Electromagnetism
with Spacetime Algebra

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$$\overset{\Rightarrow}{\nabla}\mathcal{F} = \overset{\Rightarrow}{J}$$

$$\overset{\Rightarrow}{F} = \frac{d\overset{\Rightarrow}{P}}{d\tau} = q\overset{\Rightarrow}{V} \cdot \mathcal{F}$$

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Chapter 1

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Before developing the necessary mathematics, survey the crucial physics.

1.1 Action at a Distance

By Coulomb's law [1], the electric force on a stationary point charge q due to a stationary charge q' separated by a displacement $\vec{\boldsymbol{\mathscr{r}}}$ is

$$\vec{F} = q\frac{q'}{4\pi\boldsymbol{\mathscr{r}}^2}\hat{\boldsymbol{\mathscr{r}}}, \tag{1.1}$$

where the direction $\hat{\boldsymbol{\mathscr{r}}} = \vec{\boldsymbol{\mathscr{r}}}/\boldsymbol{\mathscr{r}}$ (read "script r hat equals script r vector over script r"). In natural units, both the free space permittivity ϵ_0 and permeability μ_0 (and hence light speed c) are one. Coulomb's law tells "how" and not "why". Understand it as following from something more fundamental, such as a symmetry.

The denominator $4\pi\boldsymbol{\mathscr{r}}^2$ reflects the dilution of the source charge over a sphere of area $4\pi\boldsymbol{\mathscr{r}}^2$. The unit vector $\hat{\boldsymbol{\mathscr{r}}}$ in the numerator reflects the isotropy of space: if no preferred or special direction exists, only the imaginary line joining the two charges q' and q singles out the direction for the electrical force.

However, these arguments break down if (say) the source charge q' is moving, because its velocity vector introduces another, independent direction. In fact, the force between two electric charges in arbitrary motion is complicated by velocity, acceleration, and time delay effects. The force need not even lie along the line joining the two charges.

Suppose a source charge q' is at position $\vec{r}\,'$ with velocity $\vec{v}\,'$ and acceleration $\vec{\mathsf{a}}\,'$, and a test charge q is at position $\vec{r}$ with velocity $\vec{v}$ and acceleration $\vec{\mathsf{a}}$, as in Fig. 1.1. Let their separation $\vec{\boldsymbol{\mathscr{r}}} = \vec{r} - \vec{r}\,'$. Define the velocity of the electromagnetic "news" traveling from q' to q to be $\vec{c} = \hat{\boldsymbol{\mathscr{r}}}c = \hat{\boldsymbol{\mathscr{r}}}$. If the velocity $\vec{u}\,' = \vec{c} - \vec{v}\,'$, then the force

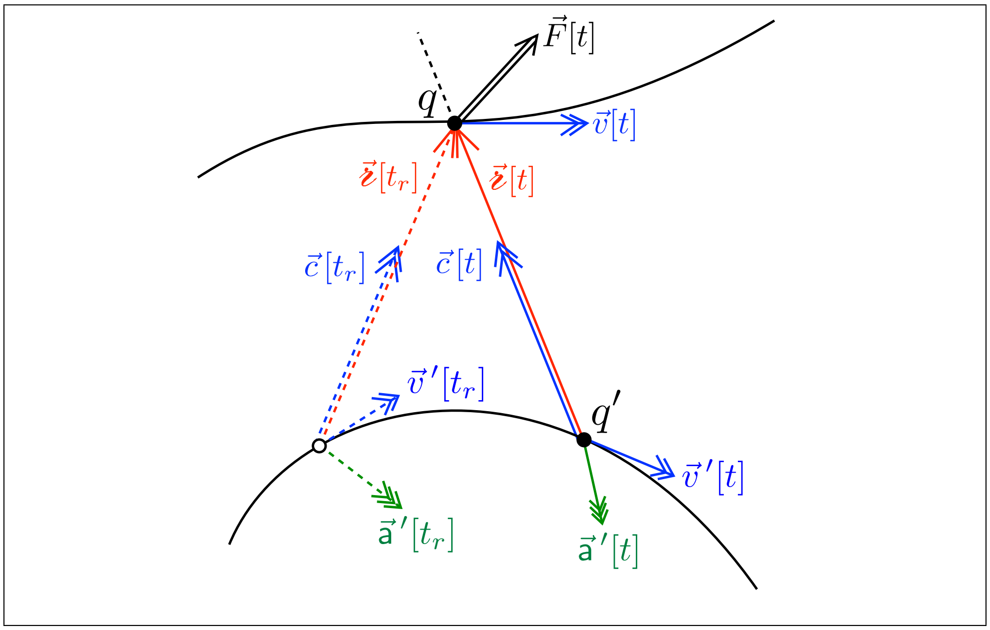

Figure 1.1: The force between two electric charges in arbitrary motion is complicated by velocity, acceleration, and time delay effects.

on the test charge q is

$$\vec{F} = q\frac{q'}{4\pi \mathscr{r}^2}\left(\frac{1}{\hat{\mathscr{r}}\cdot\vec{u}\,'}\right)^3 \Bigg(\left(1-v'^2\right)\vec{u}\,' + \vec{\mathscr{r}}\times\left(\vec{u}\,'\times\vec{\mathsf{a}}\,'\right) + \vec{v}\times\left(\hat{\mathscr{r}}\times\left(\left(1-v'^2\right)\vec{u}\,' + \vec{\mathscr{r}}\times\left(\vec{u}\,'\times\vec{\mathsf{a}}\,'\right)\right)\right)\Bigg), \tag{1.2}$$

where the kinematical variables $\vec{\mathscr{r}}$, $\vec{v}\,'$, and $\vec{\mathsf{a}}\,'$ are evaluated at the earlier, "retarded" time defined implicitly by $t - t_r = \mathscr{r}[t_r]/c = \mathscr{r}[t_r]$.

If the test charge velocity $\vec{v} = \vec{0}$, then the **magnetic** terms vanish; if the source charge acceleration $\vec{\mathsf{a}}\,' = \vec{0}$, then the **radiation** terms vanish; if, in addition, the source charge velocity $\vec{v}\,' = \vec{0}$, then the velocity $\vec{u}\,' = \hat{\mathscr{r}}$, and Eq. 1.2 reduces to the **electric** term of Eq. 1.1.

It is sometimes instructive to explicitly include factors of $c = 1$. For example, restoring c to the generalized Coulomb's law of Eq. 1.2 gives

$$\epsilon_0\vec{F} = q\frac{q'}{4\pi \mathscr{r}^2}\left(\frac{1}{\hat{\mathscr{r}}\cdot\vec{u}\,'/c}\right)^3 \Bigg(\left(1-v'^2/c^2\right)\vec{u}\,'/c + \vec{\mathscr{r}}\times\left(\vec{u}\,'\times\vec{\mathsf{a}}\,'\right)/c^3 + \vec{v}/c\times\left(\hat{\mathscr{r}}\times\left(\left(1-v'^2/c^2\right)\vec{u}\,'/c + \vec{\mathscr{r}}\times\left(\vec{u}\,'\times\vec{\mathsf{a}}\,'\right)/c^3\right)\right)\Bigg), \tag{1.3}$$

which demonstrates that the magnetic and radiation terms are typically much smaller than the electric term, as typically $v', v \ll c$.

This generalized Coulomb's law is *impressive*, as it summarizes all fundamental knowledge of electromagnetism in a single equation involving only elementary notation, but it is also *complicated*. Seek a more elegant description via Maxwell's field equations.

1.2 Classical Field theory

Experience has shown that the **field paradigm** is a better way. Decompose Coulomb's law of Eq. 1.1 into two parts by introducing the **electric field** $\vec{\mathcal{E}}$ with

$$\vec{F} = q\vec{\mathcal{E}}, \tag{1.4a}$$

$$\vec{\mathcal{E}} = \frac{Q}{4\pi \boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}}. \tag{1.4b}$$

This replaces the action-at-a-distance paradigm of

$$Q \Leftrightarrow q \tag{1.5}$$

with the field paradigm of

$$Q \Leftrightarrow \vec{\mathcal{E}} \Leftrightarrow q. \tag{1.6}$$

Is the field real or artifice? Show later that the field is endowed with energy E, linear momentum $\vec{p}$, and angular momentum $\vec{L}$, and hence is as real as atoms.

Problem

1. Check that each term in Eq. 1.3 has the correct SI units.

2. Write the generalized Coulomb's law of Eq. 1.2 as

$$\vec{F} = q\frac{q'}{4\pi \boldsymbol{\imath}^2}\left(\frac{1}{1-\hat{\boldsymbol{\imath}}\cdot\vec{v}\,'}\right)^2 \Big(\vec{w}\,'/\gamma^2 + \vec{\boldsymbol{\imath}} \times (\vec{w}\,' \times \vec{\mathsf{a}}\,') + \vec{v} \times \left(\hat{\boldsymbol{\imath}} \times \left(\vec{w}\,'/\gamma^2 + \vec{\boldsymbol{\imath}} \times (\vec{w}\,' \times \vec{\mathsf{a}}\,')\right)\right) \Big), \tag{1.7}$$

Find $\vec{w}\,'$ and γ and interpret them physically using the special theory of relativity.

Chapter 2

Geometric Algebra of Space

The classical field equations of electromagnetism, Maxwell's equations, are traditionally expressed in **vector algebra**. However, **geometric algebra** [2] subsumes vector algebra by, for example, combining the dot and cross product into a single geometric product. Maxwell's equations find their most elegant expression in geometric algebra.

2.1 Geometric Product

Let $\vec{u}$, $\vec{v}$, $\vec{w}$ be **3-vectors**, vectors in 3-dimensional Euclidean space. Assume a **geometric product** denoted by juxtaposition

$$\vec{u}\vec{v} \neq \vec{u} \cdot \vec{v} \neq \vec{u} \times \vec{v}, \tag{2.1}$$

which is associative

$$\vec{u}(\vec{v}\vec{w}) = \vec{u}\vec{v}\vec{w} = (\vec{u}\vec{v})\vec{w} \tag{2.2}$$

but not necessarily commutative (so expect $\vec{u}\vec{v} \neq \vec{v}\vec{u}$ at least sometimes).

2.2 Normalized, Antisymmetric Basis

Expand a vector $\vec{v}$ in the basis or (reference) frame $\{\hat{e}_k\}$ with components v^k to get

$$\vec{v} = \sum_{k=1}^{3} \hat{e}_k v^k, \tag{2.3}$$

where the superscript k is an index rather than an exponent. (Save the subscript for a future index.) The conventional letter "e" for unit vectors comes from the German *einheit*, which means unit or unity. Adopt the **Einstein summation convention** and shorten this to

$$\vec{v} = \hat{e}_k v^k, \tag{2.4}$$

read "v vector equals e sub k hat times v super k", with an **implied sum** over repeated raised and lowered indices. Latin indices always range from 1 to 3. Explicitly expand $\vec{v}$ in alternate notations as

$$\begin{aligned}\vec{v} &= \hat{e}_1 v^1 + \hat{e}_2 v^2 + \hat{e}_3 v^3 \\ &= \hat{e}_x v^x + \hat{e}_y v^y + \hat{e}_z v^z \\ &= \hat{x} v_x + \hat{y} v_y + \hat{z} v_z. \end{aligned} \tag{2.5}$$

Assume the basis vectors

$$\{\hat{e}_k\} = \{\hat{e}_1, \hat{e}_2, \hat{e}_3\} \tag{2.6}$$

satisfy the **abstract algebra**

$$\hat{e}_k \hat{e}_l + \hat{e}_l \hat{e}_k = 2\delta_{kl}, \tag{2.7}$$

where the **Kronecker delta**

$$\delta_{kl} = \begin{cases} 1, & k = l, \\ 0, & k \neq l, \end{cases} \tag{2.8}$$

is also the **flat space metric**, and δ_{kl} occupies the kth row and lth column of the square array

$$\delta_{kl} \leftrightarrow \boxed{\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}}. \tag{2.9}$$

Thus, under the geometric product, the basis vectors $\{\hat{e}_k\}$ are **antisymmetric** and **normalized**; for example, $\hat{e}_1\hat{e}_2 = -\hat{e}_2\hat{e}_1$ and $\hat{e}_3\hat{e}_3 = (\hat{e}_3)^2 = 1$, and so on.

2.3 Multivectors

Unlike traditional vector algebra, geometric algebra combines scalars and vectors to form **multivectors**. This is analogous to combining real and imaginary numbers to form complex numbers. (In fact, complex numbers are a **subalgebra** of geometric algebra.) The geometric algebra of space is spanned by the $1 + 3 + 3 + 1 = 8 = 2^3$ multivectors of Table 2.1.

Table 2.1: Basis for geometric algebra of 3-dimensional space.

multivector	number	name	grade
1	1	scalar	0
$\{\hat{e}_k\}$	3	vector	1
$\{\hat{e}_k\hat{e}_l\} = \{\mathcal{I}\hat{e}_m\}$	3	bivector = pseudovector	2
$\hat{e}_1\hat{e}_2\hat{e}_3 = \mathcal{I}$	1	trivector = pseudoscalar	3

The most general such multivector

$$M = s + \vec{v} + B + T \tag{2.10}$$

or

$$M = M^0 + M^1\hat{e}_1 + M^2\hat{e}_2 + M^3\hat{e}_3 + M^4\hat{e}_1\hat{e}_2 + M^5\hat{e}_2\hat{e}_3 + M^6\hat{e}_3\hat{e}_1 + M^7\hat{e}_1\hat{e}_2\hat{e}_3 \tag{2.11}$$

has grade-0 **scalar**, grade-1 **vector**, grade-2 **bivector**, and grade-3 **trivector** parts:

$$s = \langle M\rangle_0 = M^0, \tag{2.12a}$$
$$\vec{v} = \langle M\rangle_1 = M^1\hat{e}_1 + M^2\hat{e}_2 + M^3\hat{e}_3, \tag{2.12b}$$
$$B = \langle M\rangle_2 = M^4\hat{e}_1\hat{e}_2 + M^5\hat{e}_2\hat{e}_3 + M^6\hat{e}_1\hat{e}_1, \tag{2.12c}$$
$$T = \langle M\rangle_3 = M^7\hat{e}_1\hat{e}_2\hat{e}_3. \tag{2.12d}$$

Attempt to create a grade-4 **quadvector** (or higher grade multivector) by multiplying a trivector by a vector, and it contracts to form a bivector instead. Geometrically interpret scalars as directionless points, vectors as directed lines, bivectors as directed areas, and trivectors as directed volumes.

2.4 Pseudoscalar

Because of the antisymmetry and normalization of the $\{\hat{e}_k\}$, the trivector $\mathcal{I} = \hat{e}_1\hat{e}_2\hat{e}_3$ commutes with all basis vectors,

$$\begin{aligned}
\mathcal{I}\hat{e}_1 &= \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_1 = -\hat{e}_1\hat{e}_2\hat{e}_1\hat{e}_3 = +\hat{e}_1\hat{e}_1\hat{e}_2\hat{e}_3 = +\hat{e}_1\hat{e}_1\hat{e}_2\hat{e}_3 = \hat{e}_1\mathcal{I},\\
\mathcal{I}\hat{e}_2 &= \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_2 = -\hat{e}_1\hat{e}_2\hat{e}_2\hat{e}_3 = -\hat{e}_1\hat{e}_2\hat{e}_2\hat{e}_3 = +\hat{e}_2\hat{e}_1\hat{e}_2\hat{e}_3 = \hat{e}_2\mathcal{I},\\
\mathcal{I}\hat{e}_3 &= \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_3 = +\hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_3 = -\hat{e}_1\hat{e}_3\hat{e}_2\hat{e}_3 = +\hat{e}_3\hat{e}_1\hat{e}_2\hat{e}_3 = \hat{e}_3\mathcal{I},
\end{aligned} \tag{2.13}$$

and hence commutes with all vectors $\vec{v} = \hat{e}_k v^k$,

$$\mathcal{I}\,\vec{v} = \vec{v}\,\mathcal{I}. \tag{2.14}$$

It also squares to negative one,

$$\mathcal{I}^2 = \mathcal{I}\mathcal{I} = \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_1\hat{e}_2\hat{e}_3 = +\hat{e}_2\hat{e}_3\hat{e}_2\hat{e}_3 = -\hat{e}_3\hat{e}_3 = -1. \tag{2.15}$$

Since the trivector $\mathcal{I}$ acts like an (imaginary) scalar, refer to it as a **pseudoscalar**.

2.5 Pseudovectors

The pseudoscalar $\mathcal{I}$ relates vectors and bivectors via the **duality** transformations

$$\begin{aligned}
\mathcal{I}\hat{e}_1 = \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_1 = -\hat{e}_1\hat{e}_2\hat{e}_1\hat{e}_3 = +\hat{e}_1\hat{e}_1\hat{e}_2\hat{e}_3 &= +\hat{e}_2\hat{e}_3,\\
\mathcal{I}\hat{e}_2 = \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_2 = -\hat{e}_1\hat{e}_2\hat{e}_2\hat{e}_3 &= -\hat{e}_1\hat{e}_3,\\
\mathcal{I}\hat{e}_3 = \hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_3 &= +\hat{e}_1\hat{e}_2,
\end{aligned} \tag{2.16}$$

or, by further permutations,

$$\begin{aligned}\hat{e}_1\mathcal{I} &= \mathcal{I}\hat{e}_1 = \hat{e}_2\hat{e}_3,\\ \hat{e}_2\mathcal{I} &= \mathcal{I}\hat{e}_2 = \hat{e}_3\hat{e}_1,\\ \hat{e}_3\mathcal{I} &= \mathcal{I}\hat{e}_3 = \hat{e}_1\hat{e}_2.\end{aligned} \tag{2.17}$$

Since bivectors are **dual** to vectors in this one-to-one (or isomorphic) way, refer to them as **pseudovectors**.

2.6 Inner & Outer Product Decomposition

Form the geometric product of two generic vectors

$$\vec{u}\vec{v} = (\hat{e}_k u^k)(\hat{e}_l v^l) = \hat{e}_k\hat{e}_l\, u^k v^l. \tag{2.18}$$

Explicitly expand the implied sums to find

$$\begin{aligned}\vec{u}\vec{v} = &+ \hat{e}_1\hat{e}_1 u^1 v^1 + \hat{e}_1\hat{e}_2 u^1 v^2 + \hat{e}_1\hat{e}_3 u^1 v^3\\ &+ \hat{e}_2\hat{e}_1 u^2 v^1 + \hat{e}_2\hat{e}_2 u^2 v^2 + \hat{e}_2\hat{e}_3 u^2 v^3\\ &+ \hat{e}_3\hat{e}_1 u^3 v^1 + \hat{e}_3\hat{e}_2 u^3 v^2 + \hat{e}_3\hat{e}_3 u^3 v^3.\end{aligned} \tag{2.19}$$

Invoke the anti-commutation and normalization of the basis vectors to segregate the symmetric and antisymmetric parts and write

$$\begin{aligned}\vec{u}\vec{v} = &+ u^1 v^1 + u^2 v^2 + u^3 v^3\\ &+ \hat{e}_2\hat{e}_3(u^2 v^3 - u^3 v^2) + \hat{e}_3\hat{e}_1(u^3 v^1 - u^1 v^3) + \hat{e}_1\hat{e}_2(u^1 v^2 - u^2 v^1),\end{aligned} \tag{2.20}$$

which is a scalar plus a bivector

$$\begin{aligned}\vec{u}\vec{v} &= \vec{u}\cdot\vec{v} + \vec{u}\wedge\vec{v}\\ &= \langle\vec{u}\vec{v}\rangle_0 + \langle\vec{u}\vec{v}\rangle_2.\end{aligned} \tag{2.21}$$

The symmetric, scalar part is the **inner** or **dot** or **scalar** product

$$\langle\vec{u}\vec{v}\rangle_0 = \vec{u}\cdot\vec{v} = u^1 v^1 + u^2 v^2 + u^3 v^3 \tag{2.22}$$

and the antisymmetric, bivector part is the **outer** or **wedge** or **bivector** product

$$\begin{aligned}\langle\vec{u}\vec{v}\rangle_2 = \vec{u}\wedge\vec{v} &= \hat{e}_2\hat{e}_3(u^2 v^3 - u^3 v^2) + \hat{e}_3\hat{e}_1(u^3 v^1 - u^1 v^3) + \hat{e}_1\hat{e}_2(u^1 v^2 - u^2 v^1)\\ &= \mathcal{I}\,\hat{e}_1(u^2 v^3 - u^3 v^2) + \mathcal{I}\,\hat{e}_2(u^3 v^1 - u^1 v^3) + \mathcal{I}\,\hat{e}_3(u^1 v^2 - u^2 v^1)\\ &= \mathcal{I}\,\vec{u}\times\vec{v},\end{aligned} \tag{2.23}$$

which is the dual of the traditional **cross** or **vector** product

$$\vec{u}\times\vec{v} = \hat{e}_1(u^2 v^3 - u^3 v^2) + \hat{e}_2(u^3 v^1 - u^1 v^3) + \hat{e}_3(u^1 v^2 - u^2 v^1). \tag{2.24}$$

Unlike the cross product, which exists in only 3 dimensions (as only in 3 dimensions exists a unique perpendicular to a plane), the outer product generalizes to any number of dimensions, including the $3 + 1 = 4$ dimensions of spacetime.

Use these identifications to recover the Eq. 2.21 decomposition of a generic geometric product,

$$\begin{aligned}\vec{u}\vec{v} &= (\hat{e}_k u^k)(\hat{e}_l v^l) \\ &= \hat{e}_k \hat{e}_l\, u^k v^l \\ &= (\hat{e}_k \cdot \hat{e}_l + \hat{e}_k \wedge \hat{e}_l) u^k v^l \\ &= \hat{e}_k \cdot \hat{e}_l\, u^k v^l + \hat{e}_k \wedge \hat{e}_l\, u^k v^l \\ &= \delta_{kl}\, u^k v^l + \hat{e}_k \wedge \hat{e}_l\, u^k v^l \\ &= \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}, \end{aligned} \tag{2.25}$$

where

$$\vec{u} \cdot \vec{v} = \delta_{kl}\, u^k v^l, \tag{2.26}$$

$$\vec{u} \wedge \vec{v} = \hat{e}_k \wedge \hat{e}_l\, u^k v^l. \tag{2.27}$$

2.7 Symmetries

By Eq. 2.22 and Eq. 2.23, the inner product is symmetric and the outer product is antisymmetric,

$$\vec{u} \cdot \vec{v} = +\vec{v} \cdot \vec{u}, \tag{2.28}$$

$$\vec{u} \wedge \vec{v} = -\vec{v} \wedge \vec{u}. \tag{2.29}$$

(where the latter implies $\vec{v} \wedge \vec{v} = 0$). Use these symmetries to write the Eq. 2.21 fundamental decomposition

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v} \tag{2.30}$$

as

$$\begin{aligned}\vec{v}\vec{u} &= \vec{v} \cdot \vec{u} + \vec{v} \wedge \vec{u} \\ &= \vec{u} \cdot \vec{v} - \vec{u} \wedge \vec{v}, \end{aligned} \tag{2.31}$$

and add and subtract to solve for

$$\langle \vec{u}\vec{v} \rangle_0 = \vec{u} \cdot \vec{v} = \frac{\vec{u}\vec{v} + \vec{v}\vec{u}}{2} = +\vec{v} \cdot \vec{u}, \tag{2.32}$$

$$\langle \vec{u}\vec{v} \rangle_2 = \vec{u} \wedge \vec{v} = \frac{\vec{u}\vec{v} - \vec{v}\vec{u}}{2} = -\vec{v} \wedge \vec{u}. \tag{2.33}$$

Beware that the algebraic signs in these equations alternate as the order of the multivectors increases. For example, if B is a bivector, then

$$\langle \vec{u}B \rangle_1 = \vec{u} \cdot B = \frac{\vec{u}B - B\vec{u}}{2} = -B \cdot \vec{u}, \tag{2.34}$$

$$\langle \vec{u}B \rangle_3 = \vec{u} \wedge B = \frac{\vec{u}B + B\vec{u}}{2} = +B \wedge \vec{u}. \tag{2.35}$$

As an example, set the bivector $B = \vec{v}\mathcal{I}$ to show that the pseudoscalar $\mathcal{I} = \hat{e}_1\hat{e}_2\hat{e}_3$ can interchange inner and outer products,

$$\vec{u}\cdot(\vec{v}\mathcal{I}) = \frac{\vec{u}\vec{v}\mathcal{I} - \vec{v}\mathcal{I}\vec{u}}{2} = \frac{\vec{u}\vec{v}\mathcal{I} - \vec{v}\vec{u}\mathcal{I}}{2} = \frac{\vec{u}\vec{v} - \vec{v}\vec{u}}{2}\mathcal{I} = \vec{u}\wedge\vec{v}\mathcal{I}, \tag{2.36}$$

and

$$\vec{u}\wedge(\vec{v}\mathcal{I}) = \frac{\vec{u}\vec{v}\mathcal{I} + \vec{v}\mathcal{I}\vec{u}}{2} = \frac{\vec{u}\vec{v}\mathcal{I} + \vec{v}\vec{u}\mathcal{I}}{2} = \frac{\vec{u}\vec{v} + \vec{v}\vec{u}}{2}\mathcal{I} = \vec{u}\cdot\vec{v}\mathcal{I}, \tag{2.37}$$

These formulas generalize to any multivector M,

$$\vec{u}\cdot(M\mathcal{I}) = \vec{u}\wedge M\mathcal{I}, \tag{2.38}$$

$$\vec{u}\wedge(M\mathcal{I}) = \vec{u}\cdot M\mathcal{I}, \tag{2.39}$$

As another example, show that a bivector is perpendicular to its dual vector. If

$$B = B^1\hat{e}_2\hat{e}_3 + B^2\hat{e}_3\hat{e}_1 + B^3\hat{e}_1\hat{e}_2 = \mathcal{I}(B^1\hat{e}_1 + B^2\hat{e}_2 + B^3\hat{e}_3) = \mathcal{I}\vec{b}, \tag{2.40}$$

then

$$\vec{b}\cdot B = \frac{\vec{b}B - B\vec{b}}{2} = \frac{\vec{b}\mathcal{I}\vec{b} - \mathcal{I}\vec{b}\vec{b}}{2} = \mathcal{I}\frac{\vec{b}\vec{b} - \vec{b}\vec{b}}{2} = 0. \tag{2.41}$$

Without loss of generality, rotate and scale the coordinates so that $\vec{b} = \hat{e}_3$ and hence $B = \hat{e}_1\hat{e}_2 = \hat{e}_1\cdot\hat{e}_2 + \hat{e}_1\wedge\hat{e}_2 = \hat{e}_1\wedge\hat{e}_2$. Figure 2.1 illustrates the orthogonality of the duality.

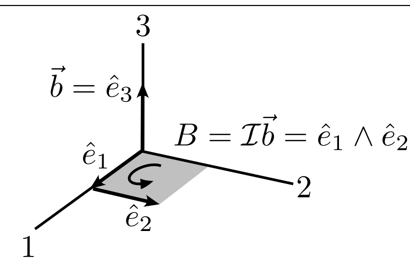

Figure 2.1: A bivector $B = \mathcal{I}\vec{b}$ is perpendicular to its dual vector $\vec{b}$.

2.8 Parallel & Perpendicular

If vectors $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u}\wedge\vec{v} = 0$ and $\vec{u}\vec{v} = \vec{u}\cdot\vec{v}$. If they are perpendicular, then $\vec{u}\cdot\vec{v} = 0$ and $\vec{u}\vec{v} = \vec{u}\wedge\vec{v}$. These important special cases are summarized in Table 2.2.

If $\hat{n}$ is a unit vector and $\vec{v}$ is a general vector, then

$$\vec{v} = \hat{n}^2\vec{v} = (\hat{n}\hat{n})\vec{v} = \hat{n}\hat{n}\vec{v} = \hat{n}(\hat{n}\vec{v}) = \hat{n}(\hat{n}\cdot\vec{v} + \hat{n}\wedge\vec{v}) = \vec{v}_\parallel + \vec{v}_\perp \tag{2.42}$$

Table 2.2: Special cases of geometric product of two vectors.

case	product
generic	$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$
$\vec{u} \parallel \vec{v}$	$\vec{u}\vec{v} = +\vec{v}\vec{u} = \vec{u} \cdot \vec{v}$
$\vec{u} \perp \vec{v}$	$\vec{u}\vec{v} = -\vec{v}\vec{u} = \vec{u} \wedge \vec{v}$

where

$$\vec{v}_{\parallel} = \hat{n}(\hat{n} \cdot \vec{v}) \tag{2.43}$$

is the **projection** and

$$\vec{v}_{\perp} = \hat{n}(\hat{n} \wedge \vec{v}) \tag{2.44}$$

is the **rejection** of $\vec{v}$, parallel and perpendicular to $\hat{n}$.

Figure 2.2 depicts a specific example, with

$$\hat{n} = \hat{e}_1, \tag{2.45}$$
$$\vec{v} = \hat{e}_1 + 2\hat{e}_2, \tag{2.46}$$

so that the dot and wedge products are

$$\hat{n} \cdot \vec{v} = 1, \tag{2.47}$$
$$\hat{n} \wedge \vec{v} = 2\hat{e}_1\hat{e}_2, \tag{2.48}$$

and the projection and rejection are

$$\vec{v}_{\parallel} = \hat{e}_1, \tag{2.49}$$
$$\vec{v}_{\perp} = 2\hat{e}_2. \tag{2.50}$$

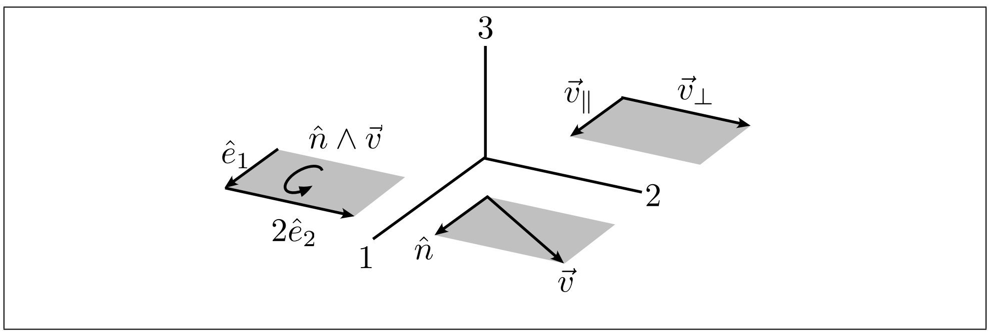

Figure 2.2: The projection and rejection of the vector $\vec{v}$ on the vector $\hat{n}$.

2.9 Magnitudes

If $\vec{v}$ is a vector, then

$$\vec{v}^{\,2} = \vec{v} \cdot \vec{v} + \vec{v} \wedge \vec{v} = \vec{v} \cdot \vec{v} = |\vec{v}|^2 \tag{2.51}$$

where the magnitude $|\vec{v}|$ is its scalar length.

Suppose that the angle between the vectors $\vec{u}$ and $\vec{v}$ is θ, as in Fig. 2.3. Since the inner product is the familiar dot or scalar product, its magnitude

$$|\vec{u} \cdot \vec{v}| = |\vec{u}||\vec{v}||\cos\theta| \tag{2.52}$$

is the length of the projection of $\vec{u}$ on $\vec{v}$.

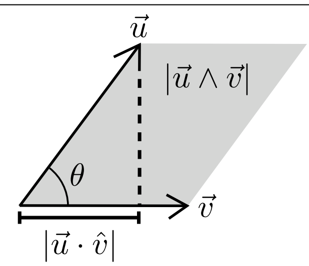

Figure 2.3: The magnitudes of the inner and outer products, the projected length and the shaded area.

Since the outer product is the dual of the familiar cross or vector product, expect its magnitude

$$|\vec{u} \wedge \vec{v}| = |\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}||\sin\theta| \tag{2.53}$$

to be the area of the parallelogram framed by $\vec{u}$ on $\vec{v}$. To verify this, compute the square of the outer product $\vec{u} \wedge \vec{v}$,

$$\begin{aligned}
(\vec{u} \wedge \vec{v})^2 &= (\vec{u}\vec{v} - \vec{u} \cdot \vec{v})(\vec{u}\vec{v} - \vec{u} \cdot \vec{v}) \\
&= \vec{u}\vec{v}\,\vec{u}\vec{v} - (\vec{u} \cdot \vec{v})\vec{u}\vec{v} - \vec{u}\vec{v}(\vec{u} \cdot \vec{v}) + (\vec{u} \cdot \vec{v})^2 \\
&= \vec{u}\vec{v}(-\vec{v}\vec{u} + 2\vec{u} \cdot \vec{v}) - 2(\vec{u} \cdot \vec{v})\vec{u}\vec{v} + (\vec{u} \cdot \vec{v})^2 \\
&= -\vec{u}|\vec{v}|^2\vec{u} + (\vec{u} \cdot \vec{v})^2 \\
&= -|\vec{u}|^2|\vec{v}|^2 + |\vec{u}|^2|\vec{v}|^2\cos^2\theta \\
&= -|\vec{u}|^2|\vec{v}|^2\sin^2\theta,
\end{aligned} \tag{2.54}$$

which implies Eq. 2.53.

Consider the trivector

$$T = \vec{u} \wedge (\vec{v} \wedge \vec{w}) = (\vec{u} \wedge \vec{v}) \wedge \vec{w} = \vec{u} \wedge \vec{v} \wedge \vec{w}. \tag{2.55}$$

Express the corresponding volume more traditionally by multiplying by the pseudoscalar $\mathcal{I}$ and using Eq. 2.39 and Eq. 2.23 to convert wedge products to dot products and cross products. If $B = \vec{v} \wedge \vec{w} = \mathcal{I}\vec{v} \times \vec{w}$, then

$$T = \vec{u} \wedge B = -\vec{u} \cdot (B\mathcal{I})\mathcal{I} = -\vec{u} \cdot (\mathcal{I}\vec{v} \times \vec{w}\mathcal{I})\mathcal{I} = \vec{u} \cdot \vec{v} \times \vec{w}\,\mathcal{I} = |T|\mathcal{I}, \tag{2.56}$$

where the magnitude

$$|T| = |\vec{u} \wedge \vec{v} \wedge \vec{w}| = \vec{u} \cdot \vec{v} \times \vec{w} \tag{2.57}$$

is the volume of the parallelepiped framed by $\vec{u}$ on $\vec{v}$ on $\vec{w}$.

2.10 Geometric Interpretations

Visualize a scalar s as a point, a vector $\vec{u}$ as a directed line or arrow, a bivector $B = \vec{u} \wedge \vec{v}$ as a directed plane, and a trivector $T = (\vec{u} \wedge \vec{v}) \wedge \vec{w}$ as a directed solid, as in Fig. 2.4. Use the wedge product to create multivectors of any order.

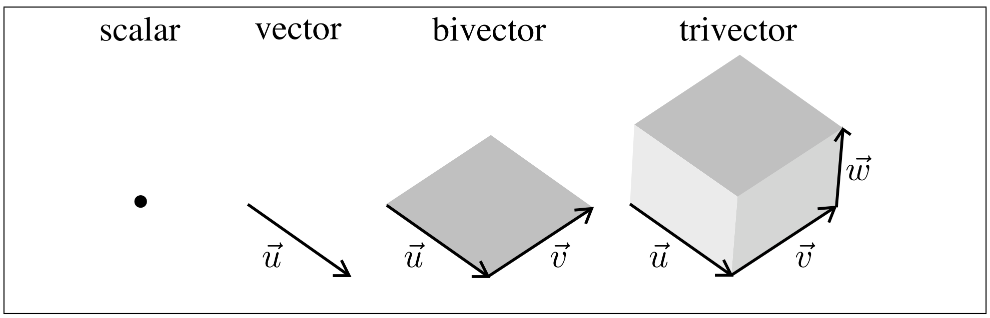

Figure 2.4: Visualization of the lowest grade multivectors.

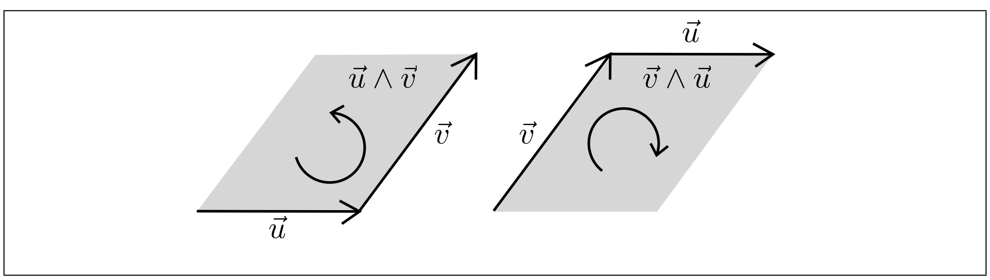

Figure 2.5: Visualization of bivectors as oriented areas. Interchanging the vectors reverses the orientation of the area.

Think of the outer or wedge product $B = \vec{u} \wedge \vec{v}$ as the parallelogram formed by sweeping $\vec{v}$ along $\vec{u}$, as in Fig. 2.5. Think of the wedge product $-B = \vec{v} \wedge \vec{u}$ as the parallelogram formed by sweeping $\vec{u}$ along $\vec{v}$. Interchanging the vectors reverses the orientation of the area, due to the antisymmetry of the wedge product. Similarly, think of the wedge product $T = (\vec{u} \wedge \vec{v}) \wedge \vec{w}$ as the parallelepiped formed by sweeping $\vec{u} \wedge \vec{v}$ along $\vec{w}$. Again, permuting the vectors cycles the orientation.

2.11 Metric

Associate with any (possibly nonorthonormal) reference frame $\{\vec{e}_k\}$ a second (possibly nonorthonormal) **reciprocal** frame $\{\vec{e}^k\}$ mutually orthonormal to the first,

$$\vec{e}^k \cdot \vec{e}_l = \delta^k{}_l. \tag{2.58}$$

Expand a vector in both frames by

$$\vec{v} = v^k \vec{e}_k = v_l \vec{e}^l, \tag{2.59}$$

where the **contravariant** and **covariant** components

$$\vec{v} \cdot \vec{e}^k = v^l \vec{e}_l \cdot \vec{e}^k = v^l \delta_l{}^k = v^k, \tag{2.60a}$$

$$\vec{v} \cdot \vec{e}_k = v_l \vec{e}^l \cdot \vec{e}^k = v_l \delta^l{}_k = v_k \tag{2.60b}$$

(remember "co is low") imply the identities

$$\vec{v} = \vec{v} \cdot \vec{e}_k \vec{e}^k = \vec{v} \cdot \vec{e}^l \vec{e}_l. \tag{2.61}$$

Define the symmetric covariant **metric tensor** and its contravariant inverse to be the dot product of the same-frame basis vectors,

$$g_{kl} = \vec{e}_k \cdot \vec{e}_l = \vec{e}_l \cdot \vec{e}_k = g_{lk}, \tag{2.62a}$$

$$g^{kl} = \vec{e}^k \cdot \vec{e}^l = \vec{e}^l \cdot \vec{e}^k = g^{lk}, \tag{2.62b}$$

where

$$g^{kl} g_{lm} = \vec{e}^k \cdot \vec{e}^l \vec{e}_l \cdot \vec{e}_m = \vec{e}^k \cdot \vec{e}_m = \delta^k{}_m. \tag{2.63}$$

Write the dot product with or without the metrics as

$$\vec{u} \cdot \vec{v} = u^k \vec{e}_k \cdot v^l \vec{e}_l = u^k v^l \; \vec{e}_k \cdot \vec{e}_l = u^k v^l g_{kl} = u_m v_n g^{mn}, \tag{2.64a}$$

$$\vec{u} \cdot \vec{v} = u^k \vec{e}_k \cdot v_l \vec{e}^l = u^k v_l \; \vec{e}_k \cdot \vec{e}^l = u^k v_l \; \delta_k{}^l = u^k v_k = u_m v^m. \tag{2.64b}$$

By comparison, the metrics raise or lower component indices

$$v^m = g^{mn} v_n, \tag{2.65a}$$

$$v_k = g_{kl} v^l, \tag{2.65b}$$

and by substituting into the Eq. 2.59 vector expansion, they raise or lower basis vector indices

$$\vec{e}^m = g^{mn} \vec{e}_n, \tag{2.66a}$$

$$\vec{e}_m = g_{mn} \vec{e}^n. \tag{2.66b}$$

The covariant and contravariant basis vectors $\vec{e}_k$ and $\vec{e}^k$ are the same in Euclidean frames where the metric $g_{kl} = \delta_{kl}$. However, they differ in nonorthonormal

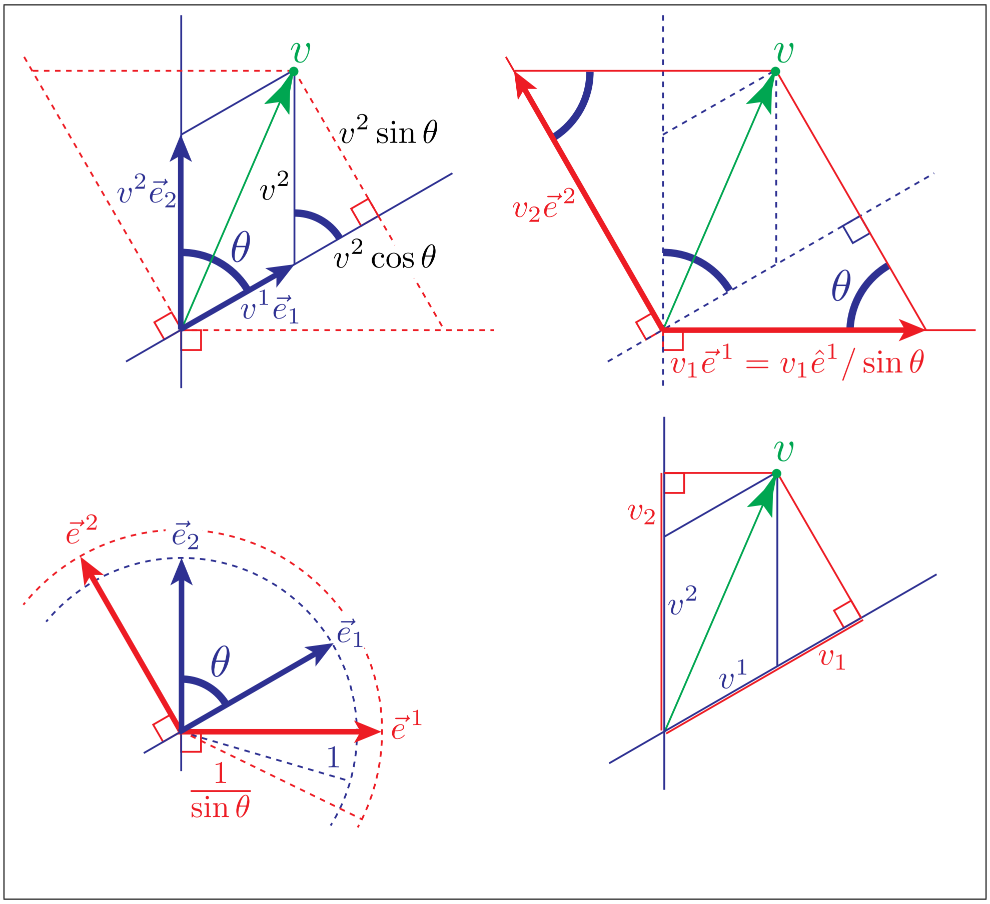

Figure 2.6: Acute skew frame (blue) and its obtuse reciprocal frame (red).

frames. Consider a flat-space skew frame whose axes are separated by an angle θ, as in Fig. 2.6. From the geometry, the squared length of the vector $\vec{v}$ is

$$\begin{aligned}
\vec{v} \cdot \vec{v} = \vec{v}^2 &= (v^1 + v^2 \cos\theta)^2 + (v^2 \sin\theta)^2 \\
&= v^1 v^1 + v^1 v^2 \cos\theta + v^2 v^1 \cos\theta + v^2 v^2 \\
&= g_{kl} v^k v^l
\end{aligned} \tag{2.67}$$

so the metric and its inverse

$$g_{kl} \leftrightarrow \boxed{\begin{matrix} 1 & \cos\theta \\ \cos\theta & 1 \end{matrix}}, \tag{2.68a}$$

$$g^{kl} \leftrightarrow \boxed{\begin{matrix} 1 & -\cos\theta \\ -\cos\theta & 1 \end{matrix}} \, \frac{1}{\sin^2\theta}. \tag{2.68b}$$

The reciprocal bases vectors

$$\vec{e}^{\,1} = g^{1n}\vec{e}_n = (\vec{e}_1 - \vec{e}_2\cos\theta)\frac{1}{\sin^2\theta}, \tag{2.69a}$$

$$\vec{e}^{\,2} = g^{2n}\vec{e}_n = (-\vec{e}_1\cos\theta + \vec{e}_2)\frac{1}{\sin^2\theta}, \tag{2.69b}$$

and the reciprocal components

$$v_1 = g_{1n}v^n = v^1 + v^2\cos\theta, \tag{2.70a}$$

$$v_2 = g_{2n}v^n = v^1\cos\theta + v^2. \tag{2.70b}$$

The skew frame basis vectors are of unit length, but the reciprocal bases vector are not,

$$\sqrt{\vec{e}_1\cdot\vec{e}_1} = \sqrt{g_{11}} = 1 = \sqrt{g_{22}} = \sqrt{\vec{e}_2\cdot\vec{e}_2}, \tag{2.71a}$$

$$\sqrt{\vec{e}^{\,1}\cdot\vec{e}^{\,1}} = \sqrt{g^{11}} = \frac{1}{\sin\theta} = \sqrt{g^{22}} = \sqrt{\vec{e}^{\,2}\cdot\vec{e}^{\,2}}, \tag{2.71b}$$

so the contravariant unit vectors

$$\hat{e}^{\,1} = \vec{e}^{\,1}\sin\theta, \tag{2.72a}$$

$$\hat{e}^{\,2} = \vec{e}^{\,2}\sin\theta. \tag{2.72b}$$

Thus, by Fig. 2.6, the contravariant components v^n correspond to parallel projection and the covariant components v_n to perpendicular projection onto the original frame.

2.12 Rotations

To rotate a vector $\vec{v} = \hat{e}_1$ through an angle θ in the plane $B = \hat{e}_1\wedge\hat{e}_2 = \hat{e}_1\hat{e}_2$, multiply by the **rotor**

$$R = e^{-\hat{e}_1\hat{e}_2\theta/2} = e^{-B\theta/2} \tag{2.73}$$

on the right and by its **reversion**

$$\tilde{R} = R^\dagger = e^{-\hat{e}_2\hat{e}_1\theta/2} = e^{+B\theta/2} \tag{2.74}$$

on the left, with the exponentials defined by their infinite series expansions like

$$\begin{aligned} R^\dagger &= e^{+\hat{e}_1\hat{e}_2\theta/2} \\ &= 1 + \left(\hat{e}_1\hat{e}_2\,\frac{\theta}{2}\right) + \frac{1}{2!}\left(\hat{e}_1\hat{e}_2\,\frac{\theta}{2}\right)^2 + \frac{1}{3!}\left(\hat{e}_1\hat{e}_2\,\frac{\theta}{2}\right)^3 + \cdots \\ &= \left(1 - \frac{1}{2!}\left(\frac{\theta}{2}\right)^2 + \cdots\right) + \hat{e}_1\hat{e}_2\left(\left(\frac{\theta}{2}\right) - \frac{1}{3!}\left(\frac{\theta}{2}\right)^3 + \cdots\right) \\ &= \cos\left[\frac{\theta}{2}\right] + \hat{e}_1\hat{e}_2\sin\left[\frac{\theta}{2}\right], \end{aligned} \tag{2.75}$$

as $i^2 = (\hat{e}_1\hat{e}_2)^2 = -1$. Hence, the rotated vector is

$$
\begin{aligned}
\vec{v}\,' &= R\,\vec{v}\,R^\dagger = e^{-B\theta/2}\,\vec{v}\,e^{+B\theta/2} = e^{-\hat{e}_1\hat{e}_2\theta/2}\,\vec{v}\,e^{+\hat{e}_1\hat{e}_2\theta/2} \\
&= \left(\cos\left[\frac{\theta}{2}\right] - \hat{e}_1\hat{e}_2 \sin\left[\frac{\theta}{2}\right]\right) \hat{e}_1 \left(\cos\left[\frac{\theta}{2}\right] + \hat{e}_1\hat{e}_2 \sin\left[\frac{\theta}{2}\right]\right) \\
&= \left(\cos\left[\frac{\theta}{2}\right] - \hat{e}_1\hat{e}_2 \sin\left[\frac{\theta}{2}\right]\right) \left(\hat{e}_1 \cos\left[\frac{\theta}{2}\right] + \hat{e}_2 \sin\left[\frac{\theta}{2}\right]\right) \\
&= \hat{e}_1 \left(\cos^2\left[\frac{\theta}{2}\right] - \sin^2\left[\frac{\theta}{2}\right]\right) + \hat{e}_2 \left(2\sin\left[\frac{\theta}{2}\right]\cos\left[\frac{\theta}{2}\right]\right) \\
&= \hat{e}_1 \cos\theta + \hat{e}_2 \sin\theta, \qquad (2.76)
\end{aligned}
$$

as illustrated in Fig. 2.7.

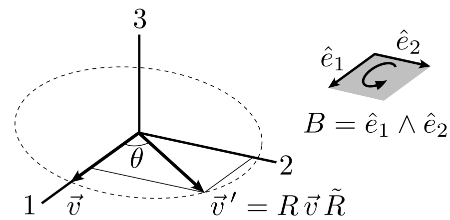

Figure 2.7: The rotor R rotates a vector $\vec{v}$ through an angle θ in the plane of the bivector B.

Such rotation in a plane generalizes to any dimension, whereas the traditional notion of rotation about a line is confined to three dimensions. Rotate any multivector in the same way.

Problems

1. Reduce the following to the simplest multivector form of Eq. 2.11.

   (a) $(\hat{e}_1\hat{e}_2 + 2\hat{e}_1\hat{e}_3)(\hat{e}_1 - \hat{e}_1\hat{e}_2)$.

   (b) $(3 - \hat{e}_1 + 2\hat{e}_1\hat{e}_2\hat{e}_3)(\hat{e}_1 + 3\hat{e}_1\hat{e}_2)$.

   (c) $\hat{e}_2\hat{e}_1\hat{e}_3\hat{e}_2\hat{e}_1\hat{e}_2\hat{e}_2\hat{e}_3$.

2. Let $\vec{u} = \hat{e}_1 - 2\hat{e}_2$, $\vec{v} = \hat{e}_1 + \hat{e}_2 - \hat{e}_3$, and $\vec{w} = 3\hat{e}_1 + \hat{e}_3$. Reduce the following products to simplest form.

   (a) $\vec{u} \cdot \vec{v}$.

   (b) $\vec{u} \wedge \vec{v}$.

   (c) $(\vec{u} \wedge \vec{v}) \wedge \vec{w}$.

   (d) $\vec{u} \wedge (\vec{v} \wedge \vec{w})$.

3. Assume the vectors of Problem 2, and define the bivector $B = \langle \vec{v}\vec{w} \rangle_2$. Verify the following identities. (Hint: Because the dot product is grade lowering, $\vec{u} \cdot B = \langle \vec{u}B \rangle_1$, and so on.)

   (a) By Eq. 2.34, $\vec{u} \cdot B = (\vec{u}B - B\vec{u})/2$.

   (b) By Eq. 2.35, $\vec{u} \wedge B = (\vec{u}B + B\vec{u})/2$.

4. Derive the **group composition law** for rotors by showing that if rotor R_1 rotates vector $\vec{u}$ into vector $\vec{v}$ and rotor R_2 rotates vector $\vec{v}$ into vector $\vec{w}$, then the composite rotor $R = R_2 R_1$ rotates vector $\vec{u}$ into vector $\vec{w}$. (Hint: Like the matrix transpose or adjoint operations, the reversion of a product of rotors is the product of the reversions in reverse order.)

5. In mechanics, **Euler angles** $\{\phi, \theta, \psi\}$ traditionally parameterize rotations in three-dimensional space: First rotate anticlockwise through an angle ϕ about the direction $\hat{e}_3$, next rotate anticlockwise through an angle θ about the transformed direction $\hat{e}'_1$, finally rotate anticlockwise through an angle ψ about the transformed direction $\hat{e}''_3$. Find bivectors B_1, B_2, B_3 such that

$$R = e^{-B_1\phi/2} e^{-B_2\theta/2} e^{-B_3\psi/2}. \tag{2.77}$$

   (Hint: Use the Problem 4 composition law and the Eq. 2.40 orthogonality of bivectors and their dual vectors.)

Chapter 3

Geometric Calculus of Space

3.1 Geometric Derivative

Use the flat space metric $g_{kl} = \delta_{kl}$ to define equivalent basis vectors $\hat{e}_l = g_{lk}\hat{e}^k$. With orthonormal frame $\{\hat{e}^k\}$ and position $\vec{x} = \hat{e}_k x^k$, expand the **geometric** or **vector** derivative $\vec{\nabla}$ in components $\nabla_k = \partial_k$ to get

$$\vec{\nabla} = \sum_{k=1}^{3} \hat{e}^k \frac{\partial}{\partial x^k} = \hat{e}^k \partial_k, \tag{3.1}$$

read "big del vector equals e hat super k times del sub k". (Although ∇ is sometimes referred to as the "nabla" operator, pronounce the pseudo-letters ∇ and ∂ "del" in analogy with pronouncing the Greek letters Δ and δ "delta".) Expand $\vec{\nabla}$ in alternate notations as

$$\begin{aligned}\vec{\nabla} &= \hat{e}_1\partial_1 + \hat{e}_2\partial_2 + \hat{e}_3\partial_3,\\ &= \hat{e}_1\frac{\partial}{\partial x^1} + \hat{e}_2\frac{\partial}{\partial x^2} + \hat{e}_3\frac{\partial}{\partial x^3},\\ &= \hat{e}_x\frac{\partial}{\partial x} + \hat{e}_y\frac{\partial}{\partial y} + \hat{e}_z\frac{\partial}{\partial z},\\ &= \hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}.\end{aligned} \tag{3.2}$$

Apply $\vec{\nabla}$ to a generic vector field $\vec{v}[\vec{x}]$ to get

$$\vec{\nabla}\vec{v} = (\hat{e}^k\partial_k)(\hat{e}_l v^l), \tag{3.3}$$

and use $\hat{e}^1 = \hat{e}_1$, and so on, to expand this into

$$\begin{aligned}\vec{\nabla}\vec{v} = &+ \hat{e}_1\hat{e}_1\partial_1 v^1 + \hat{e}_1\hat{e}_2\partial_1 v^2 + \hat{e}_1\hat{e}_3\partial_1 v^3\\ &+ \hat{e}_2\hat{e}_1\partial_2 v^1 + \hat{e}_2\hat{e}_2\partial_2 v^2 + \hat{e}_2\hat{e}_3\partial_2 v^3\\ &+ \hat{e}_3\hat{e}_1\partial_3 v^1 + \hat{e}_3\hat{e}_2\partial_3 v^2 + \hat{e}_3\hat{e}_3\partial_3 v^3.\end{aligned} \tag{3.4}$$

By invoking the anti-commutation and normalization of the basis vectors, segregate the symmetric and antisymmetric parts, to find

$$\begin{aligned}\vec{\nabla}\vec{v} = &+ \partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3 \\ &+ \hat{e}_2\hat{e}_3(\partial_2 v^3 - \partial_3 v^2) + \hat{e}_3\hat{e}_1(\partial_3 v^1 - \partial_1 v^3) + \hat{e}_1\hat{e}_2(\partial_1 v^2 - \partial_2 v^1)\end{aligned} \tag{3.5}$$

or

$$\vec{\nabla}\vec{v} = \vec{\nabla}\cdot\vec{v} + \vec{\nabla}\wedge\vec{v}, \tag{3.6}$$

where the symmetric, scalar part is the **interior derivative** or **divergence**

$$\vec{\nabla}\cdot\vec{v} = \partial_1 v^1 + \partial_2 v^2 + \partial_3 v^3, \tag{3.7}$$

and the antisymmetric, bivector part is the **exterior derivative**

$$\begin{aligned}\vec{\nabla}\wedge\vec{v} &= \hat{e}_2\hat{e}_3(\partial_2 v^3 - \partial_3 v^2) + \hat{e}_3\hat{e}_1(\partial_3 v^1 - \partial_1 v^3) + \hat{e}_1\hat{e}_2(\partial_1 v^2 - \partial_2 v^1) \\ &= \mathcal{I}\,\hat{e}_1(\partial_2 v^3 - \partial_3 v^2) + \mathcal{I}\,\hat{e}_2(\partial_3 v^1 - \partial_1 v^3) + \mathcal{I}\,\hat{e}_3(\partial_1 v^2 - \partial_2 v^1) \\ &= \mathcal{I}\,\vec{\nabla}\times\vec{v},\end{aligned} \tag{3.8}$$

which is the dual of the traditional **curl**

$$\vec{\nabla}\times\vec{v} = \hat{e}_1(\partial_2 v^3 - \partial_3 v^2) + \hat{e}_2(\partial_3 v^1 - \partial_1 v^3) + \hat{e}_3(\partial_1 v^2 - \partial_2 v^1). \tag{3.9}$$

Unlike the curl, which exists in only 3 dimensions, the exterior derivative generalizes to any number of dimensions.

3.2 Differentiating Scalar Fields

Given a *constant* vector $\vec{a} = \hat{e}^k a_k$, consider a scalar field

$$\varphi[\vec{x}] = \vec{a}\cdot\vec{x} = \hat{e}^k a_k \cdot \hat{e}^l x_l = \hat{e}^k\cdot\hat{e}^l a_k x_l = \delta^{kl} a_k x_l = a_l x^l, \tag{3.10}$$

whose gradient is

$$\begin{aligned}\vec{\nabla}(\vec{a}\cdot\vec{x}) &= \hat{e}^k\partial_k(a_l x^l) \\ &= \hat{e}^k a_l\,\partial_k x^l \\ &= \hat{e}^k a_l \delta^l_k \\ &= \hat{e}^k a_k \\ &= \vec{a},\end{aligned} \tag{3.11}$$

where, from the definition of partial differentiation,

$$\partial_k x^l = \frac{\partial x^l}{\partial x^k} = \delta^l_k. \tag{3.12}$$

This is analogous to $d(3x)/dx = 3$ in ordinary one-dimensional calculus. Set $\vec{a} = \hat{e}^k$ to find that the gradient of a coordinate is the corresponding basis vector,

$$\vec{\nabla}x^k = \hat{e}^k. \tag{3.13}$$

This is analogous to $d(x)/dx = 1$.

Next consider the scalar field

$$\theta[\vec{x}] = \vec{x}^{\,2} = \vec{x}\vec{x} = \vec{x} \cdot \vec{x} = x^l \hat{e}_l \cdot x^m \hat{e}_m = x^l x^m \hat{e}_l \cdot \hat{e}_m = x^l x^m \delta_{lm} = x^l x_l, \tag{3.14}$$

whose gradient is

$$\begin{aligned}
\vec{\nabla}(\vec{x}^{\,2}) &= \hat{e}^k \partial_k (x^l x_l) \\
&= \hat{e}^k (\partial_k x^l) x_l + \hat{e}^k x^l (\partial_k x_l) \\
&= \hat{e}^k \delta^l_k x_l + \hat{e}^k x^l \delta_{kl} \\
&= \hat{e}^k x_k + \hat{e}^k x_k \\
&= 2\vec{x},
\end{aligned} \tag{3.15}$$

according to the product rule and the Eq. 3.12 partial differentiation identity. This is analogous to $d(x^2)/dx = 2x$.

3.3 Differentiating Vector Fields

Consider a vector field $\vec{v}[\vec{x}]$. The **interior derivative** or **divergence** of the vector field $\vec{v}[\vec{x}]$ is the symmetric

$$\vec{\nabla} \cdot \vec{v} = \hat{e}^k \partial_k \cdot \hat{e}_l v^l = \hat{e}^k \cdot \hat{e}_l \, \partial_k v^l = \delta^k_l \partial_k v^l = \partial_k v^k. \tag{3.16}$$

The **exterior derivative** of $\vec{v}[\vec{x}]$ is the antisymmetric

$$\vec{\nabla} \wedge \vec{v} = \hat{e}^k \partial_k \wedge \hat{e}_l v^l = \hat{e}^k \wedge \hat{e}_l \, \partial_k v^l = \hat{e}^k \wedge \hat{e}^l \, \partial_k v_l \tag{3.17}$$

For example, the geometric vector derivative of $\vec{x}$ is

$$\vec{\nabla}\vec{x} = \vec{\nabla} \cdot \vec{x} + \vec{\nabla} \wedge \vec{x} = 3 + 0 = 3, \tag{3.18}$$

which is the dimension of space. This is analogous to $d(x)/dx = 1$ in ordinary one-dimensional calculus.

If M is a generic multivector, then

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge M) = \hat{e}^k \partial_k \wedge \left(\hat{e}^l \partial_l \wedge M\right) = \hat{e}^k \wedge \hat{e}^l \wedge (\partial_k \partial_l M) = A^{kl} \wedge S_{kl}. \tag{3.19}$$

Because $S^{kl} = S^{lk}$ is symmetric but $A_{kl} = -A_{lk}$ is antisymmetric, the double sum of Eq. 3.19 vanishes. (Expand explicitly to verify.) Hence,

$$\vec{\nabla} \wedge (\vec{\nabla} \wedge M) = 0 \tag{3.20}$$

and so the exterior derivative of an exterior derivative always vanishes.

Insert $N = M\mathcal{I}^{-1}$ into Eq. 3.19, multiply by the pseudoscalar $\mathcal{I}$, and use the dot-wedge duality of Eq. 2.39 to convert it to

$$0 = \vec{\nabla} \wedge (\vec{\nabla} \wedge N)\mathcal{I} = \vec{\nabla} \cdot (\vec{\nabla} \wedge N\,\mathcal{I}) = \vec{\nabla} \cdot (\vec{\nabla} \cdot (N\mathcal{I})) \tag{3.21}$$

or

$$\vec{\nabla} \cdot (\vec{\nabla} \cdot M) = 0. \tag{3.22}$$

Thus, the interior derivative of the interior derivative always vanishes. (For consistency, and by extension, assume that the dot product of a vector and a scalar vanishes.)

Problems

1. Let $\vec{u}[\vec{x}] = \hat{e}_1 2x^1 - \hat{e}_2 x^1 x^2 x^3$, $\vec{v}[\vec{x}] = \hat{e}_1 3x^1 + \hat{e}_2 2x^2 - \hat{e}_3 x^3$, and $\vec{w}[\vec{x}] = \hat{e}_1 2x^2 x^3 + \hat{e}_3 x^1 x^2$. Reduce the following derivatives to simplest form.

   (a) $\vec{\nabla} \cdot \vec{v}$.

   (b) $\vec{\nabla} \wedge \vec{v}$.

   (c) $\vec{\nabla} \cdot \vec{u}$.

   (d) $\vec{\nabla} \vec{w}$.

2. Assume the vector fields of Problem 1, and define the bivector field $B[\vec{x}] = \langle \vec{v}[\vec{x}]\, \vec{w}[\vec{x}] \rangle_2$. Verify the following identities.

   (a) By Eq. 3.20, $\vec{\nabla} \wedge (\vec{\nabla} \wedge \vec{u}) = 0$.

   (b) By Eq. 3.22, $\vec{\nabla} \cdot (\vec{\nabla} \cdot B) = 0$.

Chapter 4

Geometric Algebra of Spacetime

In the **theory of relativity**, Albert Einstein placed space and time on an equal footing, but it was Hermann Minkowski who first recognized that the geometry of spacetime was non-Euclidean. Incorporate spacetime geometry into geometric algebra by adding a single time-like basis vector $\hat{e}_0$.

4.1 Basis

Expand a spacetime vector $\vec{\vec{v}}$ of $3+1=4$ dimensions in the basis or (reference) frame $\{\hat{e}_\alpha\}$ with components v^α to get

$$\vec{\vec{v}} = \sum_{\alpha=0}^{3} \hat{e}_\alpha v^\alpha = \hat{e}_\alpha v^\alpha, \tag{4.1}$$

read "v bold vector equals e sub alpha hat times v super alpha", with an implied sum over repeated raised and lowered indices. Unlike Latin indices that range from 1 to 3, Greek indices range from 0 to 3. Explicitly expand $\vec{\vec{v}}$ in alternate notations as

$$\vec{\vec{v}} = \hat{e}_\alpha v^\alpha = \hat{e}_0 v^0 + \hat{e}_k v^k = \hat{e}_t v^t + \vec{v}. \tag{4.2}$$

Assume the basis vectors

$$\{\hat{e}_\alpha\} = \{\hat{e}_0, \hat{e}_1, \hat{e}_2, \hat{e}_3\} \tag{4.3}$$

satisfy the abstract algebra

$$\hat{e}_\alpha \hat{e}_\beta + \hat{e}_\beta \hat{e}_\alpha = 2\eta_{\alpha\beta}, \tag{4.4}$$

or, alternately,

$$\hat{e}_\alpha \cdot \hat{e}_\beta = \frac{\hat{e}_\alpha \hat{e}_\beta + \hat{e}_\beta \hat{e}_\alpha}{2} = \eta_{\alpha\beta}, \tag{4.5}$$

where the **flat spacetime metric**

$$\eta_{\alpha\beta} \leftrightarrow \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{4.6}$$

The **signature** of the spacetime metric is the sequence $\{-,+,+,+\}$ of algebraic signs on the main diagonal, which is consistent with the flat space signature, so that $\{\hat{e}_k\}$ still obey the Eq. 2.7 subalgebra. (However, the opposite signature of $\{+,-,-,-\}$ is also widely used, often with the frame denoted by $\{\hat{\gamma}_\alpha\}$.) Consequently, under the geometric product, $\{\hat{e}_\alpha\}$ are antisymmetric and normalized like $\{\hat{e}_k\}$ with the crucial exception that $\hat{e}_0\hat{e}_0 = (\hat{e}_0)^2 = -1$; while space and time are indeed on an equal footing, they are not completely interchangeable – else one could walk back to yesterday!

The three space basis vectors $\hat{e}_k$ share the same algebra as the Pauli spin matrices σ_k from non-relativistic quantum mechanics. The four spacetime basis vectors $\hat{e}_\alpha$ share the same algebra as the Dirac gamma matrices γ_α from relativistic quantum mechanics.

4.2 Multivectors

The corresponding **spacetime algebra** is spanned by the $1+4+6+4+1 = 16 = 2^4$ multivectors of Table 4.1.

Table 4.1: Basis for geometric algebra of 4-dimensional spacetime.

multivector	number	name	grade
1	1	scalar	0
$\{\hat{e}_\alpha\}$	4	vector	1
$\{\hat{e}_\alpha\hat{e}_\beta\}$	6	bivector	2
$\{\hat{e}_\alpha\hat{e}_\beta\hat{e}_\gamma\} = \{\mathscr{I}\hat{e}_\delta\}$	4	trivector = pseudovector	3
$\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3 = \mathscr{I}$	1	quadvector = pseudoscalar	4

The most general such spacetime multivector

$$M = s + \overset{\Rightarrow}{v} + B + T + Q \tag{4.7}$$

or

$$\begin{aligned} M = {} & M^0 \\ & + M^1\hat{e}_0 + M^2\hat{e}_1 + M^3\hat{e}_2 + M^4\hat{e}_3 \\ & + M^5\hat{e}_0\hat{e}_1 + M^6\hat{e}_0\hat{e}_2 + M^7\hat{e}_0\hat{e}_3 + M^8\hat{e}_1\hat{e}_2 + M^9\hat{e}_1\hat{e}_3 + M^{10}\hat{e}_2\hat{e}_3 \\ & + M^{11}\hat{e}_0\hat{e}_1\hat{e}_2 + M^{12}\hat{e}_0\hat{e}_1\hat{e}_3 + M^{13}\hat{e}_0\hat{e}_2\hat{e}_3 + M^{14}\hat{e}_1\hat{e}_2\hat{e}_3 \\ & + M^{15}\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3 \end{aligned} \tag{4.8}$$

has grade-0 scalar, grade-1 vector, grade-2 bivector, grade-3 trivector, and grade-4 quadvector parts:

$$\begin{aligned}
s &= \langle M \rangle_0 = M^0, \\
\vec{\vec{v}} &= \langle M \rangle_1 = M^1 \hat{e}_0 + M^2 \hat{e}_1 + M^3 \hat{e}_2 + M^4 \hat{e}_3 \ , \\
B &= \langle M \rangle_2 = M^5 \hat{e}_0 \hat{e}_1 + M^6 \hat{e}_0 \hat{e}_2 + M^7 \hat{e}_0 \hat{e}_3 + M^8 \hat{e}_1 \hat{e}_2 + M^9 \hat{e}_1 \hat{e}_3 + M^{10} \hat{e}_2 \hat{e}_3, \\
T &= \langle M \rangle_3 = M^{11} \hat{e}_0 \hat{e}_1 \hat{e}_2 + M^{12} \hat{e}_0 \hat{e}_1 \hat{e}_3 + M^{13} \hat{e}_0 \hat{e}_2 \hat{e}_3 + M^{14} \hat{e}_1 \hat{e}_2 \hat{e}_3, \\
Q &= \langle M \rangle_4 = M^{15} \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3.
\end{aligned} \tag{4.9}$$

Attempt to create a grade-5 **quintvector** (or higher grade multivector) by multiplying a quadvector by a vector, and it contracts to form a trivector instead.

4.3 Pseudoscalars

Unlike the trivector $\mathcal{I} = \hat{e}_1 \hat{e}_2 \hat{e}_3$ that commutes with all the basis vectors, the quadvector $\mathscr{I} = \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3$ *anticommutes* with all basis vectors,

$$\begin{aligned}
\mathscr{I} \hat{e}_0 &= \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_0 = -\hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_0 \hat{e}_3 = +\hat{e}_0 \hat{e}_1 \hat{e}_0 \hat{e}_2 \hat{e}_3 = -\hat{e}_0 \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_0 \mathscr{I}, \\
\mathscr{I} \hat{e}_1 &= \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_1 = -\hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_1 \hat{e}_3 = +\hat{e}_0 \hat{e}_1 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_1 \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_1 \mathscr{I}, \\
\mathscr{I} \hat{e}_2 &= \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_2 = -\hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_2 \hat{e}_3 = +\hat{e}_0 \hat{e}_2 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_2 \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_2 \mathscr{I}, \\
\mathscr{I} \hat{e}_3 &= \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_3 = -\hat{e}_0 \hat{e}_1 \hat{e}_3 \hat{e}_2 \hat{e}_3 = +\hat{e}_0 \hat{e}_3 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_3 \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 = -\hat{e}_3 \mathscr{I},
\end{aligned} \tag{4.10}$$

and hence anticommutes with all spacetime vectors $\vec{\vec{v}} = \hat{e}_\alpha v^\alpha$,

$$\mathscr{I} \vec{\vec{v}} = -\vec{\vec{v}} \mathscr{I}. \tag{4.11}$$

(The extra dimension necessitates an extra permutation that interchanges the algebraic signs of the products.) However, like the trivector $\mathcal{I}$, the quadvector $\mathscr{I}$ also squares to negative one,

$$\mathscr{I}^2 = \mathscr{I} \mathscr{I} = \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 = +\hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_1 \hat{e}_2 \hat{e}_3 = +\hat{e}_2 \hat{e}_3 \hat{e}_2 \hat{e}_3 = -\hat{e}_3 \hat{e}_3 = -1. \tag{4.12}$$

Since the quadvector $\mathscr{I}$ acts like an imaginary (although anti-commuting) scalar, refer to it as a spacetime pseudoscalar. Anti-commuting numbers called **Grassman variables** are used to describe **fermionic** fields in **quantum field theory**. Table 4.2 summarizes the pseudoscalars.

Table 4.2: Pseudoscalar comparison.

symbol	content	pronunciation	grade
i	$\hat{e}_1 \hat{e}_2$	"i"	2
$\mathcal{I}$	$\hat{e}_1 \hat{e}_2 \hat{e}_3$	"big i"	3
$\mathscr{I}$	$\hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3$	"script big i"	$1 + 3 = 4$

4.4 Pseudovectors

The spacetime pseudoscalar $\mathscr{I}$ relates spacetime vectors and spacetime trivectors via the duality transformations

$$\begin{aligned}\mathscr{I}\hat{e}_0 &= \hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_0 = -\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_0\hat{e}_3 = +\hat{e}_0\hat{e}_1\hat{e}_0\hat{e}_2\hat{e}_3 = -\hat{e}_0\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3 = +\hat{e}_1\hat{e}_2\hat{e}_3,\\ \mathscr{I}\hat{e}_1 &= \hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_1 = -\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_1\hat{e}_3 = +\hat{e}_0\hat{e}_1\hat{e}_1\hat{e}_2\hat{e}_3 = +\hat{e}_0\hat{e}_2\hat{e}_3,\\ \mathscr{I}\hat{e}_2 &= \hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_2 = -\hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_2\hat{e}_3 = -\hat{e}_0\hat{e}_1\hat{e}_3,\\ \mathscr{I}\hat{e}_3 &= \hat{e}_0\hat{e}_1\hat{e}_2\hat{e}_3\hat{e}_3 = +\hat{e}_0\hat{e}_1\hat{e}_2,\end{aligned} \tag{4.13}$$

or, by further permutations,

$$\begin{aligned}-\hat{e}_0\mathscr{I} &= \mathscr{I}\hat{e}_0 = +\hat{e}_1\hat{e}_2\hat{e}_3,\\ -\hat{e}_1\mathscr{I} &= \mathscr{I}\hat{e}_1 = +\hat{e}_2\hat{e}_3\hat{e}_0,\\ -\hat{e}_2\mathscr{I} &= \mathscr{I}\hat{e}_2 = -\hat{e}_3\hat{e}_0\hat{e}_1,\\ -\hat{e}_3\mathscr{I} &= \mathscr{I}\hat{e}_3 = +\hat{e}_0\hat{e}_1\hat{e}_2.\end{aligned} \tag{4.14}$$

Since spacetime trivectors are dual to spacetime vectors in this one-to-one (or isomorphic) way, refer to them as spacetime pseudovectors.

4.5 Spacetime Metric

Given the frame $\{\hat{e}_\alpha\}$, consider a second **reciprocal** frame $\{\hat{e}^\alpha\}$, where

$$\hat{e}_\alpha = \eta_{\alpha\beta}\hat{e}^\beta = \eta_{\alpha\gamma}\hat{e}^\gamma \tag{4.15}$$

and

$$\hat{e}^\alpha = \eta^{\alpha\beta}\hat{e}_\beta = \eta^{\alpha\beta}\eta_{\beta\gamma}\hat{e}^\gamma. \tag{4.16}$$

Consistency demands

$$\eta^{\alpha\beta}\eta_{\beta\gamma} = \delta^\gamma_\alpha, \tag{4.17}$$

so that the flat spacetime metric $\eta_{\alpha\beta}$ is its own inverse $\eta^{\alpha\beta}$,

$$\eta^{\alpha\beta} \leftrightarrow \boxed{\begin{matrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{matrix}} \leftrightarrow \eta_{\alpha\beta}. \tag{4.18}$$

Therefore,

$$\hat{e}^\alpha\cdot\hat{e}_\beta = \eta^{\alpha\gamma}\hat{e}_\gamma\cdot\hat{e}_\beta = \eta^{\alpha\gamma}\eta_{\gamma\beta} = \delta^\alpha_\beta. \tag{4.19}$$

and the reciprocal frames are mutually orthonormal.

Expand a spacetime vector in both frames,

$$\vec{\vec{v}} = v^\alpha\hat{e}_\alpha = v_\alpha\hat{e}^\alpha, \tag{4.20}$$

where

$$v_\alpha = \eta_{\alpha\beta} v^\beta = \eta_{\alpha\beta}\eta^{\beta\gamma} v_\gamma. \tag{4.21}$$

The spatial basis vectors $\hat{e}_k$ and $\hat{e}^k$ are the same, but the temporal basis vectors $\hat{e}_0$ and $\hat{e}^0$ are negatives of each other. Similarly, the spatial components v_k and v^k are the same, but the temporal components v_0 and v^0 are negatives of each other. For example, $\hat{e}_1 = \hat{e}^1$ and $v_1 = v^1$, but $\hat{e}_0 = -\hat{e}^0$ and $v_0 = -v^0$.

The metric can appear explicitly in dot products, as in

$$\vec{\vec{u}} \cdot \vec{\vec{v}} = (\hat{e}_\alpha u^\alpha) \cdot (\hat{e}_\beta v^\beta) = \hat{e}_\alpha \cdot \hat{e}_\beta \; u^\alpha v^\beta = \eta_{\alpha\beta} \; u^\alpha v^\beta, \tag{4.22}$$

or the components of the reciprocal frames can absorb the metric, as in

$$\vec{\vec{u}} \cdot \vec{\vec{v}} = (\hat{e}_\alpha u^\alpha) \cdot (\hat{e}^\beta v_\beta) = \hat{e}_\alpha \cdot \hat{e}^\beta \; u^\alpha v_\beta = \delta_\alpha^\beta \; u^\alpha v_\beta = u^\alpha v_\alpha = u^\beta v_\beta. \tag{4.23}$$

Similarly, the metric can appear in vector lengths or be absorbed,

$$\begin{aligned} \left|\vec{\vec{v}}\right|^2 = \vec{\vec{v}} \cdot \vec{\vec{v}} = \vec{\vec{v}}\vec{\vec{v}} &= \eta_{\alpha\beta} \; v^\alpha v^\beta = v^\alpha v_\alpha \\ &= v^0 v_0 + v^k v_k = -(v^t)^2 + \vec{v}^{\,2}. \end{aligned} \tag{4.24}$$

Table 4.3: Important spacetime vectors, with metric signature $\{-,+,+,+\}$ and light speed $c = 1$.

name	spacetime split	length squared
event	$\vec{\vec{x}} = \hat{e}_t c t + \vec{x}$	$\vec{\vec{x}}^2 = -c^2 \tau^2$
velocity	$\vec{\vec{V}} = \hat{e}_t \gamma c + \gamma \vec{v}$	$\vec{\vec{V}}^2 = -c^2$
momentum	$\vec{\vec{P}} = \hat{e}_t \gamma m c + \gamma m \vec{v}$	$\vec{\vec{P}}^2 = -m^2 c^2$
derivative	$\vec{\vec{\nabla}} = -\hat{e}_t \partial_{ct} + \vec{\nabla}$	$\vec{\vec{\nabla}}^2 = \Box^2$

4.6 Important Spacetime Vectors

Locate an **event** in spacetime by

$$\vec{\vec{x}} = \hat{e}_\alpha x^\alpha = \hat{e}_0 x^0 + \hat{e}_k x^k = \hat{e}_t t + \vec{x}, \tag{4.25}$$

where $\hat{e}_t = \hat{e}_0$. Denote the relative **spacetime displacement** of two nearby events by

$$d\vec{\vec{x}} = \hat{e}_\alpha dx^\alpha = \hat{e}_0 dx^0 + \hat{e}_k dx^k = \hat{e}_t dt + d\vec{x}. \tag{4.26}$$

The invariant length of the spacetime displacement is the **proper time** $d\tau$ between the events, as

$$\begin{aligned} d\overset{\Rightarrow}{x}^2 &= (\hat{e}_t dt + d\vec{x})(\hat{e}_t dt + d\vec{x}) \\ &= \hat{e}_t dt \hat{e}_t dt + d\vec{x} d\vec{x} + \hat{e}_t dt d\vec{x} + d\vec{x} \hat{e}_t dt \\ &= (\hat{e}_t)^2 (dt)^2 + (d\vec{x})^2 \\ &= -(dt)^2 + (d\vec{x})^2 \\ &= -(dt)^2 (1 - \vec{v}^{\,2}) \\ &= -\frac{(dt)^2}{\gamma^2} \\ &= -(d\tau)^2 \end{aligned} \tag{4.27}$$

or

$$d\tau = \frac{dt}{\gamma} \leq dt, \tag{4.28}$$

where the **relativistic stretch**

$$\gamma = \frac{1}{\sqrt{1 - \vec{v}^{\,2}}} \geq 1, \tag{4.29}$$

and the space velocity $\vec{v} = d\vec{x}/dt$.

By Eq. 4.26, the relative **spacetime velocity** of a massive particle is

$$\overset{\Rightarrow}{V} = \frac{d\overset{\Rightarrow}{x}}{d\tau} = \hat{e}_t \frac{dt}{d\tau} + \frac{d\vec{x}}{d\tau} = \hat{e}_t \gamma + \vec{V}, \tag{4.30}$$

where $\vec{V} = \gamma \vec{v}$. The time component $V^t = \gamma$ is the relativistic stretch. The invariant length of the spacetime velocity is always one (or light speed), as

$$\overset{\Rightarrow}{V}^2 = (\hat{e}_t \gamma + \vec{V})(\hat{e}_t \gamma + \vec{V}) = -(V^t)^2 + \vec{V}^2 = -\gamma^2 (1 - \vec{v}^{\,2}) = -1. \tag{4.31}$$

The relative **spacetime momentum** of a particle of mass m is

$$\overset{\Rightarrow}{P} = m \overset{\Rightarrow}{V} = \hat{e}_t \gamma m + m \vec{V} = \hat{e}_t E + \vec{P}, \tag{4.32}$$

where $E = \gamma m$ and $\vec{P} = \gamma \vec{p} = \gamma m \vec{v}$. The time component $P^t = E$ is the energy. (If the mass is at relative rest, then $\gamma = 1$ and $E = m = mc^2$.) The invariant length of the spacetime momentum is the mass, as

$$\overset{\Rightarrow}{P}^2 = (\hat{e}_t E + \vec{P})(\hat{e}_t E + \vec{P}) = -E^2 + \vec{P}^2 = -\gamma^2 m^2 (1 - \vec{v}^{\,2}) = -m^2 \tag{4.33}$$

The relative **spacetime derivative** is

$$\overset{\Rightarrow}{\nabla} = \hat{e}^\alpha \partial_\alpha = \hat{e}^0 \partial_0 + \hat{e}^k \partial_k = \hat{e}^t \partial_t + \vec{\nabla} = -\hat{e}_t \partial_t + \vec{\nabla}. \tag{4.34}$$

The invariant length of the spacetime derivative is the **d'Alembert operator**,

$$\vec{\vec{\nabla}}^2 = (-\hat{e}_t\partial_t + \vec{\nabla})(-\hat{e}_t\partial_t + \vec{\nabla}) = -\partial_t^2 + \vec{\nabla}^2 = \Box^2. \tag{4.35}$$

Just as the three sides of the nabla operater ∇ can represent the three dimensions of space, the four sides of the d'Alembert operator can represent the four dimensions of spacetime. Table 4.3 summarizes the spacetime vectors central to the theory of relativity.

4.7 Spacetime Rotations

A **spacetime rotation** known as a **Lorentz transformation** relates observations of (reference) frames in relative motion. Parameterize the relative speed

$$|\vec{v}| = \tanh w, \tag{4.36}$$

by the hyperbolic angle or **rapidity** w, so that the relativistic stretch

$$\gamma = \frac{1}{\sqrt{1-\vec{v}^{\,2}}} = \cosh w. \tag{4.37}$$

To **boost** the event $\vec{\vec{x}} = \hat{e}_t t + \hat{e}_x x$ to a frame moving at space velocity $\vec{v} = \hat{e}_x|\vec{v}|$, multiply by the rotor

$$R = e^{\hat{e}_t\hat{e}_x w/2} \tag{4.38}$$

on the right and by its reversion

$$R^\dagger = e^{\hat{e}_x\hat{e}_t w/2} \tag{4.39}$$

on the left, with the exponentials defined by their infinite series expansions like

$$\begin{aligned} R &= e^{\hat{e}_t\hat{e}_x w/2} \\ &= 1 + \left(\hat{e}_t\hat{e}_x\frac{w}{2}\right) + \frac{1}{2!}\left(\hat{e}_t\hat{e}_x\frac{w}{2}\right)^2 + \frac{1}{3!}\left(\hat{e}_t\hat{e}_x\frac{w}{2}\right)^3 + \cdots \\ &= \left(1 + \frac{1}{2!}\left(\frac{w}{2}\right)^2 + \cdots\right) + \hat{e}_t\hat{e}_x\left(\left(\frac{w}{2}\right) + \frac{1}{3!}\left(\frac{w}{2}\right)^3 + \cdots\right) \\ &= \cosh\left[\frac{w}{2}\right] + \hat{e}_t\hat{e}_x\sinh\left[\frac{w}{2}\right], \end{aligned} \tag{4.40}$$

as $(\hat{e}_t\hat{e}_x)^2 = 1$. Hence, the boosted event is

$$
\begin{aligned}
\overset{\Rightarrow}{x}{}' &= R\,\overset{\Rightarrow}{x}\,R^\dagger = e^{\hat{e}_t\hat{e}_x}\,\overset{\Rightarrow}{x}\,e^{\hat{e}_x\hat{e}_t}\\
&= \left(\cosh\left[\frac{w}{2}\right] + \hat{e}_t\hat{e}_x \sinh\left[\frac{w}{2}\right]\right)(\hat{e}_t t + \hat{e}_x x)\left(\cosh\left[\frac{w}{2}\right] + \hat{e}_x\hat{e}_t \sinh\left[\frac{w}{2}\right]\right)\\
&= \left(\cosh\left[\frac{w}{2}\right] + \hat{e}_t\hat{e}_x \sinh\left[\frac{w}{2}\right]\right)\left(\hat{e}_t\left(t\cosh\left[\frac{w}{2}\right] + x\sinh\left[\frac{w}{2}\right]\right) + \right.\\
&\qquad\qquad \left.\hat{e}_x\left(t\sinh\left[\frac{w}{2}\right] + x\cosh\left[\frac{w}{2}\right]\right)\right)\\
&= \hat{e}_t\left(t\left(\cosh^2\left[\frac{w}{2}\right] + \sinh^2\left[\frac{w}{2}\right]\right) + x\left(2\sinh\left[\frac{w}{2}\right]\cosh\left[\frac{w}{2}\right]\right)\right)\\
&+ \hat{e}_x\left(t\left(2\sinh\left[\frac{w}{2}\right]\cosh\left[\frac{w}{2}\right] + x\left(\cosh^2\left[\frac{w}{2}\right] + \sinh^2\left[\frac{w}{2}\right]\right)\right)\right)\\
&= \hat{e}_t\,(t\cosh w + x\sinh w) + \hat{e}_x\,(x\cosh w + t\sinh w)\\
&= \hat{e}_t t' + \hat{e}_x x', \qquad (4.41)
\end{aligned}
$$

where the components

$$t' = t\cosh w + x\sinh w = \cosh w\,(t + x\tanh w), \qquad (4.42a)$$
$$x' = x\cosh w + t\sinh w = \cosh w\,(x + t\tanh w), \qquad (4.42b)$$

which, by Eq. 4.36 and Eq. 4.37, form the familiar Lorentz transformation

$$t' = \gamma(t + |\vec{v}|x), \qquad (4.43a)$$
$$x' = \gamma(x + |\vec{v}|t), \qquad (4.43b)$$

as illustrated by the spacetime diagram of Fig. 4.1.

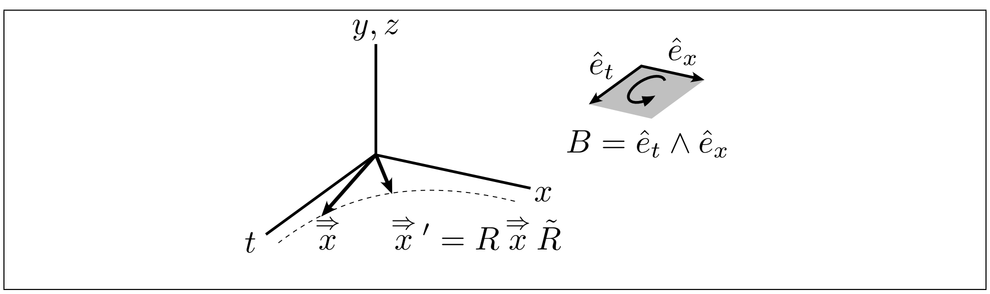

Figure 4.1: An active Lorentz boost is a spacetime rotation.

Problems

1. Reduce the following to the simplest multivector form of Eq. 4.8.

   (a) $(\hat{e}_0\hat{e}_2 + 4\hat{e}_1\hat{e}_3\hat{e}_0)(\hat{e}_0 - \hat{e}_1\hat{e}_2)$.

   (b) $(3 - \hat{e}_1\hat{e}_3 + 2\hat{e}_0\hat{e}_2\hat{e}_3)(\hat{e}_1 + 3\hat{e}_0\hat{e}_2\hat{e}_0)$.

   (c) $\hat{e}_2\hat{e}_1\hat{e}_3\hat{e}_2\hat{e}_0\hat{e}_2\hat{e}_2\hat{e}_0\hat{e}_1\hat{e}_3$.

2. If one event occurs at $\overset{\Rightarrow}{x}_1 = \hat{e}_0 + 2\hat{e}_1 - 3\hat{e}_2 - \hat{e}_3$ and a second event occurs at $\overset{\Rightarrow}{x}_2 = 3\hat{e}_0 + \hat{e}_1 - 2\hat{e}_2 - 2\hat{e}_3$, what is the proper time between them?

3. Suppose a particle has spacetime momentum $\overset{\Rightarrow}{P} = 3\hat{e}_0 + 2\hat{e}_1 + \hat{e}_2 - \hat{e}_3$.

   (a) What is its mass m?

   (b) What is its space speed $|\vec{v}|$?

Chapter 5

The Foundation of Electromagnetism

Spacetime algebra distills all of electromagnetism, including optics, into two simple equations. The **Maxwell field equation** defines the electromagnetic field due to electric charges; the **Lorentz force equation** defines the force on electric charges moving in an electromagnetic field.

5.1 Field

Define electric $\vec{\mathcal{E}}[\vec{x}]$ and magnetic $\vec{\mathcal{B}}[\vec{x}]$ space vector fields by

$$\vec{\mathcal{E}} = \hat{e}_k \mathcal{E}^k, \tag{5.1a}$$

$$\vec{\mathcal{B}} = \hat{e}_k \mathcal{B}^k. \tag{5.1b}$$

Combine these into a single electromagnetic field, the **Faraday** spacetime bivector

$$\mathcal{F} = (\vec{\mathcal{E}} + \mathscr{I}\vec{\mathcal{B}})\hat{e}_0 = \vec{\mathcal{E}}\hat{e}_0 - \mathcal{I}\vec{\mathcal{B}} \tag{5.2}$$

and expand it to

$$\begin{aligned}\mathcal{F} &= \hat{e}_k \mathcal{E}^k \hat{e}_0 + \hat{e}_0 \hat{e}_1 \hat{e}_2 \hat{e}_3 \mathcal{B}^k \hat{e}_k \hat{e}_0 \\ &= \mathcal{E}^k \hat{e}_k \hat{e}_0 - \mathcal{B}^k \hat{e}_1 \hat{e}_2 \hat{e}_3 \hat{e}_k \\ &= \mathcal{E}^1 \hat{e}_1 \hat{e}_0 + \mathcal{E}^2 \hat{e}_2 \hat{e}_0 + \mathcal{E}^3 \hat{e}_3 \hat{e}_0 \\ &\quad - \mathcal{B}^1 \hat{e}_2 \hat{e}_3 - \mathcal{B}^2 \hat{e}_3 \hat{e}_1 - \mathcal{B}^3 \hat{e}_1 \hat{e}_2. \end{aligned} \tag{5.3}$$

The electric field components $\mathcal{E}^k$ are the coefficients of the spacetime bivectors $\hat{e}_k \hat{e}_0$, while the magnetic field components $\mathcal{B}^k$ are the coefficients of the space bivectors $\hat{e}_k \hat{e}_l$, as depicted in Fig. 5.1. Also write Faraday as

$$\mathcal{F} = \frac{1}{2}\mathcal{F}^{\alpha\beta}\hat{e}_\alpha \hat{e}_\beta = \frac{1}{2}\mathcal{F}^{\alpha\beta}\hat{e}_\alpha \wedge \hat{e}_\beta, \tag{5.4}$$

where the antisymmetric components

$$\mathcal{F}^{\alpha\beta} \leftrightarrow \boxed{\begin{matrix} 0 & -\mathcal{E}^1 & -\mathcal{E}^2 & -\mathcal{E}^3 \\ \mathcal{E}^1 & 0 & -\mathcal{B}^3 & \mathcal{B}^2 \\ \mathcal{E}^2 & \mathcal{B}^3 & 0 & -\mathcal{B}^1 \\ \mathcal{E}^3 & -\mathcal{B}^2 & \mathcal{B}^1 & 0 \end{matrix}} \leftrightarrow -\mathcal{F}^{\beta\alpha}. \tag{5.5}$$

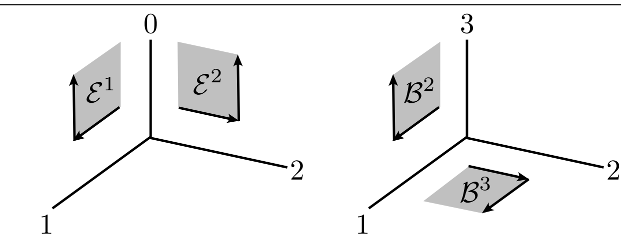

Figure 5.1: Four of the six components of Faraday, the electromagnetic field bivector. Each electric bivector has one timelike edge $\hat{e}_0$ and one spacelike edge $\hat{e}_k$, while each magnetic bivectors has two spacelike edges.

The Faraday spacetime bivector field $\mathcal{F}$ decomposes into different electric $\vec{\mathcal{E}}$ and magnetic $\vec{\mathcal{B}}$ space vector fields for (reference) frames in relative motion. For example, if one frame has a pure electric field

$$\mathcal{F} = \mathcal{E}^1\hat{e}_1\hat{e}_0 + \mathcal{E}^2\hat{e}_2\hat{e}_0 = \hat{e}_x\hat{e}_t\mathcal{E}_x + \hat{e}_y\hat{e}_t\mathcal{E}_y, \tag{5.6}$$

then, using the Sec. 4.7 boost, another frame moving at space velocity $\vec{v} = \hat{e}_x|\vec{v}| = \hat{e}_x \tanh w$ has a mix of electric and magnetic fields

$$\begin{aligned} \mathcal{F}' &= R\,\mathcal{F}\,R^\dagger \\ &= e^{\hat{e}_t\hat{e}_x w/2}\left(\hat{e}_x\hat{e}_t\mathcal{E}_x + \hat{e}_y\hat{e}_t\mathcal{E}_y\right)e^{\hat{e}_x\hat{e}_t w/2} \\ &= e^{\hat{e}_t\hat{e}_x w/2}\hat{e}_x\hat{e}_t e^{-\hat{e}_t\hat{e}_x w/2}\mathcal{E}_x + e^{\hat{e}_t\hat{e}_x w/2}\hat{e}_y\hat{e}_t e^{-\hat{e}_t\hat{e}_x w/2}\mathcal{E}_y \\ &= \hat{e}_x\hat{e}_t\mathcal{E}_x + \hat{e}_y\hat{e}_t\mathcal{E}_y e^{-\hat{e}_t\hat{e}_x w}, \end{aligned} \tag{5.7}$$

as $\hat{e}_x\hat{e}_t$ and $\hat{e}_t\hat{e}_x$ commute but $\hat{e}_y\hat{e}_t$ and $\hat{e}_t\hat{e}_x$ anticommute. Hence,

$$\begin{aligned} \mathcal{F}' &= \hat{e}_x\hat{e}_t\mathcal{E}_x + \hat{e}_y\hat{e}_t\mathcal{E}_y\left(\cosh w - \hat{e}_t\hat{e}_x \sinh w\right) \\ &= \hat{e}_x\hat{e}_t\mathcal{E}_x + \hat{e}_y\hat{e}_t\mathcal{E}_y \cosh w + \hat{e}_y\hat{e}_x\mathcal{E}_y \cosh w \tanh w \\ &= \hat{e}_x\hat{e}_t\mathcal{E}_x + \hat{e}_y\hat{e}_t\mathcal{E}_y\gamma - \hat{e}_x\hat{e}_y\mathcal{E}_y\gamma|\vec{v}| \\ &= \hat{e}_x\hat{e}_t\mathcal{E}'_x + \hat{e}_y\hat{e}_t\mathcal{E}'_y - \hat{e}_x\hat{e}_y\mathcal{B}'_z \end{aligned} \tag{5.8}$$

where

$$\mathcal{E}'_x = \mathcal{E}_x, \tag{5.9a}$$

$$\mathcal{E}'_y = \gamma\mathcal{E}_y, \tag{5.9b}$$

$$\mathcal{B}'_z = \gamma|\vec{v}|\mathcal{E}_y. \tag{5.9c}$$

5.2 Source

Define the electric charge density ρ as charge per unit space volume,

$$\rho = \frac{dQ}{dx\,dy\,dz} = \frac{dQ^0}{dx^1\,dx^2\,dx^3}, \tag{5.10}$$

and the electric current density $\vec{J}$ as current per unit perpendicular area,

$$J^1 = J_x = \frac{I_x}{dy\,dz} = \frac{dQ_x}{dt\,dy\,dz} = \frac{dQ^1}{dx^0\,dx^2\,dx^3}, \tag{5.11}$$

$$J^2 = J_y = \frac{I_y}{dx\,dz} = \frac{dQ_y}{dx\,dt\,dz} = \frac{dQ^2}{dx^0\,dx^1\,dx^3}, \tag{5.12}$$

$$J^3 = J_z = \frac{I_z}{dy\,dz} = \frac{dQ_z}{dx\,dy\,dt} = \frac{dQ^3}{dx^0\,dx^1\,dx^2}. \tag{5.13}$$

For a charge density ρ moving with velocity $\vec{v}$, the current density $\vec{J} = \rho\vec{v}$.

Current densities well describe current carrying wires, for example, but they also naturally extend charge density from space to spacetime, as in Fig. 5.2. Combine both into a single electromagnetic **source**, the spacetime vector

$$\overset{\Rightarrow}{J} = \hat{e}_t\rho + \hat{e}_k J^k = \hat{e}_\alpha J^\alpha, \tag{5.14}$$

where the components

$$J^\alpha \leftrightarrow \boxed{\begin{matrix}\rho\\ J_x\\ J_y\\ J_z\end{matrix}}\,. \tag{5.15}$$

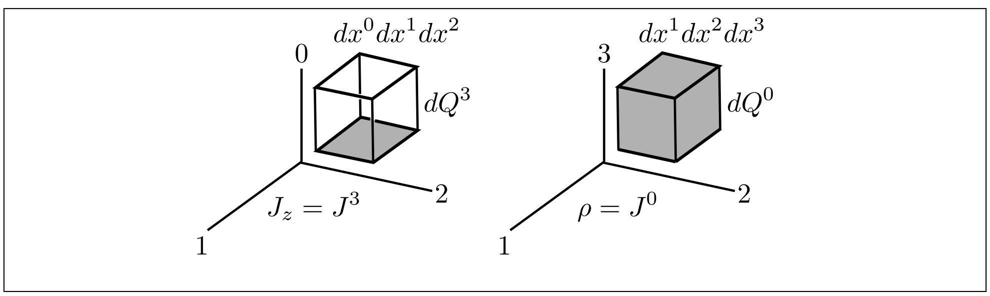

Figure 5.2: Two of the four components of the electromagnetic source spacetime vector.

Because of **charge conservation**, if charge density decreases in a region, charge density currents must diverge from that region. Using the Section 6.1 definition of divergence as flux or outflow per unit volume, write

$$\partial_t\rho = -\vec{\nabla}\cdot\vec{J} \tag{5.16}$$

or

$$0 = \partial_t \rho + \vec{\nabla} \cdot \vec{J} = (-\hat{e}_0 \partial_0 + \vec{\nabla}) \cdot (\hat{e}_0 \rho + \vec{J}) = \vec{\vec{\nabla}} \cdot \vec{\vec{J}}, \tag{5.17}$$

which is the **continuity equation**.

5.3 The Maxwell Field Equation

In spacetime algebra, using the geometric product, the traditional Maxwell field equations reduce to a single equation: the spacetime derivative of the electromagnetic field is its source,

$$\vec{\vec{\nabla}} \mathcal{F} = \vec{\vec{J}}, \tag{5.18}$$

read "big del spacetime vector times script big f equals big j spacetime vector". Expand this to

$$(-\hat{e}_0 \partial_0 + \vec{\nabla})(\vec{\mathcal{E}} + \mathscr{I} \vec{\mathcal{B}}) \hat{e}_0 = \hat{e}_0 \rho + \vec{J}, \tag{5.19}$$

or

$$\begin{aligned} &(-\hat{e}_0 \partial_0 + \hat{e}_1 \partial_1 + \hat{e}_2 \partial_2 + \hat{e}_3 \partial_3)(\hat{e}_1 \hat{e}_0 \mathcal{E}^1 + \hat{e}_2 \hat{e}_0 \mathcal{E}^2 + \hat{e}_3 \hat{e}_0 \mathcal{E}^3 \\ &-\hat{e}_2 \hat{e}_3 \mathcal{B}^1 + \hat{e}_1 \hat{e}_3 \mathcal{B}^2 - \hat{e}_1 \hat{e}_2 \mathcal{B}^3) = \hat{e}_0 \rho + \hat{e}_1 J^1 + \hat{e}_2 J^2 + \hat{e}_3 J^3, \end{aligned} \tag{5.20}$$

or even

$$\begin{aligned} &-\hat{e}_0 \hat{e}_1 \hat{e}_0 \partial_0 \mathcal{E}^1 - \hat{e}_0 \hat{e}_2 \hat{e}_0 \partial_0 \mathcal{E}^2 - \hat{e}_0 \hat{e}_3 \hat{e}_0 \partial_0 \mathcal{E}^3 + \hat{e}_0 \hat{e}_2 \hat{e}_3 \partial_0 \mathcal{B}^1 - \hat{e}_0 \hat{e}_1 \hat{e}_3 \partial_0 \mathcal{B}^2 + \hat{e}_0 \hat{e}_1 \hat{e}_2 \partial_0 \mathcal{B}^3 \\ &+\hat{e}_1 \hat{e}_1 \hat{e}_0 \partial_1 \mathcal{E}^1 + \hat{e}_1 \hat{e}_2 \hat{e}_0 \partial_1 \mathcal{E}^2 + \hat{e}_1 \hat{e}_3 \hat{e}_0 \partial_1 \mathcal{E}^3 - \hat{e}_1 \hat{e}_2 \hat{e}_3 \partial_1 \mathcal{B}^1 + \hat{e}_1 \hat{e}_1 \hat{e}_3 \partial_1 \mathcal{B}^2 - \hat{e}_1 \hat{e}_1 \hat{e}_2 \partial_1 \mathcal{B}^3 \\ &+\hat{e}_2 \hat{e}_1 \hat{e}_0 \partial_2 \mathcal{E}^1 + \hat{e}_2 \hat{e}_2 \hat{e}_0 \partial_2 \mathcal{E}^2 + \hat{e}_2 \hat{e}_3 \hat{e}_0 \partial_2 \mathcal{E}^3 - \hat{e}_2 \hat{e}_2 \hat{e}_3 \partial_2 \mathcal{B}^1 + \hat{e}_2 \hat{e}_1 \hat{e}_3 \partial_1 \mathcal{B}^2 - \hat{e}_2 \hat{e}_1 \hat{e}_2 \partial_2 \mathcal{B}^3 \\ &+\hat{e}_3 \hat{e}_1 \hat{e}_0 \partial_3 \mathcal{E}^1 + \hat{e}_3 \hat{e}_2 \hat{e}_0 \partial_3 \mathcal{E}^2 + \hat{e}_3 \hat{e}_3 \hat{e}_0 \partial_3 \mathcal{E}^3 - \hat{e}_3 \hat{e}_2 \hat{e}_3 \partial_3 \mathcal{B}^1 + \hat{e}_3 \hat{e}_1 \hat{e}_3 \partial_3 \mathcal{B}^2 - \hat{e}_3 \hat{e}_1 \hat{e}_2 \partial_3 \mathcal{B}^3 \\ &= \hat{e}_0 \rho + \hat{e}_1 J^1 + \hat{e}_2 J^2 + \hat{e}_3 J^3. \end{aligned} \tag{5.21}$$

Use the Eq. 4.13 duality transformations and the Eq. 4.4 algebra of the spacetime basis vectors to rewrite this as

$$\begin{aligned} &-\hat{e}_1 \partial_0 \mathcal{E}^1 - \hat{e}_2 \partial_0 \mathcal{E}^2 - \hat{e}_3 \partial_0 \mathcal{E}^3 + \mathscr{I} \hat{e}_1 \partial_0 \mathcal{B}^1 + \mathscr{I} \hat{e}_2 \partial_0 \mathcal{B}^2 + \mathscr{I} \hat{e}_3 \partial_0 \mathcal{B}^3 \\ &+\hat{e}_0 \partial_1 \mathcal{E}^1 + \mathscr{I} \hat{e}_3 \partial_1 \mathcal{E}^2 - \mathscr{I} \hat{e}_2 \partial_1 \mathcal{E}^3 - \mathscr{I} \hat{e}_0 \partial_1 \mathcal{B}^1 + \hat{e}_3 \partial_1 \mathcal{B}^2 - \hat{e}_2 \partial_1 \mathcal{B}^3 \\ &-\mathscr{I} \hat{e}_3 \partial_2 \mathcal{E}^1 + \hat{e}_0 \partial_2 \mathcal{E}^2 + \mathscr{I} \hat{e}_1 \partial_2 \mathcal{E}^3 - \hat{e}_3 \partial_2 \mathcal{B}^1 - \mathscr{I} \hat{e}_0 \partial_1 \mathcal{B}^2 + \hat{e}_1 \partial_2 \mathcal{B}^3 \\ &+\mathscr{I} \hat{e}_2 \partial_3 \mathcal{E}^1 - \mathscr{I} \hat{e}_1 \partial_3 \mathcal{E}^2 + \hat{e}_0 \partial_3 \mathcal{E}^3 + \hat{e}_2 \partial_3 \mathcal{B}^1 - \hat{e}_1 \partial_3 \mathcal{B}^2 + \mathscr{I} \hat{e}_0 \partial_3 \mathcal{B}^3 \\ &= \hat{e}_0 \rho + \hat{e}_1 J^1 + \hat{e}_2 J^2 + \hat{e}_3 J^3. \end{aligned} \tag{5.22}$$

Equate timelike vectors $\hat{e}_0$ to write

$$\partial_1 \mathcal{E}^1 + \partial_2 \mathcal{E}^2 + \partial_3 \mathcal{E}^3 = \rho. \tag{5.23}$$

Equate spacelike vectors $\hat{e}_k$ to write

$$
\begin{aligned}
\partial_2\mathcal{B}^3 - \partial_3\mathcal{B}^2 - \partial_0\mathcal{E}^1 &= J^1,\\
\partial_3\mathcal{B}^1 - \partial_1\mathcal{B}^3 - \partial_0\mathcal{E}^2 &= J^2,\\
\partial_1\mathcal{B}^2 - \partial_2\mathcal{B}^1 - \partial_0\mathcal{E}^3 &= J^3.
\end{aligned}
\tag{5.24}
$$

Equate timelike trivectors $\mathscr{I}\hat{e}_0$ to write

$$\partial_1\mathcal{B}^1 + \partial_2\mathcal{B}^2 + \partial_3\mathcal{B}^3 = 0. \tag{5.25}$$

Equate spacelike trivectors $\mathscr{I}\hat{e}_k$ to write

$$
\begin{aligned}
\partial_2\mathcal{E}^3 - \partial_3\mathcal{E}^2 + \partial_0\mathcal{B}^1 &= 0,\\
\partial_3\mathcal{E}^1 - \partial_1\mathcal{E}^3 + \partial_0\mathcal{B}^2 &= 0,\\
\partial_1\mathcal{E}^2 - \partial_2\mathcal{E}^1 + \partial_0\mathcal{B}^3 &= 0.
\end{aligned}
\tag{5.26}
$$

Write these equations traditionally as

$$
\begin{aligned}
\vec{\nabla}\cdot\vec{\mathcal{E}} &= \rho,\\
\vec{\nabla}\times\vec{\mathcal{B}} - \partial_t\vec{\mathcal{E}} &= \vec{J},\\
\vec{\nabla}\cdot\vec{\mathcal{B}} &= 0,\\
\vec{\nabla}\times\vec{\mathcal{E}} + \partial_t\vec{\mathcal{B}} &= \vec{0}.
\end{aligned}
\tag{5.27}
$$

The timelike multivectors form the divergence equations and the spacelike multivectors form the curl equations; the vectors form the inhomogeneous (source) equations and the trivectors form the homogeneous (source-free) equations.

Accomplish the same decomposition at a higher level. Begin with

$$\overset{\Rightarrow}{J} = \overset{\Rightarrow}{\nabla}\mathcal{F}, \tag{5.28}$$

and expand to write

$$
\begin{aligned}
\hat{e}_0\rho + \vec{J} &= (-\hat{e}_0\partial_0 + \vec{\nabla})(\vec{\mathcal{E}}\hat{e}_0 - \mathcal{I}\vec{\mathcal{B}})\\
&= -\partial_0\vec{\mathcal{E}} + \mathscr{I}\partial_0\vec{\mathcal{B}} + \hat{e}_0\vec{\nabla}\vec{\mathcal{E}} - \mathcal{I}\vec{\nabla}\vec{\mathcal{B}}.
\end{aligned}
\tag{5.29}
$$

Decompose the geometric product into dot and cross products using the Eq. 2.39 duality of the wedge and cross products to write

$$\vec{\nabla}\vec{\mathcal{E}} = \vec{\nabla}\cdot\vec{\mathcal{E}} + \vec{\nabla}\wedge\vec{\mathcal{E}} = \vec{\nabla}\cdot\vec{\mathcal{E}} + \mathcal{I}\vec{\nabla}\times\vec{\mathcal{E}}, \tag{5.30}$$

$$\vec{\nabla}\vec{\mathcal{B}} = \vec{\nabla}\cdot\vec{\mathcal{B}} + \vec{\nabla}\wedge\vec{\mathcal{B}} = \vec{\nabla}\cdot\vec{\mathcal{B}} + \mathcal{I}\vec{\nabla}\times\vec{\mathcal{B}}. \tag{5.31}$$

Hence,

$$\hat{e}_0\rho + \vec{J} = -\partial_0\vec{\mathcal{E}} + \mathscr{I}\partial_0\vec{\mathcal{B}} + \hat{e}_0\vec{\nabla}\cdot\vec{\mathcal{E}} + \mathscr{I}\vec{\nabla}\times\vec{\mathcal{E}} - \mathcal{I}\vec{\nabla}\cdot\vec{\mathcal{B}} + \vec{\nabla}\times\vec{\mathcal{B}}, \tag{5.32}$$

which again results in the traditional Maxwell equations of Eq. 5.27.

Like all differential equations, provide suitable boundary conditions to uniquely specify a solution to the Maxwell equation. For example, require that the field $\mathcal{F}$ decay sufficiently rapidly at infinity.

5.4 The Lorentz Force Equation

The spacetime force on a test charge moving in an electromagnetic field is the product of the charge and the dot product of the spacetime velocity with the field,

$$\vec{\vec{F}} = \frac{d\vec{\vec{P}}}{d\tau} = q\vec{\vec{V}} \cdot \mathcal{F}, \tag{5.33}$$

read "d big p spacetime vector over d tau equals q times big v spacetime vector dot script big f". Use $\vec{\vec{V}} \cdot \mathcal{F} = \left\langle \vec{\vec{V}}\mathcal{F} \right\rangle_1$ to expand this to

$$\gamma \frac{d}{dt}(E\hat{e}_0 + \vec{P}) = q\left\langle (\gamma\hat{e}_0 + \gamma\vec{v})(\vec{\mathcal{E}}\hat{e}_0 - \mathcal{I}\vec{\mathcal{B}}) \right\rangle_1, \tag{5.34}$$

$$\frac{dE}{dt}\hat{e}_0 + \frac{d\vec{P}}{dt} = q\left\langle \vec{\mathcal{E}} - \mathscr{I}\vec{\mathcal{B}} + \vec{v}\vec{\mathcal{E}}\hat{e}_0 - \mathcal{I}\vec{v}\vec{\mathcal{B}} \right\rangle_1. \tag{5.35}$$

Decompose the geometric product into dot and cross products to write

$$\vec{v}\vec{\mathcal{E}} = \vec{v} \cdot \vec{\mathcal{E}} + \vec{v} \wedge \vec{\mathcal{E}} = \vec{v} \cdot \vec{\mathcal{E}} + \mathcal{I}\vec{v} \times \vec{\mathcal{E}}, \tag{5.36}$$

$$\vec{v}\vec{\mathcal{B}} = \vec{v} \cdot \vec{\mathcal{B}} + \vec{v} \wedge \vec{\mathcal{B}} = \vec{v} \cdot \vec{\mathcal{B}} + \mathcal{I}\vec{v} \times \vec{\mathcal{B}}. \tag{5.37}$$

Hence,

$$\frac{dE}{dt}\hat{e}_0 + \frac{d\vec{P}}{dt} = q\left\langle \vec{\mathcal{E}} - \mathscr{I}\vec{\mathcal{B}} + \vec{v} \cdot \vec{\mathcal{E}}\hat{e}_0 + \mathscr{I}\vec{v} \times \vec{\mathcal{E}}\hat{e}_0 - \mathcal{I}\vec{v} \cdot \vec{\mathcal{B}} + \vec{v} \times \vec{\mathcal{B}} \right\rangle_1, \tag{5.38}$$

$$\frac{dE}{dt}\hat{e}_0 + \frac{d\vec{P}}{dt} = q\vec{\mathcal{E}} + q\vec{v} \cdot \vec{\mathcal{E}}\hat{e}_0 + q\vec{v} \times \vec{\mathcal{B}}, \tag{5.39}$$

which results in the traditional form

$$\frac{dE}{dt} = q\vec{v} \cdot \vec{\mathcal{E}}, \tag{5.40}$$

$$\frac{d\vec{P}}{dt} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}). \tag{5.41}$$

The timelike vectors form the **power equation**, while the spacelike vectors from the traditional **Lorentz force**. To elucidate the former, extend it by noting $\vec{v} \cdot \vec{v} \times \vec{\mathcal{B}} = 0$ and so

$$\frac{dE}{dt} = \vec{v} \cdot \vec{F}, \tag{5.42}$$

$$\vec{F} = \frac{d\vec{P}}{dt} = q(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}). \tag{5.43}$$

Problems

1. Verify the Faraday components of Eq. 5.5.

2. Prove the following decomposition of the Faraday bivector, and try to interpret it physically.

   (a) $\hat{e}_0 \cdot \mathcal{F} = -\mathcal{F} \cdot \hat{e}_0 = \vec{\mathcal{E}}$.

   (b) $\hat{e}_0 \wedge \mathcal{F} = \mathcal{F} \wedge \hat{e}_0 = -\mathscr{I}\vec{\mathcal{B}}$.

3. Consider parallel plate capacitors moving parallel and perpendicular to their plates. Use elementary relativity and electromagnetism (including simple forms of Gauss's and Ampère's laws) to qualitatively check the boosted fields of Eq. 5.9.

4. Suppose a particle of mass m and charge Q moves with spacetime velocity

$$\begin{aligned}\overset{\Rightarrow}{V} &= 2\hat{e}_0 - \sqrt{3}\,\hat{e}_1 \sin x^0 + \sqrt{3}\,\hat{e}_2 \cos x^0 \\ &= 2\,\hat{t} - \sqrt{3}\,\hat{x} \sin t + \sqrt{3}\,\hat{y} \cos t\end{aligned}$$

   in the electromagnetic field

$$\begin{aligned}\mathcal{F} &= e^{-r}(x^1\hat{e}_1\hat{e}_0 + x^2\hat{e}_2\hat{e}_0 + x^3\hat{e}_3\hat{e}_0) \\ &= e^{-r}(x\,\hat{x}\hat{t} + y\,\hat{y}\hat{t} + z\,\hat{z}\hat{t}) \\ &= e^{-r}\vec{r}\,\hat{t}\end{aligned}$$

   where $r = |\vec{x}| = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} = \sqrt{x^2 + y^2 + z^2}$.

   (a) What is the charge's space speed $|\vec{v}|$?

   (b) Sketch its orbit in both space and spacetime.

   (c) What is the spacetime force $\overset{\Rightarrow}{F}$ on the charge?

   (d) What is the corresponding Newtonian force $\vec{f}$ on the charge?

   (e) What spacetime source $\overset{\Rightarrow}{J}$ can produce this field? (Hint: Use and show $\vec{\nabla} r = \hat{r}$ and $\vec{\nabla} \cdot \vec{r} = 3$.)

   (f) What are the corresponding charge and current densities, ρ and $\vec{J}$?

Chapter 6

Interpreting Maxwell's Equations

Descend from four-dimensional spacetime to three-dimensional space, where the traditional Maxwell's equations, in either differential or integral form, balance elegance and practicality and admit powerful and visualizable geometrical interpretations. Seek such interpretations by relating the derivative concepts of divergence and curl to the integral concepts of flux and circulation. For simplicity, denote Euclidean coordinates and basis vectors by

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} = x\hat{e}_x + y\hat{e}_y + z\hat{e}_z = x^1\hat{e}_1 + x^2\hat{e}_2 + x^3\hat{e}_3 = x^k\hat{e}_k = \vec{x}. \tag{6.1}$$

6.1 Divergence & Flux

Let $\vec{v}[\vec{r}]$ be a vector field, such as the velocity field of a flowing fluid. The **flux** Φ of the vector field through an area a is

$$\Phi = \iint_a \vec{v} \cdot d\vec{a} = \langle v_\perp \rangle \, a, \tag{6.2}$$

where $d\vec{a}$ is perpendicular to the area element, and $\langle v_\perp \rangle$ is the average perpendicular component of $\vec{v}$. With dimensions of volume per time, the flux could represent the number of gallons per minutes that flows through a pipe. The flux Φ of the vector field through a *closed* area a bounding a volume V is

$$\Phi = \oiint_{a=\partial V} \vec{v} \cdot d\vec{a}, \tag{6.3}$$

where $d\vec{a}$ is the *outward* area element and the symbol ∂ (without an index) is the **boundary operator**.

Consider the flux (or outflow) of the vector field from an infinitesimal cube $dV = dx\,dy\,dz$ at $\{x, y, z\}$, as in Fig. 6.1. Sum the flux through the six sides of the

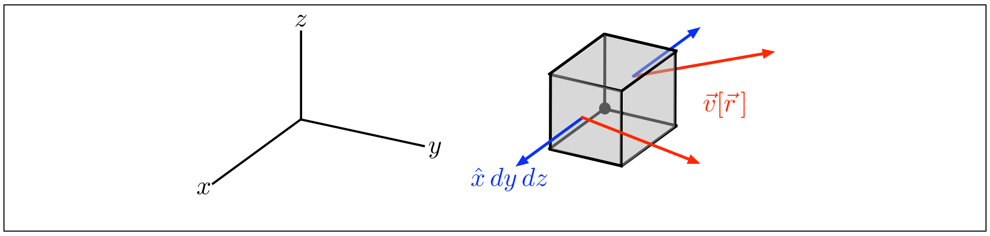

Figure 6.1: Flux of a vector field $\vec{v}[\vec{x}]$ through and infinitesimal cube $dx\, dy\, dz$.

cube, pairing the fluxes through opposite sides, to get

$$\begin{aligned} d\Phi &= + v_x[x+dx, y, z]\, dy\, dz - v_x[x, y, z]\, dy\, dz \\ &\quad + v_y[x, y+dy, z]\, dz\, dx - v_y[x, y, z]\, dz\, dx \\ &\quad + v_z[x, y, z+dz]\, dx\, dy - v_z[x, y, z]\, dx\, dy \\ &= \left(\frac{\partial v_x}{\partial x} dx\right) dy\, dz + \left(\frac{\partial v_y}{\partial y} dy\right) dz\, dx + \left(\frac{\partial v_z}{\partial z} dz\right) dx\, dy \\ &= (\partial_x v_x + \partial_y v_y + \partial_z v_z)\, dx\, dy\, dz. \\ &= \vec{\nabla} \cdot \vec{v}\, dV. \end{aligned} \tag{6.4}$$

Thus, the divergence of a vector field $\vec{v}[\vec{x}]$ is the flux per unit volume,

$$\operatorname{div} \vec{v} = \vec{\nabla} \cdot \vec{v} = \frac{d\Phi}{dV}. \tag{6.5}$$

Figure 6.2 depicts a couple of examples. If the vector field is the position $\vec{v}[\vec{r}] = \vec{r}$, then $\vec{\nabla} \cdot \vec{v} = 3$ and net flux exists everywhere (not just at the origin). If the vector field is a unit vector, say $\vec{v}[\vec{r}] = \hat{x}$, then $\vec{\nabla} \cdot \vec{v} = 0$ and net flux exists nowhere, as what enters on the left exits on the right.

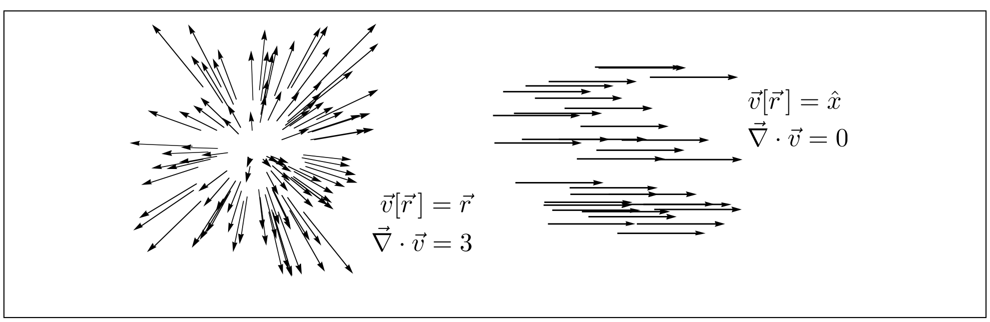

Figure 6.2: Vector fields $\vec{v}[\vec{r}]$ with simple divergences $\vec{\nabla} \cdot \vec{v}$. Random sampling represents the fields.

6.2 Curl & Circulation

Given the same vector field $\vec{v}[\vec{r}]$, the **circulation** Γ around a line ℓ bounding an area a is

$$\Gamma = \oint_{\ell=\partial a} \vec{v} \cdot d\vec{\ell} = \langle v_{\parallel} \rangle \, \ell. \tag{6.6}$$

where $d\vec{\ell}$ is parallel (or tangent) to the line element, and $\langle v_{\parallel} \rangle$ is the average parallel component of $\vec{v}$.

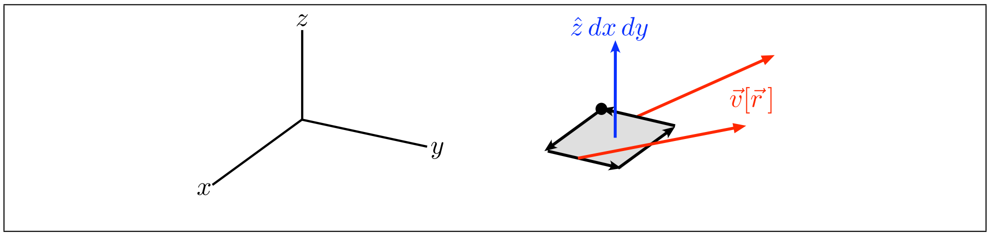

Figure 6.3: Circulation of a vector field $\vec{v}[\vec{r}]$ around an infinitesimal square $\hat{z}\, dx\, dy$.

Consider the circulation of the vector field around an infinitesimal area $da_z = dx\, dy$ at $\{x, y, z\}$, as in Fig. 6.3. Sum the circulations around the four sides of the square, pairing the contributions along opposite sides, to get

$$\begin{aligned} d\Gamma_z &= + v_x[x, y, z]\, dx - v_x[x, y + dy, z]\, dx \\ &\quad + v_y[x + dx, y, z]\, dy - v_y[x, y, z]\, dy \\ &= \left(-\frac{\partial v_x}{\partial y} dy\right) dx + \left(\frac{\partial v_y}{\partial x} dx\right) dy \\ &= (\partial_x v_y - \partial_y v_x)\, dx\, dy \\ &= \left(\vec{\nabla} \times \vec{v}\right)_z da_z. \end{aligned} \tag{6.7}$$

Thus,

$$\left(\vec{\nabla} \times \vec{v}\right)_z = \frac{d\Gamma_z}{da_z} = \left(\frac{d\vec{\Gamma}}{da}\right)_z. \tag{6.8}$$

and the curl of a vector field is the circulation per unit area,

$$\operatorname{curl} \vec{v} = \vec{\nabla} \times \vec{v} = \frac{d\vec{\Gamma}}{da}. \tag{6.9}$$

Figure 6.4 depicts a couple of examples. If the vector field is the linear velocity field of a wheel spinning at $\vec{\omega} = \omega \hat{x}$, then

$$\vec{v}[\vec{r}] = \vec{\omega} \times \vec{r} = \omega \hat{x} \times (y\hat{y} + z\hat{z}) = \omega y \hat{z} - \omega z \hat{y}, \tag{6.10}$$

and

$$\vec{\nabla} \times \vec{v} = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z) \times (\omega y \hat{z} - \omega z \hat{y}) = \omega \hat{x} + \omega \hat{x} = 2\vec{\omega}. \tag{6.11}$$

So, drop a stick in a stream and watch it spin; the curl of the water's velocity field is twice the angular velocity of the stick. In fact, if the vector field is the parabolic velocity profile of a river, fast near the center and slow at the banks,

$$\vec{v}[\vec{r}] = \hat{x}y(1-y) = \hat{x}(y-y^2) \tag{6.12}$$

then

$$\vec{\nabla} \times \vec{v} = (\hat{x}\partial_x + \hat{y}\partial_y + \hat{z}\partial_z) \times (\hat{x}(y-y^2)) = -\hat{z}(1-2y) = \hat{z}2(y-1/2). \tag{6.13}$$

The stick doesn't spin at all at the stream's center $y = 1/2$ but spins one way near the left bank $y < 1/2$ and the opposite way near the right bank $y > 1/2$, as expected.

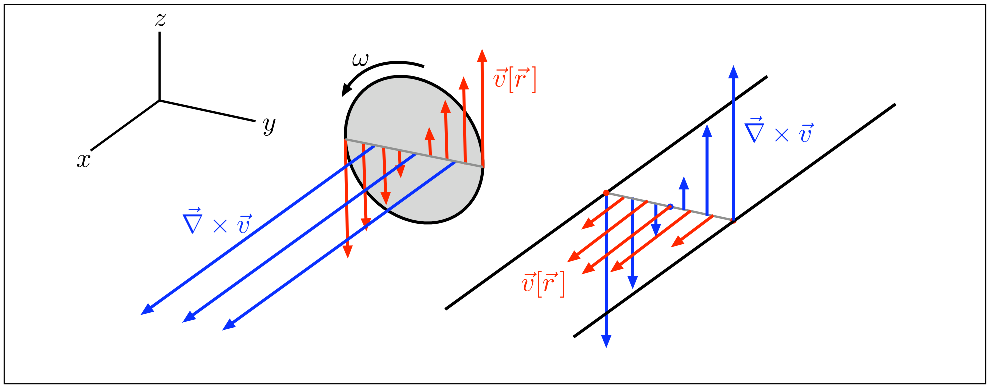

Figure 6.4: Vector fields $\vec{v}[\vec{r}]$ with simple circulations $\vec{\nabla} \times \vec{v}$, spinning wheel (left) and river (right). Regular sampling represents the fields.

6.3 Integral Form

According to the **fundamental theorem of calculus**, the integral of an ordinary derivative of a function *is* the function. For the derivatives of space vector calculus, the fundamental theorem takes different forms. The integral of the divergence of a vector field motivates the **divergence theorem**,

$$\iiint_V \operatorname{div} \vec{v}\, dV = \iiint_V \vec{\nabla} \cdot \vec{v}\, dV = \iiint_V \frac{d\Phi}{dV} dV = \int d\Phi = \Phi = \oint\!\!\oint_{a=\partial V} \vec{v} \cdot d\vec{a}, \tag{6.14}$$

because the infinitesimal fluxes $d\Phi$ across all interior boundaries cancel in pairs (as each exit is also an entrance), as in Fig. 6.5.

Similarly, the integral of the curl of a vector field motivates the **curl theorem**,

$$\iint_a \operatorname{curl} \vec{v} \cdot d\vec{a} = \iint_a \vec{\nabla} \times \vec{v} \cdot d\vec{a} = \iint_a \frac{d\vec{\Gamma}}{da} \cdot d\vec{a} = \int d\Gamma = \Gamma = \oint_{\ell=\partial a} \vec{v} \cdot d\vec{\ell}, \tag{6.15}$$

because the infinitesimal flows $d\Gamma$ along interior boundaries cancel in pairs, as in Fig. 6.6. Both the divergence and the curl theorem are special cases, in different dimensions, of the generalized **Stokes' theorem**.

The integral of the gradient of a scalar field $S[\vec{r}]$ motivates yet another example,

$$\int_{\vec{r}_1}^{\vec{r}_2} \operatorname{grad} S \cdot d\vec{\ell} = \int_{\vec{r}_1}^{\vec{r}_2} \vec{\nabla} S \cdot d\vec{\ell} = \int_{\vec{r}_1}^{\vec{r}_2} \frac{d\vec{S}}{d\ell} \cdot d\vec{\ell} = \int dS = \Delta S = S[\vec{r}_2] - S[\vec{r}_1]. \quad (6.16)$$

For an ordinary function $f[x]$, the lowest dimensional example is

$$\int_{x_1}^{x_2} f'[x] dx = \int_{x_1}^{x_2} \frac{df}{dx} dx = \int df = \Delta f = f[x_2] - f[x_1]. \quad (6.17)$$

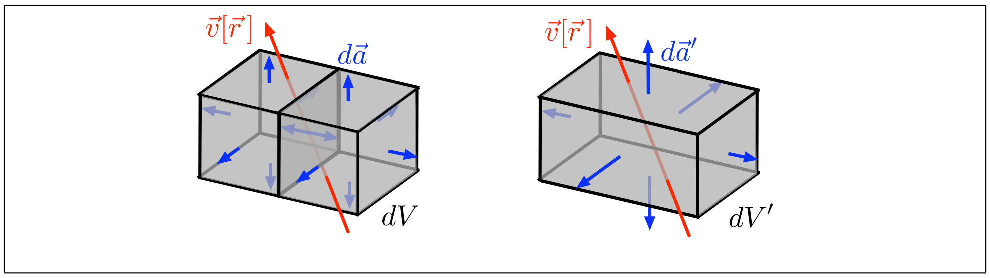

Figure 6.5: Interior fluxes $\vec{v} \cdot d\vec{a}$ cancel in pairs leaving only the flux through the exterior, so $\int d\Phi = \Phi$.

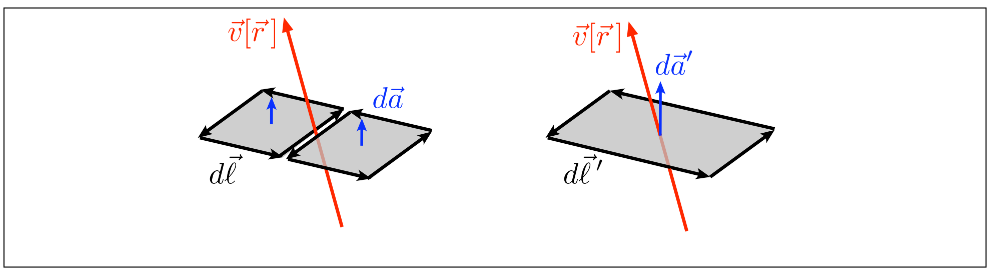

Figure 6.6: Interior circulations $\vec{v} \cdot d\vec{\ell}$ cancel in pairs leaving only the circulation around the exterior, so $\int d\Gamma = \Gamma$.

Use these fundamental theorems of space vector calculus to convert Maxwell's equations from differential form to integral form. Begin with the **Helmholtz form**, where specifying divergence and curl of the fields determines the fields everywhere,

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = 0, \qquad \vec{\nabla} \times \vec{\mathcal{E}} = -\partial_t \vec{\mathcal{B}},$$
$$\vec{\nabla} \cdot \vec{\mathcal{E}} = \rho, \qquad \vec{\nabla} \times \vec{\mathcal{B}} = +\partial_t \vec{\mathcal{E}} + \vec{J}. \quad (6.18)$$

Integrate the divergence equations over a volume bounded by a closed surface and integrate the curl equations over an area bounded by a closed curve to get

$$\iiint_V \vec{\nabla} \cdot \vec{\mathcal{B}}\, dV = 0, \qquad \iint_a \vec{\nabla} \times \vec{\mathcal{E}} \cdot d\vec{a} = -\frac{d}{dt} \iint_a \vec{\mathcal{B}} \cdot d\vec{a},$$
$$\iiint_V \vec{\nabla} \cdot \vec{\mathcal{E}}\, dV = \iiint_V \rho\, dV, \quad \iint_a \vec{\nabla} \times \vec{\mathcal{B}} \cdot d\vec{a} = +\frac{d}{dt} \iint_a \vec{\mathcal{E}} \cdot d\vec{a} + \iint_a \vec{J} \cdot d\vec{a}. \quad (6.19)$$

Apply the divergence and curl theorems to write

$$\oiint_{a=\partial V} \vec{\mathcal{B}} \cdot d\vec{a} = 0, \qquad \oint_{\ell=\partial a} \vec{\mathcal{E}} \cdot d\vec{\ell} = -\frac{d}{dt}\iint_a \vec{\mathcal{B}} \cdot d\vec{a},$$

$$\oiint_{a=\partial V} \vec{\mathcal{E}} \cdot d\vec{a} = \iiint_V \rho\, dV, \qquad \oint_{\ell=\partial a} \vec{\mathcal{B}} \cdot d\vec{\ell} = +\frac{d}{dt}\iint_a \vec{\mathcal{E}} \cdot d\vec{a} + \iint_a \vec{J} \cdot d\vec{a}, \tag{6.20}$$

or, more succinctly,

$$\Phi_{\mathcal{B}} = 0, \qquad \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}},$$

$$\Phi_{\mathcal{E}} = Q, \qquad \Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}} + I, \tag{6.21}$$

where Q is the charge inside the volume V, I is the current passing through the area a, and the overdots denote time differentiation. The fluxes Φ on the left are over closed surfaces, but the fluxes on the right are over open surfaces. The fluxes Φ have the dimensions of charge Q, the circulations Γ have the dimensions of current I.

6.4 Interpretations

Each Maxwell equation now has a simple, visualizable meaning. In the differential form of Eq. 6.18, **Gauss's law** for the magnetic field, $\vec{\nabla} \cdot \vec{\mathcal{B}} = 0$, means that the magnetic field never diverges (forming loops instead). **Gauss's law** for the electric field, $\vec{\nabla} \cdot \vec{\mathcal{E}} = \rho$, means that the electric field diverges from positive charge density and converges on negative charge density. **Faraday's law**, $\vec{\nabla} \times \vec{\mathcal{E}} = -\partial_t \vec{\mathcal{B}}$, means that an electric anti-curl accompanies a time varying magnetic field. **Ampère's law** (as extended by Maxwell), $\vec{\nabla} \times \vec{\mathcal{B}} = +\partial_t \vec{\mathcal{E}} + \vec{J}$, means that a magnetic curl accompanies a time varying electric field or an electric current density. (The special, stationary case of Ampère's original law, $\vec{\nabla} \times \vec{\mathcal{B}} = \vec{J}$, means that a magnetic curl accompanies a positive current density and a magnetic anti-curl accompanies a negative current density.)

In integral form of Eq. 6.29, **Gauss's law** for the magnetic field, $\Phi_{\mathcal{B}} = 0$, means that the flux of the magnetic field through any closed surface vanishes, so whatever goes in must come out. **Gauss's law** for the electric field, $\Phi_{\mathcal{E}} = Q$, means that the flux of the electric field through any closed surface is the electric charge inside. **Faraday's law**, $\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}$, means that the circulation of the electric field around any closed loop is *minus* the time rate of change of the magnetic flux through any surface bounded by the loop. **Ampère's law** (as extended by Maxwell), $\Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}} + I$, means that the circulation of the magnetic field around any closed loop is the time rate of change of the electric flux through any surface bounded by the loop *plus* the electric current passing through the loop. (The special, stationary case of Ampère's original law, $\Gamma_{\mathcal{B}} = I$, means the circulation of the magnetic field around any closed loop is the electric current passing though the loop.)

6.5 Maxwell's Form

Maxwell worked before the advent of vector algebra and so explicitly wrote Eq. 6.18 in terms of components,

$$\begin{aligned}
\partial_x \mathcal{B}_x + \partial_y \mathcal{B}_y + \partial_z \mathcal{B}_z &= 0, \\[1ex]
\partial_y \mathcal{E}_z - \partial_z \mathcal{E}_y + \partial_t \mathcal{B}_x &= 0, \\
\partial_z \mathcal{E}_x - \partial_x \mathcal{E}_z + \partial_t \mathcal{B}_y &= 0, \\
\partial_x \mathcal{E}_y - \partial_y \mathcal{E}_x + \partial_t \mathcal{B}_z &= 0, \\[1ex]
\partial_x \mathcal{E}_x + \partial_y \mathcal{E}_y + \partial_z \mathcal{E}_z &= \rho, \\[1ex]
\partial_y \mathcal{B}_z - \partial_z \mathcal{B}_y - \partial_t \mathcal{E}_x &= J_x, \\
\partial_z \mathcal{B}_x - \partial_x \mathcal{B}_z - \partial_t \mathcal{E}_y &= J_y, \\
\partial_x \mathcal{B}_y - \partial_y \mathcal{B}_x - \partial_t \mathcal{E}_z &= J_z,
\end{aligned} \tag{6.22}$$

where they reveal more of their internal structure and symmetry, as in Sec. 5.3.

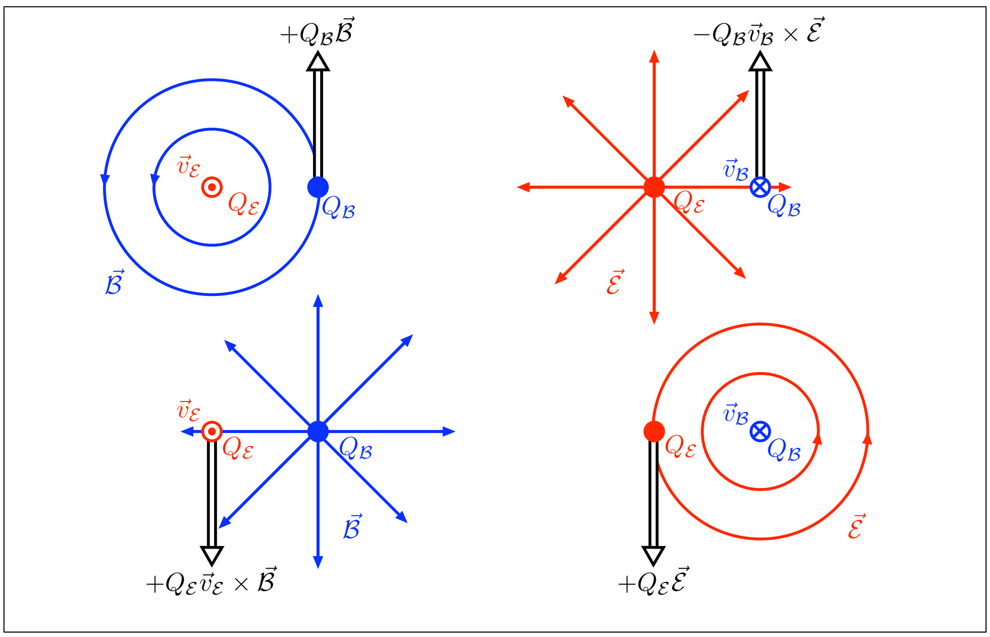

Figure 6.7: Passing electric and magnetic charges $Q_\mathcal{E}$ and $Q_\mathcal{B}$, with relative velocities $\vec{v}_\mathcal{E} = -\vec{v}_\mathcal{B}$ (perpendicular to the page), from two different references frames (left and right), illustrate the consistency of the sign conventions in Eq. 6.23 and Eq. 6.24.

6.6 Magnetic Monopole Form

Include magnetic charges $Q_\mathcal{B}$ and currents $I_\mathcal{B}$ to further symmetrize Maxwell's equations,

$$\begin{aligned} \Phi_\mathcal{B} &= Q_\mathcal{B}, \qquad \Gamma_\mathcal{E} = -\dot{\Phi}_\mathcal{B} - I_\mathcal{B} = -\frac{d}{dt}(\Phi_\mathcal{B} + Q_\mathcal{B}), \\ \Phi_\mathcal{E} &= Q_\mathcal{E}, \qquad \Gamma_\mathcal{B} = +\dot{\Phi}_\mathcal{E} + I_\mathcal{E} = +\frac{d}{dt}(\Phi_\mathcal{E} + Q_\mathcal{E}). \end{aligned} \tag{6.23}$$

(Note again that the fluxes Φ on the left side are over closed areas, while those on the right side are over open areas.) Such **magnetic monopoles** have not yet been observed but are predicted by **Grand Unified Theories**. For a particle with electric charge $q_\mathcal{E}$ and magnetic charge $q_\mathcal{B}$ moving with relative velocity $\vec{v}$, the Lorentz force law becomes

$$\vec{F} = q_\mathcal{E}\,\vec{\mathcal{E}} + q_\mathcal{B}\,\vec{\mathcal{B}} + q_\mathcal{E}\,\vec{v}\times\vec{\mathcal{B}} - q_\mathcal{B}\,\vec{v}\times\vec{\mathcal{E}}, \tag{6.24}$$

where the lone minus sign can be derived by demanding consistency between the forces on passing electric and magnetic charges from each of their reference frames, as illustrated by Fig. 6.7.

6.7 Matter Fields

Electric fields from free charges Q_f can polarize matter and create bound surface charges Q_b, as in the dielectric material within the Fig. 6.8 capacitor. Magnetic fields from free currents I_f can magnetize matter and create bound surface currents I_b, as in the paramagnetic material within the Fig. 6.8 solenoid. Conventionally decompose the total electric field as the **difference** of the free electric field and the polarization field,

$$\vec{\mathcal{E}} = \vec{\mathcal{D}} - \vec{\mathcal{P}}. \tag{6.25}$$

Decompose the total magnetic field as the **sum** of the free magnetic field and the magnetization field,

$$\vec{\mathcal{B}} = \vec{\mathcal{H}} + \vec{\mathcal{M}}. \tag{6.26}$$

For simple materials only, the total electric and magnetic fields are proportional to the free electric and magnetic fields,

$$\vec{\mathcal{E}} = \vec{\mathcal{D}}/\epsilon \tag{6.27}$$

and

$$\vec{\mathcal{B}} = \vec{\mathcal{H}}\mu, \tag{6.28}$$

where ϵ and μ are the **permittivity** and **permeability** of the material. A paraelectric material like a ceramic with $\epsilon > 1$ **weakens** the electric field; a paramagnetic material like aluminum with $\mu > 1$ **strengthens** the magnetic field. For complex **hysteretic** materials, like ferroelectrics and ferromagnets, ϵ and μ are not constants

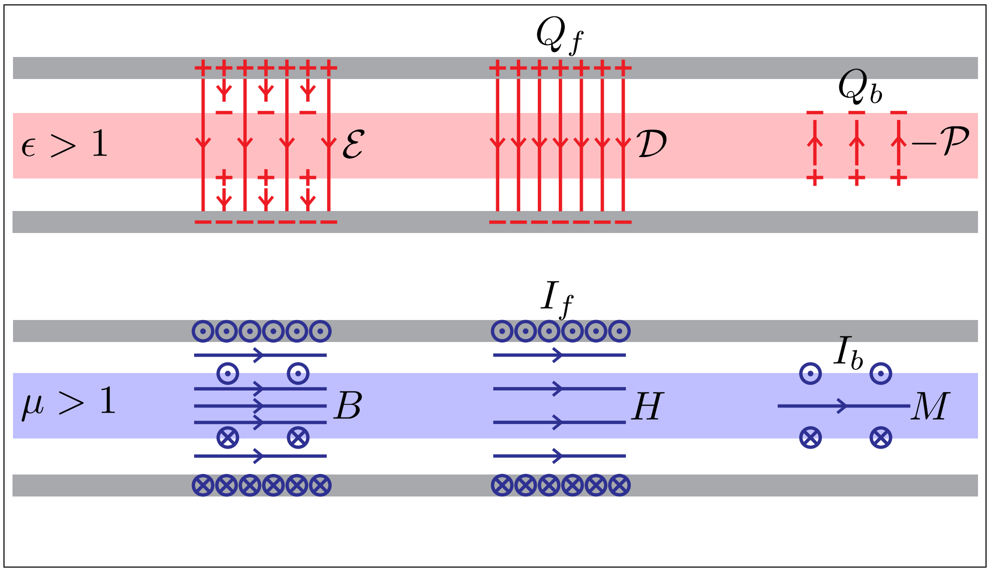

Figure 6.8: Cross sections of capacitor & dielectric (top) and solenoid & paramagnet (bottom). The total electric field is the free electric field minus the polarization, $\vec{\mathcal{E}} = \vec{\mathcal{D}} - \vec{\mathcal{P}}$; the total magnetic field is the free magnetic field plus the magnetization, $\vec{\mathcal{B}} = \vec{\mathcal{H}} + \vec{\mathcal{M}}$.

but are variables that depend on the **history** of the material. In natural units, the permittivity and permeability of the vacuum are one, and the total and free fields are the same, $\vec{\mathcal{E}} = \vec{\mathcal{D}}$ and $\vec{\mathcal{B}} = \vec{\mathcal{H}}$.

Experimentalists often control the free charges and currents, rather than the bound charges and currents, and so prefer the free electric and magnetic fields $\vec{\mathcal{D}}$ and $\vec{\mathcal{H}}$ over the total electric and magnetic fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$. So, in matter, modify the two Maxwell source equations to get

$$\begin{aligned} \varPhi_{\mathcal{B}} &= 0, & \varGamma_{\mathcal{E}} &= -\dot{\varPhi}_{\mathcal{B}}, \\ \varPhi_{\mathcal{D}} &= Q_f, & \varGamma_{\mathcal{H}} &= +\dot{\varPhi}_{\mathcal{D}} + I_f, \end{aligned} \tag{6.29}$$

where $Q_f = Q - Q_b$ is the free charge and $I_f = dQ_f/dt$ is the free current.

Problems

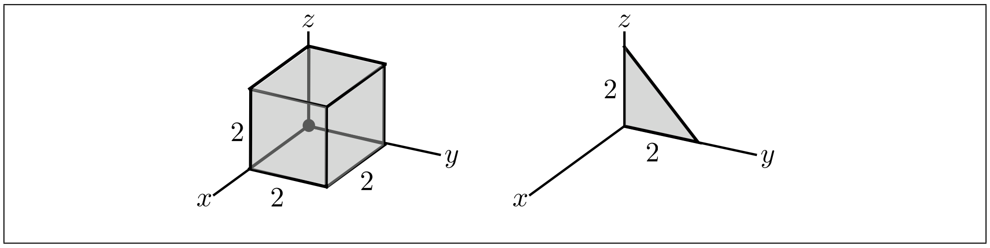

Figure 6.9: Cube and triangle for Problem 1.

1. Assume the vector field $\vec{v} = xy\,\hat{x} + 2yz\,\hat{y} + 3zx\,\hat{z}$.

    (a) Verify the divergence theorem of Eq. 6.14 for the cube of Fig. 6.9 by separately computing the volume integral of the divergence of the field inside the cube and the surface integral of the field over the boundary of the cube.

    (b) Verify the curl theorem of Eq. 6.15 for the triangle of Fig. 6.9 and by separately computing the area integral of the curl of the field inside the triangle and the line integral of the field along the boundary of the triangle.

2. Consider a circular electric current intersecting a (hypothetical) circular magnetic current at right angles, so that each current passes through the center of the other current. Show that the sign conventions adopted in the Eq. 6.23 Maxwell integral monopole equations are necessary to avoid a diverging solution. (Hint: Imagine a situation where the fluxes change arbitrarily slowly and focus on the circulation equations.)

Chapter 7

Electrostatics

In a world of static electric charges, irrotational electric fields diverge from charge densities, and Maxwell's equations reduce to

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = \rho, \qquad \vec{\nabla} \times \vec{\mathcal{E}} = \vec{0}, \tag{7.1}$$

or

$$\Phi_{\mathcal{E}} = Q, \qquad \Gamma_{\mathcal{E}} = 0. \tag{7.2}$$

An electric force is proportional to an electric charge and an electric field, and Lorentz's equation reduces to

$$\vec{F} = q\vec{\mathcal{E}}. \tag{7.3}$$

7.1 Coulomb's Law

Consider the electric field of a point charge Q at the origin $\vec{r} = 0$. Symmetry demands that the field $\vec{\mathcal{E}}[\vec{r}] = \mathcal{E}[r]\hat{r}$ must be radial and depend only on its distance. Apply Gauss's law to a sphere of radius r concentric with the charge to obtain

$$Q = \Phi_{\mathcal{E}} = \oiint_{a=\partial V} \vec{\mathcal{E}} \cdot d\vec{a} = \oiint_{r'=r} \mathcal{E}[r']da' = \mathcal{E}[r] \oiint_{r'=r} da' = \mathcal{E}[r]4\pi r^2. \tag{7.4}$$

or

$$\vec{\mathcal{E}}[\vec{r}] = \frac{Q}{4\pi r^2}\hat{r}, \tag{7.5}$$

which is **Coulomb's law**. The charge's electric field spreads over the spherical surface area $4\pi r^2$ and dilutes with distance r by that same factor. If the charge is at $\vec{r}\,'$, then

$$\vec{\mathcal{E}} = \frac{Q}{4\pi \mathscr{r}^2}\hat{\mathscr{r}}, \tag{7.6}$$

where the displacement vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$ points from the source point to the field point.

The **principle of superposition** is implicit in the vector notation used to express Coulomb's law. For a **discrete** charge distribution Q_k at $\vec{r}_k$, the displacement vectors are $\vec{\mathscr{r}}_k = \vec{r} - \vec{r}_k$, and the electric field at the point $\vec{r}$ is

$$\vec{\mathcal{E}} = \sum_k \vec{\mathcal{E}}_k = \sum_k \frac{Q_k}{4\pi \mathscr{r}_k^2} \hat{\mathscr{r}}_k. \tag{7.7}$$

For a continuous charge distribution,

$$\vec{\mathcal{E}} = \int d\vec{\mathcal{E}} = \int \frac{dQ}{4\pi \mathscr{r}^2} \hat{\mathscr{r}}, \tag{7.8}$$

where the infinitesimal charge

$$dQ = \lambda d\ell = \sigma da = \rho dV \tag{7.9}$$

and λ, σ, ρ are line, surface, and volume charge densities.

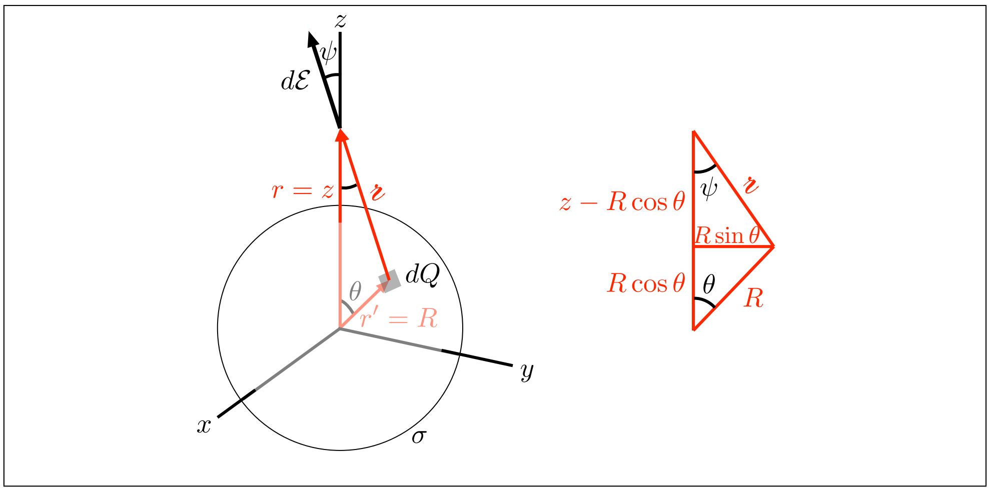

Figure 7.1: A spherical shell of radius R holds an surface charge density σ (left) and the geometry of the position triangle rotated into the plane of the page (right).

As an example, compute the electric field $\vec{\mathcal{E}}$ of a spherical shell of charge density σ and radius R. By symmetry, assume the shell is centered at the origin, the field point $\vec{r} = z\hat{z}$ is on the z-axis, and the source point $\vec{r}' = \vec{R}$ is on the shell, as in Fig. 7.1. Let ψ be the angle between $d\vec{\mathcal{E}}$ and $\hat{z}$. Employ spherical coordinates $\{r, \theta, \phi\}$, where θ is the co-latitude and ϕ is the longitude. By symmetry, write

$$\mathcal{E} = \mathcal{E}_z = \int d\mathcal{E}_z = \int d\mathcal{E} \cos\psi. \tag{7.10}$$

Expand by geometry, and perform the trivial ϕ integration to get

$$\begin{aligned}
\mathcal{E} &= \int \frac{dQ}{4\pi \mathscr{r}^2}\left(\frac{z - R\cos\theta}{\mathscr{r}}\right) \\
&= \frac{1}{4\pi}\iint \frac{\sigma da(z - R\cos\theta)}{\left(\vec{z}-\vec{R}\right)^3} \\
&= \frac{\sigma}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{(Rd\theta)(R\sin\theta d\phi)(z - R\cos\theta)}{(z^2 - 2\vec{z}\cdot\vec{R} + R^2)^{3/2}} \\
&= \frac{\sigma}{2}\int_0^{\pi} \frac{R^2(z - R\cos\theta)\sin\theta\, d\theta}{(z^2 - 2zR\cos\theta + R^2)^{3/2}}.
\end{aligned} \tag{7.11}$$

First make the variable substitution $u = \cos\theta$ and then the parameter substitution $\zeta = z/R$ to show

$$\begin{aligned}
\mathcal{E} &= \frac{\sigma}{2}\int_{+1}^{-1} \frac{R^2(z - Ru)(-du)}{(z^2 - 2zRu + R^2)^{3/2}} \\
&= \frac{\sigma}{2}\int_{-1}^{+1} \frac{R^2(z - Ru)du}{(z^2 - 2uzR + R^2)^{3/2}} \\
&= \frac{\sigma}{2}\int_{-1}^{+1} \frac{(\zeta - u)du}{(\zeta^2 - 2u\zeta + 1)^{3/2}}.
\end{aligned} \tag{7.12}$$

Make the variable substitution $v = \zeta^2 - 2u\zeta + 1$ to conclude

$$\begin{aligned}
\mathcal{E} &= \frac{\sigma}{2}\frac{1}{(2\zeta)^2}\int_{(\zeta+1)^2}^{(\zeta-1)^2} \frac{\left(\zeta - \frac{\zeta^2+1-v}{2\zeta}\right)\left(-\frac{dv}{2\zeta}\right)}{v^{3/2}} \\
&= \frac{\sigma}{2}\frac{1}{4\zeta^2}\int_{(\zeta-1)^2}^{(\zeta+1)^2} \left((\zeta^2-1)v^{-3/2} + v^{-1/2}\right) dv \\
&= \frac{\sigma}{2}\frac{1}{2\zeta^2}\left(-(\zeta^2-1)v^{-1/2} + v^{1/2}\right)\Big|_{(\zeta-1)^2}^{(\zeta+1)^2} \\
&= \frac{\sigma}{2}\frac{1}{2\zeta^2}\left(\frac{-\zeta^2+1+v}{\sqrt{v}}\right)\Big|_{(\zeta-1)^2}^{(\zeta+1)^2} \\
&= \frac{\sigma}{2}\frac{1}{\zeta^2}\left(\frac{\zeta+1}{|\zeta+1|} + \frac{\zeta-1}{|\zeta-1|}\right).
\end{aligned} \tag{7.13}$$

Two cases exist. In the **interior** of the shell, $z < R$ and $\zeta < 1$, so

$$\mathcal{E}[z < R] = \frac{\sigma}{2}\frac{1}{\zeta^2}(1 - 1) = 0, \tag{7.14}$$

and the field vanishes identically. In the **exterior** of the shell, $z > R$ and $\zeta > 1$, so

$$\mathcal{E}[z > R] = \frac{\sigma}{2}\frac{1}{\zeta^2}(1 + 1) = \frac{\sigma R^2}{z^2} = \frac{Q}{4\pi z^2}, \tag{7.15}$$

and the field is that of a point charge $Q = \sigma\, 4\pi R^2$ at the center of the sphere! Summarize this by

$$\vec{\mathcal{E}}[\vec{r}] = \begin{cases} 0, & r < R, \\ \dfrac{Q}{4\pi r^2}\hat{r}, & r > R. \end{cases} \tag{7.16}$$

This is the electricity version of Newton's famous gravity **shell theorem**. Rapidly confirm this result by applying Gauss's law to concentric spheres inside and outside the shell.

7.2 Boundary Conditions

Electrostatic fields are discontinuous at charge layers, like the shell of charge. Consider a surface with normal $\hat{n}$ and surface charge density σ, as in Fig. 7.2. Let $\vec{\mathcal{E}}^-$ be the electric field just below the surface and $\vec{\mathcal{E}}^+$ be the electric field just above the surface. Apply $\Phi_\mathcal{E} = Q$ to a cylinder straddling the surface of cross sectional area a and vanishing height $h \to 0$ to get

$$\mathcal{E}_\perp^+ a - \mathcal{E}_\perp^- a - 0 = \sigma a \tag{7.17}$$

or

$$\Delta\mathcal{E}_\perp = \mathcal{E}_\perp^+ - \mathcal{E}_\perp^- = \sigma. \tag{7.18}$$

Apply $\Gamma_\mathcal{E} = 0$ to a rectangular loop straddling the surface of length ℓ and vanishing height $h \to 0$ to get

$$\mathcal{E}_\parallel^+ \ell + 0 - \mathcal{E}_\parallel^- \ell - 0 = 0 \tag{7.19}$$

or

$$\Delta\mathcal{E}_\parallel = \mathcal{E}_\parallel^+ - \mathcal{E}_\parallel^- = 0. \tag{7.20}$$

Thus, the normal component of the electrostatic field is discontinuous by the charge density,

$$\Delta\vec{\mathcal{E}} = \vec{\mathcal{E}}^+ - \vec{\mathcal{E}}^- = \sigma\hat{n}. \tag{7.21}$$

For example, this is true for the Eq. 7.16 shell of charge, where $\sigma = Q/4\pi R^2$ and $\hat{n} = \hat{r}$.

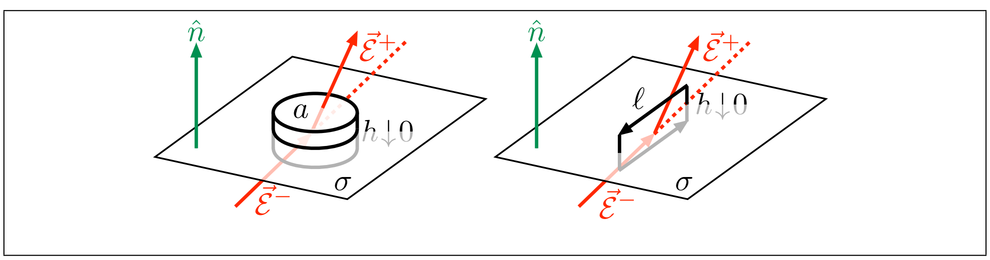

Figure 7.2: A closed cylinder (left) and rectangular loop (right) straddle a charge layer to compute the discontinuity in an electric field $\vec{\mathcal{E}}$ due to the charge density σ.

7.3 Point Charges and the Dirac Delta

If $\rho[\vec{r}]$ is the electric charge density of a point charge Q at $\vec{r} = \vec{0}$, it must satisfy both

$$\iiint \rho[\vec{r}]dV = Q, \tag{7.22}$$

and

$$\rho[\vec{r} \neq \vec{0}] = 0, \tag{7.23}$$

No normal function vanishes everywhere except a point and still has a nonzero integral. Instead write

$$\rho[\vec{r}] = Q\,\delta^3[\vec{r}] = Q\,\delta[x]\delta[y]\delta[z], \tag{7.24}$$

where the **Dirac delta** is a **generalized function** with just those properties.

The discrete analogue of the Kronecker delta δ_{kl}, the Dirac delta $\delta[x-a]$ is like an infinite spike bounding a unit area Think of it as the limit

$$\delta[x] = \lim_{\epsilon \to 0} \delta_\epsilon[x] \tag{7.25}$$

of "top-hat" functions

$$\delta_\epsilon[x] = \begin{cases} 1/\epsilon, & |x| < \epsilon/2, \\ 0, & \epsilon/2 \leq |x|. \end{cases} \tag{7.26}$$

Apparently,

$$\delta[x] = \begin{cases} \infty & x = 0, \\ 0, & x \neq 0, \end{cases} \tag{7.27}$$

such that

$$\int_{-\infty}^{\infty} \delta[x]dx = 1. \tag{7.28}$$

Since the Dirac delta vanishes everywhere except at a point, when it occurs in an integral, replace the the rest of the integrand by its value at that point. For example,

$$\int_{-\infty}^{\infty} f[x]\delta[x]dx = \int_{-\infty}^{\infty} f[0]\delta[x]dx = f[0]\int_{-\infty}^{\infty} \delta[x]dx = f[0]. \tag{7.29}$$

More generally, the Dirac delta exhibits the **sifting property**

$$\int_{-\infty}^{\infty} f[x]\delta[x-a]dx = f[a]. \tag{7.30}$$

For example,

$$\int_0^3 x^3\delta[x-2]dx = 8 \tag{7.31}$$

but only if it is nonzero within the limits of integration,

$$\int_0^1 x^3\delta[x-2]dx = 0. \tag{7.32}$$

Use a change of variable to prove

$$\int_{-\infty}^{\infty} f[x]\delta[kx]dx = \int_{-\infty}^{\infty} f\left[\frac{\xi}{k}\right]\delta[\xi]\frac{d\xi}{k} = \frac{1}{k}f[0]\int_{-\infty}^{\infty}\delta[\xi]d\xi = \frac{1}{k}f[0], \tag{7.33}$$

where $k > 0$. More generally, show

$$\int_{-\infty}^{\infty} f[x]\delta[kx]dx = \frac{1}{|k|}f[0] = \int_{-\infty}^{\infty} f[x]\frac{1}{|k|}\delta[x]dx. \tag{7.34}$$

Since this is true for all sufficiently nice "test" functions $f[x]$, write

$$\delta[kx] = \frac{1}{|k|}\delta[x]. \tag{7.35}$$

Hence, the dimensions of the Dirac delta must be inverse to those of its argument. Furthermore, $k = -1$ implies that the Dirac delta is symmetric,

$$\delta[-x] = \delta[x]. \tag{7.36}$$

Return to the motivating example of the point charge of Eq. 7.24, where Gauss's law in differential form $\vec{\nabla}\cdot\vec{\mathcal{E}} = \rho$ implies

$$\vec{\nabla}\cdot\left(\frac{Q}{4\pi r^2}\hat{r}\right) = Q\delta^3[\vec{r}], \tag{7.37}$$

or

$$\vec{\nabla}\cdot\left(\frac{\hat{r}}{r^2}\right) = 4\pi\delta^3[\vec{r}], \tag{7.38}$$

which is an important differential identity. For example, given the electric field

$$\vec{\mathcal{E}}[\vec{r}] = \iiint \frac{\rho[\vec{r}\,']dV'}{4\pi \mathscr{r}^2}\hat{\mathscr{r}}, \tag{7.39}$$

its divergence

$$\vec{\nabla}\cdot\vec{\mathcal{E}}[\vec{r}] = \iiint \frac{\rho[\vec{r}\,']dV'}{4\pi}\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = \iiint \frac{\rho[\vec{r}\,']dV'}{4\pi}4\pi\delta^3[\vec{r}-\vec{r}\,'] = \rho[\vec{r}\,] \tag{7.40}$$

as expected from Gauss's law.

7.4 Electric Potential

Consider the circulation of an electrostatic field around the closed loop joining points $\vec{a}$ and $\vec{b}$ in Fig. 7.4. By the Eq. 7.2 Maxwell's equations,

$$0 = \Gamma_{\mathcal{E}} = \oint_{\ell}\vec{\mathcal{E}}\cdot d\vec{\ell} = \underbrace{\int_{\vec{a}}^{\vec{b}}}_{\text{low}}\vec{\mathcal{E}}\cdot d\vec{\ell} + \underbrace{\int_{\vec{b}}^{\vec{a}}}_{\text{high}}\vec{\mathcal{E}}\cdot d\vec{\ell} \tag{7.41}$$

or

$$\int_{\vec{a}\atop \text{low}}^{\vec{b}} \vec{\mathcal{E}}\cdot d\vec{\ell} = -\int_{\vec{b}\atop \text{high}}^{\vec{a}} \vec{\mathcal{E}}\cdot d\vec{\ell} = +\int_{\vec{a}\atop \text{high}}^{\vec{b}} \vec{\mathcal{E}}\cdot d\vec{\ell}, \tag{7.42}$$

so that the line integral of the electrostatic field is the same over both the high and low paths. Because those paths are generic, the electrostatic field line integrals are path independent. Thus, given a reference point $\vec{\mathcal{O}}$, define an **electric scalar field** or **electric potential** by

$$\varphi[\vec{r}\,] = -\int_{\vec{\mathcal{O}}}^{\vec{r}} \vec{\mathcal{E}}[\vec{r}\,']\cdot d\vec{r}\,' = -\int_{\vec{\mathcal{O}}}^{\vec{r}} \vec{\mathcal{E}}\cdot d\vec{\ell}, \tag{7.43}$$

where the minus sign is conventional (so that positive charges move from high to low potential). Show later that the electric potential is a potential energy per unit charge.

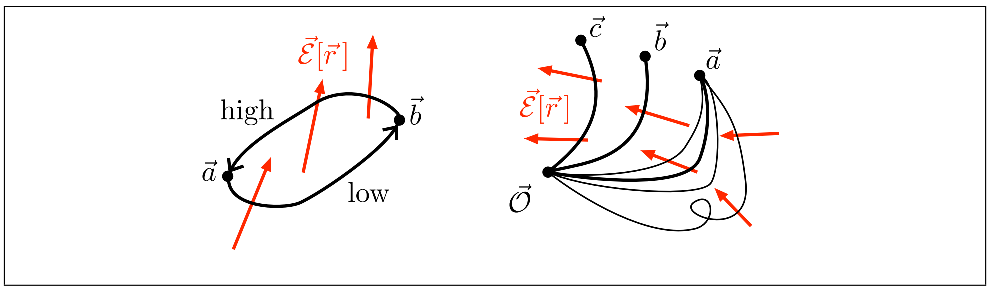

Figure 7.3: A path independent electric vector field $\vec{\mathcal{E}}$ and a reference point $\vec{\mathcal{O}}$ enables the definition of an electric scalar field or electric potential φ.

Next, consider the electric potential difference between the points $\vec{u}$ and $\vec{v}$,

$$\begin{aligned}
\varphi[\vec{v}\,] - \varphi[\vec{u}\,] &= -\int_{\vec{\mathcal{O}}}^{\vec{v}} \vec{\mathcal{E}}\cdot d\vec{\ell} + \int_{\vec{\mathcal{O}}}^{\vec{u}} \vec{\mathcal{E}}\cdot d\vec{\ell} \\
&= -\int_{\vec{\mathcal{O}}}^{\vec{v}} \vec{\mathcal{E}}\cdot d\vec{\ell} - \int_{\vec{u}}^{\vec{\mathcal{O}}} \vec{\mathcal{E}}\cdot d\vec{\ell} \\
&= -\int_{\vec{u}}^{\vec{\mathcal{O}}} \vec{\mathcal{E}}\cdot d\vec{\ell} - \int_{\vec{\mathcal{O}}}^{\vec{v}} \vec{\mathcal{E}}\cdot d\vec{\ell} \\
\Delta\varphi &= -\int_{\vec{u}}^{\vec{v}} \vec{\mathcal{E}}\cdot d\vec{\ell}.
\end{aligned} \tag{7.44}$$

By the Eq. 6.16 fundamental theorem of calculus for the gradient, write this also as

$$\Delta\varphi = \int_{\vec{u}}^{\vec{v}} \vec{\nabla}\varphi\cdot d\vec{\ell}, \tag{7.45}$$

and so identify the electric field as the negative gradient of the electric potential,

$$\vec{\mathcal{E}} = -\vec{\nabla}\varphi, \tag{7.46}$$

which is the inverse of the Eq. 7.43 definition.

Immediately check that

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\vec{\nabla} \times \vec{\nabla}\varphi = \vec{0}, \tag{7.47}$$

because the curl of any gradient vanishes identically. In fact, this constraint on the components of the *vector* field $\vec{\mathcal{E}}[\vec{r}]$ precisely enables the same information to be encoded in the *scalar* field $\varphi[\vec{r}]$.

Shifting the reference point $\vec{\mathcal{O}}$ shifts the potential $\varphi[\vec{r}]$ without changing the electric field $\vec{\mathcal{E}}[\vec{r}]$. For example, if

$$\varphi_1[\vec{r}] = -\int_{\vec{\mathcal{O}}_1}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell}, \tag{7.48}$$

then

$$\varphi_2[\vec{r}] = -\int_{\vec{\mathcal{O}}_2}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell} = -\int_{\vec{\mathcal{O}}_2}^{\vec{\mathcal{O}}_1} \vec{\mathcal{E}} \cdot d\vec{\ell} - \int_{\vec{\mathcal{O}}_1}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell} = k + \varphi_1[\vec{r}], \tag{7.49}$$

and so

$$\vec{\mathcal{E}}_2 = -\vec{\nabla}\varphi_2 = \vec{0} - \vec{\nabla}\varphi_1 = \vec{\mathcal{E}}_1 \tag{7.50}$$

A good reference point is often at infinity (except for problems with idealized charge distributions that extend to infinity).

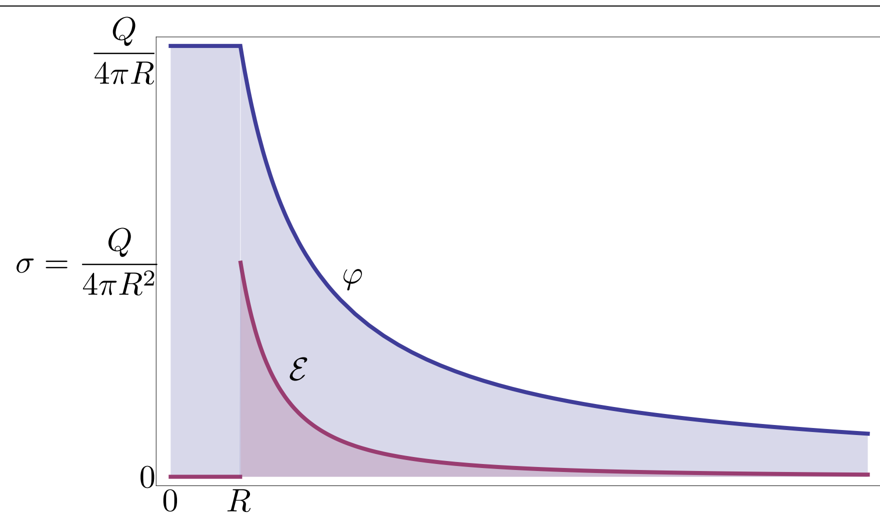

Figure 7.4: Electric field magnitude $\mathcal{E}$ and electric potential φ as a function of radial distance r for a spherical shell of radius R and charge Q.

As an example, compute the electric potential φ of a spherical shell of charge Q and radius R centered at the origin. By symmetry, $\varphi[\vec{r}] = \varphi[r]$. Set the reference point at infinity and choose a simple radial path $d\vec{\ell} = \hat{r}dr$ to integrate over the electric field of Eq. 7.16. Two cases exist. In the exterior of the shell $r > R$ and

$$\varphi[r] = -\int_{\infty}^{r>R} \mathcal{E}[r']dr' = -\int_{\infty}^{r} \frac{Q}{4\pi r'^2}dr' = +\frac{Q}{4\pi r'}\bigg|_{\infty}^{r} = \frac{Q}{4\pi r}. \tag{7.51}$$

In the interior of the shell $r < R$ and

$$\varphi[r] = -\int_{\infty}^{r<R} \mathcal{E}[r']dr' = -\int_{\infty}^{R} \frac{Q}{4\pi r'^2}dr' - \int_{R}^{r} 0\,dr' = \frac{Q}{4\pi R}. \tag{7.52}$$

Summarize this by

$$\varphi[r] = \begin{cases} \dfrac{Q}{4\pi R}, & r \leq R, \\ \dfrac{Q}{4\pi r}, & r \geq R. \end{cases} \tag{7.53}$$

Unlike the electric field, which is discontinuous at the shell, the electric potential is continuous everywhere, as in Fig. 7.4.

As another example, compute the electric potential φ of a line of charge density λ. First find the electric field $\vec{\mathcal{E}}$ by applying Gauss's law $\Phi_{\mathcal{E}} = Q$ to a closed coaxial cylinder of radius $r_\perp = s$ and length ℓ to get

$$0 + \mathcal{E}(2\pi s)\ell + 0 = \ell\lambda \tag{7.54}$$

so that

$$\mathcal{E} = \frac{\lambda}{2\pi s} \tag{7.55}$$

or

$$\vec{\mathcal{E}} = \frac{\lambda}{2\pi s}\hat{s}. \tag{7.56}$$

Then integrate the electric field $\vec{\mathcal{E}}$ along a path perpendicular to the line to get

$$\varphi = -\int_{s_0}^{s} \frac{\lambda}{2\pi s'}ds' = -\frac{\lambda}{2\pi}\log s'\bigg|_{s_0}^{s} = \frac{\lambda}{2\pi}\log\frac{s_0}{s}, \tag{7.57}$$

where $s_0 < \infty$ is some constant fiducial distance.

The electric potential φ inherits superposition from the electric field $\vec{\mathcal{E}}$. For a **discrete** charge distribution Q_k at $\vec{r}_k$, the displacement vectors are $\vec{\boldsymbol{\imath}}_k = \vec{r} - \vec{r}_k$, and the electric potential at the point $\vec{r}$ is

$$\varphi = \sum_k \varphi_k = \sum_k \frac{Q_k}{4\pi \boldsymbol{\imath}_k}. \tag{7.58}$$

For a continuous charge distribution

$$\varphi = \int d\varphi = \int \frac{dQ}{4\pi \boldsymbol{\imath}}, \tag{7.59}$$

where $dQ = \lambda d\ell = \sigma da = \rho dV$. Compared to the electric field integral of Coulomb's law

$$\vec{\mathcal{E}} = \int d\vec{\mathcal{E}} = \int \frac{dQ}{4\pi \boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}}, \tag{7.60}$$

the electric potential integral lacks that pesky $\hat{\boldsymbol{\imath}}$ unit vector, making it often easier to find the electric potential first and subsequently differentiate to find the electric field than to find the electric field directly via Coulomb's law.

As an example, (re)compute the electric field $\vec{\mathcal{E}}$ of a spherical shell of charge Q (and charge density σ) and radius R centered at the origin, as in Fig. 7.1, by first computing the electric potential. By symmetry, take the field point $\vec{r} = z\hat{z}$ to be on the z-axis. The source point $\vec{r}\,' = \vec{R}$ will be on the shell. The electric potential

$$\begin{aligned}
\varphi &= \int \frac{dQ}{4\pi \boldsymbol{\imath}} \\
&= \frac{1}{4\pi}\iint \frac{\sigma da}{|\vec{z} - \vec{R}|} \\
&= \frac{\sigma}{4\pi}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} \frac{(Rd\theta)(R\sin\theta d\phi)}{(z^2 - 2zR\cos\theta + R^2)^{1/2}} \\
&= \frac{\sigma R^2}{2}\int_{\theta=0}^{\pi} \frac{\sin\theta d\theta}{(z^2 - 2zR\cos\theta + R^2)^{1/2}} \\
&= \frac{\sigma R^2}{2}\frac{1}{zR}(z^2 - 2zR\cos\theta + R^2)^{1/2}\Big|_{\theta=0}^{\pi} \\
&= \frac{R\sigma}{2z}(|z+R| - |z-R|)
\end{aligned} \tag{7.61}$$

Two cases exist. In the interior of the shell $z \leq R$ and

$$\varphi[z \leq R] = R\sigma = \frac{Q}{4\pi R}. \tag{7.62}$$

In the exterior of the shell $z \geq R$ and

$$\varphi[z \geq R] = \frac{R^2\sigma}{z} = \frac{Q}{4\pi z}. \tag{7.63}$$

Summarize this by

$$\varphi[\vec{r}] = \begin{cases} \dfrac{Q}{4\pi R}, & r \leq R, \\ \dfrac{Q}{4\pi r}, & r \geq R, \end{cases} \tag{7.64}$$

which is the same as the Eq. 7.53 line integral result. Finally, $\vec{\mathcal{E}} = -\vec{\nabla}\varphi$ readily recovers the Eq. 7.16 surface integral result.

7.5 Work & Energy

Energy E stored in a charge configuration is the work W done to assemble it. Recover the stored energy as kinetic energy by letting the charges fly apart. Move a test charge q from a reference point $\vec{\mathcal{O}}$, typically at infinity, at constant velocity so that the applied force balances the electric force, $\vec{F} + q\vec{\mathcal{E}} = \vec{0}$. The work done (and energy stored) is

$$W = \int_{\vec{\mathcal{O}}}^{\vec{r}} \vec{F} \cdot d\vec{\ell} = -q \int_{\vec{\mathcal{O}}}^{\vec{r}} \vec{\mathcal{E}} \cdot d\vec{\ell} = q\,\varphi[\vec{r}\,] = E. \tag{7.65}$$

Thus, the electric potential φ is the "potential" energy per unit charge stored in the charge distribution. Alternately, think of the electric potential φ as the potential energy per unit charge stored in the corresponding electric field $\vec{\mathcal{E}}$.

Imagine assembling a collection of electric charges $\{Q_k\}$. The work to assemble the first charge in the absence of any others is

$$W_1 = 0. \tag{7.66}$$

The work to assemble the second charge in the presence of the first is

$$W_2 = Q_2 \varphi_1[\vec{r}_2\,] = Q_2 \frac{Q_1}{4\pi \mathscr{r}_{12}}. \tag{7.67}$$

The work to assemble the third charge in the presence of the first two is

$$W_3 = Q_3 \varphi_{12}[\vec{r}_3\,] = Q_3 \left(\frac{Q_1}{4\pi \mathscr{r}_{13}} + \frac{Q_2}{4\pi \mathscr{r}_{23}} \right). \tag{7.68}$$

The work to assemble the fourth charge in the presence of the first three is

$$W_4 = Q_4 \varphi_{123}[\vec{r}_4\,] = Q_4 \left(\frac{Q_1}{4\pi \mathscr{r}_{14}} + \frac{Q_2}{4\pi \mathscr{r}_{24}} + \frac{Q_3}{4\pi \mathscr{r}_{34}} \right). \tag{7.69}$$

The *total* work to assemble all four charges is

$$\begin{aligned} W &= W_1 + W_2 + W_3 + W_4 \\ &= \frac{Q_1 Q_2}{4\pi \mathscr{r}_{12}} + \frac{Q_1 Q_3}{4\pi \mathscr{r}_{13}} + \frac{Q_1 Q_4}{4\pi \mathscr{r}_{14}} + \frac{Q_2 Q_3}{4\pi \mathscr{r}_{23}} + \frac{Q_2 Q_4}{4\pi \mathscr{r}_{24}} + \frac{Q_3 Q_4}{4\pi \mathscr{r}_{34}}. \end{aligned} \tag{7.70}$$

More generally,

$$W = \sum_{\text{pairs}} \frac{Q_k Q_l}{4\pi \mathscr{r}_{kl}}. \tag{7.71}$$

Be careful not to double count pairs when alternately writing

$$W = \sum_{k=1}^{N} \sum_{l=1}^{k-1} \frac{Q_k Q_l}{4\pi \mathscr{r}_{kl}} = \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l<k}}^{N} \frac{Q_k Q_l}{4\pi \mathscr{r}_{kl}} = \sum_{l<k} \frac{Q_k Q_l}{4\pi \mathscr{r}_{kl}} \tag{7.72}$$

or half the double count while still avoiding the self-energies with

$$W = \frac{1}{2} \sum_{k=1}^{N} \sum_{\substack{l=1 \\ l\neq k}}^{N} \frac{Q_k Q_l}{4\pi \mathscr{r}_{kl}} = \frac{1}{2} \sum_{l\neq k} \frac{Q_k Q_l}{4\pi \mathscr{r}_{kl}}. \tag{7.73}$$

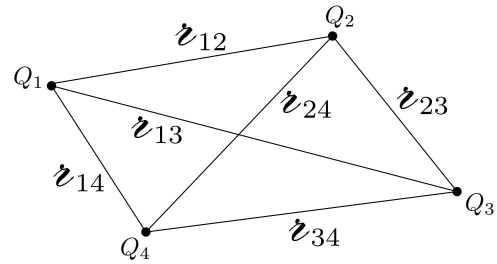

Figure 7.5: With work W, assemble four charges whose configuration stores energy $E = W$.

For a continuous charge distribution, re-introduce the electric potential as

$$W = \frac{1}{2}\sum_{k=1}^{N} Q_k \sum_{\substack{l=1\\ l\neq k}}^{N} \frac{Q_l}{4\pi \mathscr{r}_{kl}} = \frac{1}{2}\sum_{k=1}^{N} Q_k \varphi[\vec{r}_k] \tag{7.74}$$

and generalize to

$$W = \frac{1}{2}\int dQ\,\varphi = \frac{1}{2}\int \varphi\, dQ = \frac{1}{2}\iiint_V \varphi\rho\, dV, \tag{7.75}$$

where the integral is over the volume V containing the charge. Use Gauss's Law $\vec{\nabla}\cdot\vec{\mathcal{E}} = \rho$ to eliminate the charge density,

$$W = \frac{1}{2}\iiint_V \varphi\vec{\nabla}\cdot\vec{\mathcal{E}}\, dV, \tag{7.76}$$

and integrate by parts using the Eq. A-8 product rule

$$\vec{\nabla}\cdot\left(\varphi\vec{\mathcal{E}}\right) = \left(\vec{\nabla}\varphi\right)\cdot\vec{\mathcal{E}} + \varphi\,\vec{\nabla}\cdot\vec{\mathcal{E}} \tag{7.77}$$

to write

$$\varphi\,\vec{\nabla}\cdot\vec{\mathcal{E}} = -\left(\vec{\nabla}\varphi\right)\cdot\vec{\mathcal{E}} + \vec{\nabla}\cdot\left(\varphi\vec{\mathcal{E}}\right) \tag{7.78}$$

and

$$\begin{aligned}\iiint_V \varphi\,\vec{\nabla}\cdot\vec{\mathcal{E}}\, dV &= -\iiint_V \left(\vec{\nabla}\varphi\right)\cdot\vec{\mathcal{E}}\, dV + \iiint_V \vec{\nabla}\cdot\left(\varphi\vec{\mathcal{E}}\right) dV\\ &= \iiint_V \vec{\mathcal{E}}\cdot\vec{\mathcal{E}}\, dV + \oiint_{a=\partial V} \varphi\vec{\mathcal{E}}\cdot d\vec{a}\\ &= \iiint_{r<R} \mathcal{E}^2\, dV + \oiint_{r=R} \varphi\vec{\mathcal{E}}\cdot d\vec{a}.\end{aligned} \tag{7.79}$$

Any physical charge distribution is localized about the experimenter, so the final "boundary" term in Eq. 7.79, the integral over a large spherical surface of radius R, is of order

$$\oiint_{r=R} \varphi\vec{\mathcal{E}}\cdot d\vec{a} \sim \frac{1}{R}\frac{1}{R^2}R^2 = \frac{1}{R} \to 0 \tag{7.80}$$

as $R \to \infty$. Thus, the work required to assemble (and the energy stored in) a continuous charge distribution is

$$W = \frac{1}{2} \iiint \mathcal{E}^2 \, dV = E, \tag{7.81}$$

where the integration is implicitly over all space. More briefly, the integration by parts amounts to moving the derivative in the integrand from one factor to the other, incurring a minus sign,

$$W = \frac{1}{2} \iiint \varphi \vec{\nabla} \cdot \vec{\mathcal{E}} \, dV = -\frac{1}{2} \iiint \vec{\nabla} \varphi \cdot \vec{\mathcal{E}} \, dV = \frac{1}{2} \iiint \mathcal{E}^2 \, dV. \tag{7.82}$$

One subtlety of the Eq. 7.81 work is that it diverges for a point charge, where

$$W = \frac{1}{2} \iiint \mathcal{E}^2 \, dV \sim \left(\frac{1}{r^2}\right)^2 r^3 = \frac{1}{r} \to \infty \tag{7.83}$$

as $r \to 0$. This is because of the infinite **self energy** needed to create the charge. This complication sneaks into the calculations when $\varphi[\vec{r}_k]$ is replaced by $\varphi[\vec{r}]$ in going from Eq. 7.73 to Eq. 7.74.

The work required to assemble the charge distribution is also the energy stored in the corresponding electric field. Therefore, from Eq. 7.81, the energy density stored in an electrostatic field is

$$\frac{dE}{dV} = \frac{1}{2} \mathcal{E}^2. \tag{7.84}$$

Table 7.1 compares and contrasts similar but different electrostatic formulas. Memorize these essential prototypes.

Table 7.1: Summary of similar but different electrostatic formulas.

single charge	pair of charges
$\varphi = \dfrac{Q^1}{4\pi r^1}$	$W = \dfrac{Q^2}{4\pi r^1}$
$\mathcal{E} = \dfrac{Q^1}{4\pi r^2}$	$F = \dfrac{Q^2}{4\pi r^2}$

7.6 Poisson's Equation

Since the electric field is the gradient of the electric potential,

$$\vec{\mathcal{E}} = -\vec{\nabla} \varphi, \tag{7.85}$$

the curl of the electric field vanishes,

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\vec{\nabla} \times \vec{\nabla}\varphi = \vec{0}, \tag{7.86}$$

because the curl of any gradient is identically zero. The *other* Maxwell electrostatic equation becomes

$$\rho = \vec{\nabla} \cdot \vec{\mathcal{E}} = \vec{\nabla} \cdot (-\vec{\nabla}\varphi) = -\nabla^2\varphi, \tag{7.87}$$

or

$$\nabla^2\varphi = -\rho, \tag{7.88}$$

which is **Poisson's equation** with solution

$$\varphi = \iiint_V \frac{\rho\, dV}{4\pi \mathscr{r}}. \tag{7.89}$$

In this way, explicitly solve the Eq. 7.88 inhomogeneous partial differential equation!

7.7 Laplace's Equation

In regions where the charge density vanishes, Poisson's equation becomes

$$\nabla^2\varphi = 0, \tag{7.90}$$

which is **Laplace's equation**. Solutions to Laplace's equation are **harmonic functions**. These arise in many places in physics, including gravitation, heat flow, and soap films. The value of a harmonic function at any point is the **average** of the function over any sphere centered on that point,

$$\varphi_\odot = \bar{\varphi}. \tag{7.91}$$

It follows that a harmonic function has no local maxima or minima, and so its extrema are on its boundary.

In one dimension, Laplace's equation

$$\varphi''[x] = \partial_x^2\varphi[x] = 0 \tag{7.92}$$

has the explicit straight-line solution

$$\varphi[x] = mx + b, \tag{7.93}$$

where m and b are integration constants, or

$$\varphi - \varphi_1 = \frac{\varphi_2 - \varphi_1}{x_2 - x_1}(x - x_1), \tag{7.94}$$

where $\{\varphi_1, \varphi_2\}$ are the solution's values at its endpoints $\{x_1, x_2\}$. The averaging property is obviously

$$\varphi_\odot = \varphi[x] = \frac{\varphi[x-R] + \varphi[x+R]}{2} = \bar{\varphi}. \tag{7.95}$$

In two dimensions, Laplace's equation

$$\partial_x^2 \varphi[x,y] + \partial_y^2 \varphi[x,y] = 0 \tag{7.96}$$

is much more interesting. No closed form general solution is possible, but the averaging property becomes

$$\varphi_{\odot} = \varphi[x,y] = \frac{\oint \varphi \, d\ell}{\oint d\ell} = \frac{1}{2\pi R} \oint_{r=R} \varphi \, d\ell = \bar{\varphi}. \tag{7.97}$$

A plot of $\varphi[x,y]$ is a two-dimensional surface with no hills and no valleys, no bumps and no dents. A ball on the surface rolls to one side and falls off.

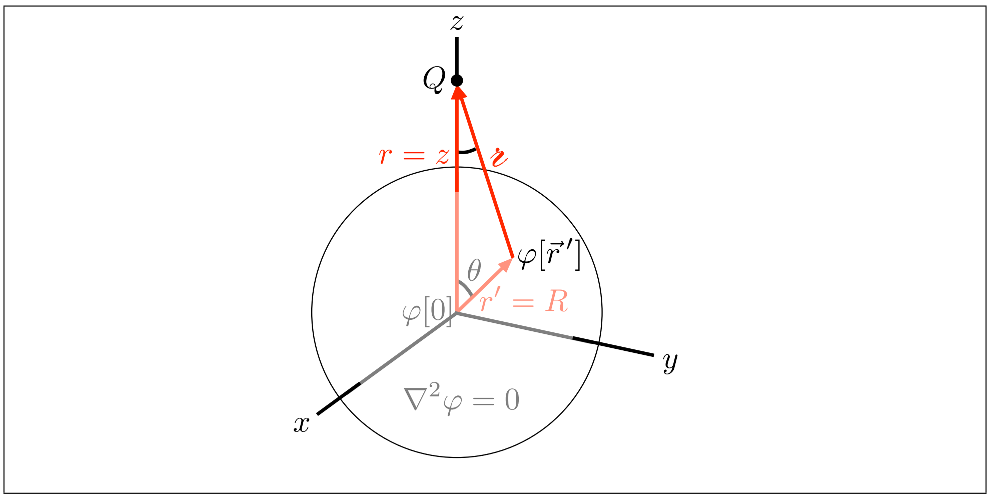

Figure 7.6: The electric potential φ at a point $\vec{0}$ due to a charge Q at $\vec{r} = z\hat{z}$ is the average of the potential over a spherical surface $\vec{r}' = \vec{R}$ centered on the point.

In three dimensions, Laplace's equation

$$\partial_x^2 \varphi[x,y,z] + \partial_y^2 \varphi[x,y,z] + \partial_z^2 \varphi[x,y,z] = 0 \tag{7.98}$$

again has no closed form general solution, and the averaging property becomes

$$\varphi_{\odot} = \varphi[x,y,z] = \frac{1}{4\pi R^2} \oiint_{r=R} \varphi \, da = \bar{\varphi}. \tag{7.99}$$

To prove this, consider a point charge Q at a position $r = z$ external to an imaginary sphere of radius R, as in Fig. 7.6. The average electric potential over the sphere is

$$\begin{aligned}
\bar{\varphi} &= \frac{1}{4\pi R^2} \oiint_{r=R} \varphi \, da \\
&= \frac{1}{4\pi R^2} \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{Q}{4\pi \mathscr{r}} (R d\theta)(R \sin\theta d\phi) \\
&= \frac{1}{4\pi R^2} \frac{Q}{4\pi} R^2 \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \frac{\sin\theta \, d\theta d\phi}{\sqrt{z^2 - 2zR\cos\theta + R^2}} \\
&= \frac{Q}{8\pi} \int_{\theta=0}^{\pi} (z^2 - 2zR\cos\theta + R^2)^{-1/2} \sin\theta d\theta \\
&= \frac{Q}{8\pi} \frac{1}{zR} (z^2 - 2zR\cos\theta + R^2)^{1/2} \Big|_{\theta=0}^{\pi} \\
&= \frac{Q}{8\pi} \frac{|z+R| - |z-R|}{zR} \\
&= \frac{Q}{4\pi z},
\end{aligned} \tag{7.100}$$

as $z > R$. Hence, $\bar{\varphi} = \varphi_{\odot}$. Generalize to arbitrary charge distributions by superposition.

The harmonic averaging property implies **Earnshaw's theorem**, which states that a charged particle cannot be held in a *stable* equilibrium by electrostatic forces alone. Because charges are forced along the gradient of the potential, $\vec{F} = -q\vec{\nabla}\varphi$, a stable equilibrium must be characterized by a local minimum of the potential, which is impossible for the harmonic function solutions of Laplace's equation.

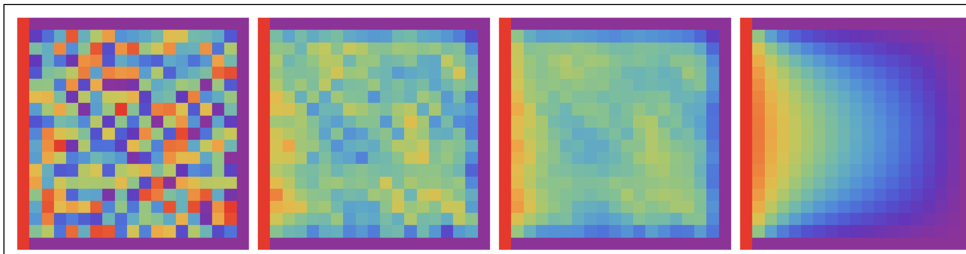

Figure 7.7: The left side of the box is held at a different potential than the other three sides, but from random guesses for the potentials at an interior grid of points (left), the relaxation algorithm quickly relaxes to the equilibrium solution (right).

The harmonic averaging property is also the basis of the **relaxation algorithm** for numerically solving Laplace's equation for the electric potential φ. First, fix φ on the boundaries and guess φ on a grid of points inside the boundaries. Then replace φ at each grid point by the average of it neighbors and repeat until φ stops changing, as in Fig. 7.7. The updates can be synchronous or asynchronous, and so the relaxation algorithm is trivial to parallelize.

7.8 Separation of Variables

A popular technique for solving Laplace's equations involves seeking solutions that are products of functions each of which depends on only one of the coordinates. For example, if

$$\varphi[\vec{r}] = \varphi[x, y, z] = X[x]Y[y]Z[z], \tag{7.101}$$

then Laplace's equation becomes

$$0 = \nabla^2\varphi = X''[x]Y[y]Z[z] + X[x]Y''[y]Z[z] + X[x]Y[y]Z''[z], \tag{7.102}$$

where the primes indicate differentiation with respect to the single arguments. Divide Eq. 7.102 by Eq. 7.101 to get

$$0 = \frac{\nabla^2\varphi}{\varphi} = \frac{X''[x]}{X[x]} + \frac{Y''[y]}{Y[y]} + \frac{Z''[z]}{Z[z]}. \tag{7.103}$$

The only way the sum of these three terms can vanish for all x, y, z, is if each term itself is a constant (else, for example, varying x would change the first term and not the others, thereby changing the sum). These constants must satisfy

$$0 = c_x + c_y + c_z, \tag{7.104}$$

where

$$\begin{aligned} X''[x] &= c_x X[x], \\ Y''[y] &= c_y Y[y], \\ Z''[z] &= c_z Z[z]. \end{aligned} \tag{7.105}$$

The Eq. 7.101 **separation of variables** transforms a partial differential equation into three ordinary differential equations! However, in specific applications boundary conditions may necessitate the superposition of infinitely many such solutions.

As an example, use separation of variables to find the electric potential between two parallel semi-infinite grounded conductors terminated orthogonally by an electrode held at a fixed potential, as in Fig. 7.8. If the conductors and electrode are infinite in the z direction, symmetry implies

$$\varphi[\vec{r}] = \varphi[x, y, z] = \varphi[x, y]. \tag{7.106}$$

Furthermore, since no z dependence exists, $c_z = 0$, and so by Eq. 7.104, $c_x = -c_y = k^2$, where k is another constant. Laplace's equation reduces to

$$\begin{aligned} X''[x] &= +k^2 X[x], \\ Y''[y] &= -k^2 Y[y], \end{aligned} \tag{7.107}$$

which have solutions

$$\begin{aligned} X[x] &= a_x \exp[-kx] + b_x \exp[kx], \\ Y[y] &= a_y \sin[ky] + b_y \cos[ky], \end{aligned} \tag{7.108}$$

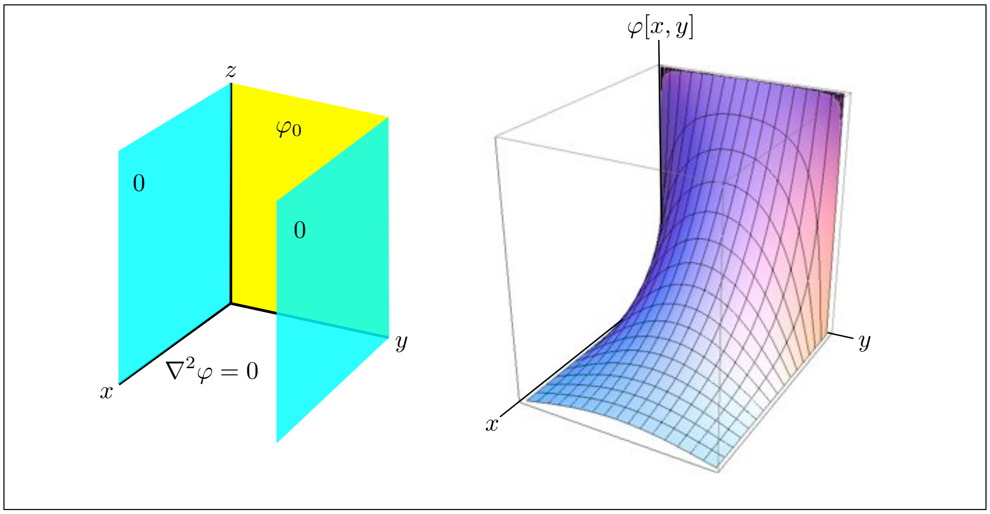

Figure 7.8: The electric potential $\varphi[x, y]$ between two parallel semi-infinite grounded conductors (cyan) terminated orthogonally by an electrode (yellow) held at a fixed potential φ_0.

where a_x, b_x, a_y, b_y are constants. Take the boundary conditions to be

$$\varphi[x, 0] = 0, \tag{7.109}$$
$$\varphi[x, d] = 0, \tag{7.110}$$
$$\varphi[\infty, y] = 0, \tag{7.111}$$
$$\varphi[0, y] = \varphi_0, \tag{7.112}$$

where d is the distance between the conductors. The Eq. 7.109 and Eq. 7.110 boundary conditions imply $b_y = 0$ and $kd = n\pi$, where n is a natural number, so

$$Y[y] = a_y \sin\left[-n\pi \frac{y}{d}\right]. \tag{7.113}$$

The Eq. 7.111 boundary condition implies $b_x = 0$, so

$$X[x] = a_x \exp\left[-n\pi \frac{x}{d}\right]. \tag{7.114}$$

Hence,

$$\varphi_n[x, y] = X[x]Y[y] = a_n \exp\left[-n\pi \frac{x}{d}\right] \sin\left[n\pi \frac{y}{d}\right], \tag{7.115}$$

where $a_n = a_x a_y$ is yet another constant. Finally, to satisfy the Eq. 7.112 boundary condition, form the infinite superposition

$$\varphi[x, y] = \sum_{n=1}^{\infty} \varphi_n[x, y] = \sum_{n=1}^{\infty} a_n \exp\left[-n\pi \frac{x}{d}\right] \sin\left[n\pi \frac{y}{d}\right] \tag{7.116}$$

and demand

$$\varphi_0 = \varphi[0, y] = \sum_{n=1}^{\infty} a_n \sin\left[n\pi\frac{y}{d}\right]. \tag{7.117}$$

Use **Fourier's trick** to solve for the constant coefficients a_n by exploiting the **orthogonality of sinusoids**,

$$\begin{aligned}
&\int_0^d \sin\left[m\pi\frac{y}{d}\right]\sin\left[n\pi\frac{y}{d}\right] dy \\
&\quad = \int_0^d \frac{1}{2}\left(\cos\left[(m-n)\pi\frac{y}{d}\right] - \cos\left[(m+n)\pi\frac{y}{d}\right]\right) dy \\
&\quad = \frac{1}{2}\left(\frac{d}{(m-n)\pi}\sin\left[(m-n)\pi\frac{y}{d}\right] - \frac{d}{(m+n)\pi}\sin\left[(m+n)\pi\frac{y}{d}\right]\right)\Big|_0^d \\
&\quad = \begin{cases} (0-0)-(0-0)=0, & m \neq n, \\ (d/2-0)-(0-0)=d/2, & m=n, \end{cases} \\
&\quad = \frac{d}{2}\delta_{mn}.
\end{aligned} \tag{7.118}$$

Multiply both sides of Eq. 7.117 by $\sin[m\pi y/d]$ and integrate to get

$$\begin{aligned}
\int_0^d \sin\left[m\pi\frac{y}{d}\right]\left(\varphi_0 = \sum_{n=1}^{\infty} a_n \sin\left[n\pi\frac{y}{d}\right]\right) dy, \\
\varphi_0 \int_0^d \sin\left[m\pi\frac{y}{d}\right] dy = \sum_{n=1}^{\infty} a_n \int_0^d \sin\left[m\pi\frac{y}{d}\right]\sin\left[n\pi\frac{y}{d}\right] dy, \\
\varphi_0 \frac{d}{m\pi}(1-\cos[m\pi]) = \sum_{n=1}^{\infty} a_n \frac{d}{2}\delta_{mn} = a_m \frac{d}{2}.
\end{aligned} \tag{7.119}$$

Hence, the coefficients

$$a_m = \varphi_0 \frac{2}{m\pi}\left(1-(-1)^m\right) = \begin{cases} \varphi_0 \dfrac{4}{m\pi}, & m \text{ odd}, \\ 0, & m \text{ even}. \end{cases} \tag{7.120}$$

The final solution is

$$\varphi[x, y] = \varphi_0 \frac{4}{\pi}\sum_{n \text{ odd}} \frac{1}{n}\exp\left[-n\pi\frac{x}{d}\right]\sin\left[n\pi\frac{y}{d}\right], \tag{7.121}$$

or more explicitly,

$$\varphi[x, y] = \varphi_0 \frac{4}{\pi}\sum_{\ell=1}^{\infty} \frac{1}{2\ell-1}\exp\left[-(2\ell-1)\pi\frac{x}{d}\right]\sin\left[(2\ell-1)\pi\frac{y}{d}\right], \tag{7.122}$$

which is graphed in Fig. 7.8.

Problems

1. Find the electric field $\vec{\mathcal{E}}$ at distance r along the perpendicular axis of the following charge distributions of size R. Expand the result in powers of R/r to show that the field is asymptotically like that of a point charge when $r \gg R$.
   (a) A ring of radius R and linear charge density λ.
   (b) A disk of radius R and surface charge density σ.
2. Find the electric flux $\Phi_{\mathcal{E}}$ through one face of a cube with a charge Q at a vertex on the opposite face. Integrate – or just write the answer using symmetry.
3. Evaluate the following integrals.
   (a) $\int_2^6 (3x^2 - 2x - 1)\delta[x-3]dx$.
   (b) $\int_0^5 \cos x\, \delta[x-\pi]dx$.
   (c) $\int_0^3 x^3 \delta[x+1]dx$.
   (d) $\int_{-\infty}^{\infty} \log[x+3]\, \delta[x+2]dx$.
4. Show that the derivative of the **step function**,
$$\theta[x] = \begin{cases} 0, & x \leq 0, \\ 1, & 0 < x, \end{cases}$$
   is the Dirac delta,
$$\frac{d\theta}{dx} = \delta[x].$$
5. Find the charge densities $\rho[\vec{r}]$ for the following charge distributions. Check that the volume integral of the charge density is the total charge.
   (a) An electric **dipole** consisting of a point charge $-Q$ at $-\vec{r}_0$ and a point charge $+Q$ at $+\vec{r}_0$.
   (b) A charged shell centered on the origin of radius R and total charge Q.
6. Prove the following vector differential identities. (Hint: Partial derivatives commute for the smooth functions typically assumed in physics.)
   (a) The curl of any gradient is zero, $\vec{\nabla} \times \vec{\nabla} s = \vec{0}$.
   (b) The divergence of any curl is zero, $\vec{\nabla} \cdot \vec{\nabla} \times \vec{v} = 0$.
7. Find the electric field $\vec{\mathcal{E}}$ both inside and outside the following charge distributions using Gauss's law in integral form. Sketch the electric field magnitude $\mathcal{E}$ as a function of the radial distance r.
   (a) A solid sphere of radius R and constant volume charge density ρ.
   (b) A solid sphere of radius R and radially increasing volume charge density $\rho[r] = kr$.

8. Find the electric potential φ both inside and outside the charged solid sphere of Problem 7a via the techniques below.

   (a) Volume integration using Eq. 7.59.

   (b) Line integration using Eq. 7.43.

9. Find the energy $E = W$ stored in the solid sphere of Problem 7a via the techniques below. Express the answer in terms of the total charge Q in the sphere.

   (a) Volume integration of the electric potential using Eq. 7.75.

   (b) Volume integration of the electric field using Eq. 7.81.

10. Consider an empty cube with identical charges at each of its vertices. Consistent with Earnshaw's theorem, show that the electric potential at the cube's center exhibits a three-dimensional "saddle point": The center is a *stable* equilibrium along the body diagonals (so small diagonal displacements of a test charge cause it to oscillate about the center) but is an *unstable* equilibrium perpendicular to the faces (so small perpendicular displacements of a test charge cause it to move far from the center).(Hint: Show that the corresponding forces .)

11. Use separation of variables to find the electric potential everywhere inside a cubical box of volume ℓ^3, which has five sides grounded (at zero potential) and one side held at a constant potential φ_0. Express the result as a *doubly* infinite sum. Create two-dimensional surface or scatter plots of the potential in three orthogonal cross sections of the box. (Hint: Assume sinusoidal solutions in two directions and exponential solutions in the third, and relate the two sinusoidal frequencies to the exponential decay constant. The final solution is neatest when expressed in terms of sines and hyperbolic sines.)

Chapter 8

Magnetostatics

In a world of stationary electric currents, divergenceless magnetic fields curl around current densities, and Maxwell's equations reduce to

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = 0, \qquad \vec{\nabla} \times \vec{\mathcal{B}} = \vec{J}, \tag{8.1}$$

or

$$\Phi_{\mathcal{B}} = 0, \qquad \Gamma_{\mathcal{B}} = I. \tag{8.2}$$

A magnetic force is proportional to an electric charge and the charge's velocity crossed with a magnetic field, and Lorentz's equation reduces to

$$\vec{F} = q\,\vec{v} \times \vec{\mathcal{B}}. \tag{8.3}$$

8.1 Biot-Savart's Law

Consider the circulating magnetic field $d\vec{\mathcal{B}}$ of an infinitesimal charge dQ of length $d\ell$ moving with velocity $\vec{v}$ in a current I, as in Fig. 8.1. Displace a magnetic charge $Q_{\mathcal{B}}$ a vector $\vec{\boldsymbol{\imath}}$ from dQ so that at dQ it produces a magnetic field

$$\vec{\mathcal{B}} = \frac{Q_{\mathcal{B}}}{4\pi \boldsymbol{\imath}^2}(-\hat{\boldsymbol{\imath}}) = -\frac{Q_{\mathcal{B}}}{4\pi \boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}}, \tag{8.4}$$

which results in an infinitesimal force

$$+\,d\vec{F} = dQ\,\vec{v} \times \vec{\mathcal{B}} = -dQ\,\vec{v} \times \frac{Q_{\mathcal{B}}}{4\pi \boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}} \tag{8.5}$$

on dQ due to $Q_{\mathcal{B}}$. This is equal in magnitude but opposite in direction to the infinitesimal force

$$-\,d\vec{F} = Q_{\mathcal{B}} d\vec{\mathcal{B}} \tag{8.6}$$

on $Q_{\mathcal{B}}$ due to dQ and its circulating magnetic field $d\vec{\mathcal{B}}$. Combine Eq. 8.5 and Eq. 8.6 to get

$$d\vec{\mathcal{B}} = \vec{v} \times \frac{dQ}{4\pi \boldsymbol{\imath}^2}\hat{\boldsymbol{\imath}}, \tag{8.7}$$

where

$$v\,dQ = \frac{d\ell}{dt}dQ = \frac{dQ}{dt}d\ell = Id\ell, \tag{8.8}$$

and more generally

$$\vec{v}\,dQ = \vec{I}d\ell = \vec{K}da = \vec{J}dV, \tag{8.9}$$

and $\vec{K}$ is (surface) current per cross sectional line and $\vec{J}$ is (volume) current per cross sectional area. Integrate to find the magnetic field

$$\vec{\mathcal{B}} = \int \vec{v} \times \frac{dQ}{4\pi \mathscr{r}^2}\hat{\mathscr{r}} = \int_\ell \frac{Id\vec{\ell} \times \hat{\mathscr{r}}}{4\pi \mathscr{r}^2} \tag{8.10}$$

due to the entire current, where $Id\vec{\ell} = \vec{I}d\ell$. This is **Biot-Savart's law**. Delete the velocity cross product to obtain Coulomb's law,

$$\vec{\mathcal{E}} = \int \frac{dQ}{4\pi \mathscr{r}^2}\hat{\mathscr{r}}. \tag{8.11}$$

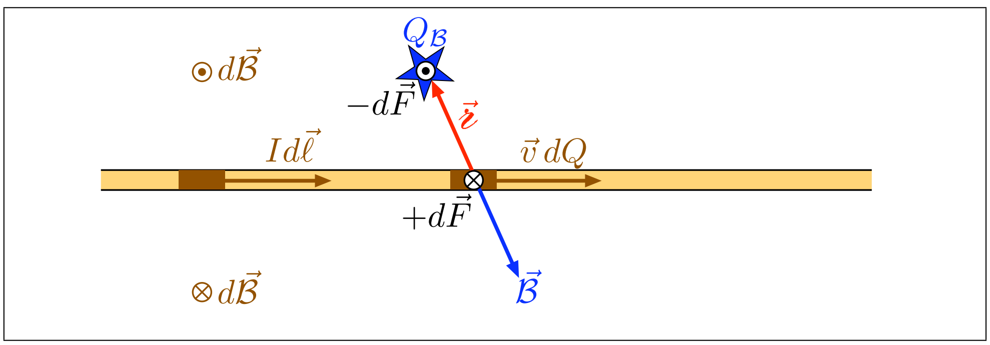

Figure 8.1: Magnetic charge $Q_{\mathcal{B}}$ (blue) used to derive Biot-Savart's law for the magnetic field $d\vec{\mathcal{B}}$ (brown) due to a current element $\vec{v}\,dQ = Id\vec{\ell}$.

As an example, compute the magnetic field $\vec{\mathcal{B}}$ of an long straight current I. By symmetry, assume the current coincides with the z-axis, the field point $\vec{r} = x\hat{x}$ is on the x-axis, and the source point $\vec{r}\,' = z\hat{z}$ is on the z-axis, as in Fig. 8.2. Let ψ be the angle between $\vec{r}$ and $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$. Employ cylindrical coordinates $\{s, \phi, z\}$, where s is the perpendicular distance to the z-axis and ϕ is the longitude. By symmetry, write

$$\mathcal{B} = \mathcal{B}_y = \int d\mathcal{B}_y = \int \frac{Idz\sin[\pi/2 + \psi]}{4\pi \mathscr{r}^2}. \tag{8.12}$$

Eliminate the variables z and $\mathscr{r}$ in favor of the angle ψ using

$$\mathscr{r} = \frac{x}{\cos\psi} = x\sec\psi \tag{8.13}$$

and

$$z = x\tan\psi \tag{8.14}$$

or

$$dz = x \sec^2 \psi \, d\psi \tag{8.15}$$

to get

$$\begin{aligned} \mathcal{B} &= \int_{-\pi/2}^{\pi/2} \frac{Ix \sec^2 \psi \, d\psi \cos \psi}{4\pi x^2 \sec^2 \psi} \\ &= \frac{I}{4\pi x} \int_{-\pi/2}^{\pi/2} \cos \psi \, d\psi \\ &= \frac{I}{2\pi x}. \end{aligned} \tag{8.16}$$

Generalize this to

$$\vec{\mathcal{B}}[\vec{r}] = \frac{I}{2\pi s} \hat{\phi}. \tag{8.17}$$

The denominator reflects the dilution of the source current over a circle of circumference $2\pi s$.

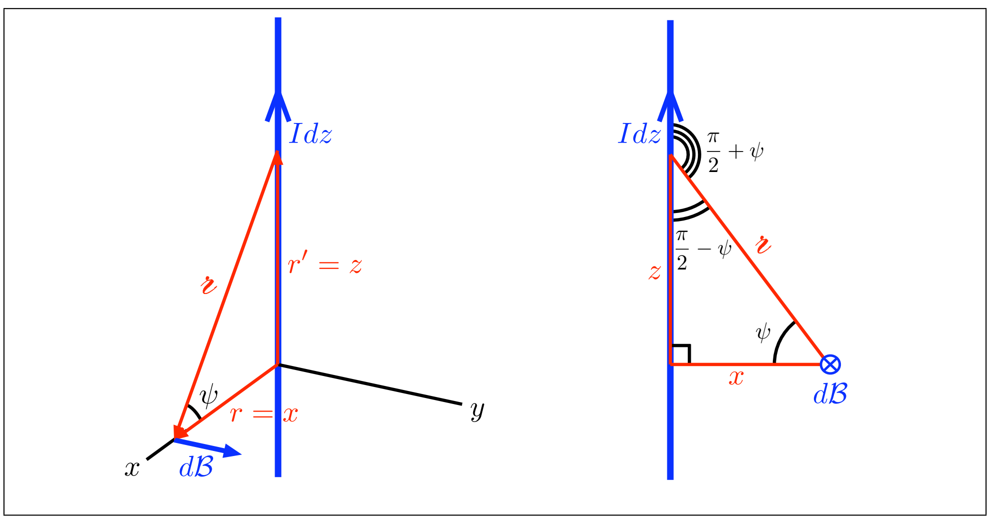

Figure 8.2: A long straight current (left) and the geometry of the position triangle rotated into the plane of the page (right).

To compute the magnetic field on the plane bisecting a *short* straight current of length $2h$, calculate as before, except use finite limits of integration to get

$$\begin{aligned} \mathcal{B} &= \frac{I}{4\pi s} \int_{-\Psi}^{\Psi} \cos \psi \, d\psi \\ &= \frac{I}{2\pi s} \sin \Psi \\ &= \frac{I}{2\pi s} \frac{h}{\sqrt{s^2 + h^2}}. \end{aligned} \tag{8.18}$$

If the current is relatively long, then $h \gg s$ and

$$\mathcal{B} = \frac{I}{2\pi s}\left(1+\left(\frac{s}{h}\right)^2\right)^{-1/2} \sim \frac{I}{2\pi s}\left(1-\frac{1}{2}\left(\frac{s}{h}\right)^2\right) \sim \frac{I}{2\pi s}, \tag{8.19}$$

which is the previous result for the long straight current.

8.2 Boundary Conditions

Magnetostatic fields are discontinuous at current layers. Consider a surface with normal $\hat{n}$ and surface current density $\vec{K}$, as in Fig. 8.3. Let $\vec{\mathcal{B}}^-$ be the magnetic field just below the surface and $\vec{\mathcal{B}}^+$ be the magnetic field just above the surface. Apply $\varPhi_{\mathcal{B}} = 0$ to a cylinder straddling the surface of cross sectional area a and vanishing height $h \to 0$ to get

$$\mathcal{B}_\perp^+ a - \mathcal{B}_\perp^- a - 0 = 0 \tag{8.20}$$

or

$$\varDelta\mathcal{B}_\perp = \mathcal{B}_\perp^+ - \mathcal{B}_\perp^- = 0. \tag{8.21}$$

Apply $\varGamma_{\mathcal{B}} = I$ to a rectangular loop straddling the surface of length ℓ and vanishing height $h \to 0$ to get

$$\mathcal{B}_\parallel^+ \ell + 0 - \mathcal{B}_\parallel^- \ell - 0 = K\ell \tag{8.22}$$

or

$$\varDelta\mathcal{B}_\parallel = \mathcal{B}_\parallel^+ - \mathcal{B}_\parallel^- = K. \tag{8.23}$$

Thus, the parallel component of the magnetostatic field is discontinuous by the current density,

$$\varDelta\vec{\mathcal{B}} = \vec{\mathcal{B}}^+ - \vec{\mathcal{B}}^- = \vec{K} \times \hat{n}. \tag{8.24}$$

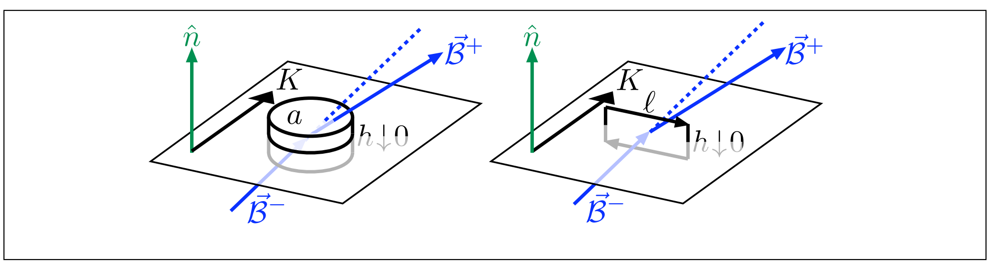

Figure 8.3: A closed cylinder (left) and rectangular loop (right) straddle a current layer to compute the discontinuity in a magnetic field $\vec{\mathcal{B}}$ due to the current density $\vec{K}$.

8.3 Magnetic Potential

In analogy with the electric potential

$$\varphi = \iiint_V \frac{\rho\, dV}{4\pi \mathscr{r}}, \tag{8.25}$$

define the **magnetic potential**

$$\vec{\mathcal{A}} = \iiint_V \frac{\vec{J} dV}{4\pi \mathscr{r}}. \tag{8.26}$$

The scalar nature of the electrostatic source ρ suggests a scalar potential φ in the same way that the vector nature of the magnetostatic source $\vec{J}$ suggests a vector potential $\vec{\mathcal{A}}$. By Eq. 8.9, write

$$\vec{\mathcal{A}} = \iiint_V \frac{\vec{J} dV}{4\pi \mathscr{r}} = \iint_a \frac{\vec{K} da}{4\pi \mathscr{r}} = \int_\ell \frac{\vec{I} d\ell}{4\pi \mathscr{r}} = \int \vec{v} \frac{dQ}{4\pi \mathscr{r}} = \int \vec{v}\, d\varphi. \tag{8.27}$$

Differentiate the potential to find the field. For the electric potential, the derivative must be a gradient. Remembering that the displacement vector $\vec{\mathscr{r}} = \vec{r} - \vec{r}\,'$, compute

$$\begin{aligned}
\vec{\nabla}\varphi &= \vec{\nabla}_{\vec{r}} \iiint \frac{\rho[\vec{r}\,'] d^3\vec{r}\,'}{4\pi \mathscr{r}} \\
&= \iiint \frac{\rho[\vec{r}\,'] d^3\vec{r}\,'}{4\pi} \vec{\nabla}_{\vec{r}} \left(\frac{1}{\mathscr{r}} \right) \\
&= \iiint \frac{\rho[\vec{r}\,'] d^3\vec{r}\,'}{4\pi} \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \\
&= -\iiint \frac{\rho[\vec{r}\,'] d^3\vec{r}\,'}{4\pi \mathscr{r}^2} \hat{\mathscr{r}} \\
&= -\vec{\mathcal{E}},
\end{aligned} \tag{8.28}$$

by Coulomb's law. The electric potential is unique up to a constant, because φ and $\varphi + \varphi_0$ represent the same electric field $\vec{\mathcal{E}}$ provided $\vec{\nabla}\varphi_0 = \vec{0}$.

For the magnetic potential, the derivative must be a curl. Compute

$$
\begin{aligned}
\vec{\nabla} \times \vec{\mathcal{A}} &= \vec{\nabla}_{\vec{r}} \times \iiint \frac{\vec{J}[\vec{r}\,']d^3\vec{r}\,'}{4\pi \mathscr{r}} \\
&= \iiint \frac{d^3\vec{r}\,'}{4\pi} \vec{\nabla}_{\vec{r}} \times \left(\frac{1}{\mathscr{r}} \vec{J}[\vec{r}\,'] \right) \\
&= \iiint \frac{d^3\vec{r}\,'}{4\pi} \vec{\nabla}_{\vec{r}} \left(\frac{1}{\mathscr{r}} \right) \times \vec{J}[\vec{r}\,'] \\
&= \iiint \frac{d^3\vec{r}\,'}{4\pi} \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \times \vec{J}[\vec{r}\,'] \\
&= \iiint \frac{\vec{J}[\vec{r}\,']d^3\vec{r}\,' \times \hat{\mathscr{r}}}{4\pi \mathscr{r}^2} \\
&= \vec{\mathcal{B}},
\end{aligned}
\tag{8.29}
$$

by Biot-Savart's law. The magnetic potential is unique up to a curlless vector field, because $\vec{\mathcal{A}}$ and $\vec{\mathcal{A}} + \vec{\mathcal{A}}_0$ represent the same magnetic field $\vec{\mathcal{B}}$ provided $\vec{\nabla} \times \vec{\mathcal{A}}_0 = \vec{0}$.

If the magnetic source is a line current, then $\vec{J}\,dV = \vec{I}\,d\ell$. Hence, curl both sides of Eq. 8.27 to get

$$
\begin{aligned}
\vec{\mathcal{B}} &= \vec{\nabla} \times \vec{\mathcal{A}} \\
&= \vec{\nabla} \times \int_\ell \frac{\vec{I}\,d\ell}{4\pi \mathscr{r}} \\
&= \int_\ell \frac{d\ell}{4\pi} \vec{\nabla} \times \left(\frac{1}{\mathscr{r}} \vec{I} \right) \\
&= \int_\ell \frac{d\ell}{4\pi} \vec{\nabla} \left(\frac{1}{\mathscr{r}} \right) \times \vec{I} \\
&= \int_\ell \frac{d\ell}{4\pi} \left(-\frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) \times \vec{I} \\
&= \int_\ell \frac{\vec{I} d\ell \times \hat{\mathscr{r}}}{4\pi \mathscr{r}^2},
\end{aligned}
\tag{8.30}
$$

which is another derivation of Biot-Savart's law.

As an example, compute the magnetic potential $\vec{\mathcal{A}}$ of a long straight current. If the current coincides with the z-axis, as in Fig. 8.2, that will also be the direction of the magnetic potential, whose magnitude at the perpendicular distance $x = s$ is

$$\begin{aligned}
\mathcal{A} &= \int_\ell \frac{I d\ell}{4\pi \mathscr{r}} \\
&= \frac{I}{4\pi} \int_{-\infty}^{\infty} \frac{dz}{\sqrt{s^2 + z^2}} \\
&= \frac{I}{4\pi} \int_{-\pi/2}^{\pi/2} \frac{s \sec^2 \psi \, d\psi}{s \sec \psi} \\
&= \frac{I}{4\pi} \int_{-\pi/2}^{\pi/2} \sec \psi \, d\psi \\
&= \frac{I}{4\pi} \log[\tan \psi + \sec \psi] \bigg|_{-\pi/2}^{\pi/2} \\
&= \infty - \infty?
\end{aligned} \tag{8.31}$$

Attempt to handle the infinities by replacing the angles by distances and using limits to write

$$\begin{aligned}
\mathcal{A} &= \frac{I}{4\pi} \lim_{L \to \infty} \log \left[\frac{z}{s} + \frac{\sqrt{z^2 + s^2}}{s} \right] \Bigg|_{-L}^{L} \\
&= \frac{I}{4\pi} \lim_{L \to \infty} \log \left[\frac{L + \sqrt{L^2 + s^2}}{-L + \sqrt{L^2 + s^2}} \times \frac{L + \sqrt{L^2 + s^2}}{L + \sqrt{L^2 + s^2}} \right] \\
&= \frac{I}{4\pi} \lim_{L \to \infty} \log \left[\left(\frac{L + \sqrt{L^2 + s^2}}{s} \right)^2 \right] \\
&= \frac{I}{2\pi} \lim_{L \to \infty} \log \left[\frac{L + \sqrt{L^2 + s^2}}{s} \right] \\
&= \frac{I}{2\pi} \lim_{L \to \infty} \log \left[\frac{2L}{s} \right] \\
&= \lim_{L \to \infty} \frac{I}{2\pi} \log[2L] - \frac{I}{2\pi} \log s.
\end{aligned} \tag{8.32}$$

The infinite constant does not affect the derivative, which is the field. Hence, **renormalize** A by having it absorb the infinity, and write

$$\mathcal{A} = \frac{I}{2\pi} \log s_0 - \frac{I}{2\pi} \log s = \frac{I}{2\pi} \log \left[\frac{s_0}{s} \right], \tag{8.33}$$

or, more generally,

$$\vec{\mathcal{A}}[\vec{r}] = \frac{I}{2\pi} \log \left[\frac{s_0}{s} \right] \hat{z} = \frac{\vec{I}}{2\pi} \log \left[\frac{s_0}{s} \right], \tag{8.34}$$

where s_0 is some constant fiducial distance. (This is similar to the Eq. 7.57 electric potential of a line charge because similar geometry forces similar physics.) The

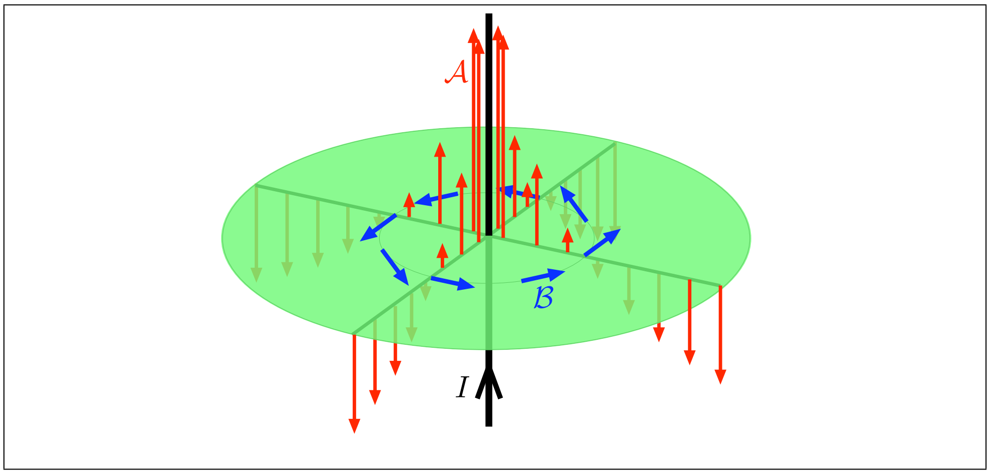

Figure 8.4: The magnetic field $\vec{\mathcal{B}}$ of the long straight current I is the curl of the magnetic potential $\vec{\mathcal{A}}$.

magnetic field of the long straight current is the curl of this magnetic potential, $\vec{\mathcal{B}} = \vec{\nabla} \times \vec{\mathcal{A}}$, as in Fig. 8.4.

Another important example is the magnetic potential $\vec{\mathcal{A}}$ for a constant magnetic field $\vec{\mathcal{B}}$, namely

$$\vec{\mathcal{A}} = -\frac{1}{2}\vec{r} \times \vec{\mathcal{B}}, \tag{8.35}$$

as illustrated by Fig. 8.5. Use the Eq. A-11 product rule to check that $\vec{\mathcal{A}}$ curls to $\vec{B}$,

$$\begin{aligned}
\vec{\nabla} \times \vec{\mathcal{A}} &= -\frac{1}{2}\vec{\nabla} \times \left(\vec{r} \times \vec{\mathcal{B}}\right) \\
&= -\frac{1}{2}\left(\left(\vec{\mathcal{B}} \cdot \vec{\nabla}\right)\vec{r} - \vec{\mathcal{B}}\left(\vec{\nabla} \cdot \vec{r}\right) + \vec{r}\left(\vec{\nabla} \cdot \vec{\mathcal{B}}\right) - \left(\vec{r} \cdot \vec{\nabla}\right)\vec{\mathcal{B}}\right) \\
&= -\frac{1}{2}\left(\vec{\mathcal{B}} - 3\vec{\mathcal{B}} + \vec{0} - \vec{0}\right) \\
&= \vec{\mathcal{B}},
\end{aligned} \tag{8.36}$$

where the last term vanishes because $\vec{\mathcal{B}}$ is a constant. Use the Eq. A-9 product rule to note that $\vec{\mathcal{A}}$ is divergenceless,

$$\begin{aligned}
\vec{\nabla} \cdot \vec{\mathcal{A}} &= -\frac{1}{2}\vec{\nabla} \cdot \left(\vec{r} \times \vec{\mathcal{B}}\right) \\
&= -\frac{1}{2}\left(\vec{\mathcal{B}} \cdot \vec{\nabla} \times \vec{r} - \vec{r} \cdot \vec{\nabla} \times \vec{\mathcal{B}}\right) \\
&= -\frac{1}{2}\left(0 - 0\right) \\
&= 0.
\end{aligned} \tag{8.37}$$

However, since the same nonzero curl is present not just at the "center" but everywhere, translate the potential through an arbitrary vector to $\vec{\mathcal{A}} + \vec{\mathcal{A}}_0$ and still generate the same field.

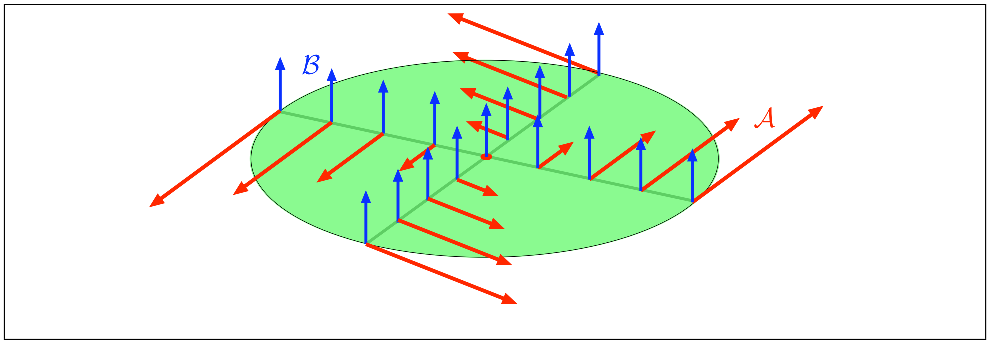

Figure 8.5: A constant magnetic field $\vec{\mathcal{B}}$ has an everywhere curling potential $\vec{\mathcal{A}}$.

8.4 Poisson's Equation

Since the magnetic field is the curl of the magnetic potential,

$$\vec{\mathcal{B}} = \vec{\nabla} \times \vec{\mathcal{A}}, \tag{8.38}$$

the divergence of the magnetic field vanishes,

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = \vec{\nabla} \cdot \vec{\nabla} \times \vec{\mathcal{A}} = 0, \tag{8.39}$$

because the divergence of any curl is identically zero. Shifting the magnetic potential $\vec{\mathcal{A}}$ by the gradient of a scalar field $\vec{\nabla}\lambda$ doesn't change the magnetic field $\vec{\mathcal{B}}$ because the curl of a gradient is identically zero. Use this **gauge freedom** to simplify the *other* Maxwell magnetostatic equation,

$$\vec{\mathcal{J}} = \vec{\nabla} \times \vec{\mathcal{B}} = \vec{\nabla} \times (\vec{\nabla} \times \vec{\mathcal{A}}) = \vec{\nabla}(\vec{\nabla} \cdot \vec{\mathcal{A}}) - (\vec{\nabla} \cdot \vec{\nabla})\vec{\mathcal{A}}, \tag{8.40}$$

by choosing

$$\vec{\mathcal{A}}' = \vec{\mathcal{A}} + \vec{\nabla}\lambda \tag{8.41}$$

such that

$$0 = \vec{\nabla} \cdot \vec{\mathcal{A}}' = \vec{\nabla} \cdot \vec{\mathcal{A}} + \vec{\nabla} \cdot \vec{\nabla}\lambda = \vec{\nabla} \cdot \vec{\mathcal{A}} + \nabla^2\lambda \tag{8.42}$$

or

$$\nabla^2\lambda = -\vec{\nabla} \cdot \vec{\mathcal{A}}, \tag{8.43}$$

which is Poisson's equation with solution

$$\lambda = \iiint_V \frac{\vec{\nabla} \cdot \vec{\mathcal{A}}}{4\pi \mathscr{r}} dV. \tag{8.44}$$

With the Eq. 8.42 **Coulomb gauge** choice, Eq. 8.40 implies

$$\nabla^2 \vec{\mathcal{A}}' = -\vec{J}, \tag{8.45}$$

which is a vector form of Poisson's equation with solution

$$\vec{\mathcal{A}}' = \iiint_V \frac{\vec{J}\, dV}{4\pi \mathscr{r}}. \tag{8.46}$$

8.5 Statics Comparison

Magnetostatics and electrostatics have comparable structures, as in Table 8.1. However, with its deflecting, curling nature, magnetostatics is richer for the extra cross product.

Table 8.1: Comparison of electrostatics and magnetostatics (with the Coulomb gauge choice $\vec{\nabla} \cdot \vec{\mathcal{A}} = 0$).

Electrostatics	Magnetostatics
$\vec{\nabla} \cdot \vec{\mathcal{E}} = \rho, \quad \vec{\nabla} \times \vec{\mathcal{E}} = \vec{0}$ $\Phi_{\mathcal{E}} = Q, \quad \Gamma_{\mathcal{E}} = 0$ $\vec{F} = q\vec{\mathcal{E}}$	$\vec{\nabla} \cdot \vec{\mathcal{B}} = 0, \quad \vec{\nabla} \times \vec{\mathcal{B}} = \vec{J}$ $\Phi_{\mathcal{B}} = 0, \quad \Gamma_{\mathcal{B}} = I$ $\vec{F} = q\, \vec{v} \times \vec{\mathcal{B}}$
$\vec{\mathcal{E}} = \int \frac{dQ}{4\pi \mathscr{r}^2} \hat{\mathscr{r}}$ $\vec{\mathcal{E}} = \frac{Q}{4\pi r^2} \hat{r}$	$\vec{\mathcal{B}} = \int \vec{v} \times \frac{dQ}{4\pi \mathscr{r}^2} \hat{\mathscr{r}}$ $\vec{\mathcal{B}} = \frac{I}{2\pi s} \hat{\phi}$
$\vec{\mathcal{E}} = -\vec{\nabla}\varphi$ $\nabla^2 \varphi = -\rho$ $\varphi = \iiint_V \frac{\rho\, dV}{4\pi \mathscr{r}}$	$\vec{\mathcal{B}} = \vec{\nabla} \times \vec{\mathcal{A}}$ $\nabla^2 \vec{\mathcal{A}} = -\vec{J}$ $\vec{\mathcal{A}} = \iiint_V \frac{\vec{J}\, dV}{4\pi \mathscr{r}}$

Problems

1. Consider a circular loop of radius R carrying a current I.

   (a) Find the magnetic field $\vec{\mathcal{B}}$ everywhere on the perpendicular axis of the loop using Biot-Savart's law.

   (b) What is the ratio of the magnetic field at the loop's center to the magnetic field a distance R from a long straight current I?

2. Find the magnetic field $\vec{\mathcal{B}}$ at the center of the following loops of current I.

   (a) A square with side $2R$ and inscribed circular radius R.

   (b) An n-sided polygon of inscribed circular radius R.

   (c) What happens as the number n of sides in Problem 2b goes to infinity?

3. Find the magnetic field $\vec{\mathcal{B}}$ both inside and outside the following current distributions using Ampère's law in integral form.

   (a) A solid cylinder of radius R and constant current density $\vec{J} = J\hat{z}$.

   (b) A solid cylinder of radius R and radially increasing current density $\vec{J}[s] = ks\hat{z}$.

4. Explicitly curl the Eq. 8.34 magnetic potential to get the magnetic field of the long straight current. (Hint: Use the Eq. A-22 curl operator in cylindrical coordinates.)

Chapter 9

Electrodynamics

Describe a world of dynamic fields and charges by the full Maxwell equations

$$\begin{aligned}\vec{\nabla}\cdot\vec{\mathcal{B}} &= 0, \qquad \vec{\nabla}\times\vec{\mathcal{E}} = -\partial_t\vec{\mathcal{B}},\\ \vec{\nabla}\cdot\vec{\mathcal{E}} &= \rho, \qquad \vec{\nabla}\times\vec{\mathcal{B}} = +\partial_t\vec{\mathcal{E}} + \vec{J},\end{aligned} \tag{9.1}$$

or

$$\begin{aligned}\Phi_{\mathcal{B}} &= 0, \qquad \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}},\\ \Phi_{\mathcal{E}} &= Q, \qquad \Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}} + I,\end{aligned} \tag{9.2}$$

and the full Lorentz equation

$$\vec{F} = q(\vec{\mathcal{E}} + \vec{v}\times\vec{\mathcal{B}}). \tag{9.3}$$

9.1 Faraday's Law

A linear **electric generator** consists of a constant magnetic field $-\mathcal{B}\hat{z}$ and a conducting cross bar $\ell\hat{y}$ at $x\hat{x}$ sliding on parallel conductors that terminate at a fixed transverse conductor, as in Fig. 9.1. Kick the cross bar so its initial velocity is $v_0\hat{x}$. If the area element $d\vec{a} = da\,\hat{z}$, then the magnetic flux through the loop is

$$\Phi_{\mathcal{B}} = \iint_a \vec{\mathcal{B}}\cdot d\vec{a} = -\iint_a \mathcal{B}\,da = -\mathcal{B}(\ell x) < 0. \tag{9.4}$$

and it is decreasing

$$\dot{\Phi}_{\mathcal{B}} = -\mathcal{B}\ell\dot{x} = -\mathcal{B}\ell v_x < 0. \tag{9.5}$$

The changing magnetic flux induces a circulating electric field that drives current around the loop formed by the cross bar, parallel conductors, and resistor, like a delocalized battery. Assuming the circuit is an **ohmic device** of resistance $\mathcal{R}$,

$$\Gamma_{\mathcal{E}} = \oint_l \vec{\mathcal{E}}\cdot d\vec{\ell} = \Delta\varphi = I\mathcal{R}, \tag{9.6}$$

where a right-hand-rule relates the positive sense of circulation around the loop to the positive direction of the area bounded by the loop. From Eq. 9.6 and Eq. 9.5, Faraday's law $\Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}$ implies

$$I\mathcal{R} = -(-\mathcal{B}\ell v_x) \tag{9.7}$$

or

$$I = \frac{\mathcal{B}\ell}{\mathcal{R}} v_x > 0. \tag{9.8}$$

The induced current (counterclockwise from above) generates a magnetic field (upward through the loop) that opposes the change in the magnetic flux (increasing downward). This electromagnetic "inertia" is know as **Lenz's rule** and is embodied mathematically in the minus sign in Faraday's law. Assume the initial speed v_0 is small enough that the induced current's magnetic field contributes negligibly to the magnetic flux though the loop.

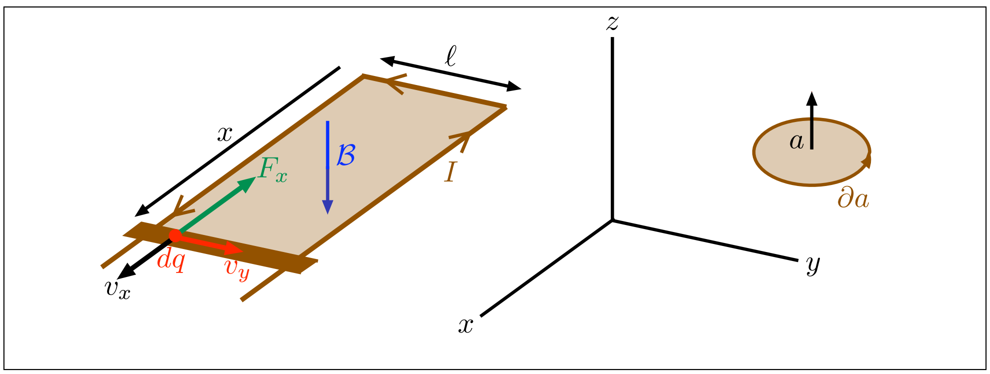

Figure 9.1: From a kicked start, the sliding cross bar of this linear electric generator exponentially slows, dissipating energy as heat in the rails.

The magnetic field $-\mathcal{B}\hat{z}$ deflects the current $\vec{I} = I\hat{y}$ flowing in the cross bar with a force

$$\vec{F} = \int dq\, \vec{v} \times \vec{\mathcal{B}} = \int I d\vec{\ell} \times \vec{\mathcal{B}}, \tag{9.9}$$

whose x-component

$$F_x = -\int I dy\, \mathcal{B} = -I\mathcal{B} \int dy = -I\mathcal{B}\ell = -\frac{\mathcal{B}^2\ell^2}{\mathcal{R}} v_x < 0 \tag{9.10}$$

resists its motion. If m is the mass of the cross bar, then **Newton's second law** implies

$$-\frac{\mathcal{B}^2\ell^2}{\mathcal{R}} v_x = F_x = m a_x = m\frac{dv_x}{dt} \tag{9.11}$$

or

$$\frac{dv_x}{dt} = -\frac{\mathcal{B}^2\ell^2}{m\mathcal{R}} v_x, \tag{9.12}$$

which has solution

$$v_x[t] = v_0\, e^{-t/\tau}, \tag{9.13}$$

where

$$\tau = \frac{m\mathcal{R}}{\mathcal{B}^2\ell^2} \tag{9.14}$$

is the decay time of the exponential slowing of the cross bar.

The sliding cross bar dissipates its kinetic energy as heat in the rails. The power dissipation is

$$\frac{dE}{dt} = \mathcal{P} = I^2\mathcal{R} = \frac{\mathcal{B}^2\ell^2}{\mathcal{R}^2}v_x^2\mathcal{R} = \frac{\mathcal{B}^2\ell^2}{m\mathcal{R}}mv_0^2\, e^{-2t/\tau} = \frac{1}{\tau}mv_0^2\, e^{-2t/\tau}, \tag{9.15}$$

and so the total energy dissipated

$$E = \int_0^\infty \mathcal{P}\, dt = \frac{1}{\tau}mv_0^2 \int_0^\infty e^{-2t/\tau} dt = \frac{1}{\tau}mv_0^2 \left(\frac{e^{-2t/\tau}}{-2/\tau}\right)\Bigg|_0^\infty = \frac{1}{2}mv_0^2. \tag{9.16}$$

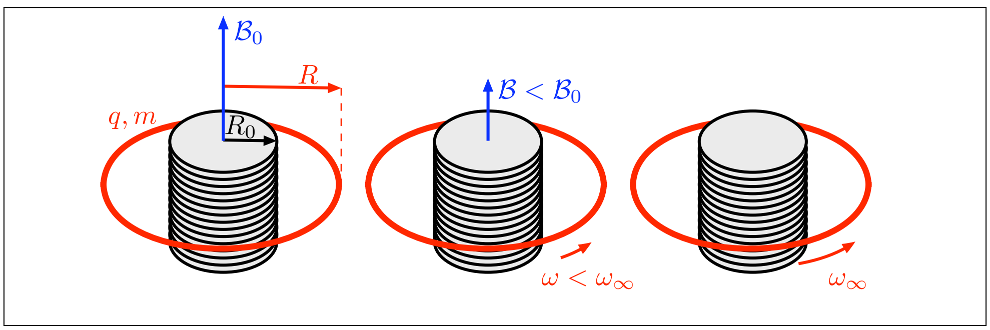

Figure 9.2: Turn off the solenoid and the coaxial charged ring begins to rotate!

Next consider a ring of radius R, charge q, and mass m coaxial with a solenoid of radius R_0 and internal magnetic field $\mathcal{B}_0$. Turn off the solenoid, and the ring begins rotating, as in Fig. 9.2, even thought magnetic field *at the ring* is always zero! This is a version of the **Feynman disk paradox**.

Qualitatively, a changing magnetic field induces a circulating electric field that forces the ring to rotate. The rotating ring of charge constitutes an electric current whose magnetic field opposes the reduction of the initial magnetic flux through the disk bounded by the ring. Quantitatively, Faraday's law implies

$$\mathcal{E}(2\pi R) = \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}} = -\frac{d}{dt}\left(\mathcal{B}(\pi R_0^2)\right) = -\pi R_0^2\frac{d\mathcal{B}}{dt}. \tag{9.17}$$

The resulting torque on the ring is

$$\vec{\tau} = \int d\vec{\tau} = \int \vec{R}\times d\vec{F} = \int \vec{R}\times dq\vec{\mathcal{E}} = \oint \vec{R}\times\lambda d\ell\vec{\mathcal{E}}, \tag{9.18}$$

where $\lambda = q/2\pi R$ is the ring's line charge density. Hence the magnitude of the torque is

$$\tau = \oint R\lambda\, d\ell\, \mathcal{E} = R\lambda\mathcal{E} \oint d\ell = R\lambda\, \mathcal{E}(2\pi R), \tag{9.19}$$

or, using the Eq. 9.17 electric circulation,

$$\tau = R\frac{q}{2\pi R}\left(-\pi R_0^2 \frac{d\mathcal{B}}{dt}\right) = -\frac{1}{2} q R_0^2 \frac{d\mathcal{B}}{dt} > 0. \tag{9.20}$$

The final angular momentum is

$$L_\infty = \int_{-\infty}^{\infty} \tau\, dt = -\frac{1}{2} q R_0^2 \int_{\mathcal{B}_0}^{0} d\mathcal{B} = \frac{1}{2\pi} q\pi R_0^2 \mathcal{B}_0 = \frac{q\Phi_{\mathcal{B}_0}}{2\pi}. \tag{9.21}$$

The rotational inertia of the ring about its center is $I_c = mR^2$, so the final angular speed of the ring is

$$\omega_\infty = \frac{L}{I_c} = \frac{1}{2}\frac{q}{m}\frac{R_0^2}{R^2}\mathcal{B}_0 \le \frac{1}{2}\frac{q}{m}\mathcal{B}_0. \tag{9.22}$$

This result is independent of how the magnetic field is extinguished, provided it does not change so fast as to introduce radiation and time delay effects. What is the source of the angular momentum? Show later that it is initially stored in crossed electric and magnetic fields inside the solenoid!

9.2 Multiply Connected Regions

Consider a box [3] with two electrodes protruding from opposite sides. Connect identical volt meters across the electrodes but on opposite sides of the box. The voltmeters register *different* voltages $\Delta\varphi_L \neq \Delta\varphi_R$! How can this be?

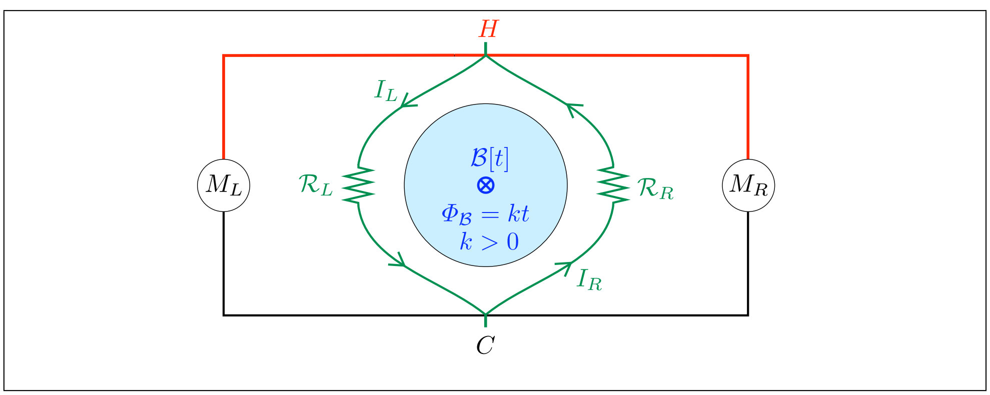

Figure 9.3: The counter-clockwise circulating electric field induced by the solenoid's changing magnetic flux creates an asymmetry in the volt meter readings.

Inside the box is a solenoid generating a time-varying magnetic flux

$$\Phi_\mathcal{B} = kt, \tag{9.23}$$

where $k > 0$, along with two resistors $\mathcal{R}_L$ and $\mathcal{R}_R$, as in Fig. 9.3. The voltages differ even if the resistors are identical, because the time-varying magnetic flux induces a counter-clockwise circulating electric field that breaks the symmetry.

Some loops in in the plane of the volt meters can shrink to points but others – those encircling the solenoid – cannot. **Topologically** this resembles a torus rather than a sphere. Be careful when deploying Faraday's law in such **multiply connected regions**. The circulation of the electric field around any closed loop depends on how the loop is wound about the solenoid. For a single clockwise winding, $\Gamma_\mathcal{E} = -\dot{\Phi}_\mathcal{B} = -k$. More generally,

$$\oint_\ell \vec{\mathcal{E}} \cdot d\vec{\ell} = \Gamma_\mathcal{E} = nk, \tag{9.24}$$

where $n = 0, \pm 1, \pm 2, \ldots$ is the loop's **winding number**, as in Fig. 9.4. The electric field is irrotational and path independent only for those loops that do not enclose the solenoid.

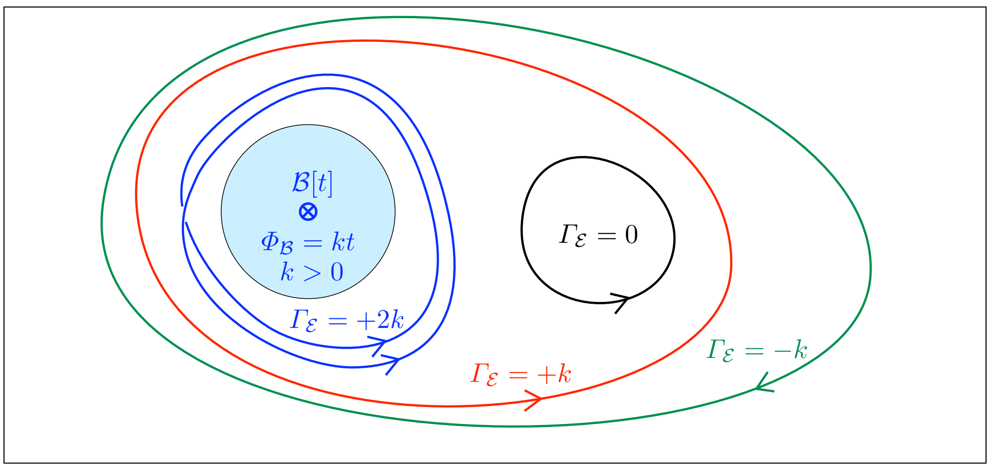

Figure 9.4: The electric field is irrotational only for those loops that *do not* enclose the solenoid.

A volt meter measures the line integral of the electric field from its "hot" (red) lead to its "cold" (black) lead,

$$\int_H^C \vec{\mathcal{E}} \cdot d\vec{\ell} = -\int_H^C \vec{\nabla}\varphi \cdot d\vec{\ell} = -\int_H^C d\varphi = \varphi_H - \varphi_C = \Delta\varphi > 0. \tag{9.25}$$

Thus, the left meter measures

$$\Delta\varphi_L = \underset{\text{via } M_L}{\int_H^C} \vec{\mathcal{E}} \cdot d\vec{\ell}, \tag{9.26}$$

and the right meter measures

$$\Delta\varphi_R = \underset{\text{via } M_R}{\int_H^C} \vec{\mathcal{E}} \cdot d\vec{\ell}. \tag{9.27}$$

The difference in these measurements is

$$\begin{aligned}\Delta\varphi_L - \Delta\varphi_R &= \int_{\substack{H\\ \text{via } M_L}}^{C} \vec{\mathcal{E}}\cdot d\vec{\ell} - \int_{\substack{H\\ \text{via } M_R}}^{C} \vec{\mathcal{E}}\cdot d\vec{\ell} \\ &= \int_{\substack{H\\ \text{via } M_L}}^{C} \vec{\mathcal{E}}\cdot d\vec{\ell} + \int_{\substack{C\\ \text{via } M_R}}^{H} \vec{\mathcal{E}}\cdot d\vec{\ell} \\ &= \oint_{\text{ccw}} \vec{\mathcal{E}}\cdot d\vec{\ell} \\ &= k, \end{aligned} \tag{9.28}$$

which is nonzero unless $k = 0$ and the solenoid stops varying its magnetic field.

For Ohmic devices of electrical resistance $\mathcal{R}$, the current $I = \Delta\varphi/\mathcal{R}$. Thus, the current through the left resistor is

$$I_L = \frac{1}{\mathcal{R}_L}\int_{\substack{H\\ \text{via } \mathcal{R}_L}}^{C} \vec{\mathcal{E}}\cdot d\vec{\ell} = \frac{1}{\mathcal{R}_L}\int_{\substack{H\\ \text{via } \mathcal{M}_L}}^{C} \vec{\mathcal{E}}\cdot d\vec{\ell} = +\frac{\Delta\varphi_L}{\mathcal{R}_L}, \tag{9.29}$$

and the current through the right resistor is

$$I_R = \frac{1}{\mathcal{R}_R}\int_{\substack{C\\ \text{via } \mathcal{R}_R}}^{H} \vec{\mathcal{E}}\cdot d\vec{\ell} = \frac{1}{\mathcal{R}_R}\int_{\substack{C\\ \text{via } \mathcal{M}_R}}^{H} \vec{\mathcal{E}}\cdot d\vec{\ell} = -\frac{1}{\mathcal{R}_R}\int_{\substack{H\\ \text{via } \mathcal{M}_R}}^{C} \vec{\mathcal{E}}\cdot d\vec{\ell} = -\frac{\Delta\varphi_R}{\mathcal{R}_R}. \tag{9.30}$$

Since the volt meters draw negligible current by design, $I_L = I_R$, and so

$$\frac{\Delta\varphi_L}{\mathcal{R}_L} = -\frac{\Delta\varphi_R}{\mathcal{R}_R}. \tag{9.31}$$

or

$$\mathcal{R}_R\Delta\varphi_L + \mathcal{R}_L\Delta\varphi_R = 0. \tag{9.32}$$

Solve Eq. 9.28 and Eq. 9.32 simultaneously to find

$$\Delta\varphi_L = +\frac{\mathcal{R}_L}{\mathcal{R}_L + \mathcal{R}_R}k \tag{9.33}$$

and

$$\Delta\varphi_R = -\frac{\mathcal{R}_R}{\mathcal{R}_L + \mathcal{R}_R}k \tag{9.34}$$

and hence

$$I_L = I_R = \frac{1}{\mathcal{R}_L + \mathcal{R}_R}k. \tag{9.35}$$

The linearly increasing magnetic flux of Eq. 9.23 is impractical. Instead drive the solenoid current with a triangular-wave generator. Replace the volt meters by two channels of a multiple-trace oscilloscope. Use a third channel to monitor the current through a small resistor in series with the solenoid windings. Enjoy experimenting with different windings of the leads.

9.3 Ampère-Maxwell's Law

Consider a steady current I between a source charge $Q[t]$ and a sink charge $-Q[t]$. Since the source charge is decreasing, $\dot{Q} < 0$, the current $I = -\dot{Q} > 0$. Assume that the source and sink are symmetrically placed on the z-axis at a height h above and below the xy-plane, as in Fig. 9.5.

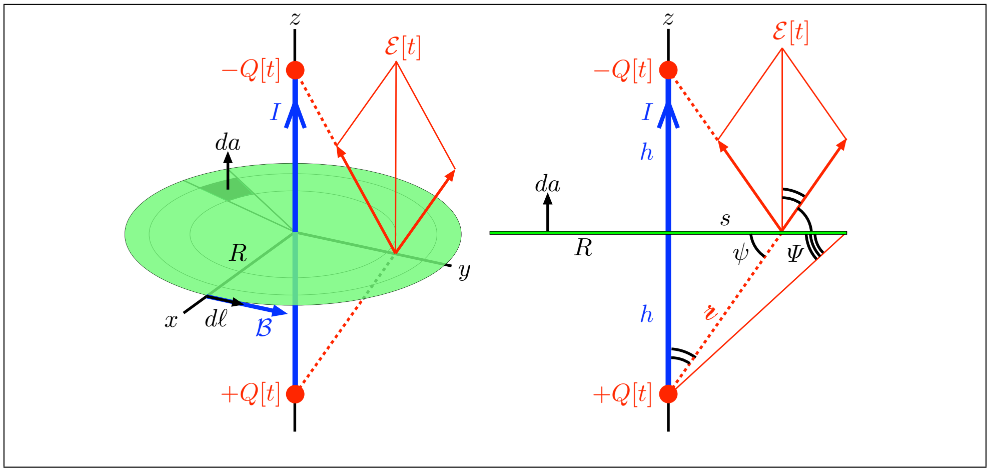

Figure 9.5: A steady current I moves current from a source charge $Q[t]$ to a sink charge $-Q[t]$. As the charges decrease in magnitude, the electric flux $\Phi_{\mathcal{E}}$ through the disk decreases.

Compute the corresponding magnetic field $\vec{\mathcal{B}}$ by evaluating Ampère-Maxwell's law, $\Gamma_{\mathcal{B}} = +\dot{\Phi}_{\mathcal{E}} + I$, about the disk and the circle bounding it, which bisect the source and sink. The electrical field at a radius s on the disk is

$$\vec{\mathcal{E}} = 2\frac{Q}{4\pi \mathscr{r}^2}\cos\left[\frac{\pi}{2}-\psi\right]\hat{z} = 2\frac{Q}{4\pi \mathscr{r}^2}\frac{h}{\mathscr{r}}\hat{z} = \frac{Q}{2\pi}\frac{h}{(s^2+h^2)^{3/2}}\hat{z}. \tag{9.36}$$

The electric flux through the disk is

$$\begin{aligned}
\Phi_{\mathcal{E}} &= \iint_a \vec{\mathcal{E}}\cdot d\vec{a} \\
&= \int_{s=0}^{R}\int_{\phi=0}^{2\pi}\frac{Q}{2\pi}\frac{h}{(s^2+h^2)^{3/2}}(sd\phi)ds \\
&= Qh\int_0^R \frac{sds}{(s^2+h^2)^{3/2}} \\
&= -Qh\frac{1}{\sqrt{s^2+h^2}}\Bigg|_0^R \\
&= Q\left(1-\frac{h}{\sqrt{R^2+h^2}}\right).
\end{aligned} \tag{9.37}$$

Differentiate with respect to time to find

$$\dot{\Phi}_{\mathcal{E}} = \dot{Q}\left(1 - \frac{h}{\sqrt{R^2+h^2}}\right) = -I\left(1 - \frac{h}{\sqrt{R^2+h^2}}\right). \tag{9.38}$$

The circulation of the magnetic field around the circle bounding the disk is

$$\Gamma_{\mathcal{B}} = \oint_{\ell=\partial a} \vec{\mathcal{B}} \cdot d\vec{\ell} = B(2\pi R). \tag{9.39}$$

Hence, the Ampère-Maxwell law demands

$$B(2\pi R) = -I\left(1 - \frac{h}{\sqrt{R^2+h^2}}\right) + I. \tag{9.40}$$

or

$$B = \frac{I}{2\pi R}\frac{h}{\sqrt{R^2+h^2}}, \tag{9.41}$$

which agrees with the Eq. 8.18 magnetic field of a short, straight current obtained using Biot-Savart's law.

9.4 Dynamic Potentials

The Maxwell's equations

$$\vec{\nabla} \cdot \vec{\mathcal{E}} = \rho, \tag{9.42a}$$

$$\vec{\nabla} \cdot \vec{\mathcal{B}} = 0, \tag{9.42b}$$

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\partial_t \vec{\mathcal{B}}, \tag{9.42c}$$

$$\vec{\nabla} \times \vec{\mathcal{B}} = +\partial_t \vec{\mathcal{E}} + \vec{J}, \tag{9.42d}$$

are $1+1+3+3=8$ differential equations (plus boundary conditions) in $3+3=6$ unknowns $\{\vec{\mathcal{E}}, \vec{\mathcal{B}}\}$ given $1+3=4$ sources $\{\rho, \vec{J}\}$. Achieve a more compact representation by re-expressing Maxwell's equation in terms of potentials, which need to be extended for electrodynamics.

To satisfy the Eq. 9.42b magnetic Gauss's law, take

$$\vec{\mathcal{B}} = \vec{\nabla} \times \vec{\mathcal{A}}, \tag{9.43}$$

as the divergence of any curl vanishes. Thus, in dynamics as in statics the magnetic field is the derivative of the magnetic potential. The Eq. 9.42c Faraday's law then requires

$$\vec{\nabla} \times \vec{\mathcal{E}} = -\partial_t \vec{\mathcal{B}} = -\partial_t \vec{\nabla} \times \vec{\mathcal{A}} = -\vec{\nabla} \times \partial_t \vec{\mathcal{A}} \tag{9.44}$$

or

$$\vec{\nabla} \times \left(\vec{\mathcal{E}} + \partial_t \vec{\mathcal{A}}\right) = \vec{0}. \tag{9.45}$$

To satisfy this, take

$$\vec{\mathcal{E}} + \partial_t \vec{\mathcal{A}} = -\vec{\nabla}\varphi, \tag{9.46}$$

as the curl of any gradient vanishes. Thus, in dynamics the electric field is the derivative of both the electric *and* magnetic potentials,

$$\vec{\mathcal{E}} = -\vec{\nabla}\varphi - \partial_t \vec{\mathcal{A}}. \tag{9.47}$$

The Eq. 9.42a Gauss's law then requires

$$-\nabla^2\varphi - \vec{\nabla}\cdot\partial_t\vec{\mathcal{A}} = \rho. \tag{9.48}$$

Finally, the Eq. 9.42d Ampère-Maxwell's equation then requires

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathcal{A}}\right) = +\partial_t\left(-\vec{\nabla}\varphi - \partial_t\vec{\mathcal{A}}\right) + \vec{J} \tag{9.49}$$

or

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathcal{A}}\right) - \nabla^2\vec{\mathcal{A}} = -\vec{\nabla}\partial_t\varphi - \partial_t^2\vec{\mathcal{A}} + \vec{J} \tag{9.50}$$

Consolidate Eq. 9.48 and Eq. 9.50 as Maxwell's equations in potential form,

$$\nabla^2\varphi + \partial_t\vec{\nabla}\cdot\vec{\mathcal{A}} = -\rho, \tag{9.51a}$$

$$\left(\nabla^2\vec{\mathcal{A}} - \partial_t^2\vec{\mathcal{A}}\right) - \vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathcal{A}} + \partial_t\varphi\right) = -\vec{J}, \tag{9.51b}$$

which are $1+3=4$ differential equations (plus boundary conditions) in $1+3=4$ unknowns $\{\varphi, \vec{\mathcal{A}}\}$ given $1+3=4$ sources $\{\rho, \vec{J}\}$. Unfortunately, they are not yet ready for T-shirt display. Instead, note that our definitions do not completely define our potentials. Quantify the freedom in our potentials, and exploit it to simplify these equations.

9.5 Gauge Transformations

Suppose $\{\varphi, \vec{\mathcal{A}}\}$ and $\{\varphi', \vec{\mathcal{A}}'\}$ correspond to the same $\{\vec{\mathcal{E}}, \vec{\mathcal{B}}\}$ fields. By how much can they differ? Let

$$\varphi' = \varphi + \alpha, \tag{9.52a}$$

$$\vec{\mathcal{A}}' = \vec{\mathcal{A}} + \vec{\beta}, \tag{9.52b}$$

where α and $\vec{\beta}$ are scalar and vector fields, which can change, but not independently. To show this, first differentiate to find the corresponding magnetic fields,

$$\vec{\nabla}\times\vec{\mathcal{A}}' = \vec{\nabla}\times\vec{\mathcal{A}} + \vec{\nabla}\times\vec{\beta}, \tag{9.53}$$

or

$$\vec{\mathcal{B}}' = \vec{\mathcal{B}} + \vec{\nabla}\times\vec{\beta}. \tag{9.54}$$

For $\vec{\mathcal{B}}' = \vec{\mathcal{B}}$, it is sufficient that

$$\vec{\beta} = \vec{\nabla}\lambda, \tag{9.55}$$

where λ is a scalar field. Similarly, differentiate to find the corresponding electric fields,

$$-\vec{\nabla}\varphi' - \partial_t \vec{\mathcal{A}}' = -\vec{\nabla}\varphi - \vec{\nabla}\alpha - \partial_t \vec{\mathcal{A}} - \partial_t \vec{\beta}, \tag{9.56}$$

or

$$\vec{\mathcal{E}}' = \vec{\mathcal{E}} - \vec{\nabla}\left(\alpha + \partial_t \lambda\right). \tag{9.57}$$

For $\vec{\mathcal{E}}' = \vec{\mathcal{E}}$, it is sufficient that

$$\alpha + \partial_t \lambda = 0 \tag{9.58}$$

or

$$\alpha = -\partial_t \lambda. \tag{9.59}$$

Therefore,

$$\varphi' = \varphi - \partial_t \lambda, \tag{9.60a}$$

$$\vec{\mathcal{A}}' = \vec{\mathcal{A}} + \vec{\nabla}\lambda. \tag{9.60b}$$

Subtracting a scalar field $\lambda[t, \vec{r}]$ from the electric potential φ while simultaneously adding the gradient of the same scalar field to the magnetic potential $\vec{\mathcal{A}}$ does not change the fields. Refer to this as **gauge freedom**, because the gauges of old-fashioned analog volt meters could be freely reset without affecting currents and charges.

9.6 Coulomb Gauge

The **Coulomb gauge** is convenient for statics. Choose the scalar field $\lambda[t, \vec{r}]$ such that

$$\vec{\nabla} \cdot \vec{\mathcal{A}} = 0. \tag{9.61}$$

The Eq. 9.51a Maxwell's equations in potential form reduce to

$$\nabla^2 \varphi = -\rho, \tag{9.62a}$$

$$\left(\nabla^2 \vec{\mathcal{A}} - \partial_t^2 \vec{\mathcal{A}}\right) - \vec{\nabla}\partial_t \varphi = -\vec{J}. \tag{9.62b}$$

If in addition the fields are static, then Maxwell's equations further reduce to

$$\nabla^2 \varphi = -\rho, \tag{9.63a}$$

$$\nabla^2 \vec{\mathcal{A}} = -\vec{J}, \tag{9.63b}$$

which are identical, explicitly integrable Poisson's equations.

Is the Coulomb gauge choice always possible? Suppose

$$\vec{\nabla} \cdot \vec{\mathcal{A}} = \Theta \neq 0. \tag{9.64}$$

Use Eq. 9.60 to change gauge such that

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \vec{\mathcal{A}}' \\ &= \vec{\nabla} \cdot \vec{\mathcal{A}} + \vec{\nabla} \cdot \vec{\nabla}\lambda \\ &= \Theta + \nabla^2 \lambda, \end{aligned} \tag{9.65}$$

or

$$\nabla^2 \lambda = -\Theta, \tag{9.66}$$

which is an explicitly integrable Poisson's equation.

9.7 Lorentz Gauge

The **Lorentz gauge** is convenient for dynamics. Choose the scalar field $\lambda[t, \vec{r}]$ such that

$$\vec{\nabla} \cdot \vec{\mathcal{A}} + \partial_t \varphi = 0. \tag{9.67}$$

The Eq. 9.51a Maxwell's equations in potential form reduce to

$$\nabla^2 \varphi - \partial_t^2 \varphi = -\rho, \tag{9.68a}$$

$$\nabla^2 \vec{\mathcal{A}} - \partial_t^2 \vec{\mathcal{A}} = -\vec{J}, \tag{9.68b}$$

or

$$\Box^2 \varphi = -\rho, \tag{9.69a}$$

$$\Box^2 \vec{\mathcal{A}} = -\vec{J}, \tag{9.69b}$$

which are identical spacetime Poisson's equations! This form is ready for the T-shirt.

Is the Lorentz gauge choice always possible? Suppose

$$\vec{\nabla} \cdot \vec{\mathcal{A}} + \partial_t \varphi = \Theta \neq 0. \tag{9.70}$$

Use Eq. 9.60 to change gauge such that

$$\begin{aligned} 0 &= \vec{\nabla} \cdot \vec{\mathcal{A}}' + \partial_t \varphi' \\ &= \vec{\nabla} \cdot \vec{\mathcal{A}} + \vec{\nabla} \cdot \vec{\nabla} \lambda + \partial_t \varphi - \partial_t \partial_t \lambda \\ &= \left(\vec{\nabla} \cdot \vec{\mathcal{A}} + \partial_t \varphi \right) + \left(\nabla^2 \lambda - \partial_t^2 \lambda \right) \\ &= \Theta + \Box^2 \lambda, \end{aligned} \tag{9.71}$$

or

$$\Box^2 \lambda = -\Theta, \tag{9.72}$$

which is the spacetime Poisson's equation. Show later that this is explicitly solvable. Wear that T-shirt now.

9.8 Field Momentum

In terms of the electromagnetic potentials, the force on a charge q moving with velocity $\vec{v}$ is

$$\vec{F} = \frac{d\vec{p}}{dt} = q \left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}} \right) = q \left(-\nabla \varphi - \partial_t \vec{\mathcal{A}} + \vec{v} \times \left(\vec{\nabla} \times \vec{\mathcal{A}} \right) \right). \tag{9.73}$$

The velocity $\vec{v}\,[t]$ is a function of time and not space. Hence,

$$\vec{v} \times \left(\vec{\nabla} \times \vec{\mathcal{A}}\right) = \vec{\nabla}\left(\vec{v} \cdot \vec{\mathcal{A}}\right) - \left(\vec{v} \cdot \vec{\nabla}\right)\vec{\mathcal{A}}, \tag{9.74}$$

and so

$$\frac{d\vec{p}}{dt} = -q\left(\partial_t \vec{\mathcal{A}} + \left(\vec{v} \cdot \vec{\nabla}\right)\vec{\mathcal{A}} + \vec{\nabla}\left(\varphi - \vec{v} \cdot \vec{\mathcal{A}}\right)\right). \tag{9.75}$$

The vector field $\vec{\mathcal{A}}\left[t, x[t], y[t], z[t]\right] = \vec{\mathcal{A}}\left[t, \vec{r}\,[t]\right]$ at the charge q can change because the field varies in time, or because it varies in space, or both. Hence, the **total** or **convective derivative**

$$\frac{d\vec{\mathcal{A}}}{dt} = \frac{\partial \vec{\mathcal{A}}}{\partial t} + \frac{\partial x}{\partial t}\frac{\partial \vec{\mathcal{A}}}{\partial x} + \frac{\partial y}{\partial t}\frac{\partial \vec{\mathcal{A}}}{\partial y} + \frac{\partial z}{\partial t}\frac{\partial \vec{\mathcal{A}}}{\partial z} = \frac{\partial \vec{\mathcal{A}}}{\partial t} + \vec{v} \cdot \vec{\nabla}\,\vec{\mathcal{A}}, \tag{9.76}$$

and so

$$\frac{d\vec{p}}{dt} = -q\left(\frac{d\vec{\mathcal{A}}}{dt} + \vec{\nabla}\left(\varphi - \vec{v} \cdot \vec{\mathcal{A}}\right)\right), \tag{9.77}$$

or

$$\frac{d}{dt}\left(\vec{p} + q\vec{\mathcal{A}}\right) = -q\left(\vec{\nabla}\left(\varphi - \vec{v} \cdot \vec{\mathcal{A}}\right)\right). \tag{9.78}$$

Express this as

$$\frac{d\vec{p}_{\text{tot}}}{dt} = -\vec{\nabla}U, \tag{9.79}$$

where the **potential energy**

$$U = q\left(\varphi - \vec{v} \cdot \vec{\mathcal{A}}\right) \tag{9.80}$$

and the **total** or **canonical momentum**

$$\vec{p}_{\text{tot}} = \vec{p} + q\vec{\mathcal{A}}. \tag{9.81}$$

Interpret $\vec{p}$ as the **mechanical momentum** and $q\vec{\mathcal{A}}$ as the **field momentum**. Thus, the magnetic potential $\vec{\mathcal{A}}$ (in the appropriate gauge) is the momentum per unit charge, just as in electrostatics the electric potential φ is the energy per unit charge.

9.9 Energy & Momentum Density

A charge density ρ moving at velocity $\vec{v}$ is a current density $\vec{J} = \rho\vec{v}$ and is accompanied by a radial electric field $\vec{\mathcal{E}}$ and a circulating magnetic field $\vec{\mathcal{B}}$. What is the rate of work done on the sources by the fields? The force is

$$\vec{F} = \int dq\left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right). \tag{9.82}$$

The work dW through a distance $d\vec{\ell} = \vec{v}\,dt$ is

$$\begin{aligned} dW &= \vec{F} \cdot d\vec{\ell} \\ &= \int dq \left(\vec{\mathcal{E}} + \vec{v} \times \vec{\mathcal{B}}\right) \cdot \vec{v}\,dt \\ &= \iiint_V \rho\, dV \vec{\mathcal{E}} \cdot \vec{v}\,dt + 0 \\ &= dt \iiint_V \vec{\mathcal{E}} \cdot \rho\vec{v}\,dV, \end{aligned} \tag{9.83}$$

where the deflecting magnetic field makes no contribution as it is always orthogonal to the displacement. Divide by the time step dt to find

$$\frac{dW}{dt} = \iiint_V \vec{\mathcal{E}} \cdot \vec{J}\,dV. \tag{9.84}$$

Use Ampère-Maxwell's law $\vec{\nabla} \times \vec{\mathcal{B}} = +\partial_t \vec{\mathcal{E}} + \vec{J}$ to eliminate the source from the power density

$$\vec{\mathcal{E}} \cdot \vec{J} = \vec{\mathcal{E}} \cdot \vec{\nabla} \times \vec{\mathcal{B}} - \vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{E}} = \vec{\mathcal{E}} \cdot \vec{\nabla} \times \vec{\mathcal{B}} - \vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{E}}. \tag{9.85}$$

Use the Eq. A-9 product rule and Faraday's law $\vec{\nabla} \times \vec{\mathcal{E}} = -\partial_t \vec{\mathcal{B}}$ to put the fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ on equal footing,

$$\begin{aligned} \vec{\mathcal{E}} \cdot \vec{J} &= \vec{\mathcal{B}} \cdot \vec{\nabla} \times \vec{\mathcal{E}} - \vec{\nabla} \cdot \left(\vec{\mathcal{E}} \times \vec{\mathcal{B}}\right) - \vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{E}} \\ &= \vec{\mathcal{B}} \cdot \left(-\partial_t \vec{\mathcal{B}}\right) - \vec{\nabla} \cdot \left(\vec{\mathcal{E}} \times \vec{\mathcal{B}}\right) - \vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{E}} \\ &= -\vec{\mathcal{B}} \cdot \partial_t \vec{\mathcal{B}} - \vec{\nabla} \cdot \left(\vec{\mathcal{E}} \times \vec{\mathcal{B}}\right) - \vec{\mathcal{E}} \cdot \partial_t \vec{\mathcal{E}} \\ &= -\partial_t \frac{\mathcal{B}^2 + \mathcal{E}^2}{2} - \vec{\nabla} \cdot \left(\vec{\mathcal{E}} \times \vec{\mathcal{B}}\right) \\ &= -\partial_t \epsilon - \vec{\nabla} \cdot \vec{S}, \end{aligned} \tag{9.86}$$

where $\epsilon = (\mathcal{E}^2 + \mathcal{B}^2)/2$ and $\vec{S} = \vec{\mathcal{E}} \times \vec{\mathcal{B}}$. Finally, use the Eq. 9.86 power density to express the Eq. 9.84 rate of work as

$$\begin{aligned} \frac{dW}{dt} &= \iiint_V \left(-\partial_t \epsilon - \vec{\nabla} \cdot \vec{S}\right) dV \\ &= -\iiint_V \partial_t \epsilon\, dV - \iiint_V \vec{\nabla} \cdot \vec{S}\, dV \\ &= -\frac{d}{dt} \iiint_V \epsilon\, dV - \oiint_{a=\partial V} \vec{S} \cdot d\vec{a}, \end{aligned} \tag{9.87}$$

which is **Poynting's theorem**,

$$\frac{dW}{dt} = -\frac{dE}{dt} - \oiint_{a=\partial V} \vec{S} \cdot d\vec{a}, \tag{9.88}$$

where the **energy density**

$$\frac{dE}{dV} = \epsilon = \frac{\mathcal{E}^2 + \mathcal{B}^2}{2}, \tag{9.89}$$

and the **energy flux density** is the **momentum density**

$$\frac{d\vec{p}}{dV} = \vec{S} = \vec{\mathcal{E}} \times \vec{\mathcal{B}}, \tag{9.90}$$

because

$$S = \frac{d\mathcal{P}}{da_\perp} = \frac{dE}{da_\perp dt} = \frac{dE}{da_\perp d\ell_{||}} \frac{d\ell_{||}}{dt} = \frac{dE}{dV} c = \frac{dp}{dV} c^2 = \frac{dp}{dV}, \tag{9.91}$$

where $E = \sqrt{p^2 + m^2} = p$ for the massless photons that are the quanta of the electromagnetic field. ($\vec{S}$ "poynts" in the direction of momentum density and energy flux.) The Eq. 9.89 electromagnetic energy density generalizes the Eq. 7.84 electric energy density. The Eq. 9.88 Poynting theorem implies that the work done in a unit time on the sources by the electromagnetic force is the decrease in the energy stored in the field less the energy that flows away.

As an example, consider a variation of the Sec. 9.1 Feynman disk paradox. A charge q of mass m is a distance s from the axis of a solenoid of radius R and internal magnetic field $\vec{\mathcal{B}}$. Turn off the solenoid, and the charge receives an impulse, as in the left side of Fig. 9.6. Qualitatively, a changing magnetic field induces a circulating electric field that forces the charge to move.

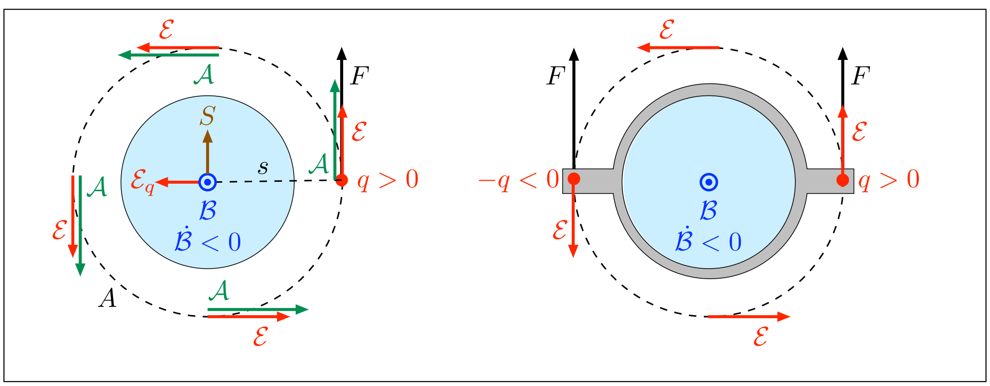

Figure 9.6: Suddenly turn off the solenoid, and the charges receive impulses from momentum stored in the fields. Variation on the right is a model for a reactionless drive spacecraft.

Quantitively, Faraday's law implies

$$\mathcal{E}(2\pi s) = \Gamma_{\mathcal{E}} = -\dot{\Phi}_{\mathcal{B}}, \tag{9.92}$$

and the circulating electric field has magnitude

$$\mathcal{E} = -\frac{1}{2\pi s}\dot{\Phi}_{\mathcal{B}}. \tag{9.93}$$

The charge experiences a force

$$\frac{dp}{dt} = F = q\mathcal{E} = -\frac{q}{2\pi s}\frac{d\Phi_\mathcal{B}}{dt}. \tag{9.94}$$

Assume the charge doesn't move far in the time the magnetic field decays, and integrate

$$\int_0^{\Delta p} dp = -\frac{q}{2\pi s}\int_{\Phi_\mathcal{B}}^0 d\Phi_\mathcal{B} \tag{9.95}$$

to find the impulse

$$\Delta p = \frac{q}{2\pi s}\Phi_\mathcal{B} \tag{9.96}$$

and velocity change

$$\Delta v = \frac{q}{m}\frac{\Phi_\mathcal{B}}{2\pi s}. \tag{9.97}$$

What is the source of the momentum? Initially, momentum is distributed through the crossed $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ inside the solenoid. Alternately, **potential momentum** exists *at the charge* in the $\vec{\mathcal{A}}$ field.

Using the Eq. 9.93 circulating electric field in the absence of the charge, $\vec{\mathcal{E}} = -\vec{\nabla}\varphi - \partial_t\vec{\mathcal{A}}$, implies

$$-\frac{\dot{\Phi}_\mathcal{B}}{2\pi s}\hat{\phi} = -\vec{0} - \partial_t\vec{\mathcal{A}}, \tag{9.98}$$

which integrates to

$$\vec{\mathcal{A}} = \frac{\Phi_\mathcal{B}}{2\pi s}\hat{\phi}. \tag{9.99}$$

Check this by noting that for $s > R$, the magnetic flux

$$\Phi_\mathcal{B} = \iint_a \vec{\mathcal{B}}\cdot d\vec{a} = \iint_a \vec{\nabla}\times\vec{\mathcal{A}}\cdot d\vec{a} = \oint_{\ell=\partial a}\vec{\mathcal{A}}\cdot d\vec{\ell} = \mathcal{A}_\phi(2\pi s). \tag{9.100}$$

Hence, the field momentum

$$q\mathcal{A} = \frac{q}{2\pi s}\Phi_\mathcal{B} = \Delta p. \tag{9.101}$$

In another variation, suppose two charges of opposite sign are attached symmetrically across the diameter of the solenoid, as in the right side of Fig. 9.6. Change the magnetic flux through the solenoid by $\Delta\Phi_\mathcal{B}$, and the *entire* apparatus of mass M changes velocity by (a typically very small)

$$\Delta v = 2\frac{q}{M}\frac{\Phi_\mathcal{B}}{2\pi s}. \tag{9.102}$$

Rotate the charges (slowly to avoid radiation effects) and vary the magnetic flux to move freely through space with a **reactionless drive**, shuffling momentum between the spacecraft and its electromagnetic fields!

Problems

1. In the plane of a square conducting loop of side ℓ and electrical resistance $\mathcal{R}$, place a long straight current I parallel to the near side of the loop and a distance s away.
   (a) Find the magnetic flux $\Phi_\mathcal{B}$ through the loop due to the current.
   (b) Pull the loop perpendicular from the wire at a speed v. What is the magnitude and direction of the electrical current in the near side of the loop?
   (c) Pull the loop parallel to the wire at a speed v. What is the loop current in this case?

2. Suspend vertically a rectangular conducting loop of width ℓ, mass m, and electrical resistance $\mathcal{R}$, with its upper portion penetrated perpendicularly by a horizontal magnetic field $\vec{\mathcal{B}}$.
   (a) Release the loop, and compute its speed as a function of time as it falls out of the magnetic field.
   (b) What is the **terminal speed** of the loop, assuming it doesn't completely exit the magnetic field?

3. **Thomson's dipole** consists of an electrical charge $Q_\mathcal{E} = e$ and a magnetic charge $Q_\mathcal{B} = b$ separated by a distance δ.
   (a) Imagine balancing the dipole on one end and then gently pushing it over. Qualitatively but carefully explain what happens and why.
   (b) Find the total angular momentum stored in the dipole's electromagnetic fields, both magnitude and direction. (Hint: Place e at the origin and b a distance δ along the z-axis, and integrate the angular momentum density $\vec{r} \times d\vec{p}/dV$ over all space.)
   (c) Quantum mechanically, the dipole's angular momentum is quantized to $L = n\hbar/2$, where $n = 1, 2, 3, \ldots$. What does this imply for the range of possible electric and magnetic charges? (This is a famous argument due to Dirac.)

Chapter 10

Electromagnetic Radiation

Accelerate a charge and ripples in its fields propagate outward at light speed carrying energy and momentum. Discover the relevant electric and magnetic potentials and then differentiate to find the electric and magnetic fields. First examine the propagation of electromagnetic waves.

10.1 Electromagnetic Waves

In a vacuum, where charges and currents are absent, Maxwell's equations

$$\begin{aligned} \vec{\nabla}\cdot\vec{\mathcal{E}} &= 0, \qquad \vec{\nabla}\times\vec{\mathcal{E}} = -\partial_t\vec{\mathcal{B}}, \\ \vec{\nabla}\cdot\vec{\mathcal{B}} &= 0, \qquad \vec{\nabla}\times\vec{\mathcal{B}} = +\partial_t\vec{\mathcal{E}}, \end{aligned} \tag{10.1}$$

are **coupled**, first-order partial differential equations for the electric and magnetic fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$. **Decouple** them by curling the curl equations to get

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathcal{E}}\right) = \vec{\nabla}\times\left(-\partial_t\vec{\mathcal{B}}\right), \tag{10.2}$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathcal{E}}\right) - \nabla^2\vec{\mathcal{E}} = -\partial_t\left(\vec{\nabla}\times\vec{\mathcal{B}}\right), \tag{10.3}$$

$$\vec{\nabla}\vec{0} - \nabla^2\vec{\mathcal{E}} = -\partial_t\left(+\partial_t\vec{\mathcal{E}}\right), \tag{10.4}$$

$$\nabla^2\vec{\mathcal{E}} = \partial_t^2\vec{\mathcal{E}}, \tag{10.5}$$

and, symmetrically,

$$\vec{\nabla}\times\left(\vec{\nabla}\times\vec{\mathcal{B}}\right) = \vec{\nabla}\times\left(+\partial_t\vec{\mathcal{E}}\right), \tag{10.6}$$

$$\vec{\nabla}\left(\vec{\nabla}\cdot\vec{\mathcal{B}}\right) - \nabla^2\vec{\mathcal{B}} = +\partial_t\left(\vec{\nabla}\times\vec{\mathcal{E}}\right), \tag{10.7}$$

$$\vec{\nabla}\vec{0} - \nabla^2\vec{\mathcal{B}} = +\partial_t\left(-\partial_t\vec{\mathcal{B}}\right), \tag{10.8}$$

$$\nabla^2\vec{\mathcal{B}} = \partial_t^2\vec{\mathcal{B}}. \tag{10.9}$$

In summary, the spacetime Poisson's equations

$$(\nabla^2 - \partial_t^2)\vec{\mathcal{E}} = \Box^2\vec{\mathcal{E}} = \vec{0},$$
$$(\nabla^2 - \partial_t^2)\vec{\mathcal{B}} = \Box^2\vec{\mathcal{B}} = \vec{0}, \tag{10.10}$$

are uncoupled, second-order partial differential equations for the electric and magnetic fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$. (The higher-order is the price of decoupling.) These are simple enough to solve by guessing.

Seek sinusoidal solutions

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)},$$
$$\vec{\mathcal{B}} = \vec{\mathcal{B}}_0 e^{i(\vec{k}\cdot\vec{r}-\omega t)}, \tag{10.11}$$

where the spatial frequency $k = 2\pi/\lambda$, the temporal frequency $\omega = 2\pi/T$, and the speed $v = \lambda/T = \omega/k$. Describe the physical waves by the real parts of the complex $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$.

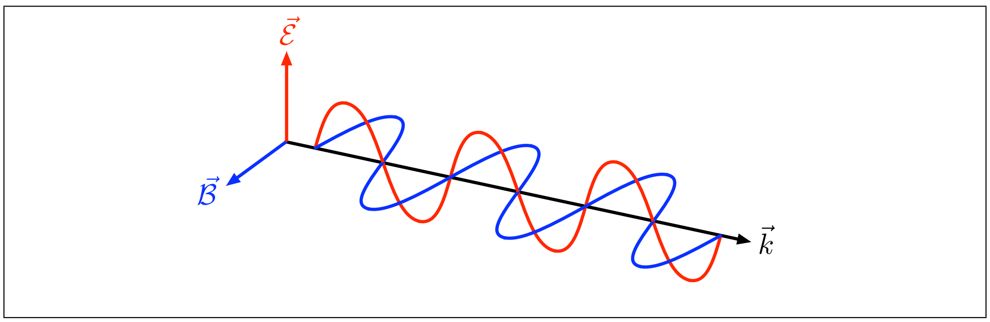

Figure 10.1: Maxwell's equations support the propagation of transverse electromagnetic waves in a vacuum.

For exponentials, differentiating reduces to multiplication; for example, if $f = f_0 e^{\alpha x}$, then $\partial_x f = \alpha f$. Hence, the Eq. 10.10 spacetime Poisson's equations become

$$(k^2 - \omega^2)\vec{\mathcal{E}} = \vec{0},$$
$$(k^2 - \omega^2)\vec{\mathcal{B}} = \vec{0}, \tag{10.12}$$

which imply $k = \omega$ and $v = \omega/k = 1 = c$. The Eq. 10.1 sourceless Maxwell's equations become

$$\vec{k}\cdot\vec{\mathcal{E}} = 0, \qquad \vec{k}\times\vec{\mathcal{E}} = +\omega\vec{\mathcal{B}},$$
$$\vec{k}\cdot\vec{\mathcal{B}} = 0, \qquad \vec{k}\times\vec{\mathcal{B}} = -\omega\vec{\mathcal{E}}. \tag{10.13}$$

Divide the cross product equations by $k = \omega$ to obtain

$$\hat{k}\times\vec{\mathcal{E}} = +\vec{\mathcal{B}},$$
$$\hat{k}\times\vec{\mathcal{B}} = -\vec{\mathcal{E}}. \tag{10.14}$$

The square of the magnetic field

$$\mathcal{B}^2 = \vec{\mathcal{B}} \cdot \vec{\mathcal{B}} = \vec{\mathcal{B}} \cdot \hat{k} \times \vec{\mathcal{E}} = \vec{\mathcal{E}} \cdot \vec{\mathcal{B}} \times \hat{k} = \vec{\mathcal{E}} \cdot \vec{\mathcal{E}} = \mathcal{E}^2, \tag{10.15}$$

so $\mathcal{B} = \mathcal{E}$. The scalar product of the electric and magnetic field

$$\vec{\mathcal{E}} \cdot \vec{\mathcal{B}} = \vec{\mathcal{E}} \cdot \hat{k} \times \vec{\mathcal{E}} = 0, \tag{10.16}$$

and the vector product

$$\vec{\mathcal{E}} \times \vec{\mathcal{B}} = \vec{\mathcal{E}} \times \left(\hat{k} \times \vec{\mathcal{E}}\right) = \hat{k}\mathcal{E}^2 - \left(\hat{k} \cdot \vec{\mathcal{E}}\right) \vec{\mathcal{E}} = \hat{k}\mathcal{E}^2. \tag{10.17}$$

Divide by $\mathcal{E}\mathcal{B} = \mathcal{E}^2$ to obtain

$$\hat{\mathcal{E}} \times \hat{\mathcal{B}} = \hat{k}. \tag{10.18}$$

Together, these results define **transverse electromagnetic waves** that propagate at light speed with electric and magnetic fields perpendicular to each other and to the direction of propagation $\hat{k}$, as in Fig. 10.1.

10.2 Retarded Potentials

Recall that if the fields are the derivative of the potentials,

$$\begin{aligned}\vec{\mathcal{E}} &= -\vec{\nabla}\varphi - \partial_t \vec{\mathcal{A}}, \\ \vec{\mathcal{B}} &= \vec{\nabla} \times \vec{\mathcal{A}},\end{aligned} \tag{10.19}$$

with the Lorentz gauge constraint

$$\vec{\nabla} \cdot \vec{\mathcal{A}} + \partial_t \varphi = 0, \tag{10.20}$$

then Maxwell's equations in potential form reduce to the spacetime Poisson's equations

$$\begin{aligned}\Box^2 \varphi &= -\rho, \\ \Box^2 \vec{\mathcal{A}} &= -\vec{J}.\end{aligned} \tag{10.21}$$

In the **static case**, these further reduce to the Poisson's equations

$$\begin{aligned}\nabla^2 \varphi &= -\rho, \\ \nabla^2 \vec{\mathcal{A}} &= -\vec{J},\end{aligned} \tag{10.22}$$

which have the solutions

$$\begin{aligned}\varphi[\vec{r}] &= \iiint \frac{\rho[\vec{r}\,']dV'}{4\pi \mathscr{r}}, \\ \vec{\mathcal{A}}[\vec{r}] &= \iiint \frac{\vec{J}[\vec{r}\,']dV'}{4\pi \mathscr{r}},\end{aligned} \tag{10.23}$$

where the integrals are over all space and the sources vanish at infinity.

The **non-static case** must account for the fact that "electromagnetic news" travels at light speed. It is not the condition of the sources *now* that matters, but rather it is the condition of the sources at the earlier **retarded time**, defined *implicitly* by

$$t_r = t - \mathscr{r} = t - \mathscr{r}[t_r]/c, \tag{10.24}$$

that matters. The sources *there and then* determine the fields *here and now*. For example, see light from stars in the night sky that left them at the retarded times corresponding to their distances. See most of the stars of the "Big Dipper" not as they are now, but as they were about a human lifetime (~ 75 years) ago.

Hence, write the solutions to the spacetime Poisson's equations as

$$\begin{aligned}\varphi[t, \vec{r}] &= \iiint \frac{\rho[t_r, \vec{r}\,']dV'}{4\pi \mathscr{r}[t_r, \vec{r}\,']},\\ \vec{\mathcal{A}}[t, \vec{r}] &= \iiint \frac{\vec{J}[t_r, \vec{r}\,']dV'}{4\pi \mathscr{r}[t_r, \vec{r}\,']}.\end{aligned} \tag{10.25}$$

Since the integrands are evaluated at the retarded times (with the most distant charge and current elements evaluated at the earliest times), refer to these as the **retarded potentials**.

To check, differentiate the potentials to find the fields. For example, if

$$\varphi = \iiint \frac{\rho\, dV}{4\pi \mathscr{r}} = \frac{1}{4\pi} \iiint \rho \frac{1}{\mathscr{r}} dV \tag{10.26}$$

where ρ is implicitly evaluated at the retarded time t_r and integrated over the source points $\vec{r}\,'$, then

$$\vec{\nabla}\varphi = \frac{1}{4\pi} \iiint \left(\left(\vec{\nabla}\rho\right) \frac{1}{\mathscr{r}} + \rho\, \vec{\nabla} \left(\frac{1}{\mathscr{r}}\right)\right) dV. \tag{10.27}$$

But

$$\vec{\nabla}\rho = \frac{\partial}{\partial \vec{r}} \rho[t_r, \vec{r}\,'] = \frac{\partial \rho}{\partial t_r} \frac{\partial t_r}{\partial \vec{r}} = \dot{\rho} \frac{\partial}{\partial \vec{r}} (t - \mathscr{r}) = -\dot{\rho}\, \vec{\nabla} \mathscr{r} = -\dot{\rho} \hat{\mathscr{r}}. \tag{10.28}$$

and

$$\vec{\nabla} \left(\frac{1}{\mathscr{r}}\right) = -\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}, \tag{10.29}$$

so

$$\vec{\nabla}\varphi = \frac{1}{4\pi} \iiint \left(-\dot{\rho} \frac{\hat{\mathscr{r}}}{\mathscr{r}} - \rho \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} \right) dV. \tag{10.30}$$

Furthermore,

$$\nabla^2\varphi = \frac{1}{4\pi} \iiint \left(-\left(\vec{\nabla}\dot{\rho}\right) \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}} - \dot{\rho}\, \vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) - \left(\vec{\nabla}\rho\right) \cdot \frac{\hat{\mathscr{r}}}{\mathscr{r}^2} - \rho\, \vec{\nabla} \cdot \left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right)\right) dV. \tag{10.31}$$

But

$$\vec{\nabla}\dot{\rho} = \frac{\partial}{\partial \vec{r}}\dot{\rho}[t_r, \vec{r}\,'] = \frac{\partial \dot{\rho}}{\partial t_r}\frac{\partial t_r}{\partial \vec{r}} = \ddot{\rho}\frac{\partial}{\partial \vec{r}}(t - \mathscr{r}) = -\ddot{\rho}\,\vec{\nabla}\mathscr{r} = -\ddot{\rho}\hat{\mathscr{r}}. \tag{10.32}$$

and

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}}\right) = \vec{\nabla}\cdot\left(\frac{1}{\mathscr{r}^2}\vec{\mathscr{r}}\right) = \vec{\nabla}\left(\frac{1}{\mathscr{r}^2}\right)\cdot\vec{\mathscr{r}} + \frac{1}{\mathscr{r}^2}\vec{\nabla}\cdot\vec{\mathscr{r}} = -\frac{2}{\mathscr{r}^3}\hat{\mathscr{r}}\cdot\vec{\mathscr{r}} + \frac{1}{\mathscr{r}^2}3 = \frac{1}{\mathscr{r}^2}, \tag{10.33}$$

and

$$\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{\mathscr{r}^2}\right) = 4\pi\vec{\nabla}\cdot\left(\frac{\hat{\mathscr{r}}}{4\pi\mathscr{r}^2}\right) = 4\pi\vec{\nabla}\cdot\vec{\mathcal{E}}_1 = 4\pi\rho_1 = 4\pi\delta^3[\vec{\mathscr{r}}], \tag{10.34}$$

so

$$\begin{aligned}
\nabla^2\varphi &= \frac{1}{4\pi}\iiint\left(+\frac{\ddot{\rho}}{\mathscr{r}} - \frac{\dot{\rho}}{\mathscr{r}^2} + \frac{\dot{\rho}}{\mathscr{r}^2} - 4\pi\delta^3[\mathscr{r}]\right)dV \\
&= \iiint\frac{\ddot{\rho}\,dV}{4\pi\mathscr{r}} - \iiint\rho\,\delta^3[\vec{\mathscr{r}}]dV \\
&= \partial_t^2\iiint\frac{\rho\,dV}{4\pi\mathscr{r}} - \iiint\rho\big[t - |\vec{r} - \vec{r}\,'|, \vec{r}\,'\big]\delta^3[\vec{r} - \vec{r}\,']d^3\vec{r}\,' \\
&= \partial_t^2\varphi - \rho[t, \vec{r}],
\end{aligned} \tag{10.35}$$

or

$$\Box^2\varphi = \nabla^2\varphi - \partial_t^2\varphi = -\rho, \tag{10.36}$$

which is the spacetime Poisson's equation for the electric potential.

Use an analogous argument to demonstrate that not only do the Eq. 10.25 retarded potentials solve Maxwell's equations, but so too do the **advanced potentials**

$$\begin{aligned}
\varphi_a[t, \vec{r}] &= \iiint\frac{\rho[t_a, \vec{r}\,']dV'}{4\pi\mathscr{r}[t_a, \vec{r}\,']}, \\
\vec{\mathcal{A}}_a[t, \vec{r}] &= \iiint\frac{\vec{J}[t_a, \vec{r}\,']dV'}{4\pi\mathscr{r}[t_a, \vec{r}\,']},
\end{aligned} \tag{10.37}$$

where the **advanced time**

$$t_a = t + \mathscr{r} = t + \mathscr{r}[t_r]/c, \tag{10.38}$$

The advanced potentials *in the present* depend on the sources *in the future*. The advanced solutions violate a naive **causality** and are typically discarded. However, the **Wheeler-Feynman absorber theory** [4] is a time-symmetric interpretation of electrodynamics that uses equally the retarded *and* advanced solutions to Maxwell's equations. The absorber theory has inspired a **transactional interpretation** [5] of quantum mechanics wherein present events are jointly determined by past and future events.

10.3 Liénard-Weichert Potentials

The electric potential of a point charge Q moving along an arbitrary trajectory $\vec{r}\,' = \vec{r}_Q[t]$ is

$$
\begin{aligned}
\varphi[t, \vec{r}] &= \iiint \frac{\rho[t_r, \vec{r}\,']dV'}{4\pi \mathscr{r}[t_r, \vec{r}\,']} \\
&= \iiint \frac{\rho[t_r, \vec{r}\,']dV'}{4\pi |\vec{r} - \vec{r}_Q[t_r]|} \\
&= \frac{1}{4\pi |\vec{r} - \vec{r}_Q[t_r]|} \iiint \rho[t_r, \vec{r}\,']dV', \qquad (10.39)
\end{aligned}
$$

but the remaining integral is *not* the total charge! Integrate the charge density at different retarded times for different parts to get

$$\iiint \rho[t_r, \vec{r}\,']dV' \neq Q, \qquad (10.40)$$

which effectively smears the moving charge along its trajectory. By contrast, integrate the charge density at one instant of time to get

$$\iiint \rho[t, \vec{r}\,']dV' = Q. \qquad (10.41)$$

Show below that this complication does *not* disappear for point charges, which are the limit of extended charges as their sizes shrink to zero.

Fundamentally, a moving object appears slightly longer than it really is because the light received from its rear left earlier, when the object was further away, than the light received *simultaneously* from its front, when the object was closer. This *nonrelativistic* effect depends only on electromagnetic news traveling at the *finite* light speed.

For example, consider a sphere of diameter ℓ moving with speed v so that in time Δt it moves a distance $\Delta\ell$ directly toward an observer, as in the left part of Fig. 10.2. Light from the rear of the sphere must travel an extra distance $\ell' = \ell + \Delta\ell$ and leave a time $\Delta t = \ell'/c$ earlier to arrive at the observer simultaneously with light from the front. Thus,

$$\ell' = \frac{\ell'}{c} = \Delta t = \frac{\Delta\ell}{v} = \frac{\ell' - \ell}{v} \qquad (10.42)$$

or

$$\ell' = \ell \frac{1}{1 - v} > \ell \qquad (10.43)$$

and the apparent length of the sphere is $1/(1 - v)$ times greater than the actual length of the sphere.

Similarly, if the sphere is moving at at an angle α with respect to the observer, as in the right part of Fig. 10.2, then light from the rear of the sphere must travel and extra distance $\ell' \cos\alpha$ and leave a time $\Delta t = \ell' \cos\alpha/c$ earlier to arrive at the observer simultaneously with light from the front. Thus,

$$\ell' \cos\alpha = \frac{\ell' \cos\alpha}{c} = \Delta t = \frac{\Delta\ell}{v} = \frac{\ell' - \ell}{v} \qquad (10.44)$$

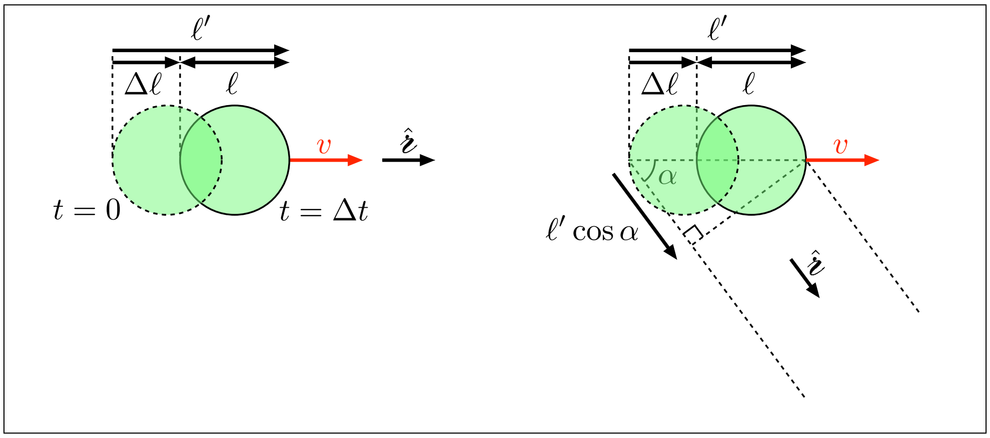

Figure 10.2: A moving object appears smeared along its trajectory due to the finite light speed, whether approaching directly (left) or obliquely (right).

or

$$\ell' = \ell \frac{1}{1 - v\cos\alpha} = \ell \frac{1}{1 - \hat{\boldsymbol{\imath}} \cdot \vec{v}} > \ell \tag{10.45}$$

and the apparent length of the sphere is $1/(1 - \hat{\boldsymbol{\imath}} \cdot \vec{v})$ times greater than the actual length of the sphere.

Since the motion is forward-backward and not left-right or up-down, the apparent volume of the sphere is also $1/(1 - \hat{\boldsymbol{\imath}} \cdot \vec{v})$ times greater than the actual volume of the sphere. Since this correction factor is independent of the size of the sphere, apply it also to the Eq. 10.40 volume integral of the retarded charge density for the point charge in arbitrary motion to get

$$\iiint \rho[t_r, \vec{r}\,']dV' = Q \frac{1}{1 - \hat{\boldsymbol{\imath}}[t_r] \cdot \vec{v}[t_r]} > Q. \tag{10.46}$$

Hence, completing Eq. 10.39, the electric potential for a moving point charge is

$$\varphi[t, \vec{r}] = \frac{Q}{4\pi \boldsymbol{\imath}[t_r]} \frac{1}{1 - \hat{\boldsymbol{\imath}}[t_r] \cdot \vec{v}[t_r]}. \tag{10.47}$$

For a moving charge density, $\vec{J} = \rho\vec{v}$, and so

$$\begin{aligned} \vec{\mathcal{A}}[t, \vec{r}] &= \iiint \frac{\vec{J}[t_r, \vec{r}\,']dV'}{4\pi \boldsymbol{\imath}[t_r, \vec{r}\,']} \\ &= \iiint \frac{\rho[t_r, \vec{r}\,']\vec{v}[t_r]dV'}{4\pi|\vec{r} - \vec{r}_Q[t_r]|} \\ &= \frac{\vec{v}[t_r]}{4\pi|\vec{r} - \vec{r}_Q[t_r]|} \iiint \rho[t_r, \vec{r}\,']dV' \\ &= \frac{Q\vec{v}[t_r]}{4\pi \boldsymbol{\imath}[t_r]} \frac{1}{1 - \hat{\boldsymbol{\imath}}[t_r] \cdot \vec{v}[t_r]}. \end{aligned} \tag{10.48}$$

More succinctly, the **Liénard-Wiechert potentials** for a moving point charge are

$$\varphi = \frac{Q}{4\pi \mathscr{r}} \frac{1}{1 - \hat{\mathscr{r}} \cdot \vec{v}}\,,$$
$$\vec{\mathcal{A}} = \frac{Q\vec{v}}{4\pi \mathscr{r}} \frac{1}{1 - \hat{\mathscr{r}} \cdot \vec{v}}\,, \tag{10.49}$$

where the relative displacement $\vec{\mathscr{r}}$ and the velocity $\vec{v}$ are implicitly evaluated at the retarded time. If the velocity $\vec{v} = \vec{0}$, then $\varphi = Q/4\pi\mathscr{r}$ and $\vec{\mathcal{A}} = \vec{0}$, as in electrostatics.

10.4 Constant Velocity

As an important example, for which the retarded time can be explicitly calculated, compute the electric potential of a point charge moving at constant velocity,

$$\vec{r}_Q[t] = \vec{v}t. \tag{10.50}$$

First, compute the retarded time. Square the retarded time relation

$$t - t_r = \mathscr{r}[t_r] = |\vec{r} - \vec{r}_Q[t_r]| = |\vec{r} - \vec{v}t_r| \tag{10.51}$$

to get

$$t^2 - 2tt_r + t_r^2 = r^2 - 2\vec{r} \cdot \vec{v}t_r + v^2 t_r^2. \tag{10.52}$$

Arrange this as a quadratic equation in standard form

$$(1 - v^2)t_r^2 - 2(t - \vec{r} \cdot \vec{v})t_r + (t^2 - r^2) = 0, \tag{10.53}$$

and use the **quadratic formula** to solve for

$$t_r^{\pm} = \frac{+2(t - \vec{r} \cdot \vec{v}) \pm \sqrt{4(t - \vec{r} \cdot \vec{v})^2 - 4(1 - v^2)(t^2 - r^2)}}{2(1 - v^2)}. \tag{10.54}$$

Examine a convenient special case to choose the sign. If the speed $v = 0$, then Eq. 10.54 reduces to $t_r^{\pm} = t \pm r$, where the minus sign corresponds to the retarded solution (and the plus sign corresponds to the advanced solution). Cancel a factor of 2 and write the retarded time as

$$t_r = t_r^{-} = \frac{(t - \vec{r} \cdot \vec{v}) - \sqrt{(t - \vec{r} \cdot \vec{v})^2 - (1 - v^2)(t^2 - r^2)}}{1 - v^2}. \tag{10.55}$$

Next divide the retarded displacement vector

$$\vec{\mathscr{r}} = \vec{r} - \vec{r}_Q[t_r] = \vec{r} - \vec{v}t_r \tag{10.56}$$

by its Eq. 10.51 magnitude

$$\mathscr{r} = t - t_r \tag{10.57}$$

to form its direction

$$\hat{\mathscr{r}} = \frac{\vec{r} - \vec{v}t_r}{t - t_r}. \tag{10.58}$$

In the denominator of the Eq. 10.49 electric potential is

$$\begin{aligned}
\tilde{r} &= \mathscr{r}(1 - \hat{\mathscr{r}} \cdot \vec{v}) \\
&= (t - t_r)\left(1 - \left(\frac{\vec{r} - \vec{v}t_r}{t - t_r}\right) \cdot \vec{v}\right) \\
&= t - t_r - \vec{r} \cdot \vec{v} + v^2 t_r \\
&= (t - \vec{r} \cdot \vec{v}) - t_r(1 - v^2) \\
&= \sqrt{(t - \vec{r} \cdot \vec{v})^2 - (1 - v^2)(t^2 - r^2)},
\end{aligned} \tag{10.59}$$

via the Eq. 10.55 retarded time. Hence, the electric potential

$$\varphi[t, \vec{r}] = \frac{Q}{4\pi\tilde{r}} = \frac{Q}{4\pi\sqrt{(t - \vec{r} \cdot \vec{v})^2 - (1 - v^2)(t^2 - r^2)}}. \tag{10.60}$$

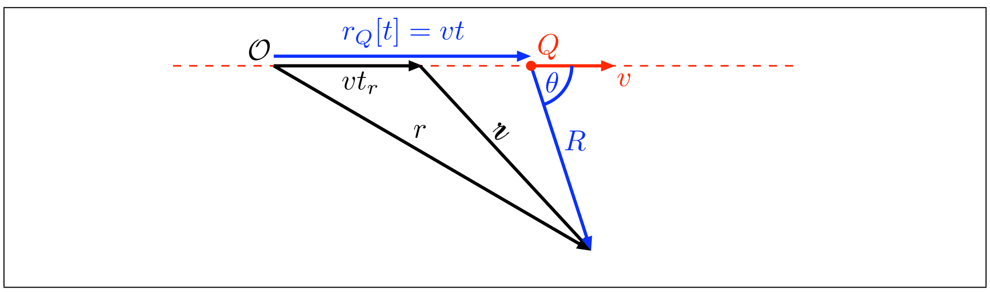

Figure 10.3: Simplify the retarded electric potential φ for a point charge Q moving at constant velocity $\vec{v}$ using the *present* displacement vector $\vec{R}$.

Exploit the predictability of uniform motion to simplify the potential by expressing it in terms of the *present* displacement vector

$$\vec{R} = \vec{\mathscr{r}}[t] = \vec{r} - \vec{v}t, \tag{10.61}$$

as in Fig. 10.3. Use this to eliminate t and $\vec{r}$ from the Eq. 10.59 $\tilde{r}$ in two steps. First, eliminate t by substituting $\vec{v}t = \vec{r} - \vec{R}$ into

$$\begin{aligned}
\tilde{r}^2 &= (t - \vec{r} \cdot \vec{v})^2 - (1 - v^2)(t^2 - r^2) \\
&= t^2 - 2t\vec{r} \cdot \vec{v} + (\vec{r} \cdot \vec{v})^2 - t^2 + r^2 + v^2t^2 - v^2r^2 \\
&= -2\vec{r} \cdot (\vec{r} - \vec{R}) + (\vec{r} \cdot \vec{v})^2 + r^2 + (\vec{r} - \vec{R})^2 - v^2r^2 \\
&= -2r^2 + 2\vec{r} \cdot \vec{R} + (\vec{r} \cdot \vec{v})^2 + r^2 + r^2 - 2\vec{r} \cdot \vec{R} + R^2 - v^2r^2 \\
&= (\vec{r} \cdot \vec{v})^2 - (rv)^2 + R^2.
\end{aligned} \tag{10.62}$$

Next, eliminate $\vec{r}$ by substituting $\vec{r} = \vec{R} + \vec{v}t$ into the **constant of the motion**,

$$\begin{aligned}(\vec{r}\cdot\vec{v})^2 - (rv)^2 =& \big((\vec{R}+\vec{v}t)\cdot\vec{v}\big)^2 - \big(\vec{R}+\vec{v}t\big)^2 v^2 \\ =& \big(\vec{R}\cdot\vec{v}\big)^2 + 2\big(\vec{R}\cdot\vec{v}\big)v^2 t + v^4 t^2 \\ & - (Rv)^2 - 2\big(\vec{R}\cdot\vec{v}\big)tv^2 - v^4 t^2 \\ =& \big(\vec{R}\cdot\vec{v}\big)^2 - (Rv)^2. \end{aligned} \tag{10.63}$$

If the angle between the present displacement $\vec{R}$ and the velocity $\vec{v}$ is θ, then the constant of the motion

$$\big(\vec{R}\cdot\vec{v}\big)^2 - (Rv)^2 = (Rv\cos\theta)^2 - (Rv)^2 = -R^2v^2\sin^2\theta \tag{10.64}$$

and so

$$\tilde{r}^2 = -R^2v^2\sin^2\theta + R^2 = R^2(1 - v^2\sin^2\theta). \tag{10.65}$$

Hence, the electric potential for a charge in uniform motion

$$\varphi = \frac{Q}{4\pi R\sqrt{1-(v\sin\theta)^2}}. \tag{10.66}$$

If the speed $v = 0$ (or approach angle $\theta = 0$), then $\varphi = Q/4\pi R$, as in electrostatics.

10.5 Arbitrary Motion

Use Eq. 10.19 to differentiate the potentials φ and $\vec{\mathcal{A}}$ to obtain the fields $\vec{\mathcal{E}}$ and $\vec{\mathcal{B}}$ for a point charge Q in arbitrary motion $\vec{r}\,' = \vec{r}_Q[t]$. The result is worth the effort.

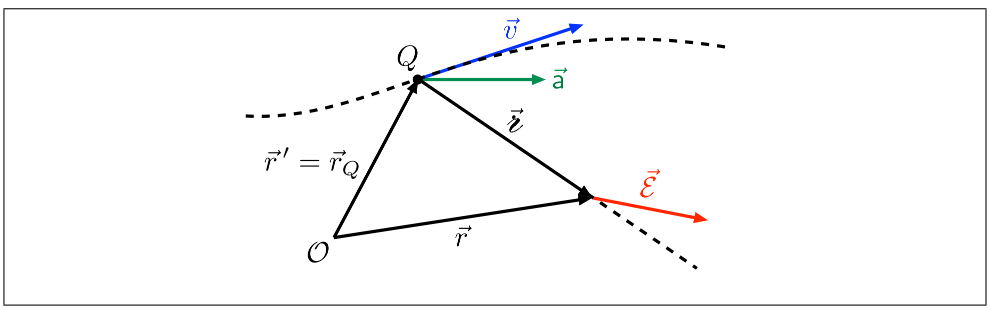

Figure 10.4: Electric field $\vec{\mathcal{E}}$ for a charge Q in arbitrary motion with retarded position $\vec{r}_Q$, retarded velocity $\vec{v}$, and retarded acceleration $\vec{a}$.

First, use **implicit differentiation** to compute the time derivative of the retarded time, $\partial_t t_r$. Square the retarded time relation $t - t_r = \mathscr{r}$ to obtain

$$(t-t_r)^2 = \mathscr{r}^2 = \vec{\mathscr{r}}\cdot\vec{\mathscr{r}}, \tag{10.67}$$

and differentiate with respect to time to find

$$2(t-t_r)(1-\partial_t t_r) = 2\vec{\mathscr{r}}\cdot\partial_t\vec{\mathscr{r}} \tag{10.68}$$

or

$$1 - \partial_t t_r = \hat{\mathscr{r}} \cdot \partial_t \vec{\mathscr{r}}. \tag{10.69}$$

The chain rule implies

$$\partial_t \vec{\mathscr{r}} = \partial_t (\vec{r} - \vec{r}_Q) = -\partial_t \vec{r}_Q = -\frac{\partial t_r}{\partial t} \frac{d\vec{r}_Q}{dt_r} = -(\partial_t t_r)\vec{v}. \tag{10.70}$$

Hence, substitute Eq. 10.70 into Eq. 10.69 to obtain

$$1 - \partial_t t_r = -\hat{\mathscr{r}} \cdot \vec{v}\, \partial_t t_r \tag{10.71}$$

and solve for

$$\partial_t t_r = \frac{1}{1 - \hat{\mathscr{r}} \cdot \vec{v}} = \frac{\mathscr{r}}{\mathscr{r} - \vec{\mathscr{r}} \cdot \vec{v}}. \tag{10.72}$$

Next, use implicit differentiation to compute the space derivative of the retarded time, $\vec{\nabla} t_r$. Gradient the Eq. 10.67 square of the retarded time relation to find

$$2(t - t_r)(\vec{0} - \vec{\nabla} t_r) = \vec{\nabla}(\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}) \tag{10.73}$$

and use the Eq. A-7 product rule to obtain

$$-\vec{\nabla} t_r = \frac{1}{2\mathscr{r}} \vec{\nabla}(\vec{\mathscr{r}} \cdot \vec{\mathscr{r}}) = \frac{1}{2\mathscr{r}} 2\left((\vec{\mathscr{r}} \cdot \vec{\nabla})\vec{\mathscr{r}} + \vec{\mathscr{r}} \times (\vec{\nabla} \times \vec{\mathscr{r}})\right). \tag{10.74}$$

The chain rule implies

$$(\vec{\mathscr{r}} \cdot \vec{\nabla})\vec{\mathscr{r}} = (\vec{\mathscr{r}} \cdot \vec{\nabla})\vec{r} - (\vec{\mathscr{r}} \cdot \vec{\nabla})\vec{r}_Q = \mathscr{r} - \vec{\mathscr{r}} \cdot \frac{\partial}{\partial \vec{r}} \vec{r}_Q = \mathscr{r} - \vec{\mathscr{r}} \cdot \frac{\partial t_r}{\partial \vec{r}} \frac{d\vec{r}_Q}{dt_r} = \mathscr{r} - (\mathscr{r} \cdot \vec{\nabla} t_r)\vec{v} \tag{10.75}$$

and

$$\vec{\nabla} \times \vec{\mathscr{r}} = \vec{\nabla} \times \vec{r} - \vec{\nabla} \times \vec{r}_Q = \vec{0} - \frac{\partial}{\partial \vec{r}} \times \vec{r}_Q = -\frac{\partial t_r}{\partial \vec{r}} \times \frac{d\vec{r}_Q}{dt_r} = -\vec{\nabla} t_r \times \vec{v} \tag{10.76}$$

and so

$$\vec{\mathscr{r}} \times (\vec{\nabla} \times \vec{\mathscr{r}}) = -\vec{\mathscr{r}} \times (\vec{\nabla} t_r \times \vec{v}) = -(\vec{\mathscr{r}} \cdot \vec{v})\vec{\nabla} t_r + (\mathscr{r} \cdot \vec{\nabla} t_r)\vec{v}. \tag{10.77}$$

Hence, substitute the Eq. 10.75 directional derivative and the Eq. 10.77 double product into Eq. 10.74 to obtain

$$-\vec{\nabla} t_r = \frac{1}{\mathscr{r}} \left((\vec{\mathscr{r}} - (\vec{\mathscr{r}} \cdot \vec{v})\vec{\nabla} t_r)\right) = \hat{\mathscr{r}} - (\hat{\mathscr{r}} \cdot \vec{v})\vec{\nabla} t_r, \tag{10.78}$$

and solve for

$$\vec{\nabla} t_r = \frac{-\hat{\mathscr{r}}}{1 - \hat{\mathscr{r}} \cdot \vec{v}} = \frac{-\vec{\mathscr{r}}}{\mathscr{r} - \vec{\mathscr{r}} \cdot \vec{v}}. \tag{10.79}$$

Next, compute the gradient of the electric potential, $\vec{\nabla}\varphi$. Gradient

$$\varphi = \frac{Q}{4\pi\mathscr{r}} \frac{1}{1 - \hat{\mathscr{r}} \cdot \vec{v}} = \frac{Q}{4\pi} \frac{1}{\mathscr{r} - \vec{\mathscr{r}} \cdot \vec{v}} \tag{10.80}$$

to obtain

$$\vec{\nabla}\varphi = \frac{Q}{4\pi}\frac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v})^2}\left(-\vec{\nabla}\mathscr{r} + \vec{\nabla}(\vec{\mathscr{r}}\cdot\vec{v})\right), \tag{10.81}$$

where

$$\vec{\nabla}\mathscr{r} = \vec{\nabla}(t - t_r) = -\vec{\nabla}t_r, \tag{10.82}$$

and, by the Eq. A-7 product rule,

$$\vec{\nabla}(\vec{\mathscr{r}}\cdot\vec{v}) = (\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v} + (\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}} + \vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v}) + \vec{v}\times(\vec{\nabla}\times\vec{\mathscr{r}}), \tag{10.83}$$

where the directional derivatives

$$(\vec{\mathscr{r}}\cdot\vec{\nabla})\vec{v} = \vec{\mathscr{r}}\cdot\frac{\partial}{\partial\vec{r}}\vec{v} = \vec{\mathscr{r}}\cdot\frac{\partial t_r}{\partial\vec{r}}\frac{d\vec{v}}{dt_r} = (\vec{\mathscr{r}}\cdot\vec{\nabla}t_r)\vec{\mathsf{a}}, \tag{10.84}$$

and

$$(\vec{v}\cdot\vec{\nabla})\vec{\mathscr{r}} = (\vec{v}\cdot\vec{\nabla})\vec{r} - (\vec{v}\cdot\vec{\nabla})\vec{r}_Q = \vec{v} - \vec{v}\cdot\frac{\partial}{\partial\vec{r}}\vec{r}_Q = \vec{v} - \vec{v}\cdot\frac{\partial t_r}{\partial\vec{r}}\frac{d\vec{r}_Q}{dt_r} = \vec{v} - (\vec{v}\cdot\vec{\nabla}t_r)\vec{v}, \tag{10.85}$$

and the curls

$$\vec{\nabla}\times\vec{v} = \frac{\partial}{\partial\vec{r}}\times\vec{v} = \frac{\partial t_r}{\partial\vec{r}}\times\frac{d\vec{v}}{dt_r} = \vec{\nabla}t_r\times\vec{\mathsf{a}} \tag{10.86}$$

and

$$\vec{\nabla}\times\vec{\mathscr{r}} = \vec{\nabla}\times\vec{r} - \vec{\nabla}\times\vec{r}_Q = -\frac{\partial}{\partial\vec{r}}\times\vec{r}_Q = -\frac{\partial t_r}{\partial\vec{r}}\times\frac{d\vec{r}_Q}{dt_r} = -\vec{\nabla}t_r\times\vec{v}. \tag{10.87}$$

and so the double products

$$\vec{\mathscr{r}}\times(\vec{\nabla}\times\vec{v}) = \vec{\mathscr{r}}\times(\vec{\nabla}t_r\times\vec{\mathsf{a}}) = (\vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\vec{\nabla}t_r - (\vec{\mathscr{r}}\cdot\vec{\nabla}t_r)\vec{\mathsf{a}} \tag{10.88}$$

and

$$\vec{v}\times(\vec{\nabla}\times\vec{\mathscr{r}}) = -\vec{v}\times(\vec{\nabla}t_r\times\vec{v}) = -v^2\vec{\nabla}t_r + (\vec{v}\cdot\vec{\nabla}t_r)\vec{v}. \tag{10.89}$$

Hence, substitute the Eq. 10.82 gradient of the relative displacement, the Eq. 10.84 and Eq. 10.85 directional derivatives, and the Eq. 10.88 and Eq. 10.89 double products, into the Eq. 10.81 gradient of the potential to obtain

$$\begin{aligned}\vec{\nabla}\varphi = \frac{Q}{4\pi}\frac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v})^2}\Big(&+\nabla t_r \\ &+(\vec{\mathscr{r}}\cdot\vec{\nabla}t_r)\vec{\mathsf{a}} + \vec{v} - (\vec{v}\cdot\vec{\nabla}t_r)\vec{v} \\ &+(\vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\vec{\nabla}t_r - (\vec{\mathscr{r}}\cdot\vec{\nabla}t_r)\vec{\mathsf{a}} - v^2\vec{\nabla}t_r + (\vec{v}\cdot\vec{\nabla}t_r)\vec{v}\Big)\end{aligned} \tag{10.90}$$

and simplify to

$$\begin{aligned}\vec{\nabla}\varphi &= \frac{Q}{4\pi}\frac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v})^2}\left(\vec{v} + (1 - v^2 + \vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\vec{\nabla}t_r\right) \\ &= \frac{Q}{4\pi}\frac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v})^2}\left(\vec{v} + (1 - v^2 + \vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\frac{-\vec{\mathscr{r}}}{\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}}\right) \\ &= \frac{Q}{4\pi}\frac{1}{(\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v})^3}\left((\mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v})\vec{v} - (1 - v^2 + \vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\vec{\mathscr{r}}\right)\end{aligned} \tag{10.91}$$

or finally

$$-\vec{\nabla}\varphi = \frac{Q}{4\pi \boldsymbol{\imath}^2} \frac{1}{(1-\hat{\boldsymbol{\imath}}\cdot\vec{v})^3}\left((1-v^2+\vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}})\hat{\boldsymbol{\imath}} - (1-\hat{\boldsymbol{\imath}}\cdot\vec{v})\,\vec{v}\right). \tag{10.92}$$

Next, compute the time derivative of the magnetic potential, $\partial_t\vec{\mathcal{A}}$. Time differentiate

$$\vec{\mathcal{A}} = \varphi\vec{v} = \frac{Q}{4\pi}\frac{1}{\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v}}\vec{v} \tag{10.93}$$

to obtain

$$\partial_t\vec{\mathcal{A}} = (\partial_t\varphi)\vec{v} + \varphi(\partial_t\vec{v}), \tag{10.94}$$

where

$$\partial_t\vec{v} = \frac{\partial t_r}{\partial t}\frac{d\vec{v}}{dt_r} = (\partial_t t_r)\vec{\mathsf{a}} \tag{10.95}$$

and

$$\partial_t\varphi = \frac{Q}{4\pi}\frac{-1}{(\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})^2}\left(\partial_t\boldsymbol{\imath} - (\partial_t\vec{\boldsymbol{\imath}})\cdot\vec{v} - \vec{\boldsymbol{\imath}}\cdot(\partial_t\vec{v})\right), \tag{10.96}$$

where

$$\partial_t\boldsymbol{\imath} = \partial_t(t-t_r) = 1 - \partial_t t_r, \tag{10.97}$$

and

$$\partial_t\vec{\boldsymbol{\imath}} = \partial_t(\vec{r}-\vec{r}_Q) = 0 - \frac{\partial}{\partial t}\vec{r}_Q = -\frac{\partial t_r}{\partial t}\frac{d\vec{r}_Q}{dt_r} = -(\partial_t t_r)\vec{v}. \tag{10.98}$$

Hence,

$$\begin{aligned}
\partial_t\vec{\mathcal{A}} &= \frac{Q}{4\pi}\frac{-1}{(\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})^2}\left((1-\partial_t t_r + v^2\partial_t t_r - \vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}}\,\partial_t t_r)\vec{v} - (\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})(\partial_t t_r)\vec{\mathsf{a}}\right) \\
&= \frac{Q}{4\pi}\frac{-1}{(\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})^2}\left(\vec{v} - \left((1-v^2+\vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}})\vec{v} + (\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})\vec{\mathsf{a}}\right)\partial_t t_r\right) \\
&= \frac{Q}{4\pi}\frac{-1}{(\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})^2}\left(\vec{v} - \left((1-v^2+\vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}})\vec{v} + (\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})\vec{\mathsf{a}}\right)\frac{\boldsymbol{\imath}}{\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v}}\right) \\
&= \frac{Q}{4\pi}\frac{-1}{(\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})^3}\left((\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})\vec{v} - (1-v^2+\vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}})\boldsymbol{\imath}\vec{v} - (\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})\boldsymbol{\imath}\vec{\mathsf{a}}\right) \\
&= \frac{Q}{4\pi\boldsymbol{\imath}^2}\frac{-1}{(1-\hat{\boldsymbol{\imath}}\cdot\vec{v})^3}\left((1-\hat{\boldsymbol{\imath}}\cdot\vec{v})\vec{v} - (1-v^2+\vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}})\vec{v} - (\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})\vec{\mathsf{a}}\right),
\end{aligned} \tag{10.99}$$

or finally,

$$-\partial_t\vec{\mathcal{A}} = \frac{Q}{4\pi\boldsymbol{\imath}^2}\frac{1}{(1-\hat{\boldsymbol{\imath}}\cdot\vec{v})^3}\left((v^2 - \hat{\boldsymbol{\imath}}\cdot\vec{v} - \vec{\boldsymbol{\imath}}\cdot\vec{\mathsf{a}})\vec{v} - (\boldsymbol{\imath} - \vec{\boldsymbol{\imath}}\cdot\vec{v})\vec{\mathsf{a}}\right). \tag{10.100}$$

Next, combine Eq. 10.92 and Eq. 10.100 to compute the electric field

$$\begin{aligned}
\vec{\mathcal{E}} &= -\vec{\nabla}\varphi - \partial_t \vec{\mathcal{A}} \\
&= \frac{Q}{4\pi \mathscr{r}^2} \frac{1}{(1-\hat{\mathscr{r}}\cdot\vec{v})^3}\Big((1-v^2+\vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\hat{\mathscr{r}} - (1-v^2+\mathscr{r}\cdot\vec{\mathsf{a}})\vec{v} - (\mathscr{r}-\vec{\mathscr{r}}\cdot\vec{v})\vec{\mathsf{a}}\Big) \\
&= \frac{Q}{4\pi \mathscr{r}^2} \frac{1}{(1-\hat{\mathscr{r}}\cdot\vec{v})^3}\Big((1-v^2)(\hat{\mathscr{r}}-\vec{v}) + (\vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\hat{\mathscr{r}} - (\vec{\mathscr{r}}\cdot\vec{\mathsf{a}})\vec{v} - \mathscr{r}\vec{\mathsf{a}} + (\vec{\mathscr{r}}\cdot\vec{v})\vec{\mathsf{a}}\Big) \\
&= \frac{Q}{4\pi \mathscr{r}^2} \frac{1}{(1-\hat{\mathscr{r}}\cdot\vec{v})^3}\Big((1-v^2)(\hat{\mathscr{r}}-\vec{v}) + \vec{\mathscr{r}}\times(\hat{\mathscr{r}}\times\vec{\mathsf{a}}) - \vec{\mathscr{r}}\times(\vec{v}\times\vec{\mathsf{a}})\Big) \\
&= \frac{Q}{4\pi \mathscr{r}^2} \frac{1}{(1-\hat{\mathscr{r}}\cdot\vec{v})^3}\Big((1-v^2)(\hat{\mathscr{r}}-\vec{v}) + \vec{\mathscr{r}}\times\big((\hat{\mathscr{r}}-\vec{v})\times\vec{\mathsf{a}}\big)\Big), \qquad (10.101)
\end{aligned}$$

where the right side is implicitly evaluated at the retarded time.

Introduce the velocity

$$\vec{u} = \vec{c} - \vec{v} = \hat{\mathscr{r}}c - \vec{v} = \hat{\mathscr{r}} - \vec{v} \qquad (10.102)$$

so that

$$u_{\parallel} = \hat{\mathscr{r}}\cdot\vec{u} = 1 - \hat{\mathscr{r}}\cdot\vec{v}, \qquad (10.103a)$$

$$\vec{\mathscr{r}}\cdot\vec{u} = \mathscr{r} - \vec{\mathscr{r}}\cdot\vec{v}, \qquad (10.103b)$$

and the relativistic stretch

$$\gamma = \frac{1}{\sqrt{1-v^2/c^2}} = \frac{1}{\sqrt{1-v^2}}, \qquad (10.104)$$

and rewrite the electric field as

$$\vec{\mathcal{E}} = \frac{Q}{4\pi}\frac{1}{u_{\parallel}^3}\left(\frac{\vec{u}}{\gamma^2\mathscr{r}^2} + \frac{\hat{\mathscr{r}}\times(\vec{u}\times\vec{\mathsf{a}})}{\mathscr{r}}\right). \qquad (10.105)$$

At large distances, the acceleration-dependent **radiation term**, which decreases like $1/\mathscr{r}$, dominates the acceleration-independent **Coulomb term**, which decreases like $1/\mathscr{r}^2$.

Next, compute the curl of the magnetic potential, $\vec{\nabla}\times\vec{\mathcal{A}}$. Curl

$$\vec{\mathcal{A}} = \varphi\vec{v} = \frac{Q}{4\pi\mathscr{r}}\frac{1}{1-\hat{\mathscr{r}}\cdot\vec{v}}\vec{v} \qquad (10.106)$$

to obtain the magnetic field

$$\vec{\mathcal{B}} = \vec{\nabla}\times\vec{\mathcal{A}} = (\vec{\nabla}\varphi)\times\vec{v} + \varphi\vec{\nabla}\times\vec{v} = -\vec{v}\times\vec{\nabla}\varphi + \varphi\vec{\nabla}\times\vec{v}, \qquad (10.107)$$

where

$$\vec{\nabla}\times\vec{v} = \frac{\partial}{\partial\vec{r}}\times\vec{v} = \frac{\partial t_r}{\partial\vec{r}}\times\frac{d\vec{v}}{dt_r} = \vec{\nabla}t_r\times\vec{\mathsf{a}} = \frac{-\hat{\mathscr{r}}\times\vec{\mathsf{a}}}{1-\hat{\mathscr{r}}\cdot\vec{v}}, \qquad (10.108)$$

and recall the Eq. 10.92

$$-\vec{\nabla}\varphi = \frac{Q}{4\pi \text{ᴤ}^2}\frac{1}{(1-\hat{\text{ᴤ}}\cdot\vec{v})^3}\left((1-v^2+\vec{\text{ᴤ}}\cdot\vec{\mathsf{a}})\hat{\text{ᴤ}} - (1-\hat{\text{ᴤ}}\cdot\vec{v})\,\vec{v}\right). \tag{10.109}$$

Substitute these into the Eq. 10.107 magnetic field to obtain

$$\vec{\mathcal{B}} = \frac{Q}{4\pi \text{ᴤ}^2}\frac{1}{(1-\hat{\text{ᴤ}}\cdot\vec{v})^3}\left((1-v^2+\vec{\text{ᴤ}}\cdot\vec{\mathsf{a}})\vec{v}\times\hat{\text{ᴤ}} + \vec{0} - \text{ᴤ}(1-\hat{\text{ᴤ}}\cdot\vec{v})\hat{\text{ᴤ}}\times\vec{\mathsf{a}}\right). \tag{10.110}$$

Introduce $\vec{u} = \hat{\text{ᴤ}} - \vec{v}$ and the resulting $\hat{\text{ᴤ}}\times\vec{u} = \vec{0} - \hat{\text{ᴤ}}\times\vec{v} = \vec{v}\times\hat{\text{ᴤ}}$ to write

$$\begin{aligned}
\vec{\mathcal{B}} &= \frac{Q}{4\pi \text{ᴤ}^2}\frac{1}{(1-\hat{\text{ᴤ}}\cdot\vec{v})^3}\left((1-v^2+\vec{\text{ᴤ}}\cdot\vec{\mathsf{a}})\hat{\text{ᴤ}}\times\vec{u} - \text{ᴤ}(1-\hat{\text{ᴤ}}\cdot\vec{v})\hat{\text{ᴤ}}\times\vec{\mathsf{a}}\right)\\
&= \hat{\text{ᴤ}}\times\frac{Q}{4\pi \text{ᴤ}^2}\frac{1}{(1-\hat{\text{ᴤ}}\cdot\vec{v})^3}\left((1-v^2+\vec{\text{ᴤ}}\cdot\vec{\mathsf{a}})\vec{u} - (\text{ᴤ}-\vec{\text{ᴤ}}\cdot\vec{v})\vec{\mathsf{a}}\right)\\
&= \hat{\text{ᴤ}}\times\frac{Q}{4\pi \text{ᴤ}^2}\frac{1}{(1-\hat{\text{ᴤ}}\cdot\vec{v})^3}\left((1-v^2)\vec{u} + (\vec{\text{ᴤ}}\cdot\vec{\mathsf{a}})\vec{u} - (\vec{\text{ᴤ}}\cdot\vec{u})\vec{\mathsf{a}}\right)\\
&= \hat{\text{ᴤ}}\times\frac{Q}{4\pi \text{ᴤ}^2}\frac{1}{(1-\hat{\text{ᴤ}}\cdot\vec{v})^3}\left((1-v^2)\vec{u} + \vec{\text{ᴤ}}\times(\vec{u}\times\vec{\mathsf{a}})\right)\\
&= \hat{\text{ᴤ}}\times\frac{Q}{4\pi}\frac{1}{u_\parallel^3}\left(\frac{\vec{u}}{\gamma^2\text{ᴤ}^2} + \frac{\hat{\text{ᴤ}}\times(\vec{u}\times\vec{\mathsf{a}})}{\text{ᴤ}}\right).
\end{aligned} \tag{10.111}$$

Substitute the Eq. 10.105 electric field to show

$$\vec{\mathcal{B}} = \hat{\text{ᴤ}}\times\vec{\mathcal{E}}. \tag{10.112}$$

Thus, the magnetic field of a point charge in arbitrary motion is always perpendicular to the electric field *and* to the vector from the retarded source. The Sec. 10.1 transverse electromagnetic waves are a special case.

10.6 Arbitrary Force

Suppose a source charge $q' = Q$ is at position $\vec{r}\,'$ with velocity $\vec{v}\,'$ and acceleration $\vec{\mathsf{a}}\,'$, and a test charge q is at position $\vec{r}$ with velocity $\vec{v}$, as in Fig. 1.1. Define the relative separation $\vec{\text{ᴤ}} = \vec{r} - \vec{r}\,'$, velocity $\vec{u}\,' = \vec{c} - \vec{v}\,' = c\,\hat{\text{ᴤ}} - \vec{v}\,'$, and projection $u'_\parallel = \hat{\text{ᴤ}}\cdot\vec{u}\,'$. The force on the test charge q is

$$\vec{F} = q\left(\vec{\mathcal{E}} + \vec{v}\times\vec{\mathcal{B}}\right), \tag{10.113}$$

or using the Eq. 10.105 electric field and the Eq. 10.111 magnetic field,

$$\vec{F} = q\frac{q'}{4\pi}\frac{1}{u_\parallel'^3}\left(\frac{\vec{u}\,'}{\gamma^2\text{ᴤ}^2} + \frac{\hat{\text{ᴤ}}\times(\vec{u}\,'\times\vec{\mathsf{a}}\,')}{\text{ᴤ}} + \vec{v}\times\left(\hat{\text{ᴤ}}\times\left(\frac{\vec{u}\,'}{\gamma^2\text{ᴤ}^2} + \frac{\hat{\text{ᴤ}}\times(\vec{u}\,'\times\vec{\mathsf{a}}\,')}{\text{ᴤ}}\right)\right)\right), \tag{10.114}$$

where the source variables are evaluated at the retarded time. This is the generalized Coulomb's law of the Sec. 1.1 advertisement.

10.7 Feynman's Formula

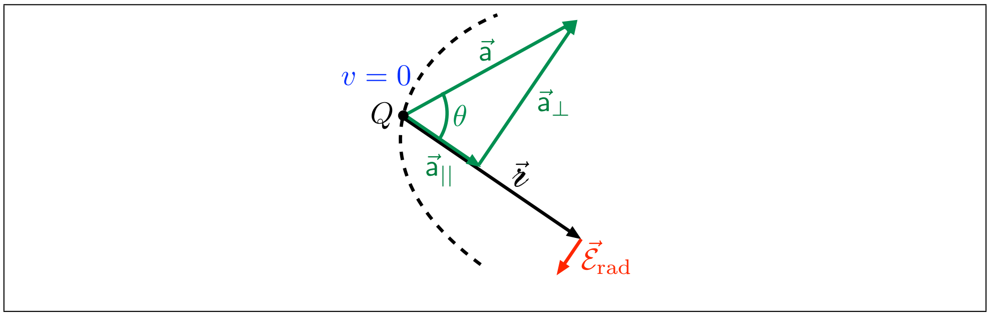

Figure 10.5: Radiation electric field $\vec{\mathcal{E}}_{\text{rad}}$ for an accelerating charge Q momentarily at rest at the retarded time.

Decompose the Eq. 10.105 electric field into its Coulomb and radiation parts,

$$\vec{\mathcal{E}} = \vec{\mathcal{E}}_{\text{cou}} + \vec{\mathcal{E}}_{\text{rad}}, \tag{10.115}$$

where

$$\begin{aligned} \vec{\mathcal{E}}_{\text{cou}} &= \frac{Q}{4\pi \mathscr{r}^2} \frac{\vec{u}}{\gamma^2 u_{\|}^3}, \\ \vec{\mathcal{E}}_{\text{rad}} &= \frac{Q}{4\pi \mathscr{r}} \hat{\mathscr{r}} \times \left(\frac{\vec{u}}{u_{\|}^3} \times \vec{a} \right). \end{aligned} \tag{10.116}$$

If the charge Q is instantaneously at rest at the retarded time, as in Fig. 10.5, then $v = 0$ and so $\vec{u} = \hat{\mathscr{r}}$, $u_{\|} = 1$, $\gamma = 1$, and

$$\begin{aligned} \vec{\mathcal{E}}_{\text{cou}} \Big|_{v=0} &= \frac{Q}{4\pi \mathscr{r}^2} \hat{\mathscr{r}}, \\ \vec{\mathcal{E}}_{\text{rad}} \Big|_{v=0} &= \frac{Q}{4\pi \mathscr{r}} \hat{\mathscr{r}} \times \left(\hat{\mathscr{r}} \times \vec{a} \right). \end{aligned} \tag{10.117}$$

Expand the radiation field to write

$$\vec{\mathcal{E}}_{\text{rad}} \Big|_{v=0} = \frac{Q}{4\pi \mathscr{r}} \left((\hat{\mathscr{r}} \cdot \vec{a}) \hat{\mathscr{r}} - (\hat{\mathscr{r}} \cdot \hat{\mathscr{r}}) \vec{a} \right) = \frac{Q}{4\pi \mathscr{r}} \left(\vec{a}_{\|} - \vec{a} \right) = -\frac{Q}{4\pi \mathscr{r}} \vec{a}_{\perp}, \tag{10.118}$$

and show that the radiation electric field is proportional to the projection of the retarded acceleration transverse to the line of sight. Thus, if a charge oscillates along the line of sight, at the extremes of its motion, when it is momentarily at rest (at the retarded time), no radiation exists. **Feynman's formula**

$$\vec{\mathcal{E}}_{\text{rad}} = -\frac{Q}{4\pi \mathscr{r}} \vec{a}_{\perp} \tag{10.119}$$

is exact for $v = 0$ and a very good approximation for $v \ll 1$.

10.8 Larmor's Formula

As an accelerating charge moves on its trajectory, it drags along some energy in its near fields. However, the rest of its energy detaches itself from the charge and propagates to infinity as **radiation**. Recalling Eq. 9.91, the radiated power

$$\mathcal{P} = \iint_a \vec{S} \cdot d\vec{a}, \tag{10.120}$$

where the Poynting vector

$$\vec{S} = \vec{\mathcal{E}} \times \vec{\mathcal{B}} = \vec{\mathcal{E}} \times \left(\hat{\boldsymbol{r}} \times \vec{\mathcal{E}} \right) = \mathcal{E}^2 \hat{\boldsymbol{r}} - \left(\hat{\boldsymbol{r}} \cdot \vec{\mathcal{E}} \right) \vec{\mathcal{E}}. \tag{10.121}$$

Use the Eq. 10.115 decomposition of the electric field and note that $\mathcal{E}_{\rm cou} \sim 1/\boldsymbol{r}^2$, $\mathcal{E}_{\rm rad} \sim 1/\boldsymbol{r}$, and $\vec{\mathcal{E}}_{\rm rad} \perp \hat{\boldsymbol{r}}$. Since the area of a sphere centered on the charge increases like $\boldsymbol{r}^2$ as $\boldsymbol{r} \to \infty$, and because the electric fields appear quadratically in the Poynting vector, only $\mathcal{E}_{\rm rad}$ will make a nonzero contribution to the power. Hence, take

$$\vec{S}_{\rm rad} = \mathcal{E}_{\rm rad}^2 \hat{\boldsymbol{r}}. \tag{10.122}$$

If the charge is momentarily at rest, use the Eq. 10.119 Feynman's formula with $\mathsf{a}_\perp = \mathsf{a} \sin\theta$ to write

$$\vec{S}_{\rm rad}\Big|_{v=0} = \left(\frac{Q}{4\pi \boldsymbol{r}} \mathsf{a} \sin\theta \right)^2 \hat{\boldsymbol{r}}. \tag{10.123}$$

The total power radiated is

$$\begin{aligned}
\mathcal{P}\Big|_{v=0} &= \iint_a \vec{S}_{\rm rad}\Big|_{v=0} \cdot d\vec{a} \\
&= \int_{\phi=0}^{2\pi} \int_{\theta=0}^{\pi} \left(\frac{Q}{4\pi \boldsymbol{r}} \mathsf{a} \sin\theta \right)^2 \hat{\boldsymbol{r}} \cdot \Big(\hat{\boldsymbol{r}} (\boldsymbol{r} d\theta)(\boldsymbol{r} \sin\theta d\phi) \Big) \\
&= \left(\frac{Q}{4\pi} \mathsf{a} \right)^2 \int_{\phi=0}^{2\pi} d\phi \int_{\theta=0}^{\pi} \sin^3\theta \, d\theta \\
&= \frac{Q^2 \mathsf{a}^2}{16\pi^2} \, 2\pi \, \frac{4}{3} \\
&= \frac{1}{6\pi} Q^2 \mathsf{a}^2.
\end{aligned} \tag{10.124}$$

Thus, the power radiated is proportional to the square of the charge and the square of the acceleration. **Larmor's formula**

$$\mathcal{P} = \frac{1}{6\pi} Q^2 \mathsf{a}^2 \tag{10.125}$$

is exact for $v = 0$ and a very good approximation for $v \ll 1$.

Problems

1. Explicitly differentiate the Eq. 10.37 advanced potentials to verify that they solve the Eq. 9.69 Maxwell's equations in potential form.

2. Consider a point charge Q moving with *constant* velocity $\vec{v}$.

   (a) Find and sketch its electric field $\vec{\mathcal{E}}$, by starting with Eq. 10.105, but expressing the result in terms of the Fig. 10.3 angle θ.

   (b) Find and sketch the corresponding magnetic field $\vec{\mathcal{B}}$.

Appendix A

Coordinate Systems

Multiple coordinate systems are useful in electromagnetism to solve problems of different symmetries, including rectangular, spherical, and cylindrical.

A-1 Curvilinear Coordinates

Consider a general curvilinear coordinate system $\{u_1, u_2, u_3\}$ whose axes are orthogonal at point. An infinitesimally small cube with edges parallel to the local curvilinear coordinate directions has edges of lengths h_1du_1, h_2du_2, and h_2du_2, as in Fig. A.1.

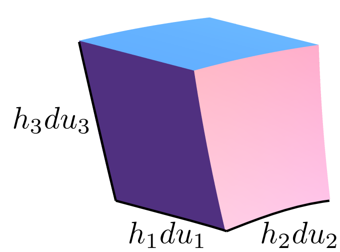

Figure A.1: Generic coordinate system $\{u_1, u_2, u_3\}$ and infinitesimal volume element of size h_1du_1 by h_2du_2 by h_3du_3.

The square of the distance across opposite corners of the cube is

$$ds^2 = (h_1du_1)^2 + (h_2du_2)^2 + (h_3du_3)^2 = h_1^2du_1^2 + h_2^2du_2^2 + h_3^2du_3^2. \tag{A-1}$$

The volume of the cube is

$$dV = (h_1du_1)(h_2du_2)(h_3du_3) = h_1h_2h_3\, du_1du_2du_3. \tag{A-2}$$

A component of the gradient of a scalar field $S[\vec{r}]$ is the change of the scalar field along one edge of the infinitesimal cube divided by the length of that edge. Hence,

$$\vec{\nabla} S = \hat{u}_1 \frac{1}{h_1} \frac{\partial S}{\partial u_1} + \hat{u}_2 \frac{1}{h_2} \frac{\partial S}{\partial u_2} + \hat{u}_3 \frac{1}{h_3} \frac{\partial S}{\partial u_3}. \tag{A-3}$$

The divergence of a vector field $\vec{v}[\vec{r}]$ is the flux of the vector field through the faces of the infinitesimal cube divided by the volume of the cube. Hence,

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} (h_2 h_3\, v_1) + \frac{\partial}{\partial u_2} (h_3 h_1\, v_2) + \frac{\partial}{\partial u_3} (h_1 h_2\, v_3) \right). \tag{A-4}$$

The Laplacian of a vector field is the divergence of the gradient, so

$$\nabla^2 S = \frac{1}{h_1 h_2 h_3} \left(\frac{\partial}{\partial u_1} \left(\frac{h_2 h_3}{h_1} \frac{\partial S}{\partial u_1} \right) + \frac{\partial}{\partial u_2} \left(\frac{h_3 h_1}{h_2} \frac{\partial S}{\partial u_2} \right) + \frac{\partial}{\partial u_3} \left(\frac{h_1 h_2}{h_3} \frac{\partial S}{\partial u_3} \right) \right). \tag{A-5}$$

A component of the curl of a vector field is the circulation of the vector field around a face of the the infinitesimal cube divided by the area of that face. Hence,

$$\begin{aligned} \vec{\nabla} \times \vec{v} = \hat{u}_1 & \frac{1}{h_2 h_3} \left(\frac{\partial}{\partial u_2} (h_3 v_3) - \frac{\partial}{\partial u_3} (h_2 v_2) \right) + \\ \hat{u}_2 & \frac{1}{h_3 h_1} \left(\frac{\partial}{\partial u_3} (h_1 v_1) - \frac{\partial}{\partial u_1} (h_3 v_3) \right) + \\ \hat{u}_3 & \frac{1}{h_1 h_2} \left(\frac{\partial}{\partial u_1} (h_2 v_2) - \frac{\partial}{\partial u_2} (h_1 v_1) \right). \end{aligned} \tag{A-6}$$

A-2 Polar Spherical Coordinates

Define spherical coordinates $\{u_1, u_2, u_3\} = \{r, \theta, \phi\}$ by

$$\begin{aligned} x &= r \sin\theta \cos\phi, \\ y &= r \sin\theta \sin\phi, \\ z &= r \cos\theta, \end{aligned} \tag{A-7}$$

where θ is the co-latitude and ϕ is the longitude, as in Fig. A.2. By inspection, the scale factors

$$\begin{aligned} h_1 &= 1, \\ h_2 &= r, \\ h_3 &= r \sin\theta. \end{aligned} \tag{A-8}$$

Hence, the diagonal square distance

$$ds^2 = dr^2 + (r\, d\theta)^2 + (r \sin\theta\, d\phi)^2 = dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta\, d\phi^2, \tag{A-9}$$

and the elemental volume

$$dV = (dr)(r\, d\theta)(r \sin\theta\, d\phi) = r^2 \sin\theta\, dr\, d\theta\, d\phi. \tag{A-10}$$

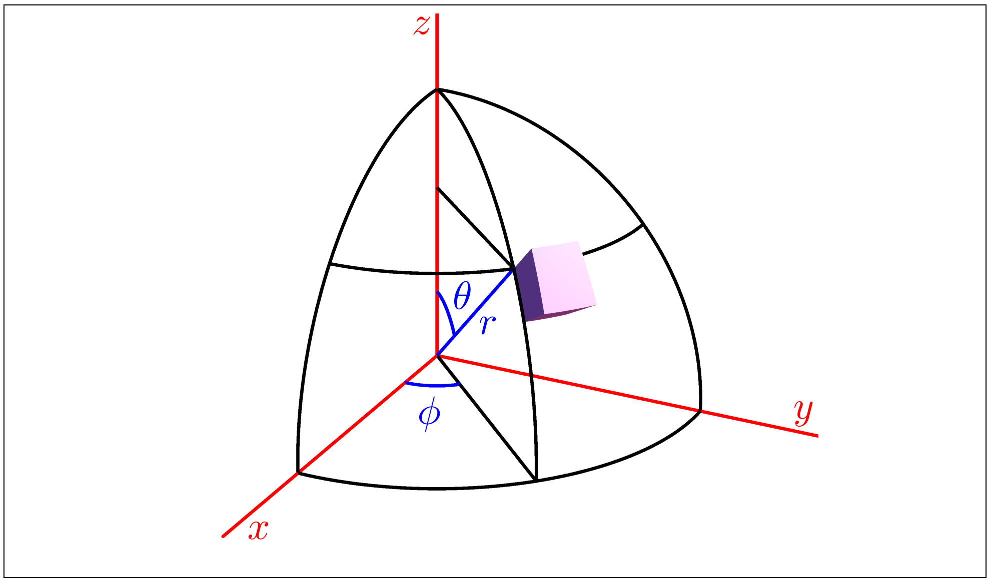

Figure A.2: Polar spherical coordinate system $\{r, \theta, \phi\}$ and infinitesimal volume element of size dr by $r\,d\theta$ by $r \sin\theta\, d\phi$.

The spherical gradient

$$\vec{\nabla} S = \hat{r}\frac{\partial S}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial S}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial S}{\partial \phi}. \tag{A-11}$$

The spherical divergence

$$\vec{\nabla}\cdot\vec{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\, v_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, v_\theta) + \frac{1}{r\sin\theta}\frac{\partial v_\phi}{\partial\phi}. \tag{A-12}$$

The spherical Laplacian

$$\nabla^2 S = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial S}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi}\left(\frac{\partial S}{\partial\phi}\right). \tag{A-13}$$

The spherical curl

$$\begin{aligned}\vec{\nabla}\times\vec{v} = \hat{r}\frac{1}{r\sin\theta}&\left(\frac{\partial}{\partial\theta}(\sin\theta\, v_\phi) - \frac{\partial v_\theta}{\partial\phi}\right) + \\ \hat{\theta}\frac{1}{r}&\left(\frac{1}{\sin\theta}\frac{\partial v_r}{\partial\phi} - \frac{\partial}{\partial r}(r v_\phi)\right) + \\ \hat{\phi}\frac{1}{r}&\left(\frac{\partial}{\partial r}(r v_\theta) - \frac{\partial v_r}{\partial\theta}\right).\end{aligned} \tag{A-14}$$

In the $\theta = \pi/2$ equatorial plane, polar spherical coordinates become polar coordinates $\{r, \phi\}$.

A-3 Cylindrical Coordinates

Define cylindrical coordinates $\{u_1, u_2, u_3\} = \{s, \phi, z\}$ by

$$\begin{aligned} x &= s\cos\phi, \\ y &= s\sin\phi, \\ z &= z, \end{aligned} \tag{A-15}$$

where $s = r_\perp$ is the perpendicular distance form the axis and ϕ is the longitude, as in Fig. A.3. By inspection, the scale factors

$$\begin{aligned} h_1 &= 1, \\ h_2 &= s, \\ h_3 &= 1. \end{aligned} \tag{A-16}$$

Hence, the diagonal square distance

$$ds^2 = dr^2 + (s\,d\phi)^2 + dz^2 = dr^2 + s^2 d\phi^2 + dz^2 \tag{A-17}$$

and the elemental volume

$$dV = (ds)(s\,d\phi)(dz) = s\,dsd\phi\,dz. \tag{A-18}$$

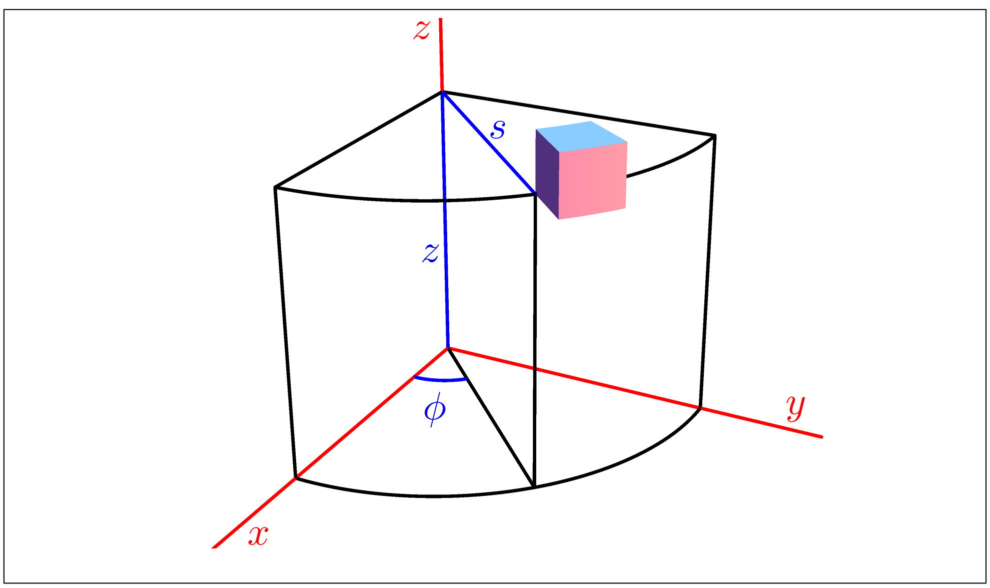

Figure A.3: Cylindrical coordinate system $\{s, \phi, z\}$ and infinitesimal volume element of size ds by $s\,d\phi$ by dz.

The cylindrical gradient

$$\vec{\nabla} S = \hat{s}\frac{\partial S}{\partial s} + \hat{\phi}\frac{1}{s}\frac{\partial S}{\partial \theta} + \hat{z}\frac{\partial S}{\partial z}. \tag{A-19}$$

The cylindrical divergence

$$\vec{\nabla} \cdot \vec{v} = \frac{1}{s}\frac{\partial}{\partial s}(s\, v_s) + \frac{1}{s}\frac{\partial v_\theta}{\partial \phi} + \frac{\partial v_z}{\partial \phi}. \tag{A-20}$$

The cylindrical Laplacian

$$\nabla^2 S = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial S}{\partial s}\right) + \frac{1}{s^2}\frac{\partial}{\partial \phi}\left(\frac{\partial S}{\partial \phi}\right) + \frac{\partial}{\partial z}\left(\frac{\partial S}{\partial z}\right). \tag{A-21}$$

The cylindrical curl

$$\begin{aligned}\vec{\nabla} \times \vec{v} = \hat{s}&\left(\frac{1}{s}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right) + \\ \hat{\phi}&\left(\frac{\partial v_s}{\partial z} - \frac{\partial v_z}{\partial s}\right) + \\ \hat{z}\,\frac{1}{s}&\left(\frac{\partial}{\partial s}(s v_\phi) - \frac{\partial v_s}{\partial \phi}\right).\end{aligned} \tag{A-22}$$

Appendix B

Product Rules

In **one-dimensional calculus**, the product rule for differentiation is

$$\frac{d}{dx}(fg) = \frac{df}{dx}g + f\frac{dg}{dx}. \tag{A-1}$$

In **three-dimensional calculus**, different kinds of products and derivatives resulting in six different product rules exist. Derive these by carefully employing the mixed "box" product

$$\vec{A} \cdot \left(\vec{B} \times \vec{C}\right) = \vec{C} \cdot \left(\vec{A} \times \vec{B}\right) = \vec{B} \cdot \left(\vec{C} \times \vec{A}\right), \tag{A-2}$$

the vector double product

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}\left(\vec{A} \cdot \vec{C}\right) - \vec{C}\left(\vec{A} \cdot \vec{B}\right) \tag{A-3}$$

or

$$\vec{B}\left(\vec{A} \cdot \vec{C}\right) = \vec{A} \times \left(\vec{B} \times \vec{C}\right) + \vec{C}\left(\vec{A} \cdot \vec{B}\right), \tag{A-4}$$

and the linearity of the derivative

$$\vec{\nabla} = \vec{\nabla}_A + \vec{\nabla}_B. \tag{A-5}$$

The **gradient** of the product of two scalar fields is

$$\vec{\nabla}\left(fg\right) = \left(\vec{\nabla} f\right) g + f\left(\vec{\nabla} g\right). \tag{A-6}$$

The gradient of the scalar product of two vector fields is

$$\begin{aligned}
\vec{\nabla}\left(\vec{A} \cdot \vec{B}\right) &= \vec{\nabla}_A\left(\vec{A} \cdot \vec{B}\right) + \vec{\nabla}_B\left(\vec{A} \cdot \vec{B}\right) \\
&= \vec{\nabla}_A\left(\vec{A} \cdot \vec{B}\right) + \vec{\nabla}_B\left(\vec{B} \cdot \vec{A}\right) \\
&= \vec{B} \times \left(\vec{\nabla}_A \times \vec{A}\right) + \left(\vec{B} \cdot \vec{\nabla}_A\right)\vec{A} + \vec{A} \times \left(\vec{\nabla}_B \times \vec{B}\right) + \left(\vec{A} \cdot \vec{\nabla}_B\right)\vec{B} \\
&= \vec{B} \times \left(\vec{\nabla} \times \vec{A}\right) + \left(\vec{B} \cdot \vec{\nabla}\right)\vec{A} + \vec{A} \times \left(\vec{\nabla} \times \vec{B}\right) + \left(\vec{A} \cdot \vec{\nabla}\right)\vec{B}
\end{aligned} \tag{A-7}$$

The **divergence** of the product of a scalar field and a vector field is

$$\vec{\nabla} \cdot \left(f\vec{A} \right) = \left(\vec{\nabla} f \right) \cdot \vec{A} + f \left(\vec{\nabla} \cdot \vec{A} \right). \tag{A-8}$$

The divergence of the vector product of two vector fields is

$$\begin{aligned} \vec{\nabla} \cdot \left(\vec{A} \times \vec{B} \right) &= \vec{\nabla}_A \cdot \left(\vec{A} \times \vec{B} \right) + \vec{\nabla}_B \cdot \left(\vec{A} \times \vec{B} \right) \\ &= \vec{\nabla}_A \cdot \left(\vec{A} \times \vec{B} \right) - \vec{\nabla}_B \cdot \left(\vec{B} \times \vec{A} \right) \\ &= \vec{B} \cdot \left(\vec{\nabla}_A \times \vec{A} \right) - \vec{A} \cdot \left(\vec{\nabla}_B \times \vec{B} \right) \\ &= \vec{B} \cdot \left(\vec{\nabla} \times \vec{A} \right) - \vec{A} \cdot \left(\vec{\nabla} \times \vec{B} \right) \end{aligned} \tag{A-9}$$

The **curl** of the product of a scalar field and a vector field is

$$\vec{\nabla} \times \left(f\vec{A} \right) = \left(\vec{\nabla} f \right) \times \vec{A} + f \left(\vec{\nabla} \times \vec{A} \right). \tag{A-10}$$

The curl of the vector product of two vector fields is

$$\begin{aligned} \vec{\nabla} \times \left(\vec{A} \times \vec{B} \right) &= \vec{\nabla}_A \times \left(\vec{A} \times \vec{B} \right) + \vec{\nabla}_B \times \left(\vec{A} \times \vec{B} \right) \\ &= \vec{\nabla}_A \times \left(\vec{A} \times \vec{B} \right) + \vec{\nabla}_B \times \left(\vec{A} \times \vec{B} \right) \\ &= \left(\vec{B} \cdot \vec{\nabla}_A \right) \vec{A} - \vec{B} \left(\vec{\nabla}_A \cdot \vec{A} \right) + \vec{A} \left(\vec{\nabla}_B \cdot \vec{B} \right) - \left(\vec{A} \cdot \vec{\nabla}_B \right) \vec{B} \\ &= \left(\vec{B} \cdot \vec{\nabla} \right) \vec{A} - \vec{B} \left(\vec{\nabla} \cdot \vec{A} \right) + \vec{A} \left(\vec{\nabla} \cdot \vec{B} \right) - \left(\vec{A} \cdot \vec{\nabla} \right) \vec{B} \end{aligned} \tag{A-11}$$

Problems

1. Explicitly verify the following vector derivative identities by expanding both sides in components.
   (a) Eq. A-9.
   (b) Eq. A-11.

Appendix C

Electromagnetic Units

The "natural" units used in this text best reflect the geometric unity of spacetime: if distances and durations are both measured in meters (for example), then light speed is one, velocities are dimensionless ratios, and electric and magnetic fields share the same dimension (of charge per length squared). Table A-1 summarizes familiar electromagnetic equations in three commonly used systems of units, while Table A-2 summarizes the natural dimensions of important variables.

Table A-1: Important electromagnetic equations in three common systems of units.

Natural	Gaussian	SI
$\vec{\nabla}\cdot\vec{\mathcal{E}}=\rho$ $\vec{\nabla}\cdot\vec{\mathcal{B}}=0$ $\vec{\nabla}\times\vec{\mathcal{E}}=-\partial_t\vec{\mathcal{B}}$ $\vec{\nabla}\times\vec{\mathcal{B}}=+\partial_t\vec{\mathcal{E}}+\vec{J}$	$\vec{\nabla}\cdot\vec{\mathcal{E}}=4\pi\rho$ $\vec{\nabla}\cdot\vec{\mathcal{B}}=0$ $\vec{\nabla}\times\vec{\mathcal{E}}=-\frac{1}{c}\partial_t\vec{\mathcal{B}}$ $\vec{\nabla}\times\vec{\mathcal{B}}=+\frac{1}{c}\partial_t\vec{\mathcal{E}}+\frac{4\pi}{c}\vec{J}$	$\vec{\nabla}\cdot\vec{\mathcal{E}}=\frac{1}{\epsilon_0}\rho$ $\vec{\nabla}\cdot\vec{\mathcal{B}}=0$ $\vec{\nabla}\times\vec{\mathcal{E}}=-\partial_t\vec{\mathcal{B}}$ $\vec{\nabla}\times\vec{\mathcal{B}}=+\mu_0\epsilon_0\partial_t\vec{\mathcal{E}}+\mu_0\vec{J}$
$\vec{F}=q\left(\vec{\mathcal{E}}+\vec{v}\times\vec{\mathcal{B}}\right)$	$\vec{F}=q\left(\vec{\mathcal{E}}+\frac{\vec{v}}{c}\times\vec{\mathcal{B}}\right)$	$\vec{F}=q\left(\vec{\mathcal{E}}+\vec{v}\times\vec{\mathcal{B}}\right)$
$\mathcal{E}=\frac{Q}{4\pi r^2}$ $\mathcal{B}=\frac{I}{2\pi s}$	$\mathcal{E}=\frac{Q}{r^2}$ $\mathcal{B}=\frac{2}{c}\frac{I}{s}$	$\mathcal{E}=\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}$ $\mathcal{B}=\frac{\mu_0}{2\pi}\frac{I}{s}$
$\vec{\mathcal{E}}=\vec{\mathcal{D}}-\vec{\mathcal{P}}$ $\vec{\mathcal{B}}=\vec{\mathcal{H}}+\vec{\mathcal{M}}$	$\vec{\mathcal{E}}=\vec{\mathcal{D}}-4\pi\vec{\mathcal{P}}$ $\vec{\mathcal{B}}=\vec{\mathcal{H}}+4\pi\vec{\mathcal{M}}$	$\vec{\mathcal{E}}=\frac{1}{\epsilon_0}\vec{\mathcal{D}}-\frac{1}{\epsilon_0}\vec{\mathcal{P}}$ $\vec{\mathcal{B}}=\mu_0\vec{\mathcal{H}}+\mu_0\vec{\mathcal{M}}$

Table A-2: Dimensions of important electromagnetic variables in natural units.

Name	Symbol(s)	Dimensions
time	t, T	$\mathcal{L}$
position	$r, R, \mathscr{r}, s$	$\mathcal{L}$
area	a	$\mathcal{L}^2$
volume	V	$\mathcal{L}^3$
velocity	v, u	1
acceleration	a	$\mathcal{L}^{-1}$
mass	m, M	$\mathcal{M}$
force	f, F	$\mathcal{M}\mathcal{L}^{-1}$
torque	τ	$\mathcal{M}$
energy	E	$\mathcal{M}$
work	W	$\mathcal{M}$
power	$\mathcal{P}$	$\mathcal{M}\mathcal{L}^{-1}$
linear momentum	p, P	$\mathcal{M}$
angular momentum	L	$\mathcal{M}\mathcal{L}$
charge	q, Q	$\mathcal{Q}$
current	I	$\mathcal{Q}\mathcal{L}^{-1}$
charge density	ρ	$\mathcal{Q}\mathcal{L}^{-3}$
current density	J	$\mathcal{Q}\mathcal{L}^{-3}$
electric field	$\mathcal{E}$	$\mathcal{Q}\mathcal{L}^{-2}$
magnetic field	$\mathcal{B}$	$\mathcal{Q}\mathcal{L}^{-2}$
flux	$\Phi_{\mathcal{E}}, \Phi_{\mathcal{B}}$	$\mathcal{Q}$
circulation	$\Gamma_{\mathcal{E}}, \Gamma_{\mathcal{B}}$	$\mathcal{Q}\mathcal{L}^{-1}$
electric potential	φ	$\mathcal{Q}\mathcal{L}^{-1}$
magnetic potential	$\mathcal{A}$	$\mathcal{Q}\mathcal{L}^{-1}$
ohmic resistance	$\mathcal{R}$	1

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