac{y^3}{3} = D.$$

*

Example 2.6. Solve $\left(3x^2 + 2xy^2\right) dx + \left(2x^2y\right) dy = 0$, where $y(2) = -3$. *

Solution. We have

$$\int \left(3x^2 + 2xy^2\right) dx = x^3 + x^2y^2 + C$$

for some arbitrary constant C. Since C is arbitrary, we equivalently have $x^3 + x^2y^2 = C$. With the initial condition in mind, we have

$$8 + 4 \cdot 9 = C \Longrightarrow C = 44.$$

Therefore, $x^3 + x^2y^2 = 44$ and it follows that

$$y = \frac{\pm\sqrt{44 - x^3}}{x^2}.$$

But with the restriction that $y(2) = -3$, the only solution is

$$y = -\frac{\sqrt{44 - x^3}}{x^2}$$

on the domain $\left(-\sqrt[3]{44}, \sqrt[3]{44}\right) \setminus \{0\}$. ◊

Let $\omega = F\,dx + G\,dy$. Let $y = s(x)$ be the solution of the DE $\omega = 0$, i.e., $F + Gs'(x) = 0$. Let $y_0 = s(x_0)$ and let γ be the piece of the graph of $y = s(x)$ from (x_0, y_0) to (x, y). Figure 2.1 shows this idea. Since $y = s(x)$ is a solution to $\omega = 0$, we must have $\omega = 0$ along γ. Therefore, $\int_\gamma \omega = 0$. This can be seen

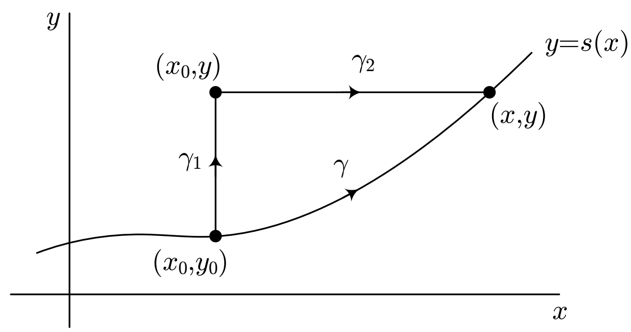


Figure 2.1: The graph of $y = s(x)$ with γ connecting (x_0, y_0) to (x, y).

by parameterizing γ by $\gamma(x) = (x, s(x))$, thereby giving us

$$\int_\gamma \omega = \int_{x_0}^{x} F\,dx + Gs'(x)\,dx = \int_{x_0}^{x} 0\,dx = 0.$$

This much holds for any ω.

Now suppose that ω is exact. Then the integral is independent of the path. Therefore

$$\begin{aligned} 0 &= \int_\gamma \omega = \int_{\gamma_1} F\,dx + G\,dy + \int_{\gamma_2} F\,dx + G\,dy \\ &= \int_{y_0}^{y} G(x_0, y)\,dy + \int_{x_0}^{x} F(x, y)\,dx. \end{aligned}$$

We can now solve Example 2.6 with this new method.

Solution (Alternate solution to Example 2.6). We simply have

$$\begin{aligned} 0 &= \int_{-3}^{4} 2 \cdot 2^2 y\,dy + \int_{2}^{x} \left(3x^2 + 2xy^2\right) dx \\ &= 4y^2 - 4\left(-3\right)^2 + x^3 + x^2y^2 - 2^3 - 2^2y^2 \\ &= 4y^2 - 36 + x^3 + x^2y^2 - 8 - 4y^2, \end{aligned}$$

finally giving us $x^3 + x^2y^2 = 44$, which agrees with our previous answer. ♢

Remark. Separable equations are actually a special case of exact equations, that is,

$$f(y)y' = g(x) \Longrightarrow -g(x)\,dx + f(y)\,dy = 0 \Longrightarrow \frac{\partial}{\partial x} f(y) = 0 = \frac{\partial}{\partial y}\left(-g(x)\right).$$

So the equation is exact. ◊

2.3 Integrating Factors

Consider the equation $\omega = 0$. Even if ω is not exact, there may be a function $I(x, y)$ such that $I\omega$ is exact. So $\omega = 0$ can be solved by multiplying both sides by I. The function I is called an *integrating factor* for the equation $\omega = 0$.

Example 2.7. Solve $y/x^2 + 1 + y'/x = 0$. ∗

Solution. We have

$$\left(\frac{y}{x^2} + 1\right) dx + \frac{1}{x}\, dy = 0.$$

We see that

$$\left[\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{x^2}\right] \neq \left[\frac{1}{x^2} = \frac{\partial}{\partial y}\left(\frac{y}{x^2} + 1\right)\right].$$

So the equation is not exact. Multiplying by x^2 gives us

$$\begin{aligned} \left(y + x^2\right) dx + x\, dy &= 0, \\ d\left(xy + \frac{x^3}{3}\right) &= 0, \\ xy + \frac{x^3}{3} &= C \end{aligned}$$

for some arbitrary constant C. Solving for y finally gives us

$$y = \frac{C}{x} - \frac{x^3}{3}.$$

◊

There is, in general, no algorithm for finding integrating factors. But the following may suggest where to look. It is important to be able to recognize common exact forms:

$$\begin{aligned} x\, dy + y\, dx &= d(xy)\,, \\ \frac{x\, dy - y\, dx}{x^2} &= d\left(\frac{y}{x}\right), \\ \frac{x\, dx + y\, dy}{x^2 + y^2} &= d\left(\frac{\ln\left(x^2 + y^2\right)}{2}\right), \\ \frac{x\, dy - d\, dx}{x^2 + y^2} &= d\left(\tan^{-1}\left(\frac{y}{x}\right)\right), \\ x^{a-1}y^{b-1}\left(ay\, dx + bx\, dy\right) &= d\left(x^a y^b\right). \end{aligned}$$

Example 2.8. Solve $\left(x^2y^2 + y\right)dx + \left(2x^3y - x\right)dy = 0$. $*$

Solution. Expanding, we have

$$x^2y^2\,dx + 2x^3y\,dy + y\,dx - x\,dy = 0.$$

Here, $a = 1$ and $b = 2$. Thus, we wish to use

$$d\left(xy^2\right) = y^2\,dx + 2xy\,dy.$$

This suggests dividing the original equation by x^2 which gives

$$y^2\,dx + 2xy\,dy + \frac{y\,dx - x\,dy}{x^2} = 0.$$

Therefore,

$$xy^2 + \frac{y}{x} = C, \quad x \neq 0,$$

where C is an arbitrary constant. Additionally, $y = 0$ on the domain $\mathbb{R}$ is a solution to the original equation. $\diamondsuit$

Example 2.9. Solve $y\,dx - x\,dy - \left(x^2 + y^2\right)dx = 0$. $*$

Solution. We have

$$\frac{y\,dx - x\,dy}{x^2 + y^2} - dx = 0,$$

unless $x = 0$ and $y = 0$. Now, it follows that

$$\begin{aligned}
-\tan^{-1}\left(\frac{y}{x}\right) - x &= C,\\
\tan^{-1}\left(\frac{y}{x}\right) &= -C - x,\\
\tan^{-1}\left(\frac{y}{x}\right) &= D - x, \quad (D = -C)\\
\frac{y}{x} &= \tan(D - x)\,,\\
y &= x\tan(D - x)\,,
\end{aligned}$$

where C is an arbitrary constant and the domain is

$$D - x \neq (2n + 1)\,\frac{\pi}{2}, \quad x \neq (2n + 1)\,\frac{\pi}{2}$$

for any integer n. Also, since the derivation of the solution is based on the assumption that $x \neq 0$, it is unclear whether or not 0 should be in the domain, i.e., does $y = x\tan(D - x)$ satisfy the equation when $x = 0$? We have $y - xy' -$

$\left(x^2 + y^2\right) = 0$. If $x = 0$ and $y = x\tan(D - x)$, then $y = 0$ and the equation is satisfied. Thus, 0 is in the domain. $\lozenge$

Proposition 2.10
Let $\omega = dg$. Then for any function $P : \mathbb{R} \to \mathbb{R}$, $P(g)$ is exact.

Proof. Let $Q = \int P(t)\,dy$. Then $d(Q(g)) = P(g)\,dg = P(g)\omega$. $\square$

To make use of Proposition 2.10, we can group together some terms of ω to get an expression you recognize as having an integrating factor and multiply the equation by that. The equation will now look like $dg + h = 0$. If we can find an integrating factor for h, it will not necessarily help, since multiplying by it might mess up the part that is already exact. But if we can find one of the form $P(g)$, then it will work.

Example 2.11. Solve $\left(x - yx^2\right) dy + y\,dx = 0$. $*$

Solution. Expanding, we have

$$\underbrace{y\,dx + x\,dy}_{d(xy)} - yx^2\,dy = 0.$$

Therefore, we can multiply teh equation by any function of xy without disturbing the exactness of its first two terms. Making the last term into a function of y alone will make it exact. So we multiply by $(xy)^{-2}$, giving us

$$\frac{y\,dx + x\,dy}{x^2y^2} - \frac{1}{y}\,dy = 0 \Longrightarrow -\frac{1}{xy} - \ln(|y|) = C,$$

where C is an arbitrary constant. Note that $y = 0$ on the domain $\mathbb{R}$ is also a solution. $\lozenge$

Given

$$M\,dx + N\,dy = 0, \tag{$*$}$$

we want to find I such that $IM\,dx + IN\,dy$ is exact. If so, then

$$\underbrace{\frac{\partial}{\partial x}(IN)}_{I_xN + IN_x} = \underbrace{\frac{\partial}{\partial y}(IM)}_{I_yM + IM_y}.$$

If we can find any particular solution $I(x, y)$ of the PDE

$$I_x N + I N_x = I_y M + I M_y, \qquad (**)$$

then we can use it as an integrating factor to find the general solution of $(*)$. Unfortunately, $(**)$ is usually even harder to solve than $(*)$, but as we shall see, there are times when it is easier.

Example 2.12. We could look for an I having only x's and no y's? For example, consider $I_y = 0$. Then

$$I_x N + I N_x = I M_y \text{ implies } \frac{I_x}{I} = \frac{M_y - N_x}{N}.$$

This works if $(M_y - N_x)/N$ happens to be a function of x alone. Then

$$I = e^{\int \frac{M_y - N_x}{N}\,dx}.$$

Similarly, we can also reverse the role of x and y. If $(N_x - M_y)/M$ happens to be a function of y alone, then

$$e^{\int \frac{N_x - M_y}{M}\,dy}$$

works. $*$

2.4 Linear First Order Equations

A first order linear equation $(n = 1)$ looks like

$$y' + P(x)y = Q(x).$$

An integrating factor can always be found by the following method. Consider

$$\begin{aligned} dy + P(x)y\,dx &= Q(x)\,dx, \\ \underbrace{(P(x)y - Q(x))}_{M(x,y)}\,dx + \underbrace{dy}_{N(x,y)} &= 0. \end{aligned}$$

We use the DE for the integrating factor $I(x, y)$. The equation $IM\,dx + IN\,dy$ is exact if

$$I_x N + I N_x = I_y M + I M_y.$$

In our case,

$$I_x + 0 = I_y\,(P(x)y - Q(x)) + IP(x). \qquad (*)$$

We need only one solution, so we look for one of the form $I(x)$, i.e., with $I_y = 0$. Then $(*)$ becomes

$$\frac{dI}{dx} = IP(x).$$

This is separable. So

$$\begin{aligned}
\frac{dI}{I} &= P(x)\,dx,\\
\ln(|I|) &= \int P(x)\,dx + C,\\
|I| &= e^{\int P(x)\,dx}, \quad e^x > 0\\
I &= e^{\int P(x)\,dx}.
\end{aligned}$$

We conclude that $e^{\int P(x)\,dx}$ is an integrating factor for $y' + P(x)y = Q(x)$.

Example 2.13. Solve $y' - (1/x)\,y = x^3$, where $x > 0$. $*$

Solution. Here $P(x) = -1/x$. Then

$$I = e^{\int P(x)\,dx} = e^{-\int \frac{1}{x}\,dx} = e^{-\ln(|x|)dx} = \frac{1}{|x|} = \frac{1}{x},$$

where $x > 0$. Our differential equation is

$$\frac{x\,dy - y\,dx}{x} = x^3\,dx.$$

Multiplying by the integrating factor $1/x$ gives us

$$\frac{x\,dy - y\,dx}{x^2} = x^2\,dx.$$

Then

$$\begin{aligned}
\frac{y}{x} &= \frac{x^3}{3} + C,\\
y &= \frac{x^3}{3} + Cx
\end{aligned}$$

on the domain $(0, \infty)$, where C is an arbitrary constant ($x > 0$ is given). $\Diamond$

In general, given $y' + P(x)y = Q(x)$, multiply by $e^{\int P(x)\,dx}$ to obtain

$$\underbrace{e^{\int P(x)\,dx}y' + e^{\int P(x)\,dx}P(x)y}_{d\left(ye^{\int P(x)\,dx}\right)/dx} = Q(x)e^{\int P(x)\,dx}.$$

Therefore,

$$\begin{aligned} ye^{\int P(x)\,dx} &= \int Q(x)e^{\int P(x)\,dx}\,dx + C, \\ y &= e^{-\int P(x)\,dx}\int Q(x)e^{\int P(x)\,dx}\,dx + Ce^{-\int P(x)\,dx}, \end{aligned}$$

where C is an arbitrary constant.

Example 2.14. Solve $xy' + 2y = 4x^2$. *

Solution. What should $P(x)$ be? To find it, we put the equation in standard form, giving us

$$y' + \frac{2}{x}y = 4x.$$

Therefore, $P(x) = 2/x$. Immediately, we have

$$I = e^{\int (2/x)dx} = e^{\ln\left(x^2\right)} = x^2.$$

Multiplying the equation by x^2 gives us

$$\begin{aligned} x^2y' + 2xy &= 4x^3, \\ x^2y &= x^4 + C, \\ y &= x^2 + \frac{C}{x^2}, \end{aligned}$$

where C is an arbitrary constant and $x \neq 0$. ◊

Example 2.15. Solve $e^{-y}\,dy + dx + 2x\,dy = 0$. *

Solution. This equation is linear with x as a function of y. So what we have is

$$\frac{dx}{dy} + 2x = -e^{-y},$$

where $I = e^{\int 2\,dy} = e^{2y}$. Therefore,

$$\begin{aligned} e^{2y}\frac{dx}{dy} + 2xe^{2y} &= -e^{y}, \\ xe^{2y} &= -e^{y} + C, \end{aligned}$$

where C is an arbitrary constant. We could solve explicitly for y, but it is messy. The domain is not easy to determine. ♢

2.5 Substitutions

In many cases, equations can be put into one of the standard forms discussed above (separable, linear, etc.) by a substitution.

Example 2.16. Solve $y'' - 2y' = 5$. *

Solution. This is a first order linear equation for y'. Let $u = y'$. Then the equation becomes

$$u' - 2u = 5.$$

The integration factor is then $I = e^{-\int 2\,dx} = e^{-2x}$. Thus,

$$\begin{aligned} u'e^{-2x} - 2ue^{-2x} &= 5e^{-2x}, \\ ue^{-2x} &= -\frac{5}{2}e^{-2x} + C, \end{aligned}$$

where C is an arbitrary constant. But $u = y'$, so

$$y = -\frac{5}{2}x + \frac{C}{2}e^{2x} + C_1 = -\frac{5}{2}x + C_1e^{2x} + C_2$$

on the domain $\mathbb{R}$, where C_1 and C_2 are arbitrary constants. ♢

We now look at standard substitutions.

2.5.1 Bernoulli Equation

The Bernoulli equation is given by

$$\frac{dy}{dx} + P(x)y = Q(x)y^{n}.$$

Let $z = y^{1-n}$. Then

$$\frac{dz}{dx} = (1-n)\,y^{-n}\frac{dy}{dx},$$

giving us

$$\begin{aligned}
y^{-n}\frac{dy}{dx} + P(x)y^{1-n} &= Q(x), \\
\frac{1}{1-n}\frac{dz}{dx} + P(x)z &= Q(x), \\
\frac{dz}{dx} + (1-n)\,P(x)z &= (1-n)\,Q(x),
\end{aligned}$$

which is linear in z.

Example 2.17. Solve $y' + xy = xy^3$. *

Solution. Here, we have $n = 3$. Let $z = y^{-2}$. If $y \neq 0$, then

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}.$$

Therefore, our equation becomes

$$\begin{aligned}
-\frac{y^3 z'}{2} + xy &= xy^3, \\
-\frac{z'}{2} + xy^{-2} &= x, \\
z' - 2xy &= -2x.
\end{aligned}$$

We can readily see that $I = e^{-\int 2x\,dx} = e^{-x^2}$. Thus,

$$\begin{aligned}
e^{-x^2}z' - 2xe^{-x^2} &= -2xe^{-x^2}, \\
e^{-x^2}z &= e^{-x^2} + C, \\
z &= 1 + Ce^{x^2},
\end{aligned}$$

where C is an arbitrary constant. But $z = y^{-2}$. So

$$y = \pm\frac{1}{\sqrt{1 + Ce^{x^2}}}.$$

The domain is

$$\begin{cases} \mathbb{R}, & C > -1, \\ |x| > \sqrt{-\ln(-C)}, & C \leq -1. \end{cases}$$

An additional solution is $y = 0$ on $\mathbb{R}$. ♢

2.5.2 Homogeneous Equations

Definition (Homogeneous function of degree n)
A function $F(x, y)$ is called *homogeneous of degree* n if $F(\lambda x, \lambda y) = \lambda^n F(x, y)$. For a polynomial, homogeneous says that all of the terms have the same degree.

Example 2.18. The following are homogeneous functions of various degrees:

$$
\begin{aligned}
3x^6 + 5x^4y^2 &\qquad \text{homogeneous of degree 6,} \\
3x^6 + 5x^3y^2 &\qquad \text{not homogeneous,} \\
x\sqrt{x^2 + y^2} &\qquad \text{homogeneous of degree 2,} \\
\sin\left(\frac{y}{x}\right) &\qquad \text{homogeneous of degree 0,} \\
\frac{1}{x+y} &\qquad \text{homogeneous of degree } -1.
\end{aligned}
$$
*

If F is homogeneous of degree n and G is homogeneous of degree k, then F/G is homogeneous of degree $n - k$.

Proposition 2.19
If F is homogeneous of degree 0, then F is a function of y/x.

Proof. We have $F(\lambda x, \lambda y) = F(x, y)$ for all λ. Let $\lambda = 1/x$. Then $F(x, y) = F(1, y/x)$. □

Example 2.20. Here are some examples of writing a homogeneous function of degree 0 as a function of y/x.

$$
\begin{aligned}
\frac{\sqrt{5x^2 + y^2}}{x} &= \sqrt{5 + \left(\frac{y}{x}\right)^2}, \\
\frac{y^3 + x^2y}{x^2y + x^3} &= \frac{(y/x)^3 + (y/x)}{(y/x) + 1}.
\end{aligned}
$$
*

Consider $M(x, y)\,dx + N(x, y)\,dy = 0$. Suppose M and N are both homogeneous and of the same degree. Then

$$\frac{dy}{dx} = -\frac{M}{N},$$

This suggests that $v = y/x$ (or equivalently, $y = vx$) might help. In fact, write

$$-\frac{M(x,y)}{N(x,y)} = R\left(\frac{y}{x}\right).$$

Then

$$\underbrace{\frac{dy}{dx}}_{v+x\frac{dv}{dx}} = R\left(\frac{y}{x}\right) = R(v).$$

Therefore,

$$x\frac{dv}{dx} = R(v) - v,$$
$$\frac{dv}{R(v) - v} = \frac{dx}{x},$$

which is separable. We conclude that if M and N are homogeneous of the same degree, setting $y = vx$ will give a separable equation in v and x.

Example 2.21. Solve $xy^2\,dy = \left(x^3 + y^3\right) dx$. $*$

Solution. Let $y = vx$. Then $dy = v\,dx + x\,dv$, and our equation becomes

$$xv^2x^2\left(v\,dx + x\,dv\right) = \left(x^3 + v^3x^2\right) dx,$$
$$x^3v^3\,dx + x^4v^2\,dv = x^3\,dx + v^3x^3\,dx.$$

Therefore, $x = 0$ or $v^2\,dv = dx/x$. So we have

$$\frac{v^3}{3} = \ln(|x|) + C = \ln(|x|) + \underbrace{\ln(|A|)}_{C} = \ln(|Ax|) = \ln(Ax).$$

where the sign of A is the opposite of the sign of x. Therefore, the general solution is $y = x\left(3\ln(Ax)\right)^{1/3}$, where A is a nonzero constant. Every $A > 0$ yields a solution on the domain $(0, \infty)$; every $A < 0$ yields a solution on $(-\infty, 0)$. In addition, there is the solution $y = 0$ on the domain $\mathbb{R}$. $\diamondsuit$

2.5.3 Substitution to Reduce Second Order Equations to First Order

A second order DE has the form

$$F(y'', y', y, x) = 0.$$

If it is independent of y, namely, $F(y'', y', x) = 0$, then it is really just a first order equation for y' as we saw in earlier examples.

Consider now the case where it is independent of x, namely, $F(y'', y', y) = 0$. Substitute $v = dy/dx$ for x, i.e., eliminate x between the equations

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0$$

and $v = dy/dx$. Then

$$\frac{d^2y}{dx^2} = \frac{dv}{dx} = \frac{dv}{dy}\frac{dy}{dx} = \frac{dv}{dy}v.$$

Therefore,

$$F\left(\frac{d^2y}{dx^2}, \frac{dy}{dx}, y\right) = 0 \rightsquigarrow F\left(\frac{dv}{dy}v, v, y\right) = 0.$$

This is a first order equation in v and y.

Example 2.22. Solve $y'' = 4\,(y')^{3/2}\,y$. *

Solution. Let $v = dy/dx$. Then

$$\frac{d^2y}{dx^2} = \frac{dv}{dy}v$$

and our equation becomes

$$\begin{aligned}
\frac{dv}{dy}v &= 4v^{3/2}y,\\
\frac{dv}{\sqrt{v}} &= 4y\,dy, \quad v \geq 0,\\
2\sqrt{v} &= 2y^2 + C_1,\\
\sqrt{v} &= y^2 + C_2,
\end{aligned}$$

where C_1 is an arbitrary constant and $C_2 = C_1/2$. But $v = dy/dx$, so we have

$$\begin{aligned}
\frac{dy}{dx} &= \left(y^2 + C_2\right)^2, \\
dx &= \frac{dy}{\left(y^2 + C_2\right)^2}, \\
x &= \int \frac{dy}{\left(y^2 + C_2\right)^2} \\
&= \begin{cases} \frac{1}{2C_2^{3/2}} \left(\tan^{-1}\left(\frac{y}{\sqrt{C_2}}\right) + \frac{\sqrt{C_2} y}{y^2 + C_2}\right) + C_3, & C_2 > 0, \\ -\frac{1}{3y^3} + C_3, & C_2 = 0, \\ -\frac{1}{2(-C_2)^{3/2}} \cdot \frac{y^2}{y^2 + C_2} + C_3, & C_2 < 0. \end{cases}
\end{aligned} \qquad \Diamond$$

Next consider second order linear equations. That is,

$$P(x)y'' + Q(x)y' + R(x)y = 0.$$

We can eliminate y by letting $y = e^v$. Then $y' = e^v v'$ and $y'' = e^v \left(v'\right)^2 + e^v v''$. The equation then becomes

$$P(x)\left(e^v \left(v'\right)^2 + e^v v''\right) + Q(x)e^v v' + R(x)e^v = 0,$$

which is a first order equation in v'.

Example 2.23. Solve $x^2 y'' + \left(x - x^2\right) y' - e^{2x} y = 0$. *

Solution. Let $y = e^v$. Then the equation becomes

$$x^2 e^v \left(v'\right)^2 + x^2 e^v v'' + \left(x - x^2\right) e^v v' - e^{2x} e^v = 0.$$

Write $z = v'$. Then

$$x^2 z' + x^2 z^2 + (1 - x)\, xz = e^{2x}.$$

Now we are on our own—there is no standard technique for this case. Suppose we try $u = xy$. Then $z = u/x$ and

$$z' = -\frac{u}{x^2} + \frac{1}{x} u'.$$

Then it follows that

$$xu' + u^2 - xu = e^{2x}.$$

This is a bit simpler, but it is still nonstandard. We can see that letting $u = se^x$

will give us some cancellation. Thus, $u' = s'e^x + se^x$ and our equation becomes

$$\begin{aligned} xs'e^x + \cancel{xse^x} + s^2e^{2x} - \cancel{xse^x} &= e^{2x}, \\ xs' + s^2e^x &= e^x, \\ xs' &= e^x\left(1 - s^2\right), \\ \frac{s'}{1-s^2} &= \frac{e^x}{x}, \\ \frac{1}{2}\ln\left(\left|\frac{1+s}{1-s}\right|\right) &= \int \frac{e^x}{x}\,dx. \end{aligned}$$

Working our way back through the subsitutions we find that $s = zxe^{-x}$ so our solution becomes

$$\frac{1}{2}\ln\left(\left|\frac{1+zxe^{-x}}{1-zxe^{-x}}\right|\right) = \int \frac{e^x}{x}\,dx.$$

Using algebra, we could solve this equation for z in terms of x and then integrate the result to get v which then determines $y = e^v$ as a function of x. The algebraic solution for z is messy and we will not be able to find a closed form expression for the antidervative v, so we will abandon the calculation at this point. In practical applications one would generally use power series techniques on equations of this form and calculate as many terms of the Taylor series of the solution as are needed to give the desired accuracy. ♢

Next consider equations of the form

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0.$$

If $c_1 = c_2 = 0$, the equation is homogeneous of degree 1. If not, try letting $\bar{x} = x - h$ and $\bar{y} = y - k$. We try to choose h and k to make the equation homogeneous. Since h and k are constants, we have $d\bar{x} = dx$ and $d\bar{y} = dy$. Then our equation becomes

$$(a_1\bar{x} + a_1h + b_1\bar{y} + b_1k + c_1)\,d\bar{x} + (a_2\bar{x} + a_2h + b_2\bar{y} + b_2k + c_2)\,d\bar{y} = 0.$$

We want $a_1h + b_1k = -c_1$ and $a_2h + b_2k = -c_2$. We can always solve for h and k, unless

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

So suppose

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0.$$

Then $(a_2, b_2) = m\,(a_1, b_1)$. Let $z = a_1 x + b_1 y$. Then $dz = a_1\,dx + b_1\,dy$. If $b_1 \neq 0$, we have

$$\begin{aligned} dy &= \frac{dz - a_1\,dx}{b_1}, \\ (z + c_1)\,dx + (mz + c_2)\,\frac{dz - a_1\,dx}{b_1} &= 0, \\ \left(z + c_1 + \frac{a_1}{b_1}\right) dx + \left(\frac{mz + c_2}{b_1}\right) dx &= 0, \\ b_1\,dx &= -\frac{mz + c_2}{z + c_1 + a_1/b_1}\,dz. \end{aligned}$$

This is a separable equation.

If $b_1 = 0$ but $b_2 \neq 0$, we use $z = a_2 x + b_2 y$ instead. Finally, if both $b_1 = 0$ and $b_2 = 0$, then the original equation is separable.

Chapter 3

Applications and Examples of First Order Ordinary Differential Equations

3.1 Orthogonal Trajectories

An equation of the form $f(x, y, C) = 0$ determines a family of curves, one for every value of the constant C. Figure 3.1 shows curves with several values of C. Differentiating $f(x, y, C) = 0$ with respect to x gives a second equation

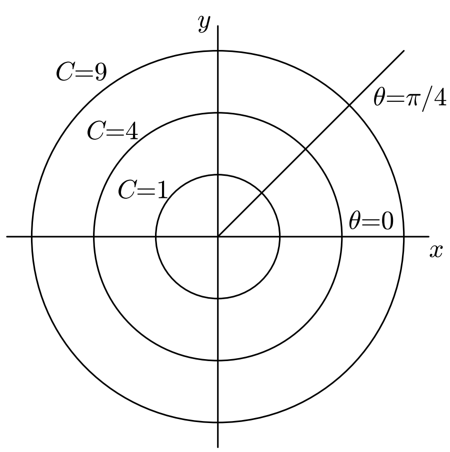


Figure 3.1: Graphs of $x^2 + y^2 - C = 0$ with various values of C.

$g(x, y, C, dy/dx) = 0$, which is explicitly given by

$$g = \frac{\partial f}{\partial x}(x, y, C) + \frac{\partial f}{\partial y}(x, y, C)\,\frac{dy}{dx}.$$

Example 3.1. We have

$$x^2 + y^3 + Cx^4 = 0 \text{ implies } 2x + 3y^2\frac{dy}{dx} + 4Cx^2 = 0.$$

We can eliminate C between the equations $f = 0$ and $g = 0$, that is,

$$f = 0 \Longrightarrow C = -\frac{x^2 + y^2}{x^4}.$$

Therefore,

$$g = 0 \Longrightarrow 2x - 3y^2\frac{dy}{dx} - 4\frac{x^2 + y^3}{x} = 0.$$

This yields a first order DE, where the solution includes the initial family of curves. *

So just as a first order DE usually determines a one-parameter family of curves (its solutions), a one-parameter family of curves determines a DE.

A coordinate system (Cartesian coordinates, polar coordinates, etc.) consists of two families of curves which always intersect at right angles. Two families of curves which always intersect at right angles are called *orthogonal trajectories* of each other. Given a one-parameter family of curves, we wish to find another family of curves which always intersects it at right angles.

Example 3.2. Find orthogonal trajectories of $y = C\sin(x)$. *

Solution. Figure 3.2 shows the plot of $y = C\sin(x)$ with several values of C. We have $C = y/\sin(x)$, so it follows that

$$\frac{dy}{dx} = \underbrace{C}_{y/\sin(x)}\cos(x) = y\frac{\cos(x)}{\sin(x)} = y\cot(x).$$

Two curves meet at right angles if the product of their slopes is -1, i.e., $m_{\text{new}} = -1/m_{\text{old}}$. So orthogonal trajectories satisfy the equation

$$\frac{dy}{dx} = -\frac{1}{y\cot(x)} = -\frac{\tan(x)}{y}.$$

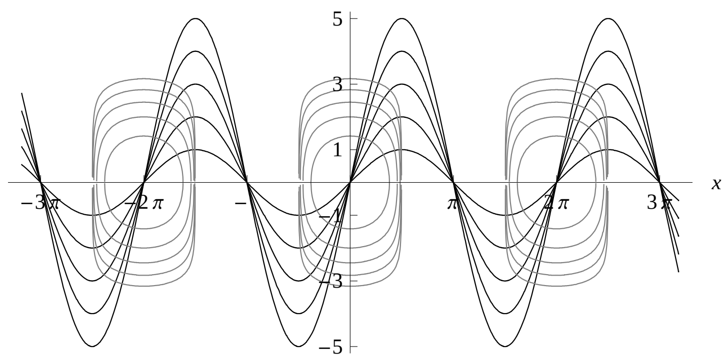


Figure 3.2: A plot of the family $y = C\sin(x)$ (in black) and its orthogonal trajectories (in gray).

This is a separable equation, and we quickly have

$$\begin{aligned} y\,dy &= -\tan(x), \\ \frac{y^2}{2} &= \ln(|\cos(x)|) + C, \\ y &= \pm\sqrt{2\ln(|\cos(x)|) + C_1}, \end{aligned}$$

where $C_1 = 2C$ is also an arbitrary constant. ◊

3.2 Exponential Growth and Decay

There are many situations where the rate of change of some quantity x is proportional to the amount of that quantity, i.e., $dx/dt = kx$ for some constant k. The general solution is $x = Ae^{kt}$.

1. *Radium gradually changes into uranium.* If $x(t)$ represents the amount of radium present at time t, then $dx/dt = kx$. In this case, $k < 0$.

2. *Bank interest.* The amount of interest you receive on your bank account is proportional to your balance. In this case, $k > 0$. Actually, you do not do quite this well at the bank because they compound only daily rather than continuously, e.g., if we measure this time in years, our model predicts that $x = x_0e^{kt}$, where x_0 is your initial investment and k is the interest rate. Actually, you get

$$x = x_0\left(1 + \frac{k}{n}t\right)^n,$$

where n is the number of interest periods throughout the year. Typically, $n = 365$ (daily interest), but the difference is small when $n = 365$. Note that $\lim_{n\to\infty}(1 + kt/n)^n = e^{kt}$.

3. *Population growth of rabbits.* The number of rabbits born at any time is roughly proportional to the number of rabbits present. If $x(t)$ is the number of rabbits at time t, then $dx/dt = kx$ with $k > 0$. Obviously, this model is not accurate as it ignores deaths.

Example 3.3. The half-life of radium is 1600 years, i.e., it takes 1600 years for half of any quantity to decay. If a sample initially contains 50 g, how long will it be until it contains 45 g? *

Solution. Let $x(t)$ be the amount of radium present at time t in years. Then we know that $dx/dt = kx$, so $x = x_0 e^{kt}$. With $x(0) = 50$, we quickly have $x = 50e^{kt}$. Solving for t gives $t = \ln(x/50)/k$. With $x(1600) = 25$, we have $25 = 50e^{1600k}$. Therefore,

$$1600k = \ln\left(\frac{1}{2}\right) = -\ln(2),$$

giving us $k = -\ln(2)/1600$. When $x = 45$, we have

$$\begin{aligned} t &= \frac{\ln(x/50)}{k} = \frac{\ln(45/50)}{-\ln(2)/1600} = -1600 \cdot \frac{\ln(8/10)}{\ln(2)} = 1600 \cdot \frac{\ln(10/8)}{\ln(2)} \\ &\approx 1600 \cdot \frac{0.105}{0.693} \approx 1600 \times 0.152 \approx 243.2. \end{aligned}$$

Therefore, it will be approximately 243.2 years until the sample contains 45 g of radium. ◊

3.3 Population Growth

Earlier, we discussed population growth where we considered only births. We now consider deaths also. Take, as our model, the following.

Let $N(t)$ be the number of people at time t. Assume that the land is intrinsically capable of supporting L people and that the rate of increase is proportional to both N and $L - N$. Then

$$\frac{dN}{dt} = kN(L - N),$$

and the solution is

$$N = \frac{L}{1 + (L/N_0 - 1)\, e^{-kLt}},$$

where k is the proportionality constant.

3.4 Predator-Prey Models

Example 3.4 (Predator-Prey). Consider a land populated by foxes and rabbits, where the foxes prey upon the rabbits. Let $x(t)$ and $y(t)$ be the number of rabbits and foxes, respectively, at time t. In the absence of predators, at any time, the number of rabbits would grow at a rate proportional to the number of rabbits at that time. However, the presence of predators also causes the number of rabbits to decline in proportion to the number of encounters between a fox and a rabbit, which is proportional to the product $x(t)y(t)$. Therefore, $dx/dt = ax - bxy$ for some positive constants a and b. For the foxes, the presence of other foxes represents competition for food, so the number declines proportionally to the number of foxes but grows proportionally to the number of encounters. Therefore $dy/dt = -cx + dxy$ for some positive constants c and d. The system

$$\begin{aligned} \frac{dx}{dt} &= ax - bxy, \\ \frac{dy}{dt} &= -cy + dxy \end{aligned}$$

is our mathematical model.

If we want to find the function $y(x)$, which gives the way that the number of foxes and rabbits are related, we begin by dividing to get the differential equation

$$\frac{dy}{dx} = \frac{-cy + dxy}{ax - bxy}$$

with $a, b, c, d, x(t), y(t)$ positive.

This equation is separable and can be rewritten as

$$\frac{(a - by)\, dy}{y} = \frac{(-c + dx)\, dx}{x}.$$

Integrating gives

$$a \ln(y) - by = -c \ln(x) + dx + C,$$

or equivalently

$$y^a e^{-by} = kx^{-c} e^{dx} \tag{3.1}$$

for some positive constant $k = e^C$.

The graph of a typical solution is shown in Figure 3.3.

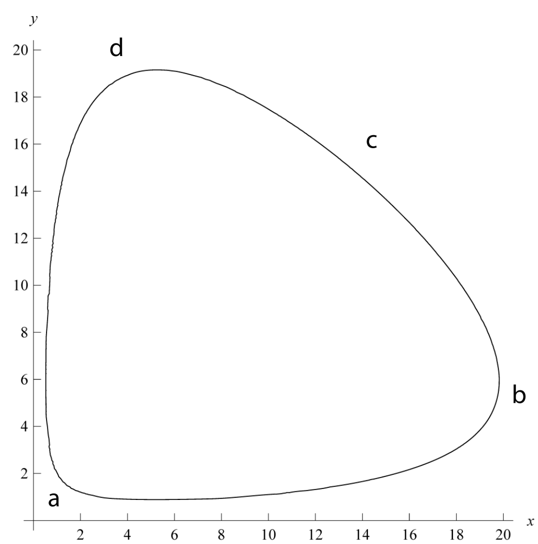


Figure 3.3: A typical solution of the Predator-Prey model with $a = 9.4$, $b = 1.58$, $c = 6.84$, $d = 1.3$, and $k = 7.54$.

Beginning at a point such as a, where there are few rabbits and few foxes, the fox population does not initially increase much due to the lack of food, but with so few predators, the number of rabbits multiplies rapidly. After a while, the point b is reached, at which time the large food supply causes the rapid increase in the number of foxes, which in turn curtails the growth of the rabbits. By the time point c is reached, the large number of predators causes the number of rabbits to decrease. Eventually, point d is reached, where the number of rabbits has declined to the point where the lack of food causes the fox population to decrease, eventually returning the situation to point a. $*$

3.5 Newton's Law of Cooling

Let T and T_s be the temperature of an object and its surroundings, respectively. Let T_0 and T_{s_0} be initial temperatures. Then Newton's Law of Cooling states that

$$\frac{dT}{dt} = k\left(T_s - T\right), \quad k > 0.$$

As T changes, the object gives or takes heat from its surroundings. We now have two cases.

CASE 1: *T_s is constant.* This occurs either because the heat given off is transported elsewhere or because the surroundings are so large that the contribution is negligible. In such a case, we have

$$\frac{dT}{dt} + kT = kT_s,$$

which is linear, and the solution is

$$T = T_0 e^{-kt} + T_s\left(1 - e^{-kt}\right).$$

CASE 2: *The system is closed.* All heat lost by the object is gained by its surroundings. We need to find T_s as a function of T. We have

$$\text{change in temperature} = \frac{\text{heat gained}}{\text{weight} \times \text{specific heat capacity}}.$$

Therefore,

$$\begin{aligned} \frac{T - T_0}{wc} &= -\frac{T_s - T_{s_0}}{w_s c_s}, \\ T_s &= T_{s_0} + \frac{wc}{w_s c_s}\left(T_0 - T\right), \\ \frac{dT}{dt} + k\left(1 + \frac{wc}{w_s c_s}\right)T &= k\left(T_{s_0} + \frac{wc}{w_s c_s}T_0\right). \end{aligned}$$

This is linear with different constants than the previous case. The solution is

$$T = T_0 + \left(\frac{T_{s_0} + \frac{wc}{w_s c_s}T_0}{1 + \frac{wc}{w_s c_s}}\right)\left(1 - e^{-k\left(1 + \frac{wc}{w_s c_s}\right)t}\right).$$

3.6 Water Tanks

Example 3.5. A tank contains a salt water solution consisting initially of 20 kg of salt dissolved into 10 ℓ of water. Fresh water is being poured into the tank at a rate of 3 ℓ/min and the solution (kept uniform by stirring) is flowing out at 2 ℓ/min. Figure 3.4 shows this setup. Find the amount of salt in the tank after 5 minutes. *

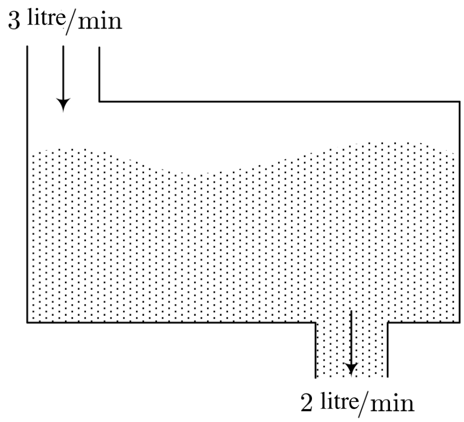


Figure 3.4: Fresh water is being poured into the tank as the well-mixed solution is flowing out.

Solution. Let $Q(t)$ be the amount (in kilograms) of salt in the tank at time t (in minutes). The volume of water at time t is

$$10 + 3t - 2t = 10 + t.$$

The concentration at time t is given by

$$\frac{\text{amount of salt}}{\text{volume}} = \frac{Q}{10+t}$$

kg per litre. Then

$$\frac{dQ}{dt} = -\,(\text{rate at which salt is leaving}) = -\frac{Q}{10+t} \cdot 2 = -\frac{2Q}{10+t}.$$

Thus, the solution to the problem is the solution of

$$\frac{dQ}{dt} = -\frac{2Q}{10+t}$$

evaluated at $t = 5$. We see that it is simply a separable equation.

To solve it, we have

$$\begin{aligned}
\frac{dQ}{Q} &= -2\frac{dt}{10+t}, \\
\int \frac{dQ}{Q} &= -2\int \frac{dt}{10+t}, \\
\ln(|Q|) &= -2\ln(|10+t|), \\
\ln(|Q|) &= \ln\Big(|10+t|^{-2}\Big) + C,
\end{aligned}$$

where C is a constant. Thus,

$$\begin{aligned}
\ln(|Q|) &= \ln\Big(|10+t|^{-2}\Big) + C, \\
&= \ln\Big(A\,|10+t|^{-2}\Big), \\
|Q| &= A\,|10+t|^{-2}
\end{aligned}$$

where $A = e^C$. But $Q \geq 0$ (we cannot have a negative amount of salt) and $t \geq 0$ (we do not visit the past), so we remove absolute value signs, giving us

$$Q = A\,(10+t)^{-2}.$$

Initially, i.e., at $t = 0$, we know that the tank contains 20 kg of salt. Thus, the initial condition is $Q(0) = 20$, and we have

$$Q(20) = 0 \Longrightarrow \frac{A}{(10+0)^2} = 20 \Longrightarrow \frac{A}{100} = 20 \Longrightarrow A = 2000.$$

Our equation therefore becomes

$$Q(t) = \frac{2000}{(10+t)^2}.$$

Figure 3.5 shows a plot of the solution. Evaluating at $t = 5$ gives

$$Q(5) = \frac{2000}{15^2} = \frac{80}{225} = \frac{80}{9} \approx 8.89.$$

Therefore, after 5 minutes, the tank will contain approximately 8.89 kg of salt.$\Diamond$

Additional conditions desired of the solution ($Q(0) = 20$) in the above example) are called *boundary conditions* and a differential equations together with boundary conditions is called a *boundary-value problem* (BVP).

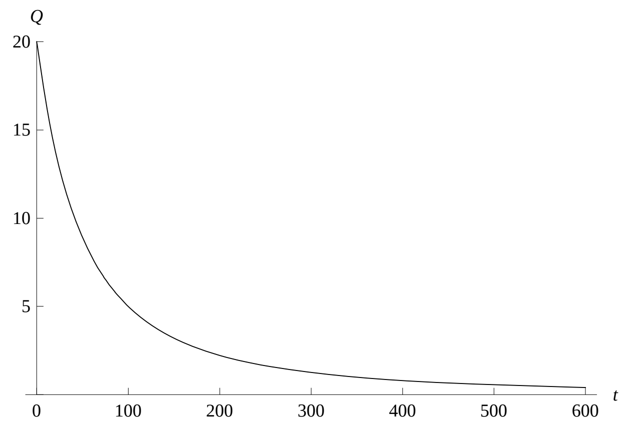


Figure 3.5: The plot of the amount of salt $Q(t)$ in the tank at time t shows that salt leaves slower as time moves on.

$A = \frac{A}{(10+t)^2}$ is the general solution to the DE $\frac{dQ}{dt} = -\frac{2dt}{10+t}$. $Q = \frac{2000}{(10+t)^2}$ is the solution to the boundary value problem $\frac{dQ}{dt} = -\frac{2dt}{10+t}$.; $Q(0) = 200$. Boundary-value problems like this one where the boundary conditions are initial values of various quantities are also called *initial-value problems* (IVP).

3.7 Motion of Objects Falling Under Gravity with Air Resistance

Let $x(t)$ be the height at time t, measured positively on the downward direction. If we consider only gravity, then

$$a = \frac{d^2x}{dt^2}$$

is a constant, denoted g, the acceleration due to gravity. Note that $F = ma = mg$. Air resistance encountered depends on the shape of the object and other things, but under most circumstances, the most significant effect is a force opposing the motion which is proportional to a power of the velocity. So

$$\underbrace{F}_{ma} = mg - kv^n$$

and

$$\frac{d^2x}{dt^2} = g - \frac{k}{m}\left(\frac{dx}{dt}\right)^n,$$

which is a second order DE, but there is no x term. So it is first order in x'. Therefore,

$$\frac{dv}{dt} = g - \frac{k}{m}v^n.$$

This is not easy to solve, so we will make the simplifying approximation that $n = 1$ (if v is small, there is not much difference between v and v^n). Therefore, we have

$$\begin{aligned}\frac{dv}{dt} &= g - \frac{k}{m}v,\\ \frac{dv}{dt} + \frac{k}{m}v &= g,\end{aligned}$$

The integrating factor is

$$I = e^{\int \frac{k}{m}\,dt} = e^{kt/m}.$$

Therefore,

$$\begin{aligned}\left(\frac{dv}{dt} + \frac{k}{m}v\right)e^{kt/m} &= ge^{kt/m},\\ e^{kt/m}v &= \frac{gm}{k}e^{kt/m} + C,\\ v &= \frac{mg}{k} + Ce^{-kt/m},\end{aligned}$$

where C is an arbitrary constant. Note that

$$v(0) = v_0 \Longrightarrow v_0 = \frac{mg}{k} + C \Longrightarrow C = v_0 - \frac{mb}{k}.$$

So we have

$$\underbrace{v}_{dx/dt} = \frac{mg}{k} + \left(v_0 - \frac{mg}{k}\right)e^{-kt/m}.$$

Note that

$$\underbrace{\int_{x_0}^{x} dx}_{x-x_0} = \int_0^t v\,dt = \frac{mg}{k}t - \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)\left(e^{-kt/m} - 1\right).$$

Thus, we finally have

$$x = x_0 + \frac{mg}{k}t + \frac{m}{k}\left(v_0 - \frac{mg}{k}\right)\left(1 - e^{-kt/m}\right).$$

3.8 Escape Velocity

Let $x(t)$ be the height of an object at time t measured positively upward (from the centre of the earth). Newton's Law of Gravitation states that

$$\underbrace{F}_{ma} = -\frac{kmM}{x^2},$$

where m is the mass of the object, M is the mass of the Earth, and $k > 0$. Note that

$$x'' = -\frac{kM}{x^2}.$$

We define the constant g known as the "acceleration due to gravity" by $x'' = -g$ when $x = R$, where R = radius of the earth. So $k = gR^2/M$. Therefore $x'' = -gR^2/x^2$. Since t does not appear, letting $v = dx/dt$ so that

$$\frac{d^2x}{dt^2} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v$$

will reduce it to first order. Thus,

$$\begin{aligned}
v\frac{dv}{dx} &= -\frac{gR^2}{x^2}, \\
v\,dv &= -\frac{gR^2}{x^2}\,dx, \\
\int v\,dv &= \int -\frac{gR^2}{x^2}\,dx, \\
\frac{v^2}{2} &= \frac{gR^2}{x} + C,
\end{aligned}$$

for some arbitrary constant C. Note that v decreases as x increases, and if v is not sufficiently large, eventually $v = 0$ at some height $x_{\max}$. So $C = -gR^2/x_{\max}$ and

$$v^2 = 2R^2g\left(\frac{1}{x} - \frac{1}{x_{\max}}\right).$$

Suppose we leave the surface of the Earth starting with velocity V. How far will we get? ($v = V$ when $x = R$). We have

$$\begin{aligned} V^2 &= 2R^2 g \left(\frac{1}{R} - \frac{1}{x_{\max}} \right), \\ \frac{V^2}{2R^2 g} &= \frac{1}{R} - \frac{1}{x_{\max}}, \\ \frac{1}{x_{\max}} &= \frac{1}{R} - \frac{V^2}{2R^2 g} = \frac{2Rg - V^2}{2R^2 g}, \\ x_{\max} &= \frac{2R^2 g}{2Rg - V^2}. \end{aligned}$$

That is, if $V^2 < 2Rg$, we will get as far as

$$\frac{2R^2 g}{2Rg - V^2}$$

and then fall back. To escape, we need $V \geq \sqrt{2Rg}$. Thus, the escape velocity is $\sqrt{2Rg} \approx 6.9\,\mathrm{mi/s}$, or roughly 25000 mph.

3.9 Planetary Motion

Newton's law of gravitation states that

$$\underbrace{F}_{m\mathbf{a}} = -\frac{km}{r^2}\mathbf{r}.$$

Suppose we let $a_r = -k/r^2$ and $a_0 = 0$. Then

$$a_r = a_x \cos(\theta) + a_y \sin(\theta), \qquad a_\theta = -a_x \sin(\theta) + a_y \cos(\theta).$$

Let $x = r\cos(\theta)$ and $y = r\sin(\theta)$. Then

$$\begin{aligned} x' &= r'\cos(\theta) - r\sin(\theta)\theta', \\ a_x = x'' &= r''\cos(\theta) - r'\sin(\theta)\theta' - r'\sin(\theta)\theta' - r\cos(\theta)\left(\theta'\right)^2 - r\sin(\theta)\theta'', \\ y' &= r'\sin(\theta) + r\cos(\theta)\theta', \\ a_y = y'' &= r''\sin(\theta) + r'\cos(\theta)\theta' + r'\cos(\theta)\theta' - r\sin(\theta)\left(\theta'\right)^2 + r\cos(\theta)\theta'' \\ &= r''\sin(\theta) + 2r'\theta'\cos(\theta) - \left(\theta'\right)^2 r\sin(\theta) + \theta'' r\cos(\theta). \end{aligned}$$

Now a_r becomes

$$\begin{aligned} a_r &= r'' \cos^2(\theta) - 2r'\theta' \sin(\theta)\cos(\theta) - (\theta')^2 r \cos^2(\theta) - \theta'' \sin(\theta)\cos(\theta) \\ &\quad + r'' \sin^2(\theta) + 2r'\theta' \sin(\theta)\cos(\theta) - (\theta')^2 r \sin^2(\theta) + \theta'' r \sin(\theta)\cos(\theta) \\ &= r'' - (\theta')^2 r \end{aligned}$$

and a_θ becomes

$$\begin{aligned} a_\theta &= -r'' \sin(\theta)\cos(\theta) + 2r'\theta' \sin^2(\theta) + (\theta')^2 r \sin(\theta)\cos(\theta) + \theta'' r \sin^2(\theta) \\ &\quad + r'' \sin(\theta)\cos(\theta) + 2r'\theta' \cos^2(\theta) - (\theta')^2 r \sin(\theta)\cos(\theta) + \theta'' r \cos^2(\theta) \\ &= 2r'\theta' + \theta'' r. \end{aligned}$$

Note that

$$r'' - (\theta')^2 r = \frac{k}{r^2}, \qquad (*)$$

$$2r'\theta' + \theta'' r = 0. \qquad (**)$$

Equation $(**)$ implies that*

$$\underbrace{2rr'\theta' + \theta'' r^2}_{d(r^2\theta')} = 0 \Longrightarrow r^2\theta' = h.$$

We want the path of the planet, so want an equation involving r and θ. Therefore, we eliminate t between Equation $(*)$ and $r^2\theta' = h$. Thus,

$$\begin{aligned} r' &= \frac{dr}{d\theta}\theta' = \frac{dr}{d\theta}\frac{h}{r^2}, \\ r'' &= \frac{d^2r}{d\theta^2}\theta' \frac{h}{r^2} - 2\frac{dr}{d\theta}\frac{h}{r^2} r' = \frac{d^2r}{d\theta^2}\frac{h}{r^2}\frac{h}{r^2} - 2\frac{dr}{d\theta}\frac{h}{r^3}\frac{dr}{d\theta}\frac{h}{r^2} \\ &= -\frac{h^2}{r^2}\left(\frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^2}\frac{d^2r}{d\theta^2}\right). \end{aligned}$$

*Note that

$$A = \int_{\theta_0}^{\theta}\int_0^{r(\theta)} r\,dr\,d\theta = \int_{\theta_0}^{\theta} \frac{r'(\theta)^2}{2}\,d\theta,$$

so $dA/d\theta = r(\theta)^2/2$. So in our case, $A' = h/2$. Therefore, $A = ht/2$, which is Kepler's Second Law (area swept out is proportional to time).

Let $u = 1/r$. Then

$$\begin{aligned}
\frac{du}{d\theta} &= -\frac{1}{r^2}\frac{dr}{d\theta}, \\
\frac{d^2u}{d\theta^2} &= \frac{2}{r^3}\left(\frac{dr}{d\theta}\right)^2 - \frac{1}{r^2}\frac{d^2r}{d\theta^2} = -\frac{r^2r''}{h^2} = -\frac{r''}{h^2u^2}.
\end{aligned}$$

Therefore, Equation $(*)$ implies that

$$\begin{aligned}
-h^2u^2\frac{d^2u}{d\theta^2} - \left(\frac{h}{r^2}\right)^2 r &= -\frac{k}{r^2}, \\
-h^2u^2\frac{d^2u}{d\theta^2} - u^3h^2 &= -ku^2, \quad u \neq 0, \\
\frac{d^2u}{d\theta^2} + u &= \frac{k}{h^2}.
\end{aligned}$$

Therefore,

$$\underbrace{u}_{1/r} = B\cos(\theta - \delta) + \frac{k}{h^2} = C_1\sin(\theta) + C_2\cos(\theta) + \frac{k}{h^2},$$

where

$$r = \frac{1}{k/h^2 + B\cos(\theta-\delta)} = \frac{1}{k/h^2 + C_1\sin(\theta) + C_2\cos(\theta)}.$$

Therefore,

$$\begin{aligned}
\overbrace{\frac{k}{h^2}r + C_1r\sin(\theta) + C_2r\cos(\theta)}^{\frac{k}{h^2}r + C_1x + C_2y} &= 1, \\
\frac{k}{h^2}r &= 1 - C_1x - C_2x, \\
\frac{k^2}{h^4}\left(x^2+y^2\right) &= \left(1 - C_1x - C_2y\right)^2,
\end{aligned}$$

which is the equation of an ellipse.

3.10 Particle Moving on a Curve

Let $x(t)$ and $y(t)$ denote the x and y coordinates of a point at time t, respectively. Pick a reference point on the curve and let $s(t)$ be the arc length of the piece of the curve between the reference point and the particle. Figure 3.6 illustrates this idea. Then

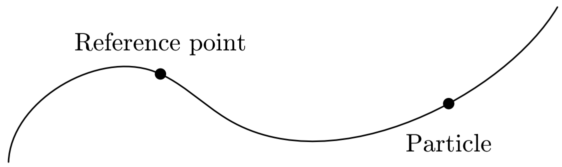


Figure 3.6: A particle moving on a curve.

$$v = \frac{ds}{dt}, \qquad \|\mathbf{a}\| = a = \frac{d^2 s}{dt^2}.$$

The vector $\mathbf{v}$ is in the tangent direction. The goal is to find components $\mathbf{a}$ in the tangential and normal directions. Let θ be the angle between the tangent line and the x axis. Then

$$\cos(\theta) = \frac{dx}{ds}, \qquad \sin(\theta) = \frac{dy}{ds}.$$

In the standard basis, we have $\mathbf{a} = (a_x, a_y)$, where

$$a_x = \frac{d^2 x}{dt^2}, \qquad a_y = \frac{d^2 y}{dt^2}.$$

To get components in $\mathbf{T}, \mathbf{n}$ basis, apply the matrix representing the change of basis from $\mathbf{T}, \mathbf{n}$ to the standard basis, i.e., rotation by $-\theta$, i.e.,

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} a_x \\ a_y \end{bmatrix} = \begin{bmatrix} a_x \cos(\theta) + a_y \sin(\theta) \\ -a_x \sin(\theta) + a_y \cos(\theta) \end{bmatrix},$$

i.e.,

$$\begin{aligned} a_{\mathbf{T}} &= a_x \cos(\theta) + a_y \sin(\theta) = a_x \frac{dx}{ds} + a_y \frac{dy}{ds} \\ &= \left(a_x \frac{dx}{dt} + a_y \frac{dy}{dt} \right) \frac{dt}{ds} = \frac{a_x v_x + a_y v_y}{v} \end{aligned}$$

and

$$a_n = -a_x \sin(\theta) + a_y \cos(\theta) = \frac{a_y v_x - a_x v_y}{v}.$$

Also, we have $v = \sqrt{v_x^2 + v_y^2}$, so

$$\frac{d^2 s}{dt^2} = \frac{dv}{dt} = \frac{2 v_x \frac{dv_x}{dt} + 2 v_y \frac{dv_y}{dt}}{2\sqrt{v_x^2 + v_y^2}} = \frac{v_x a_x + v_y a_y}{v} = a_{\mathbf{T}}.$$

But this was clear to start with because it is the component of **a** in the tangential direction that affects ds/dt.

We now have

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{v_y}{v_x},$$
$$y'' = \frac{v_x \frac{dv_y}{dx} - v_y \frac{dv_x}{dx}}{v_x^2} = \frac{v_x \frac{dv_y}{dt}\frac{dt}{dx} - v_y \frac{dv_x}{dt}\frac{dt}{dx}}{v_x^2} = \frac{v_x a_y - v_y a_x}{v_x^3} = \frac{a_{\mathbf{n}} v}{v_x^3},$$

where $a_{\mathbf{n}} = y'' v_x^3 / v$ and

$$v_x = \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{ds} v,$$

so

$$a_{\mathbf{n}} = \frac{y'' \left(\frac{dx}{ds}\right)^3 v^3}{v} = \frac{y'' v^2}{(ds/dx)^3} = \frac{y''}{\left(1 + (y')^2\right)^{3/2}} v^2.$$

Let

$$R = \frac{\left(1 + (y')^2\right)^{3/2}}{y''} \tag{3.2}$$

so that $a_{\mathbf{n}} = v^2/R$. The quantity R is called the *radius of curvature*. It is the radius of the circle which mostly closely fits the curve at that point. This is shown in Figure 3.7. If the curve is a circle, then R is a constant equal to the

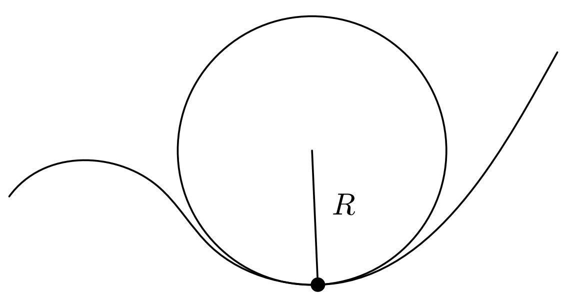


Figure 3.7: The radius of curvature is the circle whose edge best matches a given curve at a particular point.

radius.

Conversely, suppose that R is a constant. Then

$$Ry'' = \left(1 + (y')^2\right)^{3/2}.$$

Let $z = y'$. Then

$$Rz' = \left(1 + z^2\right)^{3/2}$$

and

$$dx = \frac{R\,dz}{(1+z^2)^{3/2}},$$

so

$$\begin{aligned}
x &= \int \frac{R}{(1+z^2)^{3/2}}\,dz = \frac{Rz}{\sqrt{1+z^2}} + A,\\
x - A &= \frac{R}{\sqrt{(1/z)^2+1}},\\
\left(\frac{1}{z}\right)^2 + 1 &= \frac{R^2}{(x-A)^2},\\
\left(\frac{1}{z}\right)^2 &= \frac{R^2}{(x-A)^2} - 1 = \frac{R^2-(x-A)^2}{(x-A)^2},\\
z^2 &= \frac{(x-A)^2}{R^2-(x-A)^2},\\
\underbrace{z}_{dy/dx} &= \pm\frac{x-A}{\sqrt{R^2-(x-A)^2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= \pm\int \frac{x-A}{\sqrt{R^2-(x-A)^2}}\,dx\\
&= \pm\sqrt{R^2-(x-A)^2} + B,\\
(y-B)^2 &= R^2-(x-A)^2,\\
(x-A)^2 + (y-B)^2 &= R^2.
\end{aligned}$$

Suppose now that the only external force $\mathbf{F}$ acting on the particle is gravity, i.e., $\mathbf{F} = (0, -mg)$. Then in terms of the $\mathbf{T}, \mathbf{n}$ basis, the components of $\mathbf{F}$ are

$$\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 0 \\ -mg \end{bmatrix} = \begin{bmatrix} -mg\sin(\theta) \\ -mg\cos(\theta) \end{bmatrix} = \begin{bmatrix} -mg\frac{dy}{ds} \\ -mg\frac{dx}{ds} \end{bmatrix}.$$

Equating the tangential components gives us

$$\begin{aligned}
\cancel{m}\frac{d^2s}{dt^2} &= -\cancel{m}g\frac{dy}{ds},\\
\frac{ds}{dt} &= v,\\
\frac{d^2s}{dt^2} &= \frac{dv}{dt} = \frac{ds}{dt} = \frac{dv}{ds}v,\\
\frac{dv}{ds}v &= -g\frac{dy}{ds},\\
v\,dv &= -g\,dy,\\
\frac{v^2}{2} &= -g(y-y_0) = g(y_0-y),\\
\frac{ds}{dt} &= v = \pm\sqrt{2g(y_0-y)}.
\end{aligned}$$

To proceed, we must know $y(s)$, which depends on the curve. Therefore,

$$\begin{aligned}
dt &= \pm\frac{ds}{\sqrt{2g(y_0-y(s))}},\\
t &= \pm\int\frac{ds}{\sqrt{2g(y_0-y(s))}}.
\end{aligned}$$

If the ma_n ever becomes greater than the normal component of $\mathbf{F}$, the particle will fly off the curve. This will happen when

$$\begin{aligned}
\cancel{m}\frac{y''v_x^3}{v} &= -\cancel{m}g\frac{dx}{ds} = -g\frac{dx}{dt}\frac{dt}{ds} = -g\frac{v_x}{v},\\
y''v_x^2 &= -g,\\
v_x &= \frac{dx}{dt} = \frac{dx}{ds}\frac{ds}{dt} = \frac{dx}{ds}v.
\end{aligned}$$

Therefore,

$$\left(\begin{aligned}
-\cancel{g} &= y''\left(\frac{dx}{ds}\right)^2 v^2 = \frac{y''v_2}{(ds/dx)^2}\\
&= \frac{y''v^2}{1+(y')^2} = \frac{y''2\cancel{g}(y_0-y)}{1+(y')^2}
\end{aligned}\right) \Longrightarrow 2y''(y-y_0) = 1+(y')^2.$$

Example 3.6. A particle is placed at the point $(1,1)$ on the curve $y = x^3$ and released. It slides down the curve under the influence of gravity. Determine whether or not it will ever fly off the curve, and if so, where. *

Solution. First note that

$$y = x^3, \qquad y' = 3x^2, \qquad y'' = 6x, \qquad y_0 = 1.$$

Therefore,

$$\begin{aligned} 2y''(y - y_0) &= 1 - (y')^2, \\ 12x\left(x^3 - 1\right) &= 1 + 9x^4, \\ 12x^4 - 12x &= 1 + 9x^4, \\ 3x^4 - 12x - 1 &= 0. \end{aligned} \qquad \Diamond$$

Solving for the relevant value of x gives us $x \approx -0.083$, which corresponds to $y \approx -0.00057$. Therefore, the particle will fly off the curve at $(-0.083, -0.00057)$. Figure 3.8 shows the graph.

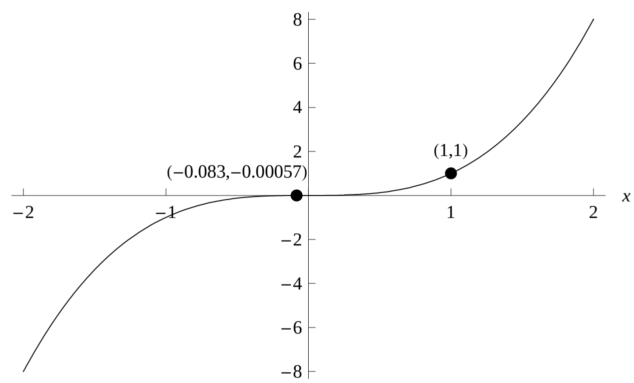


Figure 3.8: A particle released from $(1, 1)$, under the influence of gravity, will fly off the curve at approximately $(-0.083, -0.00057)$.

Chapter 4

Linear Differential Equations

Although we have dealt with a lot of manageable cases, first order equations are difficult in general. But second order equations are much worse. The most manageable case is linear equations. We begin with the general theory of linear differential equations. Specific techniques for solving these equations will be given later.

Definition (nth order linear differential equation)
An nth order linear differential equation is an equation of the form*

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + P_2(x)\frac{d^{n-2}y}{dx^{n-2}} + \cdots + P_n(x)y = Q(x).$$

An nth order linear equation can be written as a linear system (see Chapter 9) as follows. Let

$$y_1(x) = y(x), \quad y_2(x) = \frac{dy}{dx}, \quad y_3(x) = \frac{d^2y}{dx^2}, \quad \ldots, \quad y_n(x) = \frac{d^{n-1}y}{dx^{n-1}}.$$

Then

$$\frac{dy_1}{dx} = y_2, \quad \frac{dy_2}{dx} = y_3, \quad \ldots, \quad \frac{dy_{n-1}}{dx} = y_n$$

*If the coefficient of $d^n y/dx^n$ is not 1, we can divide the equation through this coefficient to write it in *normal form*. A linear equation is in normal form when the coefficient of $y^{(n)}$ is 1.

and we have

$$\begin{aligned}\frac{dy_n}{dx} = \frac{d^n y}{dx^n} &= -P_1(x)\frac{d^{n-1}y}{dx^{n-1}} - \cdots - P_n(x)y + Q(x)\\ &= -P_1(x)y_n - \cdots - P_n(x)y_1 + Q(x).\end{aligned}$$

Therefore,

$$\frac{d\mathbf{Y}}{dx} = \mathbf{A}(x)\mathbf{Y} + \mathbf{B}(x),$$

where

$$\mathbf{A}(x) = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -P_n(x) & -P_{n-1}(x) & -P_{n-2}(x) & \cdots & -P_1(x) \end{bmatrix}$$

and

$$\mathbf{B}(x) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ Q(x) \end{bmatrix}.$$

Theorem 4.1
If $P_1(x), \ldots, P_n(x), Q(x)$ are continuous throughout some interval I and x_0 is an interior point of I, then for any numbers $W_0, \ldots, W_{n-1}$, there exists a unique solution* throughout I of

$$\frac{d^n y}{dx^n} + P_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + P_n(x)y = Q(x)$$

satisfying

$$y(x_0) = w_0, \quad y'(x_0) = w_1, \quad \ldots, \quad y^{(n-1)}(x_0) = w_{n-1}.$$

Proof. The proof is given in Chapter 10. □

4.1 Homogeneous Linear Equations

To begin, consider the case where $Q(x) \equiv 0$ called the *homogeneous case*, where

$$y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \cdots + P_n(x)y = 0. \tag{4.1}$$

Proposition 4.2

1. If $r(x)$ is a solution of Equation (4.1), then so is $ar(x)$ for all $r \in \mathbb{R}$.
2. If $r(x)$ and $s(x)$ are solutions of Equation (4.1), then so is $r(x)+s(x)$.

Proof.

1. Substituting $y = ar$ into the LHS gives

$$ar^{(n)} + P_1(x)ar^{(n-1)} + \cdots + P_n(x)ar = a\left(r^{(n)} + \cdots + P_n(x)r\right).$$

But since r is a solution, we have

$$a\left(r^{(n)} + \cdots + P_n(x)r\right) = a \cdot 0.$$

Therefore, ar is a solution.

2. Substituting $y = r + s$ in the LHS gives

$$\left(\begin{array}{c} r^{(n)} + s^{(n)} \\ +P_1(x)\left(r^{(n-1)} + \cdots + s^{(n-1)}\right) + \cdots \\ +P_n(x)\left(r+s\right) \end{array}\right) = \left(\begin{array}{c} r^{(n)} + \cdots + P_n(x)r \\ +s^{(n)} + \cdots + P_n(x)s \end{array}\right)$$
$$= 0 + 0 = 0.$$

Therefore, $r + s$ is a solution. □

Corollary 4.3
The set of solutions to Equation (4.1) forms a vector space.

Proof. Let V be the set of solutions to Equation (4.1) and let $y_1, y_2, \ldots, y_n \in V$. Further, let $x_0 \in \mathbb{R}$. Set

$$W(y_1, y_2, \ldots, y_n)(x) = \begin{vmatrix} y_1(x) & y_2(x) & \cdots & y_n(x) \\ y_1'(x) & y_2'(x) & \cdots & y_n'(x) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & y_2^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix},$$

the Wronskian of $y_1, y_2, \ldots, y_n$. Then

$$\frac{dW}{dx} = \overbrace{\begin{vmatrix} y_1'(x) & \cdots & y_n'(x) \\ y_1'(x) & \cdots & y_n'(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}}^{\text{two equal rows}} + \overbrace{\begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ y_1''(x) & \cdots & y_n''(x) \\ y_1''(x) & \cdots & y_n''(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x) & \cdots & y_n^{(n-1)}(x) \end{vmatrix}}^{\text{two equal rows}} + \cdots$$

$$+ \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix} = \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix}$$

$$= \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ -\sum_{k=1}^n P_k(x) y_1^{(n-k)} & \cdots & -\sum_{k=1}^n P_k(x) y_n^{(n-k)} \end{vmatrix}$$

$$= -\sum_{k=1}^n \left(P_k(x) \begin{vmatrix} y_1(x) & \cdots & y_n(x) \\ \vdots & \ddots & \vdots \\ y_1^{(n-2)}(x) & \cdots & y_n^{(n-2)}(x) \\ y_1^{(n)}(x) & \cdots & y_n^{(n)}(x) \end{vmatrix} \right) = -P_1(x)W.$$

Therefore, $W = Ae^{-\int P_1(x)\,dx}$. This is Abel's formula. There are two cases:

$$A = 0 \Longrightarrow W(x) \equiv 0,$$
$$A \neq 0 \Longrightarrow W(x) \neq x$$

for all x. ∎

Theorem 4.4
Let V be the set of solutions to Equation (4.1) and suppose that $y_1, y_2, \ldots, y_n \in V$. Then $y_1, y_2, \ldots, y_n$ are linearly independent if and only if $W \neq 0$.

Proof. Let V be the set of solutions to Equation (4.1) and suppose that $y_1, y_2, \ldots, y_n \in V$. Suppose that $y_1, y_2, \ldots, y_n$ are linearly dependent. Then there exists constants $c_1, c_2, \ldots, c_n$ such that

$$\begin{aligned} c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) &= 0, \\ c_1 y_1'(x) + c_2 y_2'(x) + \cdots + c_n y_n'(x) &= 0, \\ &\vdots \\ c_1 y_1^{(n-1)}(x) + c_2 y_2^{(n-1)}(x) + \cdots + c_n y_n^{(n-1)}(x) &= 0. \end{aligned}$$

Since there exists a nonzero solution for $(c_1, c_2, \ldots, c_n)$ of the equation, we have

$$M \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = 0,$$

so it follows that $|M| = 0 = W$.

Now suppose that $W = 0$. Select an $x_0 \in \mathbb{R}$. Since $W(x) \equiv 0$, we have $W(x_0) = 0$. Thus, there exists constants $c_1, c_2, \ldots, c_n$ not all zero such that

$$M \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix},$$

where

$$M = \begin{bmatrix} y_1(x_0) & \cdots & y_n(x_0) \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix},$$

that is,

$$\begin{aligned}
c_1y_1(x_0) + c_2y_2(x_0) + \cdots + c_ny_n(x_0) &= 0, \\
c_1y_1'(x_0) + c_2y_2'(x_0) + \cdots + c_ny_n'(x_0) &= 0, \\
&\vdots \\
c_1y_1^{(n-1)}(x_0) + c_2y_2^{(n-1)}(x_0) + \cdots + c_ny_n^{(n-1)}(x_0) &= 0.
\end{aligned}$$

Let $y(x) = c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x)$. Since $y_1, y_2, \ldots, y_n$ are solutions to Equation (4.1), so is f, and f satisfies

$$f(x_0) = 0, \quad f'(x_0) = 0, \quad f''(x_0) = 0, \quad \ldots, \quad f^{(n-1)}(x_0) = 0.$$

But $y = 0$ also satisfies these conditions, and by the uniqueness part of Theorem 4.1, it is the only function satisfying them. Therefore, $f \equiv 0$, that is,

$$c_1y_1(x_0) + c_2y_2(x_0) + \cdots + c_ny_n(x_0) \equiv 0.$$

Therefore, $y_1, y_2, \ldots, y_n$ are linearly dependent. □

Corollary 4.5
Let V be the set of solutions to Equation (4.1). Then we have $\dim(V) = n$.

Proof. Let V be the set of solutions to Equation (4.1). Pick a point $x_0 \in \mathbb{R}$. Then by Theorem 4.1, given any numbers $w_1, w_2, \ldots, w_n$, there exists a unique solution of Equation (4.1) satisfying $y^{(j-1)}(x_0)$ for $j = 1, 2, \ldots, n$. Let y_1 be such a solution with $(w_1, w_2, \ldots, w_n) = (0, 0, \ldots, 0)$, let y_2 be such a solution with $(w_1, w_2, \ldots, w_n) = (0, 1, \ldots, 0)$, etc., finally letting y_n be such a solution with $(w_1, w_2, \ldots, w_n) = (0, 0, \ldots, 1)$.

We claim that $y_1, y_2, \ldots, y_n$ are linearly independent. Suppose that

$$c_1y_1(x) + c_2y_2(x) + \cdots + c_ny_n(x) \equiv 0.$$

Then differentiating gives

$$c_1y_1^{(j)}(x) + c_2y_2^{(j)}(x) + \cdots + c_ny_n^{(j)}(x) \equiv 0$$

for all j. In particular,

$$c_1 y_1^{(j)}(x_0) + c_2 y_2^{(j)}(x_0) + \cdots + c_n y_n^{(j)}(x_0) \equiv 0$$

for all j. Therefore, $y_1, y_2, \ldots, y_n$ are linearly independent and $\dim(V)) \geq n$.

Conversely, let $y_1, \ldots, y_n$ be linearly independent. Then $W(y_1, \ldots, y_n)(x) \neq 0$ for all x. Suppose that $y \in V$ and let

$$M = \begin{bmatrix} y_1(x_0) & y_2(x_0) & \cdots & y_n(x_0) \\ y_1'(x_0) & y_2'(x_0) & \cdots & y_n'(x_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(x_0) & y_2^{(n-1)}(x_0) & \cdots & y_n^{(n-1)}(x_0) \end{bmatrix}.$$

Therefore, $|M| = |W(x_0)| \neq 0$. So there exists constants $(c_1, c_2, \ldots, c_n)$ such that

$$M \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} y(x_0) \\ y'(x_0) \\ \vdots \\ y^{(n-1)}(x_0) \end{bmatrix},$$

that is,

$$y^{(j)}(x_0) = c_1 y_1^{(j)} + c_2 y_2^{(j)} + \cdots + c_1 y_1^{(j)}$$

for $0 \leq i \leq n-1$. Let $f(x) = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$. Then f is a solution to Equation (4.1) and $y^{(j)}(x_0) = f^{(j)}(x_0)$ for $j = 0, \ldots, n-1$. By uniqueness, $y = f$, i.e., $y(x) \equiv c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$. This means that y is a linear combination of $y_1, \ldots, y_n$. So $y_1, \ldots, y_n$ forms a basis for V. Therefore, $\dim(V) = n$. □

We conclude that, to solve Equation (4.1), we need to find n linearly independent solutions $y_1, \ldots, y_n$ of Equation (4.1). Then the general solution to Equation (4.1) is $c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$.

Example 4.6. Solve $y'' + 5y' + 6y = 0$. ∗

Solution. It is easy to check that $y = e^{3x}$ and $y = e^{2x}$ each satisfies $y'' + 5y' + 6y = 0$. Therefore, the general solution is $c_1 e^{3x} + c_2 e^{2x}$, where c_1 and c_2 are arbitrary constants. ◊

4.1.1 Linear Differential Equations with Constant Coefficients

The next problem concerns how to find the n linearly independent solutions. For $n > 1$, there is no good method in general. There is, however, a technique which works in the special cases where all the coefficients are constants. Consider the constant coefficient linear equation

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0. \tag{4.2}$$

We first look for solutions of the form $y = e^{\lambda x}$. If $y = e^{\lambda x}$ is a solution, then

$$a_0 \lambda^n e^{\lambda x} + a_1 \lambda^{n-1} e^{\lambda x} + \cdots + a_n e^{\lambda x} = 0.$$

Now since $e^{\lambda x} \neq 0$, we have

$$a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n = 0.$$

Let

$$p(\lambda) = a_0 \lambda^n + a_1 \lambda^{n-1} + \cdots + a_n. \tag{4.3}$$

If Equation (4.3) has n distinct real roots $r_1, r_2, \ldots, r_n$, then we have n linearly independent solutions $e^{r_1 x}, e^{r_2 x}, \ldots, e^{r_n x}$. We would like one solution for each root like this.

Suppose that Equation (4.3) has a repeated real root r. Then we do not have enough solutions of the form $e^{\lambda x}$. But consider $y = xe^{rx}$. Then

$$\begin{aligned}
y' &= re^{rx} + rxe^{rx} = (rx + 1)\, e^{rx}, \\
y'' &= re^{rx} + r\,(rx + 1)\, e^{rx} = \left(r^2 x + 2r\right) e^{rx}, \\
y''' &= r^2 e^{rx} + r\left(r^2 x + 2r\right) e^{rx} = \left(r^3 x + 3r^2\right) e^{rx}, \\
&\ \vdots \\
y^{(n)} &= \left(r^n x + n r^{n-1}\right) e^{rx}.
\end{aligned}$$

Substituting $y = xe^{rx}$ in Equation (4.2) gives us

$$\begin{aligned}\sum_{k=0}^{n-1} a_k \left(r^{n-k}x + (n-k)\, r^{n-k}\right) &= \left(\sum_{k=0}^{n} a_k r^{n-k}\right) xe^{rx} \\ &\quad + \left(\sum_{k=0}^{n-1} a_k \,(n-k)\, r^{n-k-1}\right) e^{rx} \\ &= p(r)xe^{rx} + p'(r)e^{rx}.\end{aligned}$$

Since r is a repeated root of p, both $p(r) = 0$ and $p'(r) = 0$, that is,

$$\begin{aligned}p(\lambda) = (\lambda - r)^2\, q(\lambda) &\Longrightarrow p'(\lambda) = 2\,(\lambda - r)\, q(\lambda) + (\lambda - r)^2\, q'(\lambda) \\ &\Longrightarrow p'(r) = 0.\end{aligned}$$

Therefore, xe^{rx} is also a solution of Equation (4.2).

What about complex roots? They come in pairs, i.e., $r_1 = a + ib$ and $r_2 = a - ib$, so $e^{(a+ib)x}$ and $e^{(a-ib)x}$ satisfy Equation (4.2). But we want real solutions, not complex ones. But

$$\begin{aligned}e^{(a+ib)x} &= e^{ax}\left(\cos(bx) + i\sin(bx)\right), \\ e^{(a-ib)x} &= e^{ax}\left(\cos(bx) - i\sin(bx)\right).\end{aligned}$$

Let

$$\begin{aligned}z_1 &= e^{ax}\left(\cos(bx) + i\sin(bx)\right), \\ z_2 &= e^{ax}\left(\cos(bx) - i\sin(bx)\right).\end{aligned}$$

If z_1 and z_2 are solutions, then

$$\frac{z_1 + z_2}{2} = e^{ax}\cos(bx), \qquad \frac{z_1 - z_2}{2} = e^{ax}\sin(bx)$$

are solutions. So the two roots $a + ib$ and $a - ib$ give two solutions $e^{ax}\cos(bx)$ and $e^{ax}\sin(bx)$. For repeated complex roots, use $xe^{ax}\cos(bx)$, $xe^{ax}\sin(bx)$, etc. So in all cases, we get n solutions.

The expression $c_1e^{ax}\cos(bx) + e^{ax}c_2\sin(bx)$ can be rewritten as follows. Let $A = \sqrt{c_1^2 + c_2^2}$. Then

$$\theta = \cos^{-1}\left(\frac{c_1}{\sqrt{c_1^2 + c_2^2}}\right),$$

that is,

$$\cos(\theta) = \frac{c_1}{\sqrt{c_1^2 + c_2^2}}, \qquad \sin(\theta) = \sqrt{1 - \cos^2(\theta)} = \frac{c_2}{\sqrt{c_1^2 + c_2^2}}.$$

Therefore,

$$\begin{aligned} c_1 e^{ax}\cos(bx) + e^{ax} c_2 \sin(bx) &= A e^{ax}\left(\frac{c_1}{A}\cos(bx) + \frac{c_2}{A}\sin(bx)\right) \\ &= A e^{ax}\left(\cos(\theta)\cos(bx) + \sin(\theta)\sin(bx)\right) \\ &= A e^{ax}\left(\cos(bx - \theta)\right). \end{aligned}$$

Example 4.7. Solve $y^{(5)} - 4y^{(4)} + 5''' + 6'' - 36y' + 40y = 0$. *

Solution. We have

$$\begin{aligned} \lambda^5 - 4\lambda^4 + 5\lambda^3 + 6\lambda^2 - 36\lambda + 40 &= 0, \\ (\lambda - 2)^2(\lambda + 2)\left(\lambda^2 - 2\lambda + 5\right) &= 0. \end{aligned}$$

Therefore, $\lambda = 2, 2, -2, 1 \pm 2i$ and

$$y = c_1 e^{2x} + c_2 x e^{2x} + c_3 e^{-2x} + c_4 e^{x}\cos(2x) + c_5 \sin(2x),$$

where c_1, c_2, c_3, c_4, c_5 are arbitrary constants. ◊

4.2 Nonhomogeneous Linear Equations

A nonhomogeneous linear equation is an equation of the form

$$y^{(n)} + p_1(x)y^{(n-1)} + p_2(x)y^{(n-2)} + \cdots + p_n(x)y = q(x). \tag{4.4}$$

Let Equation (4.1) (p. 47) be the corresponding homogeneous equation. If u and v are solutions of Equation (4.4), then $u - v$ is a solution of Equation (4.1). Conversely, if u is a solution of Equation (4.4) and v is a solution of Equation (4.1), then $u + v$ is a solution of Equation (4.4).

Let $y_1, y_2, \ldots, y_n$ be n linearly independent solutions of Equation (4.1) and let y_p be a solution of Equation (4.4). If y is any solution to Equation (4.4), then $y - y_p$ is a solution to Equation (4.1), so

$$y - y_p = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n.$$

Therefore,

$$y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n + y_p$$

is the general solution to Equation (4.4).

Example 4.8. Solve $y'' - y = x$. $*$

Solution. The corresponding homogeneous equation is $y'' - y = 0$. It then follows that $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$. Therefore,

$$y_1 = e^x, \qquad y_2 = e^{-x}.$$

By inspection, $y_p = x$ solves the given equation. Therefore, the general solution is

$$y = c_1 e^x + c_2 e^{-x} - x,$$

where c_1 and c_2 are arbitrary constants. $\Diamond$

We will consider more systematically how to find y_p later.

Chapter 5

Second Order Linear Equations

A second order linear equation is an equation of the form

$$y'' + P(x)y' + Q(x)y = R(x). \tag{5.1}$$

To find the general solution of Equation (5.1), we need

- *Two solutions:* y_1 and y_2 of the corresponding homogeneous equation $y'' + P(x)y' + Q(x) = 0$.
- *One solution:* y_p of Equation (5.1).

Then the general solution is $y = c_1y_1 + c_2y_2 + y_p$, where c_1 and c_2 are arbitrary constants.

Except in the case where $P(x)$ and $Q(x)$ are constants (considered earlier on p. 52), there is no general method of finding y_1, but

1. given y_1, there exists a method of finding y_2.
2. given y_1 and y_2, there exists a method of finding y_p.

We will discuss (1) first.

5.1 Reduction of Order

As mentioned earlier, we can find a second solution from the first. We first develop the tools we need.

Lemma 5.1
Let $f(x)$ be a twice differentiable function on a closed bounded interval J. If $\{x \in J : f(x) = 0\}$ is infinite, then there exists an $x_0 \in J$ such that $f(x_0) = 0$ and $f'(x_0) = 0$.

Proof. Let $S = \{x \in J : f(x) = 0\}$. If $|S|$ is infinite, then by the Bolzano-Weierstrass Theorem, S has an accumulation point x_0, i.e., there exists an $x_0 \in S$ such that every interval about x_0 contains another point of S. Therefore, by Rolle's Theorem, every interval about x_0 contains a point where f' is 0. Therefore, by continuity, $f'(x_0) = 0$.* □

Corollary 5.2
Let $y_1(x)$ be a solution of $y'' + P(x)y' + Q(x)y = 0$ on a closed bounded interval J. If y_1 is not the zero solution, then $y_1(x) = 0$ for at most finitely many x in J.

Proof. Let V be the set of solutions of $y'' + P(x)y' + Q(x)y = 0$ on J. Then V is a two-dimensional vector space. So if $y_1 \not\equiv 0$, then there exists a $y_2 \in V$ such that y_1 and y_2 are linearly independent. Therefore, $W(y_1, y_2) \neq 0$, i.e., $W(y_1, y_2)$ is never zero (the Wronskian is either always 0 or never 0). But if $y_1(x) = 0$ for infinitely many x in J, by Lemma 5.1, there exists an $x_0 \in J$ such that $y_1(x_0) = 0$ and $y_1'(x_0) = 0$. Therefore,

$$W(y_1, y_2)(x_0) = \begin{vmatrix} y_1(x_0) & y_2(x_0) \\ y_1'(x_0) & y_2'(x_0) \end{vmatrix} = \begin{vmatrix} 0 & y_2(x_0) \\ 0 & y_2'(x_0) \end{vmatrix} = 0.$$

This is a contradiction. Therefore, $y_1(x) = 0$ for at most finitely many x in J. □

Let y_1 a solution to $y'' + P(x)y' + Q(x)y = 0$ on a closed bounded interval J. We want to find a linearly independent second solution y_2.

We break J into a finite collection of subintervals so that $y_1(x) \neq 0$ on the interior of each subinterval. We will describe how to find the second solution y_2

*Note that f' is necessarily—but certainly not sufficiently—continuous if f' is differentiable.

on the interior of each interval. This will define y_2 throughout J except for a few isolated points.

Consider a subinterval I on which $y_1(x)$ is never zero. So $y_1(x) > 0$ or $y_2(x) < 0$ throughout I. We have

$$W(y_1, y_2) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}.$$

If y_2 is also a solution to $y'' + P(x)y' + Q(x)y = 0$, then $W = Ae^{-\int P(x)\,dx}$ for some constant A (Albel's formula).* Suppose we choose $A = 1$. Then

$$y_1 y_2' - y_1' y_2 = e^{-\int P(x)\,dx},$$

where y_1 is known. This is a linear first order equation for y_2. In standard form, we have

$$y_2' - \frac{y_1'}{y_1} y_2 = \frac{e^{-\int P(x)\,dx}}{y_1},$$

where $y_1 \neq 0$ throughout I. The integrating factor is

$$e^{\int -\frac{y_1'}{y_1}} = e^{-\ln(|y_1|)+C} = \frac{C'}{|y_1|},$$

where $C' = \pm 1$, depending on the sign of y_1. We use $1/y_1$. Therefore,

$$\begin{aligned} \frac{y_2'}{y_1} - \frac{y'}{y_1^2} y_2 &= \frac{e^{-\int P(x)\,dx}}{y_1^2}, \\ \frac{y_2}{y_1} &= \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx, \end{aligned}$$

finally giving us

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx. \tag{5.2a}$$

This defines y_2 on J except at a few isolated points where $y_1 = 0$. At those points, we have

$$y_1 y_2' - y_1' y_2 = e^{-\int P(x)\,dx},$$

so

$$y_2 = -\frac{e^{-\int P(x)\,dx}}{y_1'}. \tag{5.2b}$$

This defines y_2 are the boundary points of the subintervals.

*Different A's correspond to different y_2's.

Example 5.3. Observe that $y = x$ is a solution of

$$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = 0$$

and solve

$$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = xe^x.$$

*

Solution. Using the fact that $y_1 = x$, we have

$$\begin{aligned} y_2 &= x\int \frac{e^{-\int\left(-\frac{x+2}{x}\right)dx}}{x^2}\,dx = x\int \frac{e^{\int\left(1+\frac{2}{x}\right)dx}}{x^2}\,dx = x\int \frac{e^{x+2\ln(|x|)}}{x^2}\,dx \\ &= x\int \frac{e^x e^{2\ln(|x|)}}{x^2}\,dx = x\int \frac{e^x\left|x\right|^2}{x^2}\,dx = x\int e^x\,dx = xe^x. \end{aligned}$$

Therefore,

$$W = \begin{vmatrix} x & xe^x \\ 1 & e^x + xe^x \end{vmatrix} = xe^x + x^2e^x - xe^x = x^2e^x.$$

From this, we have

$$\begin{aligned} v_1' &= -\frac{y_2R}{W} = -\frac{xe^x xe^x}{x^2e^x} = -e^x \Longrightarrow v_1 = -e^x, \\ v_2' &= \frac{y_1R}{W} = \frac{xxe^x}{x^2e^x} = 1 \Longrightarrow v_2 = x. \end{aligned}$$

Therefore,

$$y_p = v_1y_1 + v_2y_2 = -xe^x + x^2e^x,$$

and the general solution is

$$y = C_1x + C_2xe^x - xe^x + x^2e^x = C_1x + C_2'xe^x + x^2e^x, \qquad x \neq 0,$$

where C_1, C_2, and C_2' are arbitrary constants. ◊

5.2 Undetermined Coefficients

Suppose that y_1 and y_2 are solutions of $y'' + P(x)y' + Q(x)y = 0$. The general method for finding y_p is called *variation of parameters*. We will first consider another method, the method of *undetermined coefficients*, which does not always work, but it is simpler than variation of parameters in cases where it does work.

The method of undetermined coefficients has two requirements, that

1. we have constant coefficients, i.e., $P(x) = p$ and $Q(x) = q$ for some constants p and q.

2. $R(x)$ is "nice".

Example 5.4. Solve $y'' - y = e^{2x}$. $*$

Solution. We have $\lambda^2 - 1 = 0$. So $y_1 = e^x$ and $y_2 = e^{-x}$. Since $R(x) = e^{2x}$, we look for a solution of the form $y = Ae^{2x}$. Then

$$y' = 2Ae^{2x}, \qquad y'' = 4Ae^{2x}.$$

Then according to the differential equation, we have

$$\underbrace{4Ae^{2x} - Ae^{2x}}_{3Ae^{2x}} = e^{2x} \Longrightarrow A = \frac{1}{3}.$$

Therefore, the general solution is

$$y = c_1 e^x + c_2 e^{-x} + \frac{1}{3} e^{2x},$$

where c_1 and c_2 are arbitrary constants. $\Diamond$

Undetermined coefficients works when $P(x)$ and $Q(x)$ are constants and $R(x)$ is a sum of the terms of the forms shown in Table 5.1.

Table 5.1: The various forms of y_p ought to be given a particular $R(x)$.

$R(x)$	Form of y_p
Cx^n	$A_0x^n + A_1x^{n-1} + \cdots + A_{n-1}x + A_n$
Ce^{rx}	Ae^{rx}
$Ce^{rx}\cos(kx)$ $Ce^{rx}\sin(kx)$	$Ae^{rx}\cos(kx) + Be^{rx}\sin(kx)$

The success of undetermined coefficients depends on the derivative of $R(x)$ having some basic form as $R(x)$. There is an easy way to deal with a complication: whenever a term in the trial y_p is already part of the general solution to the homogeneous equation, multiply it by x. For example, if we have $y'' - y = e^x$, as before, we have $y_1 = e^x$ and $y_2 = e^{2x}$. Thus, we try $y_p = Axe^{2x}$ and find A.

Example 5.5. Solve $y'' + 3y' + 2y = e^x - 3$. $*$

Solution. We have $\lambda^2 + 3\lambda + 2 = 0$. Factoring gives $(\lambda + 2)(\lambda + 1) = 0$, so $\lambda \in \{-1, -2\}$, and we have $y_1 = e^{-x}$ and $y_2 = e^{-2x}$. Let $y = Ae^x + B$. Then $y' = Ae^x$ and $y'' = Ae^x$, giving us

$$y'' + 3y' + 2y = Ae^x + 3Ae^x + 2Ae^x + 2B = 6Ae^x + 2B.$$

Comparing coefficients with $e^x - 3$, we have

$$6A = 1 \Longrightarrow A = \frac{1}{6},$$
$$2B = -3 \Longrightarrow B = -\frac{3}{2}.$$

Therefore, $y_p = e^x/6 - 3/2$, and we have

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{e^x}{6} - \frac{3}{2},$$

where c_1 and c_2 are arbitrary constants. ♢

Example 5.6. Solve $y'' + y = \sin(x)$. $*$

Solution. We have $\lambda^2 + 1 = 0$, so $\lambda = \pm i$, and it follows that $y_1 = \cos(x)$ and $y_2 = \sin(x)$. Let $y = Ax\sin(x) + Bx\cos(x)$. Then

$$\begin{aligned} y' &= A\sin(x) + Ax\cos(x) + B\cos(x) - Bx\sin(x) \\ &= (A - Bx)\sin(x) + (Ax + B)\cos(x), \\ y'' &= -B\sin(x) + (A - Bx)\cos(x) + A\cos(x) - (Ax + B)\sin(x) \\ &= (-Ax - 2B)\sin(x) + (2A - Bx)\cos(x). \end{aligned}$$

Therefore,

$$y'' + y = (-Ax - 2B)\sin(x) + (2A - Bx)\cos(x) + Ax\sin(x) + Bx\cos(x).$$

Comparing coefficients with $\sin(x)$, we see that

$$-2B = 1 \Longrightarrow B = -\frac{1}{2},$$
$$2A = 0 \Longrightarrow A = 0.$$

Therefore, $y_p = -x\cos(x)/2$, and the general solution is

$$y = c_1 \sin(x) + c_2 \cos(x) - \frac{x\cos(x)}{2},$$

where c_1 and c_2 are arbitrary constants. ♢

Example 5.7. Solve $y'' - 4y = x^2$. $*$

Solution. We have $\lambda^2 - 4 = 0$, so $\lambda = \pm 2$, and it follows that $y_1 = e^{2x}$ and $y_2 = e^{-2x}$. Let $y = Ax^2 + Bx + C$. Then $y' = 2Ax + B$ and $y'' = 2A$. Therefore,

$$y'' - 4y = 2A - 4Ax^2 - 4B - 4C = -4Ax^2 + 0x + (2A - 4B - 4C)\,.$$

Comparing coefficients with x^2, we immediately see that

$$\begin{aligned} -4A = 1 &\Longrightarrow A = -\frac{1}{4}, \\ -4B = 0 &\Longrightarrow B = 0, \\ 2A - 4C = 0 &\Longrightarrow C = \frac{A}{2} = \frac{-1/4}{2} = -\frac{1}{8}. \end{aligned}$$

Therefore, $y_p = -x^2/4 - 1/8$, and the general solution is

$$y = c_1 e^{2x} + c_2 e^{-2x} - \frac{x^2}{4} - \frac{1}{8},$$

where c_1 and c_2 are arbitrary constants. ♢

Example 5.8. Solve $y'' - y' = x$. $*$

Solution. We have $\lambda^2 - \lambda = 0$, so $\lambda \in \{1, 0\}$, and it follows that $y_1 = e^x$ and $y_2 = 1$. Let $y = Ax^2 + Bx$. Then $y' = 2Ax + B$ and $y'' = 2A$. Therefore,

$$y'' - y = 2A - 2Ax - B.$$

Comparing coefficients with x, we immediately see that

$$\begin{aligned} -2A = 1 &\Longrightarrow A = -\frac{1}{2}, \\ 2A - B = 0 &\Longrightarrow B = -1. \end{aligned}$$

Therefore, $y_p = -x^2/2 - x$, and the general solution is

$$y = c_1 e^x + c_2 - \frac{x^2}{2} - x,$$

where c_1 and c_2 are arbitrary constants. $\Diamond$

5.2.1 Shortcuts for Undetermined Coefficients

Suppose we have

$$y'' + ay' + by = R(x).$$

Let $p(\lambda) = \lambda^2 + a\lambda + b$. We now consider two cases for $R(x)$.

Case I: $R(x) = ce^{\alpha x}$. It follows that $y_p = Ae^{\alpha x}$, so $y' = \alpha Ae^{\alpha x}$ and $y'' = \alpha Ae^{\alpha x}$. Then

$$\begin{gathered} y'' + ay' + by = ce^{\alpha x} \\ \left(\alpha^2 + a\alpha + b\right) Ae^{\alpha x} = p(\alpha)Ae^{\alpha x}. \end{gathered}$$

Therefore, $A = c/p(\alpha)$, unless $p(\alpha) = c$.

What if $p(\alpha) = 0$? Then $e^{\alpha x}$ is a solution to the homogeneous equation. Let $y = Axe^{\alpha x}$. Then

$$\begin{aligned} y' &= Ae^{\alpha x} + A\alpha xe^{\alpha x}, \\ y'' &= A\alpha e^{\alpha x} + A\alpha e^{\alpha x} + A\alpha^2 xe^{\alpha x} = 2A\alpha e^{\alpha x} + A\alpha^2 xe^{\alpha x}. \end{aligned}$$

Thus,

$$\begin{aligned} y'' + y' + y &= A\left[\left(\alpha^2 x + a\alpha x + b\right) + 2\alpha + a\right] e^{\alpha x} \\ &= A\left(p(\alpha)xe^{\alpha x} + p'(\alpha)e^{\alpha x}\right) \\ &= Ap'(\alpha)e^{\alpha x}. \end{aligned}$$

Therefore, $A = c/p'(\alpha)$, unless $p'(\alpha) = 0$ too.

If $p(\alpha) = 0$ and $p'(\alpha) = 0$, then $e^{\alpha x}$ and $xe^{\alpha x}$ both solve the homogeneous equation. Letting $y = Ax^2e^{\alpha x}$, we have $A = c/p''(\alpha)$.

Case II: $R(x) = ce^{rx}\cos(bx)$ or $ce^{rx}\sin(bx)$. We use complex numbers. Set $\alpha = r + ib$. Then

$$ce^{\alpha x} = ce^{rx}\cos(bx) + ice^{rx}\sin(bx).$$

Let $z = y + iw$. Consider the equation

$$z'' + az' + bz = ce^{\alpha x}. \tag{5.3}$$

The real part and imaginary parts are

$$y'' + ay' + by = ce^{rx}\cos(bx), \tag{5.4a}$$

$$w'' + aw' + bw = ce^{rx}\sin(bx), \tag{5.4b}$$

respectively. The solutions of Equation (5.3,5.4a,5.4b) are then

$$z = \frac{ce^{\alpha x}}{p(\alpha)}, \qquad y = \mathrm{Re}\left(\frac{ce^{\alpha x}}{p(\alpha)}\right), \qquad w = \mathrm{Im}\left(\frac{ce^{\alpha x}}{p(\alpha)}\right),$$

respectively.

Example 5.9. Solve $y'' + 8y' + 12y = 3\cos(2x)$. $*$

Solution. We have $p(\lambda) = \lambda^2 + 8\lambda + 12 = (\lambda+2)(\lambda+6)$. Therefore, $\lambda \in \{-2, -6\}$ and it follows that $y_1 = e^{-2x}$ and $y_2 = e^{-6x}$. So

$$\begin{aligned} z &= \frac{3e^{2ix}}{p(2i)} = \frac{3e^{2ix}}{(2i)^2 + 16i + 12} = \frac{3e^{2ix}}{-4 + 16i + 12} \\ &= \frac{3e^{2ix}}{8 + 16i} = \frac{3}{8} \cdot \frac{1}{1+2i} e^{2ix} = \frac{3}{8} \cdot \frac{1-2i}{(1-2i)(1+2i)} e^{2ix} \\ &= \frac{3}{8} \cdot \frac{1-2i}{1+2i} e^{2ix} = \frac{3}{40}(1-2i)\, e^{2ix} \\ &= \frac{3}{40}(1-2i)(\cos(2x) + i\sin(2x)) \\ &= \frac{3}{40}\left[\cos(2x) + 2\sin(2x) + i\left(-2\cos(2x) + \sin(2x)\right)\right]. \end{aligned}$$

Therefore,

$$\mathrm{Re}(z) = \frac{3}{40}(\cos(2x) + 2\sin(2x)), \qquad \mathrm{Im}(z) = \frac{3}{40}(-2\cos(x) + \sin(2x)).$$

Hence, the solution is

$$y = c_1 e^{-2x} + c_2 e^{-6x} + \frac{3}{40}(\cos(2x) + 2\sin(2x)),$$

where c_1 and c_2 are arbitrary constants. $\Diamond$

We conclude that there is no shortcut when $R(x)$ is a polynomial or any multiple of a polynomial.

5.3 Variation of Parameters

Consider the equation

$$y'' + P(x)y' + Q(x)y = R(x), \tag{I}$$

where $P(x)$ and $Q(x)$ are not necessarily constant. Let y_1 and y_2 be linearly independent solutions of the auxiliary equation

$$y'' + P(x)y' + Q(x)y = 0. \tag{H}$$

So for any constants C_1 and C_2, $C_1y_1 + C_2y_2$ satisfies Equation (H).

The idea of *variation of parameters* is to look for functions $v_1(x)$ and $v_2(x)$ such that $v_1y_1 + v_2y_2$ satisfies Equation (I). Let y_1 and y_2. If we want y to be a solution to Equation (I), then substitution into Equation (I) will give one condition on v_1 and v_2.

Since there is only one condition to be satisfied and two functions to be found, there is lots of choice, i.e., there are many pairs v_1 and v_2 which will do. So we will arbitrarily decide only to look for v_1 and v_2 which also satisfy $v_1'y_1 + v_2'y_2 = 0$. We will see that this satisfaction condition simplifies things. So consider

$$\begin{aligned}
y &= v_1y_1 + v_2y_2, \\
y' &= v_1y_1 + v_2y_2' + v_1'y_1 + v_2'y_2 = v_1y_1' + v_2y_2', \\
y'' &= v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2'.
\end{aligned}$$

To solve Equation (I) means that

$$\begin{pmatrix} v_1y_1'' + v_2y_2'' + v_1'y_1' + v_2'y_2' + P(x)v_1y_1' \\ +P(x)v_2y_2' + Q(x)v_1y_1 + Q(x)v_2y_2 \end{pmatrix} = R(x),$$

$$\begin{pmatrix} v_1\left(y_1'' + P(x)y_1' + Q(x)y_1\right) \\ +v_2\left(y_2'' + P(x)y_2' + Q(x)y_2\right) \\ +v_1'y_1' + v_2'y_2' \end{pmatrix} = R(x).$$

Therefore,

$$\begin{aligned}
v_1'y_1' + v_2'y_2' &= R(x), \\
v_1'y_1' + v_2'y_2' &= 0,
\end{aligned}$$

and it follows that

$$v_1' = -\frac{y_2 R}{W}, \qquad v_2' = \frac{y_1 R}{W},$$

where

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} \neq 0,$$

i.e., it is never zero regardless of x, since y_1 and y_2 are linearly independent.

Example 5.10. Solve $y'' - 5y' + 6y = x^2 e^{3x}$. *

Solution. The auxiliary equation is

$$y'' - 5y' + 6y = 0 \rightsquigarrow \lambda^2 - 5\lambda + 6 = 0 \Longrightarrow \lambda = 3, 2.$$

Therefore, the solutions are $y_1 = e^{3x}$ and $y_2 = e^{2x}$. Now,

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^{3x} & e^{2x} \\ 3e^{3x} & 2e^{2x} \end{vmatrix} = 2e^{5x} - 3e^{5x} = -e^{5x}.$$

Therefore,

$$v_1' = -\frac{y_2 R}{W} = -\frac{e^{2x} x^2 e^{3x}}{-e^{5x}} = x^2 \Longrightarrow v_1 = \frac{x^3}{3}.$$

We can use any function v_1 with the right v_1', so choose the constant of integration to be 0.

For v_2, we have

$$v_2' = \frac{y_1 R}{W} = -\frac{e^{3x} x^2 e^{3x}}{-e^{5x}} = -x^2 e^x,$$

so

$$\begin{aligned} v_2 &= -\underbrace{\int x^2 e^x \, dx}_{\text{by parts}} = -x^2 e^x + 2 \underbrace{\int x e^x \, dx}_{\text{by parts}} \\ &= -x^2 e^x + 2xe^x - 2 \int e^x \, dx \\ &= x^2 e^x + 2xe^x - 2e^x \\ &= -\left(x^2 - 2x + 2\right) e^x. \end{aligned}$$

Finally,

$$y_p = \frac{x^3}{3} e^{3x} - \left(x^2 - 2x + 2\right) e^x e^{2x} = e^{3x} \left(\frac{x^3}{3} - x^2 + 2x - 2 \right),$$

and the general solution is

$$\begin{aligned}
y &= C_1 e^{3x} + C_2 e^{2x} + e^{2x}\left(\frac{x^3}{3} - x^2 + 2x - 2\right) \\
&= e^{3x}\left(\frac{x^3}{3} - x^2 + 2x - 2 + C_1\right) + C_2 e^{2x} \\
&= \left(\frac{x^3}{3} - x^2 + 2x + C_1\right) e^{3x} + C_2 e^{2x},
\end{aligned}$$

where C_1 and C_2 are arbitrary constants. ♢

Example 5.11. Solve $y'' + 3y' + 2y = \sin(e^x)$. *

Solution. We immediately have $\lambda^2 + 3\lambda + 2 = 0$, which gives $\lambda \in \{-1, -2\}$. Therefore, $y_1 = e^{-x}$ and $y_2 = e^{-2x}$. Now,

$$W = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix} = -2e^{-3x} + e^{-3x} = -e^{-3x}.$$

Therefore,

$$v_1' = -\frac{e^{-2x}\sin(e^x)}{-e^{-3x}} = e^x \sin(e^x) \Longrightarrow v_1 = -\cos(e^x).$$

For v_2, we have

$$v_2' = \frac{e^{-x}\sin(e^x)}{-e^{-3x}} = -e^{2x}\sin(e^x).$$

Therefore,

$$\begin{aligned}
v_2 &= -\underbrace{\int e^{2x}\sin(e^x)\,dx}_{\text{substitute } t = e^x} = -\underbrace{\int t\sin(t)\,dt}_{\text{by parts}} \\
&= -\left(-t\cos(t) + \int \cos(t)\,dt\right) \\
&= -\left(-t\cos(t) + \sin(t)\right) \\
&= e^x\cos(e^x) - \sin(e^x).
\end{aligned}$$

Finally,

$$\begin{aligned} y_p &= v_1 y_1 + v_2 y_2 \\ &= -e^{-x}\cos(e^x) + e^{-2x}\left(e^x\cos(e^x) - \sin(e^x)\right) \\ &= -e^{-x}\cos(e^x) + e^{-x}\cos(e^x) - e^{-2x}\sin(e^x) \\ &= -e^{-2x}\sin(e^x)\,, \end{aligned}$$

and the general solution is

$$y = C_1 e^{-x} + C_2 e^{-2x} - e^{-2x}\sin(e^x)\,,$$

where C_1 and C_2 are arbitrary constants. ◊

Chapter 6

Applications of Second Order Differential Equations

We now look specifically at two applications of first order DE's. We will see that they turn out to be analogs to each other.

6.1 Motion of Object Hanging from a Spring

Figure 6.1 shows an object hanging from a spring with displacement d.

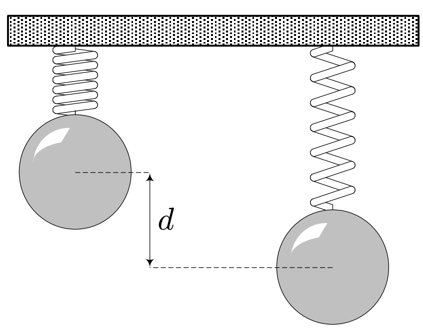


Figure 6.1: An object hanging from a spring with displacement d.

The force acting is gravity, spring force, air resistance, and any other external forces. Hooke's Law states that

$$\mathbf{F}_{\text{spring}} = kd, \tag{6.1}$$

where $\mathbf{F}$ is the spring force, d is the displacement from the spring to the natural length, and $k > 0$ is the spring constant. As we lower the object, the spring force increases. There is an equilibrium position at which the spring and gravity cancel. Let x be the distance of the object from this equilibrium position measured positively downwards. Let s be the displacement of the spring from the natural length at equilibrium. Then it follows that $ks = mg$.

If the object is located at x, then the forces are

1. gravity.
2. spring.
3. air resistance.
4. external.

Considering just gravity and the spring force, we have*

$$\mathbf{F}_{\text{gravity}} + \mathbf{F}_{\text{spring}} = mg - k\,(x + s) = -kx.$$

Suppose air resistance is proportional to velocity. Then

$$\mathbf{F}_{\text{air}} = -rv, \qquad r > 0.$$

In other words, the resistance opposes motion, so the force has the opposite sign of velocity. We will suppose that any external forces present depend only on time but not on position, i.e., $\mathbf{F}_{\text{external}} = F(t)$. Then we have

$$\underbrace{\mathbf{F}_{\text{total}}}_{ma} = -kx - rv + F(t),$$

so it follows that

$$m\frac{d^2x}{dt^2} + r\frac{dx}{dt} + kx = F(t). \tag{6.2}$$

Therefore, $m\lambda^2 + r\lambda + k = 0$ and we have

$$\lambda = \frac{-r \pm \sqrt{r^2 - 4mk}}{2m} = -\frac{r}{2m} \pm \sqrt{\left(\frac{r}{2m}\right)^2 - \frac{k}{m}}.$$

*Intuitively, when $x = 0$, they balance. Changing x creates a force in the opposite direction attempting to move the object towards the equilibrium position.

We now have three cases we need to consider.

CASE 1: $4mk > r^2$. Let $\alpha = r/2m$, let $\omega = \sqrt{k/m}$, and let

$$\beta = \sqrt{\frac{k}{m} - \left(\frac{r}{2m}\right)^2} = \sqrt{\omega^2 - \alpha^2}.$$

Then $\lambda = -\alpha \pm i\beta$ and

$$x = e^{-\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right) = Ae^{-\alpha t} \cos(\beta t - \theta), \tag{6.3}$$

where C_1 and C_2 are arbitrary constants. Figure 6.2a shows how this solution qualitatively looks like.

Damped vibrations is the common case. The oscillations die out in time, where the period is $2\pi/\beta$. If $r = 0$, then $\alpha = 0$. In such a case, there is no resistance, and it oscillates forever.

CASE 2: $4mk = r^2$. In such a case, we simply have

$$x = e^{-\frac{r}{2m}t}\left(C_1 + C_2 t\right), \tag{6.4}$$

where C_1 and C_2 are arbitrary constants. Figure 6.2b shows how this solution qualitatively looks like.

CASE 3: $4mk < r^2$. Let $a = r/2m$ and

$$b = \sqrt{\left(\frac{r}{2m}\right) - \frac{k}{m}}.$$

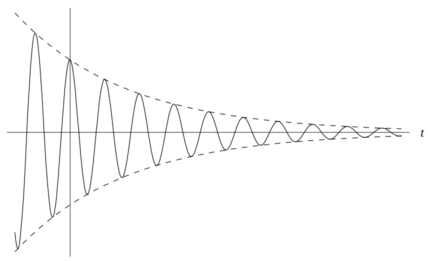
Then the solution is

$$x = C_1 e^{-(a-b)t} + C_2 e^{-(a+b)t}, \tag{6.5}$$

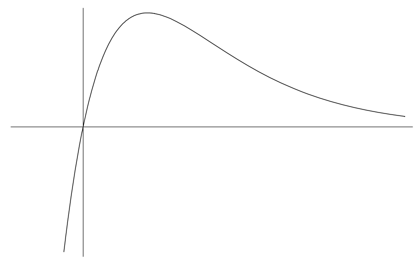
where C_1 and C_2 are arbitrary constants. This is the overdamped case. The resistance is so great that the object does not oscillate (imagine everything immersed in molasses). It will just gradually return to the equilibrium position. Figure 6.2c shows how this solution qualitatively looks like.

Consider Equation (6.2) again. We now consider special cases of $F(t)$, where

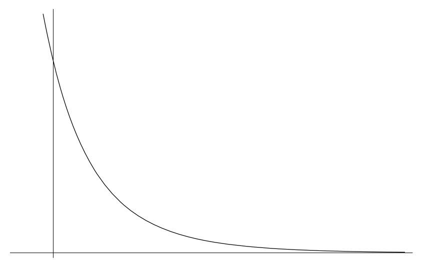
$$F(t) = \begin{cases} F_0 \cos(\omega t), \\ F_0 \sin(\omega t). \end{cases}$$



(a) $4mk > r^2$



(b) $4mk = r^2$



(c) $4mk < r^2$

Figure 6.2: The various cases for a spring-mass system.

Use undetermined coefficients to get the solution of the form

$$x_p = B\cos(\omega t) + C\sin(\omega t).$$

For $F(t) = F_0\cos(\omega t)$, we get

$$x_p = \frac{F_0}{\left(k - m\omega^2\right)^2 + \left(r\omega\right)^2}\left[(k - m\omega)\cos(\omega t) + r\omega\sin(\omega t)\right] = A\cos(\omega t - \delta)\,,$$

where

$$A = \frac{F_0}{\sqrt{\left(k - m\omega^2\right)^2 + \left(r\omega\right)^2}},$$
$$\cos(\delta) = \frac{k - m\omega^2}{\sqrt{\left(k - m\omega^2\right)^2 + \left(r\omega\right)^2}},$$
$$\sin(\delta) = \frac{r\omega}{\sqrt{\left(k - m\omega^2\right)^2 + \left(r\omega\right)^2}}.$$

For $F(t) = F_0\sin(\omega t)$, we get $x_p = A\sin(\omega t - \delta)$, where A and δ is as above.

In any solution

$$e^{-\alpha t}\,(\star) + A\sin(\omega t - \delta)\,,$$

the first term eventually dies out, leaving $A\sin(\omega t - \delta)$ as the steady state solution.

Here we consider a phenomenon known as *resonance*. The above assumes that $\cos(\omega t)$ and $\sin(\omega t)$ are not part of the solution to the homogeneous equation. Consider now

$$m\frac{d^2x}{dt^2} + kx = F_0\cos(\omega t),$$

where $\omega = \sqrt{k/m}$ (the external force frequency coincides with the natural frequency). Then the solution is*

$$x_p = \frac{F_0}{2\sqrt{km}}t\sin(\omega t).$$

Another phenomenon we now consider is *beats*. Consider $r = 0$ with ω near but not equal to $\sqrt{k/m}$. Let $\tilde{\omega} = \sqrt{k/m}$. Then the solution, with $x(0) = 0 = x'(0)$, is

$$\begin{aligned}
x &= \frac{F_0}{m\left(\tilde{\omega}^2 - \omega^2\right)}\cos(\omega t - \tilde{\omega} t) \\
&= -\frac{2F_0}{\left(\tilde{\omega} + \omega\right)\left(\tilde{\omega} - \omega\right)}\sin\left(\frac{\left(\omega - \tilde{\omega}\right)t}{2}\right)\sin\left(\frac{\left(\omega + \tilde{\omega}\right)t}{2}\right) \\
&\approx -\frac{F_0}{2\omega\varepsilon m}\sin(\varepsilon t)\sin(\omega t),
\end{aligned}$$

where $\varepsilon = \left(\omega - \tilde{\omega}\right)/2$.

6.2 Electrical Circuits

Consider the electrical circuit shown in Figure 6.3.

Let $Q(t)$ be the charge in the capacitor at time t (Coulombs). Then dT/dt is called the current, denoted I. The battery produces a voltage (potential difference) resulting in current I when the switch is closed. The resistance R results in a voltage drop of RI. The coil of wire (inductor) produces a magnetic field resisting change in the current. The voltage drop created is $L\,(dI/dt)$. The

*The amplitude of the vibrations is

$$\frac{F_0}{2\sqrt{km}}t,$$

which increases with time.

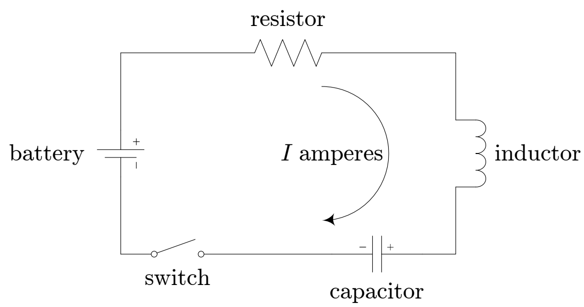


Figure 6.3: An electrical circuit, where resistance is measured in Ohms, capacitance is measured in Farads, and the inductance is measured in Henrys.

capacitor produces a voltage drop of Q/C. Unless R is too large, the capacitor will create sine and cosine solutions and, thus, an alternating flow of current. Kirkhoff's Law states that the sum of the voltage changes around a circuit is zero, so

$$E(t) + RI + L\frac{dI}{dt} + \frac{Q}{C},$$

so we have

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = E(t). \tag{6.6}$$

The solution is the same as in the case of a spring. The analogs are shown in Table 6.1.

Table 6.1: Analogous terms between a spring-mass system and an electrical circuit.

Circuit	Spring
Charge Q	Displacement x
Inductance I	Mass m
Resistance R	Friction r
Capacitance inverse $1/C$	Spring constant k
Voltage generated by battery $E(t)$	External force $F(t)$
Amperes I	Velocity v

An example of this application is choosing a station on an old-fashioned radio. The radio has a capacitor with two charged metal bars. When the user turns the tuning dial, it changes the distance between the bars, which in turn changes the capacitance C. This changes the frequency of the solution of the homogeneous equation. When it agrees with the frequency $E(t)$ of some incoming signal, resonance results so that the amplitude of the current produced from this signal is much greater than any other.

Chapter 7

Higher Order Linear Differential Equations

In this section we generalize some of the techniques for solving second order linear equations discussed in Chapter 5 so that they apply to linear equations of higher order. Recall from Chapter 4 that an nth order linear differential equation has the form

$$y^{(n)} + P_1(x)y^{(n-1)} + P_2(x)y^{(n-2)} + \cdots + P_n(x)y = Q(x). \tag{7.1}$$

The general solution is

$$y = C_1y_1 + C_2y_2 + \cdots + C_ny_n + y_p,$$

where $y_1, y_2, \ldots, y_n$ are linearly independent solutions to the auxiliary homogeneous equation and y_p is a solution of Equation (7.1).

So given $y_1, y_2, \ldots, y_n$, how do we find y_p? We again consider the two methods we have looked at in §5.2 and §5.3 (p. 60 and p. 66, respectively).

7.1 Undetermined Coefficients

If $P_1(x), P_2(x), \ldots, P_n(x)$ are constants and $R(x)$ is "nice", we can use undetermined coefficients.

Example 7.1. Solve $y^{(4)} - y = 2e^{2e} + 3e^x$. *

Solution. We have

$$p(\lambda) = \lambda^4 - 1 = (\lambda - 1)(\lambda + 1)\left(\lambda^2 + 1\right).$$

Therefore,

$$y_1 = e^x, \qquad y_2 = e^{-x}, \qquad y_3 = \cos(x), \qquad y_4 = \sin(x).$$

Let $y_p = Ae^{3x} + Bxe^{3x}$. Then by the short cut method (§5.2.1, p. 64), we have

$$A = \frac{2}{p(2)} = \frac{2}{16-1} = \frac{2}{15},$$
$$B = \frac{3}{p'(1)} = \frac{3}{4 \cdot 1^3} = \frac{3}{4}.$$

Therefore, the general solution is

$$y = C_1 e^x + C_2 e^{-x} + C_3 \cos(x) + C_4 \sin(x) + \frac{2}{15} e^{2x} + \frac{3}{4} x e^x.$$ ◊

7.2 Variation of Parameters

Suppose that $y_1, y_2, \ldots, y_n$ are solutions of the auxiliary homogeneous equation. We look for solutions of Equation (7.1) of the form

$$y = v_1 y_1 + v_2 y_2 + \cdots + v_n y_n,$$

where $v_1, v_2, \ldots, v_n$ are functions of x. One condition on $v_1, v_2, \ldots, v_n$ is given by substituting into Equation (7.1). We choose $n - 1$ conditions to be

$$v_1' y_1 + v_2' y_2 + \cdots + v_n' y_n = 0, \tag{1}$$

$$v_1' y_1' + v_1' y_2' + \cdots + v_n' y_n' = 0, \tag{2}$$

$$\vdots \qquad \qquad \vdots$$

$$v_1' y_1^{(n-2)} + v_2' y_2^{(n-2)} + \cdots + v_n' y_n^{(n-2)} = 0. \tag{$n-1$}$$

If conditions $(1), \dots, (n-1)$ are satisfied, then

$$
\begin{aligned}
y' &= \sum_{i=1}^{n} (v_i y_i)' = \sum_{i=1}^{n} v_i y_i' + \sum_{i=1}^{n} v_i' y_i = \sum_{i=1}^{n} v_i y_i', \\
y'' &= \sum_{i=1}^{n} (v_i y_i')' = \sum_{i=1}^{n} v_i y_i'' + \sum_{i=1}^{n} v_i' y_i' = \sum_{i=1}^{n} v_i y_i'', \\
&\ \vdots \\
y^{(n-1)} &= \sum_{i=1}^{n} v_i y_i^{(n-1)} + \sum_{i=1}^{n} v_i' y^{(n-2)} = \sum_{i=1}^{n} v_i y_i^{(n-1)}, \\
y^{(n)} &= \sum_{i=1}^{n} v_i y_i^{(n)} + \sum_{i=1}^{n} v_i' y_i^{(n-1)}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
y^{(n)} + P_1(x) y^{(n-1)} + \cdots + P_n(x) y &= \sum_{i=1}^{n} v_i y_i^{(n)} + \sum_{i=1}^{n} v_i' y_i^{(n-1)} \\
&\quad + P_1(x) \sum_{i=1}^{n} v_i y_i^{(n-1)} + \cdots + P_n(x) \sum_{i=1}^{n} v_i y_i \\
&= \sum_{i=1}^{n} v_i' y_i^{(n-1)} \\
&\quad + \sum_{i=1}^{n} v_i \begin{pmatrix} y_i^{(n)} + P_1(x) y_i^{(n-1)} + \cdots \\ + P_n(x) y_i \end{pmatrix} \\
&= \sum_{i=1}^{n} v_i' y_i^{(n-1)} + \cancel{\sum_{i=1}^{n} v_i \cdot 0} \\
&= \sum_{i=1}^{n} v_i' y_i^{(n-1)}.
\end{aligned}
$$

Therefore, Equation (7.1) becomes the first condition

$$
\sum_{i=1}^{n} v_i' y_i^{(n-1)} y = R(x). \tag{n}
$$

Conditions $(1), \ldots, (n-1)$ can be written as

$$M \begin{bmatrix} v_1' \\ v_2' \\ \vdots \\ v_n' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ R(x) \end{bmatrix},$$

where

$$M = \begin{bmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ \vdots & \ddots & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}.$$

Therefore, $|M| = W \neq 0$.

7.3 Substitutions: Euler's Equation

Consider the equation

$$x^n y^{(n)} + a_1 x^{n-1} y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = R(x), \tag{7.2}$$

where $a_1, a_2, \ldots, a_n$ are constants. Consider $x > 0$. Let $x = e^u$ so that $u = \ln(x)$ (for $x < 0$, we must use $u = \ln(-x)$). Then

$$\begin{aligned}
\frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{1}{x}, \\
\frac{d^2y}{dx^2} &= \frac{d^2y}{du^2}\frac{1}{x^2} - \frac{dy}{du}\frac{1}{x^2} = \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)\frac{1}{x^2}, \\
\frac{d^3y}{dx^3} &= \left(\frac{^3 y dx^3}{d} - \frac{d^2y}{du^2}\right)\frac{1}{x^3} - 2\left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)\frac{1}{x^3} \\
&= \left(\frac{d^3y}{du^3} - 3\frac{d^2y}{du^2} + 2\frac{dy}{du}\right)\frac{1}{x^3}, \\
&\vdots \\
\frac{d^ny}{dx^n} &= \left(\frac{d^ny}{du^n} + C_1\frac{d^{n-1}y}{du^{n-1}} + \cdots + C_n\frac{dy}{du}\right)\frac{1}{x^n}
\end{aligned}$$

for some constants $C_1, C_2, \ldots, C_n$. Therefore, Equation (7.2) becomes

$$\frac{d^n y}{du^n} + b_1 \frac{d^{n-1} y}{du^{n-1}} + \cdots + b_{n-1} \frac{dy}{du} + b_n y = R(e^u)$$

for some constants $b_1, b_2, \ldots, b_n$.

Example 7.2. Solve $x^2 y'' + xy' - y = x^3$ for $x > 0$. $*$

Solution. Let $x = e^u$ so that $u = \ln(x)$. Then

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\frac{1}{x}, \\ \frac{d^2y}{dx^2} &= \frac{d^2y}{du^2}\frac{1}{x^2} - \frac{dy}{du}\frac{1}{x^2} = \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right)\frac{1}{x^2}. \end{aligned}$$

Therefore,

$$\begin{aligned} \left(\frac{d^2y}{du^2} - \frac{dy}{du}\right) + \frac{dy}{du} - y &= e^{3u}, \\ \frac{d^2u}{du^2} - y &= e^{3u}. \end{aligned}$$

Considering the homogeneous case, we have $\lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$. Therefore, $y_1 = e^u$ and $y_2 = e^{-u}$. Using undetermined coefficients, let $y_p = Ae^{3u}$. Then $y' = 3Ae^{3u}$ and $y'' = 9Ae^{3u}$. Substitution gives

$$\underbrace{9Ae^{3u} - Ae^{3u}}_{8Ae^{3u}} = e^{3u}.$$

We can immediately see that $A = 1/8$. Therefore, the general solution is

$$\begin{aligned} y &= C_1 e^u + C_2 e^{-u} + \frac{1}{8} e^{3u} \\ &= C_1 e^{\ln(x)} + C_2 e^{-\ln(x)} + \frac{1}{8} e^{3\ln(x)} \\ &= C_1 x + \frac{C_2}{x} + \frac{x^3}{8}, \end{aligned}$$

where C_1 and C_2 are arbitrary constants. $\diamondsuit$

We can also consider a direct method. Solving

$$\frac{d^2y}{dx^2} + p\frac{dy}{du} + qy = 0$$

involves looking for solutions of the form $y = e^{\lambda u} = (e^u)^\lambda$ and finding λ. Since $x = e^u$, we could directly look for solutions of the form x^λ and find the right λ.

With this in mind and considering Example 7.2 again, we have

$$\begin{aligned} y &= x^\lambda, \\ y' &= \lambda x^{\lambda-1}, \\ y'' &= \lambda(\lambda-1)\, x^{\lambda-2}. \end{aligned}$$

Therefore,

$$\begin{aligned} x^2y'' + xy' - y = 0 &\Longrightarrow \lambda(\lambda-1)\, x^\lambda + \lambda x^\lambda - x^\lambda = 0 \\ &\Longrightarrow \lambda(\lambda-1) + \lambda - 1 = 0 \\ &\Longrightarrow \lambda^2 - 1 = 0 \\ &\Longrightarrow \lambda = \pm 1, \end{aligned}$$

as before.

Chapter 8

Power Series Solutions to Linear Differential Equations

8.1 Introduction

Consider the equation

$$y^{(n)} + P_1(x)y^{(n-1)} + \cdots + P_n(x)y = R(x). \tag{8.1}$$

Given solutions $y_1, \ldots, y_n$ to the homogeneous equation

$$y^{(n)} + P_1(x)y + \cdots + P_n y = 0,$$

we know how to solve Equation (8.1), but so far there is no method of finding $y_1, \ldots, y_n$ unless $P_1, \ldots, P_n$ are constants. In general, there is no method of finding $y_1, \ldots, y_n$ exactly, but we can find the power series approximation.

Example 8.1. Solve $y'' - xy' - 2y = 0$. $*$

Solution. Let

$$y = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n.$$

Then it follows that

$$\begin{aligned} y' &= a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + (n+1)\, a_{n+1} x^n + \cdots \\ &= \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{k=0}^{\infty} (k+1)\, a_{k+1} x^k, \end{aligned}$$

where $k = n - 1$. Relabeling, we have

$$y' = \sum_{n=0}^{\infty} (n+1)\, a_{n+1} x^n.$$

By the same procedure, we have

$$y'' = \sum_{n=0}^{\infty} n\,(n+1)\, a_{n+1} x^{n-1} = \sum_{n=0}^{\infty} (n+1)\,(n+2)\, a_{n+2} x^n.$$

Now considering $y'' - xy' - 2y = 0$, we have

$$\begin{aligned} \overbrace{\sum_{n=0}^{\infty} (n+1)\,(n+2)\, a_{n+2} x^n}^{y''} - x \overbrace{\sum_{n=0}^{\infty} n a_n x^{n-1}}^{y'} - 2 \overbrace{\sum_{n=0}^{\infty} a_n x^n}^{y} &= 0, \\ \sum_{n=0}^{\infty} \left[(n+1)\,(n+2)\, a_{n+2} - n a_n - 2a_n \right] x^n &= 0. \end{aligned}$$

This gives us the recurrence relation

$$\begin{aligned} (n+1)\,(n+2)\, a_{n+2} - n a_n - 2a_n &= 0, \\ (n+1)\,(n+2)\, a_{n+2} - (n+2)\, a_n &= 0, \\ (n+1)\, a_{n+2} &= a_n, \\ a_{n+2} &= \frac{a_n}{n+1} \end{aligned}$$

for all $n \geq 0$. Therefore,

$$\begin{aligned} a_2 &= \frac{a_0}{1}, \quad a_4 = \frac{a_2}{3} = \frac{a_0}{1 \cdot 3}, \quad a_6 = \frac{a_4}{5} = \frac{a_0}{1 \cdot 3 \cdot 5}, \quad \ldots, \quad a_{2n} = \frac{a_0}{\prod_{k=1}^{n} (2k-1)}, \\ a_3 &= \frac{a_1}{2}, \quad a_5 = \frac{a_3}{4} = \frac{a_1}{2 \cdot 4}, \quad a_7 = \frac{a_5}{6} = \frac{a_1}{2 \cdot 4 \cdot 6}, \quad \ldots, \quad a_{2n+1} = \frac{a_1}{\prod_{k=1}^{n} 2k}. \end{aligned}$$

Therefore,

$$\begin{aligned}
y &= a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n + \cdots \\
&= a_0 + a_1 x + a_0 x^2 + \frac{a_1}{2} x^3 + \cdots + \frac{a_0}{\prod_{k=1}^{n}(2k-1)} x^{2n} + \frac{a_1}{\prod_{k=1}^{n} 2k} x^{2n+1} + \cdots \\
&= a_0 \left(1 + x^2 + \cdots + \frac{x^{2n}}{\prod_{k=1}^{n}(2k-1)} + \cdots \right) + a_1 \left(x + \frac{x^3}{2} + \cdots + \frac{x^{2n+1}}{\prod_{k=1}^{n} 2k} + \cdots \right) \\
&= a_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{\prod_{k=1}^{n}(2k-1)} + a_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!}.
\end{aligned}$$

That is, we have $y = C_1 y_1 + C_2 y_2$, where $C_1 = a_0$ and $C_2 = a_1$ such that

$$y_1 = \sum_{n=0}^{\infty} \frac{x^{2n}}{\prod_{k=1}^{n}(2k-1)}, \qquad y_2 = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!}.$$

Can we recognize y_1 and y_2 as elementary functions? In general, no. But in this case, we have

$$y_2 = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{2^n n!} = x \sum_{n=0}^{\infty} \frac{x^{2n}}{2^n n!} = x \sum_{n=0}^{\infty} \frac{\left(x^2\right)^n}{2^n n!} = x \sum_{n=0}^{\infty} \frac{\left(x^2/2\right)^n}{n!} = x e^{x^2/2}.$$

Having one solution in closed form, we can try to find another (see §5.1, p. 57) by

$$\begin{aligned}
y_1 = y_2 \int \frac{e^{-\int P(x)\,dx}}{y_2^2} &= x e^{x^2/2} \int \frac{e^{-\int P(x)\,dx}}{x^2 e^{x^2}}\,dx \\
&= x e^{x^2/2} \int \frac{e^{x^2/2}}{x^2 e^{x^2}}\,dx = x e^{x^2/2} \int \frac{1}{x^2 e^{x^2/2}}\,dx,
\end{aligned}$$

which is not an elementary function. ◊

8.2 Background Knowledge Concerning Power Series

Theorem 8.2
Consider the power series $\sum_{n=0}^{\infty} a_n (z-p)^n$.

1. There exists an R such that $0 \leq R < \infty$ called the *radius of convergence* such that $\sum_{n=0}^{\infty} a_n (z-p)^n$ converges for $|z-p| < R$ and diverges for $|z-p| > R$. The radius of convergence is given by

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right| \tag{8.2}$$

   if the limit exists (ratio test).

2. The function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ defined for $|z-p| < R$ can be differentiated and integrated term-wise within its radius of convergence.

Proof. Proof is given in `MATA37` or `MATB43`. □

Remark. The series may or may not converge when $|z-p| = R$. ◊

Using (2) of Theorem 8.2, $f'(z)$ exists for $|z-p| < R$ with $f'(z) = \sum_{n=0}^{\infty} n a_n z^{n-1}$. Similarly,

$$\begin{aligned}
f(z) &= \sum_{n=0}^{\infty} a_n (z-p)^n, & z = p &\Longrightarrow a_0 = f(p), \\
f'(z) &= \sum_{n=0}^{\infty} n a_n (z-p)^{n-1}, & z = p &\Longrightarrow a_1 = f'(p), \\
f''(z) &= \sum_{n=0}^{\infty} n(n-1) a_n (z-p)^{n-2}, & z = p &\Longrightarrow a_2 = \frac{f''(p)}{2}, \\
&\vdots \\
f^{(n)}(z) &= \sum_{n=0}^{\infty} n!\, a_n, & z = p &\Longrightarrow a_n = \frac{f^{(n)}(p)}{n!}.
\end{aligned}$$

Conversely, if f is infinitely differentiable at p, we define $a_n = f^{(n)}(p)/n!$. Then $\sum_{n=0}^{\infty} a_n (z-p)^n$ is called the *Taylor series* of f about p.

From the above, if any power series converges to f near p, it must be its Taylor series. But

1. the Taylor series might not converge.

2. if it does converge, it does not have to converge to f.

Definition (Analytic function, ordinary/singular point)
A function f is called *analytic* at p if $\sum_{n=0}^{\infty} a_n (z-p)^n$ converges to $f(z)$ in some neighbourhood of p (i.e., if there exists an $r > 0$ such that it converges for $|z-p| < r$). If f is analytic at p, then p is called an *ordinary* point of f. If f is not analytic at p, then p is called a *singular* point of f.

Theorem 8.3 (Theorem from complex analysis)
Let $f(z)$ be analytic at p. Then the radius of convergence of the Taylor series of f at p equals the distance from p to the closest singularity of f in the complex plane.

Proof. Proof is given in `MATC34`. □

Example 8.4. Consider $f(x) = 1/\left(x^2+1\right)$. The singularities are $\pm i$. Therefore, the radius of convergence of the Taylor series about the origin is 1, with the series given by

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^{2n}.$$

The radius of convergence of the Taylor series about $x = 1$ is $\sqrt{2}$. *

8.3 Analytic Equations

Consider $y'' + P(x)y' + Q(x)y = 0$. If both $P(x)$ and $Q(x)$ are analytic at p, then p is called an *ordinary point* of the DE. Otherwise, p is called a *singular* *point* or a *singularity* of the DE. When looking at a DE, do not forget to write it in the standard form. For example,

$$x^2y'' + 5y' + xy = 0$$

has a singularity at $x = 0$, since the standard form is

$$y'' + \left(\frac{5}{x^2}\right) y' + \left(\frac{1}{x}\right) y = 0.$$

Theorem 8.5
Let $x = p$ be an ordinary point of $y'' + P(x)y' + Q(x)y = 0$. Let R be the distance from p to the closest singular point of the DE in the complex plane. Then the DE has two series $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$ and $y_2(x) = \sum_{n=0}^{\infty} b_n x^n$ which converge to linearly independent solutions to the DE on the interval $|x - p| < R$.

Remark. The content of the theorem is that the solutions are analytic within the specified interval. The fact that solutions exist on domains containing at least this interval is a consequence of theorems in Chapter 10. ♢

Proof. The proof is given in Chapter 10. □

Example 8.6. Consider $\left(x^2 + 1\right) y'' + 2xy' + 2y = 0$. Dividing by $x^2 + 1$, we have

$$y'' + \frac{3x}{x^2+1} y' + \frac{2}{x^2+1} y = 0.$$

The singularities are $\pm i$. If $p = 0$, then $r = |i - 0| = 1$. Therefore, analytic solutions about $x = 0$ exist for $|x| < 1$. If $p = 1$, then $r = |i - 1| = \sqrt{2}$. Therefore, there exist two linearly independent series of the form $\sum_{n=0}^{\infty} a_n \left(x - 1\right)^n$ converging to the solutions for at least $|x - 2| < \sqrt{2}$.

Note that the existence and uniqueness theorem (see Chapter 10) guarantees solutions from $-\infty$ to ∞ (since $x^2 + 1$ has no real zeros), but only in a finite subinterval can we guarantee that the solutions are expressible as convergent power series. ∗

Example 8.7. Consider $y'' - xy' - 2y = 0$. It has no singularities. Therefore, a series solution converging on $\mathbb{R}$ exists about any point. ∗

8.4 Power Series Solutions: Levels of Success

When using power series techniques to solve linear differential equations, ideally one would like to have a closed form solution for the final answer, but most often one has to settle for much less. There are various levels of success, listed below, and one would like to get as far down the list as one can.

For simplicity, we will assume that we have chosen to expand about the point $x = 0$, although the same considerations apply to $x = c$ for any c.

Suppose 0 is an ordinary point of the equation $L(x)y'' + M(x)y' + N(x)y = 0$ and let $y = \sum_{n=0}^{\infty} a_n x^n$ be a solution. One might hope to:

1. find the coefficients a_n for a few small values of n; (For example, find the coefficients as far as a_5 which thereby determines the $5th$ degree Taylor approximation to $y(x)$);
2. find a recursion formula which for any n gives a_{n+2} in terms of a_{n-1} and a_n;
3. find a general formula a_n in terms of n;
4. "recognize" the resulting series to get a closed form formula for $y(x)$.

In general, level (1) can always be achieved. If $L(x)$, $M(x)$, and $N(x)$ are polynomials, then level (2) can be achieved. Having achieved level (2), it is sometimes possible to get to level (3) but not always. Cases like the one in Example 8.1 where one can achieve level (4) are rare and usually beyond expectation.

8.5 Level 1: Finding a finite number of coefficients

Suppose c is an ordinary point of Equation (8.3). One approach is to finding power series solutions is as follows. Expand $P(x)$ and $Q(x)$ into Taylor series

$$P(x) = \sum_{n=0}^{\infty} p_n (x-c)^n , \qquad Q(x) = \sum_{n=0}^{\infty} q_n (x-c)^n .$$

and set $y = \sum_{n=0}^{\infty} a_n (x-c)^n$ solves the equation. Then

$$y' = \sum_{n=0}^{\infty} n a_n (x-c)^{n-1} , \qquad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2},$$

and Equation (8.3) becomes

$$\overbrace{\sum_{n=0}^{\infty} n\left(n-1\right)a_n x^{n-2}}^{y''} + \overbrace{\left(\sum_{n=0}^{\infty} p_n \left(x-c\right)^n\right)}^{P(x)}\overbrace{\left(\sum_{n=0}^{\infty} n a_n \left(x-c\right)^{n-1}\right)}^{y'} + \underbrace{\left(\sum_{n=0}^{\infty} q_n \left(x-c\right)^n\right)}_{Q(x)}\underbrace{\left(\sum_{n=0}^{\infty} a_n \left(x-c\right)^n\right)}_{y} = 0.$$

Since p_n's and q_n's are known (as many as desired can be computed from derivatives of $P(x)$ and $Q(x)$), in theory we can inductively find the a_n's one after another.

In the special case where $P(x)$ and $Q(x)$ are polynomials, we can get a recursion formula by writing a_{n+2} in terms of a_n and a_{n-1} (level 2) and, under favourable circumstances, we can find the general formula for a_n (level 3) as in Example 8.1. But in general, computation gets impossibly messy after a while and, although a few a_n's can be found, the general formula cannot.

But if you only want a few a_n's, there is an easier way. We have $a_n = y^{(n)}(c)/n!$, so

$$\begin{aligned}
a_2 &= \frac{y''(c)}{2!} = -\frac{P(x)y'(c) + Q(c)y(c)}{2} = \frac{-a_1 P(c) - a_0 Q(c)}{2}, \\
a_3 &= \frac{y'''(c)}{3!} = \frac{1}{6}\left(-P'(c)y'(c) - P(c)y'' - Q'(c)y(c) - Q(c)y'(c)\right) \\
&= \frac{1}{6}\left(-P'(c)a_1 + \frac{a_1 P(c)^2 + a_0 P(c)Q(c)}{2} - a_0 Q'(c) - a_1 Q(c)\right), \\
a_4 &= \cdots,
\end{aligned}$$

and so on.

Example 8.8. Compute the Taylor expansion about 0 as far as degree 4 for the solution of $y'' - e^{7x}y' + xy = 0$, which satisfies $y(0) = 2$ and $y'(0) = 1$. $*$

Solution. Rearranging the equation, we have $y'' = e^{5x}y' - xy$. Therefore

$$\begin{aligned}
y''' &= e^{5x}y'' + 5e^{5x}y' - xy' - y \\
y^{(4)} &= e^{5x}y''' + 5e^{5x}y'' + 5e^{5x}y'' - xy'' - y' - y' \\
&= e^{5x}y''' + 10e^{5x}y'' - xy'' - 2y'.
\end{aligned}$$

Substituting gives

$$\begin{aligned}
y''(0) &= e^0 y'(0) - 0 = 1 \\
y'''(0) &= e^0 y''(0) + 5e^0 y'(0) - 0 - y(0) = 1 + 5 - 2 = 4 \\
y^{(4)}(0) &= e^0 y'''(0) + 10e^0 y''(0) - 0 - 2y'(0) = 4 + 10 - 2 = 12
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= 2 + x + \frac{x^2}{2!} + \frac{4x^3}{3!} + \frac{12x^4}{4!} + \cdots \\
&= 2 + x + \frac{1}{2}x^2 + \frac{2}{3}x^3 - \frac{1}{2}x^4 + \cdots .
\end{aligned}$$

◊

Example 8.9. Find the series solution for $y'' + (x+3)\,y' - 2y = 0$, where $y(0) = 1$ and $y'(0) = 2$. *

Solution. Rearranging the equation, we have $y'' = -(x+3)\,y' + 2y$. Then

$$\begin{aligned}
y''(0) &= -(0+3)\,y'(0) + 2y(0) = -3 \cdot 2 + 2 \cdot 1 = -4, \\
y''' &= -y' - (x+3)\,y'' + 2y' = -(x+3)\,y'' + y', \\
y'''(0) &= -3y''(0) + y'(0) = (-3)(-4) + 2 \cdot 1 = 12 + 2 = 14, \\
y^{(4)} &= \cdots = -34.
\end{aligned}$$

Therefore,

$$\begin{aligned}
y &= 1 + 2x - \frac{4x^2}{2!} + \frac{14x^3}{3!} - \frac{34x^4}{4!} + \cdots \\
&= 1 + 2x - 2x^2 + \frac{7}{3}x^3 - \frac{17}{12}x^4 + \cdots .
\end{aligned}$$

In this case, since the coefficients were polynomials, it would have been possible to achieve level 2, (although the question asked only for a level 1 solution). However even had we worked out the recursion formula (level 2) it would have been too messy to use it to find a general formula for a_n. ◊

8.6 Level 2: Finding the recursion relation

Example 8.10. Consider $\left(x^2 - 5x + 6\right) y'' - 5y' - 2y = 0$, where $y(0) = 1$ and $y'(0) = 1$. Find the series solution as far as x^3 and determine the lower bound on its radius of convergence from the recursion formula. $*$

Solution. We have

$$\begin{aligned}
2y &= \sum_{n=0}^{\infty} 2a_n x^n, \\
5y' &= \sum_{n=0}^{\infty} 5na_n x^{n-1} = \sum_{n=0}^{\infty} 5\left(n+1\right) a_{n+1} x^n = \sum_{n=0}^{\infty} 5\left(n+1\right) a_{n+1} x^n, \\
6y'' &= \sum_{n=0}^{\infty} 6n\left(n-1\right) a_n x^{n-2} = \sum_{n=0}^{\infty} 6\left(n+2\right)\left(n+1\right) a_{n+2} x^n, \\
5xy'' &= \sum_{n=0}^{\infty} 5n\left(n-1\right) a_n x^{n-1} = \sum_{n=0}^{\infty} 5\left(n+1\right) n a_{n+1} x^n, \\
x^2 y'' &= \sum_{n=0}^{\infty} n\left(n-1\right) a_n x^n.
\end{aligned}$$

Therefore,

$$\sum_{n=0}^{\infty} \left(\begin{array}{l} n\left(n-1\right) a_n \\ -5\left(n+1\right) n a_{n+1} \\ +6\left(n+2\right)\left(n+1\right) a_{n+2} \\ -5\left(n+1\right) a_{n+1} - 2a_n \end{array} \right) x^n = \sum_{n=0}^{\infty} \left(\begin{array}{l} 6\left(n+2\right)\left(n+1\right) a_{n+2} \\ -5\left(n+1\right)\left(n+1\right) a_n \\ +\left(n^2 - n - 2\right) a_n \end{array} \right) x^n,$$

so it follows that

$$\begin{gathered}
6\left(n+2\right)\left(n+1\right) a_{n+2} - 5\left(n+1\right)^2 a_{n+1} + \left(n+1\right)\left(n-2\right) a_n = 0, \\
a_{n+2} = \frac{5}{6} \cdot \frac{n+1}{n+2} a_{n+1} - \frac{1}{6} \cdot \frac{n-2}{n+2} a_n.
\end{gathered}$$

With $a_0 = y(0) = 1$ and $a_1 = y'(0) = 1$, we have

$$\begin{aligned}
a_2 &= \frac{5}{6} \cdot \frac{1}{2} a_1 - \frac{1}{6} \cdot \frac{-2}{2} a_0 = \frac{5}{12} + \frac{2}{12} = \frac{7}{12}, \\
a_3 &= \frac{5}{6} \cdot \frac{2}{3} - \frac{1}{6} \cdot \frac{-1}{3} a_1 = \frac{5}{9} \cdot \frac{7}{12} + \frac{1}{6} \cdot \frac{1}{3} = \frac{35}{108} + \frac{6}{108} = \frac{41}{108}.
\end{aligned}$$

Therefore,

$$y = 1 + x + \frac{7}{12} x^2 + \frac{41}{108} x^3 + \cdots.$$

Notice that although we need to write our equation in standard form (in this case dividing by $x^2 - 5x + 6$ to determine that 0 is an ordinary point of the equation), it is **not** necessary to use standard form when computing the recursion relation. Instead we want to use a form in which our coefficients are polynomials. Although our recursion relation is too messy for us to achieve find a general formula for a_n (level 3) there are nevertheless some advantages in computing it. In particular, we can now proceed to use it to find the radius of convergence.

To determine the radius of convergence, we use the ratio test $R = 1/|L|$, where $L = \lim_{n\to\infty}(a_{n+1}/a_n)$, provided that the limit exists. It is hard to prove that the limit exists, but assuming that

$$\frac{a_{n+2}}{a_{n+1}} = \frac{5}{6} \cdot \frac{n+1}{n+2} - \frac{1}{6} \cdot \frac{n-2}{n+2} \cdot \frac{a_n}{a_{n+1}},$$

taking the limit gives

$$L = \frac{5}{6} \cdot 1 - \frac{1}{6} \cdot \frac{1}{L} = \frac{5}{6} - \frac{1}{6} \cdot \frac{1}{L}.$$

So $6L^2 = 5L - 1$, and solving for L gives $L = 1/2$ or $L = 1/3$. So $R = 2$ or $R = 3$. The worst case is when $R = 2$. Therefore, the radius of convergence is at least 2. Theorem 8.3 also gives this immediately. $\diamondsuit$

Example 8.11. Consider $\left(x^2 - 5x + 6\right) y'' - 5y' - 2y = 0$. Find two linearly independent series solutions y_1 and y_2 as far as x^3. $*$

Solution. Let $y_1(x)$ be the solution satisfying $y_1(0) = 1$; $y_1'(0) = 0$ and let $y_2(x)$ be the solution satisfying $y_2(0) = 0$; $y_2'(0) = 1$. Then $y_1(x)$ and $y_2(x)$ will be linearly independent solutions. We already computed the recursion relation in Example 8.10. For $y_1(x)$, we have $a_0 = y(0) = 1$ and $a_1 = y'(0) = 0$. Thus,

$$a_2 = \frac{5}{6} \cdot \frac{1}{2} a_1 - \frac{1}{6}\left(-\frac{2}{2}\right) a_0 = \frac{1}{6},$$
$$a_3 = \frac{5}{6} \cdot \frac{2}{3} a_2 - \frac{1}{6} \cdot \frac{1}{3} a_1 = \frac{5}{54}.$$

Therefore,

$$y_1 = 1 + \frac{x^2}{6} + \frac{5}{54} x^3 + \cdots .$$

For y_2, we have $a_0 = y(0) = 0$ and $a_1 = y'(0) = 1$, so

$$a_2 = \frac{5}{6} \cdot \frac{1}{2} a_1 - \frac{2}{2} a_0 = \frac{5}{12},$$
$$a_3 = \frac{5}{6} \cdot \frac{2}{3} a_2 - \left(-\frac{1}{3}\right) a_1 = \frac{5}{6} \cdot \frac{2}{3} \cdot \frac{5}{12} + \frac{1}{6 \cdot 3} = \frac{25}{108} + \frac{6}{108} = \frac{31}{108}.$$

Therefore,

$$y_2 = x + \frac{5}{12} x^2 + \frac{61}{108} x^3 + \cdots .$$

With

$$y = 1 + x + \frac{7}{12} x^2 + \frac{41}{108} x^3 + \cdots ,$$

Looking at the initial values, the solution $y(x)$ of Example 8.10 ought to be given by $y = y_1 + y_2$, and comparing are answers we see that they are consistent with this equation. ♢

Summarizing: Let p be a regular point of the equation

$$y'' + P(x)y' + Q(x)y = 0, \tag{8.3}$$

Let D be the distance from p to the closest singularities of P or Q in the complex plane. Then Equation (8.3) has two linearly independent solutions expressible as a convergent power series $\sum_{n=0}^{\infty} a_n (x-p)^n$ with $|x - p| < D$. Theorem 8.5 guarantees that the radius of convergence is at least D, but it might be larger.

Given $a_0 = y(p)$ and $a_1 = y'(p)$, to find a_n, we have $a_n = y^{(n)}(p)/n!$. We can find as many a_n's as desired, but in general we cannot find the formula. Since we have $y'' = -P(x)y' - Q(x)y$, it can be differentiated to give $y'''(p)$ and higher derivatives can be found by further differentiation. Substituting the values at p gives formulas for the a_n's.

In the special case where $P(x)$ and $Q(x)$ are polynomials, we can instead find a recurrence relation. From the recurrence relation there is a chance to get the formula for a_n (if so, then there is a chance to actually recognize the series). Even without the formula, we can determine the radius of convergence from the recurrence relation (even if $P(x)$ and $Q(x)$ are not polynomials, we could at least try to get a recurrence relation by expanding them into Taylor series,but in practice this is not likely to be successful).

8.7 Solutions Near a Singular Point

A singularity a of $y'' + P(x)y' + Q(x)y = 0$ is called a *regular* singularity if $(x-a)\,P(x)$ and $(x-a)^2\,Q(x)$ are analytic at $x = a$. To simplify notation, we assume that $a = 0$ (we can always change the variable by $v = x - a$ to move the singularity to the origin). We assume that $x > 0$. For $x < 0$, we substitute $t = -x$ and apply below to t.

Our primary goal in this section is to consider the *Method of Frobenius*. We look for solutions of the form

$$y = x^{\alpha}\left(a_0 + a_1 x + \cdots + a_n x^n + \cdots\right) = a_0 x^{\alpha} + a_1 x^{\alpha+1} + \cdots + a_n x^{\alpha+n} + \cdots$$

for some α, where $a_0 \neq 0$. Then

$$\begin{aligned} y' &= a_1 \alpha x^{\alpha-1} + a_1\left(\alpha+1\right) x^{\alpha} + \cdots + a_n\left(\alpha+n\right) x^{\alpha+n-1} + \cdots, \\ y'' &= a_1 \alpha\left(\alpha-1\right) x^{\alpha-2} + a_1 \alpha\left(\alpha+1\right) x^{\alpha-1} + \cdots \\ &\quad + a_n\left(\alpha+n\right)\left(\alpha+n-1\right) x^{\alpha+n-2} + \cdots, \\ xP(x) &= p_0 + p_1 x + \cdots + p_n x^n + \cdots, \\ x^2 Q(x) &= q_0 + q_1 x + \cdots + q_n x^n + \cdots. \end{aligned}$$

So if $x \neq 0$, then

$$\begin{aligned} a_0 \alpha\left(\alpha-1\right) x^{\alpha-2} + a_1\left(\alpha+1\right) \alpha x^{\alpha-1} + \cdots + a_n\left(\alpha+n\right)\left(\alpha+n-1\right) x^{\alpha+n-2} + \cdots \\ + x^{\alpha-2}\left(a_0 \alpha + a_1\left(\alpha+1\right) x + \cdots\right)\left(p_0 + p_1 x + \cdots + p_n x^n + \cdots\right) \\ + x^{\alpha-2}\left(a_0 + a_1 x + \cdots + a_n x^n\right)\left(q_0 + q_1 x + \cdots + q_n x^n + \cdots\right) = 0. \end{aligned}$$

We have

$$a_0 \alpha\left(\alpha-1\right) + a_0 \alpha p_0 + a_0 q_0 = 0, \qquad (0)$$

$$a_1\left(\alpha+1\right)\alpha + a_1\left(\alpha+1\right) p_0 + a_0 \alpha p_1 + a_0 q_1 + a_1 q_0 = 0, \qquad (1)$$

$$\vdots \qquad\qquad \vdots$$

$$\cdots \qquad (n)$$

Note that Equation (0) implies

$$\begin{aligned} \alpha\left(\alpha-1\right) + \alpha p_0 + q_0 &= 0, \\ \alpha^2 + \left(p_0 - 1\right)\alpha + q_0 &= 0. \end{aligned} \qquad (8.4)$$

Equation (8.4) is known as an *indicial equation*. Solving for α gives us two solutions. Then Equation (1) gives a_1 in terms of a_0, Equation (2) gives a_2 in terms of a_1, etc. We will get the solution $y = a_0 x^{\alpha} (\star)$. Other α gives a second solution $y = b_0 x^{\beta} (\star)$. The plan is to find a_n from Equation (n) for $n \geq 1$. Equation (n) looks like

$$(\star)\, a_n + (\star)\, a_{n-1} + \cdots + (\star)\, a_0 = 0,$$

where the coefficients are in terms of $\alpha, p_0, \ldots, p_n, q_0, \ldots, q_n$. There is one possible problem: what happens if the coefficient of a_n is zero? Note that

$$y'' + P(x)y' + Q(x)y = y'' + xP(x)\frac{y'}{x} + x^2 Q(x)\frac{y}{x^2},$$

so we have

$$\begin{aligned} y'' + xP(x)\frac{y'}{x} + x^2Q(x)\frac{y}{x^2} &= \sum_{n=0}^{\infty} a_n (\alpha+n)(\alpha+n-1)\, x^{\alpha+n-2} \\ &\quad + \left(p_0 + p_1 x + p_2 x^2 + \cdots\right) \sum_{n=0}^{\infty} a_n (\alpha+n)\, x^{\alpha+n-2} \\ &\quad + \left(q_0 + q_1 x + q_2 x^2 + \cdots\right) \sum_{n=0}^{\infty} a_n x^{\alpha+n-2}. \end{aligned}$$

In Equation (n), the coefficient of a_n is

$$(\alpha+n)(\alpha+n-1) + p_0(\alpha+n) + q_0.$$

Suppose that

$$(\alpha+n)(\alpha+n-1) + p_0(\alpha+n) + q_0 = 0.$$

Let $\beta = \alpha + n$. Then

$$\beta(\beta-1) + p_0\beta + q_0 = 0$$

is the indicial equation again, i.e., β is the other solution to the indicial equation. We conclude that if two solutions of the indicial equation do not differ by an integer, the method of Frobenius provides two solutions.* If the two solutions do differ by an integer, then the larger one will give a solution by this method, but the smaller one may fail.

Let $F(\lambda) = \lambda(\lambda-1) + p_0\lambda + q_0$. Suppose the solutions to $F(\lambda) = 0$ are r

*This includes the case where the solutions are complex, i.e., in this case, we get a complex solution $z = x^{a+ib}(c_0 + c_1 x + \cdots)$, so $y_1 = \mathrm{Re}(x)$ and $y_2 = \mathrm{Im}(z)$ give two real solutions.

and $r+k$, where $k \in \mathbb{Z}^+$. Given $f(x)$, set

$$L(f)(x) = f''(x) + P(x)f'(x) + Q(x)f(x).$$

Let $w = \sum_{n=0}^{\infty} a_n x^n$, where a_n is to be chosen. For any given α, we have

$$L(x^\alpha w) = \sum_{n=0}^{\infty} c_n x^{\alpha-2+n},$$

where $c_n = F(\alpha+n)a_n + (\star)\, a_{n-1} + \cdots + (\star)\, a_0$. Chose $a_0 = 1$. Then for $n \geq 1$, we can always solve $c_n = 0$ for a_n in terms of its predecessors unless $F(\alpha+n) = 0$. Provided that $\alpha \neq s-1, s-2, \ldots$, we have $F(\alpha+n) \neq 0$ for any n, so we can solve it. Suppose that we call the solution $a_n(\alpha)$, i.e., $a_0(\alpha) = 1$ for any α and a_n is chosen to make $c_n = 0$ for this α. Set $W(\alpha, x) \equiv \sum_{n=0}^{\infty} a_n(\alpha) x^n$. Then

$$L(x^\alpha w(\alpha, x)) = \sum_{n=0}^{\infty} c_n x^{\alpha-2+n} = c_0 x^{\alpha-2}.$$

But since $c_n = 0$ for $n \geq 0$ by the choice of $a_n(\alpha)$, we have

$$L(x^\alpha w(\alpha, x)) = F(\alpha) x^{\alpha-2}.$$

Since $F(s) = 0$, $y_1(x) = x^s w(s, x)$ is the solution to the DE. We now need a second solution. Unfortunately, we cannot set $\alpha = r$ since our definition of $a_n(\alpha)$ depends on $\alpha \neq s$ (positive integer). Instead, set

$$L(x^\alpha w(\alpha, x)) = F(\alpha) x^{\alpha-2}$$

and differentiated with respect to α. Note that for $g(\alpha, x)$, we have

$$\frac{\partial}{\partial x}(Lg) = L\left(\frac{\partial g}{\partial x}\right)$$

since differentiation with respect to x commutes with differentiation with respect to α (cross derivatives equal). Therefore,

$$L\left(x^\alpha \ln(x) w(\alpha, x) + x^\alpha \frac{\partial w}{\partial \alpha}\right) = F'(\alpha) x^{\alpha-2} + F(\alpha) x^{\alpha-2} \ln(x).$$

But note that

$$\alpha = s \Longrightarrow L\left(\ln(x)x^s w(s,x) + x^s \left(\left.\frac{\partial w}{\partial \alpha}\right|_{\alpha=s}\right)\right) = \overbrace{F'(s)}^{k} x^{s-2} + \overbrace{F(s)}^{0} x^{s-2}\ln(x)$$
$$\Longrightarrow L\left(\ln(x)y_1 + \left(\left.\frac{\partial w}{\partial \alpha}\right|_{\alpha=s}\right) x^s\right) = kx^{s-2}.$$

If $k = 0$, then

$$y_2 = \ln(x)y_1 + \left(\left.\frac{\partial w}{\partial \alpha}\right|_{\alpha=s}\right) x^s$$

is the second solution to our DE.

If $k > 0$, consider

$$L\left(x^r \sum_{n=0}^{\infty} a_n x^n\right) = L\left(x^r \sum_{n=0}^{\infty} c_n x^{r+n-2}\right),$$

where a_n's are to be chosen and $c_n = F(r+n)a_n + \cdots$. We have $c_0 = F(r) = 0$, where r is the root of the indicial equation. Select $a_0 = 1$ and inductively choose $a_1, \ldots, a_n$ such that $c_1, \ldots, c_n = 0$. With this choice, we have

$$L\left(x^r \sum_{n=0}^{\infty} a_n x^n\right) = \sum_{n=k}^{\infty} c_n x^{r+n-2} = c_k x^{s-2} + \sum_{n=k+1}^{\infty} c_n(r) x^{r+n-2},$$

where c_k is independent of the choice of a_k. Set $A = c_k/k$. Then

$$L\left(x^r \sum_{n=0}^{\infty} a_n x^n - A\left[\ln(x)y_1(x) + x^s \left(\left.\frac{\partial w}{\partial \alpha}\right|_{\alpha=s}\right)\right]\right) = \sum_{n=k+1}^{\infty} c_n x^{r+n-2},$$
$$L\left(\begin{array}{c} x^r \left(\begin{array}{c} a_0 + a_1 x + \cdots + a_{k-1}x^{k-1} \\ + \left(a_k - A \left.\frac{\partial w}{\partial \alpha}\right|_{\alpha=s}\right) x^k \end{array}\right) \\ + \displaystyle\sum_{n=k+1}^{\infty} a_n x^n - A\ln(x)y_1(x) \end{array}\right) = \sum_{n=k+1}^{\infty} c_n x^{r+n-2}.$$

The choice of a_k does not matter—we can choose any. Then inductively solve for $a_{k+1}, a_{k+2}, \ldots$ in terms of predecessor $a_\star$'s to make $c_n = 0$ for $n > k$. This gives a solution to the DE of the form

$$y_2 = x^r \sum_{n=0}^{\infty} a_n x^n - A\ln(x)y_1(x).$$

We conclude that if 0 is a regular singularity, then there are always two linearly independent solutions (converging near 0)

$$y_1 = x^s \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = x^r \sum_{n=0}^{\infty} b_n x^n + B(\ln(x)) y_1,$$

where r and s are the two solutions to the indicial equation. We have $B = 0$, unless $s = r + k$ for some $k \in \{0, 1, 2, \ldots\}$.

Example 8.12. Solve $4xy'' + 2y' - y = 0$. $*$

Solution. Let $y = \sum_{n=0}^{\infty} a_n x^{\alpha+n}$. Then

$$y' = \sum_{n=0}^{\infty} (\alpha + n) a_n x^{\alpha+n-1}, \qquad y'' = \sum_{n=0}^{\infty} (\alpha + n) (\alpha + n - 1) a_n x^{\alpha+n-2}.$$

Our differential equation now becomes

$$\sum_{n=0}^{\infty} 4 (\alpha + n) (\alpha + n - 1) a_n x^{\alpha+n-1} + \sum_{n=0}^{\infty} 2 (\alpha + n) a_n x^{\alpha+n-1} - \sum_{n=0}^{\infty} a_n x^{\alpha+n} = 0,$$
$$\sum_{n=0}^{\infty} [4 (\alpha + n) (\alpha + n - 1) + 2 (\alpha + n)] a_n x^{\alpha+n-1} - \sum_{n=1}^{\infty} a_{n-1} x^{\alpha+n-1} = 0.$$

Note that

$$\begin{aligned} 4 (\alpha + n) (\alpha + n - 1) + 2 (\alpha + n) &= 2 (\alpha + n) (2\alpha + 2n - 2 + 1) \\ &= 2 (\alpha + n) (2\alpha + 2n - 1). \end{aligned}$$

Therefore,

$$2\alpha (2\alpha - 1) a_0 x^{\alpha-1} + \sum_{n=1}^{\infty} [2 (\alpha + n) (2\alpha + 2n - 1) - a_{n-1}] x^{\alpha+n-1}.$$

Therefore, the indicial equation is $2\alpha (\alpha - 1) \Rightarrow \alpha \in \{0, 1/2\}$ and the recursion relation is

$$2 (\alpha + n) (2\alpha + 2n - 1) a_n - a_{n-1} = 0$$

for $n \geq 1$.

For $\alpha = 0$, we have

$$2n (2n - 1) a_n - a_{n-1} = 0 \Longrightarrow a_n = \frac{1}{2n (2n - 1)} a_{n-1}.$$

Thus,

$$a_1 = \frac{1}{2\cdot 1}a_0,$$
$$a_2 = \frac{1}{4\cdot 3}a_1 = \frac{1}{4!}a_0,$$
$$\vdots$$
$$a_n = \frac{1}{(2n)\,(2n-1)}a_{n-1} = \frac{1}{2n\,(2n-1)\,(2n-2)\cdots} = \frac{1}{(2n)!}a_0.$$

Therefore,

$$\begin{aligned} y &= x^0 a_0 \left(1 + \frac{x}{2!} + \frac{x^2}{4!} + \cdots + \frac{x^n}{(2n)}! + \cdots\right) \\ &= a_0 \left(1 + \frac{x}{2!} + \frac{x^2}{4!} + \cdots + \frac{x^n}{(2n)}! + \cdots\right) \\ &= a_0 \cosh\left(\sqrt{x}\right). \end{aligned}$$

For $\alpha = 1/2$, we have

$$2\left(\frac{1}{2}+n\right)(1+2n-1)\,b_n = b_{n-1} \Longrightarrow b_n = \frac{1}{(2n+1)\,(2n)}b_{n-1}.$$

Thus,

$$b_1 = \frac{1}{3\cdot 2}b_0,$$
$$b_2 = \frac{1}{5\cdot 4}b_1 = \frac{1}{5!}b_0,$$
$$\vdots$$
$$b_n = \frac{1}{(2n+1)!}b_0.$$

Therefore,

$$\begin{aligned} y &= x^{1/2} b_0 \left(1 + \frac{x}{3!} + \frac{x^2}{5!} + \cdots + \frac{x^n}{(2n+1)!} + \cdots\right) \\ &= b_0 \left(x^{1/2} + \frac{x^{3/2}}{3!} + \frac{x^{5/2}}{5!} + \cdots + \frac{x^{(2n+1)/2}}{(2n+1)!} + \cdots\right) \\ &= b_0 \sinh\left(\sqrt{x}\right). \end{aligned}$$

In general, however, it is possible to only calculate a few low terms. We cannot even get a general formula for a_n, let alone recognize the series. ♢

As mentioned, the above methods work near a by using the above technique after the substitution $v = x - a$. We can also get solutions "near infinity", i.e., for large x, we substitute $t = 1/x$ and look at $t = 0$. Then

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dx}\left(-\frac{1}{x^2}\right) = -\frac{1}{x^2}\frac{dy}{dt} = -t^2\frac{dy}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{2}{x^3}\frac{dy}{dt} = -\frac{1}{x^2}\frac{d^2y}{dt^2}\frac{dt}{dx} = \frac{2}{x^3}\frac{dy}{dt} - \frac{1}{x^2}\frac{d^2y}{dt^2}\left(-\frac{1}{x^2}\right) = 2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}.$$

Then

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = 0 \Longrightarrow t^4\frac{d^2y}{dt^2} + 2t^3\frac{dy}{dt} - P\left(\frac{1}{t}\right)t^2\frac{dy}{dt} + Q\left(\frac{1}{t}\right)y = 0.$$

Example 8.13 (Legendre's equation). Find the behaviour of the solutions for large x of $\left(1 - x^2\right)y'' - 2xy' + n\left(n+1\right)y = 0$, where n is a constant. $*$

Solution. Let $t = 1/x$ so that $x = 1/t$. Then our differential equation becomes

$$\begin{aligned} \left(1 - \frac{1}{t^2}\right)\left(2t^3\frac{dy}{dt} + t^4\frac{d^2y}{dt^2}\right) - \frac{2}{t}\left(-t^2\frac{dy}{dt}\right) + n\left(n+1\right)y = 0, \\ \left(t^4 - t^2\right)\frac{d^2y}{dt^2} + \left(2t^3 \cancel{-2t + 2y}\right)\frac{dy}{dt} + n\left(n+1\right)y = 0, \\ \frac{d^2y}{dt^2} + \frac{2t}{t^2-1}\frac{dy}{dt} + \frac{n\left(n+1\right)}{t^4 - t^2}y = 0. \end{aligned}$$

We see that we have a regular singularity at $t = 0$. Therefore, there exists a solution $y = t^\alpha \sum_{n=0}^{\infty} a_n t^n$ near $t = 0$, or equivalently, since $t = 1/x$,

$$y = \frac{1}{x^\alpha}\sum_{n=0}^{\infty}\frac{1}{x^n}$$

for large x. ◊

Example 8.14. Find the solutions of $y'' + xy' + y/x = 0$ for positive x near 0.*

Solution. Let

$$y = x^{\alpha} \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} a_n x^{n+\alpha}.$$

Then

$$\begin{aligned} y' &= \sum_{n=0}^{\infty} a_n (n+\alpha) x^{n+\alpha-1}, \\ y'' &= \sum_{n=0}^{\infty} a_n (n+\alpha)(n+\alpha-1) x^{n+\alpha-2}, \\ xy' &= \sum_{n=0}^{\infty} a_n x^{n+\alpha} = \sum_{n=2}^{\infty} a_{n-2} x^{n+\alpha-2}, \\ \frac{y}{x} &= \sum_{n=0}^{\infty} a_n x^{n+\alpha-1} = \sum_{n=1}^{\infty} a_{n-1} x^{n+\alpha-2}. \end{aligned}$$

Note that

$$\overbrace{a_0 \alpha (\alpha - 1) = 0}^{\text{Indicial equation}}, \tag{0}$$

$$a_1 \alpha (\alpha+1) \alpha + a_0 = 0, \tag{1}$$

$$\vdots \qquad\qquad \vdots$$

$$a_n (n+\alpha)(n+\alpha-1) + a_{n-1} + a_{n-2} = 0. \tag{n}$$

Note that Equation (0) implies that $\alpha = 0, 1$. The roots differ by an integer. The largest always gives the solution. For $\alpha = 1$, Equation (1) implies that $2a_1 + a_0 = 0$, so $a_1 = -a_0/2$. Equation (n) then becomes

$$a_n (n+1) n + a_{n-1} + a_{n-2} = 0 \Longrightarrow a_n = \frac{-a_{n-1} - a_{n-2}}{n(n+1)}.$$

Equation (2) implies that

$$6a_2 = -a_0 - a_1 = a_0 \left(-1 + \frac{1}{2}\right) = -\frac{a_0}{2} \Longrightarrow a_2 = -\frac{a_0}{12};$$

Equation (3) implies that

$$12a_3 = a_0\left(\frac{1}{2}+\frac{1}{12}\right) = a_0\frac{7}{12} \Longrightarrow a_3 = a_0\frac{7}{144}.$$

Therefore,

$$y_1 = a_0 x\left(1-\frac{1}{2}x-\frac{1}{12}x^2+\frac{7}{144}x^3+\cdots\right).$$

As for the second solution, we have

$$\begin{aligned}
y &= x^0\left(b_0+b_1x+b_2x^2+\cdots\right)+C\ln(x)y_1,\\
y' &= b_1+2b_2x+\cdots+\frac{C}{x}y_1+C\ln(x)y_1',\\
y'' &= 2b_2+6b_3x+\cdots-\frac{C}{x^2}y_1+\frac{C}{x}y_1'+\frac{C}{x}y_1'+C\ln(x)y_1''.
\end{aligned}$$

Therefore, our differential equation becomes

$$y''+xy'+\frac{y}{x}=0,$$

$$\left(\begin{array}{l}
\left(2b_2+6b_3x+24b_4x^2+\cdots\right)-\frac{C}{x^2}y_1+\frac{2C}{x}y_1'+C\ln(x)y_1''\\
+\left(b_1x+2b_2x^2+\cdots\right)+Cy_1+C\ln(x)xy_1'\\
+\left(\frac{b_0}{x}+b_1+b_2x+\cdots\right)+C\ln(x)\frac{y_1}{x}
\end{array}\right)=0,$$

$$\left(\begin{array}{l}
\left(2b_2+6b_3x+24b_4x^2+\cdots\right)-\frac{C}{x^2}y_1+\frac{2C}{x}y_1'\\
+\left(b_1x+2b_2x^2+\cdots\right)+Cy_1+\left(\frac{b_0}{x}+b_1+b_2x+\cdots\right)\\
+\underbrace{C\ln(x)y_1''+C\ln(x)xy_1'+C\ln(x)\frac{y_1}{x}}_{0}
\end{array}\right)=0,$$

$$\left(\begin{array}{l}
\left(2b_2+6b_3x+24b_4x^2+\cdots\right)-\frac{C}{x}\left(1-\frac{1}{2}x-\frac{1}{12}x^2+\frac{7}{144}x^3+\cdots\right)\\
+\frac{2C}{x}\left(1-x+\frac{1}{4}x^2+\frac{7}{36}x^3+\cdots\right)\\
+\left(b_1x+2b_2x^2+\cdots\right)+C\left(x-\frac{1}{2}x^2-\frac{1}{12}x^3+\frac{7}{144}x^4+\cdots\right)\\
+\left(\frac{b_0}{x}+b_1+b_2x+\cdots\right)
\end{array}\right)=0.$$

Looking at the coefficient of x^{-1}, we have $b_0 - C + 2C = 0 \Rightarrow C = -b_0$. Note that b_1 is arbitrary, so choose $b_0 = 0$. Then the coefficient of x^0 implies that $2b_2 + C/2 - 2C = 0 \Rightarrow b_2 = -(3/4)\,b_0$; the coefficient of x^1 implies that

$6b_3 + C/12 + C/2 + C + b_2 = 0 \Rightarrow b_3 = (7/18)\, b_0$. Therefore,

$$y_2 = b_0 \left[\left(1 - \frac{3}{4}x^2 + \frac{7}{18}x^2 + \cdots \right) - \ln(x) \left(x - \frac{1}{2}x^2 - \frac{1}{12}x^3 + \frac{7}{144}x^4 + \cdots \right) \right].$$

$\Diamond$

Example 8.15. Solve $x^2y'' - xy' + 10e^x y = 0$. $*$

Solution. Dividing out by x^2, we have

$$y'' - \frac{y'}{x} + \frac{10e^x}{x^2} y = 0.$$

We have a singularity at $x = 0$. Note that $xP(x) = -1$ and $x^2Q(x) = 10e^x = 10 \sum_{n=0}^{\infty} x^n/n!$. Therefore,

$$y = \sum_{n=0}^{\infty} a_n x^{\alpha+n}, \qquad \frac{y}{x^2} = \sum_{n=0}^{\infty} a_n x^{\alpha+n-2},$$

$$y' = \sum_{n=0}^{\infty} (\alpha+n)\, a_n x^{\alpha+n-1}, \qquad \frac{y'}{x} = \sum_{n=0}^{\infty} (\alpha+n)\, a_n x^{\alpha+n-2},$$

$$y'' = \sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1)\, a_n x^{\alpha+n-2}.$$

With these components in place, our differential equation becomes

$$\sum_{n=0}^{\infty} (\alpha+n)(\alpha+n-1)\, a_n x^{\alpha+n-2} - \sum_{n=0}^{\infty} (\alpha+n)\, a_n x^{\alpha+n-2} + 10 \left(\sum_{n=0}^{\infty} \frac{x^n}{n!} \right) \left(\sum_{n=0}^{\infty} a_n x^{\alpha+n-2} \right) = 0.$$

We now have

$$\alpha(\alpha-1)\, a_0 - \alpha a_0 + 10a_0 = 0.$$

Therefore, $F(\alpha) = \alpha(\alpha-1) - \alpha + 10 = 0 \Rightarrow \alpha = 1 \pm 3i$. For $1 + 3i$, we have

$$\begin{aligned} (\alpha+1)\,\alpha a_1 - (\alpha+1)\, a_1 + 10\,(a_0 + a_1) &= 0, \\ F(\alpha+1)a_1 + 10a_1 &= 0. \end{aligned}$$

Therefore,

$$a_1 = -\frac{10}{F(\alpha+1)} a_0 = -\frac{10}{F(2+3i)} a_0$$

and

$$\begin{aligned}F(2+3i) &= (2+3i)^2 - 2(2+3i) + 10 \\ &= 4 + 12i - 9 - 4 - 6i + 10 \\ &= 1 + 6i.\end{aligned}$$

Therefore,

$$a_1 = -\frac{10}{1+6i}a_0 = -\frac{10(1-6i)}{(1+6i)(1-6i)}a_0 = -\frac{10-60i}{1+36}a_0 = -\frac{10-60i}{37}a_0.$$

Also note that

$$(\alpha+2)(\alpha+1)a_2 - (\alpha+2)a_2 + 10\left(\frac{a_0}{2!} + a_1 + a_2\right) = 0,$$

and we have

$$\begin{aligned}F(\alpha+2)a_2 + 10a_1 + 5a_0 &= F(\alpha+2)a_2 - \frac{100-600i}{37}a_0 + \frac{175}{37}a_0 \\ &= F(\alpha+2)a_2 + \frac{75+600i}{37}a_0.\end{aligned}$$

Therefore,

$$a_2 = \frac{75+600i}{37F(\alpha+2)}a_0$$

and we have

$$z = x^{1+3i}a_0\left(1 - \left(\frac{10-60i}{37}\right)x + \cdots\right).$$

But

$$x^{1+3i} = xx^{3i} = xe^{3i\ln(x)} = x\left[\cos(3\ln(x)) + i\sin(3\ln(x))\right],$$

so

$$\begin{aligned}z &= a_0 x\cos(3\ln(x)) + a_0 i\sin(3\ln(x)) \\ &\quad + a_0\left(-\frac{10}{37}x^2\cos(3\ln(x)) - \frac{60}{37}x^2\sin(3\ln(x)) + \frac{60}{37}\sin(3\ln(x))\right) \\ &\quad + \frac{60}{37}ix^2\cos(3\ln(x)) - \frac{10}{37}ix^2\sin(3\ln(x)) + \cdots.\end{aligned}$$

Therefore,

$$\operatorname{Re}(z) = a_0 \begin{pmatrix} x\cos(3\ln(x)) \\ +x^2\left(-\dfrac{10}{37}\cos(3\ln(x)) - \dfrac{60}{37}\sin(3\ln(x))\right) \\ +x^3\left[\star\cos(3\ln(x)) + \star\sin(3\ln(x))\right] \end{pmatrix} + \cdots$$

◊

Example 8.16. Solve $xy'' + (1-x)\,y' - y = 0$. *

Solution. Note that

$$p_0 = \lim_{x\to 0} xP(x) = \lim_{x\to 0} x\frac{1-x}{x} = 1$$

and

$$q_0 = \lim_{x\to 0} x^2 Q(x) = \lim_{x\to 0} x^2\left(-\frac{1}{x}\right) = 0.$$

Therefore, 0 is a regular singular point. The indicial equation is

$$\begin{aligned} \alpha(\alpha-1) + p_0\alpha + q_0 &= 0, \\ \alpha(\alpha-1) + \alpha &= 0, \\ \alpha^2 - \alpha + \alpha &= 0, \\ \alpha^2 &= 0. \end{aligned}$$

Therefore, $\alpha = 0$ is a double root and

$$y_1 \sum_{n=0}^{\infty} x^n, \qquad y_2 = \sum_{n=0}^{\infty} b_n x^n + C\ln(x)y_1, \quad x > 0.$$

To find y_1, we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^n, \\ y' &= \sum_{n=0}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1)\,a_{n+1}x^n, \\ y'' &= \sum_{n=0}^{\infty} (n+1)\,n x^{n-1}. \end{aligned}$$

Therefore, our differential equation becomes

$$\begin{aligned} xy'' + (1-x)\,y' - y &= 0, \\ \sum_{n=0}^{\infty} \left[(n+1)\,na_{n+1} + (n+1)\,a_{n+1} - na_n - a_n\right] x^n &= 0. \end{aligned}$$

From this, we have

$$\begin{aligned} (n+1)\,na_{n+1} + (n+1)\,a_{n+1} - na_n - a_n &= 0, \\ (n+1)^2\,a_{n+1} &= (n+1)\,a_n, \\ a_{n+1} &= \frac{a_n}{n+1}. \end{aligned}$$

Therefore,

$$a_1 = \frac{a_0}{1}, \quad a_2 = \frac{a_1}{2} = \frac{a_0}{2}, \quad a_3 = \frac{a_2}{2} = \frac{a_0}{3!}, \quad \ldots, \quad a_n = \frac{a_0}{n!}.$$

So we have

$$y = a_0 \sum_{n=0}^{\infty} \frac{x^n}{n!} = a_0 e^x.$$

Immediately, $y_1 = e^x$. To find y_1 (see §5.1, p. 57), we have

$$\begin{aligned} y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2},\, dx &= e^x \int \frac{e^{-\int \frac{1-x}{x}\,dx}}{e^{2x}}\,dx \\ &= e^x \int \frac{e^{-\int \left(\frac{1}{x}-1\right)dx}}{e^{2x}}\,dx = e^x \int \frac{e^{-(\ln(x)-x)}}{e^{2x}}\,dx \\ &= e^x \int \frac{e^x/x}{e^{2x}}\,dx = e^x \int \frac{e^{-x}}{x}\,dx, \end{aligned}$$

which is not an elementary function. ♢

Example 8.17. Solve $\left(x^2+2x\right)y'' - 2\left(x^2+2x-1\right)y' + \left(x^2+2x-2\right)y = 0.$*

Solution. Let $y = \sum_{n=0}^{\infty} a_n x^{\alpha+n}$. Then

$$y' = \sum_{n=0}^{\infty} (\alpha+n)\,a_n x^{\alpha+n-1}, \qquad y'' = \sum_{n=0}^{\infty} (\alpha+n)\,(\alpha+n-1)\,a_n x^{\alpha+n-2}.$$

The DE then becomes

$$\left(\begin{array}{l}\sum_{n=0}^{\infty}(\alpha+n)(\alpha+n-1)a_nx^{\alpha+n}+2\sum_{n=0}^{\infty}(\alpha+n)(\alpha+n-1)a_nx^{\alpha+n-1}\\ -2\sum_{n=0}^{\infty}(\alpha+n)a_nx^{\alpha+n+1}-4\sum_{n=0}^{\infty}(\alpha+n)a_nx^{\alpha+n}\\ +2\sum_{n=0}^{\infty}(\alpha+n)a_nx^{\alpha+n-1}+\sum_{n=0}^{\infty}a_nx^{\alpha+n+2}+2\sum_{n=0}^{\infty}a_nx^{\alpha+n+1}\\ -2\sum_{n=0}^{\infty}a_nx^{\alpha+n}\end{array}\right)=0,$$

$$\left(\begin{array}{l}\sum_{n=1}^{\infty}(\alpha+n-1)(\alpha+n-2)a_{n-1}x^{\alpha+n-1}\\ +2\sum_{n=0}^{\infty}(\alpha+n)(\alpha+n-1)a_nx^{\alpha+n-1}-2\sum_{n=2}^{\infty}(\alpha+n-2)a_{n-2}x^{\alpha+n-1}\\ -4\sum_{n=1}^{\infty}(\alpha+n-1)a_{n-1}x^{\alpha+n-1}+2\sum_{n=0}^{\infty}(\alpha+n)a_nx^{\alpha+n-1}\\ +\sum_{n=3}^{\infty}a_{n-3}x^{\alpha+n-1}+2\sum_{n=2}^{\infty}a_{n-2}x^{\alpha+n-1}-2\sum_{n=1}^{\infty}a_{n-1}x^{\alpha+n-1}\end{array}\right)=0,$$

$$\left[\sum_{n=0}^{\infty}\left(\begin{array}{c}[2(\alpha+n)(\alpha+n-1)+2(\alpha+n)]a_n\\ +[(\alpha+n-1)(\alpha+n-2)-4(\alpha+n-1)-2]a_{n-1}\\ +[-2(\alpha+n-2)+2]a_{n-2}+a_{n-3}\end{array}\right)\right]x^{\alpha+n-1}=0,$$

$$\sum_{n=0}^{\infty}\left(\begin{array}{c}2(\alpha+n)^2a_n+\left[(\alpha+n)^2-7(\alpha+n)+4\right]a_{n-1}\\ +[-2(\alpha+n)+6]a_{n-2}+a_{n-3}\end{array}\right)=0.$$

Note that $2\alpha^2=0\Rightarrow\alpha=0$, giving us a double root. Therefore we get the recursion relation

$$2n^2a_n+\left(n^2-7n+4\right)a_{n-1}+(-2n+6)a_{n-2}+a_{n-3}=0.$$

Substituting gives

$$\begin{gathered}2a_1-2a_0=0\Longrightarrow a_1=a_0,\\ 8a_2-6a_1+2a_0=0\Longrightarrow 8a_2=6a_1-2a_0=(6-2)a_0=4a_0\Longrightarrow a_0=\frac{1}{2},\\ 18a_3+(-8)a_2+(0)a_1+a_2=0\Longrightarrow 18a_3=8\cdot\frac{1}{2}-1=3\Longrightarrow a_3=\frac{1}{6},\\ 32a_4-8a_3-(-2)a_2+(0)a_1+a_2=0\Longrightarrow 32a_3=8\cdot\frac{1}{6}-2\cdot\frac{1}{2}-1=\frac{4}{3}\Longrightarrow a_3=\frac{1}{24}.\end{gathered}$$

If we our intuition leads us to guess that perhaps $a_n = 1/n!$ be can use induction to check that this guess is in fact correct. Explicitly, suppose by induction that $a_k = 1/k!$ for all $k < n$. Then the recursion relation yields

$$\begin{aligned} 2n^2 a_n &= -\frac{n^2 - 7n + 4}{(n-1)!} + \frac{2n-6}{(n-2)!} - \frac{1}{(n-3)!} \\ &= \frac{-n^2 + 7n - 4 + (2n-6)(n-1) - (n-1)(n-2)}{(n-1)!} \\ &= \frac{-n^2 + 7n - 4 + 2n^2 - 8n + 6 - n^2 + 3n - 2}{(n-1)!} \\ &= \frac{2n}{(n-1)!} \end{aligned}$$

and so $a_n = 1/n!$, completing the induction.

This gives the solution $y_1 = \sum_{n=0}^{\infty} x^n/n! = e^x$. (If we had guessed initially that this might be the solution, we could easily have checked it by direct subsitution into the differential equation.) Using reduction of order, one then finds that the solutions are

$$y_1 = e^x, \qquad y_2 = xe^x + 2\ln(x)\, y_1. \qquad \diamondsuit$$

8.8 Functions Defined via Differential Equations

8.8.1 Chebyshev Equation

The *Chebyshev equation* is given by

$$\left(1 - x^2\right) y'' - xy' + \lambda^2 y = 0 \tag{8.5}$$

of which 0 is an ordinary point. The series solution is guaranteed to converge for $|x| < 1$. We have

$$y = \sum_{n=0}^{\infty} a_n x^n,$$
$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1},$$
$$y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} = \sum_{n=0} \infty (n+2)(n+1) a_{n+2} x^n,$$

and our differential equation becomes

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1) a_{n+2} - n(n-1) a_n - n a_n + \lambda^2 a_n\right] x^n = 0.$$

From this, we see that

$$a_{n+2} = \frac{n(n-1) + n - \lambda^2}{(n+2)(n+1)} a_n = \frac{n^2 - \lambda^2}{(n+2)(n+1)} a_n.$$

Therefore,

$$a_0 = 1, \qquad a_1 = 0, \qquad a_2 = -\frac{\lambda^2}{2}, \qquad a_3 = 0,$$
$$a_4 = -\frac{\lambda^2}{4 \cdot 3} = \frac{\left(4 - \lambda^2\right)\left(-\lambda^2\right)}{4!}, \qquad a_5 = 0,$$
$$a_6 = \frac{\left(16 - \lambda^2\right)\left(4 - \lambda^2\right)\left(-\lambda^2\right)}{6!},$$
$$\vdots$$
$$a_{2n} = \frac{\left(4(n-1)^2 - \lambda^2\right)\left(4(n-2)^2 - \lambda^2\right) \cdots \left(-\lambda^2\right)}{(2n)!} a_1.$$

Also,

$$a_0 = 0, \qquad a_1 = 1, \qquad a_2 = 0, \qquad a_3 = \frac{\left(1-\lambda^2\right)}{3\cdot 2}a_1,$$
$$a_4 = 0, \qquad a_5 = \frac{\left(9-\lambda^2\right)\left(1-\lambda^2\right)}{5!}a_1,$$
$$\vdots$$
$$a_{2n+1} = \frac{\left((2n-1)^2-\lambda^2\right)\cdots\left(1-\lambda^2\right)}{(2n+1)!}a_1.$$

Note that if λ is an integer, then one of these solutions terminates, i.e., $a_k = 0$ beyond some point. In this case, one solution is a polynomial known as a *Chebyshev polynomial*.

8.8.2 Legendre Equation

The *Legendre equation* is given by

$$\left(1-x^2\right)y'' - 2xy' + \lambda\left(\lambda+1\right)y = 0 \tag{8.6}$$

of which 0 is an ordinary point. The series solution is guaranteed to converge for $|x| < 1$. Expressed as a power series, the equation becomes

$$\sum_{n=0}^{\infty} = \left[(n+2)\left(n+1\right)a_{n+2} - n\left(n-1\right)a_n - 2na_n + \lambda\left(\lambda+1\right)a_n\right]x^n = 0.$$

From this, we have

$$a_{n+2} = \frac{n\left(n-1\right)+2n-\lambda\left(\lambda+1\right)}{\left(n+2\right)\left(n+1\right)}a_n = \frac{n\left(n+1\right)-\lambda\left(\lambda+1\right)}{\left(n+2\right)\left(n+1\right)}a_n.$$

If λ is an integer, then $a_k = 0$ for $k \geq \lambda$ gives the λth Legendre polynomial. Instead, expanding around $x = 1$ gives

$$p(x) - \frac{x}{1-x^2}.$$

Then

$$p_0 = \lim_{x\to 1} (x-1)\, P(x) = \lim_{x\to 1} \frac{x}{x+1} = \frac{1}{2},$$
$$q_0 = \lim_{x\to 1} (x-1)^2\, Q(x) = -\lim_{x\to 1} (x-1)^2 \frac{\lambda^2}{x^2-1} = 0.$$

Therefore, $x = 1$ is a regular singular point. The indicial equation is

$$\begin{aligned} \alpha(\alpha-1) + p_0\alpha + q_0 &= 0, \\ \alpha(\alpha-1) + \frac{1}{2}\alpha + q_0 &= 0, \\ \alpha^2 - \alpha + \frac{1}{2}\alpha &= 0, \\ \alpha^2 - \frac{1}{2}\alpha &= 0. \end{aligned}$$

Therefore, $\alpha = 0$ or $\alpha = 1/2$, so the solutions look like

$$y_1 = \sum_{n=0}^{\infty} a_n (x-1)^n, \qquad y_2 = (x-1)^{1/2} \sum_{n=0}^{\infty} b_n (x-1)^n$$

for $x > 1$.

For a Legendre equation, around $x = 1$, we have

$$p_0 = \lim_{x\to 1} (x-1) \left(-\frac{2x}{1-x^2}\right) = 1,$$
$$q_0 = \lim_{x\to 1} (x-1)^2 \frac{\lambda(\lambda+1)}{x^2-1} = 0.$$

Therefore, $x = 1$ is a regular singular point. The indicial equation is

$$\begin{aligned} \alpha(\alpha-1) + p_0\alpha + q_0 &= 0, \\ \alpha(\alpha-1) + \alpha &= 0, \\ \alpha^2 &= 0. \end{aligned}$$

Therefore, $\alpha = 0$ is a double root and the solutions look like

$$y_1 = \sum_{n=0}^{\infty} a_n (x-1)^n, \qquad y_2 = \sum_{n=0}^{\infty} b_n (x-1)^n + C \ln(x-1)\, y_1$$

for $x > 1$.

8.8.3 Airy Equation

The *Airy equation* is given by

$$y'' - xy = 0 \tag{8.7}$$

of which 0 is an ordinary point. The solution converges for all x. Expressed as a power series, we have

$$\sum_{n=0}^{\infty} \left[(n+2)(n+1)\, a_{n+2} - a_{n-1}\right] x^n = 0.$$

From this, we see that

$$a_{n+2} = \frac{a_{n-1}}{(n+2)(n+1)},$$

therefore,

$$a_3 = \frac{a_0}{3 \cdot 2}, \quad a_6 = \frac{a_0}{6 \cdot 5}, \quad \ldots, \quad a_{3n} = \frac{a_0}{2 \cdot 3 \cdot 5 \cdot 6 \cdot 8 \cdot 9 \cdots (3n-1)(3n)},$$

$$a_4 = \frac{a_1}{4 \cdot 3}, \quad \ldots, \quad a_{3n+1} = \frac{a_1}{3 \cdot 4 \cdot 6 \cdot 7 \cdots (3n)(3n+1)},$$

$$a_2 = \frac{a_{-1}}{2} = 0, \qquad 0 = a_2 = a_5 = a_8 = \cdots.$$

8.8.4 Laguerre's Equation

*Laguerre's equation** is given by

$$xy'' - (1-x)\, y' + \lambda y = 0 \tag{8.8}$$

of which 0 is a regular singular point. Note that

$$p_0 = \lim_{x \to 0} x \left(\frac{-(1-x)}{x} \right) = -1,$$

$$q_0 = \lim_{x \to 0} x^2 \frac{\lambda}{x} = 0.$$

*Applicable in the context of the hydrogen atom.

Therefore, $x = 0$ is a regular singular point. The indicial equation is

$$\begin{aligned} \alpha(\alpha-1)-\alpha &= 0, \\ \alpha^2-\alpha-\alpha &= 0, \\ \alpha^2-2\alpha &= 0, \\ \alpha(\alpha-2) &= 0. \end{aligned}$$

Therefore, $\alpha = 0$ or $\alpha = 2$, so the solutions look like

$$y_1 = x^2 \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = \sum_{n=0}^{\infty} b_n x^n + C\ln(x) y_1$$

for $x > 0$.

8.8.5 Bessel Equation

The *Bessel equation* is given by

$$x^2 y'' + xy' + \left(x^2 - \lambda^2\right) y = 0. \tag{8.9}$$

We have

$$\begin{aligned} p_0 &= \lim_{x\to 0} x \frac{x}{x^2} = 1, \\ q_0 &= \lim_{x\to 0} x^2 \frac{x^2-\lambda^2}{x^2} = -\lambda. \end{aligned}$$

The indicial equation is then

$$\begin{aligned} \alpha(\alpha-1)+\alpha-\lambda^2 &= 0, \\ \alpha^2-\lambda^2 &= 0, \\ (\alpha+\lambda)(\alpha-\lambda) &= 0. \end{aligned}$$

Therefore, $\alpha = \pm\lambda$. If $\lambda \neq k/2$ for some integer k, then we get

$$y_1 = x^{\lambda} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = x^{-\lambda} \sum_{n=0}^{\infty} a_n x^n$$

for $x > 0$. We encounter trouble if $\lambda = k/2$. Consider $\lambda = 1/2$. Then

$$y_1 = x^{1/2} \sum_{n=0}^{\infty} a_n x^n, \qquad y_2 = x^{-1/2} \sum_{n=0}^{\infty} b_n x^n + C \ln(x) y_1.$$

For y_1, we have

$$\begin{aligned} y &= \sum_{n=0}^{\infty} a_n x^{n+1/2}, \\ y' &= \sum_{n=0}^{\infty} \left(n + \frac{1}{2}\right) a_n x^{n-1/2}, \\ y'' &= \sum_{n=0}^{\infty} \underbrace{\left(n + \frac{1}{2}\right)\left(n - \frac{1}{2}\right)}_{n^2 - 1/4} a_n x^{n-3/2}. \end{aligned}$$

Therefore, the equation $x^2 y'' + xy' + \left(x^2 - 1/4\right) y = 0$ becomes

$$\sum_{n=0}^{\infty} \left[\left(n^2 - \frac{1}{4}\right) a_n + \left(n + \frac{1}{2}\right) a_n + a_{n-2} - \frac{1}{4} a_n\right] x^{n+1/2} = 0.$$

From this, we see that

$$\begin{aligned} \left(n^2 - \frac{1}{4} + n + \frac{1}{2} - \frac{1}{4}\right) a_n + a_{n-2} = 0, \\ a_n &= -\frac{a_{n-2}}{n^2 + n} \\ &= -\frac{a_{n-2}}{n\,(n+1)}. \end{aligned}$$

Therefore,

$$a_2 = -\frac{a_0}{3 \cdot 2}, \quad a_4 = -\frac{a_2}{5 \cdot 4} = \frac{a_0}{5!}, \quad a_6 = -\frac{a_0}{7!}, \quad \ldots, \quad a_{2n} = (-1)^n \frac{a_0}{(2n+1)!}.$$

Therefore

$$y_1 = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1/2}}{(2n+1)!} = \frac{1}{\sqrt{x}} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \frac{\sin(x)}{\sqrt{x}}.$$

Example 8.18. Identify

$$y = \sum_{n=0}^{\infty} \frac{x^{3n}}{(3n)!} = 1 + \frac{x^3}{3!} + \frac{x^6}{6!} + \frac{x^9}{9!} + \cdots.$$

*

Solution. We seek to solve $y''' = y$. From this, we see that

$$\begin{aligned} \lambda^3 - 1 &= 0, \\ (\lambda - 1)\left(\lambda^2 + \lambda + 1\right) &= 0. \end{aligned}$$

Therefore, $\lambda = 1$ or

$$\lambda = \frac{-1 \pm \sqrt{1-4}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i.$$

Therefore, the general solution is

$$y = C_1 e^x + C_2 e^{-x/2} \sin\left(\frac{\sqrt{3}}{2} x\right) + C_3 e^{-x/2} \cos\left(\frac{\sqrt{3}}{2} x\right),$$

where C_1, C_2, C_3 are arbitrary constants.

To find C_1, C_2, C_3, we note that

$$y(0) = 1, \quad y'(0) = 0, \quad y''(0) = 0.$$

Therefore,

$$\begin{aligned} C_1 + 0C_2 + C_3 &= 1, \\ C_1 + \frac{\sqrt{3}}{2} C_2 - \frac{1}{2} C_3 &= 0, \\ C_1 - \frac{\sqrt{3}}{2} C_2 - \frac{1}{2} C_3 &= 0. \end{aligned}$$

Solving this system, we have

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 1 & \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ 1 & -\frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -\frac{\sqrt{3}}{2} & \frac{3}{2} & 1 \\ 0 & \frac{\sqrt{3}}{2} & \frac{3}{2} & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -\frac{\sqrt{3}}{2} & \frac{3}{2} & 1 \\ 0 & 0 & 3 & 2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & -\frac{1}{\sqrt{3}} & 1 & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & \frac{1}{\sqrt{3}} & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \frac{2}{3} \end{array}\right].$$

Therefore, $C_1 = 1/3$, $C_2 = 0$, and $C_3 = 2/3$, so

$$y = \frac{1}{3}e^x + \frac{2}{3}e^{-x/2}\cos\left(\frac{\sqrt{3}}{2}x\right).$$

◊

Chapter 9

Linear Systems

9.1 Preliminaries

Consider

$$\frac{d\mathbf{Y}}{dx} = \mathbf{AY} + B,$$

where

$$\mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

such that

$$\frac{dy_1}{dx} = ay_1 + by_2, \qquad \frac{dy_2}{dx} = cy_1 + dy_2.$$

We will first consider the homogeneous case where $B = 0$.

Only in the case when the entries of $\mathbf{A}$ are constants is there an algorithmic method of finding the solution. If $\mathbf{A} \equiv a$ is a constant, then we know that $y = Ce^{ax}$ is a solution to $dy/dx = ay$. We will define a matrix $e^{\mathbf{A}}$ such that the general solution of $d\mathbf{Y}/dx = \mathbf{AY}$ is $\mathbf{Y} = \mathbf{C}e^{x\mathbf{A}}$, where $\mathbf{C} = [C_1, C_2, \ldots, C_n]$ is a vector of arbitrary constants. We define $e^{\mathbf{T}}$ for an arbitrary $n \times n$ matrix $\mathbf{T}$ and apply $\mathbf{T} = x\mathbf{A}$. Indeed, we know that

$$e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots + \frac{t^n}{n!} + \cdots .$$

Therefore, let us analogously define

$$e^{\mathbf{T}} \equiv \mathbf{I} + \mathbf{T} + \frac{\mathbf{T}^2}{2!} + \frac{\mathbf{T}^3}{3!} + \cdots + \frac{\mathbf{T}^n}{n!} + \cdots . \tag{9.1}$$

We will consider systematically how to compute $e^{\mathbf{T}}$ later, but first we consider an ad hoc example.

Example 9.1. Solve

$$\frac{d\mathbf{Y}}{dx} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \mathbf{Y}.$$

*

Solution. First, we write

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \mathbf{I} + \mathbf{N},$$

where

$$\mathbf{N} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

so that $\mathbf{NI} = \mathbf{IN} = \mathbf{N}$ and $\mathbf{N}^2 = \mathbf{0}$. Note that

$$\begin{aligned} \mathbf{A}^n &= \overbrace{(\mathbf{I} + \mathbf{N})^n}^{\text{Binomial exp.}} \\ &= \mathbf{I}^n + \binom{n}{1}\mathbf{I}^{n-1}\mathbf{N} + \binom{n}{2}\mathbf{I}^{n-1}\mathbf{N}^2 + \cdots + \binom{n}{n-1}\mathbf{I}\mathbf{N}^{n-1} + \mathbf{N}^n \\ &= \mathbf{I} + n\mathbf{N} = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}. \end{aligned}$$

Therefore, with $\mathbf{A}^n$ properly defined, Equation (9.1) gives

$$\begin{aligned} e^{\mathbf{A}x} &= \mathbf{I} + \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \frac{x^2}{2} + \cdots + \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix} \frac{x^n}{n!} + \cdots \\ &= \begin{bmatrix} 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots & 0 + x + 2\frac{x^2}{2!} + 3\frac{x^3}{3!} + \cdots + n\frac{x^n}{n!} \\ 0 & 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots \end{bmatrix} \\ &= \begin{bmatrix} e^x & x\left(1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots\right) \\ 0 & e^x \end{bmatrix} = \begin{bmatrix} e^x & xe^x \\ 0 & e^x \end{bmatrix}. \end{aligned}$$

Therefore, the solution is

$$\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} e^x & xe^x \\ 0 & e^x \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} C_1 e^x + C_2 x e^x \\ C_2 e^x \end{bmatrix},$$

that is,

$$y_1 = C_1 e^x + C_2 x e^x, \qquad y_2 = C_2 e^x.$$

Is this in fact the solution of the equation? We have

$$\begin{aligned}\frac{d}{dx} e^{\mathbf{A}x} &= \frac{d}{dx}\left(\mathbf{I} + \mathbf{A}x + \mathbf{A}^2\frac{x^2}{2!} + \cdots + \mathbf{A}^n\frac{x^n}{n!} + \cdots\right) \\ &= \mathbf{A} + \mathbf{A}^2 x + \cdots + \mathbf{A}^n \frac{x^{n-1}}{(n-1)! + \cdots} \\ &= \mathbf{A}e^{\mathbf{A}x}.\end{aligned}$$

So if $\mathbf{Y} = \mathbf{C}e^{\mathbf{A}x}$, then

$$\frac{d\mathbf{Y}}{dx} = \mathbf{AC}e^{\mathbf{A}x} = \mathbf{AY},$$

as required. Therefore, $\mathbf{Y} = \mathbf{C}e^{\mathbf{A}x}$ is indeed the solution. ◊

9.2 Computing $e^{\mathbf{T}}$

The easiest case to do is a diagonal matrix. If

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

then

$$e^{\mathbf{D}} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}.$$

First, we consider the effect of changing bases. Suppose $\mathbf{T}$ is diagonalizable, that is, there exists a $\mathbf{P}$ such that $\mathbf{P}^{-1}\mathbf{TP}$ is diagonalizable. Therefore, $\mathbf{T} = \mathbf{PDP}^{-1}$ and

$$\mathbf{T}^2 = \left(\mathbf{PDP}^{-1}\right)\left(\mathbf{PDP}^{-1}\right) = \mathbf{PD}^2\mathbf{P}^{-1}.$$

So in general, we have $\mathbf{T}^n = \mathbf{P}\mathbf{D}^n\mathbf{P}^{-1}$. It then follows that

$$\begin{aligned} e^{\mathbf{T}} &= \mathbf{I} + \mathbf{T} + \frac{\mathbf{T}^2}{2!} + \cdots + \frac{\mathbf{T}^n}{n!} \\ &= \mathbf{P}\left(\mathbf{I} + \mathbf{D} + \frac{\mathbf{D}^2}{2!} + \cdots + \frac{\mathbf{D}^n}{n!} + \cdots\right)\mathbf{P}^{-1} \\ &= \mathbf{P}e^{\mathbf{D}}\mathbf{P}^{-1}. \end{aligned}$$

So if $\mathbf{T}$ is diagonalizable, we can compute $e^{\mathbf{T}}$.

Example 9.2. Compute $e^{\mathbf{T}}$, where

$$T = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}.$$

*

Solution. First note that

$$\begin{aligned} \begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} &= 0, \\ (4-\lambda)(2-\lambda) - 3 &= 0, \\ 8 - 6\lambda + \lambda^2 - 3 &= 0, \\ \lambda^2 - 6\lambda + 5 &= 0, \\ (\lambda - 5)(\lambda - 1) &= 0. \end{aligned}$$

Therefore, $\lambda = 1$ or $\lambda = 5$. For $\lambda = 1$, we have

$$\begin{aligned} \begin{bmatrix} 4-1 & 1 \\ 3 & 2-1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} 3a+b \\ 3a+b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

We see that $b = -3a$, so

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -3a \end{bmatrix} = a\begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

For $\lambda = 5$, we have

$$\begin{bmatrix} 4-5 & 1 \\ 3 & 2-5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$
$$\begin{bmatrix} -a+b \\ 3-3b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

We see that $a = b$, so

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

So let

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{P}^{-1} = \frac{1}{|\mathbf{P}|} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}.$$

So

$$\begin{aligned} \mathbf{PTP} &= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ -3 & 5 \end{bmatrix} \\ &= \frac{1}{4} \begin{bmatrix} 1 & 0 \\ 0 & 20 \end{bmatrix}. \end{aligned}$$

◊

What is $\mathbf{P}$? For each eigenvalue λ_j, we find its corresponding eigenvector $\mathbf{v}_j$. Then for $\mathbf{P} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, which is an $n \times n$ matrix, we have

$$\mathbf{A} = \mathbf{P} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix} \mathbf{P}^{-1}.$$

For $\mathbf{Y}' = \mathbf{AY}$, we have

$$\begin{aligned}
\mathbf{Y} = e^{\mathbf{A}x}\tilde{\mathbf{C}} = e^{\mathbf{PDP}^{-1}x}\mathbf{C} &= \mathbf{P}e^{x\mathbf{D}}\overbrace{\mathbf{P}^{-1}\tilde{\mathbf{C}}}^{\text{constant}} \\
&= P\begin{bmatrix} e^{\lambda_1 x} & & & \\ & e^{\lambda_2 x} & & \\ & & \ddots & \\ & & & e^{\lambda_n x}\end{bmatrix}\mathbf{C} \\
&= [\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n]\begin{bmatrix} e^{\lambda_1 x} & & & \\ & e^{\lambda_2 x} & & \\ & & \ddots & \\ & & & e^{\lambda_n x}\end{bmatrix}\mathbf{C} \\
&= C_1 e^{\lambda_1 x}\mathbf{v}_1 + C_2 e^{\lambda_2 x}\mathbf{v}_2 + \cdots + C_n e^{\lambda_n x}\mathbf{v}_n,
\end{aligned}$$

where $\mathbf{C} = \mathbf{P}^{-1}\tilde{\mathbf{C}}$.

Example 9.3. Solve

$$\mathbf{Y}' = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\mathbf{Y}.$$

*

Solution. First note that

$$\begin{aligned}
\begin{vmatrix} 4-\lambda & 1 \\ 3 & 2-\lambda \end{vmatrix} &= 0, \\
(4-\lambda)(2-\lambda) - 3 &= 0, \\
8 - 6\lambda + \lambda^2 - 3 &= 0, \\
\lambda^2 - 6\lambda + 5 &= 0, \\
(\lambda-5)(\lambda-1) &= 0.
\end{aligned}$$

Therefore, $\lambda = 1$ or $\lambda = 5$. For $\lambda = 1$, we have

$$\begin{aligned}
\begin{bmatrix} 4-1 & 1 \\ 3 & 2-1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\
\begin{bmatrix} 3a+b \\ 3a+b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

We see that $b = -3a$, so

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ -3a \end{bmatrix} = a \begin{bmatrix} 1 \\ -3 \end{bmatrix}.$$

For $\lambda = 5$, we have

$$\begin{aligned} \begin{bmatrix} 4-5 & 1 \\ 3 & 2-5 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \\ \begin{bmatrix} -a+b \\ 3-3b \end{bmatrix} &= \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \end{aligned}$$

We see that $a = b$, so

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ a \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix}$$

and we have

$$\begin{aligned} \mathbf{Y} &= \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} e^x & 0 \\ 0 & e^{5x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} e^x & e^{5x} \\ -3e^x & e^{5x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\ &= \begin{bmatrix} C_1 e^x + C_2 e^{5x} \\ -3C_1 e^x + C_2 e^{5x} \end{bmatrix} = C_1 e^x \begin{bmatrix} 1 \\ -3 \end{bmatrix} + C_2 e^{5x} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \end{aligned}$$

that is,

$$y_1 = C_1 e^x + C_2 e^{5x}, \qquad y_2 = -3C_1 e^x + C_2 e^{5x}.$$

◊

If $\mathbf{T}$ does not have n distinct eigenvalues, computing $e^{\mathbf{T}}$ requires greater knowledge of linear algebra.*

Property 9.4

1. $e^{\mathbf{PAP}^{-1}} = \mathbf{P}e^{\mathbf{A}}\mathbf{P}^{-1}$.
2. If $\mathbf{AB} = \mathbf{BA}$, then $e^{\mathbf{A}+\mathbf{B}} = e^{\mathbf{A}}e^{\mathbf{B}} = e^{\mathbf{B}}e^{\mathbf{A}}$.

*MATB24.

3. If

$$\mathbf{D} = \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_n \end{bmatrix},$$

then

$$e^{\mathbf{D}} = \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix}.$$

4. If $\mathbf{A}^{n+1} = \mathbf{0}$, then

$$e^{\mathbf{A}} = \mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots + \frac{\mathbf{A}^n}{n!}.$$

Proof. We have already proved (1), and (3) and (4) are trivial. What remains is (2). So note that

$$\begin{aligned} e^{\mathbf{A}} e^{\mathbf{B}} &= \left(\mathbf{I} + \mathbf{A} + \frac{\mathbf{A}^2}{2!} + \cdots + \frac{\mathbf{A}^n}{n!}\right)\left(\mathbf{I} + \mathbf{B} + \frac{\mathbf{B}^2}{2!} + \cdots + \frac{\mathbf{B}^n}{n!}\right) \\ &= \mathbf{I} + \mathbf{A} + \mathbf{B} + \frac{1}{2}\left(\mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2\right) + \cdots . \end{aligned}$$

Also

$$\begin{aligned} e^{\mathbf{A}+\mathbf{B}} &= \mathbf{I} + \mathbf{A} + \mathbf{B} + \frac{(\mathbf{A}+\mathbf{B})^2}{2!} + \frac{(\mathbf{A}+\mathbf{B})^3}{3!} + \cdots \\ &= \mathbf{I} + \mathbf{A} + \mathbf{B} + \frac{1}{2}\left(\mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2\right) + \cdots , \end{aligned}$$

which is the same as the above if $\mathbf{AB} = \mathbf{BA}$. □

Theorem 9.5
Given $\mathbf{T}$ over $\mathbb{C}$, there exists matrices $\mathbf{P}$, $\mathbf{D}$, and $\mathbf{N}$ with $\mathbf{D}$ diagonal, $\mathbf{N}^{k+1} = \mathbf{0}$ for some k, and $\mathbf{DN} = \mathbf{ND}$ such that $\mathbf{T} = \mathbf{P}(\mathbf{D} + \mathbf{N})\mathbf{P}^{-1}$.

Proof. Proof is given in MATB24. □

Corollary 9.6
We have $e^{\mathbf{T}} = \mathbf{P}e^{\mathbf{D}}e^{\mathbf{N}}\mathbf{P}^{-1}$ ($e^{\mathbf{D}}$ and $e^{\mathbf{N}}$ can be computed as above).

Proof. Proof is given in MATB24. □

The steps involved in finding $\mathbf{P}$, $\mathbf{N}$, and $\mathbf{D}$ are essentially the same as those involved in putting $\mathbf{T}$ into Jordan Canonical form (over $\mathbb{C}$).

9.3 The 2×2 Case in Detail

Theorem 9.7 (Cayley-Hamilton)
Let $p(\lambda) = |\mathbf{A} - \lambda\mathbf{I}|$. Then $p(\mathbf{A}) = \mathbf{0}$.

Proof. Proof is given in MATB24. □

The eigenvalues of $\mathbf{A}$ are the solutions of $p(\lambda) = 0$. We now consider three cases for the solutions of $p(\lambda) = 0$.

CASE 1: *$p(\lambda)$ has two distinct real roots λ_1 and λ_2.* This is the diagonalizable case considered earlier. We find eigenvectors $\mathbf{v}$ and $\mathbf{w}$ for the eigenvalues λ_1 and λ_2. Let

$$\mathbf{P} = \begin{bmatrix} \mathbf{v}_1 & \mathbf{w}_1 \\ \mathbf{v}_2 & \mathbf{w}_2 \end{bmatrix}.$$

Then

$$\mathbf{P}^{-1}\mathbf{T}\mathbf{P} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix},$$

so

$$e^{\mathbf{T}} = \mathbf{P}\begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}\mathbf{P}^{-1}.$$

The solution of $\mathbf{Y}' = \mathbf{T}\mathbf{Y}$ would be

$$\mathbf{Y} = \mathbf{P}\begin{bmatrix} e^{\lambda_1 x} & 0 \\ 0 & e^{\lambda_2 x} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$

in this case.

CASE 2: *$p(\lambda)$ has a double root r.* Let $\mathbf{S} = \mathbf{T} - r\mathbf{I}$ so that $\mathbf{T} = S + r\mathbf{I}$. Since $\mathbf{SI} = \mathbf{IS}$, we have

$$e^{\mathbf{T}} = e^{\mathbf{S}} e^{r\mathbf{I}} = e^{s} \begin{bmatrix} e^r & 0 \\ 0 & e^r \end{bmatrix}.$$

Then

$$\begin{aligned} p(\mathbf{T}) &= \mathbf{0}, \\ (\mathbf{T} - r\mathbf{I})^2 &= \mathbf{0}, \\ \mathbf{S}^2 &= \mathbf{0}. \end{aligned}$$

Therefore,

$$e^{\mathbf{S}} = \mathbf{I} + \mathbf{S} = \mathbf{I} + \mathbf{T} - r\mathbf{I} = \mathbf{T} + (1 - r)\,\mathbf{I}$$

and

$$e^{\mathbf{T}} = (\mathbf{T} + (1 - r)\,\mathbf{I}) \begin{bmatrix} e^r & 0 \\ 0 & e^r \end{bmatrix}.$$

CASE 3: *$p(\lambda)$ has complex roots $r \pm iq$, where $q \neq 0$.* Let $\mathbf{S} = \mathbf{T} - r\mathbf{I}$ so that $\mathbf{T} = \mathbf{S} + r\mathbf{I}$. Then

$$e^{\mathbf{T}} = e^{\mathbf{S}} \begin{bmatrix} e^r & 0 \\ 0 & e^r \end{bmatrix}$$

and

$$p(\lambda) = \lambda^2 - 2r\lambda + \left(r^2 + q^2\right) = (\lambda - r)^2 + q^2.$$

We then have

$$\begin{aligned} p(\mathbf{T}) &= \mathbf{0}, \\ (\mathbf{T} - r\mathbf{I})^2 + (q\mathbf{I})^2 &= \mathbf{0}, \\ \mathbf{S}^2 + (q\mathbf{I})^2 &= \mathbf{0}, \\ \mathbf{S}^2 + q^2\mathbf{I} &= \mathbf{0}. \end{aligned}$$

Therefore, $\mathbf{S}^2 - q^2\mathbf{I}$ and it follows that

$$\begin{aligned}
e^{\mathbf{S}} &= \mathbf{I} + \mathbf{S} + \frac{\mathbf{S}^2}{2!} + \frac{\mathbf{S}^3}{3!} + \cdots \\
&= \mathbf{I} + \mathbf{S} - \frac{q^2\mathbf{I}}{2!} - \frac{q^2\mathbf{S}}{3!} + \frac{q^4\mathbf{I}}{4!} + \frac{q^4\mathbf{S}}{5!} - \frac{q^6\mathbf{I}}{6!} + \cdots \\
&= \mathbf{I}\left(1 - \frac{q^2}{2!} + \frac{q^4}{4!} - \frac{q^6}{6!} + \cdots\right) + \mathbf{S}\left(1 - \frac{q^2}{3!} + \frac{q^4}{5!} - \cdots\right) \\
&= \mathbf{I}\cos(q) + \frac{\mathbf{S}}{q}\sin(q).
\end{aligned}$$

Therefore,

$$e^{\mathbf{T}} = \left(\cos(q)\mathbf{I} + \left(\frac{\mathbf{T} - r\mathbf{I}}{q}\right)\sin(q)\right)\begin{bmatrix} e^r & 0 \\ 0 & e^r \end{bmatrix}.$$

Example 9.8 (Double root). Solve

$$\mathbf{Y}' = \begin{bmatrix} -3 & -9 \\ 4 & 9 \end{bmatrix}\mathbf{Y}.$$

*

Solution. We have

$$\begin{aligned}
p(\lambda) &= 0, \\
\begin{vmatrix} -3-\lambda & -9 \\ 4 & 9-\lambda \end{vmatrix} &= 0, \\
(-3-\lambda)(9-\lambda) + 36 &= 0, \\
-27 - 6\lambda + \lambda^2 + 36 &= 0, \\
\lambda^2 - 6\lambda + 9 &= 0, \\
(\lambda - 3)^2 &= 0.
\end{aligned}$$

Therefore, $\lambda = 3$ is a double root and we have $\mathbf{Y} = e^{\mathbf{A}x}\mathbf{C}$. For $\mathbf{A}$, the eigenvalue is 3. Therefore, for $\mathbf{T}$, the eigenvalue is $r = 3x$. Hence,

$$\begin{aligned}
\mathbf{Y} &= (\mathbf{T} + (1-r)\,\mathbf{I}) \begin{bmatrix} e^r & 0 \\ 0 & e^r \end{bmatrix} \\
&= (\mathbf{A}x + (1-3x)\,\mathbf{I}) \begin{bmatrix} e^{3x} & 0 \\ 0 & e^{3x} \end{bmatrix} \mathbf{C} \\
&= \left(\begin{bmatrix} -3x & -9x \\ 4x & 9x \end{bmatrix} + \begin{bmatrix} 1-3x & 0 \\ 0 & 1-3x \end{bmatrix} \right) \begin{bmatrix} e^{3x} & 0 \\ 0 & e^{3x} \end{bmatrix} \mathbf{C} \\
&= \begin{bmatrix} 1-6x & -9x \\ 4x & 1+6x \end{bmatrix} \begin{bmatrix} e^{3x} & 0 \\ 0 & e^{3x} \end{bmatrix} \mathbf{C} \\
&= \begin{bmatrix} e^{3x} - 6xe^{3x} & -9xe^{3x} \\ 4xe^{3x} & e^{3x} + 6xe^{3x} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\
&= \begin{bmatrix} C_1 \left(e^{3x} - 6xe^{3x}\right) - 9C_2xe^{3x} \\ 4C_1xe^{3x} + C_2 \left(e^{3x} + 6xe^{3x}\right) \end{bmatrix} \\
&= C_1e^{3x} \begin{bmatrix} 1 \\ 4 \end{bmatrix} + C_1xe^{3x} \begin{bmatrix} -6 \\ 1 \end{bmatrix} + C_2e^{3x} \begin{bmatrix} -9 \\ 1 \end{bmatrix} + C_2xe^{3x} \begin{bmatrix} 9 \\ 6 \end{bmatrix}. \qquad \lozenge
\end{aligned}$$

Example 9.9 (Complex roots). Solve

$$\mathbf{Y}' = \begin{bmatrix} -1 & 4 \\ -1 & -1 \end{bmatrix} \mathbf{Y}.$$

∗

Solution. We have

$$\begin{aligned}
p(\lambda) &= 0, \\
\begin{vmatrix} -1-\lambda & 4 \\ -1 & -1-\lambda \end{vmatrix} &= 0, \\
(-1-\lambda)(-1-\lambda) + 4 &= 0, \\
(\lambda+1)^2 + 4 &= 0.
\end{aligned}$$

Therefore, $\lambda = -1 \pm 2i$. For $\mathbf{A}$, the roots are $-1 \pm 2i$, so for $\mathbf{T}$, the roots are $(-1 \pm 2i)\,x$, i.e., $r = -x$ and $q = 2x$. Note that

$$\frac{\mathbf{T} - r\mathbf{I}}{q} = \frac{1}{2x} \begin{bmatrix} 0 & 4x \\ x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}.$$

Thus,

$$\begin{aligned}
\mathbf{Y} &= e^{\mathbf{A}x}\mathbf{C} \\
&= \left(\cos(2x)\mathbf{I} + \sin(2x)\begin{bmatrix} 0 & 2 \\ \frac{1}{2} & 0 \end{bmatrix}\right)\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{bmatrix}\mathbf{C} \\
&= \begin{bmatrix} \cos(2x) & 2\sin(2x) \\ \frac{1}{2}\sin(2x) & \cos(2x) \end{bmatrix}\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{-x} \end{bmatrix}\mathbf{C} \\
&= \begin{bmatrix} e^{-x}\cos(2x) & 2e^{-x}\sin(2x) \\ \frac{1}{2}e^{-x}\sin(2x) & e^{-x}\cos(2x) \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\
&= \begin{bmatrix} C_1e^{-x}\cos(2x) + 2C_2e^{-x}\sin(2x) \\ \frac{1}{2}C_1e^{-x}\sin(2x) + C_2e^{-x}\cos(2x) \end{bmatrix}.
\end{aligned}$$

◊

9.4 The Non-Homogeneous Case

The general solution of $\mathbf{Y}' = \mathbf{AY} + \mathbf{B}$ is $\mathbf{Y} = \mathbf{Y}_h\mathbf{C} + \mathbf{Y}_p$, where $\mathbf{Y}_h\mathbf{C}$ is the general solution of $\mathbf{Y}' = \mathbf{AY}$. Therefore, $\mathbf{Y}_h'\mathbf{C} = \mathbf{AY}_h\mathbf{C}$ for all $\mathbf{C}$, giving us

$$\mathbf{Y}_h' = \mathbf{AY}_h, \tag{9.2a}$$

and $\mathbf{Y}_p$ is a particular solution to $\mathbf{Y}' = \mathbf{AY} + \mathbf{B}$, giving us

$$\mathbf{Y}_p' = \mathbf{AY}_p + \mathbf{B}. \tag{9.2b}$$

To find $\mathbf{Y}_p$, we use variation of parameters. Thus, we have

$$\mathbf{Y}_p = \mathbf{Y}_h\mathbf{V}, \tag{9.2c}$$

where

$$\mathbf{V} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}.$$

To find $\mathbf{V}$, we have

$$\begin{aligned}
\mathbf{Y}_p &= \mathbf{Y}_h\mathbf{V}, \\
\underbrace{\mathbf{Y}_p'}_{(9.2b)} &= \underbrace{\mathbf{Y}_h'}_{(9.2a)}\mathbf{V} + \mathbf{Y}_h\mathbf{V}', \\
\mathbf{A}\mathbf{Y}_p + \mathbf{B} &= \mathbf{A}\underbrace{\mathbf{Y}_h\mathbf{V}}_{(9.2c)} + \mathbf{Y}_h\mathbf{V}' = \mathbf{A}\mathbf{Y}_p + \mathbf{Y}_h\mathbf{V}'.
\end{aligned}$$

Therefore, $\mathbf{Y}_h\mathbf{V}' = \mathbf{B} \Rightarrow \mathbf{V}' = \mathbf{Y}_h^{-1}\mathbf{B}$.

Example 9.10. Solve

$$\mathbf{Y}' = \begin{bmatrix} 4 & 1 \\ 3 & 2 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} e^{5x} \\ e^{2x} \end{bmatrix}.$$

*

Solution. As before, we have $\lambda_1 = 1$ and $\lambda_2 = 5$. Thus,

$$\mathbf{V}_1 = \begin{bmatrix} \quad 1 - 3 \end{bmatrix}, \qquad \mathbf{V}_2 = \begin{bmatrix} 11 \end{bmatrix}.$$

Therefore,

$$\mathbf{P} = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}, \qquad \mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix},$$

and the solution of $\mathbf{Y}' = \mathbf{A}\mathbf{Y}$ is $\mathbf{Y} = \mathbf{P}e^{\mathbf{D}x}\mathbf{C} = \mathbf{Y}_h\mathbf{C}$. So we have

$$\begin{aligned}
\mathbf{Y}_h = \mathbf{P}e^{\mathbf{D}x} &= \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} e^x & 0 \\ 0 & e^{5x} \end{bmatrix}\begin{bmatrix} e^x & e^{5x} \\ -3e^x & e^{5x} \end{bmatrix}, \\
\mathbf{Y}_h^{-1} = e^{-\mathbf{D}x}\mathbf{P}^{-1} &= \frac{1}{|\mathbf{P}|}\begin{bmatrix} e^{-x} & 0 \\ 0 & e^{-5x} \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{-x} & -e^{-x} \\ 3e^{-5x} & e^{-5x} \end{bmatrix}.
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathbf{V}' = \mathbf{Y}_h^{-1}\mathbf{B} &= \frac{1}{4}\begin{bmatrix} e^{-x} & -e^{-x} \\ 3e^{-5x} & e^{-5x} \end{bmatrix}\begin{bmatrix} e^{5x} \\ e^{2x} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} e^{4x} - e^x \\ 3 + e^{-3x} \end{bmatrix}, \\
\mathbf{V} &= \frac{1}{4}\begin{bmatrix} \frac{1}{4}e^{4x} - e^x \\ 3x - \frac{1}{3}e^{-3x} \end{bmatrix}
\end{aligned}$$

and

$$\mathbf{Y}_p = \mathbf{Y}_h\mathbf{V} = \frac{1}{4}\begin{bmatrix} e^x & e^{5x} \\ -3e^x & e^{5x} \end{bmatrix}\begin{bmatrix} \frac{1}{4}e^{4x} - e^x \\ 3x - \frac{1}{3}e^{-3x} \end{bmatrix}$$

$$= \frac{1}{4}\begin{bmatrix} \frac{1}{4}e^{5x} - e^{2x} + 3xe^{5x} - \frac{1}{3}e^{2x} \\ -\frac{3}{4}e^{5x} + 3e^{2x} + 9xe^{5x} - \frac{1}{3}e^{2x} \end{bmatrix}$$

$$= e^{5x}\begin{bmatrix} \frac{1}{16} \\ -\frac{3}{16} \end{bmatrix} + e^{-2x}\begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} + xe^{5x}\begin{bmatrix} \frac{3}{4} \\ \frac{9}{4} \end{bmatrix} + e^{2x}\begin{bmatrix} -\frac{1}{12} \\ -\frac{1}{12} \end{bmatrix}.$$

Therefore,

$$\mathbf{Y} = \mathbf{Y}_h\mathbf{C} + \mathbf{Y}_p$$

$$= C_1e^x\begin{bmatrix} 1 \\ -3 \end{bmatrix} + C_2e^{5x}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{5x}\begin{bmatrix} \frac{1}{16} \\ -\frac{3}{16} \end{bmatrix}$$

$$+ e^{-2x}\begin{bmatrix} -\frac{1}{4} \\ \frac{3}{4} \end{bmatrix} + xe^{5x}\begin{bmatrix} \frac{3}{4} \\ \frac{9}{4} \end{bmatrix} + e^{2x}\begin{bmatrix} -\frac{1}{12} \\ -\frac{1}{12} \end{bmatrix},$$

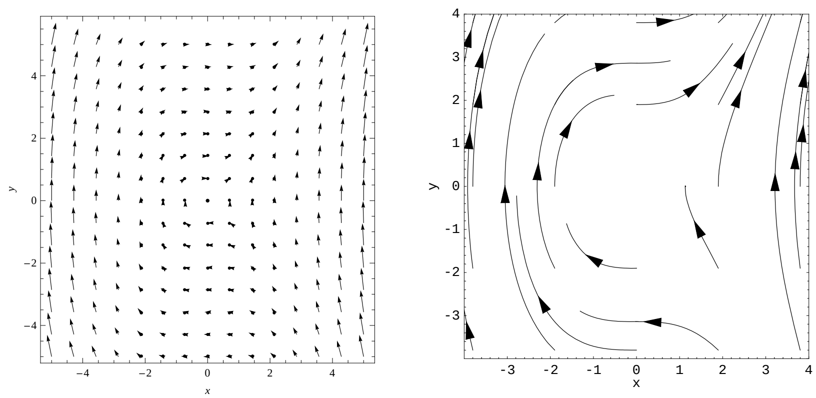
where C_1 and C_2 are arbitrary constants. ◊

9.5 Phase Portraits: Qualitative and Pictorial Descriptions of Solutions of Two-Dimensional Systems

Let $\mathbf{V} : \mathbb{R}^2 \to \mathbb{R}^2$ be a vector field. Imagine, for example, that $\mathbf{V}(x, y)$ represents the velocity of a river at the point (x, y).* We wish to get the description of the path that a leaf dropped in the river at the point (x_0, y_0) will follow. For example, Figure 9.1 shows the vector field of $\mathbf{V}(x, y) = \big(y, x^2\big)$. Let $\gamma(t) = (x(t), y(t))$ be such a path. At any time, the leaf will go in the direction that the river is flowing at the point at which it is presently located, i.e., for all t, we have $\gamma'(t) = \mathbf{V}(x(t), y(t))$. If $\mathbf{V}(F, G)$, then

$$\frac{dx}{dt} = \underbrace{F(x, y)}_{y}, \qquad \frac{dy}{dt} = \underbrace{G(x, y)}_{x^2}.$$

*We are assuming here that $\mathbf{V}$ depends only on the position (x, y) and not also upon time t.



(a) The vector field plot of (y, x^2). (b) The trajectories of (y, x^2)

Figure 9.1: The vector field plot of (y, x^2) and its trajectories.

In general, it will be impossible to solve this system exactly, but we want to be able to get the overall shape of the solution curves, e.g., we can see that in Figure 9.1, no matter where the leaf is dropped, it will head towards (∞, ∞) as $t \to \infty$.

Before considering the general case, let us look at the linear case where we can solve it exactly, i.e., $\mathbf{V} = (ax + by, cx + dy)$ with

$$\frac{dx}{dt} = ax + by, \qquad \frac{dy}{dt} = cx + dy,$$

or $\mathbf{X}' = \mathbf{A}\mathbf{X}$, where

$$\mathbf{X} = \begin{bmatrix} x \\ y \end{bmatrix}, \qquad \mathbf{A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Recall from Theorem 10.12 (p. 157) that if all the entries of $\mathbf{A}$ are continuous, then for any point (x_0, y_0), there is a unique solution of $\mathbf{X}' = \mathbf{A}\mathbf{X}$ satisfying $x(t_0) = x_0$ and $y(t_0) = y_0$, i.e., there exists a unique solution through each point; in particular, the solution curves do not cross.

The above case can be solved explicitly, where

$$\mathbf{X} = e^{\mathbf{A}t} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

is a solution passing through (x_0, y_0) at time $t = 0$. We will consider only cases where $|\mathbf{A}| \neq 0$.

9.5.1 Real Distinct Eigenvalues

Let λ_1 and λ_2 be distinct eigenvalues of $\mathbf{A}$ and let $\mathbf{v}$ and $\mathbf{w}$ be their corresponding eigenvectors. Let $\mathbf{P} = (\mathbf{v}, \mathbf{w})$. Then

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \underbrace{\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}}_{\mathbf{D}}.$$

Therefore, $\mathbf{A}t = \mathbf{P}\left(\mathbf{D}t\right)\mathbf{P}^{-1}$ and we have

$$\begin{aligned} \mathbf{X} = e^{\mathbf{A}t}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} &= \mathbf{P}e^{\mathbf{D}t}\mathbf{P}^{-1}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = (\mathbf{v}, \mathbf{w})\begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \\ &= (\mathbf{v}, \mathbf{w})\begin{bmatrix} C_1 e^{\lambda_1 t} \\ C_2 e^{\lambda_2 t} \end{bmatrix} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2 e^{\lambda_2 t}\mathbf{w}. \end{aligned}$$

Different C_1 and C_2 values give various solution curves.

Note that $C_1 = 1$ and $C_2 = 0$ implies that $\mathbf{X} = e^{\lambda_1 t}\mathbf{v}$. If $\lambda_1 < 0$, then the arrows point toward the origin as shown in Figure 9.2a in which contains a *stable node*. Note that

$$\mathbf{X} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2 e^{\lambda_2 t}\mathbf{w} = e^{\lambda_2 t}\left(C_1 e^{(\lambda_1 - \lambda_2)t}\mathbf{v} + C_2\mathbf{w}\right).$$

The coefficient of $\mathbf{v}$ goes to 0 as $t \to \infty$, i.e., as $t \to \infty$, $\mathbf{X} \to (0, 0)$, approaching along a curve whose tangent is $\mathbf{w}$. As $t \to -\infty$, $\mathbf{X} = e^{\lambda_1 t}\left(C_1\mathbf{v} + C_2 e^{(\lambda_2 - \lambda_1)t}\mathbf{w}\right)$, i.e., the curves get closer and closer to being parallel to $\mathbf{v}$ as $t \to -\infty$.

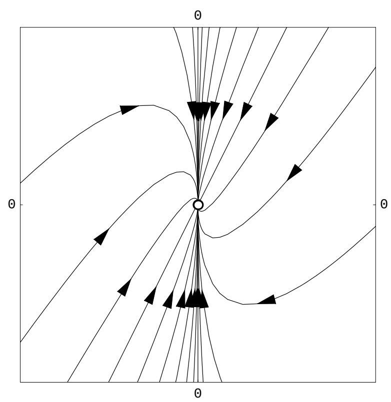
We have the degenerate case when $\lambda_1 < \lambda_2 = 0$, in which case $\mathbf{X} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2\mathbf{w}$.

The case when $\lambda_1 < 0 < \lambda_2$ gives us the phase portrait shown in Figure 9.2b in which contains a *saddle point*. This occurs when $|\mathbf{A}| < 0$. The case when $0 < \lambda_1 < \lambda_2$ gives us the phase portrait shown in Figure 9.2c in which contains an *unstable node*. We have

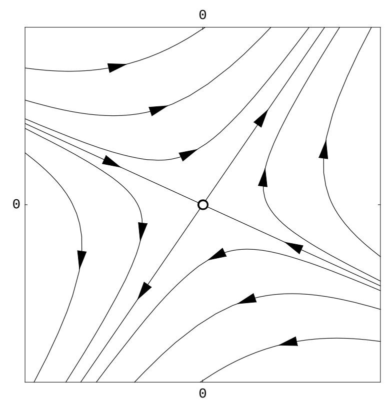
$$\mathbf{X} = C_1 e^{\lambda_1 t}\mathbf{v} + C_2 e^{\lambda_2 t}\mathbf{w} = e^{\lambda_1 t}\left(C_1\mathbf{v} + C_2 e^{(\lambda_2 - \lambda_1)t}\mathbf{w}\right).$$

Therefore, as $t \to \infty$, $\mathbf{X}$ approaches parallel to $\mathbf{w}$ asymptotically.

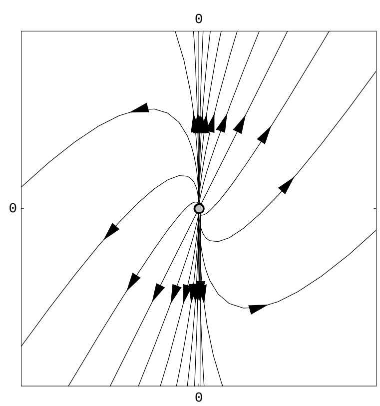
Note that in all cases, the origin itself is a fixed point, i.e., at the origin, $x' = 0$ and $y' = 0$, so anything dropped at the origin stays there. Such a point



(a) $\lambda_1 < \lambda_2 < 0$



(b) $\lambda_1 < 0 < \lambda_2$



(c) $0 < \lambda_1 < \lambda_2$

Figure 9.2: The cases for λ_1 and λ_2.

is called an equilibrium point; in a stable node, if it is disturbed, it will come back in an unstable node; if perturbed slightly, it will leave the vicinity of the origin.

9.5.2 Complex Eigenvalues

Complex eigenvalues come in the form $\lambda = \alpha \pm \beta i$, where $\beta \neq 0$. In such a case, we have

$$\mathbf{X} = C_1 \operatorname{Re}\big(e^{\lambda t}\mathbf{v}\big) + C_2 \operatorname{Im}\big(e^{\lambda t}\mathbf{v}\big)\,,$$

where $\mathbf{v} = \mathbf{p} + i\mathbf{q}$ is an eigenvector for λ. Then

$$\begin{aligned} e^{\lambda t}\mathbf{v} &= e^{\alpha t}e^{\beta i t}\,(\mathbf{p} + i\mathbf{q}) \\ &= e^{\alpha t}\,(\cos(\beta t) + i\sin(\beta t))\,(\mathbf{p} + i\mathbf{q}) \\ &= e^{\alpha t}\,(\cos(\beta t)\mathbf{p} - \sin(\beta t)\mathbf{q} + i\cos(\beta t)\mathbf{q} + i\sin(\beta t)\mathbf{p})\,. \end{aligned}$$

Therefore,

$$\begin{aligned} \mathbf{X} &= e^{\alpha t}\left[(C_1\cos(\beta t)\mathbf{p} - C_1\sin(\beta t)\mathbf{q}) + C_1\cos(\beta t)\mathbf{q} + C_2\sin(\beta t)\mathbf{q}\right] \\ &= e^{\alpha t}\begin{bmatrix} k_1\cos(\beta t) + k_2\sin(\beta t) \\ k_3\cos(\beta t) + k_4\sin(\beta t) \end{bmatrix} \\ &= e^{\alpha t}\left(\cos(\beta t)\begin{bmatrix} k_1 \\ k_3 \end{bmatrix} + \sin(\beta t)\begin{bmatrix} k_2 \\ k_4 \end{bmatrix}\right). \end{aligned}$$

Note that $\operatorname{tr}(\mathbf{A}) = 2\alpha$.* So

$$\begin{aligned} \alpha = 0 &\Longrightarrow \operatorname{tr}(\mathbf{A}) = 0, \\ \alpha > 0 &\Longrightarrow \operatorname{tr}(\mathbf{A}) < 0, \\ \alpha < 0 &\Longrightarrow \operatorname{tr}(\mathbf{A}) > 0. \end{aligned}$$

Consider first $\alpha = 0$. To consider the axes of the ellipses, first note that

$$\mathbf{X} = \underbrace{\begin{bmatrix} p_1 & q_1 \\ p_2 & q_2 \end{bmatrix}}_{\mathbf{P}}\underbrace{\begin{bmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{bmatrix}}_{\mathbf{C}}\begin{bmatrix} \cos(\beta t) \\ \sin(\beta t) \end{bmatrix}.$$

*Recall that the *trace* of a matrix is the sum of the elements on the main diagonal.

Except in the degenerate case, where $\mathbf{p}$ and $\mathbf{q}$ are linearly dependent, we have

$$\begin{bmatrix} \cos(\beta t) \\ \sin(\beta t) \end{bmatrix} = \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{X}.$$

Therefore,

$$\begin{aligned} \cos(\beta t) + \sin(\beta t) \begin{bmatrix} \cos(\beta t) \\ \sin(\beta t) \end{bmatrix} &= \mathbf{X}^t \left(\mathbf{P}^{-1}\right)^t \left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{X}, \\ \cos^2(\beta t) + \sin^2(\beta t) &= \mathbf{X}^t \left(\mathbf{P}^{-1}\right)^t \left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{X}, \\ 1 &= \mathbf{X}^t \left(\mathbf{P}^{-1}\right)^t \left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{X}. \end{aligned}$$

Note that $\mathbf{C} = \mathbf{C}^t$, so

$$\begin{aligned} \mathbf{C}\mathbf{C}^t &= \begin{bmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{bmatrix} \begin{bmatrix} C_1 & C_2 \\ C_2 & -C_1 \end{bmatrix} \\ \mathbf{C}^2 &= \begin{bmatrix} C_1^2 + C_2^2 & 0 \\ 0 & C_1^2 + C_2^2 \end{bmatrix} \\ &= \left(C_1^2 + C_2^2\right)\mathbf{I}. \end{aligned}$$

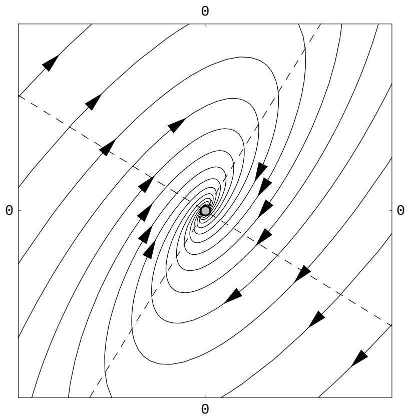
Therefore,

$$\begin{aligned} \mathbf{C}^{-1} &= \frac{\mathbf{C}}{C_1^2 + C_2^2}, \\ \left(\mathbf{C}^{-1}\right)^t = \mathbf{C}^{-1} &= \frac{\mathbf{C}}{C_1^2 + C_2^2}, \\ \left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1} = \left(\mathbf{C}^{-1}\right)^2 &= \frac{\left(C_1^2 + C_2^2\right)\mathbf{I}}{\left(C_1^2 + C_2^2\right)^2} = \frac{\mathbf{I}}{C_1^2 + C_2^2}. \end{aligned}$$

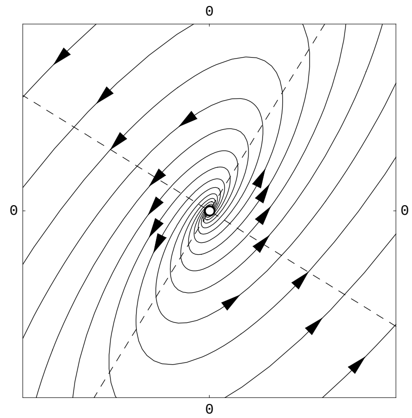
Therefore,

$$1 = \mathbf{X}^t \left(\mathbf{P}^{-1}\right)^t \left(\mathbf{C}^{-1}\right)^t \mathbf{C}^{-1}\mathbf{P}^{-1}\mathbf{X} = \frac{1}{C_1^2 + C_2^2}\mathbf{X}^t \left(\mathbf{P}^{-1}\right)^t \mathbf{P}^{-1}\mathbf{X}.$$

Let $\mathbf{T} = \left(\mathbf{P}^{-1}\right)^t \mathbf{P}^{-1}$. Then $\mathbf{X}^t\mathbf{T}\mathbf{X} = C_1^2 + C_2^2$ and $\mathbf{T} = \mathbf{T}^t$ ($\mathbf{T}$ is symmetric). Therefore, the eigenvectors of $\mathbf{T}$ are mutually orthogonal and form the axes of the ellipses. Figure 9.3 shows a stable spiral and an unstable spiral.



(a) $\alpha < 0$



(b) $\alpha > 0$

Figure 9.3: The cases for α, where we have a stable spiral when $\alpha > 0$ and an unstable spiral when $\alpha > 0$.

9.5.3 Repeated Real Roots

We have $\mathbf{N} = \mathbf{A} - \lambda\mathbf{I}$, where $\mathbf{N}^2 = \mathbf{0}$ and $\mathbf{A} = \mathbf{N} + \lambda\mathbf{I}$. So

$$e^{\mathbf{A}t} = e^{\mathbf{N}t + \lambda t\mathbf{I}} = e^{\mathbf{N}t}e^{\lambda t\mathbf{I}} = (\mathbf{I} + \mathbf{N}t)\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}.$$

Therefore,

$$\mathbf{X} = e^{\mathbf{A}t}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = (\mathbf{I} + \mathbf{N}t)\begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{\lambda t}\left(\mathbf{I} + \mathbf{N}t\right)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Note that $\mathbf{N}^2 = \mathbf{0} \Rightarrow |\mathbf{N}|^2 = 0 \Rightarrow |\mathbf{N}| = 0$. Therefore,

$$\mathbf{N} = \begin{bmatrix} n_1 & n_2 \\ \alpha n_1 & \alpha n_2 \end{bmatrix}.$$

Also, $\mathbf{N}^2 = \mathbf{0} \Rightarrow \operatorname{tr}(\mathbf{N}) = 0 \Rightarrow n_1 + \alpha n_2 = 0$. Let

$$\mathbf{V} = \begin{bmatrix} 1\alpha \end{bmatrix}.$$

Then

$$\mathbf{N}\mathbf{v} = \begin{bmatrix} n_1 + \alpha n_2 \\ \alpha\left(n_1 + \alpha n_2\right) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

So $\mathbf{A}\mathbf{v} = (\mathbf{N} + \lambda\mathbf{I})\,\mathbf{v} = \lambda\mathbf{v}$, i.e., $\mathbf{v}$ is an eigenvector for λ. Therefore,

$$\begin{aligned}\mathbf{X} = e^{\lambda t}\,(\mathbf{I} + \mathbf{N}t)\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} &= e^{\lambda t}\begin{bmatrix} C_1 + (n_1C_1 + n_2C_2)\,t \\ C_2 + \alpha\,(n_1 + n_2C_2)\,t \end{bmatrix} \\ &= e^{\lambda t}\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + (n_1C_1 + n_2C_2)\,t\begin{bmatrix} 1 \\ \alpha \end{bmatrix}\right) \\ &= e^{\lambda t}\left(\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} + (n_1C_1 + n_2C_2)\,t\mathbf{v}\right).\end{aligned}$$

If $\lambda < 0$, we have

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as $t \to \infty$. If $\lambda > 0$, then

$$\begin{bmatrix} x \\ y \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

as $t \to -\infty$. What is the limit of the slope? In other words, what line is approached asymptotically? We have

$$\lim_{t\to\infty}\frac{y}{x} = \lim_{t\to\infty}\frac{C_2 + (n_1C_1 + n_2C_2)\,t\mathbf{v}_2}{C_1 + (n_1C_1 + n_2C_2)\,t\mathbf{v}_1} = \frac{\mathbf{v}_2}{\mathbf{v}_1},$$

i.e., it approaches $\mathbf{v}$. Similarly,

$$\lim_{t\to-\infty}\frac{y}{x} = \frac{\mathbf{v}_2}{\mathbf{v}_1},$$

i.e., it also approaches $\mathbf{v}$ as $t \to -\infty$. Figure 9.4 illustrates the situation.

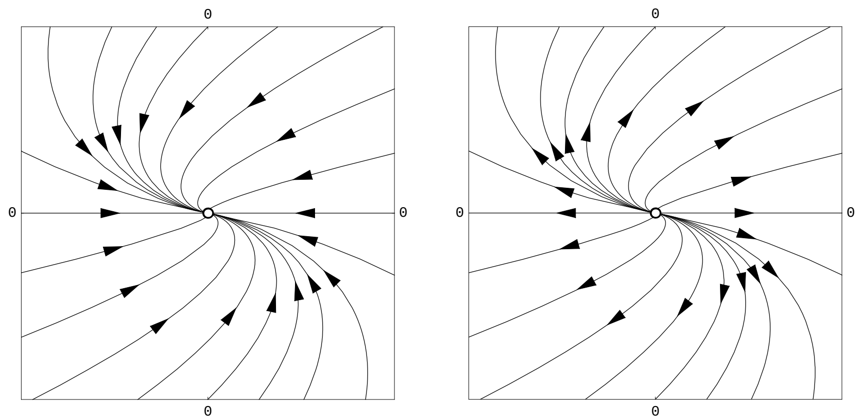
We encounter the degenerate case when $\mathbf{N} = \mathbf{0}$. This does not work, but then $\mathbf{A} = \lambda\mathbf{I}$, so

$$\mathbf{X} = e^{\mathbf{A}t}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\lambda t} \end{bmatrix}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix} = e^{\lambda t}\begin{bmatrix} C_1 \\ C_2 \end{bmatrix},$$

which is just a straight line through

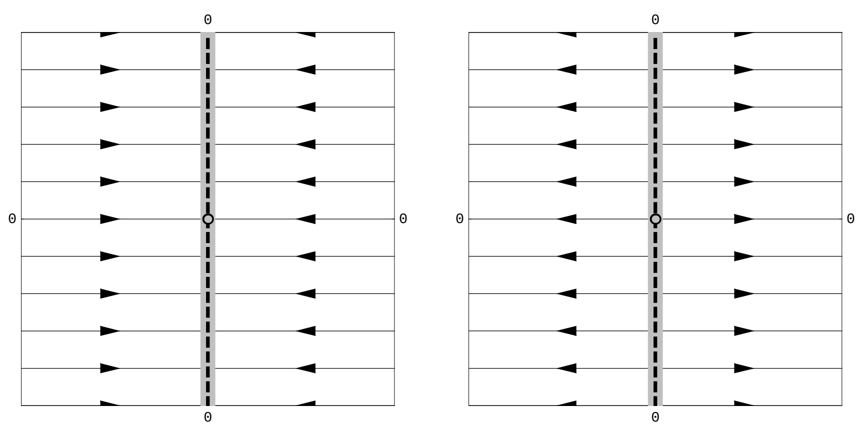
$$\begin{bmatrix} C_1 \\ C_2 \end{bmatrix}.$$

Figure 9.5 illustrates this situation.



(a) $\lambda < 0$ (b) $\lambda > 0$

Figure 9.4: The cases for λ, where we have a stable node when $\lambda < 0$ and an unstable node when $\lambda > 0$.



(a) $\lambda < 0$ (b) $\lambda > 0$

Figure 9.5: The degenerate cases for λ when $\mathbf{N} = \mathbf{0}$, where we have a stable node when $\lambda < 0$ and an unstable node when $\lambda > 0$.

Chapter 10

Existence and Uniqueness Theorems

10.1 Picard's Method

Consider the general first order IVP

$$\begin{cases} \frac{dy}{dx} = f(x,y), \\ y(x_0) = y_0. \end{cases} \tag{$*$}$$

Picard's Method is a technique for approximating the solution to Equation ($*$). If $y(x)$ is a solution of Equation ($*$), then

$$\int_{x_0}^{x} \frac{dy}{dt}\,dt = \int_{x_0}^{x} f(t, y(t))\,dt,$$
$$\int_{y_0}^{y} dy = y - y_0,$$

giving us

$$y = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt. \tag{$**$}$$

Conversely, if y satisfies ($**$), then differentiating gives

$$\frac{dy}{dx} = 0 + f(x, y(x)) = f(x, y)$$

and $y(x_0) = y_0 + 0 = y_0$. Therefore, the IVP $(*)$ is equivalent to the integral equation $(**)$.

If $p(x)$ is a function, define a new function $P(p)$ by

$$(P(p))(x) = y_0 + \int_{x_0}^{x} f(t, p(t))\, dt.$$

So a solution to $(**)$ is a function $y(x)$ such that $P(y) = y$.

Picard's Method is as follows. Start with a function $y_0(x)$ for which $y_0(x_0) = y_0$. Let $y_1 = P(y_0), y_2 = P(y_1), \ldots, y_n = P(y_{n-1}), \ldots$, and so on. Under the right conditions (discussed later), the function y_n will converge to a solution y. Intuitively, we let $y = \lim_{n\to\infty} y_n$. Since $y_{n+1} = P(y_n)$, taking limits as $n \to \infty$ gives $y = P(y)$. Later, we will attempt to make this mathematically precise.

Example 10.1. Solve $y' = -y$, where $y(0) = 1$. $*$

Solution. We begin with $y_0(x) \equiv 1$. With $x_0 = 0$, $y_0 = 1$, $f(x, y) = -y$, we have

$$\begin{aligned} y_1(x) &= y_0 + \int_{x_0}^{x} f\left(\frac{1}{x}, y_0(t)\right) dt \\ &= 1 + \int_0^x (-1)\, dt = 1 - x, \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= y_0 + \int_{x_0}^{x} f(t, y_1(t))\, dt \\ &= 1 + \int_0^x -(1 - t)\, dt = 1 - x + \frac{x^2}{2}, \end{aligned}$$

and

$$y_3(x) = 1 + \int_0^x -\left(1 - t + \frac{t^2}{2}\right) dt = 1 - x + \frac{x^2}{2} - \frac{x^3}{3!}.$$

So in general, we have

$$y_n(x) = 1 - x + \frac{x^2}{2!} + \cdots + (-1)^n \frac{x^n}{n!}.$$

Therefore, we finally have

$$y(x) = \lim_{n\to\infty} y_n(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}.$$

In this case, we recognize the solution as $y = e^{-x}$. $\Diamond$

Example 10.2. Solve $y' = x + y^2$, where $y(1) = 1$. $*$

Solution. Let $y_0(x) \equiv 1$. With $x_0 = 1$, $y_0 = 1$, and $f(x,y) = x + y^2$, we have

$$\begin{aligned} y_1(x) &= 1 + \int_1^x \left(t + 1^2\right) dt = 1 + \int_1^x \left[(t-1) + 2\right] dt \\ &= 1 + \left[\frac{(t-1)^2}{2} + 2\,(t-1)\right]_1^x = 1 + \frac{(x-1)^2}{2} + 2\,(x-1) \\ &= 1 + 2\,(x-1) + \frac{(x-1)^2}{2} \end{aligned}$$

and

$$\begin{aligned} y_2(x) &= 1 + \int_1^x \left[t + \left(1 + 2\,(t-1) + \frac{(t-1)^2}{2}\right)^2\right] dt \\ &= 1 + 2\,(x-1) + \frac{5}{2}\,(x-1)^2 + \frac{5}{3}\,(x-1)^3 + \frac{1}{4}\,(x-1)^4 + \frac{1}{20}\,(x-1)^5\,. \end{aligned}$$ $\Diamond$

Picard's Method is not convenient for actually finding solutions, but it is useful for proving that solutions exist under the right conditions.

Theorem 10.3
If $f : X \to \mathbb{R}$ is continuous, where $X \subset \mathbb{R}^n$ is closed and bounded (compact), then $f(X)$ is closed and bounded (compact). In particular, there exists an M such that $|f(x)| \le M$ for all $x \in X$.

Proof. Proof is given in `MATB43`. $\square$

Corollary 10.4
Let R be a (closed) rectangle. Let $f(x,y)$ be such that $\partial f / \partial y$ exists and is continuous throughout R. Then there exists an M such that

$$\underbrace{|f(x,y_2) - f(x,y_1)| \le M\,|y_2 - y_1|}_{\text{Lipschitz condition with respect to } y}$$

for any points (x, y_1) and (x, y_2) in R.

Proof. Let R be a (closed) rectangle. Let $f(x,y)$ be such that $\partial f/\partial y$ exists and is continuous throughout R. By Theorem 10.3, there exists an M such that

$$\left|\frac{\partial f}{\partial y}(r)\right| \leq M$$

for all $r \in R$. By the Mean Value Theorem, there exists a $c \in (y_1, y_2)$ such that

$$f(x,y_2) - f(x,y_1) = \frac{\partial f}{\partial y}(x,c)(y_2 - y_1).$$

Since R is a rectangle, we have $(x,c) \in R$, so

$$\left|\frac{\partial f}{\partial y}(x,c)\right| \leq M.$$

Therefore, $|f(x,y_2) - f(x,y_1)| \leq M\,|y_2 - y_1|$. □

Theorem 10.5
Let f be a function. Then

1. f is differentiable at x implies that f is continuous at x.
2. f is continuous on B implies that f is integrable on B, i.e., $\int_B f\,dV$ exists.

Proof. Proof is given in `MATB43`. □

Definition (Uniform convergence)
A sequence of functions $\{f_n\}$ defined on B is said to *converge uniformly* to a function f if for all $\epsilon > 0$, there exists an N such that $|f(x) - f_n(x)| < \epsilon$ for all $n \geq N$.*

---

*The point is that the same N works for all x's. If each x has an N_x that worked for it but there was no N working for all x's at once, then it would converge, but not uniformly.

Theorem 10.6
Suppose that $\{f_n\}$ is a sequence of functions that converges uniformly to f on B. if f_n is integrable for all n, then f is integrable and

$$\int_B f(x)\,dV = \lim_{n\to\infty}\int_B f_n(x)\,dV.$$

Proof. Proof is given in `MATB43`. □

Theorem 10.7
Let $\{f_n\}$ be a sequence of functions defined on B. Suppose that there exists a positive convergent series $\sum_{n=1}^{\infty} a_n$ such that $|f_n(x) - f_{n-1}(x)| \leq a_n$ for all n. Then $f(x) = \lim_{n\to\infty} f_n(x)$ exists for all $x \in B$ and $\{f_n\}$ converges uniformly to f.

Proof. Let $\{f_n\}$ be a sequence of functions defined on B. Suppose that there exists a positive convergent series $\sum_{n=1}^{\infty} a_n$ such that $|f_n(x) - f_{n-1}(x)| \leq a_n$ for all n. Let

$$\begin{aligned} f_n(x) &= f_0(x) + (f_1(x) - \cdots - f_0(x)) + \cdots + (f_n(x) - f_{n-1}(x)) \\ &= f_0(x) + \sum_{k=1}^{n} (f_k(x) - f_{k-1}(x)). \end{aligned}$$

Therefore

$$\underbrace{\lim_{n\to\infty} f_n(x)}_{f(x)} = f_0(x) + \sum_{k=1}^{\infty} (f_k(x) - f_{k-1}(x))$$

and

$$|f(x) - f_n(x)| = \sum_{k=n+1}^{\infty} (f_k(x) - f_{k-1}(x)) \leq \sum_{k=n+1}^{\infty} a_k < \epsilon$$

for n sufficiently large, since $\sum_{n=1}^{\infty} a_n$ converges. Therefore, f_n converges uniformly to f. □

10.2 Existence and Uniqueness Theorem for First Order ODE's

Theorem 10.8 (Existence and uniqueness theorem for first order ODE's*)
Let R be a rectangle and let $f(x, y)$ be continuous throughout R and satisfy the Lipschitz Condition with respect to y throughout R. Let (x_0, y_0) be an interior point of R. Then there exists an interval containing x_0 on which there exists a unique function $y(x)$ satisfying $y' = f(x, y)$ and $y(x_0) = y_0$.**

To prove the existence statement in Theorem 10.8, we will need the following lemma.

Lemma 10.9
Let $I = [x_0 - \alpha, x_0 + \alpha]$ and let $p(x)$ satisfy the following:

1. $p(x)$ is continuous on I.
2. $|p(x) - y_0| \leq a$ for all $x \in I$.

Then $\int_{x_0}^{x} f(t, p(t))\, dt$ exists for all $x \in I$ and $q(x) = y_0 + \int_{x_0}^{x} f(t, p(t))\, dt$ also satisfies (1) and (2).

Proof. Let $I = [x_0 - \alpha, x_0 + \alpha]$ and let $p(x)$ satisfy the following:

1. $p(x)$ is continuous on I.
2. $|p(x) - y_0| \leq a$ for all $x \in I$.

Immediately, (2) implies that $(t, p(t)) \in R$ for $t \in I$. We have

$$\begin{array}{ccccc} I & \longrightarrow & R & \xrightarrow{f} & \mathbb{R} \\ t & \longrightarrow & (t, p(t)) & & \end{array}$$

so f is continuous, hence integrable, i.e., $\int_{x_0}^{x} f(t, p(t))\, dt$ exists for all $x \in I$. Since $q(x)$ is differentiable on I, it is also continuous on I. Also, since

$|f(t, p(t))| \leq M$, it follows that

$$|q(x) - y_0| = \left| \int_{x_0}^{x} f(t, p(t))\, dt \right| \leq M\, |x - x_0| \leq M\alpha \leq a.$$

□

Proof (Proof of existence statement (Theorem 10.8)). Let R be a rectangle and let $f(x, y)$ be continuous throughout R that satisfy the Lipschitz Condition with respect to y throughout R. Let (x_0, y_0) be an interior point of R. Since f is continuous, there exists an M such that $|f(r)| \leq M$ for all $r \in R$. Let a be the distance from (x_0, y_0) to the boundary of R. Since (x_0, y_0) is an interior point, we have $a > 0$. Let $\alpha = \min(a, a/M)$. We will show that $y' = f(x, y)$, where $y(x_0) = y_0$, has a unique solution on the domain $I = [x_0 - \alpha, x_0 + \alpha]$.

Inductively define functions (1) and (2) of Lemma 10.9 by $y_0(x) \equiv y_0$ for all $x \in I$ and

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt.$$

By Lemma 10.9, the existence of y_{n-1} satisfying (1) and (2) guarantees the existence of y_n. By hypothesis, there exists an A such that $|f(x, v) - f(x, w)| \leq A\, |v - w|$, whenever $(x, v), (x, w) \in R$. As in the proof of Lemma 10.9, we have

$$|y_1(x) - y_0(x)| = |y_1(x) - y_0| \leq M\, |x - x_0|$$

for all $x \in I$. Likewise, we have

$$\begin{aligned}
|y_2(x) - y_1(x)| &= \left| \int_{x_0}^{x} [f(t, y_1(t)) - f(t, y_0(t))]\, dt \right| \\
&\leq \left| \int_{x_0}^{x} |f(t, y_1(t)) - f(t, y_0(t))|\, dt \right| \\
&\leq \left| \int_{x_0}^{x} A\, |y_1(t) - y_0(t)|\, dt \right| \\
&\leq \left| \int_{x_0}^{x} AM\, |t - x_0|\, dt \right| \\
&\leq MA \frac{|x - x_0|^2}{2},
\end{aligned}$$

and

$$\begin{aligned}|y_3(x) - y_2(x)| &= \left| \int_{x_0}^{x} [f(t, y_2(t)) - f(t, y_1(t))]\, dt \right| \\ &\le \left| \int_{x_0}^{x} |f(t, y_2(t)) - f(t, y_1(t))|\, dt \right| \\ &\le \left| \int_{x_0}^{x} \left(AMA \frac{|t - x_0|^2}{2} \right) dt \right| \\ &\le MA^2 \frac{|x - x_0|^3}{3!}.\end{aligned}$$

Continuing, we get

$$|y_n(x) - y_{n-1}(x)| \le \frac{MA^{n-1}\alpha^n}{n!}.$$

We have

$$\begin{aligned}y_n(x) &= y_0(x) + (y_1(x) - y_0(x)) + (y_2(x) - y_1(x)) + \cdots + (y_n(x) - y_{n-1}(x)) \\ &= y_0(x) + \sum_{k=1}^{n} (y_k(x) - y_{k-1}(x)).\end{aligned}$$

Since

$$|y_k(x) - y_{k-1}(x)| \le \frac{MA^{k-1}\alpha^k}{k!}$$

and

$$\sum_{k=1}^{\infty} \frac{MA^{k-1}\alpha^k}{k!} = \frac{M}{A}\left(e^{A\alpha} - 1\right),$$

converges, we must have

$$\lim_{n\to\infty} y_n(x) = y_0(x) + \sum_{k=1}^{\infty} (y_k(x) - y_{k-1}(x))$$

converging for all $x \in I$. Therefore, $y(x) = \lim_{n\to\infty} y_n(x)$ exists for all $x \in I$ and y_n converges uniformly to y.

Note that

$$|f(x, y_n(x)) - f(x, y_{n-1}(x))| \le A\, |y_n(x) - y_{n-1}(x)|$$

implies that $f(x, y_n(x))$ converges uniformly to $f(x, y(x))$. With

$$y_n(x) = y_0 + \int_{x_0}^{x} f(y, y_{n-1}(t))\, dt,$$

we then have

$$\begin{aligned} y(x) = \lim_{n\to\infty} y_n(x) &= \lim_{n\to\infty} \left(y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\, dt \right) \\ &= y_0 + \int_{x_0}^{x} \left(\lim_{n\to\infty} f(t, y_{n-1}(t)) \right) dt = y_0 + \int_{x_0}^{x} f(t, y(t))\, dt \end{aligned}$$

implying that y satisfies $y'(x) = f(x, y)$ and $y(x_0) = y_0$. □

To prove the uniqueness statement in Theorem 10.8, we will need the following lemma.

Lemma 10.10
If $y(x)$ is a solution to the DE with $y(x_0) = y_0$, then $|y(x) - y_0| \leq a$ for all $x \in I$. In particular, $(x, y(x)) \in R$ for all $x \in I$.

Proof. Suppose there exist an $x \in I$ such that $|y(x) - y_0| > a$. We now consider two cases.

Suppose $x > x_0$. Let $s \in \inf(\{t : t > x_0 \text{ and } |y(t) - y_0| > a\})$. So $s < x \leq x_0 + \alpha \leq x_0 + a$. By continuity, we have

$$a = |y(s) - y_0| = |y(s) - y(x_0)| = |y'(c)\,(s - x_0)| = \underbrace{|f(c, y(c))\,(s - x_0)|}_{\leq M(s-x_0)}$$

for some $c \in (x_0, s)$. However,

$$s \in I \Longrightarrow s - x_0 \leq \alpha \Longrightarrow \alpha \leq M\,(s - x_0) \leq M\alpha \leq a.$$

Therefore,

$$M\,(s - x_0) = Ma \Longrightarrow s - x_0 = a \Longrightarrow s = x_0 + a,$$

which is a contradiction because we have $s < x_0 + a$. The case with $x < x_0$ is similarly shown. □

Proof (Proof of uniqueness statement (Theorem 10.8)). Suppose $y(x)$ and $z(x)$

both satisfy the given DE and initial condition. Let $\sigma(x) = (y(x) - z(x))^2$. Then

$$\begin{aligned}\sigma'(x) &= 2\left(y(x) - z(x)\right)\left(y'(x) - z'(x)\right) \\ &= 2\left(y(x) - z(x)\right)\left(f(x, y(x)) - f(x, z(x))\right).\end{aligned}$$

Therefore for all $x \in I$,

$$\begin{aligned}\left|\sigma'(x)\right| &\le 2\left|y(x) - z(x)\right| A \left|y(x) - z(x)\right| \\ &= 2A\left(y(x) - z(x)\right)^2 \\ &= 2A\sigma(x).\end{aligned}$$

Therefore, $-2A\sigma(x) \le \sigma'(x) \le 2A\sigma(x)$, and we have

$$\begin{aligned}\sigma'(x) \le 2A\sigma(x) &\Longrightarrow \sigma'(x) - 2A\sigma(x) \le 0 \\ &\Longrightarrow e^{-2Ax}\left(\sigma'(x) - 2A\sigma(x)\right) \le 0, \quad \sigma'(x) = \frac{d\left(e^{-2Ax}\sigma(x)\right)}{dx} \\ &\Longrightarrow e^{-2Ax}\sigma(x) \text{ decreasing on } I \\ &\Longrightarrow e^{-2Ax}\sigma(x) \le e^{-2Ax_0}\sigma(x_0) \\ &\Longrightarrow \sigma(x) \le 0\end{aligned}$$

if $x \ge x_0$. Similarly,

$$\begin{aligned}\sigma'(x) \ge -2A\sigma(x) &\Longrightarrow \sigma'(x) + 2A\sigma(x) \ge 0 \\ &\Longrightarrow e^{2Ax}\left(\sigma'(x) + 2A\sigma(x)\right) \ge 0, \quad \sigma'(x) = \frac{d\left(e^{2Ax}\sigma(x)\right)}{dx} \\ &\Longrightarrow e^{2Ax}\sigma(x) \text{ increasing on } I \\ &\Longrightarrow e^{2Ax}\sigma(x) \le e^{2Ax_0}\sigma(x_0) \\ &\Longrightarrow \sigma(x) \le 0\end{aligned}$$

if $x \le x_0$. Therefore, $\sigma(x) \le 0$ for all $x \in I$. But obviously, $\sigma(x) \ge 0$ for all x from its definition. Therefore, $\sigma(x) \equiv 0$, i.e., $y(x) \equiv z(x)$. Therefore, the solution is unique. □

10.3 Existence and Uniqueness Theorem for Linear First Order ODE's

Here, we consider the special case of a linear DE

$$\frac{dy}{dx} = a(x)y + b(x).$$

Theorem 10.11 (Existence and uniqueness theorem for linear first order DE's) Let $a(x)$ and $b(x)$ be continuous throughout an interval I. Let x_0 be an interior point of I and let y_0 be arbitrary. Then there exists a unique function $y(x)$ with $y(x_0) = y_0$ satisfying $dy/dx = a(x)y + b(x)$ throughout I.*

Proof. Let $a(x)$ and $b(x)$ be continuous throughout an interval I. Let x_0 be an interior point of I and let y_0 be arbitrary. Let $A = \max(\{|a(x)| : x \in I\})$. Inductively define functions** on I by $y_0(x) = y_0$ and

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} (a(t)y_n(t) + b(t))\, dt.$$

Then

$$\begin{aligned} |y_n(x) - y_{n-1}(x)| &= \left| \begin{array}{l} \cancel{y_0} + \int_{x_0}^{x} (a(t)y_{n-1}(t) + b(t))\, dt \\ -\cancel{y_0} - \int_{x_0}^{x} (a(t)y_{n-2}(t) + b(t))\, dt \end{array} \right| \\ &= \left| \int_{x_0}^{x} \left[a(t)\left(y_{n-1}(t) - y_{n-2}(t)\right)\right] dt \right|. \end{aligned}$$

Assuming by induction that

$$|y_{n-1}(x) - y_{n-2}(x)| \le A^{n-2} \frac{|x - x_0|^n}{(n-1)!},$$

**Assuming by induction that $y_n(t)$ is continuous, $a(t)y_n(t) + b(t)$ is continuous throughout I and thus integrable, so the definition of $y_{n+1}(x)$ makes sense. Then $y_{n+1}(x)$ is also differentiable, i.e., $y'_{n+1}(x) = a(x)y_n(x) + b(x)$, and thus continuous, so the induction can proceed.

we then have

$$\left|\int_{x_0}^{x} \left[a(t)\left(y_{n-1}(t) - y_{n-2}(t)\right)\right] dt\right| \leq \left|\int_{x_0}^{x} \left(AA^{n-2}\frac{|t-x_0|^{n-1}}{(n-1)!}\right) dt\right|$$
$$\leq A^{n-1}\frac{|x-x_0|^{n}}{n!},$$

thereby completing the induction. Therefore,

$$|y_n(x) - y_{n-1}(x)| \leq \frac{A^{n-1}\alpha^n}{n!},$$

where α is the width of I. The rest of the proof is as before. □

10.4 Existence and Uniqueness Theorem for Linear Systems

A *system of differential equations* consists of n equations involving n functions and their derivatives. A *solution* of the system consists of n functions having the property that the n functional equations obtained by substituting these functions are their derivatives into the system of equations, and they hold for every point in some domain D. A *linear system of differential equations* has the form*

$$\begin{aligned}
\frac{dy_1}{dx} &= a_{11}(x)y_1 + a_{12}(x)y_2 + \cdots + a_{1n}(x)y_n + b_1(x),\\
\frac{dy_2}{dx} &= a_{21}(x)y_1 + a_{22}(x)y_2 + \cdots + a_{2n}(x)y_n + b_2(x),\\
&\vdots\\
\frac{dy_n}{dx} &= a_{n1}(x)y_1 + a_{n2}(x)y_2 + \cdots + a_{nn}(x)y_n + b_n(x).
\end{aligned}$$

We can write the system as

$$\frac{d\mathbf{Y}}{dx} = \mathbf{AY} + \mathbf{B},$$

*The point is that $y_1, y_2, \ldots, y_n$ and their derivatives appears linearly.

where

$$\mathbf{A} = \begin{bmatrix} a_{11}(x) & \cdots & a_{1n}(x) \\ a_{21}(x) & \cdots & a_{2n}(x) \\ \vdots & \ddots & \vdots \\ a_{n1}(x) & \cdots & a_{nn}(x) \end{bmatrix}, \qquad \mathbf{Y} = \begin{bmatrix} y_1(x) \\ y_2(x) \\ \vdots \\ y_n(x) \end{bmatrix}, \qquad \mathbf{B} = \begin{bmatrix} b_1(x) \\ b_2(x) \\ \vdots \\ b_n(x) \end{bmatrix}.$$

Theorem 10.12 (Existence and uniqueness theorem for linear systems of n DE's) Let $\mathbf{A}(x)$ be a matrix of functions, each continuous throughout an interval I and let $\mathbf{B}(x)$ be an n-dimensional vector of functions, each continuous throughout I. Let x_0 be an interior point of I and let $\mathbf{Y}_0$ be an arbitrary n-dimensional vector. Then there exists a unique vector of functions $\mathbf{Y}(x)$ with $\mathbf{Y}(x_0) = \mathbf{Y}_0$ satisfying

$$\frac{d\mathbf{Y}}{dx} = \mathbf{A}(x)\mathbf{Y} + \mathbf{B}(x)$$

throughout I.*

Proof. Let $\mathbf{A}(x)$ be a matrix of functions, each continuous throughout an interval I and let $\mathbf{B}(x)$ be an n-dimensional vector of functions, each continuous throughout I. Let x_0 be an interior point of I and let $\mathbf{Y}_0$ be an arbitrary n-dimensional vector.

Let $\Lambda = \max(\{\|\mathbf{A}(x)\| : x \in I\})$. Inductively define $\mathbf{Y}_0(x) = \mathbf{Y}_0$ and

$$\mathbf{Y}_{n+1}(x) + \mathbf{Y}_0 + \underbrace{\int_{x_0}^{x} (\mathbf{A}(t)\mathbf{Y}_n(t) + \mathbf{B}(t))\, dt}_{\substack{\text{vector obtained by} \\ \text{integrating componentwise}}}.$$

Then

$$\begin{aligned}
|\mathbf{Y}_n(x) - \mathbf{Y}_{n-1}(x)| &= \left| \int_{x_0}^{x} \left[\mathbf{A}(t)\left(\mathbf{Y}_{n-1}(t) - \mathbf{Y}_{n-2}(t)\right)\right] dt \right| \\
&\leq \left| \int_{x_0}^{x} \left(\Lambda \Lambda^{n-1} \frac{|t - x_0|^{n-1}}{(n-1)!} \right) dt \right| \\
&= \Lambda^n \frac{|x - x_0|^n}{n!} \\
&\leq \Lambda^n \alpha^n,
\end{aligned}$$

where α is the width of I. The rest of the proof is essentially the same as before. □

Recall Theorem 8.5 (p. 90) restated here as follows.

Theorem 10.13 (Theorem 8.5)
Let $x = p$ be an ordinary point of $y'' + P(x)y' + Q(x)y = 0$. Let R be the distance from p to the closest singular point of the DE in the complex plane. Then the DE has two series $y_1(x) = \sum_{n=0}^{\infty} a_n x^n$ and $y_2(x) = \sum_{n=0}^{\infty} b_n x^n$ which converge to linearly independent solutions to the DE on the interval $|x - p| < R$.*

Proof. Let $x = p$ be an ordinary point of $y'' + P(x)y' + Q(x)y = 0$. Let R be the distance from p to the closest singular point of the DE in the complex plane.

We first claim that if $f(z)$ is analytic at p and R is the radius of convergence of f, then for $r < R$, there exists an M (depending on r) such that

$$\frac{1}{n!} \left| f^{(n)}(p) \right| \leq \frac{M}{r^n} \qquad (*)$$

for all $n \in \mathbb{N}$. To prove this claim, since

$$\sum_{n=0}^{\infty} \frac{1}{n} f^n(p) r^n = f(r)$$

converges absolutely, let

$$M = \sum_{n=0}^{\infty} \frac{1}{n!} \left| f^n(p) \right| r^n.$$

Then
$$\frac{1}{n!}f^{(n)}(p)r^n \le M,$$
so Equation $(*)$ holds. Therefore, our claim holds.

Let $w(z)$ be a solution of $y'' + P(x)y' + Q(x)y = 0$. Being a solution to the DE, w is twice differentiable. Furthermore, differentiating the DE gives a formula for w''' in terms of w, w', w''. Continuing, w is n times differentiable for all n, so w has a Taylor series. We now wish to show that the Taylor series of w converges to w for $|z-p| < R$.

Since $|z-p| < R$, there exists an a such that $|z-p| < a < R$. As above, find constants M such that
$$\frac{1}{n!}\left|P^{(n)}(p)\right| \le \frac{M}{a^n}, \qquad \frac{1}{n!}\left|Q^{(n)}(p)\right| \le \frac{N}{a^n}$$
for all $n \in \mathbb{N}$. Let
$$A(z) = \frac{M}{1-\frac{z-p}{a}}, \qquad B(z) = \frac{N}{1-\frac{z-p}{a}}.$$
Note that
$$A(z) = M\left(1 + \frac{z-p}{a} + \left(\frac{z-p}{a}\right)^2 + \cdots + \left(\frac{z-p}{a}\right)^n + \cdots\right),$$
so
$$\begin{aligned} A(p) &= M, \\ A'(p) &= \frac{M}{a}, \\ A''(p) &= 2\frac{M}{a^2}, \\ &\vdots \\ A^{(n)}(p) &= n!\frac{M}{a^n} \ge \left|P^{(n)}(p)\right|. \end{aligned}$$
Similarly,
$$B^{(n)}(p) = n!\frac{N}{a^n} \ge \left|Q^{(n)}(p)\right|.$$
Consider
$$y'' = A(z)y' + B(z)y. \qquad (**)$$

Let $v(z)$ be a solution of Equation $(**)$ satisfying $v(p) = |w(p)|$ and $v'(p) = |w'(p)|$. In general, $w^{(n)}(p) = C_1 w'(p) + C_2 w(p)$, where C_1 and C_2 are arbitrary constants depending on P and Q and their derivatives at p, e.g.,

$$\begin{aligned} w'' &= P(p)w'(p) + Q(p)w(p), \\ w''' &= P'(p)w'(p) + P(p)w''(p) + Q'(p)w(p) + Q(p)w'(p) \\ &= P'(p)w'(p) + P(p)\left(P(p)w'(p) + Q(p)w(p)\right) + Q'(p)w(p) + Q(p)w'(p) \\ &= \left(P^2(p) + P'(p) + Q(p)\right) w'(p) + \left(P(p)Q(p) + Q'(p)\right) w(p). \end{aligned}$$

Similar formulas hold $v^{(n)}(p)$ involving derivatives of A and B at p.

Using $\left|P^{(n)}(p)\right| \le A^{(n)}(p)$ and $\left|Q^{(n)}(p)\right| \le B^{(n)}(p)$ gives $\left|w^{(n)}(p)\right| \le v^{(n)}(p)$. The Taylor series of W about p is $\sum_{n=0}^{\infty} a_n \left(z-p\right)^n$, where $a_n = w^{(n)}(p)/n!$; the Taylor series of v about p is $\sum_{n=0}^{\infty} b_n \left(z-p\right)^n$, where $b_n = v^{(n)}(p)/n!$.

Since we showed that $|a_n| \le b_n$, the first converges anywhere the second does. So we need to show that the Taylor series of $w(x)$ has a radius of convergence equal to a (this implies that the Taylor series for w converges for $n - pk$, but $a < R$ was arbitrary). We have

$$v'' = \frac{M}{1 - \frac{z-p}{a}} v' + \frac{N}{1 - \frac{z-p}{a}} v.$$

Let $u = \left(z-p\right)/a$. Then

$$\begin{aligned} v' &= \frac{dv}{du}\frac{1}{a}\frac{dv}{du}. \\ v'' &= a\frac{d^2v}{du^2}\frac{du}{dz} = \frac{1}{a^2}\frac{d^2v}{du^2}. \end{aligned}$$

Then we have

$$\begin{aligned} \frac{1}{a^2}\frac{d^2v}{du^2} &= \frac{M}{1-u}\frac{1}{a}\frac{dv}{du} + \frac{N}{1-u}v, \\ (1-u)\frac{d^2v}{du^2} &= aM\frac{dv}{du} + a^2Nv. \end{aligned}$$

Write $v = \sum_{n=0}^{\infty} \gamma_n u^n$. Using its first radius of convergence, we have

$$v = \gamma_n u^n = \gamma_n \left(\frac{z-p}{a}\right)^n = \frac{\gamma_n}{a^n}\left(z-p\right)^n = b_n \left(z-p\right)^n,$$

which implies that $\gamma_n = b_n a^n > 0$. Note that

$$\frac{dv}{du} = \underbrace{\sum_{n=1}^{\infty} n\gamma_n u^{n-1}}_{\text{adjust indices}} = \sum_{n=0}^{\infty} (n+1)\,\gamma_{n+1} u^n,$$

$$\frac{d^2v}{du^2} = \underbrace{\sum_{n=1}^{\infty} (n+1)\, n\gamma_{n+1} u^{n-1}}_{\text{adjust indices}} = \sum_{n=0}^{\infty} (n+2)\,(n+1)\,\gamma_{n+2} u^n.$$

Now we have

$$(1-u)\sum_{n=0}^{\infty} (n+2)\,(n+1)\,\gamma_{n+2} u^n = \sum_{n=0}^{\infty} aM\,(n+1)\,\gamma_{n+1} u^n + \sum_{n=0}^{\infty} a^2 N\gamma_n u^n,$$

$$\left(\begin{array}{l} \displaystyle\sum_{n=0}^{\infty} (n+2)\,(n+1)\,\gamma_{n+2} u^n \\ \displaystyle -\sum_{n=0}^{\infty} (n+2)\,(n+1)\,\gamma_{n+2} u^{n+1} \end{array}\right) = \left(\begin{array}{l} \displaystyle\sum_{n=0}^{\infty} (n+2)\,(n+1)\,\gamma_{n+2} u^n \\ \displaystyle -\sum_{n=0}^{\infty} (n+1)\, n\gamma_{n+1} u^n. \end{array}\right)$$

Therefore,

$$\sum_{n=0}^{\infty} (n+2)\,(n+1)\,\gamma_{n+2} u^n = \sum_{n=0}^{\infty} \left[(n+1)\,(n+aM)\,\gamma_{n+1} + a^2 N\gamma_n\right] u^n,$$

so it follows that

$$\begin{aligned} (n+2)\,(n+1)\,\gamma_{n+2} &= (n+1)\,(aM+n)\,\gamma_{n+1} + a^2 N\gamma_n, \\ \gamma_{n+2} &= \frac{aM+n}{n+2}\gamma_{n+1} + \frac{a^2 N\gamma_n}{n+2}, \\ \frac{\gamma_{n+2}}{\gamma_{n+1}} &= \frac{n+aM}{n+2} + \frac{a^2 N}{n+2}\frac{\gamma_n}{\gamma_{n+1}} \end{aligned}$$

for all $n \in \mathbb{N}$. Since $\gamma_n/\gamma_{n+1} < 1$, we have

$$\lim_{n\to\infty} \frac{a^2 N}{n+2}\frac{\gamma_n}{\gamma_{n+1}} = 0.$$

Therefore,

$$\lim_{n\to\infty} \frac{\gamma_{n+2}}{\gamma_{n+1}} = 1 + 0 = 1.$$

So by the ratio test (Equation (8.2) of Theorem 8.2, p. 88), the radius convergence of

$$\sum_{n=0}^{\infty} \gamma_n u^n = \sum_{n=0}^{\infty} \frac{\gamma_n}{a_n} (z-p)^n = \sum_{n=0}^{\infty} b_n \left(\frac{z-p}{a}\right)^n = 1$$

is a. □

The point of Theorem 8.5 is to state that solutions are analytic (we already know solutions exists from early theorems). The theorem gives only a lower bound on the interval. The actual radius of convergence may be larger.

Chapter 11

Numerical Approximations

11.1 Euler's Method

Consider $y' = f(x, y)$, where $y(x_0) = y_0$. The goal is to find $y(x_{\text{end}})$. Figure 11.1 shows this idea. We pick a large N and divide $[x_0, x_{\text{end}}]$ into segments of

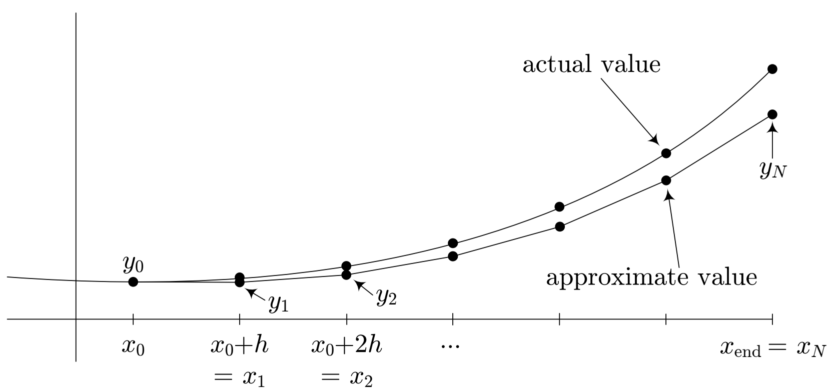


Figure 11.1: Graphical representation of Euler's method.

length $h = (x_{\text{end}} - x_0)/N$. Then

$$y(x_1) \approx y(x_0) + y'(x_0)\underbrace{(x_1 - x_0)}_{h}$$

is the linear/tangent line approximation of $y(x_1)$. Set

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) \longrightarrow y_1 \approx f(x_1), \\ y_2 &= y_1 + hf(x_1, y_1) \longrightarrow y_2 \approx f(x_2), \\ y_3 &= y_2 + hf(x_2, y_2) \longrightarrow y_3 \approx f(x_3), \\ &\vdots \\ y_{\text{answer}} &= y_{N-1} + hf(x_{N-1}, y_{N-1}) \longrightarrow y_{\text{answer}} \approx f(x_{\text{end}}). \end{aligned}$$

Example 11.1. Consider $y' = x - 2y$, where $y(0) = 1$. Approximate $y(0.3)$ using a step size of $h = 0.1$. *

Solution. First note that $x_0 = 0$. With a step size of $h = 0.1$, we have

$$x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3.$$

Therefore,

$$\begin{aligned} y_1 &= y_0 + hf(x_0, y_0) = 1 + 0.1\,(0 - 2) \\ &= 1 - 0.2 = 0.8, \\ y_2 &= 0.8 + hf(0.1, 0.8) = 0.8 + h(0.1 - 1.6) \\ &= 0.8 - 0.1 \cdot 1.5 = 0.8 - 0.15 = 0.65, \\ y_3 &= 0.65 + hf(0.2, 0.65) = 0.65 + h\,(0.2 - 1.3) \\ &= 0.65 - 0.1 \cdot 1.1 = 0.65 - 0.11 = 0.54. \end{aligned}$$

Therefore, $y(0.3) \approx 0.54$.

What about the actual value? To solve the DE, we have

$$\begin{aligned} y' + 2y &= x, \\ e^{2x}y' + 2e^{2x}y &= xe^{2x}, \\ ye^{2x} &= \int xe^{2x}\,dx = \frac{1}{2}xe^{2x} - \int \frac{1}{2}e^{2x}\,dx \\ &= \frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x} + C, \end{aligned}$$

where C is an arbitrary constant. Therefore

$$y = \frac{1}{2}x - \frac{1}{4} + Ce^{-2x}.$$

Note that

$$y(0) = 1 \Longrightarrow -\frac{1}{4} + C = 1 \Longrightarrow C = \frac{5}{4},$$

so the general solution is

$$y = \frac{1}{2}x - \frac{1}{4} + \frac{5}{4}e^{-2x}.$$

Therefore,

$$y(0.3) = 0.15 - 0.25 + \frac{5}{4}e^{-0.6} \approx -0.1 + 0.68601 \approx 0.58601,$$

so $y(0.3) = 0.58601$ is the (approximated) actual value. ♢

11.1.1 Error Bounds

The Taylor second degree polynomial of a function y with step size h is given by

$$y(a+h) = y(a) + hy'(a) + h^2\frac{y''(z)}{2}$$

for some $z \in [a, h]$, where $h^2y''(z)/2$ is the *local truncation error*, i.e., the error in each step. We have

$$h^2\frac{y''(z)}{2} \leq \frac{M}{2}h^2$$

if $|y''(z)| \leq M$ on $[a, h]$. Therefore, dividing h by, say, 3 reduces the local truncation error by 9. However, errors also accumulate due to y_{n+1} being calculated using the approximation of y_n instead of the exact value of $f(x_n)$. Reducing h increases N, since $N = (x_{\text{end}} - x_0)/h$, so this effect wipes out part of the local gain from the smaller h. The net effect is that

$$(\text{overall error} \equiv \text{global truncation error}) \leq \tilde{M}h,$$

e.g., dividing h by 3 divides the overall error only by 3, not 9.

11.2 Improved Euler's Method

To improve upon Euler's method, we consider

$$y_{n+1} = y_n h\star,$$

where, instead of using $f'(x_n)$, it is better to use the average of the left and right edges, so

$$\star = \frac{f'(x_n) + f'(x_{n+1}}{2} = \frac{f(x_n, y_n) + f(x_{n+1}, y_{n+1})}{2}.$$

Figure 11.2 illustrates this idea. However, we have one problem in that we do

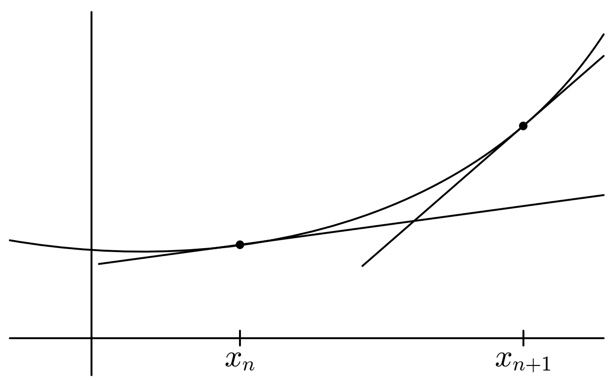


Figure 11.2: A graphical representation of the improved Euler's method.

not know y_{n+1} (that is what we are working out at this stage). Instead, we fill into the "inside" y_{n+1} from the original Euler's method, i.e.,

$$\frac{1}{2}\left(f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right),$$

i.e.,

$$y_{n+1} = y_n + \frac{h}{2}\left[f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))\right].$$

This improves the local truncation error from h^2 to h^3.

11.3 Runge-Kutta Methods

We again modify Euler's method and consider

$$y_{n+1} = y_n h \star .$$

At each step, for $\star$, we use some average of the values over the interval $[y_n, y_{n+1}]$. The most common one is $x_n + h/2$. Figure 11.3 illustrates this idea. Set

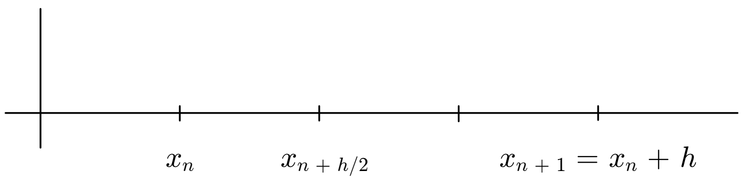


Figure 11.3: A graphical representation of the Runge-Kutta Methods.

$$\begin{aligned}
k_{n_1} &= f(x_n, y_n), \\
k_{n_2} &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n_1}\right), \\
k_{n_3} &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n_2}\right), \\
k_{n_4} &= f\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_{n_3}\right).
\end{aligned}$$

Use

$$y_{n+1} = y_n + \frac{h}{6}\left(k_{n_1} + 2k_{n_2} + 2k_{n_3} + 3k_{n_4}\right).$$

This gives a local truncation error proportional to h^5.