An Overview of Mechanisms and Patterns with Origami

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An overview of mechanisms and patterns with origami

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SUMMARY

Origami (paperfolding) has greatly progressed since its first usage for design of cult objects in Japan, and entertainment in Europe and the USA. It has now entered into artistic areas using many other materials than paper, and has been used as an inspiration for scientific and engineering realizations. This article is intended to illustrate several aspects of origami that are relevant to engineering structures, namely: geometry, pattern generation, strength of material, and mechanisms. It does not provide an exhaustive list of applications nor an in-depth chronology of development of origami patterns, but exemplifies the relationships of origami to other disciplines, with selected examples.

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Key words: geometry, tessellation, paperfolding, crease pattern, foldability, deployability

1 INTRODUCTION

Origami (from Japanese *oru*, to fold and *kami*, paper), is the ancient art of paperfolding. Traditional origami usually involves only straight folds on a (square) planar piece of paper; tearing, cutting or gluing are not allowed. Once folded, the origami constitutes a developable surface that can be unfolded as a flat plane (it is isometric to a planar surface).

Part of the activities of paperfolders worldwide is related to geometry. Starting, for instance, from an initial flat square piece of paper, its folding allows to design patterns (flat or non flat) related to geometry and/or art. The folding act itself involves the flat sheet as a particular mechanism, since the small ratio of the thickness over the size of the paper allows bending along folding lines and no stretching of the in-plane fabric. Additionally, and depending on the application, the mechanical model of a flat sheet of paper can be related to a membrane with no bending stiffness and a high stretching stiffness (its transformation therefore relies on minimal energy movement [53], [13], i.e. bending only), or to rigid plates linked along their edges (a model named as 'rigid origami'). Nevertheless, large deformations occur along folding lines that may be considered as hinges between assemblies of individual paper portions containing no fold.

Much of the realizations in origami lead to flat folded models, though some 3D models are available. This generation of 3D structures is of interest in other disciplines such as problems related to forming of steel sheets [6], architecture and furniture [51], structural engineering [50] and medical devices [37], modeling chemical 3D structures [8], micro or nano-structures [1], [2]... with designs sometimes inspired by biomechanical solutions [24].

In this article, we intend to present some overview of origami and origami-like structures and patterns, related to geometry (for which a reference book is [11]) and engineering mechanisms.

2 UNDERLYING STRUCTURE OF THE UNFOLDED PLANE

Some origami models are strongly related to Euclidean geometry [29] since the original material is a flat surface, and the folding steps may reproduce straight-edge-and-compass constructions (see [7] for instance), as a fold is similar to a straight line, and folding paper in half allows to construct distances between points a similar manner to that of a compass. Indeed, different angle values, as well as different numbers (their sines and cosines), can be constructed depending on which basic construction rules can be used, which can be extended in paper folding. A complex model can be partially described by the network of folds on the unfolded paper as discussed in the following sections, which focus on straight-edge folds and planar developable case.

2.1 Crease pattern

Viewed from one face of the unfolded paper, the folds may be classified in two sets: either mountain or valley folds, as illustrated with two basic origami bases, known as 'waterbomb' and 'preliminary' bases, in Figure 1. As expected, mountain folds locally leads to a convex 3D shape (locally in terms of spatial position, and at the beginning of the 3D shaping, i.e. when the folding angle is still small), while valley folds locally leads to concave 3D shape. Indeed, the two fold sets are interchanged when changing the point of view, i.e. the face of the paper. In such a description, the two types of folds can be considered as dual to each other.

Defining an origami model only by the representation of the creases with their type, on the unfolded flat piece of paper, is named its 'crease pattern'. The information contained in this crease pattern is generally insufficient to describe the complete folded model. The determination of a general ability of a crease pattern to fold flat is a non-trivial question [30], [4]. Moreover, if a crease pattern leads to a flat model, its uniqueness is not ensured. In the case of the two bases of Figure 1, the crease pattern is identical (to a dualization with paper reversal) and leads to two different models that can also be qualified as dual. Note that only 6 radiating folds over 8 are necessary to get a flat folded model, but that 8 folds preserve a higher symmetry in the models.

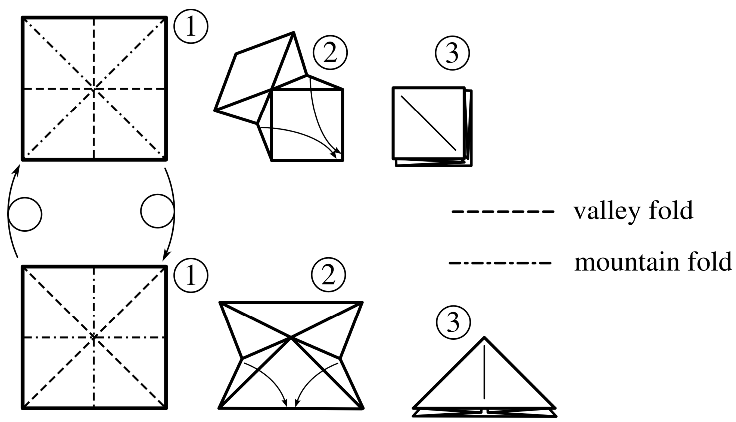

Figure 1: Crease patterns and folding steps for the preliminary base (top) and waterbomb base (bottom).

The set of folds on a flat sheet can also be split into two complementary subsets, depending on their orientation. Indeed, if starting from a square piece of paper, the boundary of the paper is of major importance on the network of creases. In traditional origami, many models emphasize creases starting from corners and splitting angles at the vertex of the paper. More recently, original models have gained from the extensive usage of creases parallel to the edges (the so-called 'box pleating' technique). As an illustration, one can compare the (simplified) crease patterns of the two models depicted in Figure 2, both from R. J. Lang[1]. They clearly make use of different networks of creases. Note that in the simple crease pattern of the previous bases of Figure 1, the two networks (i.e. radial and parallel networks) are present and are dual in their the crease type.

[1] http://www.langorigami.com

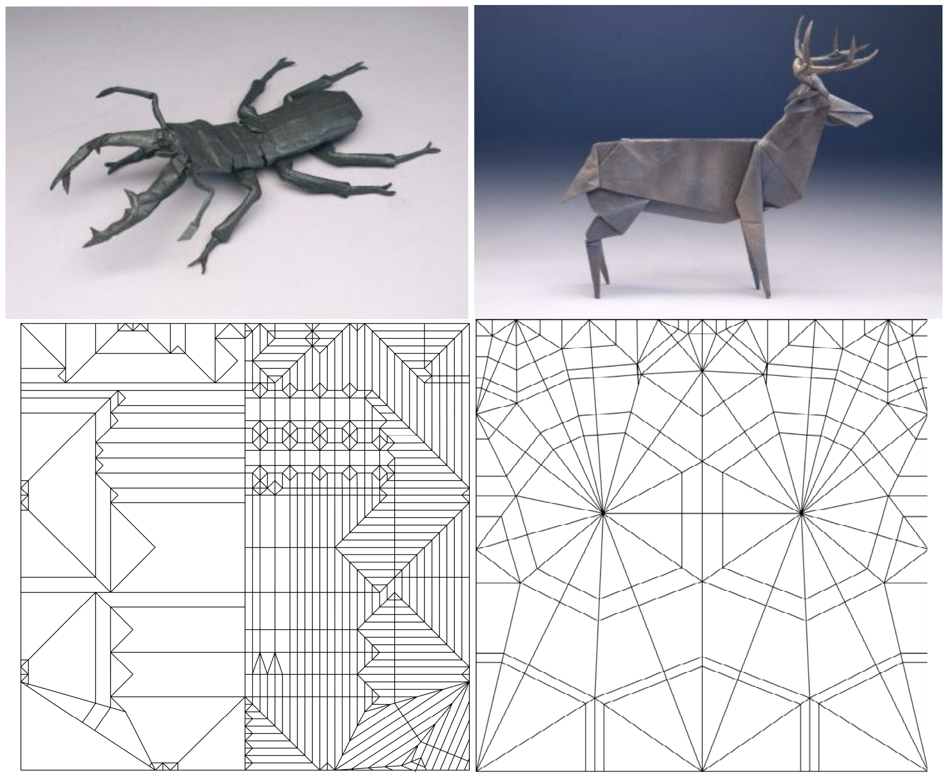

Figure 2: Top: Stag Beetle BP, opus 477 and White-tailed Deer, opus 550. Bottom: their respective crease patterns. Both models designed and folded by R. J. Lang (2008), with permission

2.2 Edge diagram

Finally, another duality using initial paper edge as a major ingredient (as a sort of edge effect) is the so-called 'color change' technique. Consider a piece of paper with its two faces with two different colors. When folding it (for instance with flat folding), the resulting color appearance can reveal special patterns. In such a case, the two faces are not considered as interchangeable one with the other, but as dual objects. An example of a challenging model is the chessboard folding problem, depicted in Figure 3, with its edge diagram (the curve of the perimeter of the starting square of paper, drawn on the final folded model). A variant is a version of the dragon curve [10], one intermediate construction step of a fractal space-filling curve. Some fractal geometry is therefore also constructible with origami [39]. Another particular use of geometry with two colored faces, dealt with equivalently, is the so-called 'iso-area' folding technique (obverse and reverse sides appear identically) [33].

Though in many cases, the edge of the initial piece of paper is tremendously used, some folded patterns do not use it at all. This is the example of the periodic tessellations (or tilings) [49], for which the pattern can be extended at will, depending on the ratio between the paper size and the elementary pattern size, since the entire plane can be covered by translating a fundamental region (or a unit cell) in each direction.

Figure 4 shows a tessellation designed by C. Bettens and called 'Château-Chinon' in [22][2] as well as its edge diagram: Since it is restricted to the edge of the model, it illustrates the possibility of assembling more tessellation modules to cover the entire plane.

[2] http://www.origamitessellations.com/

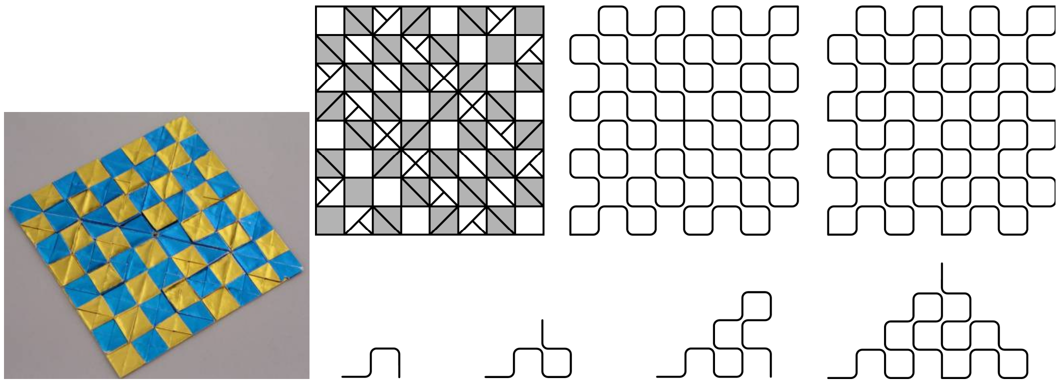

Figure 3: Two-color chessboard folded from a single square piece of paper (after [15]) and two feasible edge diagrams (top); a space-filling dragon curve generation (bottom)

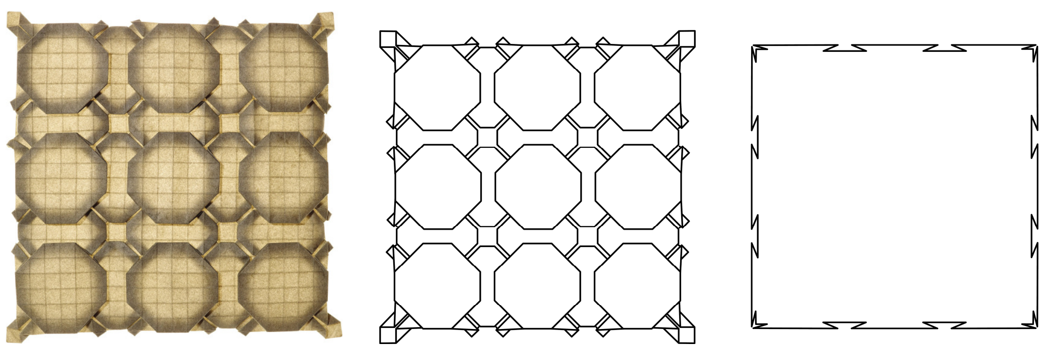

Figure 4: Left: 'Château-Chinon' planar tessellation designed by C. Bettens, model and image by E. Gjerde after [22], with permission A K Peters, Ltd. (2009). Right: its edge diagram

2.3 Textures and pattern generation

A particular application of tessellations concerns the design of pleated fabric used, for instance, for fashion designs. This is produced by steaming, pressing and drying fabric once it has been shape-folded. To do so, two cardboard 'moulds' are pre-folded and the fabric is squeezed between them when the whole laminate is folded into the desired shape. Figure 5 shows some of the fabrication steps performed in Atelier G. Lognon, Paris. The resulting small scale tessellation renders somehow as a texture on the main fabric.

An interesting feature of planar tessellations is their ability to map to a warped surface. Two single-piece origami models using this feature are depicted in Figure 6: the Koi fish of R. J. Lang[1] [38] where a regular tessellation is mapped on a mean curved surface and the pangolin of E. Joisel[3]. In this case, if a continuous spatial transformation (or distortion) is applied to the successive cells of the pattern so that each neighboring cell is slightly different one from the other, the mean surface may not be developable. The technique of cell distortion has also been applied to the tensegrity grids, with a uniform distortion in [44], to produce a spatial grid with curvatures of its mean surface.

[3] http://www.ericjoisel.com

Figure 5: Shaping pleated fabric, copyright for images owned by Jorge Ayala Studio (2008), with permission

Figure 6: Mapping non planar surfaces with tessellations made from a single piece of paper. Left: Koi opus 425 designed and folded by R. J. Lang, with permission, after [38]. Right: pangolin designed and folded by E. Joisel (approx. 2000), photography by G. Aharoni[4] (2007), with permission

Such pattern assemblies to make larger structures possible can also be designed with a different technique: the modular origami [20]. In this case, the local unit cell is obtained by folding a single piece of paper and the overall structure is the assembly of many individual non-connected origami units. Another technique uses assembly of strips of paper. In all of these cases, the stability of the folded model (and therefore part of its rigidity) depends on the 'lockings' which rely on friction between paper flaps when no gluing is allowed. The last technique also leads to applications in architecture with thin woven wooden strips [55], as shown in Figure 7. The main concern is also related to the locking of such structures, here with clamped boundary conditions, which needs to be resolved.

[4] see http://www.giladorigami.com

Figure 7: Woven structure, after [55], with permission

3 CURVATURE OF SURFACES AND FOLDS

For the previous examples, curvature was obtained with pattern assemblies; however, most origami models are composed of flat portions of paper (with no crease crossing them); when built from a single planar paper sheet, they can obviously be entirely unfolded flat. For the design of spatial structures, one can imagine flat —or curved— parts that are assembled in their final configuration which cannot be entirely unfolded flat. Some generalizations can be made from analysis of the surface curvature obtained either with a curved surface as a starting point, or with curved folds. The following sections focus on such issues, together with their limitations.

3.1 Intrinsic geometry of a single developable sheet

Even with straight-edge folds only, a sheet of paper may lead to a spatial structure if it is not flat folded. The properties a network of creases should possess to get a given target result is not trivial. Even the problem of finding if a given set of straight creases on a flat piece of paper (and even without prescribing the creases to be mountain or valley folds) will lead to a flat folded model (without paper interpenetration) is not obvious. A classical result nevertheless concerns the local flat-foldability condition: it is a necessary condition to have a flat folded state and states a condition in the vicinity of each vertex from which some straight creases radiates, as shown in Figure 8.

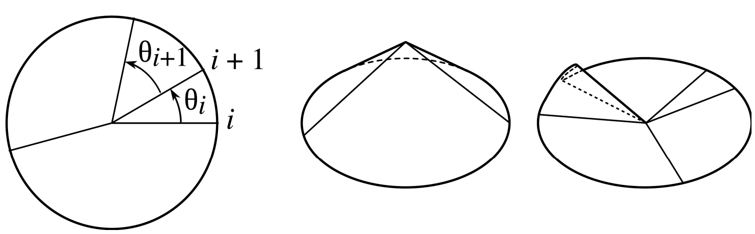

Figure 8: Illustration of local flat foldability theorem, for the flat unfolded state (left), and two non-flat states (center and right)

First, due to the presence of folds, the obtained spatial surface is not smooth enough to define its Gaussian curvature. In such a case, the so-called angle deficit is used at each fold-crossing point (or vertex). This tool is also useful if an initial non-flat sheet is used. Consider such n creases, separated by angles θ_i (i = 1, ..., n), each in $]0,2\pi[$. If the unfolded state is flat, they sum to

$$\sum_{i=1}^{n} \theta_i = 2\pi \tag{1}$$

For the case where the mountain-valley fold types has not been assigned to a crease pattern, the Kawasaki-Justin theorem claims that for a disk centered on a unique vertex to fold flat, one has to check that n is even, and

$$\sum_{1 \le 2j+1 \le n-1} \theta_{2j+1} = \sum_{2 \le 2j \le n} \theta_{2j}$$

(which is equal to π). A generalization is given in [11] for eventually non-flat unfolded states (the global sum may be greater or smaller than 2π); the condition is: n even, and

$$\sum_{i=1}^{n} (-1)^i \theta_i \in \{0, 2\pi, -2\pi\} \tag{2}$$

(one can check that the flat unfolded state is a particular case).

When the mountain-valley fold types are assigned, the constraints are stronger, and recent results concerning this situation can be found in [11]. Note that the previous conditions are not *sufficient*, though they are useful design guides. If a general mathematical result for flat foldable models is not trivial, so is the folding of 3D models [3], [9].

Concerning the design of 3D structures with curved surfaces, a particular care has been drawn for ruled surfaces and developable surfaces [23]; indeed, they do possess interesting properties from a design point of view. For instance, thin-walled concrete structures can be obtained with a framing if ruled, as in the hyperbolic paraboloid of Figure 10 (right). In this last case, the Gaussian curvature of the surface is negative (the equivalent of an angle deficit greater than 2π) and therefore the surface cannot be developed to a plane. On the other hand, an interest in developable surfaces lies in their potential design from steel sheets with a small forming energy, and particular curvature on developable surfaces can also be obtained with curved creases (though Gaussian curvature remains zero) [13].

In any case (straight or curved creases), parts of paper between folds are usually developable surfaces, i.e. for sufficiently smooth surfaces, they are locally also ruled as either conical or cylindrical surfaces [36], [14], [12]. With bending of such portions, the overall folded state will exhibit curvature. For a discussion on the relationship between curved crease geometry and surface curvature between the folds, the reader can refer to [28], [9]. Figure 9 illustrates some possibilities of surface generation from curved creases on a developable surface.

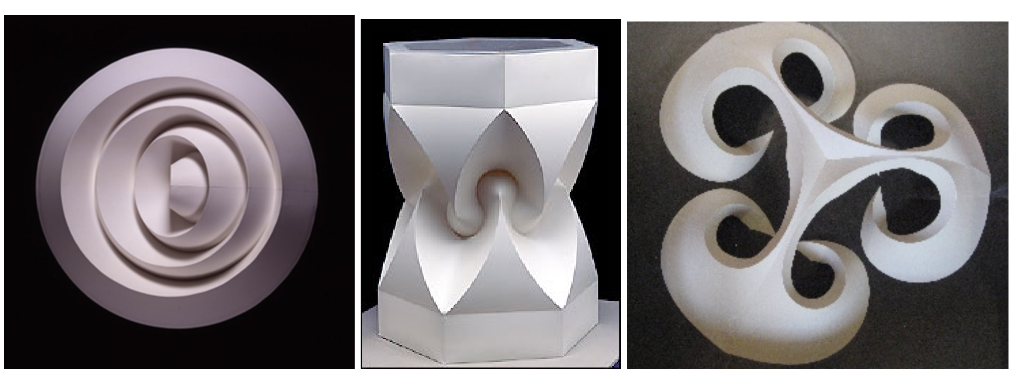

Figure 9: Curved folds and faces, designed by D. Huffman (left and middle, see [28]) and R. Resch (right)

3.2 Unwrapping

The unwrapping or unfolding problem [11] is somehow the opposite point of view: Given a spatial surface, can it be developable as a whole or by parts?

One illustration of this problem is the case of Walt Disney Concert Hall designed by F. Gehry [21], Figure 10 (on the left). Each building face is a curved developable surface that can be unfolded plane; indeed, it has been assembled from several individual bent plane panels placed edge-to-edge as in a tessellation. A matching defect can nevertheless be seen at the overall building edges and corresponds to a non-global developable surface.

Figure 10: Two usages of particular curved surfaces. Left: The Los Angeles Walt Disney Concert Hall with a stainless steel façade, piece-wise developable, image by J. Sullivan (2004), courtesy of PDPhoto.org. Right: Water tower in La-Roche-de-Glun (France), with a one-sheet hyperboloid ruled surface; image by A. Rambaud (2007), with permission

On the other hand, curved surfaces that are entirely flat unfoldable can also be obtained. Figure 11 illustrates recent results on automatic construction of developable surfaces with curved folds, matching as close as possible some given free-form surfaces.

3.3 Assemblies

Origami-like configurations using assembly of several (sometimes many) flat elementary parts have also been designed, such as in Figure 12. For these examples, only the geometry of the final structure is inspired by origami, but the folding process is not used.

4 FOLDING AND MECHANISMS

Potentially interesting structures for Civil Engineering are deployable structures [19]. Folding of an origami model involves look-a-like mechanisms: rigid origami considers rigid paper portions (not crossed by any crease) linked together with perfect hinges that model the straight creases (the case of curved creases is slightly different since their folding necessarily bends the paper; this can be considered as involving a deformation with a small energy but not zero energy as in pure mechanism movements; nevertheless, this stored energy may also be useful as an elastic spring acting as an actuator for desired 'unfolding' movement).

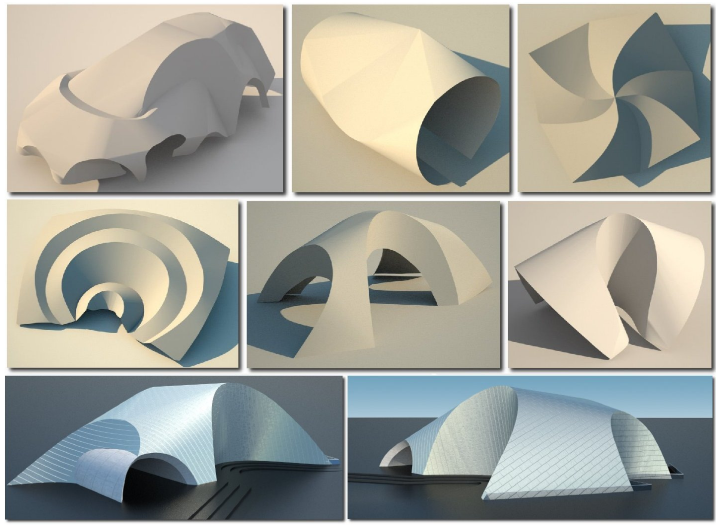

Figure 11: Automatic development of curved surfaces, with the help of computational geometry. Up: digital paper models, bottom: architectural design, after [34], with permission

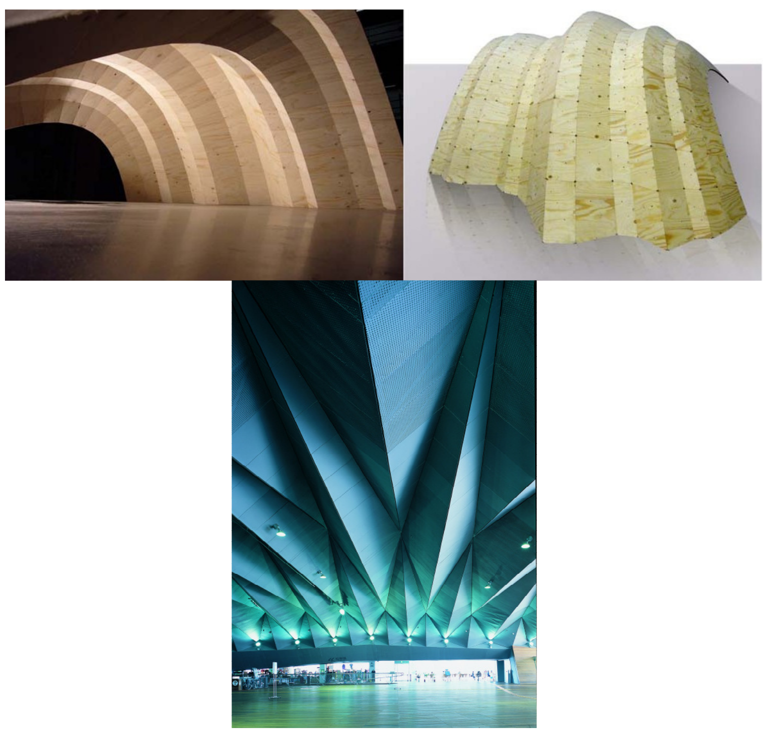

Figure 12: Top: architectural spatial structure made out of block timber panels, after [55], with permission. Bottom: The Yokohama International Port Terminal, courtesy of Foreign Office Architects, image by S. Mishima (2002), with permission

4.1 Assembly of planar developable units

As a first illustration, consider again the waterbomb base of Figure 1, and its rigid folding in Figure 13. Using the mechanism theory (see [45] for instance), the intermediate configuration on the right of Figure 13, is a mechanism with $m = 3$ independent infinitesimal degrees of freedom and a number of independent states of self-stress (or of degrees of static indeterminacy) $s = 3$, while the unfolded flat state, on the left of Figure 13, is a singular configuration for which $m = 4$ and $s = 4$.

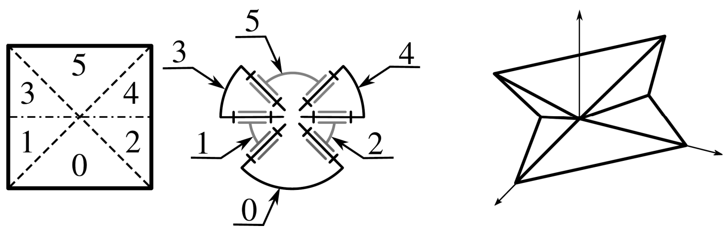

Figure 13: Waterbomb base seen as a mechanism by rigid folding: unfolded flat configuration (left) and half folded configuration (right)

The assembly of units made with these bases, in the tessellation-like design of Figure 14, left, can preserve some of the mechanisms of the elementary units, and allows the design of a medical deployable stent [37]. Some elementary mechanisms can nevertheless be eliminated by assembling several units [25], as in assembly of elementary tensegrity units [17]. Indeed, the rigid folding does have strong connection with mechanisms in truss foldable structures, for which bars are assumed rigid and linked together at their end points with spherical joints (or rotoids) [57].

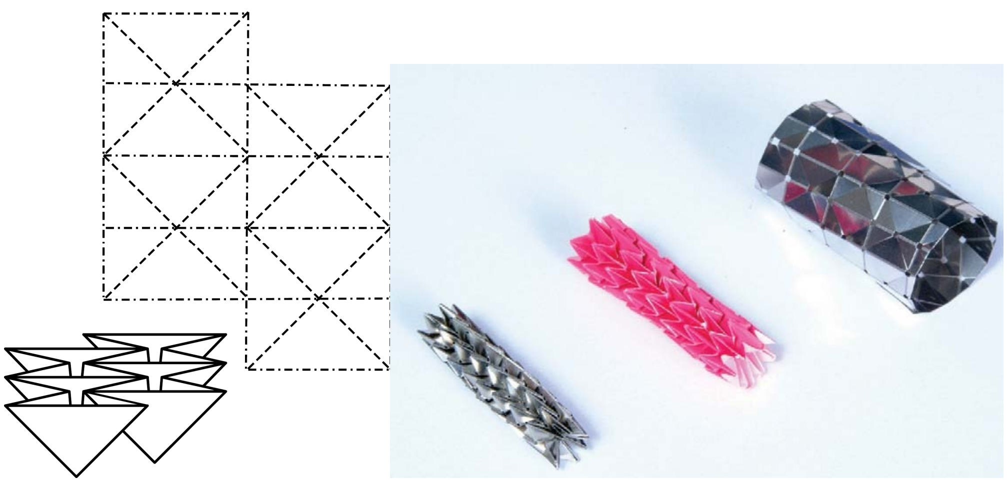

Figure 14: Assembly of waterbomb base units and partial crease pattern (left), corresponding stent design, reprinted from [58], (copyright 2007) with permission from Elsevier

Other patterns can be used, especially when one wishes to control the number of mobilities in the assembly. For instance, the Miura-Ori fold [41], designed for packing large membrane-like structures and folding/unfolding of geographic maps, uses the crease pattern of Figure 15. With the rigid folding model, and except for the two singular configurations of flat-folded and flat-unfolded states, the Miura-Ori folds are mechanisms with exactly 1 independent infinitesimal degree of freedom, whatever the number of cells is. This can be proved by recursion with the following steps:

- with 2 rows, and 2 columns, it is easy to check that the mechanism in Figure 17 (top left) is isostatic with 1 mobility;
- with 2 rows and 3 columns, 2 bodies are added, and to get an isostatic mechanism without adding any mobility, the additional 3 creases can be modeled as in Figure 17 (top right) to get 12 additional static unknowns in the linkages;
- with 3 rows and 3 columns, 3 new bodies are added, and to get an isostatic mechanism without adding any mobility, the additional 5 creases can be modeled as in Figure 17 (bottom left) to get 18 additional static unknowns;
- then, denoting with k the case with r rows and c columns, assuming that it is isostatic with 1 mobility, the case $k + 1$ corresponds to r rows and $c + 1$ columns: therefore, r bodies are added, and $5r + 2 + (r - 2) = 6r$ static unknowns are added, this is still an isostatic mechanism with 1 mobility;
- finally, the case $k + 2$ corresponds to $r + 1$ rows and $c + 1$ columns: therefore, c bodies are added, and $5c + 2 + (c - 2) = 6c$ static unknowns are added, and this is still an isostatic mechanism with 1 mobility.

The CAD model of Figure 16 illustrates this mechanism; it has been designed with an assembly of rigid solids with joints conforming with the equivalent isostatic mechanism of Figure 17. Modeling all the creases as perfect joints does not change the kinematics of the rigid mechanism, but increases its static indeterminacy up to $s = 4rc - 5(r + c) + 7$ which is also the number of geometric constraints to satisfy (here, parallel creases) to get a working rigid mechanism.

These constraints are additional ones when compared to the local flat-foldability ones (1), (2). Indeed, these last ones have to be satisfied, and the reader can check that they are in Figure 15 (bottom), where the angles surrounding each vertex are recalled. The same constraints are also satisfied for the case where irregular extensions of the Miura-Ori fold are designed, still possessing 1 mobility [47].

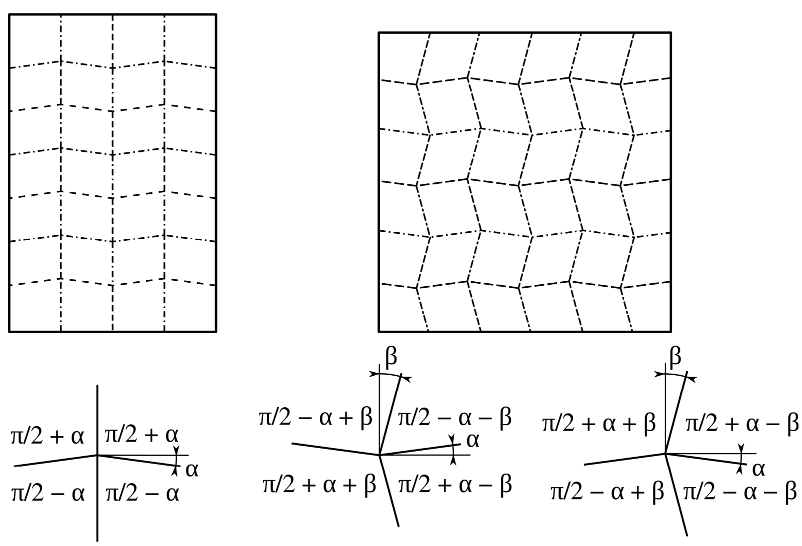

Figure 15: Miura-Ori crease patterns (upper left: genuine design, upper right: generalized design, bottom: angles surrounding each vertex)

Figure 18 is part of such a crease pattern that leads to a planar tessellation if extended in any direction. It also depicts a folded plane of paper with this pattern. One can remark that this last one can have a mean surface that is not planar: The previously described rigid folding is therefore not fulfilled here (it predicts a mechanism with 1 mobility and a planar mean surface); the obtained mean curvature arises from the small bending of the uncreased parts, which is possible because they are quadrilaterals that can be wrapped while their edges (the creases) can be straight segments.

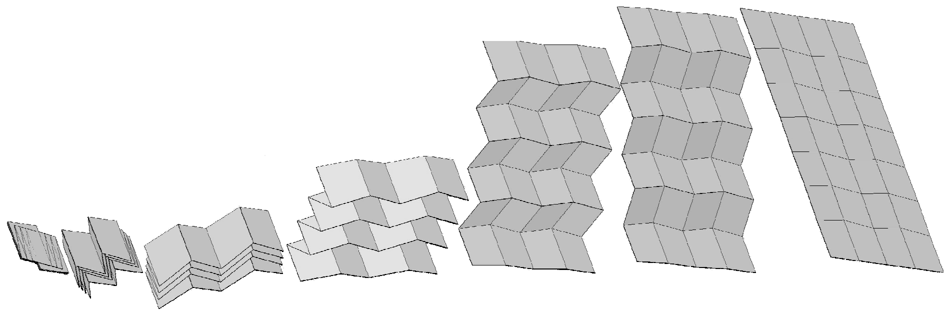

Figure 16: The single mechanism of the Miura-Ori fold (several steps of the unfolding process, from left to right, from the flat folded state to the flat unfolded state)

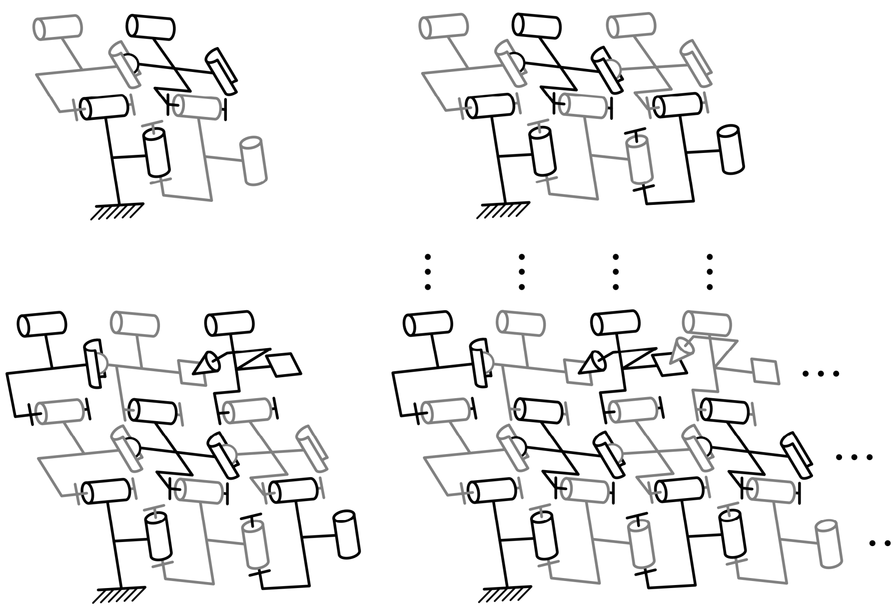

Figure 17: An isostatic mechanism equivalent to the rigid folding of the Miura-Ori. Top left: 2 rows and 2 columns; top right: adding 2 columns; bottom left: adding 3 rows; bottom right: arbitrary size.

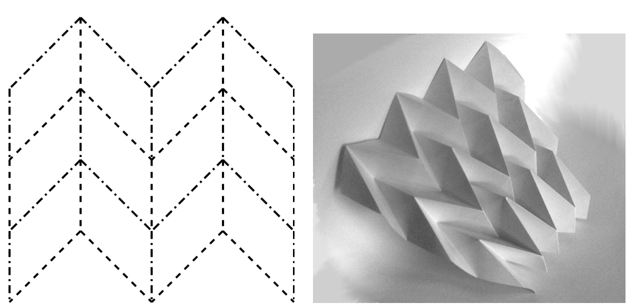

Figure 18: A tessellation similar to Miura-Ori fold (left) and a sketch of the partially folded state (right)

Other mechanisms belonging to the same family have been studied for aerospace engineering applications, as for deployability of large space telescope surfaces [27], or solar sails, such as the Ikaros project of the Japan Aerospace Exploration Agency[5].

Such technological solutions are not so different from biomechanical ones; for instance, for tree leaves unfolding [35]: see the corrugated leaves of Figure 19 with a crease pattern clearly visible.

Less micro-structured crease patterns have also been obtained to mimic biological patterns. For instance, one can refer to the 'crimp' technique[6] , together with origami, as in Figure 20.

Figure 19: Deployed Hornbeam leaves (Carpinus betulus), after Figure 2, page 148 of [35], reprinted from the *Proceedings of the Royal Society London B*, with permission of Royal Society Publishing

Figure 20: Example of a crimp model, model and design by V. Floderer (2001), image by J.-M. Nozerand, with permission

4.2 3D structures and mechanisms

Much of the previous examples concern structures that can be unfolded flat. For geometrical realizations in 3D, main works focus on polyhedral structures. Several design methods have been used, such as the

[5] http://www.jspec.jaxa.jp

[6] http://www.le-crimp.org

modular technique already mentioned, or folding from a single piece. The question of the existence of flexible polyhedra (structures exhibiting a mechanism) with rigid faces is discussed with the 'rigidity theory'. The answer concerns the case of polyhedron designed by connecting together rigid plates (for each polyhedron face) along their edges with perfectly flexible hinges: If this polyhedron is convex, then it is rigid. The bellows conjecture, now proved, states that any continuous motion of a flexible polyhedron must preserve its volume; in such a case it possesses a zero-stiffness mechanism (see discussion and references in [11]).

As for pattern assemblies, one can design particular assemblies of polyhedra in order to exhibit some mechanism. As an example, consider the ring of Figure 21 (its underlying mechanism is also referred to as a Bricard linkage, or an hexagonal kaleidocycle, or an hexaflexahedron). It is a possible rotating ring as soon as a geometric constraint is satisfied: There are 2 groups of concurring hinge axis. Such constraints are similar to those arising to ensure a mechanism for the Miura-Ori fold; they relates to the static indeterminacy level of the mechanism.

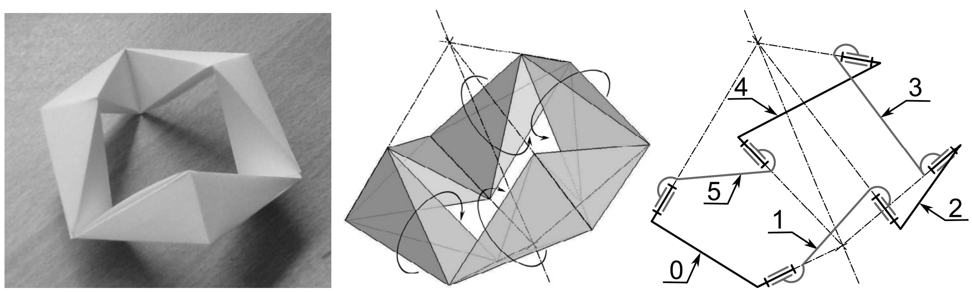

Figure 21: The complete hexaflexahedron

A particular case where similar assemblies are degenerated to lie flat corresponds to the so-called flexagons; in particular, they can reverse the folded model (and therefore its face coloring) by a succession of folding movements [40].

3D assemblies of rigid parts can also be performed throughout flexible hinges possessing some stiffness. For instance, some micro-devices based on folding have been designed using different kinds of folding mechanisms [5]: apart from designing more classical hinges between plates at small scales, many devices use local deformation in a compliant mechanisms [52] (Figure 22). The question of the rigidity and of the strength of such mechanisms is an important topic for their design. In particular, models designed with paper and conforming to rigid theory can not always be easily transposed to assemblies of rigid parts or compliant mechanisms [54], [48].

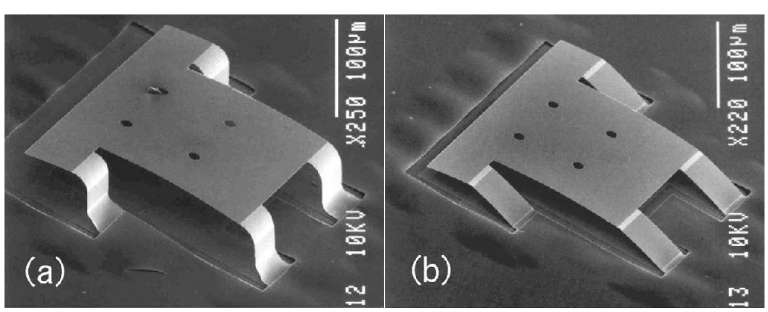

Figure 22: Scanning Electron Microscopy images of two micro-stages with different hinges lengths, reprinted from [52] (copyright 2003), with permission from Elsevier

4.3 A small insight into mechanical strength and rigidity

Once a 3D structure is obtained, the question of its stiffness can be tackled, since for engineering applications, the mechanical behavior is of interest. Apart from mechanisms (discussed in previous sections) that may exhibit zero stiffness if the loading leads the mechanism movement to develop, the stiffness is directly related to the material characteristics (mainly the Young modulus, and the thickness), and to the structural design. Indeed, it is well known that folds can rigidify a thin plate with a bending stiffness in the direction of the crease, by coupling the bending and membrane behavior in its mechanical response (as examples, one can mention simple wedges on metallic sheets, or corrugated plates [42], [18]). Another structural element that may greatly contribute to the overall stiffness is the boundary conditions (clamping) that are prescribed. This is very important for instance for large foldable structures for which the foldability feature should be prevented at will.

Another application where stiffness and strength are of interest is the packaging. Especially for carton packages, the foldability (or wrapping) is an industrial issue. The stiffness and the strength of the 3D folded state are required for the storage of the packs. A particular focus on the local strength is needed also for the packing of food-processing liquids: For sanitary reasons, the fabric is a laminated composite with impervious inner layers. Creases, and crease crossings, are the most stressed parts of the packaging, and the impervious property is mandatory. Therefore, during folding, storage and handling, the corresponding layers should not tear nor wreck. The mechanical behavior of the laminated composite close to a crease is a complex 3D behavior with possible delamination [43].

Using preferential movements with strength of flexibility as illustrated in the previous section, can have interest either to store energy in the elastic deformation, or to dissipate energy in plastic deformation. The first case may correspond to springs for maintaining movement stability [32], [46], while the second case can be related to shock absorbers or metal forming [56], [31]. For origami-inspired shock absorbers, the preferential movement is often the continuation of a stable local buckling mode. In each case, when the thickness is very small, the mechanical bending behavior tends to degenerate toward a purely geometrical problem [26].

5 CONCLUSIONS AND OUTLOOKS

Origami-like structures have connections with many areas and could inspire several design solutions from micro-structures to foldable / deployable large 3D structures. From a design point of view, the designer is also willing to get access to tools that may guide him for the design process, especially by relating global properties (for instance: foldability, presence of mechanisms...) to design constraints (for instance: particular geometry, placement of linkages). Few tools are yet available, but mention could be made of:

- automatic crease pattern generation (in particular with disk packing) [38],
- local foldability conditions (Kawasaki-Justin theorem) [11],
- mechanism theory [45].

Many recent developments have relied on assemblies of repetitive elementary structures:

- planar geometries with tessellations; if the cells are sufficiently numerous, this produces a texture on the covered surface;
- foldable / deployable structures with quasi-mechanisms with more or less strain energy involved (with elastic hinges or curved folds and bending of the material...) that leads to a micro-structured material when many cells are involved [2].

Nevertheless, producing folds at small scale is still an issue at the size of an industrial production. The example of the distorted tessellations on a warped surface is one of the most difficult cases, both for the fabrication of their components and for designing a mechanism to make them foldable and unfoldable. Nevertheless, progress have recently been made in that direction, for instance with the works of T. Tachi [47], [48], and [16].

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