M.Sc. Mathematics Part - II
Analysis - II

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Reference Books :

1) Stein and Shakarchi, Measure and Integration, Princeton Lectures in Analysis, Princeton University Press.
2) Andrew Browder, Mathematical Analysis an Introduction, Springer Undegraduate Texts In Mathematics, 1999.
3) Walter Rudin, Real and Complex Analysis, McGraw-Hill India, 1974.

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1

RIEMANN INTEGRAL - I

Unit Structure :

1.1 INTRODUCTION

The Riemann integral dealt with in calculus courses, is well suited for computations but less suited for dealing with limit processes.

Bernhard Riemann in 1868 introduced Riemann integral. He need to prove some new result about Fourier and trigonometric series. Riemann integral is based on idea of dividing. The domain of function into small units over each such unit or sub-interval we erect an approximation rectangle. The sum of the area of these rectangles approximates the area under the curve.

As the partition of the interval becomes thinner, the number of sub-interval becomes greater. The approximating rectangles become narrower and more precise. Hence area under the curve is more accurate. As limits of sub-interval tends to zero, the values of the sum of the areas of the rectangles tends to the value of an integral. Hence the area under curve to be equal to the value of the integral.

Before going for exact definition of Riemann explained the following definitions.

1.2 PARTITION

A closed rectangle in $\mathbb{R}^n$ is a subset A of $\mathbb{R}^n$ of the forms.

$A=[a_1,b_1]\times[a_2,b_2]\times....\times[a_n,b_n]$ where $a_i<b_i\in\mathbb{R}$. Note that $(x_1,x_2,....,x_n)\in A$ iff $a_i\le x_i\le b_i\forall i$.

The points $x_1, x_2, ..., x_n$ are called the partition points.

The closed interval $I_1 = [x_0, x_1], I_2 = [x_1, x_2], ..., I_n = [x_{n-1}, x_n]$ are called the component internal of $[a, b]$.

Norm : The norm of a portion P is the length of the largest sub-internal of P and is denoted by $\|P\|$.

For example : Suppose that $P_1 = t_0, t_1, ...t_k$ is a partition of $[a_1, b_1]$ and $P_2 = S_0, ..., S_r$ is a partition of $[a_2, b_2]$. Then the partition $P = (P_1.P_2)$ of $[a_1, b_1] \times [a_2, b_2]$ divides the closed rectangle $[a_1, b_1] \times [a_2, b_2]$ into Kr-gub rectangles.

In general if P_i divides $[a_i, b_i]$ into k_i sub-interval then $P = (P_1, ...P_n) [a_1, b_1] \times .... \times [a_n, b_n]$ into $K = k_1 k_2 .....k_n$ sub-rectangle. These sub-rectangles are called sub-rectangles of the partition p.

Refinement :

Definition : Let A be a rectangle in $\mathbb{R}^n$ and $f : A \to \mathbb{R}$ be a bounded function and P be partition of A for each sub-rectangles of the partition.

$$ms(f) = \inf\{f(x) : x \in S\}$$
$$= g.l.b. of\ f\ on [x_{s-1}, x_s]$$

$$Ms(f) = \sup\{f(x) : x \in S\}$$
$$= l.u.b. of\ f\ on [x_{s-1}, x_s]$$

where $S = 1, 2, ..., n$

The lower and upper sums of f for 'p' are defined by

$L(f, p) = \sum_s m_s(f) v(s)$ and $U(f, p) = \sum_s M_s(f) v(s)$

Since $m_s < M_s$ we have $L(f, p) \le U(f, p)$

Refinement of a partition : Let $P = (P_1, P_2, ..., P_n)$ and $P^* = (P_1^*, ..., P_n^*)$ be partition of a rectangle A in $\mathbb{R}^n$. We say that a partition P^* is a refinement of P if $P \subset P^*$.

If P_1 and P_2 are two partition of A then $P = P_1 \cup P_2$ is also a partition of A is called the common refinement of P_1 and P_2.

A function $f : A \to \mathbb{R}$ is called integrable on the rectangle A in $\mathbb{R}^n$ if $'f'$ is bounded $\therefore g.l.b$ of the set of all upper sum of $'f'$ and $l.u.b$ of the set of all lower sum of $'f'$ exist.

Let $U(f) = \inf\{U(f,p)\}$

$L(f) = \sup\{L(f,p)\}$

If $U(f) = L(f)$ is called $'f'$ is R-integrable over A.

$\therefore$ if can be written as $U(f) = L(f) = \int_A f$.

Theorem :

Let P and P' be partitions of a rectangle A in $\mathbb{R}^n$. If P' refines P then show that $L(f,p) \le L(f,P')$ and $U(f,P') \le U(f,p)$.

Proof :

Let a function $f : A \to \mathbb{R}$ is bounded on A P & P^* are two partition of A and P' is retinement to P.

Any subrectangle S of P' is union of some subrectangles $s_1, s_2, ..., s_k$ of P' and $V(S) = V(s_1) + V(s_2) + ... + V(s_k)$.

Now $m_s(f) = \inf\{f(x); x \in s\} \le \inf\{f(x); x \in s_i\}$

$\therefore m_s(f) \le m_{s_i}(f) \qquad \forall\, i = 1, ..., k$

$$L(f,p) = \sum_{s \in p} m_s(f) V(s)$$

$$\therefore m_s(f)V(s) = m_s(f)\big(V(s_1) + ... + V(s_k)\big)$$

$$\le m_{s_1}(f)V(s_1) + ... + m_{s_k}(f)V(s_k)$$

The sum of LHS for all subrectangle s_i of P' will get $L(f,P')$.

$\therefore L(f,p) \le L(f,p^1)$

Now, $M_s(f) = \sup\{f(x); x \in S\}$

$\ge \sup\{f(x); x \in S_i\}$

$M_s(f) \ge M_{s_i}(f) \qquad \forall\, i = 1, ..., K$

$$U(f,p)=\sum_{s\in p} m_s(f)V(s)$$

Now, $Ms_i(f)V(S)=Ms(f)\left(V(S_1)+V(S_2)+....+V(S_k)\right)$
$\le Ms(f)V(s_1)+.....+M_s(f)V(s_2)+....+M_s(f)V(s_k)$

Taking the of L.H.S. for all subrectangle S_i of P' will get $U(f,P')\therefore U(f,P)\ge U(f,P')$.

Theorem :

Let P_1 & P_2 be partitions of rectangle A & $f:A\to\mathbb{R}$ be bounded function. Show that $L(f,P_2)\le U(f,P_1)$ & $L(f,P_1)\le \bigcup(f,P_2)$.

Proof :

Let a function $f:A\to\mathbb{R}$ be a bounded find P_1 & P_2 are any two partition of A.

Let $P=P_1\bigcup P_2$

$\therefore P$ is a refinement of both P_1 & P_2

$U(f,P)\le U(f,P_1)$……….. (I)

$U(f,P)\le U(f,P_2)$……….. (II)

$L(f,P)\ge L(f,P_1)$……….. (III)

$L(f,P)\ge L(f,P_2)$ ……….. (IV)

$\therefore$ We get $U(f,P_1)\ge U(f,P)\ge L(f,P)\ge L(f,P_2)$.

Hence $U(f,P_1)\ge L(f,P_2)$

Similarly, $U(f_2,P_2)\ge U(f,P)\ge L(f,P)\ge L(f,P_1)$.

Hence, $U(f,P_2)\ge L(f,P_1)$

Theorem :

Let a function $f:A\to\mathbb{R}$ be bounded on A then for any $\in>0,\exists$a partition P on A such that $U(f,P)<U(f)+\in$ and $L(f,P)>L(f)-\in$

Proof :

Let a function $f: A \to \mathbb{R}$ be bounded on A $U(f) = \inf\{U(f,P)\}$ and $L(f) = \sup\{L(f,P)\}$ for any $\in > 0, \exists$ partitions P_1 & P_2 of A such that $U(f,P_1) < U(f) + \in$ & $L(f,P_2) > L(f) - \in$.

Let $P = P_1 \cup P_2$ the common refinement of P_1 and P_2.

$$U(f,P) \le U(f,P_1) \le U(f) + \in$$

$$L(f,P) \ge L(f,P_2) > L(f) - \in$$

$$\therefore U(f,P) < U(f) + \in$$

$$L(f,P) > L(f) - \in$$

1.3 RIEMANN CRITERION

Let A be a rectangle in $\mathbb{R}^n$ A bounded function $f: A \to \mathbb{R}$ is integrable iff for every $\in > 0$, there is a partition P of A such that $U(f,P) - L(f,P) < \in$.

Proof :

Let a function $f: A \to \mathbb{R}$ is bounded.

$$U(f) = \inf\{U(f,P)\}$$

$$L(f) = \sup\{L(f,P)\}$$

Let f be integrable of A

$\therefore U(f) = L(f)$

for any $\in > 0$, $\exists$ a partition P on A such that $U(f,p) < U(f) + \in/2$ and $L(f,p) > L(f) - \in/2$.

$\therefore U(f,p) = U(f) + \in/2$ & $-L(f,p) < -L(f) + \in/2$.

$\therefore U(f,p) - L(f,P) < U(f) + \in/2 - L(f) + \in/2$.

$\therefore U(f,p) - L(f) < \in$

Conversely,

Let for any $\in > 0$, $\exists$ a partition P on A such that $U(f,p) - L(f,P) < \in$.

$$[U(P,f) - U(f)] + [U(f) - L(f)] + [L(f) - L(f,P)] < \in$$

Since $U(f,P)-U(f)\geq o,$

$U(f)-L(f)\geq o$

and $L(f)-L(f,P)\geq o$

$\therefore$ we have, $o\leq U(f)-L(f)<\in$

Since $\in$ is arbitrary, $U(f)=L(f)$

$\therefore f$ is integrable over A.

Example 1

Let A be a rectangle in $\mathbb{R}^n$ and $f:A\to\mathbb{R}$ be a constant function. Show that f is integrable and $\int_A f = C.V(A)$ for some $C\in\mathbb{R}$.

Solution :

$f(x)=C \quad \forall x\in A$

$\therefore f$ is bounded on A

Let P be a partition of A

$m_s(f)=\inf\{f(x);x\in s\}=C$

$M_s(f)=\sup\{f(x);x\in s\}=C$

$\therefore L(f,P)=\sum_S m_s(f)V(S)=C\sum_S V(S)=CV(A)$

$U(f,P)=\sum_S M_s(f)V(S)=C\sum_S V(S)=CV(A)$

$\therefore U(f)=L(f)=C\,V(A)$

$\therefore f$ is integrable over A.

$\therefore$ by Reimann criterion, $\in<0$ s.t.

$\int_A f = C.V(A)$ for some $C\in\mathbb{R}$.

Example 2 :

Let $F:[0,1]X[0,1]\to\mathbb{R}$

$$f(x,y)=\begin{cases} o & if\ x\,is\,rational \\ 1 & if\ x\,is\,irrational \end{cases}$$

Show that 'f' is not integrable.

Solution :

Let P be a partition of $[0,1]\times[0,1]$ into S subport of P.

Take any point $\exists(x_1, y_1) \in S$ such that *x* is rational.

$\therefore f(x, y) = o$ and $\exists(x_1, y_1) \in S$ such that x_1, is irrational

$\therefore f(x_1, y_1) = 1$

$\therefore m_s(f) = \inf\{f(x); x \in S\} = 0$

$M_s(f) = \sup\{f(x); x \in S\} = 1$

$L(f, P) = \sum_S m_s(f) V(S) = 0$

$\therefore U(f, P) = \sum_S M_s(f) V(S) = 1$

$\therefore U(f) = 1, L(f) = 0$

$\therefore U(f) \neq L(f)$

$\therefore f$ is not integrable $[0,1] \times [0,1]$

1.4 PROPERTIES OF RIEMANN INTEGRAL

1) Let $f : A \to \mathbb{R}$ be integrable and $g = f$ except at finitely many points show that g is integrable and $\int_A f = \int_A g$.

Proof :

Since f is integrable over A.

$\therefore$ by Riemann Criterion, $\exists$ a partition P of A.

Such that $U(f, P) - L(f, P) < \in$ ……… (I)

Let P' be a refinement of P, such that

1) $\forall x \in A$ with $f(x) \neq g(x)$, it belongs to 2^n subrectangles of P'

2) $V(S) < \dfrac{\in}{2^{n+1} d(u - \ell)}$

Where d = numbers of points in A at which $f \neq g$

$$u = \sup_{x \in A}\{g(x)\} - \inf_{x \in A}\{f(x)\}$$

$$\ell = \inf_{x \in A}\{g(x)\} - \sup_{x \in A}\{f(x)\}$$

$\therefore P'$ is refines P, we have

$L(f, P) \leq L(f, P') \leq U(f, P') \leq U(f, P)$

$\therefore U(f, P') - L(f, P') \leq U(f, P) - L(f, P) < \in$

Now

$$U(g,P')-U(f,P')$$

$$=\sum_{i=1}^{d}\left(\sum\left(Ms_{ij}(g)-Ms_{ij}(f)\right)V(s_{ij})\right)$$

$\because$ On other rectangle, $f=g$ and so $Ms_{ij}(g)=Ms_{ij}(f)$.

$\because$ $Ms_{ij}(g)\le\sup_{x\in A}\{g(x)\}$ & $Ms_{ij}(f)\ge\inf_{x\in A}\{f(x)\}-Ms_{ij}(f)\le\inf_{x\in A}\{f(x)\}$

$$Ms_{ij}(g)-Ms_{ij}(f)\le u$$

$$\therefore U(g,P')-U(f,P')\le\sum_{i=1}^{d}\left(\sum_{j=1}^{2^n}u\right)V(S_{ij})$$

Let $V=\sup\{V(S_{ij})\}\le U(g,P^1)-U(f,P^1)\le\sum_{i=1}^{d}\sum_{j-1}^{2^n}uV\le d2^n u.v$ ……. (II)

Now similarly we get $L(g,P^1)-L(f,P^1)\ge d2^n\ell V$ ……... (III) by (II) & (III) we get.

$$U(g,P^1)-L(g,P^1)\le U(f,P^1)+d2^n u\vartheta-L(f,P^1)-d2^n\ell\vartheta$$

$$\le\frac{\in}{2}+d2^n(u-\ell)V$$

$$\le\frac{\in}{2}+\frac{d2^n\in(u-\ell)}{d2^{n+1}(u-\ell)}\frac{\in}{2}+\frac{\in}{2}=\in$$

$$\therefore U(g,P^1)-L(g,P^1)<\in$$

By Reimann Criterion G is integrable by equation (II)

$$U(g,P^1)-U(f,P^1)\le d2^n uv$$

$$\therefore U(g,P^1)\le U(f,P^1)+d2^n u\vartheta$$

Note that $\int_A g\le U(g,P^1)\le U(f,P^1)+d2^n u\vartheta$

$$\le L(f,P^1)+\frac{\in}{2}+d2^n u\vartheta$$

$$<L(f,P^1)+\in\!/2+\frac{d2^n u\in}{d2^{n+1}(u+\ell)}$$

$$< L\left(f,P^1\right)+ \in/2 + \in/2$$

$$< L\left(f,P^1\right)+ \in$$

$$< \int_A f + \in$$

This is true for any $\in > 0$

$\int_A g \le \int_A f$ …………………….. (IV)

Now $\int_A g \ge L(g,P') \ge L(f,P') + \in/2$

$$\ge U(f,P')$$

$$\ge \int_A f > \int_A f - \in/2$$

$\therefore \int_A f = \inf\{U(f,P)\}$

$\therefore \int_A g > \int_A f - \in/2$

$\therefore$ This is true for any $\in > 0$

$\therefore \int_A g \ge \int_A f$ ……… (V)

$\therefore$ from (IV) & (V) we get

$\int_A g = \int_A f$

2) Let $f : A \to \mathbb{R}$ be integrable, for any partition P of A and sub-rectangle S, show that

i) $m_s(f) + m_s(g) \le m_s(f+g)$ and

ii) $M_s(f) + M_s(g) \ge M_s(f+g)$

Deduce that

$L(f,P) + L(g,P) \le L(f+g,P)$ and

$U(f+g,P) \le U(f,P) + U(g,P)$

Solution :

Let P be a partition of A and S be a Subrectangle

$\therefore m_s(f) = \inf\{f(x); x \in S\}$

$\Rightarrow m_s(f) \le f(x) \forall x \in S$

Similarly $m_s(g) \le g(x) \forall x \in S$

$\therefore m_s(f) + m_s(g) \le f(x) + g(x) \forall x \in S$

$\Rightarrow m_s(f) + m_s(g)$ is lower bound of

$\{f(x) + g(x); x \in S\} = \{(f+g)(x); x \in S\}$

$\Rightarrow m_s(f) + m_s(g)$ is lower bound of

$\{f(x) + g(x); x \in S\} = \{(f+g)(x); x \in S\}$

$\Rightarrow m_s(f) + m_s(g) \le \inf\{(f+g)(x); x \in S\}$

$= m_s(f+g)$

$\therefore m_s(f) + m_s(g) \le m_s(f+g)$

ii) $Ms(f) = sub\{f(x); x \in s\}$

$\Rightarrow Ms(f) \ge f(x) \ \forall \ x \in s$

Similarly $Ms(g) \ge g(x) \forall x \in S$

$\therefore Ms(f) + Ms(g) \ge f(x) + g(x) \forall x \in S$

$\Rightarrow Ms(f) + Ms(g)$ is upper bound of

$\{f(x) + g(x); x \in S\} = \{(f+g)(x); x \in S\}$

$\Rightarrow Ms(f) + Ms(g) \ge \sup\{(f+g)(x); x \in S\} = Ms(f+g)$

$\therefore Ms(f) + Ms(g) \ge Ms(f+g)$

Hence,

$$L(f,P) + L(g,P) = \sum_{s \in p} \left(Ms(f) + Ms(g)\right) V(S)$$

$$\le \sum_{s \in p} \left(Ms(f+g)\right) V(S)$$

$$< L(f+g,P)$$

$\therefore L(f,P) + L(g,P) \le L(f+g,P)$

$$U(f,P) + U(g,P) = \sum_{s} \left(Ms(f) + Ms(g)\right) V(S)$$

$$\ge \sum_{s} \left(Ms(f+g)\right) V(S)$$

$$\ge U(f+g,P)$$

$U(f,P) + U(g,P) \ge U(f+g,P)$ Proved.

3) Let $f : A \to \mathbb{R}$ be integrable, & $g : A \to \mathbb{R}$ integrable than show that $f+g$ is integrable and $\int_A (f+g) = \int_A f + \int_A g$.

Proof :

Let P be any partition of A then

$U(f+g,P) - L(f+g,P) \le U(f,P) + U(g,P) - [L(f,P) + L(g,P)]$

$\le U(f,P) + U(g,P) - L(f,P) - L(g,P)$ …………………….. (I)

$\therefore f$ is integrable.

By Rieman interion for given $\in > 0, \exists$ a partition P, of A such that $U(f,P_1) - L(f,P_1) < \in/2$ ……………………………….… (II)

Similarly $\because g$ is integrable for $\in > 0, \exists$ a partition P_2 of A such that $U(g,P_2) - L(f,P_2) < \in/2$ ……………………………………… (III)

Then $P^* = P_1 \cup P_2$ is a refinement of both $P_1 \& P_2$.

$\therefore L(f,P_1) \le L(f,P^*)$; $U(f,P_1) \ge U(f,P^*)$ & $L(g,P_2) \le L(f,P^*)$; $U(g,P_2) \ge U(g,P^*)$ ………………………………………….. (IV)

$\therefore \in/2 > U(f,P_1) - L(f,P_1) \ge U(f,P^*) - L(f,P^*)$

$\in/2 > U(g,P_2) - L(g,P_2) \ge U(g,P^*) - L(g,P^*)$ ……………….. (V)

The equation I is true for any partition P of A.

In general, it is true for partition P^* of A

$\therefore U(f+g,P^*) - L(f+g,P^*)$

$\le U(f,P^*) - L(f,P^*) + U(g,P^*) - L(g,P^*)$

$< \in/2 + \in/2 = \in$

$\therefore U(f+g,P^*) - L(f+g,P^*) < \in$

By Riemann Criterian $f+g$ is integrable.

Let $\in/0$ since $\int_A f = \sup\{f,P\}$ so $\exists a$ partition P such that $\int_A f < (f,P_1) + \in/2$.

Similarly $\exists a$ partition $P_2, P_3, ...P_n$ of A S

$$\int_A g < L(g, P_2) + \in/2$$

$$U(f, P_3) < \int_A f + \in/2$$

$$U(g, P_4) < \int_A g + \in/2$$

Let $P = P_1 \cup P_2 \cup P_3 \cup P_4$.

Then $\int_A f < (f, P_1) + \in/2 \leq L(f, P) + \in/2$

Similarly $\int_A g < L(g, P) + \in/2$

$U(f, P) < \int_A f + \in/2$ and $U(g, P) < \int_A g + \in/2$

$$\int_A f + \int_A g - \in < L(f, P) + L(g, P) \leq L(f + g, P) \leq \int_A f + g$$

$$\leq U(f + g, P)$$

$$\leq U(f, P) + U(g, P)$$

$$< \int_A f + \in/2 + \int_A g + \in/2$$

$$< \int_A f + \int_A g + \in$$

$$\therefore \int_A f + \int_A g - \in \int_A f + g < \int_A f + \int_A g - \in$$

This is true for any $\in > 0$

$$\therefore \int_A f + \int_A g \leq \int_A f + g \leq \int_A f + \int_A g \Rightarrow \int_A f + g = \int_A f + \int_A g$$

4) Let $f : A \to \mathbb{R}$ be integrable for any constant C, show that $\int_A (Cf) = C\int_A f$.

Proof :

Let $C \in \mathbb{R}$

Case 1

Let $\in > 0$ and suppose $C > 0$.

Let P be a partition of A and S be a subrectangle of P.

$$M_s(Cf) = \sup\{(Cf)(x); x \in S\}$$
$$= \sup\{Cf(x); x \in S\}$$
$$= C\sup\{f(x); x \in S\}$$
$$= CMs(f)$$

Similarly,

$$ms(Cf) = Cm_s(f)$$
$$\therefore U(Cf,P) = \sum_S Ms(Cf)v(S) = C\sum_S Ms(f)v(S)$$
$$= C\ U(f,P)$$

Similarly $L(Cf,P) = C\,L(f,P)$

$\therefore f$ is integrable for above $\in < 0, \exists$ a partition P of A such that $U(f,P) - L(f,P) < \in/C$

$$\therefore U(Cf,P) - L(Cf,P) = C\,U(f,P) - C\,L(f,P)$$
$$= C[U(f,P) - L(f,P)]$$
$$= C \times \in/C = C$$

By Riemann Criteria.

(Cf) is integrable

for $\in > 0, \exists a$ partition P of A such that

$$C\int_A f - \in = C\left(\int_A f - \in/C\right) < C\,L(f,P) = L(Cf,P)$$
$$\leq \int_A Cf \leq U(Cf,P)$$
$$< C\,U(f,P) < C\left(\int_A f + \in/C\right)$$
$$\therefore \left(\int_A f - \in/C\right) < \int_A Cf < C\left(\int_A f + \in/C\right) = C\int_A f + \in$$

This is true for any $\in < 0$

$$C\int_A f \leq \int_A (Cf) \leq C\int_A f$$
$$\therefore \int_A Cf = C\int_A f$$

Case II

Now suppose $C < 0$

Let P be a partition of A and S be any subrectangle in P.

$\therefore Ms(Cf) = C\,Ms(f)$ and

$m_s(Cf) = C\,Ms(f)$

$\therefore L(Cf,P) = C\,U(f,P)$ and

$U(Cf,P) = C\,L(f,P)$

$\therefore f$ is integrable for above $\in > 0, \exists$ a partition P of A such that

$U(f,P) - L(f,P) < \in/(-C)$

$\therefore U(Cf,P) - L(Cf,P) = C\,L(f,P) - C\,U(f,P)$

$= -C[U(f,P) - L(f,P)]$

$< -C \in/-C$

$< \in$

By Riemann Criteria (Cf) is integrable.

for $\in > 0, \exists$ a partition P of A such that $C\int_A f - \in < \int_A Cf < C\int_A f + \in$.

This is true for every $\in > 0$

$C\int_A f < \int_A Cf \leq -C\int_A f$

$\therefore \int_A Cf = C\int_A f$

Example 3:

Let $f, g : A \to R$ be integrable & suppose $f \leq g$ show that $\int_A f \leq \int_A g$.

Solution :

By definition $\int_A f = \inf\{U(f,P)\}$ and $\int_A g = \inf\{U(g,P)\}$.

Let P be any partition of A & S be any subrectangle in P as $f \leq g$

$m_s(f) \leq m_s(g)$

$\therefore U(f,P) \leq U(g,P)$

$\inf\{U(f,P)\} \leq \inf\{U(g,P)\}$

This is true for any partition

$\therefore \int_A f \leq \int_A g$

Example 4:

If $f : A \to \mathbb{R}$ is integrable show that if is integrable and $\left|\int_A f\right| \le \int_A |f|$.

Solution :

$\Rightarrow$ Suppose *f* is integrable first we have to show that $|f|$ is integrable.

Let P be a partition of A & S be subrectangle of P then

$$Ms(|f|) = \sup\{|f(x)|; x \in S\}$$
$$= \sup\{|f|(x); x \in S\}$$
$$= |\sup\{f(x); x \in S\}|$$
$$= |Ms(f)|$$

Similarly

$$Ms(|f|) = |Ms(f)|$$

$$U(|f|, P) = \sum_S M_s(|f|)V(S) = \sum_S |M_s(f)|V(S)$$

$$L(|f|, P) = \sum_S |m_s(f)|V(S)$$

$$\therefore \sum_P (|M_s(f)| - |m_s(f)|)V(S) \le \sum_P (|M_s(f)| - |m_s(f)|)V(S)$$
$$\le U(f, P) - L(f, P)$$

$\therefore f$ is integrable, for $\in > 0, \exists$ a partition P such that $U(f, P) - L(f, P) < \in$.

$$\therefore U(|f|, P) - L(|f|, P) \le U(f, P) - L(f, P) < \in$$

$\therefore$ By Riemann criteria

$|f|$ is integrable over $\mathbb{R}$.

Now $\left|\int_A F\right| = \left|\inf_P \{U(f, P)\}\right|$

$$= \left|\inf_P \sum_{S \in P} M_s(f)V(S)\right|$$
$$= \left|\inf_P \sum M_s(f)V(S)\right|$$
$$= \left|\inf_P \sum_P M_s|f|V(S)\right|$$

$$\le \inf \sum_P M_s |f| V(S)$$

$$= \inf \{U(|f|, P)\}$$

$$\therefore \left|\int_A f\right| = \int_A |f|$$

Example 5:

Let $f : A \to \mathbb{R}$ and P be a partition of A show that f is integrable iff for each sub-rectangle S the function $f/_S$ which consist of f restricted to S is integrable and that in this case $\int_A f = \sum_S \int_S f/_S$.

$\Rightarrow$ Suppose $f : A \to \mathbb{R}$ is integrable.

Let P be a partition of A & S be a sub-rectangle in P.

Now to show that $f/_S ; S \to \mathbb{R}$ is integrable.

Let $\in > 0, \exists$ a partition P' of A such that $U(f, P) - L(f, P') < \in$ ($\therefore f$ is integrable)

Let $P' = P \cup P'$ then P_1 is refinement of both P & P'.

$\therefore U(f, P') \ge U(f, P_1)$ & $L(f, P') \le L(f, P_1)$

$\therefore U(f, P_1) - L(f, P_1) \le U(f, P') - L(f, P') < \in$ ………………… (I)

$\because P_1$ is refinement of P

$\because S$ is union of some subrectangle of P_1 say $S = \underset{i=1}{U} si$.

$\therefore \in > U(f, P_1) - L(f, P_1) = \sum_{S \in P_1} (M_s(f) - m_s(f)) V(S)$ for all rectangle.

$$\ge \sum_{i=1}^{k} (Ms_i(f) - m_{s_i}(f)) V(S)$$

$$= U\left(f/_S, P\right) - L\left(f/_S, P\right)$$

$\therefore$ By Riemann Criterion

$\therefore f/_S$ is integrable.

Conversely, Suppose $f/_S$ is integrable for each $S \in P$.

To show that f is integrable.

Let $\in > 0, \exists$ partition P_S of S such that

$U\left(f/S, P_S\right) - L\left(f/S, P_S\right) < \in/k$ ………………………………… (II)

$\therefore f/S$ is integrable for each $S \in P$ where K is number of rectangle in P.

Let P^1 be the partition of A obtained by taking all the subrectangle defined in the partition P_S.

There is a refinement P_S^1 of P_S containing subrectangles in P^1.

$$\therefore U\left(f/s, P_S^1\right) - L\left(f/s, P_S^1\right) < \in/k \text{ ……………………………… (III)}$$

$$\therefore U\left(f, P^1\right) - L\left(f, P^1\right) = \sum_{S^1 \in P^1} \left(M_{S^1}(f) - m_{S^1}(f)\right) V\left(S^1\right)$$

$$= \sum_{S \in P} \left(\sum_{S^1 \in P_S^1} \left(Ms^1(f) - m_{s^1}(f)\right) V\left(S^1\right) \right)$$

$$= \sum_{S \in P} \left(U\left(f/s, P_S^1\right) - L\left(f/s, P_S^1\right)\right)$$

$$< \sum_{S \in P} \in/k$$

$$< k, \in/k < \in$$

$\therefore$ By Riemann Criterian f is integrable.

Let $\in > 0$

$$\sum_{S \in P} \left(\int_S f/S - \in/k \right) < \sum_{S \in P} L\left(f/S, P_S\right)$$

$$< \sum_{S \in P} \left(\sum_{S^1 \in P_S^1} m_s^1(f) V(S') \right)$$

Let P^1 be a partition of A, obtained by taking allthe subrectangle defined in P_S.

$$\therefore \sum_{S\in P}\left(\int_S f/S - \in/k\right) < \sum_{S^1\in P^1}\left(m_{s^1}(f)\right)V\left(S^1\right)$$

$$< L\left(f, P^1\right) < \int_A f < U\left(f, P^1\right)$$

$$= \sum_{S^1\in P^1} M_{s^1}(f)V\left(S^1\right)$$

$$= \sum_{S\in P}\left(\sum_{S^1\in P^1} M_{s^1}(f)V\left(S^1\right)\right)$$

$$\therefore \sum_{S\in P}\left(U\left(f/S, P_S\right)\right) < \sum_{S\in P}\left(\int_S f/S + C/k\right)$$

$$\therefore \sum_{S\in P}\int f/S - \in C\int_A f < \sum_{S\in P}\int f/S + \in$$

This is true for all $\in > 0$

$$\therefore \sum_{S\in P}\int f/S \le \int f \le \sum_{S\in P}\int_S f/S$$

$$\therefore \int_A f = \sum_{S\in P}\int_S f/S$$

Example 6:

Let $f : A \to \mathbb{R}$ be a continues function show that f is integrable on A.

Solution :

Let $f : A \to \mathbb{R}$ be a continuous function to show that f is integrable.

Let $\in > 0$, since A is closed rectangle it is closed and bounded in $\mathbb{R}^n$.

$\therefore A$ is compact.

$\because f$ is continuous function on compact set $\Rightarrow f$ is uniformly continuously on $\mathbb{R}$.

$\therefore$ for the above $\in > 0, \exists \delta > 0$ such that $\forall x, g \in A,$ $\|x-y\| < \delta \Rightarrow |f(x) - f(y)| < \in/V(A)$.

Let P be a partition of A such that side length of each subrectangle is less than $\delta/\sqrt{n}$.

If $x, y \in S$ for some subrectangles S then

$$\|x-y\| = \sqrt{\left(x_1-y_1\right)^2 + ... + \left(x_n-y_n\right)^2}$$

$$< \sqrt{n\left(S \big/ \sqrt{n} \right)^2} = \delta$$

$$\left| f(x) - f(y) \right| < \in / V(A)$$

$\because S$ is compact
$\therefore f$ is continuous
$\therefore f$ attains its bound in S.

Let $S_1, S_2, ..., S_k$ be the subrectangle in A. Then for $1 < i < k, \exists\, x_i, y_i \in S_i$ such that $Ms_i(f) = f(x_i) m_{si}(f) = f(y_i)$.

$$\therefore U(f,P) - L(f,P) = \sum_{i=1}^{k} \left(Ms_i(f) - m_{s_i}(f) \right) V(S_i)$$

$$= \sum_{i=1}^{k} \left(f(x_i) - f(y_i) \right) V(S_i)$$

$$< \sum_{i=1}^{k} \frac{\in}{V(A)} V(S_i) < \frac{\in}{V(A)} \sum_{V(A)}^{k} V(S_i)$$

$$< \frac{\in}{V(A)} V(A) < \in$$

$\therefore$ By Riemann Criterion *f* is integrable.

1.5 REVIEW

After reading this chapter you would be knowing.

- ❖ Defining R-integral over a rectangle in $\mathbb{R}^n$
- ❖ Properties of R-integrals
- ❖ R-integrabal functions
- ❖ Continuity of functions using $\mathbb{R}$-intervals.

1.6 UNIT END EXERCISE

I) Let $f; [0,1] \times [0,1] \to \mathbb{R}$ be defined by

$$f(x,y) = 0 \; if \; 0 \le y \le 1/3$$

$$= 3 \; if \; 1/3 \le y \le 1$$

show that *f* is integrable.

II) Let Q be rectangle in $\mathbb{R}^n$ & $f; Q \to \mathbb{R}$ be any bounded function.

a) Show that for any partition P of Q $L(f,P)<U(f,P)$

b) Show that upper integral of function *f* exit.

III) Let *f* be a continuous non-negative function on $[0,1]$ and suppose there exist $x_0 \in [a,b]$ such that $f(x_0)>0$ show that $\int_0 f(x)\,dx > a$.

IV) Let *f* be integrable on $[a,b]$ and $F:[a,b]\to\mathbb{R}$ and $F^1(x)=f(x)$ then prove that $\int_a f(x)\,dx = F(b)-F(a)$

V) Which of the following functions are Riemann integrable over $[0,1]$. Justify your answer.

a) The characteristic function of the set of rational number in $[0,1]$.

b) $f(x)=x\sin y_x$ for $0<x<1$

$f(0)=3$

VI) Prove that if *f* is $\mathbb{R}$-integrable then $|f|$ is also R-integrable is the converse true? Justify your answer.

VII) Show that a monotone function defined on an interval $[a,b]$ is R-inegrable.

VIII) A function $f;[0,1]\to\mathbb{R}$ is defined as $f(x)=\frac{1}{3^{n-1}}\ \forall \frac{1}{3^n}<x\le\frac{1}{3^{n-1}}$

where $n\in\mathbb{N}$

$f(0)=0$

show that *f* is R-integrable on $[0,1]$ & calculate $-\int_0^1 f(x)\,dx$.

IX) $f(x)=x\lfloor x\rfloor\ \ \forall x\in[1,3]$ where $\lfloor x\rfloor$ denotes the greatest integer not greater than x show that *f* is R-integrable on $[1,3]$.

X) A function $f;[a,b]\to\mathbb{R}$ is continuous on $[a,b]$ $f(x)\ge 0$ $\forall x\in[a,b]$ and $\int_a^b f(x)\,dx=0$ show that $f(x)=0\ \ \forall x\in[a,b]$.

❖❖❖❖

2

MEASURE ZERO SET

Unit Structure :

2.1 INTRODUCTION

As we have seen, we cannot tell if a function is Riemann integrable or not merely by counting its discontinuities one possible alternative is to look at how much space the discontinuities take up. Our question then becomes : (i) How can one tell rigorously, how much space a set takes up. Is there a useful definition that will concide with our intuitive understanding of volume or area?

At the same time we will develop a general measure theory which serves as the basis of contemporary analysis.

In this introductory chapter we set for the some basic concepts of measure theory.

2.2 MEASURE ZERO SET

Definition :

A subset 'A' of $\mathbb{R}^n$ said to have measure 'O' if for every $\in > 0$ there is a cover $\{U_1, U_2....\}$ of A by closed rectangles such that the total volume $\sum_{i=1}^{\infty} v(Ui) < \in$.

Theorem :

A function 'f' is Riemann integrable iff 'f' is discontinuous on a set of Measure zero.

A function is said to have a property of Continuous almost everywhere if the set on which the property does not hold has measure zero. Thus, the statement of the theorem is that 'f' is Riemann integrable if and only if it is continuous atmost everywhere.

Recall positive measure : A measure function $u: M \to [0, \infty]$ such that $V\left(\bigcup_{i=1}^{\infty} u_i\right) = \sum_{i=1}^{\infty} V(u_i)$.

Example 1:

1) "Counting Measure" : Let X be any set and $M = P(X)$ the set of all subsets : If $E \subset X$ is finite, then $\mu(E) = \eta(E)$ if $E \subset X$ is infinite, then $\mu(E) = \infty$
2) "Unit mass to x_0 - Dirac delta function" : Let X be any set and $M = P(X)$ choose $x_0 \in X$ set.

   $\mu(E) = 1 \;\; if \;\; x_0 \in E$

   $\quad\;\; = 0 \;\; if \;\; x_0 \notin E$

Example 2:

Show that A has measure zero if and only if there is countable collection of open rectangle $V_1, V_2, ...$ such that $A \subseteq \bigcup V_i$ and $\sum V(v_i) < \in$.

Solution :

Suppose A has measure zero.

For $\in > 0, \exists$ countable collection of closed rectangle $V_1, V_2, ...$ such that $A \subseteq \bigcup_{i=1}^{\infty} V_i$ and $\sum_{i=1}^{\infty} V(V_i) < \frac{\in}{2}$.

For each i, choose a rectangle u_i such that $u_i \supseteq v_i$ and $V(u_i) \leq 2V(v_i)$.

Then $A \subseteq \bigcup_{i=1}^{\infty} v_i \subseteq \bigcup_{i=1}^{\infty} u_i$ and $\sum_{i=1}^{\infty} V(u_i) \leq \sum_{i=1}^{\infty} V(u_i) \leq \sum_{i=1}^{\infty} 2V(v_i)$

$\leq 2\sum_{i=1}^{\infty} v(u_i) < 2\frac{\in}{2} = \in$

Note that : u_i are open rectangles in $\mathbb{R}^n$ conversely, Suppose for $\in > 0, \exists$ countable collection of open rectangles $u_1, u_2, ...$ such that $A \subseteq \bigcup_{i=1}^{\infty} u_i$ and $\sum_{i=1}^{\infty} V(u_i) < \in$.

For each i, consider $V_i = \overline{u_i}$ then V_i is a closed rectangle and $V(v_i) = V(u_i)$.

Then $A \subseteq \bigcup_{i=1}^{\infty} u_i \subseteq \bigcup_{i=1}^{\infty} v_i$ and $\sum_{i=1}^{\infty} V(v_i) = \sum_{i=1}^{\infty} V(u_i) < \in$.

A has measure zero.

Note : Therefore we can replace closed rectangle with open rectangles in definition of measure zero sets.

Example 3:

Show that a set with finitely many points has measure zero.

Solution :

Let $A = \{a_1, ..., a_m\}$ be finite subset of $\mathbb{R}^n$.

Let $\in > 0, a_i = (a_{i1}, a_{i2}, ..., a_{in})$ and

$$Vi = \left[a_{i1} - \frac{1}{2}\left(\frac{\in}{2^{i+1}}\right)^{1/n}, a_{i1} + \frac{1}{2}\left(\frac{\in}{2^{i+1}}\right)^{1/n}\right] \times ...$$

$$... \times \left[a_{in} - \frac{1}{2}\left(\frac{\in}{2^{i+1}}\right)^{1/n}, a_{in} + \frac{1}{2}\left(\frac{\in}{2^{i+1}}\right)^{1/n}\right]$$

Then $V(Vi) = \prod_{i=1}^{n} \left(\frac{\in}{2^{i+1}}\right)^{\frac{1}{n}} = \frac{\in}{2^{i+1}}$

Clearly $a_i \in Vi$ for $1 \le i \le m$

$\therefore A \subseteq \bigcup_{i=1}^{m} Vi$ and $\sum_{i=1}^{m} V(Vi) = \sum_{i=1}^{m} \frac{\in}{2^{i+1}} < \in \cdot \sum_{i=1}^{\infty} \frac{1}{2^{i+1}} < \in \cdot \frac{1}{2} < \in$

$\therefore$ By definition of measure of zero

$\therefore$ A has measure of zero.

Example 4:

If $A = A_1 \cup A_2 \cup A_3 \cup ....$ and each Ai has measure zero, then show that A has measure zero.

Solution :

Let $\in > 0$ and $A = A_1 \cup A_2 \cup ....$ with each Ai has measure zero.

$\because$ Each Ai has measure zero for $i = 1, 2, ...$ $\exists$ a cover $\{u_{i1}, U_{i2}, ..., U_{in}\}$ of Ai

By closed rectangle such that $\sum_{i=1}^{\infty} V(u_{ii}) < \frac{\in}{2^i}, i = 1, 2, ...$

Then the collection of U_{ii} is cover A

$$\therefore \sum_{i=1}^{\infty} V(V_i) < \sum_{i=1}^{\infty} \frac{\in}{2^i} < \in$$

Thus $A = A_1 \cup A_2 \cup A_n ....$ has measure zero.

Example 5:

Let $A \subset \mathbb{R}^n$ be a Rectangle show that A does not have measure zero. But ∂A has measure zero.

Proof :

Suppose A has measure zero.

$\because$ A is a rectangle in $\mathbb{R}^n$

$\therefore V(A) > 0$

Choose $\in > 0$ such that $\in < V(A)$ …………………….. (I)

$\because$ A has measure zero

$\exists$ countable collection of open rectangle $\{u_i\}$ such that $A \subseteq \bigcup_{i=1}^{\infty} u_i$ and $\sum V(u_i) < \in$.

$\because$ A is compact

This open cover has a finite subcover after renaming. We may assume that $\{u_1, u_2, ...u_k\}$ is subcover of the cover $\{u_i\}$.

$$\therefore A \subseteq \bigcup_{i=1}^{\infty} u_i .$$

Let P be partition of A that contains all the vertices all u_i's $i = 1$ to k. Let $S_1, S_2, ..., S_n$ denote the subrectangle of partitions.

$$\therefore V(A) = \sum_{j=1}^{n} V(S_j) \le \sum_{i=1}^{k} V(u_i) < \sum_{i=1}^{\infty} V(u_i) < \in$$

which is a contradiction to (I)

$\therefore$ A does not have measure zero.

Note that ∂A is a finite union of set of the form $B=[a_1,b_1]\times[a_i,b_i]\times.....\times[a_n,b_n], \forall$. B can be covered by are closed rectangle. $B_\delta=[a_1,b_1]\times.....\times[a_i,a_{i+\delta}]\times.....\times[a_n,b_n]$.

Then $V(B_\delta)$ depend on δ and $V(B_\delta)\to 0$ as $\delta\to 0$.

$\therefore B_\delta$ has measure zero

$\therefore$ Boundary of A (∂A) is finite union of measure zero.

$\therefore \partial A$ has measur5e zero.

Example 6:

Let $A\subset\mathbb{R}^n$ with $A^\circ\neq\varnothing$. Show that A does not measure zero.

Solution :

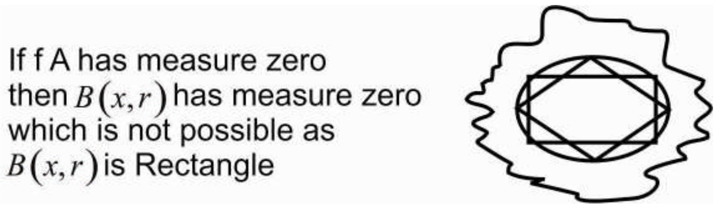
Let $A\subset\mathbb{R}^n$, with $A^\circ\neq\varnothing$

Let $x\in A^\circ$

$\therefore \exists\ r>0$, such that $B(x,r)\leq A$, But

$$B(x,r)=\{y\in A; \|y-x\|<r\}$$

$$=\left\{y\in A; \sum_{i=1}^{n}|y_i-x_i|<r\right\}$$



$\therefore A$ does not have measure zero.

Example 7:

Show that the closed interval $[a,b]$ does not have measure zero.

Solution :

Suppose $\{u_i\}_{i=1}$ be a cover of $[a,b]$ by open intervals.

$\because [a,b]$ is compact this open cover has a finite subcover.

After renaming, we may assume $\{u_1,u_2,....,u_n\}$ is the subcover of $\{u_i\}$ of $[a,b]$.

We may assume each u_i intersect $[a,b]$ (otherwise replace u_i with $u_i \cap [a,b]$)

Let $u = \bigcup_{i=1}^{n} u_i$

If u is not connected then $[a,b]$ is contained in one of connected component of u.

$\Rightarrow [a,b] \subseteq u_i$ for some i

$\therefore [a,b] \cap u_j = \varnothing$ for $i \neq j$

Which is not possible

$\therefore u$ is connected

$\Rightarrow u$ is an open interval say $u = (c,d)$ Then as $[a,b) \subseteq u = (c,d)$

$\Rightarrow \sum V(u_i) = d - c > b - a$

In particular we cannot find an open cover of $[a,b]$ with total length of the cover $< \dfrac{b-a}{2}$.

$\therefore [a,b]$ does not have measure zero.

Example 8:

If $A \subseteq [0,1]$ is the union of all open intervals (a_i, b_i) such that each rational number in (0,1) is contained in some (a_i, b_i). If $T = \sum_{i=1}^{\infty} (bi - ai) < 1$ then show that the boundary of A does not have measure zero.

Solution :

We first show that $\partial A = [0,1] \setminus A$

Note that $\partial A = \bar{A} \setminus A^\circ$

$\because A$ is open $\Rightarrow A^\circ = A$

Also $Q \cap [0,1] \subseteq A$

$\therefore \bar{Q} \cap \overline{[0,1]} \subseteq \bar{A}$

$\therefore [0,1] \subseteq \bar{A}$

But $A \subseteq [0,1] \Rightarrow \bar{A} \subseteq [0,1]$

$\therefore \bar{A} = [0,1]$

$\therefore \partial A = [0,1] \setminus A$

Let $\in = 1 - T > 0$

If ∂A has measure zero then since $\in > 0, \exists$ a cover of ∂A with open intervals such that sum of length of intervals $< 1 - T$
$\because \partial A$ is closed and bounded
$\Rightarrow \partial A$ is compact
$\Rightarrow \exists$ finite subcover $\{u_i\}_{i=1}^{n}$ for ∂A
$\therefore \sum \ell(u_i) < 1 - T$

Note that $\left\{u_i; 1 \le i \le n; (a_i, b_i)_{i=1}^{\infty}\right\}$ cover $[0,1]$ and sum of lengths of these open intervals is less than $1 - T + T = 1$ which is not possible as $[0,1] \subseteq \cup \left\{u_i;\ 1 \le i \le n;\ (a_i, b_i)_{i=1}^{\infty}\right\} \therefore \partial A$ does not have measure zero.

2.3 DEFINITION

A subset 'A' of $\mathbb{R}^n$ has content 'O' if for every $\in > 0$, there is a finite cover $\{u_1, u_2, \ldots\ldots, u_n\}$ of A by closed rectangles such that $\sum_{i=1}^{n} V(u_i) < \in$

Remark :

1) If A has content O, then A clearly has measure O.
2) Open rectangles can be used instead of closed rectangles in the definition.

Example 9:

If A is compact and has measure zero then show that A has content zero.

Solution :

Let A be a compact set in $\mathbb{R}^n$
Suppose that A has measure zero

$\therefore \exists$ a cover $\{u_1, u_2, \ldots.\}$ of A such that $\sum_{i=1}^{\infty} V(u_i) < \in$ for every $\in > 0$.
$\because A$ is compact, a finite number $u_1, u_2, \ldots\ldots, u_n$ of u_i also covers A and $\sum_{i=1}^{n} V(u_i) < \sum_{i=1}^{\infty} V(u_i) < \in$
$\therefore A$ has content zero.

Example 10 :

Give one example that a set A has measure zero but A does not have content zero.

Solution :

Let $A=[0,1]\cap Q$

Then A is countable

$\Rightarrow A$ has measure zero

Now to show that A does not have content zero.

Let $\{[a_i,b_i);1\le i\le n\}$ be cover of A

$\therefore A\subseteq[a_i,b_i]\cup....\cup[a_n,b_n]$

$\therefore \overline{A}\subseteq[a_1,b_1]\cup....\cup[a_n,b_n]$

But $\overline{A}=[0,1]$

$$\therefore \sum_{i=1}^{n}\ell([a_i,b_i))>1$$

In particular, we cannot find a finite cover for A such that $\sum_{i=1}^{n}\ell(a_i,b_i)<1/2$

$\therefore A$ does not have content zero.

Example 11:

Show that an unbounded set cannot have content zero.

Solution :

Let $A\subseteq\mathbb{R}^n$ be an unbounded set.

To show that A does not have content zero

Suppose A has content zero for $\in>0,\exists$ finite cover of closed rectangles $\{u_i\}_{i=1}^{k}$ of A such that $A\subseteq\bigcup_{i=1}^{k}u_i$ and $\sum_{i=1}^{k}V(u_i)<\in$.

Let $u_i=[a_{i1},b_{i1}]\times....\times[a_{in},b_{in}]$

Let $a_i=\min\{a_{1i},a_{2i},.....a_{ki}\}$

$b_i=\max\{b_{1i},b_{2i},.....b_{ki}\}$

then $\bigcup u_i\subseteq[a_1,b_1]\times....\times[a_n,b_n]$

$\therefore A\subseteq[a_1,b_1]\times....\times[a_n,b_n]$

$\therefore A$ is bounded

Which is contradiction

$\therefore A$ does not have content zero.

Example 12:

$f : A \to \mathbb{R}$ is non-negative and $\int_A f = 0$ where A is rectangle, then show that $\{x \in A; f(x) \neq 0\}$ has measure zero.

Solution :

For $n \in \mathbb{N}, A_n = \{x \in A; f(x) < 1/n\}$

Note that $\{x \in A, f(x) \neq 0\} = \{x \in A; F(x) > 0\}$

$\{\because f$ is non-negative$\}$

$= \bigcup_{n=1}^{\infty} \{x \in A; f(x) > 1/n\} = \bigcup_{n=1}^{\infty} A_n$

We have to show that A_n has measure zero

$\because \int_A f = 0$ and $\int_A f = \inf_P \{U(f, P)\} = 0$ for $\in > 0, \exists$ a partition P such that $U(f, P) < \in/n$

Let S be a subrectangle in P

if $S \cap A_n \neq \varnothing \Rightarrow M_s(f) \leq 1/n$

clearly $\{S \in P; S \cap A_n \neq \varnothing\}$ covers A_n and

$$\sum_{S \in P} \frac{1}{n} V(S) < \sum_{S \in P} M_s(f) V(S) \left(\because M_s(f) > \frac{1}{n} \right)$$

$$< \bigcup (f, P) < \in/n$$

$$\therefore \sum_{\substack{S \cap A_n \neq \varnothing \\ s \in p}} V(S) < \in$$

By definition A_n has content zero

$\Rightarrow A_n$ has measure zero

$\therefore \{x \in A, f(x) \neq 0\}$ is countable union of measure zero set.

$\therefore \{x \in A; f(x) \neq 0\}$ has measure zero.

* Oscillation $o(f, a)$ of ‘f’ at a

$\therefore$ for $\delta > 0$, Let $M(a, f, \delta) = \sup\{f(x); x \in A \,\&\, |x - a| < \delta\}$

$m(a, f, \delta) = \inf\{f(x); x \in A \,\&\, |x - a| < \delta\}$

The oscillation $o(f, a)$ of f at a defined by

$o(f, a) = \lim_{\delta \to o} (M(a, f, \delta) - m(a, f, \delta))$

This limit always exist since $M(a, f, \delta) - m(a, f, \delta)$ decreases as δ decreases.

Theorem :

Let A be a closed rectangle and let $f : A \to \mathbb{R}$ be a bounded function such that $O(f, x) < \in$ for all $x \in A$ show that there is a partition P of A with $U(f, P) - L(f, P) < \in \cdot V(A)$.

Proof :

Let $x \in A \Rightarrow U(f, x) < \in \Rightarrow \lim_{\delta \to O} (M(x, f, \delta) - m(x, f, \delta)) < \in$

$\therefore \exists$ a closed rectangle u_x containing x in its interior such that $M_{u_x} - M_{u_x} < \in$ by definition of oscillation.

$\therefore \{u_x; x \in A\}$ is a cover of A

$\therefore A$ is compact

$\Rightarrow$ This cover has a finite subcover say $\{u_{x1}, u_{x2}, ..., u_{xk}\}$

$\therefore A \subseteq \bigcup_{i=1}^{k} u_{x_i}$.

Let P be a partition for A such that there each subrectangle 'S' of P is contained in some u_{x_i} then $M_s(f) - m_s(f) < \in$ for each subrectangle 'S' in f

$$\therefore U(f, P) - L(f, P) = \sum_{S \in P} (M_s(f) - m_s(f)) V(S)$$

$$< \in \sum_{S \in P} V(S)$$

$$< \in \cdot V(A)$$

2.4 LEBESGUE THEOREM (ONLY STATEMENT)

Let A be a closed rectangle and $f : A \to \mathbb{R}$ is bounded function. Let $B = \{x ; f$ is not continuous at x$\}$. Then f is integrable iff B is a set of measure zero

2.5 CHARACTERISTIC FUNCTION

Let $C \subseteq \mathbb{R}^n$. The characteristics function χ_c of C is defined by

$$\chi_c(x) = 1 \quad if \ x \in C$$

$$= 0 \quad if \ x \notin C$$

If $C \subset A$ where A is a closed rectangle and $f : A \to \mathbb{R}$ is bounded then $\int_C f$ is defined as $\int_C f\chi_c$ provided $\int f \cdot \chi_c$ is integrable [i.e. if f and χ_c are integrable]

Theorem :

Let A be a closed rectangle and $C \subset A$. Show that the function $\chi_c : A \to$ □ is integrable if and only if ∂C has measure zero.

Proof :

To show that $\chi_C : A \to \mathbb{R}$ is integrable iff ∂C has measure zero.

By Lebesgue theorem, it is enough to show that $\partial C = \{x \in A : \chi_c$ is discontinuous}

Let $a \in C^\circ \Rightarrow \exists$ an open rectangle 'u' containing a such that $u \subseteq C$

$\therefore \chi_c(n) = 1 \ \forall n \in U$

$\Rightarrow \chi_c$ is continuous at a.

Let $a \in Ext(c) =$ Exterior of C

[By definition union of all open sets disjoints from C]

Ext (C) is an open set

$\exists$ an open rectangle u containing such that $U \subseteq Ext(c)$

$\therefore \chi_c(n) = 0 \ \forall n \in u$

$\Rightarrow \chi_c$ is continuous at a

If $a \notin \partial c$ then χ_c is continous at a ……………………. (I)

Let $a \in \partial c \Rightarrow$ for any open rectangle U with a in its interior contains a point $y \in C^\circ$ & a point $z \in \mathbb{R}^n \big| c$

$\therefore \chi_c(y) = 1 \& \chi_c(z) = 0$

$\therefore \chi_c$ is not continuous at a

$\therefore \partial c = \{x \in A : \chi_c$ is discontinuous at x }

$\therefore$ By Lebesgue Theorem.

χ_c is interrable if and only if ∂c has measure zero.

Theorem :

Let A be a closed rectangle and $C \subset A$

If C is bounded set of measure zero and $\int_A \chi_c$ exist then show that $\int_A \chi_c = 0$.

Proof :

$C \subseteq A$ be a bounded set with measure zero.

Suppose $\int_A \chi_c$ exist $\Rightarrow \chi_c$ is integral

To show that $\int_A \chi_c = 0$

Let P be a partition of A and S be a subrectangle in P.

$\because S$ does not have measure zero

$\Rightarrow S \not\subseteq C$

$\Rightarrow \exists x \in S$ but $x \notin C$

$\therefore \chi_c(x) = 0$

$\Rightarrow m_s(\chi_c) = 0$

This is true for any subrectangle S in P

$\therefore L(\chi_c, P) = \sum m_s(\chi_c) V(C) = 0$

This is true for any partition P

$\therefore \int_A \chi_c = \sup\{L(\chi_c, P); P$ is partition of}

$\int_A \chi_c = O$

2.6 FUBINI'S THEOREM

Fubini's Theorem reduces the computation of integrals over closed rectangles in $\mathbb{R}^n, n > 1$ to the computation of integrals over closed intervals in $\mathbb{R}$. Fubini's Theorem is critically important as it gives us a method to evaluate double integrals over rectangles without having to use the definition of a double integral directly.

If $f : A \to R$ is a bounded function on a closed rectangle then the least upper bound of all lower sum and the greatest lower bound of all upper sums exist. They are called the lower integral and upper integral of f and is denoted by $L\int_A F$ and $U\int_A F$ respectively.

Fubini's Theorem

Statement : Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$ be closed rectangles and let $f : A \times B \to \mathbb{R}$ be integrable for $x \in A$, Let $g_x : B \to \mathbb{R}$ be defined by $g_x(y) = F(x, y)$ and let

$$\ell(x) = L\int_B g_x = L\int_B f(x, y)\,dy$$

$$u(x) = U\int_B g_x = U\int_B f(x, y)\,dy$$

Then ℓ and μ are integable on A and $\int_{A\times B} f = \int_A L = \int_A \left(L\int_B f(x)\,dy \right) dx$

$$\int_{A\times B} f = \int_A u(x)\,dx = \int_A \left(U\int_B f(x, y)\,dy \right) dx$$

Proof :

Let P_A be a partition of A and P_B be a partition of B. Then $P = (P_A, P_B)$ is a partition of $A \times B$

Let S_A be a subrectangle in P_A and S_B be a subrectangle in P_B

Then by definition,

$S = S_A \times S_B$ is a subrectangle in P

$$L(f_1 P) = \sum_{S\in P} m_s(f)V(S)$$

$$= \sum_{S_B \in P_B} m_{s_A \times s_B}(f)V(S_A \times S_B)$$

$$= \sum_{S_A \in P_A} \left(\sum_{S_B \in P_B} m_{s_A \times s_B}(f)V(S_B) \right) V(S_A) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

For $x \in S_A, m_{s_A \times s_B}(f) \subseteq M_{s_B}(g_x)$

$\therefore$ For $x \in S_{A,}$

$$\therefore \sum_{S_B \in P_B} m_{s_A \times s_B} V(S_A) \cdot V(S_B) \le \sum m_{s_B}(g_x)V(S_B)$$

$$= L(g_x, P_B) \le L\int_B g_x = L(x)$$

This is true for any $x \in A$

$$\therefore L(f, P) = \sum_{S_A \in P_A} \left(\sum_{S_B \in P_B} m_{s_A \times s_B}(f)V(S_B) \right) V(S_A)$$

$$\le \sum_{S_A \in P_A} m_{s_A}(L(x))V(S_A)$$

$$= L(\ell(x), P_A) \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(II)}$$

$\therefore$ From (I) & (II)

$L(f,P) \le (L(x),P_A)$ …………………………………………… (III)

Now $U(f,P) = \sum_{S\in P} M_S(f)V(s)$

$$= \sum_{\substack{S_A\in P_A\\ S_B\in P_B}} M_{S_A\times S_B}(f)V(S_A\times S_B)$$

$$= \sum_{S_A\in P_A}\left(\sum_{S_B\in P_B} M_{S_A\times S_B}(f)V(S_B)\right)V(S_A) \quad \text{……………….. (IV)}$$

For $x\in S_A, M_{S_A\times S_B}(f) \ge M_{S_B}(g_x)$

$\therefore$For $x\in S_A$,

$$\sum_{S_B\in P_B} M_{S_A\times S_B}(f)V(S_B) \ge \sum_{S_B\in P_B} M_{S_B}(g_x)V(S_B)$$

$$= u(g_x,P_B) \ge u\int_B g_x = \mu(x)$$

This is true for any $x\in A$.

$$\sum_{S_A\in P_A}\left(\sum_{S_B\in P_B} M_{S_A\times S_B}(f)V(S_B)\right)V(S_A)$$

$$\ge \sum_{S_A\in P_A} M_{S_A}(u(x))V(S_A)$$

$$= (u(x),P_A) \quad \text{……………………………………….. (V)}$$

from (IV) & (V)

$$U(f,P) \ge U(u(x),P_A) \quad \text{……………………………. (VI)}$$

$\therefore$ By (III) & (VI)

$$L(f,P) \le L(\ell(x),P_A) \le u(L(x),P_A)$$

$$\le u(\ell(x),P_A) \le U(f,P) \quad \text{………………………… (VII)}$$

Also

$L(f,P) \le L(\ell(x),P_A) \le L(\mu(x),P_A) \le u(\ell(x),P_A)$…………. (VIII)

$\therefore f$ is integrable

$$\sup_P\{L(f,P)\} = \inf_P\{U(f,P)\} = \int_{A\times B} f$$

$$\Rightarrow \sup_{P_A}\{L(\ell(x),P_A)\} = \inf_{P_B}\{u(\ell(x),P_A)\} = \int_{A\times B} f$$

$\therefore \ell(x)$ is integrable

$$\int_{A\times B} f = \int_A \ell(x) = \int_A \left(L\int_B f(x,y) \right) dx \text{ ………………………… (IX)}$$

Also by (VIII) & (IX)

$$\sup_{P_A}\left\{L\left(L(x),P_A\right)\right\} = \inf_{P_A}\left\{U\left(u(x),P_A\right)\right\} = \int_{A\times B} f$$

$\therefore u(x)$ is integrable.

$$\Rightarrow \int_{A\times B} f = \int_A u(x)\,dx = \int_A \left(U\int_B f(x,y) \right) dx$$

Hence Proved

Remark :

The Fubini's theorem is a result which gives conditions under which it is possible to compute a double integral using interated integrals, As a consequence if allows the under integration to be changed in iterated integrals.

$$\int_{A\times B} f = \int_B \left(L\int_B f(x,y)\,dx \right) dy$$

$$= \int_B \left(U\int_A f(x,y)\,dx \right) dy$$

These integrals are called iterated integrals.

Example 13:

Using Fubini's theorem show that $D_{12}f = D_{21}f$ if $D_{12}(f)$ and $D_{21}(f)$ are continuous.

Solution :

$\Rightarrow$ Let $A \subseteq R$ and $f : A \to \mathbb{R}$ continuous

T.P.T $D_{12}f = D_{21}f$

Suppose $D_{12}f \neq D_{21}f$

$\therefore \exists\, x_0, y_0$ in domain of f such that

$\left(D_{12}f(a) - D_{21}f(a)\right) \neq 0$

without loss of generality, $\left(D_{12}f(a) - D_{21}f(a)\right) > 0$ or

$\left(D_{12}f - D_{21}f\right)(a) > 0$ ………………………………….. (I)

$$\therefore \int_A (D_{12} f - D_{21} f)(x, g) > 0$$

Let $A = [a,b] \times [c,d]$

$\therefore$ By Fubini's Theorem

$$\int_A D_{21} f(x,y) = \int_c^d \int_a^b D_{21} f(x,y)\, dx\, dy$$

$$= \int_c^d \left(D_2 f(b,y) - D_2 f(g,y)\right) dy$$

$$= f(b,d) - f(b,c) - f(a,d) + f(a,c)$$

Similarly,

$$\int_A D_{12} f(x,y) = f(b,d) - f(b,c) - f(a,d) + f(a,c)$$

$$\therefore \int_A D_{21} f(x,y) = \int_A D_{12} f(x,y)$$

$$\Rightarrow \int_A (D_{21} f - D_{12} f)(x,y) = 0$$

Which is contradiction to (I)

$\boxed{D_{12} f = D_{21} f}$ proved

Example 14:

Use Fubini's Theorem to compute the following integrals.

1) $\quad I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy.dx}{1+x^2+y^2}$

Solution :

$$I = \int_0^1 \int_0^{\sqrt{1+x^2}} \frac{dy.dx}{1+x^2+y^2}$$

$$= \int_0^1 dx \int_0^{\sqrt{1+x^2}} \frac{dy}{1+x^2+y^2}$$

$$= \int_0^1 dx \left[\frac{1}{\sqrt{1+x^2}} \tan^{-1} \frac{y}{\sqrt{1+x^2}}\right]_0^{\sqrt{1+x^2}}$$

$$= \int_0^1 dx. \frac{1}{\sqrt{1+x^2}} . \frac{\pi}{4}$$

$$= \frac{\pi}{4}\int_{0}^{1} \frac{dx}{\sqrt{1+x^2}}$$

$$= \frac{\pi}{4}\left[\log\left(x+\sqrt{1+x^2}\right]_0^1\right.$$

$$= \frac{\pi}{4}\log\left[\sqrt{x}+1\right]$$

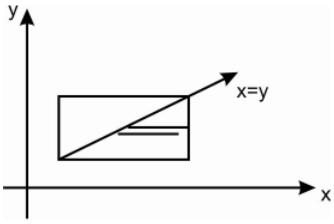
ii) $I = \int_{0}^{1} dy \int_{y}^{1} \sin\left(\frac{\pi x^2}{2}\right) dx$

Solution :

$$C = \{(x,y); y \le x < 1, 0 \le y \le 1\}$$

By Fubini's Theorem

$$I = \int_{0}^{1}\int_{y}^{1} \sin\left(\frac{\pi x^2}{2}\right) dxdy$$



$$= \int_{0}^{1}\int_{0}^{x} \sin\left(\frac{\pi x^2}{2}\right) dxdy$$

$$= \int_{0}^{1} \sin\left(\frac{\pi x^2}{2}\right) [y]_0^x \, dx$$

$$= \int_{0}^{1} x \sin\left(\frac{\pi x^2}{2}\right) dx$$

Put $\frac{\pi x^2}{2} = t,$

x	θ	1
t	0	$\pi/2$

$$\frac{2\pi x}{2} dx = dt$$

$$xdx = \frac{dt}{\pi}$$

$$I = \int_{0}^{\pi/2} \sin t \frac{dt}{\pi} = \frac{1}{\pi}\int_{0}^{\pi/2} \sin t \, dt \frac{1}{\pi}(-\cos t)_0^{\pi/2}$$

$$= \frac{1}{\pi}[-0+1] = \frac{1}{\pi}$$

2.7 REVIEWS

After reading this chapter you would be knowing.

- ❖ Definition of Measure zero set and content zero set.
- ❖ Oscillation $O(f,a)$
- ❖ Find set contain measure zero on content zero
- ❖ Statement of Lebesgue Theorem
- ❖ Definition of characteristic function & its properties.
- ❖ Fubini's Theorem & its examples.

2.8 UNIT END EXERCISES

1. If $B \subseteq A$ and A has measure zero then show that & has measure zero.
2. Show that countable set has measure zero.
3. If A is non-empty open set, then show that A is not of measure zero.
4. Give an example of a bounded set C if measure zero but ∂C does not have measure zero.
5. Show by an example that a set A has measure zero but A does not have content zero.
6. Prove that $[a_1,b_1]\times....\times[a_n,b_n]$ does not have content zero if $a_i < b_i$ for each i.
7. If C is a set of content zero show that the boundary of C has content zero.
8. Give an example of a set A and a bounded subset C of A measure zero such that $\int_A \chi_c$ does not exist.
9. If f & g are integrable, then show that f_g is integrable.
10. Let $U=[0,1]$ be the union of all open intervals (a_i,b_i) such that each rational number in $(0,1)$ is contained in some (a_i,b_i). Show that if $f=\chi_c$ except on a set of measure zero, then f is not integrable on $[0,1]$.
11. If $f:[a,b]\times[a,b]\to\mathbb{R}$ is continuous; then show that $\int_a^b \int_x^b f(x,y)\,dx\,dy = \int_a^b \int_x^b f(x,y)\,dy\,dx$
12. Use Fubini's theorem, to compute $\int_0^{\pi/2} dy \int_0^{\pi/2} \frac{\sin x}{x+y}\,dx$

13. Let $A = [-1,1] \times [0, \pi/2]$ and $f : A \to \mathbb{R}$ defined by $f(x, y) = x \sin y - ye^x$ compute $\int_A f$

14. Let $f(x, y, z) = z \sin(x + y)$ and $A = [0, \pi] \times \left[-\pi/2, \pi/2\right] \times [0,1]$ computer $\int_A f$.

❖❖❖❖

3

LEBESGUE OUTER MEASURE

Unit Structure :

3.0 OBJECTIVE

After going through this chapter you can able to know that

- Concept of σ – Algebra, Measurable set.
- Extension measure in $\mathbb{R}^n$
- Lebesgue measureable set
- Lebesgue outer measure & its properties.

3.1 INTRODUCTION

In this chapter we shall fist study such a verified theory function d-dimensional value based on the notation of a measure, and then we shall use this theory to build a stronger and more flexible theory.

Now if we want to partition the range of a function, we need same way of measuring how much of the domain is sent to a particular region of the partition, To set a feeling function what we are aiming function let us assume that we want to measure the volume of subsets $A, C\mathbb{R}^3$ and that are denote the volume of A by $\mu(A)$.

Then function we have

i) $\mu(A)$ should be non-negative number as ∞.

ii) $\mu(\varnothing)=0$ it will be convenient to assign a volume to the empty set.

iii) If $A_1, A_2, ..., A_n$ are non overlapping disjoint sets then

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right)=\sum_{n=1}^{\infty} \mu(A_n)$$

This means that the volume the whole is equal to the sum of the volume of the parts. This problems leads us to the theory of measures where we try to give a notation of measure to subsets of an Euclidean space.

Defenition :

The Euclidean norm on $\mathbb{R}^n$ is $|x|=\left(x_1^2+....+x_d^2\right)^{1/2}$.

The distance between $x, y \Rightarrow \mathbb{R}^n$ is $|x-y|$

3.2 $\sigma-$ALGEBRA

Definition :

Let X be a set. A collection A of subsets of X is called a $\sigma-$algebra of the following hold.

i) $\varnothing \in A$

ii) $A \in A \Rightarrow X/A \in A$

iii) $A_1, A_2, ..., \in A \Rightarrow \bigcup_{i=1}^{\infty} A_i \in A$

Note :

The pair (X, A) is called measurable space and elements of A are called measurable sets.

Example 1 :

Let $X=\{1,2,3\}$ and $b_1=\{\{1\},\{1,2,3\}, X, \varnothing\}$, $b_2=\{1,2,3,\{3\}, X, \varnothing\}$. Check whether b_1 and b_2 are both algebras or not.

Solution :

I) Let $X=\{1,2,3\}$ and b_1 is not $\sigma-$Algebra.

Since it does not contain $\{1\}^C$.

II) b_2 is $\sigma-$Algebra since it satisfies all condition of $\sigma-$Algebra

i.e. $X=b_1$

$\varnothing=b_2$

$\{1,2\} \in b_2$ & $\{1,2\}^C \in b_2$

$\therefore b_2$ is $\sigma-$Algebra.

Example 2 :

A measure on a topological space X whose domain is the Borel algebra is called a Borel measure.

Example : For every $x \in X$, the Dirac measure is given by
$$\delta_x(A) = \begin{cases} 1\, if\ x \in A \\ 0\, if\ x \notin A \end{cases}$$

Definition :

Let μ be a set function whose domain in a class A of subsets of a set X and whose values are non-negative extended reals, we say that μ is contably additive if $\mu\left(\bigcup_{k=1}^{\infty} A_k\right) = \sum_{k=1}^{\infty} \mu(A_k)$ whenever, (A_k) is a sequence of painoise disjoint set in A whose union is also in A.

Theorem :

Let μ be a finitely additive set function, defined on the $\sigma-$Algebra A. Then μ is countably additive iff it has the following property : if $A_n \in A$ and $A_n \subset A_{n+1}$ Anti for each positive integer n, and if $\bigcup_{n=1}^{\infty} A_n \in A$ then $\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} \mu(A_n)$.

Proof :

Suppose μ is countable additive Let $\{A_n\}$ be a sequence of elements in A s.t. $A_1 \subseteq A_2 \subseteq, ..., A = \bigcup_{i=1}^{\infty} A_i \in A$

s.t. $\mu\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \mu(A_n)$

Define $B_1 = A_1$

$B_K = A_K / A_{K-1}$ for $K \geq 2$

Examples 3:

Let $\{A_i ; i \in I\}$ be collection of $\sigma-$Algebra. Show tha $\bigcap_{i\in I} A_i$ is a $\sigma-$Algebra, but $\bigcup_{i\in I} A_i$ is not in general.

Solution :

Let $Ai = \bigcap_{i\in I} A_i$

To show that A is a $\sigma-$Algebra

a) If $\varnothing \in A$

$\because A_i$ is $\sigma-$Algebra, $\forall i \in I$

$\therefore \varnothing \in A_i \ \ \forall i \in I$

$\Rightarrow \varnothing \in \bigcap_{i\in I} A_i \Rightarrow \varnothing \in A$

b) Let $A \in \boldsymbol{A}$

$\Rightarrow A = \bigcap_{i\in I} A_i$

$\because A_i$ is $\sigma-$Algebra $\forall i \in I$

$\therefore$ For $A \in A_i \Rightarrow A^C \in A_i \ \forall i \in I$

$\therefore A^c \in \bigcap_{i\in I} A_i$

$\Rightarrow A^c \in A$

c) Let $A_k \in A, \forall k = 1, 2....$

then $A_k \in \bigcap Ai \qquad \forall i \in i$

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in A_i \quad \forall i$

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \ \bigcap_{i\in I} A_i$

$\Rightarrow \bigcup_{k=1}^{\infty} A_k \in \ A$

$A = \bigcap_{i\in I} A_i$ is a $\sigma-$Algebra

Now, we have to show that $\bigcup A_i$ is not a $\sigma-$Algebra.

Let $X = \{1, 2, 3\}$

Let $A_1 = \{\phi, X, \{1\}, \{2,3\}\}$

$A_2 = \{\phi, X, \{3\}, \{1,2\}\}$

then A_1 & A_2 are $\sigma-$Algebra but $A_1 \cup A_2$ is not $\sigma-$Algebra. $\{1\} \in A_1 \cup A_2$ but $\{1,3\} \notin A_1 \cup A_2$.

Clearly $Bi \in A \ \forall i$ and $Bi's$ are pairwise disjoint we first show that $A_k = \bigcup_{i=1}^{k} Bi$

By induction on 'k'

The result is trivial when $k = 1$

Assume the result is true for $k-1$

i.e. $A_{k-1} = \bigcup_{i=1}^{k-1} Bi$

Now $\bigcup_{i=1}^{k} Bi = \bigcup_{i=1}^{k-1} Bi \bigcup B_k$

$= A_{k-1} \bigcup (A_k / A_{k-1})$

$= A_k$

$\therefore$ The result is true for k.

$\therefore$by introduction is true for all k

$$A_k = \bigcup_{i=1}^{k} Bi \quad \forall\, k \geq 1$$

Note that $A = \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} \left(\bigcup_{i=1}^{k} Bi \right)$

$= \bigcup_{k=1}^{\infty} B_k$

$\because \mu$ is countably additive, we have

$$\mu(A) = \mu\left(\bigcup_{k=1}^{\infty} B_k \right) = \sum_{K=1}^{\infty} \mu(B_k)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} \mu(B_k)$$

$$= \lim_{n\to\infty} \left(\mu\left(\bigcup_{k=1}^{n} B_k \right) \right)$$

$$= \lim_{n\to\infty} \mu(A_n)$$

Conversely,

Suppose whenever if $A_1 \subset A_2 \subset A_3 ...., Ai \in B_r \boldsymbol{A}, \bigcup Ai \in \boldsymbol{A}$

Then $\mu\left(\bigcup_{i=1}^{\infty} Ai \right) = \lim_{n\to\infty} \mu(A_n)$

T.S.T. μ is countably additive

Let (A_n) be a pairwise disjoint sets in A.

Define $B_k = \bigcup_{i=1}^{k} Ai$ then $B_k \in \boldsymbol{A}$ and $B_1 \subseteq B_2 \subseteq ....$

$\therefore$By hypothesis, we have

$$\mu\left(\bigcup_{i=1}^{\infty} Bi \right) = \lim_{n\to\infty} \mu(B_n)$$

But $\bigcup_{i=1}^{\infty} Bi = \bigcup_{i=1}^{\infty}\left(\bigcup_{K=1}^{i} A_k\right)$

$$= \bigcup_{i=1}^{\infty} Ai$$

$$\therefore \mu\left(\bigcup_{i=1}^{\infty} Ai\right) = \mu\left(\bigcup_{i=1}^{\infty} Bi\right)$$

$$= \lim_{n\to\infty} \mu(B_n)$$

$$= \lim_{n\to\infty} \mu\left(\bigcup_{i=1}^{n} Ai\right)$$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \mu(Ai)$$

$$= \sum_{i=1}^{\infty} \mu(Ai)$$

Theorem :

Let A be a $\sigma -$Algebra, If (u,v) are measures on A, $t \in \mathbb{R}, t > 0$ and $A \in \boldsymbol{A}$ hen the following are measures on A.

a) $\mu + \vartheta$ defined by $(\mu+\vartheta)(E) = \mu(E) + \vartheta(E) E \in A$
b) $t\mu$, defined by $(t\mu)(\in) = t\mu(E), E \in A$

Proof :
a) $\mu + \vartheta$ defined by $(\mu+\vartheta)(E) = \mu(E) + \vartheta(E), E \in A$ is a measure on A.

$\therefore \mu$ & ϑ are measure on A.

$\therefore$They are countably additive non-negative set function.

$\therefore (\mu+\vartheta)(E)$ is also countably additive non-negative set function whose domain is A.

$\therefore \mu + \vartheta$ is a measure on A.

b) $(t\mu)(E) = t\mu(E)$
$\therefore \mu$ is a measure on A
$\therefore \mu$ is countable additive non negative set function whose domain in A.

$\therefore$for $E \in A$

$(t\mu)(E) = t\mu(E)$ and $t\mu$ is also countably additive non-negative set of function whose domain is A

$\therefore t\mu$ is measure on A.

3.3 EXTENSION MEASURE

Definition :

Let X be a set, A_n Exterior measure or outer measure on X is a non-negative, extended real valued function μ^* whose domain consist of all subsets of X and which satisfies :

a) $\mu^*(\phi) = 0$

b) (Monotonicity) if $A \subset B$ then $\mu^*(A) \subseteq \mu^*(B)$

c) (Countable sub-additivity)

For any sequence (A_n) of subsets of X, we have

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Theorem :

Let C be a collection of closed rectangle of $\mathbb{R}^n$, For $R \in C$, let $\vartheta(R)$ denote the volume of R. If μ^* is defined by

$$\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \vartheta(C_k); C_k \in C, \bigcup_{k=1}^{\infty}(k) \supset A\right\}$$

For $A \subset \mathbb{R}^n, A \neq \phi$ then μ^* is exterior measure on $\mathbb{R}^n$.

Proof :

T.S.T. μ^* defined by $\mu^*(A) = \inf\left\{\sum_{k=1}^{\infty} \vartheta(C_k); C_k\right\}$ is closed rectangle where $A \subset \mathbb{R}^n$ is on exterior Measure on $\mathbb{R}^n$.

We first shows that

$$\left\{\sum V(C_k); C_k \textit{ is closed set } A \subseteq C_k\right\} \neq \phi$$

Where $A \subset \mathbb{R}^n$

Let $R_k =$ rectangle with side length ‘k’ and centre origin.

Then $\bigcup_{k=1}^{\infty} R_k = \mathbb{R}^n$

$\therefore$ for any $A \subset \mathbb{R}^n = \bigcup_{k=1}^{\infty} R_k$

$\Rightarrow \{R_k\}$ covers A

$\therefore \left\{ \sum_{k=1}^{\infty} \vartheta(C_k); C_k \text{ closed rectangle } A \subseteq \bigcup_{k=1}^{\infty} C_k \right\} \neq \phi$

We now show $\mu^*(\phi) = 0$

Let $\in > 0$

Let $R = [0, \in^{y_n}] \times ..... \times [0, \in^{y_n}]$ be a rectangle in $\mathbb{R}^n$ with $\vartheta(R) = \in \;\&\; \phi \subseteq R$

$\therefore \{R\}$ covers ϕ

$\therefore$ By definition of μ^*, $\mu^*(\phi) < \in$

This is true for any $\in > 0$

$\mu^*(\phi) = 0$ ………………………………………… (1)

Let $A \subseteq B \subseteq \mathbb{R}^n$

T.S.T. $\mu^*(A) \leq \mu^*(B)$

If $\{C_k\}$ Covers B, then $\{C_k\}$ covers A

$\therefore \left\{ \sum_{k=1}^{\infty} \vartheta(C_k) : B \subseteq \bigcup_{k=1}^{\infty} C_k \right\} \subseteq \left\{ \sum \vartheta(C_k) : A \subseteq \bigcup_{k=1}^{\infty} C_k \right\}$

$\Rightarrow \inf \left\{ \sum \vartheta(C_k) : B \subseteq \bigcup_{k=1}^{\infty} C_k \right\} \geq \inf \left\{ \sum \vartheta(C_k) : A \subseteq \bigcup_{k=1}^{\infty} C_k \right\}$

$\Rightarrow \mu^*(A) \leq \mu^*(B)$ …………………………………….. (2)

Let $\{A_n\}$ be a sequence of subsets of $\mathbb{R}^n$ we show that

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n \right) \leq \sum_{n=1}^{\infty} \mu^*(A_n)$$

Let $\in > 0$ by the definition of μ^*

$\exists$ a cover $\{R_{n_i}\}_{i=1}^{\infty}$ of A_n such that

$\sum_{i=1}^{\infty} \vartheta(R_{n_i}) < \mu^*(A_n) + \in / 2^n$

Then $\bigcup_{n=1}^{\infty}\left(\bigcup_{j=1}^{\infty} R_{n_i}\right)$ covers $\bigcup_{n=1}^{\infty} A_n$

$$\mu^*\left(\bigcup_{n=1}^{\infty} A_n\right) \le \sum_{n=1}^{\infty}\left(\sum_{i=1}^{\infty} \vartheta(R_n)\right)$$

$$\le \sum_{n=1}^{\infty}\left(\mu^*(A_n) + \in\!/2^n\right)$$

$$\le \sum_{n=1}^{\infty} \mu^*(A_n) + \sum_{n=1}^{\infty} \in\!/2^n$$

$$\le \sum_{n=1}^{\infty} \mu^*(A_n) + \in$$

$$\le \sum_{n=1}^{\infty} \mu^*(A_n) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(3)}$$

From (1) (2) & (3)

μ^* is an exterior measure on $\mathbb{R}^n$

Note :

By above lemma, the exterior measure lemma attempts to describe the volume of a set $E \subseteq \mathbb{R}^n$ by approximating it from outside. The set E covered by rectangle and if the covering gets finer, with fewer rectangles overlapping the volume of E should be close to the sum of the volumes of the rectangles.

3.4 LEBESGUE OUTER MEASURE

Definition :

μ^* is called the Lebesgue exterior (or outer) measure on $\mathbb{R}^n$ and is denoted by m^*.

Now the consequences of the definition of exterior measure on $\mathbb{R}^n$.

1) If $\{R_k\}$ are countably many rectangles and $E \subset \bigcup R_k$ then $m^*(E) \le \sum V(R_k)$

2) For a given $\in > 0$ there exist countable many rectangle $\{R_k\}$ with $E \subseteq \bigcup R_k$ such that $m^*(E) \le \sum_k \vartheta(R_k) \le m^*(E) + E$.

Example 4:

Show that exterior (or outer) measure of a closed rectangle is its volume i.e. $m^*(R) = V(R)$ where R is a rectangle or a $b_0 \times in \mathbb{R}^n$.

Solution :

Let R be a closed rectangle in $\mathbb{R}^n$

$$tst\ m^*(R) = V(R)$$

Note that $\{R\}$ covers R

$\therefore$ by definition of $m^*(R)$, we get

$$m^*(R) \le V(R) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

Let $\in > 0$

By definition $m^*(R), \exists\, a$ countable cover $\{R_i\}$ of closed rectangles of R.

$$\boxed{\sum_{i=1}^{\infty} \vartheta(R_i) < m^*(R_i) + \frac{\in}{2}}$$

For each *i* choose an open rectangle S_i such that $R_i \subseteq S_i$ and

$$V(S_i) \le V(R_i) + \frac{\in}{2^{i+1}}$$

Then $R \subseteq \bigcup_{i=1}^{\infty} R_i \subseteq \bigcup_{i=1}^{\infty} S_i$

$\therefore \{S_i\}_{i=1}^{\infty}$ is an open cover of R

$\because R$ is compact this open cover has a finite sub cover say

$$R \subseteq \bigcup_{i=1}^{m} S_i \text{ (after renaming)}$$

We have

$$V(R) \le \sum_{i=1}^{m} V(S_i) \le \sum_{i=1}^{\infty} v(S_i)$$

$$\le \sum_{i=1}^{\infty} \left(V(R_i) + \frac{\in}{2^{i+1}} \right)$$

$$\le \sum_{i=1}^{\infty} V(R_i) + \in/2$$

$$< m^*(R) + \in/2 + \in/2$$

$$< m^*(R) + \in$$

This is true for any $\in > 0$

$$V(R) \le m^*(R)$$

From (1) & (2)

$$V(R) = m^*(R)$$

Example 5:

Show that exterior (or outer) measure of an open rectangle in $\mathbb{R}^n$ is volume.

Solution :

Let S_i be an open rectangle them $R_i \subseteq S_i$ where S_i is closed rectangle $\Rightarrow \{S_i\}$ is a cover of R.

$\because$ by definition $m^*(R) \leq V(S_i) = V(R)$ ……………. (1)

Let $\in > 0$ be $\{R_i\}$ be a countable cover of closed rectangle of R such that $\sum_{i=1}^{\infty} V(R_i) < m^*(R) + \in/2$ for each i choose an open rectangle S_i such that $R_i \subseteq S_i$ & $V(R_i) + \in/2^{i+1}$

Then $R \subseteq \bigcup_{i=1}^{\infty} R_i \subset \bigcup_{i=1}^{\infty} S_i$

$\therefore \{S_i\}_{i=1}^{\infty}$ is an open cover of R

$\because R$ is compact. This open cover has a sub cover say

$R \subseteq \bigcup_{i=1}^{m} S_i$ (after renaming)

We have

$$V(R) \leq \sum_{i=1}^{m} V(S_i) \leq \sum_{i=1}^{\infty} V(S_i)$$
$$\leq \sum_{i=1}^{\infty} \left(V(R_i) + \in/2^{i+1} \right)$$
$$\leq \sum_{i=1}^{\infty} V(R_i) + \in/2$$
$$< m^*(R) + \in/2 + \in/2$$
$$< m^*(R) + \in$$

This is true for any $\in > 0$

$\therefore V(R) \leq m^*(R)$ ……………………………….. (2)

From (1) & (2)

$V(R) = m^*(R)$

Example 6:

Show that exterior measure of a point in $\mathbb{R}^n$ is zero.

Solution :

Let $a=(a_1,a_2,......,a_n)\in\mathbb{R}^n$

To show that $m^*\{0\}=0$

Let $\in>0$ then the closed rectangle.

$$R=\left[a_1-\frac{\in^{1/n}}{2},a_1+\frac{\in^{1/n}}{2}\right]\times$$

$$\left[a_2-\frac{\in^{1/n}}{2},a_2+\frac{\in^{1/n}}{2}\right]\times......................$$

Covers $\{a\}$

$\therefore$ By definition of $m^*(\{a\})$, we have $m^*(\{a\})\le V(R)=\in$

This is true for any $\in>0$

$\therefore m^*(\{0\})=0$

3.5 PROPERTIES OF OUTER MEASURE

Exterior measure has the following properties.

i) (Empty set) The empty set ϕ has exterior measure $m^*(\phi)=0$.

ii) (Positivity) we have $0\le m^*(A)\le+\infty$ for every subset A of $\mathbb{R}^n$.

iii) (Monotonicity) If $A\subset B\le\mathbb{R}^n$, then $m^*(A)\le m^*(B)$.

iv) (Finite sub-additivity) If $\{A_j\}_{j\in J}$ are a finite collection of subset of $\mathbb{R}^n$ then $m^*\left(\bigcup_{j\in J}A_j\right)\le\sum_{j\in J}m^*(A_j)$

v) (Countable sub-additivity) if $\{A_j\}_{j\in J}$ are a countable collection of subsets of $\mathbb{R}^n$ then $m^*\left(\bigcup_{j\in J}A_j\right)\le\sum_{j\in J}m^*(A_j)$

vi) (Translation invariance) If E is a subset of $\mathbb{R}^n$ and $x\in\mathbb{R}^n$ then $m^*(x+\in)=m^*(\in)$.

Let $x\in\mathbb{R}^n,E\subseteq\mathbb{R}^n$

tst $m^*(x+\in)=m^*(\in)$

Let $\in>0$, by definition of $m^*(\in)$

$\exists$ a countable cover (R_i) of closed rectangles in $\mathbb{R}^n$ for s.t.

$\therefore\sum_{i=1}^{\infty}V(R_i)<m^*(E)+\in$ .... (1)

We now show that $x+E\subseteq\bigcup_{i=1}^{\infty}(x+R_i)$

Let $a\in x+E\Rightarrow a=x+y$

$\Rightarrow a-x=y\in E\subseteq\bigcup_{i=1}^{\infty}R_i$

$\Rightarrow a-x\in R_i$ for some i

$\Rightarrow a\in -x+R_i$ for some i

$\Rightarrow a\in\bigcup_{i=1}^{\infty}(x+R_i)$

$\therefore x+E\subseteq\bigcup_{i=1}^{\infty}(x+R_i)$

$\therefore$By definition of m^*, we have

$m^*(x+E)\le\sum_{i=1}^{\infty}V(x+R_i)$ …………………………………… (2)

We now show that $V(x+R_i)=V(R_i)\forall_i$

Let $R_i=[a_{iu},b_{iu}]\times.....\times[a_{iu},b_{iu}]$ then

$x+R_i=[x_1+a_{i1},x_1+b_{i1}]\times......\times[a_{in}\times x_n,b_{in}+x_n]$

$\therefore V(x+R_i)=\prod_{j=1}^{n}(b_{ij}+x_i)-(a_{ij}+x_i)$

$=\prod_{j=1}^{n}(b_{ij}-a_{ij})=V(R_i)$ ……………………………… (3)

$\therefore$ By 1,2,3 we get

$m^*(x+E)\le\sum_{i=1}^{\infty}V(x+R_i)=\sum_{i=1}^{\infty}V(R_i)<m^*(E)+\in$

$m^*(x+E)<m^*(E)+\in$

This is true for any $\in>0$

$m^*(x+E)\le m^*(E)+\in$ ………………………………….. (4)

Let $E'=x+E$ & $y=-x$

Then by (4)

$m^*(y+E')\le m^*(E')$

$\Rightarrow m^*(-x+x+E)\le m^*(x+E)$

$\Rightarrow m^*(E)\le m^*(x+E)$ …………………………………… (5)

By (4) & (5)

$\therefore m^*(x+E)=m^*(E)$

Theorem :

Show that there are uncountable subset of $\mathbb{R}$ whose exterior measure is zero.

Proof :

Define canter set as follows

Let $C_0=[0,1]$

trisect C_0 and remove the middle open interval to get C_1.

i.e. $C_1=\left[0,\frac{1}{3}\right]\cup[2/3,1]$

$=[0,1]\setminus[1/3,2/3]$

repeat this procedure for each interval in C_1 we get C_2

$$C_2=[0,1]\setminus(1/3,2/3)\setminus(1/9,2/9)\setminus(7/9,8/9)$$

$$=\left[0,{}^{1}\!/_{9}\right]\cup\left[{}^{2}\!/_{9},\frac{1}{3}\right]\cup\left[{}^{2}\!/_{3},7/9\right]$$

repeating this procedure at each stage we get a sequence of subsets C_i of $[0,1]$ for $i=0,1,2$

Note that each C_k is a compact subset of $\mathbb{R}$ and $C_0\supseteq C,\supseteq C_2$

The Cantor set 'C' is defined as $C=\bigcap_{i=0}^{\infty}C_i$

$C\neq\phi$ because all end points of each C_r is inc and also C is uncountable

We now compute

$$m^*(C_0)=1, m^*(C_1)=\frac{2}{3}=1-\frac{1}{3}$$

$$m^*(C_2)=m^*(C_1)-\frac{2}{9}$$

$$=1-\frac{1}{3}-\frac{2}{3^2}$$

$$m^*(C_3)=m^*(C_2)-\frac{2^2}{3^3}=1-\frac{1}{3}$$

$$=1-\frac{1}{3}-\frac{2}{3^2}-\frac{2^2}{3^3}$$

in general,

$$m^*(C_k) = 1 - \frac{1}{3} - \frac{2}{3^2} - \frac{2^2}{3^3} - ..... - \frac{2^{k-2}}{3^{k-1}}$$

$$= \frac{2}{3} - \frac{2}{3}\left[\frac{3}{3} + \frac{2}{3^2} + ... + \frac{2^{k-3}}{3^{k-2}}\right]$$

$$= \frac{2}{3} - \frac{2}{3}\left[\frac{\frac{1}{3}\left(1 - \left(\frac{2}{3}\right)^{k-2}\right)}{\left(1 - \frac{2}{3}\right)}\right]$$

$$= \frac{2}{3}\left[1 - 1 + \left(\frac{2}{3}\right)^{k-1}\right]$$

$$= \left(\frac{2}{3}\right)^k$$

$\because C \subseteq C_k \forall k$

$\Rightarrow m^*(C) \leq m^*(C_k) \forall k$

$\Rightarrow m^*(C_1) \leq \left(\frac{2}{3}\right) \forall k$

letting $k \to \infty$, we get

$0 \subseteq m^*(C) \leq 0$

$= m^*(C) = 0$

Theorem :

Show that exterior measure of $\mathbb{R}^n$ is infinite.

Proof :

Let $M > 0$ and R be a rectangle s.t. $V(R) = M$

note that $\mathbb{R} \subseteq \mathbb{R}^n$

$\therefore$By monotonicity of m^*

$m^*(R) \leq m^*(\mathbb{R}^n)$

But $m^*(R) = V(R) = M$

$\therefore m^*(\mathbb{R}^n) \geq M$

This is true for any $M > 0$

$\therefore m^*(\mathbb{R}^n) = \infty$

Theorem :

If E and $F \subseteq \mathbb{R}^n$ such that $d(E, F) > 0$ then show that $m^*(E \cup F) = m^*(E) + m^*(F)$.

Proof :

Let $E, F \subseteq \mathbb{R}^n$ be s.t. $d(E,F) > 0$ tst $m^*(E \cup F) = m^*(E) + m^*(F)$.
By countable subodditivity property $m^*(E \cup F) \leq m^*(E) + m^*(F)$.. (1)

Let $\in > 0$
By the definition of $m^*, \exists$ countable $\{Ri\}$ of closed rectangles in $\mathbb{R}^n$ for $E \cup F$ such that $\sum_i V(Ri) < m^*(E \cup F) + \in$. ………………. (2)

We categorize the collection $\{Ri\}$ into 3 types :

1) Those intersecting only E
2) Those interescting only F
3) Those intersecting both E & F

Note that if a rectangle $\mathbb{R}$ intersect both E & F, then $d(R) > d(E,F) > 0$ subdivide such the rectangles into rectangles whose diameter is less than $d(E,F)$.

This subrectanlges intersect either E or F not both.
$\therefore$ We can have a contable collection $\{R_2\}$ of rectangles which intersects either E or F but not both.

Let $I_1 = \{i; R_i \cap E \neq \phi\}$
$I_2 = \{i; R_i \cap F \neq \phi\}$
$\Rightarrow I_1 \cap I_2 = \phi$
$\therefore \{R_i\}_{i \in I}$, covers E, we have

$$m^*(E) \leq \sum_{i \in I_1} V(R_i)$$

Similarly, $m^*(F) \leq \sum_{i \in I_2} V(R_i)$

$$\therefore m^*(E) + m^*(F) \leq \sum_{i \in I_1} V(R_i) + \sum_{i \in I_2} V(R_i)$$
$$\leq \sum_{i=1}^{\infty} V(R_i)$$
$$< m^*(E \cup F) + \in \text{ (by (2))}$$

This is true for any $\in > 0$

$\Rightarrow m^*(E)+m^*(F)\le m^*(E\cup F)$ ……….. (3)

From (1) & (3)

$m^*(E)+m^*(F)=m^*(E\cup F)$

Theorem :

If a subset $E\subseteq\mathbb{R}^n$ is a countable unit of almost disjoint closed rectangle .

i.e. $E=\bigcup_{i=1}^{\infty}R_i$ then show that $m^*(E)=\sum_{i=1}^{\infty}\vee(R_i)$.

Proof :

Let $E=\bigcup_{i=1}^{\infty}R_i$ where R_i's are almost disjoint closed rectangles.

tpt $m^*(E)=\sum_{i=1}^{\infty}\vartheta(R_i)$

By countably subadditivity proposition of

$m^*(E)=m^*\left(\bigcup_{i=1}^{\infty}R_i\right)=\sum_{i=1}^{\infty}m^*(R_i)=\sum_{i=1}^{\infty}V(R_i)$

($\because R$ is rectangle $\Rightarrow m^*(R)=V(R)$)

Let $\in>0$, by definition of $m^*,\exists$ a countable cover $\{R_i\}$ of closed rectangle $\mathbb{R}^n$ for E s.t.

$\sum_{i=1}^{\infty}V(R_i)<m^*(E)+\in$

For each i, choose open rectangle S_i s.t. $S_i\subseteq R_i$ & $V(R_i)\le V(S_i)+\frac{\in}{2^i}$

Note that $d(S_i,S_j)>0$ for $i\ne j$

$\therefore m^*(S_i\cup S_j)=m^*(S_i)+m^*(S_j)$ for $i\ne j$ …………………….. (1)

Using (1) finite no. of times, we get $m^*\left(\bigcup_{1}^{k}S_i\right)=\sum_{i=1}^{k}m^*(S_i)$

$\because S_i\subseteq R_i\subseteq E\ \ \forall i$

$\Rightarrow\bigcup_{i=1}^{k}S_i\subseteq E$

$\therefore$By monotonicity

$$\therefore m^*(E) \geq \sum_{i=1}^{k} m^*(S_i) = \sum_{i=1}^{k} V(S_i) \,\forall\, k$$

Let $k \to \infty$

$$m^*(E) \geq \sum_{i=1}^{\infty} V(S_i) = \sum_{i=1}^{\infty}\left(V(R_i) - \in\!\!/_2 i\right)$$

$$\geq \sum_{i=1}^{\infty} V(R_i) - \in$$

This is true for any $\in > 0$

$\Rightarrow m^*(\in) \geq \sum_{i=1}^{\infty} V(R_i)$ ………………………………………….. (2)

From (1) & (2)

$$m^*(\in) = \sum_{i=1}^{\infty} V(R_i)$$

Theorem :

Show that

1) If $m^*(A) = 0$ then $m^*(A \cup B) = m^*(B)$

2) If $m^*(A \Delta B) = 0$ then show that $m^*(A) = m^*(B)$

3) $m^*(A \setminus B) \geq = m^*(A) - m^*(B)$

Proof :

1) As $B \subseteq A \cup B$

By monotonicity

$m^*(B) \leq m^*(A \cup B)$ …………………………………. (1)

Also by countable subadditive of m^*

$$m^*(A \cup B) \leq m^*(A) + m^*(B)$$

$\leq m^*(B)$ ……………………………………… (2)

From (1) & (2)

$$m^*(A \cup B) = m^*(B)$$

2) If $m^*(A \Delta B) = 0$ tst $m^*(A) = m^*(B)$

$$wk > A \Delta B = (A \setminus B) \cup (B \setminus A)$$

$$\Rightarrow m^*(A \Delta B) \leq m^*(A \setminus B) + m^*(B \setminus A)$$

given that $m^*(A \Delta B) = 0$

$$\Rightarrow m^*(A/B) + m^*(B/A) = 0$$

but $0 \le m^*(A/B) \le m^*(A\Delta B) = 0$

$\Rightarrow m^*(A/B) = 0$

$\therefore m^*(A\Delta B) \le 0 + m^*(B/A)$

WKT $m^*(A) \ge m^*(A \cap B)$

$m^*(A) = m^*(A \cap B)$

similarly we show that

$m^*(B) = m^*(A \cap B)$

$\therefore m^*(A) = m^*(B)$

3) $m^*(A \setminus B) = m^*(B) - m^*(A)$

Proof :

Since A and B are measurable sets

$\therefore A^C$ is also measurable and we have

$B = A \cup (B / A) \quad \because A \subseteq B$

$B / A = B \cap A^C$ is measurable.

$\therefore B \,\&\, A^C$ is measurable

$\therefore B = A \cup (B \setminus A)$ union of disjoint measurable sets

$\therefore m^*(A \cup B \setminus A) = m^*(A) + m^*(B \setminus A) = m^*(B)$

$\therefore m^*(B \setminus A) = m^*(B) - m^*(A)$

Theorem :

Let $E \subseteq \mathbb{R}^n$ show that $m^*(E) = \inf\left\{m^*(\Omega); \Omega \supseteq E \,\&\, \Omega\, open\right\}$

Proof :

Let $E \subseteq \mathbb{R}^n$

tst $m^*(E) = \inf\left\{m^*(\pi); \pi \supset E \text{ and } \pi \text{ open in } \mathbb{R}^n\right\}$

Let Ω be open in $\mathbb{R}^n$ s.t. $E \subseteq \Omega$

Then by monotonicity of m^*, $m^*(E) \le m^*(\Omega)$

$\therefore m^*(E)$ is lower bound of $\left\{m^*(\Omega); \Omega \supset \in, \Omega\, open\right\}$

$\therefore m^*(E) \le \inf\left\{m^*(\Omega); \Omega \supset E, \pi\, open\right\}$ ……………………. (1)

Let $\in > 0$, then by definition of m^*

$\exists$ an countable cover $\{R_i\}$ of closed rectangle of E s.t.

$$\sum_i V(R_i) \le m^*(E) + \in\!/2$$

For each im let S_i be open rectangles containing R_i s.t.
$V(R_i) < V(S_i) + \in/2\, i+1$

Let $\pi = \bigcup_1^\infty S_i$ then Ω is open & $E \subseteq \bigcup_1^\infty R_i \subseteq \bigcup_1^\infty S_i = \Omega$

$$\therefore m^*(\Omega) = m^*\left(\bigcup_1^\infty S_i\right) \le \sum_{i=1}^\infty m^*(S_i)$$
$$\le \sum_{i=1}^\infty V(S_i)$$
$$< \sum_{i=1}^\infty \left(V(R_i) + \in/2^{i+1}\right)$$
$$< \sum_1^\infty V(R_i) + \in/2$$
$$< m^*(E) + \in/2 + \in/2$$
$$< m^*(E) + \in$$

This is true for any $\in > 0$.
$\therefore m^*(\Omega) \le m^*(\in)$
$\therefore \inf\left\{m^*(\Omega); \Omega \supset \in, \Omega\, is\, open\right\}$
$\le m^*(\Omega) \le m^*(E)$

Theorem :

For every subset E of $\mathbb{R}^n, \exists$ a G_z
Subset G of $\mathbb{R}^n$ s.t. $G \ge E$ & $m^*(G) = m^*(E)$

Proof :

Let $E \subseteq \mathbb{R}^n$
we first show that
$m^*(E) = \inf\left\{m^*(\Omega)\, i\, \Omega \supset E \text{ and } \Omega \text{ is open subset of } \mathbb{R}^n\right\}$

Let $\in > 0$,
Then for each $k \in \mathbb{N}, \exists \Omega_k$ open in $\mathbb{R}^n$ & $\pi_k \ge E$ s.t.
$m^*(\pi_k) < m^*(E) + \in/2^k$

let $G = \bigcap_{k=1}^\infty \Omega_k$

$\Rightarrow G$ is G_δ-set and $G \ge E$
$\therefore$By monotonicity

$m^*(E) \le m^*(G)$ .... (1)

Note that $G \le \Omega_k \quad \forall_k$

$\Rightarrow m^*(G) \le m^*(\Omega_k) < m^*(E) + \in\!/_{2^k}$

This is true for any $\in > 0$

$\Rightarrow m^*(G) \le m^*(E)$ .... (2)

By (1) & (2)

$m^*(G) = m^*(E)$

Thoerem :

There exist a countable collection $\{A_j\}_{j \in J}$ of disjoint subset of $\mathbb{R}$ such that $m^*\left(\bigcup_{j\in J} A_j\right) \ne \sum_{j\in J} m^*(A_j)$

Solution :

Consider rational θ and realy $\mathbb{R}$

$\mathbb{R}/\theta = \{x \ne \theta; x \in \mathbb{R}\}$

We known that any two cosets are either identified or disjoint.

We now show that if $A \in \mathbb{R}/\theta$ then $A \bigcap [0,1] \ne \phi$

Let $A = x + \theta$

Let q be rational number in $[-x, -x+1]$

then $x + q \in [0,1]$

Also, $x + q \in x + \theta = A$

$\therefore x + q \in A \bigcap [0,1] \Rightarrow A \wedge [0,1] \ne \phi$

For each $A \in \mathbb{R} \setminus \theta$ choose

$x_A \in A \bigcap [0,1]$

Let $E = \{x_A; A \in \mathbb{R}/\theta\}$

By construction $E \subseteq [0,1]$

Let $X = \bigcup_{q \in \theta \bigcap [-1,1]} q + E$

We now show that

$[0,1] \subseteq X \subseteq [-1,2]$

Let $q \in [-1,1] \bigcap \theta$ Note that $E \subseteq [0,1]$

$\therefore$ for any $x \in E, \quad q + x \in [-1, 2]$

This is true for any $q \in [-1,1] \cap \theta$

Theorem :

There exist a finite collection $\{A_j\}_{j \in J}$ of disjoint subset of $\mathbb{R}$ such that $m^*\left(\bigcup_{j \in J} A_j\right) \neq \sum_{j \in J} m^*(A_j)$

Proof :

Consider θ & $\mathbb{R}$

$\mathbb{R}/\theta = \{x + \theta / x \in \mathbb{R}\}$

We known that any two cosets are either identical or disjoint.

We now show that if $A \in \mathbb{R}/\theta$ then $A \cap [0,1] \neq \phi$

Let $A = x + \theta$

Let q be a rational number in $[-x, -x+1]$ then $x + q \in [0,1]$

Also, $x + q \in x + \theta = A$

$\therefore x + q \in A \cap [0,1] \Rightarrow A \cap [0,1] \neq \phi$

For each $A \in \mathbb{R} \setminus \theta$ choose $x_A \in A \cap [0,1]$

Let $E = \{x_A / A \in \mathbb{R}/\theta\}$

By construction $E \subseteq [0,1]$

Let $X = \bigcup_{q \in \theta \cap [-1,1]} q + E$

We now show that $[0,1] \subseteq X \subseteq [-1, 2]$

Let $q \in [-1,1] \cap \theta$

Note that $E \subseteq [0,1]$

$\therefore$ for any $x \in E, \quad q + x \in [-1, 2]$

This is true for any $q \in [-1,1] \cap \theta$

There exist a finite collection $\{A_j\}_{j \in J}$ of disjoint subset of □ such that $m^*\left(\bigcup_{j \in J} A_j\right) \neq \sum_{j \in J} m^*(A_j)$

Consider $Q\{$□

$\mathbb{R}|_Q = \{x+Q \mid x \in \mathbb{R}\}$

We know that any two cosets one either identical or disjoint.

We know show that if $A \in \mathbb{R}|_Q$ then $A \cap [0,1] \neq Q$

Let $A = x+Q$

Let q be a rational number in $[-x, -x+1]$ then $x+q \in [0,1]$.

Also $x+q \in x+Q = A$

$\therefore x+q \in A \cap [0,1] \Rightarrow A \cap [0,1] \neq Q$

For each $A \in \mathbb{R}|_Q$ choose $x_A \in A \cap [0,1]$.

Let $E = \{x_A \mid A \in \mathbb{R}|_Q\}$

By construction $E \subseteq [0,1]$

Let $X = \bigcup_{q \in Q \cap [-1,1]} q + \in$

We show that $[0,1] \subseteq \times \subseteq [-1,2]$

Let $q \in [-1,1] \cap Q$

Note that $E \subseteq [0,1]$

$\therefore x \in E, q+x \in [-1,2]$

$\Rightarrow q+E \subseteq [-1,2]$

This is true for any $q \in [-1,1] \cap Q$

Let $y \in [0,1]$

Then $y \in y+0 \in y+\theta = A$ (say) but $x_A \in A$

$\therefore y - x_A = y \in \theta$

$\therefore y, x_A \in [0,1] \Rightarrow y - x_A \in [-1,1]$

$\Rightarrow q \in [-1,1] \wedge \theta$

$\therefore y \in q + x_A \in q + E$

$\therefore y \in x$

$\therefore [0,1] \subseteq X \Rightarrow [0,1] \subseteq X \subseteq [-1,2]$

$\therefore$ By monotonicity of m^*

$m^*[0,1] \leq M^*(X) \leq m^*[-1,2]$

$1 \subseteq m^*(x) \leq 3$ ………………………………………… (1)

$\because x = \bigcup_{q \in [-1,1] \wedge \theta} q + E$ by countable subadditive and translation invariance of m^*, we get.

$$m^*(X) \leq \sum_{q \in [-1,1] \cap \theta} m^*(q+E) = \sum_{q \in [-1,1] \cap \theta} m^*(E)$$

By $(1) \Rightarrow m^*(X) \neq 0$

$\Rightarrow m^*(E) \neq 0$

$\therefore$ By Aritimedian property

$\exists n \in \mathbb{N}$ s.t. $m^*(E) > \frac{1}{n}$

Let I be a finite subset of $[-1,1] \cap \theta$ with cardinality $3n$.

Then $\sum_{q \in I} m^*(E) > 3n\frac{1}{n} = 3$

$\therefore$ by (1) $m^*(x) \neq \sum_{q \in I} m^*(q+E)$

Theorem :

Let $E \subseteq \mathbb{R}^n$ & $\lambda \in \mathbb{R}(\lambda > 0)$ show that $m^*(\lambda E) = \lambda^n m^*(E)$

Proof :

To show that $m^*(\lambda E) = \lambda^n m^*(E), \lambda > 0$

Let $\in > 0$,

$\therefore$ by definition of $m^*(E), \exists$ a countable cover of $\{R_i\}$ of closed rectangle in $\mathbb{R}^n$, for E s.t. $\sum V(R_i) < m^*(E) + \in$

$$\because E \subseteq \bigcup_{i=1}^{\infty} R_i \Rightarrow \lambda E \subseteq \bigcup_{i=1}^{\infty} \lambda R_i$$

Let $R_i = [a_{i1}, b_{i1}] \times .... \times [a_{in}, b_{in}]$

$$\lambda R_i = \{\lambda(x_1, ..., x_n); x_j \in [a_{ij}, b_{ij}]\}$$

$$= \{(\lambda x_1, ..., \lambda x_n); x_j \in [a_{ij}, b_{ij}]\}$$

$$= \{(\lambda x_1, ..., \lambda x_n); \lambda x_j \in [\lambda a_{ij}, \lambda b_{ij}]\}$$

$$= [\lambda a_{i1}, \lambda b_{i1}] \times ..... \times [\lambda a_{in}, \lambda b_{in}]$$

$\Rightarrow \lambda R_i$ is a closed rectangle

$\therefore V(\lambda R_i) = \lambda^n V(R_i)$

$\therefore \lambda E \subseteq \bigcup_{i=1}^{\infty} \lambda R_i$ by monotoricity & countable additive property we get

$$m^*(\lambda E) \leq \sum_1^{\infty} m^*(\lambda R_i) = \sum_1^{\infty} V(\lambda R_i) = \sum_1^{\infty} \lambda^n V(R_i)$$
$$\leq \lambda^n \sum V(\lambda R_i) < \lambda^n m^*(E) + \in$$

This is true for any $\in > 0$

$\therefore m^*(\lambda E) \leq \lambda^n m^*(E)$ …………………………………….. (1)

let $E^1 = \lambda E$ & $\mu = \frac{1}{\lambda}$

$\therefore$ by (1)

$m^*(\mu E^1) \leq \mu^* m^*[E^1)$

$\Rightarrow m^*\left(\frac{1}{\lambda} \lambda E\right) \leq \frac{1}{\lambda^n} m^*(\lambda E)$

$\Rightarrow \lambda^n m^*(E) \leq m^*(\lambda E)$ ………………………………… (2)

From (1) & (2)

$m^*(\lambda E) = \lambda^n m^*(E)$

3.6 SUMMARY

In this chapter we have learned about.

- definition of σ-Algebra, bored algebra
- measure on a set.
- The extension Measure
- Lebesgue outer Measure (μ^*) on $\mathbb{R}^n$
- Properties of lebesgue outer measgure.

3.7 UNIT END EXERCISE

1) Let $X = \{a, b, c, d\}$ and $A_1 = \{X, \phi, \{d\}\}$ and $A_2 = \{X, \phi, \{d\}\}$, $\{a, b, c\}$ check whether A_1 & A_2 are both algebra or not. Also check wheter $A_1 \cup A_2$ is an algebra or not.

2) Show that exterior measure at any countable subset of $\mathbb{R}^n$ is zero. Justify the converse?

3) Show that the outer mesuration interval is its length.

4) Show that if $(F_\alpha)\alpha \in I$ is a collection of σ-Algebra on X then $\eta_\alpha F_\alpha$ is also a σ-Algebra on X.

5) If a subset $E \subseteq \mathbb{R}^n$ is a countable union of almost disjoint closed rectangle then show that $m^*(E) = \sum_{i=1}^{\infty} \cup(R_i)$.

6) If A_1 and A_2 are measurable subsets of the closed interval $[a,b]$ then A_1 - A_2 is measurable and if $A_1 \subseteq A_2$ then $m(A_1 - A_2) = mA_1 - mA_2$.

7) Show that for any set A, $m^* A = m^*(A+x)$ where $A + x = \{y + x ; y \in A\}$

8) Show that for any set A and any $\in > 0$, there exist an open set O such that $A \subseteq 0$ and $m^* 0 \le m^* A + \in$.

9) Compute the Lebesgue outer measure of $B = [1-2] \cup \{3\}$

10) Prove that if the boundary of $\pi \subset \mathbb{R}^k$ has outer measure zero than π is measureable.

11) Let Ω be an arbitary collection of subsets of a set. Show that for a given $A \in \sigma(C)$ there exists a countable sub-collection C_A of C depdending on A such that $A \subset \sigma(C_A)$.

12) Check that μ^* is an outer measure on R. Not

i) Let X be any seet and $\mu^* : P(X) \to [0,\infty]$ be given by

i) $\mu^*(A) = 0$ if A is countable

=1 otherwise

ii) $\left.\begin{matrix} \mu^*(A) = 0 \, if \; A \; finite \\ 1 \, if \; otherwise \end{matrix}\right\}$ then X be on infinite set

iii) $\mu^*(A) = 0$ if $A = \phi$

= 1 otherwise

❖❖❖❖

4

LEBESGUE MEASURE

Unit Structure :

4.1 OBJECTIVE

After going through this chapter you can able to know that

- Construction of Lebesgue measure in $\mathbb{R}^n$.
- Lebesgue Measurable set in $\mathbb{R}^n$.
- Properties of measurable sets.
- Existance of non-measurable sets.

4.2 INTRODUCTION

In the previous chapter we have studied about Lebesgue outer measure m^* is not countability additive and it cannot be measure. So that we have to cover with subset of $\mathbb{R}^n$ for which m^* is countably additive this subclass a collection at Measurable sets. Now we shall define lebesgue measure of a set using the lebsgue outer measure and discuss properties of lebesgue measure set.

4.3 LEBESGUE MEASURE

Definition - (Lebesgue measurability)

Let E be a subset of $\mathbb{R}^n$ we say that E is Lebesgue measurable, or measurable if we have the identity

$$m^*(A) = m^*(A \cap E) + m^*(A/E)$$

4.3.1 Properties of measurable sets :

Following are the properties of measurable sets :

a) If E is measurable, then $E^C = \mathbb{R}^n / E$ is also measurable.

b) Any set E of exterior (or outer) measure zero is measurable. In particular, any countable set is measurable.

c) If $E_1 \& E_2$ are measurable, then $E_1 \cap E_2$ and $E_1 \cup E_2$ are measurable.

d) (Boolean algebra property) If $E_1, E_2, ...E_n$ are measurable then $\bigcup_1^n E_j$ & $\bigcap_1^n E_j$ are measurable.

e) (Translation in variance) If E is measurable & $x \in \mathbb{R}^n$ then $x+E$ is also measurable, and $m(x+E) = m(E)$.

Lemma : (Finite additivity)

If $(Ei)_{i=1}^k = (E_j)_{j\in J}$ are a finite collection of disjoint measurable sets and any set A, we have

$$m^*\left(A \cap \bigcup_{j\in J} E_j\right) = \sum_{j\in J} m^*\left(A \cap E_j\right)$$

Further more we have

$$m\left(\bigcup_{j\in J} E_j\right) = \sum_{j\in J} m\left(E_j\right)$$

Proof :

We prove by induction on K

The result is trivial when K=1

Assume result is true for k-1

We prove result for K

Let $E = \bigcup_{i=1}^k E_i$

tpt $m^*(A \cap E) = \sum_{i=1}^k m^*(A \cap E_i)$

Now E_k is measurable we have for $A \cap E \subseteq \mathbb{R}^n$.

$$m^*(A \cap E) = m^*\left((A \cap E) \cap E_k\right) + m^*\left((A \cap E) \cap E_k^C\right)$$

But $(A \cap E) \cap E_k = A \cap E_k$

$(\because E_k \subseteq E)$

$$\left(A\cap E\right)\cap E_k^C = A\cap\left(E\cap E_k^C\right)$$
$$= A\cap\left(\bigcup_{i=1}^{k-1} E_i\right)$$
$$\therefore m^*\left(A\cap E\right) = m^*\left(A\cap E_k\right) + m^*\left(A\cap\left(\bigcup E_i\right)\right)$$
$$= m^*\left(A\cap E_k\right) + \sum_{i=1}^{k} m^*\left(A\cap E_i\right)$$
$$= \sum_{i=1}^{k} m^*\left(A\cap E_i\right)$$

$\therefore$ The result is true for K

By introduction, it is true for 'n'.

ii) Put $A = \mathbb{R}^n$

Theorem :

If $A\subseteq B$ are two measurable sets then B/A is also measurable & $m\left(B/A\right) = m\left(B\right) - m\left(A\right)$

Proof :

tst B/A is measurable.

Suppose A & B are measurable

$\because$ intersection of two measurable set is measurable & complement of a measurable set is measurable.

$\Rightarrow B/A = B\cap A^C$ is measurable

Note that $B = A\cup\left(B/A\right)$

which is a disjoint union.

$\because m$ is finitely additive

$m\left(B\right) = m\left(A\right) + m\left(B - A\right)$

$\Rightarrow m\left(B/A\right) = m\left(B\right) - m\left(A\right)$

Example 1 :

Let A be a measurable set of finite outer measure that is contained in B show that $m^*\left(B/A\right) = m^*\left(B\right) - m^*\left(A\right)$

$\Rightarrow$ $\because$ A is measurable

By definition for this B

$m^*\left(B\right) = m^*\left(B\cap A\right) + m^*\left(B/A\right)$

$m^*\left(B\right) = m^*\left(A\right) + m^*\left(B/A\right)$

$\because m^*\left(A\right) < \infty$ we get

$m^*\left(B/A\right) = m^*\left(B\right) - m^*\left(A\right)$

Example 2 :

Suppose $A \subseteq E \subseteq B$ where A & B are measurable sets of finite measure show that if $m(A) = m(B)$ then E is measurable.

$\Rightarrow \because$ A & B are measurable $\Rightarrow B/A \Rightarrow B \cap A^C$ is measurable.

Note that $B = A \cup (B/A) \quad (\because A \subseteq B)$.
which is a disjoint union.
$\because m$ is finitely additive, we get
$m(B) = m(A) + m(B/A)$
$m(B/A) = 0 \quad (\because m(B) = m(A))$
$\because A \subseteq E \subseteq B \Rightarrow E/A \subseteq B | A$
$m^*(E/A) \subseteq m^*(B/A) = m(B/A) = 0$
$\Rightarrow m^*(E/A) = 0$
$\Rightarrow E/A$ is measurable
$\Rightarrow E = A \cup (E/A)$ is measurable

Example 3 :

Show that if E_1 & E_2 are measurable then
$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

Solution :

Suppose E_1 & E_2 are measurable not that
$E_1 \cup E_2 = E_1 \cup (E_2/E_1)$ which is a disjoint union.

By finite additie property of 'm'
$m(E_1 \cup E_2) = m(E_1) + m(E_2/E_1)$ .... (1)
also $E_2 = (E_1 \cap E_2) \cup (E_2/E_1)$
which is a disjoint union.

By finite additivity of 'm'
$m(E_2) = m(E_1 \cap E_2) + m(E_2/E_1)$ .... (1)
$m(E_2/E_1) = m(E_2) - m(E_1 \cap E_2)$
subs in 1
$m(E_1 \cup E_2) = m(E_1) + m(E_2) - m(E_1 \cap E_2)$
$m(E_1 \cup E_2) + m(E_1 \cap E_2) = m(E_1) + m(E_2)$

Theorem :

Let $\{E_k\}_{k=1}^{\infty}$ be a countable disjoint collection of measurable sets prove that for any set A, $m^*\left(A \cap \bigcup_1^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*\left(A \cap E_k\right)$.

Proof :

Let $\{E_k\}_{k=1}^{\infty}$ be countable collection of disjoint measurable sets.

Let $A \subseteq \mathbb{R}^n$

tpt $m^*\left(A \cap \bigcup_1^{\infty} E_k\right) = \sum_{k=1}^{\infty} m^*\left(A \cap E_k\right)$.

By countable subadditivity property of m^* we get,

$$m^*\left(A \cap \left(\bigcup_1^{\infty} E_k\right)\right) = m^*\left(\bigcup_1^{\infty}\left(A \cap E_k\right)\right)$$

$$\leq \sum_{k=1}^{\infty} m^*\left(A \cap E_k\right) \text{ ………………………….. (1)}$$

Also by finite additive property of m, we get

$$m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)\right) \geq m^*\left(A \cap \bigcup_{k=1}^{\infty} E_k\right)$$

$$\geq m^*\left(\bigcup_{k=1}^{m}\left(A \cap E_k\right)\right)$$

$$\geq \sum_{k=1}^{m} m^*\left(A \cap E_k\right)$$

This is true for all ‘m’

$$m^*\left(A \cap \left(\bigcup_{k=1}^{\infty} E_k\right)\right) \geq \sum_{k=1}^{\infty} m^*\left(A \cap E_k\right) \text{ ……………………….. (2)}$$

from (1) & (2)

$$m^*\left(A \cap \left(\bigcup_1^{\infty} E_k\right)\right) = \sum_{k=1}^{\infty} m^*\left(A \cap E_k\right)$$

Theorem :

Show that the union of a countable collection of measurable set is measurable.

Proof :

Let $\{A_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets and

$$E = \bigcup_{k=1}^{\infty} A_k .$$

tst E is measurable.

Define $B_1 = A_1$ & for $k \geq 2$

$$B_k = A_k \Big| \bigcup_{1}^{k-1} A_i$$

Since finite union of complement m-set are measurable
$\Rightarrow B_k$ is measurable.

Clearly B_k 's are pairwise disjoint

$$\bigcup_{k=1}^{\infty} B_k = \bigcup_{k=1}^{\infty}\left(A_k \Big| \bigcup_{1}^{k-1} A_i\right)$$

$$= \bigcup_{k=1}^{\infty}\left(A_k \cap \left(\bigcup_{i=1}^{k-1} A_i\right)^C\right)$$

$$= \bigcup_{k=1}^{\infty}\left(A_k \cap \left(\bigcup_{1}^{k-1} A_i\right)^C\right)$$

$$= A_1 \cup \left(A_2 \cap \left(\cap A_1^C\right)\right) \cup \left[A_3 \cap A_1^C \cap A_2^C\right] \cup ....$$

$$= \bigcup_{k=1}^{\infty} A_k = E$$

Example 4 :

Show that the intersections of a countable collection of measurable set is measurable.

$\Rightarrow$ Let A be a subset of $\mathbb{R}^n$ and for $n \in \mathbb{N}$.

Define $F_n = \bigcup_{k=1}^{\infty} B_k \subseteq E$

$\therefore B_k{}'S$ are measurable

$\Rightarrow F_n$ is measurable

$\therefore$ By definition

$$m^*(A) = m^*(A \cap F_n) + m^*\left(A \cap F_n^C\right)$$

$$\because F_n \subseteq E \Rightarrow F_n^C \supseteq E^C \Rightarrow A \cap F_n^C \supseteq A \cap E^C$$

$$\Rightarrow m^*\left(A \cap E^C\right) \subseteq m^*\left(A \cap F_n^C\right)$$

$\therefore m^*(A) \geq m^*(A \cap F_n) + m^*\left(A \cap E^C\right)$ …………………….. (1)

Now

$m^*(A \cap F_n) = m^*\left(A \cup \left(\bigcup_1^n B_k\right)\right)$

$= m^*\left(\bigcup_{K=1}^n (A \cap B_k)\right)$

$= m^*\left(\bigcup_{K=1}^n (A \cap B_k)\right)$

$= \sum_{K=1}^n m^*(A \cap B_k)$

$= \sum_{K=1}^n m^*(A \cap B_k)$

$\therefore$ By (1)

$m^*(A) \geq \sum_{k=1}^n m^*(A \cap B_k) + m^*\left(A \cap E^C\right)$

$\because$ LHS is independent of n, we have

$m^*(A) \geq \sum_1^n m^*(A \cap B_k) + m^*\left(A \cap E^C\right)$

But

$m^*(A \cap E) = m^*\left(A \cap \left(\bigcup_1^\infty B_k\right)\right)$

$= m^*\left(\bigcup_1^\infty (A \cap B_k)\right)$

$\leq \sum_1^\infty m^*(A \cap B_k)$

$m^*(A) \geq m^*(A \cap E) + m^*\left(A \cap E^C\right)$

As $A = (A \cap E) \cup \left(A \cap E^C\right)$ by countable subadditivity proposition of m^*.

$m^*(A) \leq m^*(A \cap E) + m^*\left(A \cap E^C\right)$ ………………………. (3)

By (2) & (3)

$m^*(A) = m^*(A \cap E) + m^*\left(A \cap E^C\right)$

$\therefore$By definition E is measurable.

Example 5 : Countable additive

If $\{E_j\}_{j\in J}$ are a countable collection of disjoint measurable sets then $\bigcup_{j\in J} E_j$ is measurable and $m\left(\bigcup_{j\in J} E_j\right)=\sum_{j\in J} m(E_j)$

$\Rightarrow$ Without loss of generality we may assume $J=\mathbb{N}$ suppose $\{E_k\}_{k=1}^{\infty}$ be a countable collection of disjoint measurable set we first show that $E=\cup E_k$ measurable let $F_n=\cup E_k$.
then by previous exercise we get E is measurable.

We now show that

$$m(E)=\sum_{1}^{\infty} m(E_k)$$

By subadditivity proposition of m^*

$$m(E)=m^*(E)=m^*\left(\bigcup_{1}^{\infty} E_k\right)$$

$$\leq \sum_{1}^{\infty} m^*(E_k)$$

$$=\sum_{k=1}^{\infty} m(E_k) \text{ .... (*)}$$

By finite additivity property and monotonicity of m^*
we have as $F_n \supseteq E$

$$m(E)\geq m(F_n)=m\left(\bigcup_{k=1}^{n} E_k\right)$$

$$=\sum_{k=1}^{n} m(E_k)$$

$\therefore$LHS is independent of n, we get

$$m(E)\geq \sum_{k=1}^{\infty} m(E_k) \text{ .... (**)}$$

$\therefore$By countable additivity

$$m(E)\geq \sum_{k=1}^{\infty} m(E_k)$$

Example 6 :

Show that every closed and open rectangles in $\mathbb{R}^n$ are measurable.

$\Rightarrow$ Let R be a closed rectangle tst R is measurable

Let $\in > 0$, Let $A \subseteq \mathbb{R}^n$

by definition of $m^*(A)$

$\exists$ a countable collection of closed rectangle $\{R_i\}_{i=1}^{\infty}$ such that

$$A \subseteq \bigcup_{i=1}^{\infty} R_i \text{ and } \sum_{i=1}^{\infty} V(R_i) < m^*(A) + \in \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(1)}$$

we decompose each R_i into finite union of almost disjoint rectangle $\left\{R_i, S_{i_1}, \ldots., S_{ik}\right\}$ such that $R_i = R_i^1 \cup \left(\bigcup_{j=1}^{k} S_{ij} \right)$.

$R_i^1 = R_i \cap R \subseteq R$ and $S_{i_j} \subseteq R^C$

$\therefore$By finite additive property of M.

$$m(R_i) = m(R_i) + \sum_{j=1}^{k} m\left(S_{i_j}\right)$$

$$\Rightarrow V(R_i) = V\left(R_i^1\right) + \sum_{j=1}^{k} V\left(S_{i_j}\right)$$

$$\therefore \sum_{i=1}^{\infty} V(R_i) = \sum_{i=1}^{\infty} V(R_i) + \sum_{i=1}^{\infty}\left(\sum_{j=1}^{\infty} V\left(S_{i_j}\right)\right)$$

Note That $\{R_i\}_{i=1}^{\infty}$ lover $A \wedge R$

$$\left[\because A \cap R \subseteq \left(\bigcup_{i=1}^{\infty} R_i\right) \cap R = \bigcup_{i=1}^{\infty} (R_i \cap R) = \bigcup_{i=1}^{\infty} R_i^1\right]$$

$\left\{S_{i_j}\right\} i, j$ covers $A \cap R^C$

$$\sum_{1}^{\infty} V(R_i) = m^*\left(\bigcup_{1}^{\infty} R_i^1\right) \geq m^*(A \cap R)^C \text{ and } m^*\left(\bigcup_{i,j} S_{ij}\right) \leq m^*\left(A \cap R^C\right)$$

$$m^*(A \cap R) \leq m^*\left(\bigcup_{i,j} S_{ij}\right)$$

$$\leq \sum_{i,j} m^*\left(S_{ij}\right) = \sum_{i,j} V\left(S_{ij}\right)$$

$\therefore$By (1)

$$m^*(A)+\in>\sum_{i=1}^{\infty}V(R_i)$$
$$=\sum_{i=1}^{\infty}V(R_i)+\sum_{i=1}^{\infty}\sum_{j=1}^{k}V(S_{ij})$$
$$\geq m^*(A\cap R)+m^*(A\cap R^C)$$

This is true for any $\in>0$

$m^*(A)\geq m^*(A\cap R)+m^*(A\cap R^C)$

$\therefore$By definition R is measurable.

Example 7 :

Show that every open and closed subsets of $\mathbb{R}^n$ are measurable.

$\Rightarrow$ Let $K=\max\{K_i\}$

Let G be an open subset of $\mathbb{R}^n$ consider the grid of rectangle in $\mathbb{R}^n$ of side length one and whose vertices have integer co-ordinates.

TST G is measurable.

$\therefore$ Number of rectangle in grid is countable and one almost disjoint we ignore all these rectangle contained in G^C.

Now we have two types of rectangle (1) Those rectangle contained in G (2) Those rectangle intersect with G & G^C.

Let C = set of all rectangle contained in G.

We bisect type (2) rectangle into two rectangle each of its side length is ½.

Repeat the process iterating this process for arbitrarily many times we get a constable collections c of almost disjoint rectangle contained in G.

By construction $\bigcup_{R\in C}R\subseteq G$

Let $x\in G$

$\therefore G$ is open

We can choose sufficiently small rectangle in the bisection procedure that contains x is entirely contained in G.

$\therefore x \in \bigcup_{R \in C} R$

$\therefore G \subseteq \bigcup_{R \in C} R$

$\therefore G = \bigcup_{R \in C} R$

$\therefore G$ is countable union of closed rectangle and hence G is measurable.

4.4 OUTER APPROXIMATION BY OPEN SETS

Let $E \subseteq \mathbb{R}^n$ such that E is measurable iff for $\in > 0$, there is an open set Ω containing E for which $m^*(\Omega/E) < \in$.

$\Rightarrow$ Suppose E is measurable

Let $\in > 0$

Suppose $m^*(E) < \infty$

$\therefore$By the definition of $m^*(E)$

$\exists$ a countable collection of open rectangles $\{R_i\}$ such that $E \subseteq \bigcup_{i=1} R_i$

and $\sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in$.

Let $\Omega = \bigcup_{i=1}^{\infty} R_i$ which is countable union of opensets.

$\therefore \Omega$ is open in $\mathbb{R}^n$ and $E \subseteq \Omega$

$\therefore \Omega$ is open, it is measurable

$\therefore \Omega/E$ is measurable

$\Omega = E \cup (\Omega/E)$ which is a countably disjoint union

$m^*(\Omega) = m^*(E) + m^*(\Omega/E)$

$\therefore m^*(\Omega/E) = m^*(\Omega) - m^*(E)$

But

$\Omega = \bigcup_{i=1}^{\infty} R_i \Rightarrow m^*(\Omega) \leq \sum_{i=1}^{\infty} m^*(R_i) \leq \sum_{i=1}^{\infty} V(R_i)$

$\therefore m^*(\Omega/E) \leq \sum_{i=1}^{\infty} V(R_i) - m^*(E) < \in$

Suppose $m^*(E) = \infty$

For each k

$E_k = E \cap R_K$ where

R_k = rectangle with centre origin and side length K

For each k

Then $m^*(E_k) \le m^*(R_i) = V(R_i) = K^\infty < \infty$

$\therefore$ by first case for each K, $\exists\ \Omega_k$ open in $\mathbb{R}^n$ such that

$E_k \subseteq \Omega_k m^*(\Omega_k / E_k) < \frac{E}{2^k}$.

Let $\Omega = \bigcup_{k=1}^{\infty} \Omega_k$ which is countable union of open set.

$\therefore \Omega$ is open and $E \subseteq \Omega$

$$m^*(\Omega/E) = m^*\left(\Omega \cap E^C\right)$$

$$= m^*\left(\bigcup_{k=1}^{\infty} \Omega_k \cap E^C\right)$$

$$= m^*\left(\bigcup_{k=1}^{\infty} (\Omega_k / E)\right)$$

$$\le \sum_{k=1}^{\infty} m^*(\Omega_k / E)$$

$$\le \sum_{k=1}^{\infty} m^*(\Omega_k / E_k)$$

$$\le \sum_{k=1}^{\infty} \frac{\in}{2^k} = \in$$

Conversely suppose for a given $\in > 0\ \exists$ open set $\Omega \ge E$ such that $m^*(\Omega/E) < \in$.

Tst E is measurable

Let $A \subseteq \mathbb{R}^n$

$\therefore \Omega$ is open

$\Rightarrow \Omega$ is measurable

$m^{\&}(A) = m^*(A \cap \Omega) + m^*(A/\Omega)$

Note that $A/E = (A/\Omega) \cup ((A \cap \Omega/E))$ which is a disjoint union.

$$\therefore m^*(A/E) = m^*(A/\Omega) + m^*((A \cap \Omega)/E)$$

$$\therefore m^*(A \cap E) + m^*(A/E) = m^*(A \cap E) + m^*(A/\Omega) + m^*((A \cap \Omega)/E)$$

$$\le m^*(A \cap E) + m^*(A/\Omega) + m^*(A \cap \Omega)$$

$$< m^*(A) + \in$$

This is true for any $\in > 0$

$\therefore m^*(A \cap E) + m^*(A/E) \leq m^*(A)$

$\therefore E$ is measurable.

Exercise 8 :

Let $E \subseteq \mathbb{R}^n$ S.T., E is measurable iff for each $\in > 0$ there is G_s set G conlaining E for which $m^*(G/E) = 0$.

Proof : suppose E is measurable

$\therefore$By outer approximation by an open set.

For each $n \in \mathbb{N}$, $\exists$ an open set $\Omega_k \supseteq E$ s.t.

$$m^*(\Omega_k/E) < 1/k$$

Let $G = \bigcap_{k=1}^{\infty} \Omega_k$, then G is a G_δ set ant $E \subseteq G$

$$m^*\left(G/E\right) = m^*\left(\bigcap_{K=1}^{\infty} \Omega_k\, K/E\right)$$

$$= m^*\left(\left(\bigcap_{K=1}^{\infty} \Omega_K\right) \cap E^C\right)$$

$$= m^*\left(\bigcap_{K=1}^{\infty} \left(\Omega_k \cap E^C\right)\right)$$

$$\leq m^*\left(\Omega_K \cap E^C\right)$$

$$\leq m^*(\Omega_K/E)$$

$$< 1/k$$

This is true for all k

$m^*(G/E) = 0$

Conversely, suppose $\exists$ G_δ set $G \supseteq E$

s.t. $m^*(G/E) = 0$

tst E is measurable

Let $A \subseteq \mathbb{R}^n$

$\therefore G$ is countable int of measurable

Set $\Rightarrow$ G is measurable.

$\therefore$By definition

$m^*(A) = m^*(A \cap G) + m^*\left(A \cap G^C\right)$

Note that

$A/E = (A/G) \cup \left((A \cup G)/E\right)$

Which is a disjoint union

$\therefore m^*(A/E) = m^*(A/G) + m^*((A \cap G)/E)$

$\therefore m^*(A \cap E) + m^*(A/E) = +m^*(A \cap E) + (A/G) + m^*(A \cap G/E)$

$\leq m^*(A \cap G) + m^*(A/G) + m^*(G/E)$

$\leq m^*(A) + 0$

$\leq m^*(A)$

4.5 INNER APPROXIMATION BY CLOSED SETS

Theorem :

Let $E \subseteq \mathbb{R}^n$ S.T. E is measurable iff for each $\in > 0$, there is a closed set $F \subset E$ for which $m^*(E/F) < E$.

Proof :

Suppose E is measurable

$\Rightarrow E^C$ is measurable

Let $\in > 0$

$\therefore$By outer approximative by open seet $\exists$ an open set $\Omega \supset E^C$ s.t.

$m^*(\Omega/E^C) < \in$

Let $E = \Omega^C \Rightarrow F$ is closed & $F \subseteq E$.

Now $m^*(E/F) = m^*(E \cap F^C) = m^*(E \cap \Omega)$

$= m^*(\Omega \cap E) = m^*\left(\Omega \cap \left(E^C\right)^C\right)$

$= m^*(\Omega/E^C) < \in$

Conversely suppose for $\in > 0, \exists$ closed set $F \subseteq E$ such that $m^*(E/F) < E$

Tst E is measurable

Let $A \subseteq \mathbb{R}^n$

$\because$ F is measurable

By definition

$m^*(A) = m^*(A \cap F) + m^*(A/F)$

Note that

$A \cap E = ((A \cap F)/F) + \cup (A \cap F)$ which is disjoint union.

$\therefore m^*(A\cap E)=m^*(A\cap F)+m^*((A\cap E)/F)$

$\therefore m^*(A\cap E)+m^*(A/E)$

$=m^*(A\cap F)+m^*((A\cap E)/F)+m^*(A/E)$

$\leq m^*(A\cap F)+m^*(E/F)+m^*(A/F)$

$<m^*(A)+\in$

Example 9 :

Let E be a set of finite outer measure show that there is an $F\sigma$ set F & a G_δ set G s.t. $F\subseteq E\subseteq G$ & $m^*(F)=m^*(E)=m^*(G)$.

[Ans] $\therefore E$ is measurable for given each k $\exists$ open set G_k and closed set F_k such that $F_k\subseteq E\subseteq G_k$ and $m^*(G_k/F_k)<1/k$.

Let $G=\bigcap_{k=1}^{\infty}G_k$ & $F=\bigcup_{k=1}^{\infty}F_k$.

Then G is G_S set and F is $F\sigma$ set and $F\subseteq E\subseteq G$.

We now show that $m^*(G)=m^*(E)=m^*(F)$ $G=E\cup(G/E)$ which is disjoint union.

$m^*(G)=m^*(E)+m^*(G/E)$

Now $G/E=G\cap E^C$

$=\left(\bigcap_{k=1}^{\infty}G_k\cap E^C\right)$

$=\bigcap_{k=1}^{\infty}(G_k\cap E^C)$

$=\cap(G_k/E)\subseteq G_k/E$

$\subseteq G_k/F_k$

$\therefore m^*(G/E)\leq m^*(G_k/F_k)<1/k$

This is true for all k

$\therefore m^*(G/E)=0$

$\therefore m^*(G)=m^*(E)$ .... (1)

$E = F \cup (E/F)$

$m^*(E) = m^*(F) + m^*(E/F)$

$$E/F = E \cap F^C = E \cap \left(\left(\bigcup_{k=1}^{\infty} F_k\right)^C\right)$$

$$= E \cap \left(\bigcup_{k=1}^{\infty} F_k^C\right) = \cap\left(E \cap F_k^C\right)$$

$$= \bigcap_{k=1}^{\infty} (E/F_k)$$

$$\subseteq E/F_k$$

$$\subseteq G_k/F_k$$

$m^*(E/F) \leq m^*(G_k/F_k) < 1/k$

This is true for all k

$\therefore m^*(E/F) = 0$

Example 10 :

Let E be a set of finite outer measure show that if E is not measure, then there is an open set Ω containing E that has finite outer measure and for which $m^*(\Omega/E) > m^*(\Omega) - m^*(E)$.

Solution :

$\Rightarrow$ Since E is not measurable

$\Rightarrow \exists \in_0 > 0$ for any open set Ω containing E.

$m^*(\Omega/E) \geq \in_0$ …………………………………… (1)

$\therefore E$ has finite outer measure.

By definition $\exists$ a countable collection of open rectangles $\{R_i\}_{i=1}^{\infty}$ such that $E \subseteq \bigcup_{i=1}^{\infty} R_i$ and $\sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in_0$.

Let $\Omega_0 = \bigcup_{i=1}^{\infty} R_i$

$\Rightarrow E \subseteq \Omega_0$ & Ω_0 open.

$\therefore$By (1) $m^*(\Omega/E) > \in_0$ …………………………. (2)

By countable subadditivity of m^*

$$m^*(\Omega_0) \leq \sum_{i=1}^{\infty} m^*(R_i) = \sum_{i=1}^{\infty} V(R_i) < m^*(E) + \in_0$$

$$\therefore m^*(\Omega_0) - m^*(E) < \in_0 \leq m^*(\Omega_0/E)$$

$$\therefore m^*(\Omega_0/E) > m^*(\Omega_0) - m^*(E)$$

4.6 CONTINUITY FROM ABOVE

Theorem :

If $\{B_k\}_{k=1}^{\infty}$ is a descending collection of measurable set and $m(B_1) < \infty$ then $m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$

Proof :

$\Rightarrow B_1 \geq B_2 \geq$ …. Be collection of measurable sets and $m(B_1) < \infty$

tst $m\left(\bigcap_{k=1}^{\infty} B_k\right) = \lim_{k\to\infty} m(B_k)$

Let $A_k = B_1/B_k \; \forall k \geq 1$ then $A_1 \subseteq A_2 \subseteq \ldots\ldots$ and A_k ’s are measurable ($\therefore B_k$ ’s are measurable)

$$\therefore \bigcup_{k=1}^{\infty} A_k = \bigcup_{k=1}^{\infty} (B_1/B_k) = \bigcup_{k=1}^{\infty} \left(B_1 \cap B_k^C\right)$$

$$= B_1 \cap \left(\bigcup_{k=1}^{\infty} B_k^C\right)$$

$$= B_1 \cap \left(\bigcup_{k=1}^{\infty} B_k\right)^C$$

Let $B = \bigcup_{k=1}^{\infty} B_k$

$$\therefore \bigcup_{k=1}^{\infty} A_k = B_1 \cap B^C = B_1/B$$

$\therefore$ By continuity from below

$m(B_1/B) = \lim_{k\to\infty} m(A_k)$ …………………………. (*)

$\therefore B$ and B_1 are measurable

$m(B_1/B) = m(B_1) - m(B)$ and

$$m(A_k) = m(B_1/B_k)$$

$$= m(B_1) - m(B_k)$$

$\therefore$By (*)

$$m(B_1)-m(B)\lim_{k\to\infty}\left(m(B_1)-m(B_k)\right)$$

$$=m(B_1)-\lim_{k\to\infty}m(B_k)$$

$$\therefore m(B)=\lim_{k\to\infty}m(B_k)\ i.e.\left(\bigcap_{k=1}^{\infty}B_k\right)=\lim_{k\to\infty}m(B_k)$$

Example 11 :

Show by an example that for continuity from aboe the assumption $m(E_1)<\infty$ is necessary.

$\Rightarrow$Let $B_k=(k,\infty)$ then $B_1\supseteq B_2\supseteq\ldots$ and $m(B_k)=\infty\forall_k$ we now show that $\bigcap_{k=1}^{\infty}B_k=\phi$.

Let $x\in\bigcap_{k=1}^{\infty}B_k\Rightarrow x\in B_k=(k,\infty)\forall k$

$$\Rightarrow x>k,\forall\, k$$

$\Rightarrow\mathbb{N}$ is bounded by x, which is not possible.

$$\therefore\bigcap_{k=1}^{\infty}B_k=\phi$$

$$\therefore 0=m(\phi)=m(\cap B_k)\neq\infty=\lim_{k\to\infty}m(B_k)$$

Example 12 :

Show that the continuity of measure together with finite additivity of measure implies countable additivity of measure.

$\Rightarrow$ Let $\{E_k\}$ be a countable collection of disjoint measure sets.

Let $A_k=\bigcup_{i=1}^{k}E_i$

Then A_k's are measurable and $A_1\subseteq A_2\subseteq\ldots\ldots$

Also $\bigcup_{k=1}^{\infty}A_k=\bigcup_{k=1}^{\infty}\left(\bigcup_{i=1}^{k}E_i\right)=\bigcup_{k=1}^{\infty}E_k$

$\therefore$By continvity from below, $m\left(\bigcup_{k=1}^{\infty}A_k\right)=\lim_{k\to\infty}m(A_k)$.

But by the finite additive property

$$m(A_k) = m\left(\bigcup_{i=1}^{k} E_i\right) = \sum_{i=1}^{k} m(E_i)$$

$$\therefore m\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} m(A_k) = \lim_{k\to\infty} \sum_{i=1}^{k} m(E_i)$$

$$= \sum_{i=1}^{k} m(E_i)$$

$$\therefore m\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} m(E_k)$$

Definition :

For a measurable set E, we say that a property holds atmost everywhere on E, or it holds for almost all $x \in E$, provided there is a subset E_0 of E for which $m(E_0) = 0$ and the property holds for all $x \in E/E_0$.

4.7 BOREL CANTELLI LEMMA

Let $\{E_k\}_{k=1}^{\infty}$ be a countable collection of measurable sets for which $\sum_{k=1}^{\infty} m(E_k) < \infty$. Then almost all $x \in \mathbb{R}^n$ belong to Atmost finitely many of the E_k's.

Proof :

Let E_0 be the subset of $\mathbb{R}^n$ such that $E_0 = \{x \in \mathbb{R}^n : x \in E_k$ for infinitely many}

$$E_0 = \bigcap_{k=1}^{\infty}\left(\bigcup_{i=k}^{\infty} E_k\right)$$

We sow that $m(E_0) = 0$

Let $F_k = \bigcup_{k=i}^{\infty} E_k$

Then $F_1 \supseteq F_2 \geq \ldots\ldots$ and $\bigcap_{k=1}^{\infty} F_k = E_0$

Note that $\sum_{i=1}^{\infty} m(E_i) < \infty$

Let $L=\sum_{i=1}^{\infty} m(E_i)$

$$\Rightarrow m(F_1)=m\left(\bigcup_{i=k}^{\infty} E_i\right)\leq m\left(\bigcup_{i=k}^{\infty} E_k\right)$$

$$\leq \lim_{k\to\infty} m\left(\bigcup_{i=k}^{\infty} E_i\right)$$

$$\leq \lim_{k\to\infty} \sum_{i=k}^{\infty} m(E_i)$$

$$\leq \lim_{k\to\infty}\left(\sum_{i=k}^{\infty} m(E_i)-\sum_{i=1}^{k-1} m(E_i)\right)$$

$$\leq \lim_{k\to\infty}\left(L-\sum_{i=1}^{k-1} m(E_i)\right)$$

$$\leq L-\sum_{i=1}^{\infty} m(E_i)$$

$$\leq L-L$$

$$=0$$

$$\therefore m(E_0)=0$$

Example 13 :

Show that there is a non-measurable subset in $\mathbb{R}$.

Solution : $\mathbb{R}\,|\,Q=\{x+Q\,|\,x\in\mathbb{R}\}$

WKT any two cosets are either identical or disjoint.
We now show that

If $A\in\mathbb{R}\,|\,Q$ then $A\bigcap[0,1]=\phi$

Let $A=x+Q$

Let q be a rational number in $[-x,-x+1]$ then $x+q\in[0,1]$

Also $x+q\in x\in Q=A$

$\therefore x+q\in A\bigcap[0,1]$

$\Rightarrow A\bigcap[0,1]\neq\phi$

For each $A\in\mathbb{R}/Q$ choose $x_A\in A\bigcap[0,1]$

Let $E=\{x_A/A\in\mathbb{R}/Q\}$

By construction $E\subseteq[0,1]$

Let $X=\bigcup_{q\in[-1,1]\bigcap\theta} q+E$

$\therefore$ For any $x\in E, q+x\in[-1,2]$

$\Rightarrow q+E\subseteq[-1,2]$

This is true for any $q\in[-1,1]\cap Q$

Let $y\in[-1,1]$ then $y\in y+0\in y+Q=A$ (say)

but $x_A\in A$

$\therefore y-x_A=q\in Q(\because x_A\in A\Rightarrow x_A\in y+Q$ for some $q\in Q\}$

$\because y,x_A\in[0,1]$

$\Rightarrow y-x_A\in[-1,1]$

$\Rightarrow q\in[-1,1]\cap Q$

$\therefore y\in=q+x_A\in q+E$

$\therefore y\in X\Rightarrow[0,1]\subseteq X\Rightarrow[0,1]\subseteq X\subseteq[-1,2]$

$\therefore$By monotonicity of m^*

$m^*([0,1])\leq m^*(x)\subseteq m^*([-1,2])$

$1\leq m^*(x)\subseteq 3$

If E is measurable then $q+E$ is measurable and $m(E)=m(q+E)$

$$m\left(\bigcup_{q\in[-1,1]\cap Q} E\right)=\sum_{q\in[-1,1]\cap Q} m(q+E)$$

$$m(X)=\sum_{q\in[-1,1]\cap Q} m(E)$$

$\therefore 1\leq m(X)\leq 3$

$$\Rightarrow 1\sum_{q\in[-1,1]\cap Q} m(E)\leq 3$$

If $m(E)=0$ then $\sum_{q\in[-1,1]\cap Q} m(E)=0$

$\therefore 1\leq 0\leq 3$ and if $m(E)\neq 0$ then $\sum_{q\in[-1,1]\cap Q} m(E)=\infty$

Which is contradictin to (1)

$\therefore E$ is not measurable.

4.8 SUMMARY

In this chapter we have learned about.

- Lebesgue measureable sets.
- Construction of Lebesgue measurable sets in $\mathbb{R}^n$
- Properties of Lebesgue measurable sets
- Non-measurable sets

4.9 UNIT END EXERISES

1. Show that the intersection of a countable collection of measurable sets is measurable.
2. Show tht every open and closed subset of $\mathbb{R}^n$ are measurable.
3. Show that a set E is measurable if and only if for each $\in > 0$, there is a closed set F and open set Ω for which $F \subseteq E \subseteq \Omega$ and $m^*(\Omega/F) < \in$
4. Let E be a measurable set in $\mathbb{R}^n$ and $m(E) < \infty$ show that for any $\in > 0$ there exist a compact set $k \subseteq E$ such that $m^*(E/K) < E$.
5. If $\{A_k\}_{k=1}$ is an ascending collection of measurable sets then $M\left(\bigcup_{k=1}^{\infty} A_k\right) = \lim_{k\to\infty} m(A_k)$
6. The outer measure of α, the set of all rational number is ‘0’.
7. Prove that the outer measure of countable set is zero.
8. Show that the outer Measure of an interval is its length.

❖❖❖❖

5

MEASURABLE FUNCTION

Unit Structure :

5.0 Objective
5.1 Introduction
5.2 Measurable Function
5.3 Properties of Measurable Function
5.4 Egoroff's Theorem
5.5 Lusin's Theorem
5.6 Summary
5.7 Unit End Exercise

5.0 OBJECTIVE

After going through this chaper youcan able to know that

- Measurable function
- Properties of measurable function.
- Concept of simple function

5.1 INTRODUCTION

In the previous chapter we have studied about Lebesgue measure of sets of finite and infinite measures. Now we can discuss Lebesgue Measurability of functions. The definition of measurability of function applies to both bounded and unbounded functions. We also discuss simple function and its Approximation.

5.2 MEASURABLE FUNCTIONS

Definition : We say a function '*f*' on $\mathbb{R}^n$ is extended real valued if it take value on $\overline{\mathbb{R}}$.

Definition : A property is said to hold almost everywhere on a measurable set E provided it holds on E/E_0, where E_0 is a subset of E for which $m(E_0)=0$

Example 1 : Let *f* be a function defined on a measurable subset *E* of $\mathbb{R}^n$. Then the following are equivalent.

1. For each real number C, the set $\{x \in E : f(x) > C\}$ is measurable.
2. For each real number C, the set $\{x \in E; f(x) \geq C\}$ is measurable.
3. For each real number C, the set $\{x \in E; f(x) < C\}$ is measurable.
4. For each real number C, the set $\{x \in E; f(x) \leq C\}$ is measurable.

Solution :

$\Rightarrow (1) \Rightarrow (2)$

Suppose for any $C \in \mathbb{R}$

$\{x \in \mathbb{R}, f(x) > C\}$ is measurable ………. (*)

Let $C \in \mathbb{R}$

tst $\{x \in \mathbb{R}; f(x) \geq C\}$ is measurable

Note that $\{x \in E : f(x) \geq C\} = \bigcap_{n=1}^{\infty} \left\{x \in E; f(x) > C - \frac{1}{n}\right\}$ which is a measurable as countable intersection of measurable set is measurable (by (*))

$\therefore \{x \in E : f(x) \geq C\}$ is measurable

$(2) \Rightarrow (3)$

Suppose $\{x \in E : f(x) \geq C\}$ is measurable

$\{x \in E; f(n) < C\} = \{n \in E; f(x) \geq C\}^C$ which is measurable as complement of measurable set is measurable.

$\therefore \{x \in E; f(x) < C\}$ is measurable.

$(3) \Rightarrow (4)$

Suppose $\{x \in E; f(x) < C\}$ is measurable.

Let $C \in \mathbb{R}$

tst $\{x \in E; f(x) \leq C\}$ is measurable.

Note that

$\{x \in E; f(x) \leq C\} = \bigcap_{n=1}^{\infty} \left\{x \in E; f(x) < C + \frac{1}{n}\right\}$ which is measurable as countable intersection of measurable set is measurable set.

$\Rightarrow \{x \in E; f(x) \leq C\}$ is measurable.

$(4) \Rightarrow (5)$

Suppose $\{x \in E; f(x) \leq C\}$ is measurable.

tst $\{x \in E; f(x) > C\}$ is measurable.

Note that

$\{x \in E; f(x) > C\} = \{x \in E; f(x) \leq C\}^C$ which is measurable as complement of measurable set is measurable.
$\Rightarrow \{x \in E; f(x) > C\}$ is measurable.

Definition : An extended real-valued function '*f*' defined $E \subset \mathbb{R}^n$ is said to be Lebesgue measurable or measurable, if its domain E is measurable and it satisfies one of the above four statement i.e. For each real number C, the set $\{x \in E; f(x) \leq C\}$ is measurable.

Example 2 : Show that a real vaued function that is continuous on its measurable domain is measurable.

Solution :
Let '*f*' be a continuous function
tst '*f*' is measurable
Let $C \in \mathbb{R}$
Note that, $\{x \in E; f(x) > C\} = f^{-1}(C, \infty)$ but (C, ∞) is open subset of $\mathbb{R}$ and $f : E \to \mathbb{R}$ is continuous.
$\therefore f^{-1}(C, \infty)$ is open in E
$\therefore f^{-1}(C, \infty) = G \cap E$ for some G is open subset of $\mathbb{R}^n$ but any open-subset of $\mathbb{R}^n$ is measurable and E is given as measurable.
$\therefore f^{-1}(C, \infty) = G \cap E$ is measurable
$\therefore \{x \in E; f(x) > C\} = f^{-1}(C, \infty)$ is measurable
$\therefore$ By definition
f is measurable.

Example 3 : Let *f* be an extended real valued function on *E*. Sho that
1) *F* is measurable on E and $f = g$ a.e. on E then g is measurable on E.
2) For a measurable subset D of E, f is measurable on E iff the restriction of F to D and E/D are measurable.

Solution : Suppose *f* is measurable and $f = g$ a.e.
Let $A = \{x \in E : f(x) \neq g(x)\}$
Then as $f = g$ a.e. we have $m(A) = 0$
tst *g* is measurable.
Let $C \in \mathbb{R}, \{x \in E; g(x) > C\}$
$= \{x \in A; g(x) > C\} \cup \{x \in E/A; g(x) > C\}$
$= \{x \in A; g(x) > C\} \cup \{x \in E/A; f(x) > C\} (\because f = g)$

$(\therefore f = g)$

$= \{x \in A; g(x) > C\} \cup \{x \in E; f(x) > C\} \cap (E/A)\}$

But $\{x \in A; g(x) > C\} \subseteq A$ and $m(A) = 0$

$\therefore$ any subset of measure zero set is measurable

$\Rightarrow \{x \in A; g(x) > C\}$ is measurable

$\because f$ is measurable $\Rightarrow \{x \in E; f(x) > C\}$ is measurable

$\because E$ & A are measurable $(\because m(A) = 0)$

$\Rightarrow E/A$ is measurable

$\therefore \{x \in A; g(x) > C\} \cup [\{x \in E; f(x) > C\} \cap (E/A)]$ is measurable

$\Rightarrow \{x \in E; g(x) > C\}$ is measurable

$\Rightarrow g$ is measurable.

2) $\left\{x \in E; f|_D (x) > C\right\} = \{x \in D; f(x) > C\}$

$= \{x \in E; f(x) > C\} \cap D$

For $f|_{D}|_{E} = \left\{x \in E; f|_{D}|_{E} (x) > C\right\}$

$= \{x \in {}^{E}|_{D}; f(x) > C\}$

$= \{x \in E; f(x) > C\} \cap {}^{E}|_{D}$

Converse

$= \{x \in \in; f(x) > C\} = \{x \in D; f(x) > C\} \cup \{x \in E/D; f(x) > C\}$

$\Rightarrow \{x \in D; f(x) > C\}$ is measurable and $\{x \in E/D; f(x) > C\}$ is measurable.

As union of measurable set is measurable

$\Rightarrow f$ is measurable.

5.3 PROPERTIES OF MEASURABLE FUNCTION

Let f and g be measurable function on E that are finite a.e. on E show that

1) (Linearity) for any $'\alpha'$ and $'\beta'$, $\alpha F + \beta g$ is measurable on F.

2) (Product) fg is measurable on E.

Solution :

Let $E_0 = \left\{x \in E : f(x) = \pm\infty\right\}$ and $g(x) = \pm\infty$ then as f and g are finite a.e. on E we have $m(E_0) = 0$

$\therefore$ the restriction $^{(f+g)}|_{E_0}$ is measurable.

$\therefore$ any extension of $'f+g'$ as an extended real valued function to all of E is also measurable.

Without loss by generality, we may assume that ‘f’ and ‘g’ are finite all over E.

Now we first show that $'\alpha f'$ is measurable for some $\alpha \in \mathbb{R}$.

If $\underline{\underline{\alpha = 0}}$ then αf is a zero function then for any $C \in \mathbb{R}$.

$$\left\{x \in E : (\alpha F)(x) > C\right\} = \left\{x \in E : \alpha f(x) > C\right\}$$

$$= \begin{cases} \phi \ \ if\ C \geq 0 \\ E\ if\ C < 0 \end{cases}$$

$\therefore \phi$ and E are measurable $\Rightarrow \left(x \in E; (\alpha F)(x) > C\right)$ is measurable

$\Rightarrow \alpha F$ is measurable.

Suppose $\underline{\underline{\alpha \neq 0}}$

$$\left\{x \in E : (\alpha F)(x) > C\right\} = \left\{x \in E : \alpha f(x) > C\right\}$$

$$= \begin{cases} \left\{x \in E; f(x) > {}^{C}\!/\!_{\alpha}\right\} \alpha > 0 \\ \left\{x \in E; f(x) < {}^{C}\!/\!_{\alpha}\right\} \alpha < 0 \end{cases} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(*)$$

$\because f$ is measurable and C & α are red numbers.

$\therefore (*)$ is measurable

$\Rightarrow \left\{x \in E_i \, (\alpha f)(x) > C\right\}$ is measurable

$\Rightarrow (\alpha f)(x)$ is measurable

$\Rightarrow \alpha f$ is measurable ………………………. (1)

We now show that $(f+g)$ is measurable.

Let $C \in \mathbb{R}$

If $(f+g)(x) < C$

$\Rightarrow f(x) + g(x) < C$

$\Rightarrow f(x) < C - g(x)$

$\because Q$ is dense in $\mathbb{R}$, then is an $r \in Q$ such that $f(x) < r < C - g(x)$

$$\therefore \left\{x \in E; (f+g)(x) < C\right\} = \bigcup_{r \in Q} \left\{x \in E; f(x) < r\right\} \cap \left\{x \in E : g(x) < C - r\right\}$$

$\because Q$ is countable and $\{x\in E: f(x)<r\}$ is measurable & $\{x\in E: g(x)<C-r\}$ is measurable

$\therefore$ countable union of measurable set is measurable
$\Rightarrow \{x\in E:(f+g)(x)<C\}$ is measurable
$\Rightarrow f+g$ is measurable .... (2)
From (1) & (2)
$(\alpha f+\beta g)$ is measurable.

2) tpt (fg) is measurable
Note that $fg=\frac{1}{2}\left[(f+g)^2-f^2-g^2\right]$
$\because f, g$ are measurable $\Rightarrow f+g, \alpha f$ is measurable it is enough tst square of measurable function is measurable.

Let $C\geq 0$
Then
$\{x\in E; f^2(x)>C\}=\{x\in E; f(x)>\sqrt{C}\}\cup\{x\in E; f(x)<C-\sqrt{C}\}$

Which is union of two measurable set.
$\therefore$by definition, f^2 is measurable,
If $C<0$
$\{x\in E; f^2(x)>C\}=E$ which is measurable.
$\Rightarrow$In both the case f^2 is measurable
$\Rightarrow (fg)$is measurable.

*** Composition function** (fog)

Example 3:

Let g be measurable real valued function defined on E and f a continuous real valued function defined on all of $\mathbb{R}$ show that the composition fog is a measurable function on E.

Solution :
Given; Let 'g' be measurable function and 'f' be continuous function on $\mathbb{R}$.
Let $g; E\to\mathbb{R}$ be measurable and $f:\mathbb{R}\to R$ be a continuous
Let $C\in\mathbb{R}$
tst :*fog* is measurable

Note that $\{x\in E;(fog)(x)>C\}$

$\therefore (fog)^{-1}\left((C,\infty) = g^{-1}\left(f^{-1}\right)(C,\infty)\right)$

$\because (C,\infty)$ is open subset and f is continuous $\Rightarrow f^{-1}(C,\infty)$ is open in $\mathbb{R}$.

$\therefore f^{-1}(C,\infty) = 0$ for some open subset O of $\mathbb{R}$.

$\because O$ is open in $\mathbb{R}$, we can write

$$O = \bigcup_{i=1}^{\infty}(a_i, b_i)$$

$$\therefore g^{-1}\left(f^{-1}(C,\infty)\right) = g^{-1}\left(\bigcup_{i=1}^{\infty}(a_i, b_i)\right)$$

$$= \bigcup_{i=1}^{\infty}\left(g^{-1}(a_i, b_i)\right)$$

$$= \bigcup_{i=1}^{\infty}\left(\{x \in E : g(x) > a_i\} \cap \{x \in E : g(x) > b_i\}\right)$$

$\Rightarrow \{x \in E : g(x) > a_i\}$ is measurable and $\{x \in E : g(x) > b_i\}$ is measurable.

$\Rightarrow$ countable union of measurable set is measurable set.

$\{x : (fog)(x) > C\}$ is measurable

$\therefore fog$ is measurable function on E.

Check your Progress :

If f is measurable, then show that

1) f^k is measurable for all integer $K \geq 1$
2) $f + \lambda$ is measurable for a given constant $\lambda \in \mathbb{R}$
3) λf is measurable for a given constant $\lambda \in \mathbb{R}$
4) $|f|$ is measurable
5) $\sup f_n(n), \inf f_n(n), \lim_{n\to\infty} \sup_A f_n(n) \lim_{n\to\infty} \inf f_n(n)$ are measurable.

Definition :

For a sequence $\{f_n\}$ of functions with common domain E,a function f on E and a subset A of E, we say that

1) The sequence $\{f_n\}$converges to 'f' point wise E, on A provided $\lim_{n\to\infty}\{f_n\}(n) = f(x)$ for all $x \in A$
2) The sequence $\{f_n\}$ converges to 'f' point wise a.e. on A provided it converges to F pointwise on A/B where $m(B) = 0$
3) The sequence $\{f_n\}$ converges to 'f' uniformly on A provided for each $\in > 0, \exists N \in \mathbb{N}$ such that $|f - f_n| < \in$ on a for all $n \geq N$.

Theorem :

Let $\{f_n\}$ be a sequence of measurable function on E that converges point-wise a.e. on E to the function f, show that f is measurable.

Proof :

Let E_0 be a subset of E with $m(E_0)=0$ and $f_n \to f$ on E/E_0.

$\therefore m(E_0)=0$ & we have 'f' is measurable on E iff $f|_{E-E_0}$ is measurable.

$\therefore$By replacing E by $E-E_0$ we may assume that the $\{f_n\}$ converges to f on E

tst f is measurable

Let $C \in \mathbb{R}$

tst $\{x \in E; f(x) < C\}$ is measurable

$\{x \in E; f(x) < C\} = \{x \in E; \lim_{n\to\infty} f(x) < C\}$ but

$\lim_{n\to\infty} f(x) < C$ iff there are natural nos. n and k for which

$f_j(x) < C - \frac{1}{n} \quad \forall j \geq k$

$$\therefore \{x \in E; f(x) < C\} = \bigcup_{1 \le k, n < \infty} \left(\bigcap \left\{ x \in E; f_j(x) < C - \frac{1}{n} \right\} \right)$$

note that $\bigcap_{j=k}^{\infty} \left\{ x \in E; f_j(x) < C - \frac{1}{n} \right\}$ is measurable.

Countable union of measurable set is measurable

$\Rightarrow \{x \in E; f(x) < C\}$ is measurable.

Simple Functions :

Definitions :

A real-valued functions ϕ defined on a measurable set E is said to be simple if it is measurable and takes only a finite number of values.

If ϕ is simple, has domain E and takes the distinct values $C_1, \ldots, C_n$ then $\phi = \sum_{k=1}^{n} C_k \chi_{E_k}$ on E, where $E_k = \{x \in E; \phi(x) = C_k\}$.

This particular expression of ϕ is a linear combination of characteristic functions is called the canonical representation of the simple function ϕ.

Theorem : The simple Approximation Lemma

Let 'f' be a measurable real valued function on E. Assume 'f' is bound on E. Then for each $\in > 0$, there are simple function $\phi_\in$ and Ψ_E defined on E which have the following approximation properties :

$\phi_E \leq f \leq \Psi_E$ and $0 \leq \Psi_E - \phi_E < E$ on E.

Proof :

Suppose $f : E \to R$ is bounded measurable f_n

$\therefore f$ is bounded, $\exists M > 0$ such that $|f(x)| < M \quad \forall x \in E$

Let (c,d) be an open interval s.t. $f(E) \leq (c,d)$ ($\therefore f$ is bounded)

Let $\in > 0$

Consider the partition

$C = y_0 < y, < .... < y_{n=d}$ of $[c,d]$ with $y_k - y_{k-1} < \in, 1 \leq k \leq n$

Define $\phi_E = \sum_{k=1}^{n} y_{k-1}\chi_{E_k}, \Psi_E = \sum_{k=1}^{n} y\, \chi_{E_k}$ where $E_k = f^{-1}([y_{k-1}, y_k])$

Note that $E_k = f^{-1}([y_{k-1}, y_k])$

$= \{x \in E; f(x) \in [y_{k-1}, y_k]\}$

$= \{x \in E; y_{k-1} \leq f(x) < y_k\}$

$= \{x \in E; f(x) \geq y_{k-1}\} \cap \{x \in E : f(x) < y_k\}$

which is measurable. ($\therefore f$ is measurable)

$\therefore \chi_{E_k}$ are measurable, $1 \leq k \leq n$

$\Rightarrow \phi_\in$ & Ψ_E are measurable and takes only finite number of values

$\therefore \phi_\in$ & Ψ_E are simple functions.

Let $x \in E \Rightarrow f(x) \in (c,d)$

$\therefore \exists k$ s.t. $y_{k-1} \leq f(x) < y_k$

$\therefore \phi_E(x) = y_{k-1} \leq f(x) < y_k = \Psi_E(x)$ …………………. (1)

$\Rightarrow \phi_E(x) \leq f(x) \leq \Psi_E(x)$

Also by (1) $0 \leq \Psi_E(x) - \phi_E(x) = y_k - y_{k-1} < \in$

Theorem : The Simple Approximation Theorem

An extended real valued function 'f' on a measurable set E is measurable if and only if there is a sequence $\{\phi_n\}$ of simple functions on E which converges point-wise on E to f and has the property that $|\phi_n| \leq |f|$ on E for all 'n'.

If ‘f’ is non negative, we way choose $\{\phi_n\}$ to be increasing.

Proof :

Suppose f is measurable

Case (1) Assume $f \geq 0$

Let $n \in \mathbb{N}$, Define $E_n = \{x \in E; f(x) < n\}$

Then $f|_{E_n}$ is a bounded function.

$\therefore$By simple Approximation Lemma for $\in = \frac{1}{n}, \exists$ simple functions $\phi_\in$ & Ψ_E such that $\phi_\in \leq f|_{E_n} \leq \Psi_n$ and $0 \leq \Psi_n - \phi_n < 1/n$.

We extend ϕ_n on E defining $\phi_n(x) = n$ if $f(x) \geq n$ construct the sequences $\{\phi_n\}$.

We now show that $\phi_n \to f$ pointwise on E

(1) If ‘f’ is finite

$\therefore \exists N \in \mathbb{N}$ such that $f(x) < \mathbb{N}$

$\Rightarrow x \in E_N$

$\therefore \phi_N(x) \leq f(x) \leq \Psi_N(x)$

$\Rightarrow f(x) - \phi_N(x) \leq \Psi_N(x) - \phi_N(x) < \frac{1}{N}$

$\Rightarrow f(x) - \phi_N(x) < \frac{1}{n} \forall n \geq \mathbb{N}$

$\Rightarrow \phi_n(x) \to f(x)$ as $n \to \infty$

(2) If $f = \infty$

$f(x) > N$ for any $N \in \mathbb{N}$

$\Rightarrow \phi_n(x) = n$

$\Rightarrow \lim_{n \to \infty} \phi_n(x) = \infty = f$

Case (2) ‘f’ is any measurable function

Define $f^{-1}_{(x)} = \max\{f(x), 0\}$

$f^{-1}(x) = \min\{f(x), 0\}$

$\Rightarrow f(x) = f^+(x) + (f^1(x))$

$\because f^+$ and $-f^-$ are non-negative measurable function.

$\therefore$ By Case (1), $\exists$ a sequence of simple functions $\{\phi_n\}$&$\{\psi_n\}$ s.t. $\phi_n \to f^+$ pointwise and $\Psi_n \to f^-$ pointwise.

$\therefore \phi_n - \Psi_n \to f$ pointwise
$\therefore \phi_n$ and Ψ_n are simple function $\forall\, n$
$\Rightarrow \phi_n - \Psi_n$ a's also a simple function $\forall\, n$.

5.4 EGOROFF'S THEOREM

Theorem Statement (Assume E has finite measure)
Let $\{f_n\}$ be a sequence of measurable functions one that converges pointwise on E to the real valued function f. Then for each $\in > 0$ there is a closed set F contained in E for which $\{f_n\} \to f$ uniformly on F and $m(E/F) < \in$.

Proof :

Since $f_n \to f$ pointwise on E, for $\in > 0$, and $x \in E, \exists\, K \in \mathbb{N}$ such that $|f_j(x) - f(x)| < \in \ \forall\, j \geq K$ .... (1)

Since we want to get a region of uniform convergence, we accumulate all $x \in E$ for which the same N holds for a fixed E.

For any pair k & n define

$$E_k^n = \left\{ x \in E : \left| f_j(x) - f(x) \right| < \frac{1}{n},\ \forall\, j \geq K \right\}$$

Not all E_k^n are empty otherwise it will contradict pointwise converges of $\{f_n\}$ $\forall x \in E$.
$\because f_j$ and f are measurable $\Rightarrow E_k^n$ is measurable.
Note that from fixed n

$$E_k^n \subseteq E_{k+1}^n \text{ and } \bigcup_{k=1}^{\infty} E_k^n = E$$

$\therefore$ By the confinuity of measure.

$$m(E) = \lim_{K \to \infty} m\left(E_k^n\right)$$

$\because m(E)$ is finite, i.e. $m(E) < \infty$, for the above, $\in > 0$, such that

$$m(E) - M\left(E_k^n\right) < \frac{\in}{2^{n+1}}$$

$\Rightarrow m\left(E/E_{k_n}^n\right) < \dfrac{\in}{2^{n+1}}$ by countable additivity).

By construction for each $x \in E_{k_n}^n$

$$\left| f_j(x) - f(n) \right| < \frac{1}{n}\ \forall_j \geq k_n \text{ .... (2)}$$

Let $\boxed{A = \bigcap E_{k_n}^n}$

We show that $f_n \to f$ uniformly on A

Let $\in > 0$ choose $n_0 \in \mathbb{N} \frac{1}{n_0} < \in$

By (2)

$\left| f_j(x) - f(n) \right| < \frac{1}{n_0} \ \forall_j \geq k_{n_0}$ on $E_{k_{n_0}}^{n_0}$

$\because A \subseteq E_{Kn_0}^{n_0}$

$\Rightarrow \left| f_j(x) - f(n) \right| < \frac{1}{n_0} < \in \ \forall_j \geq k_{n_0}$ on A

$\therefore f_n \to f$ uniformly on A.

Now $m(E/A) = m(E \cap A^C)$

$$= m\left(E \cap \left(\bigcup \left(E_{k_n}^n\right)^C\right)\right)$$

$$= m\left(\bigcup_{n=1}^{\infty} \left(E \cap \left(E_{k_n}^n\right)^C\right)\right)$$

$$\leq \sum_{n=1}^{\infty} m\left(E / E_{k_n}^n\right)$$

$$< \sum_{n=1}^{\infty} \frac{\in}{2^{n+1}} = \frac{\in}{2}$$

$\because E_{k_n}^n$ are measurable and countable intersection of measurable set is measurable.

$\Rightarrow A$ is measurable.

$\therefore \exists$ a closed subset F of A s.t. $m(A/F) < \in/2$

$\therefore m(E/F) = m((E/A) \cup (A/F))$

$$= m(E/A) + m(A/F)$$

$$< \in/2 + \in/2 = \in$$

$\because f_n \to f$ uniformly on A & $F \subseteq A$

$\Rightarrow f_n \to f$ uniformly on F.

Examples 4 : Let f be a simple function defined on E. Then for each $\in > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on F & $m(E/F) < \in$.

Solution:

Let f be a simple function defined on $E \subseteq \mathbb{R}$

Let f takes the values $a_1, \ldots\ldots, a_n$ be the distance values taken by 'f'.

$$\therefore f = \sum_{i=1}^{n} a_i \chi_{E_i}$$

Where $E_i = \{x \in E : F(x) = a_i\}$

Note that $E = \bigcup_{i=1}^{n} E_i$

$\because a_k$'s are distinct $\Rightarrow E_k$'s are disjoint

$\because f$ is measurable $\Rightarrow F_{ks}$' are measurable

Let $\in > 0$

For each $k, 1 \le k \le n,\ E_k$ is measurable $\Rightarrow \exists$ closed subset F_k of E_k such that $m(E_k / F_k) < \frac{\in}{n}$

Let $F = \bigcup_{j=1}^{n} F_j$

$\Rightarrow F$ is closed

$$\begin{aligned}
m(E/F) &= m\left(E \cap F^C\right) \\
&= m\left(\left(\bigcup_{k=1}^{n} E_k\right) \cap F^C\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(E_k \cap F^C\right)\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(E_k \cap \left(\bigcap_{j=1}^{n} F_j^C\right)\right)\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(\bigcap_{j=1}^{n} \left(E_k \cap F_j^C\right)\right)\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(E_k \cap F_j^C\right)\right) \\
&= m\left(\bigcup_{k=1}^{n} \left(E_k / F_k\right)\right) \\
&\le \sum_{k=1}^{n} m(E_k / F_k) < \sum_{k=1}^{n} \frac{\in}{n} \\
&< \frac{\in}{n} \cdot n \\
&< \in
\end{aligned}$$

Define $g : F \to \mathbb{R}$ by $g(x) = a_i$ if $x \in F_i$

$\because E_i's$ are disjoint $\Rightarrow F_i's$ are disjoint g is well defined and $f = g$ on F we now show that 'g' is continuous on f then $F^1 = \bigcup_{i \neq k} F_i, F^1 \bigcap F_k = \phi$ and $x \in F_k$.

$\because \exists$ an open interval $I \subseteq F_K$ containing $'x'$ $I \bigcap F^1 = \phi$

$\therefore g(y) = a_k \ \forall y \in I$

$\therefore |g(y) - g(x)| = |a_k - a_k| = 0 < \in \ \forall_y \in I$

$\therefore g$ is continuous at *x*.

This is true for any $x \in F$

$\therefore g$ is continuous on F.

We can extend this continuous function 'g' on the closed set F to a continuous function on $\mathbb{R}$.

Let the new function be 'g' then 'g' is continuous on $\mathbb{R}$ and $g = F$ on f and $m(E/F) < \in$.

5.5 LUSIN'S THEOREM

Statement :

Let f be a real valued measurable function defined on E then for each $\in > 0$, there is a continuous function g on $\mathbb{R}$ and a closed set F contained in E for which $f = g$ on f and $m(E \mid f) < \in$.

Proof :

Let f be a real valued measurable function defined on E.

1) $m(E)$ is finite

$\therefore$ by simple Approximation theorem $\exists$ a sequence $\{\phi_n\}$ of simple function on E such that $\phi_n \to f$ and $|\phi_n| \leq |f|$ on $E \ \forall_n$.

$\therefore$for each $n \in \mathbb{N}$ there is a continous function $'g_n'$ on $\mathbb{R}$ and a closed set f f_n conained in E for which $\phi_n = g_n$ on f_n & $m(E/F_n) < \frac{\in}{2^{n+1}}$.

$\because \phi_n \to f$ pointwise on E

By Egoroff's theorem

$\exists$ a closed set f_0 contained in E such that $\{\phi_n\} \to F$ uniformly on F_0 and $m(E/F_0) < \in\!/2$.

Let $F = \bigcap_{h=0}^{\infty} F_n$

F is closed as countable intersection of closed sets.

Each ϕ_n is uniformly on $F\left(\because F \subseteq F_0\right)$

$\because \phi_n$ is continuous

$\Rightarrow f$ is continuous on F

i.e. $f/_F$ is continous.

We can extend $f/_F$ to a continuous function 'g' on $\mathbb{R}$.

Then $f = g$ on F

and $m(E/F) = m\left(E \cap F_n^C\right)$

$$= m\left(\bigcup_{n=0}^{\infty} E/F_n\right)$$

$$= m\left(\left(E|_{F_{n_0}}\right) \cup \left(\bigcup_{n=1}^{\infty} E/F_n\right)\right)$$

$$= m\left(\left(E/F_{n_0}\right) + \sum_{n=1}^{\infty} m\left(E/F_n\right)\right)$$

$$< \in/2 + \sum_{n=1}^{\infty} \frac{\in}{2^{n+1}}$$

$$< \in/2 + \in/2$$

$$< \in$$

5.6 SUMMARY

In this chapter we have learned about

- Concept of measurable functions.
- Properties of measurable functions
- Simple functions & ith Approximation Theorem
- Egoroffs Theorem and LUSIN Theorem of Measurable function.

5.7 UNIT END EXERCISE

1. Pure that "every continuous function is measurable".
2. Show that the sum and Product of two simple function are simple function
3. Show that if $f, [0, \infty] \rightarrow \mathbb{R}$ is differentiable, than f^1 is measurable.

4. Prove that if f is a measurable function X, than the set $f^{-1}(\infty)=\{x\in X \mid f(x)=\infty\}$ is measurable.
5. Prove that if $f:[0,1]\to\mathbb{R}$ is continous atmost everywhere than f is measurable.
6. State and prove Egoroff's Theorem of measurable function.
7. State and Prove Lusin's Theorem of real valued measurable function.
8. If 'f' is measurable then show that $f^{1}(C)$ is measurable, $C\in\mathbb{R}$.
9. If f is measurable then show that $\dfrac{\lambda f}{(-f)}$ is measurable.
10. Show that χ_A is Measurable if and only if the set A is measurable.

❖❖❖❖

6

LEBESGUE INTEGRAL

Unit Structure :

6.0 OBJECTIVES

After going through this chapter you can able to know that

- Lebesgue integral
- Lebesgue integral of a simple function
- Lebesgue integral of a bounded measurable function
- The general Lebesgue integral

6.1 INTRODUCTION

We have already learned simple functions, measurable functions. Now here we are going to discuss. Lebesgue integral on this function. Lebesgue integral over come on the class of all Riemannintegrable functions & the limitation of operations. So now we defined the general notation of the Lebesgue integral on $\mathbb{R}^n$ step by step.

6.2 LEBESGUE INTEGRAL OF SIMPLE FUNCTION

Definition :

For a simple function ϕ with canonical representation $\phi(x)=\sum_{i=1}^{n} a_i X_{E_i}$ defined on a set of finite measure E, we define the integral of ϕ over E by $\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$.

Example 1 : Let $\{E_i\}_{i=1}^{n}$ be a finite disjoint collection of measurable subset of a set of finite measure E. For $1 \le i \le n,$ Let $a_i \in \mathbb{R}$.

If $\phi = \sum_{i=1}^{n} a_i \chi_{Ei}$ on E, than $\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$.

Solution :

Let $\phi = \sum_{i=1}^{n} a_i \chi_{E_i}$ s.t. E_i's are pairwise disjoint which may not be in canonical form.

Let $\{b_j\}_{j=1}^{k}$ be distinct elements of $\{a_i, ...a_n\}$.

Define $F_j = \bigcup_{i \in I_j} E_i$ where $I_j = \{i : a_i = a_j\}$.

Note that F_j's are disjoint.

$\therefore m(F_j) = \sum_{i \in I_j} m(E_i)$

$\therefore \phi = \sum_{j=1}^{k} b_j \chi_{F_j}$ is a canonical representation of ϕ.

$\therefore$ By definition $\int_E \phi = \sum_{j=1}^{k} b_j m(F_j)$

$$= \sum_{j=1}^{k} b_j \left(\sum_{i \in I_j} m(E_i) \right)$$

$$\int_E \phi = \sum_{i=1}^{n} a_i m(E_i)$$

6.2.1 Theorem (Properties of integral simple function)

Let ϕ and Ψ be simple functions defined on a set of finite measure.

Then

1) Linearity : For any $'\alpha'$ and $'\beta'$

$$\int_E (\alpha\phi + \beta\Psi) = \alpha \int_E \phi + \beta \int_E \Psi$$

Proof :

Let $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$ and $\Psi = \sum_{j=1}^{n} b_j \chi_{B_j}$ be canonival representation of ϕ and Ψ respectively.

$C_{ij} = A_i \cap B_j, 1 \le i \le n, 1 \le j \le m$

then $\phi = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i \chi C_{ij}$ and $\Psi = \sum_{i=1}^{n}\sum_{j=1}^{m} b_i \chi C_{ij}$ ………… (1)

$\therefore$ By definition $\int_E \phi = \sum_{i=1}^{n}\sum_{j=1}^{m} a_i m(C_{ij})$ and $\int_{\in} \Psi = \sum_{i=1}^{n}\sum_{j=1}^{m} b_j m(C_{ij})$

By (1)

$$\alpha\phi + \beta\Psi = \sum_{i=1}^{n}\sum_{j=1}^{m}(\alpha a_i + \beta b_j)\chi C_{ij}$$

$\therefore$ By definition

$$\int_E \alpha\phi + \beta\Psi = \sum_{i=1}^{n}\sum_{j=1}^{m}(\alpha a_i + \beta b_j) m(C_{ij})$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\alpha a_i\ m(C_{ij}) + \sum_{i=1}^{n}\sum_{j=1}^{m}\beta b_j\ m(C_{ij})$$

$$= \alpha\left(\sum_{i=1}^{n}\sum_{j=1}^{m} a_i m(C_{ij})\right) + \beta\left(\sum_{i=1}^{n}\sum_{j=1}^{m} b_i m(C_{ij})\right)$$

$$= \alpha\int_E \phi + \beta\int_E \Psi$$

2) Monotonicity

If $\phi \le \Psi$ on E then $\int_E \phi \le \int_E \Psi$

Proof :

Suppose $\phi \le \Psi$ on E

tst $\int_E \phi \le \int_E \Psi$

Let $f = \Psi - \phi \ge 0$

$\therefore$By linearity property

$$\int_E \Psi \le \int_E \phi = \int_E (\Psi - \phi) = \int_E f \ge 0$$

$$\therefore \int_E \Psi \ge \int_E \phi$$

3) Additivity :

For any two disjoint subset $A, B \subseteq E$ with finite measure, $\int_{A\cup B} \phi \ge \int_A \phi \int_B \phi$

Solution :

$$\int_{A\cup B} \phi = \int_E \phi\chi_{A\cup B}$$

$$= \int_E \phi(\chi_A + \chi_B)$$
$$= \int_E \phi\chi_A + \int_E \phi\chi_B$$
$$= \int_A \phi + \int_B \phi$$

4) Triangle inequality : If ϕ is a simple $|\phi|$ and $\left|\int_E \phi\right| \leq \int_E |\phi|$.

Solution : Let ϕ be a simple function and $\phi = \sum_{i=1}^{n} a_i \chi_{Ai}$ be canonical representation of ϕ.

Then $|\phi| = \sum_{i=1}^{n} |a_i| \chi_{Ai}$ which is a simple function.

By Definition

$$\int_E \phi = \sum_{i=1}^{n} a_i m(Ai)$$
$$\therefore \left|\int_E \phi\right| = \left|\sum_{i=1}^{n} a_i m(Ai)\right|$$
$$\leq \sum_{i=1}^{n} |a_i m(Ai)| \text{ (by triangle inequality)}$$
$$\leq \sum_{i=1}^{n} |a_i||m(Ai)|$$
$$\leq \sum_{i=1}^{n} |a_i||m(Ai)|$$
$$\leq \int_E |a_i|$$

5) If $\phi = \Psi$ a.e. on E, then $\int_E \phi = \int_E \Psi$

Solution : Suppose $\phi = \Psi$ a.e. on F

Let $E_0 = \{x \in E; \phi(a) \neq \Psi(x)\}$

Then $m(E_0) = 0$ and on $E/E_0; \phi = \Psi$

Let $\phi = \sum_{i=1}^{n} a_i \chi_{Ai}$ and $\Psi = \sum_{j=1}^{n} b_j \chi_{Bj}$ be canonical representation of ϕ and Ψ representation.

$\therefore$By definition

$$\int_E \phi = \sum_{i=1}^{n} a_i m(Ai)$$

$$= \sum_{i=1}^{n} a_i m(A_i \cap E_0) \cup (A_i \cup E \mid E_0)$$

$$= \sum_{i=1}^{n} a_i m(A_i \cap E_0) + \sum_{i=1}^{n} a_i m\left(A_i \cap (E) E_0\right)$$

$$= O + \sum_{i=1}^{n} a_i m\left(A_i \cap (E/E_0)\right)$$

$$\int_E \phi = \int_{E|E_0} \phi$$

Similarly

$$\int_E \Psi = \int_{E/E_0} \psi$$

$\because \phi = \Psi$ on E/E_0

$$\therefore \int_E \phi = \int_E \psi$$

* Lebesgue integral of a bounded measurable function on a set of finite measure.

We now extend the notion of integral of simple function to a bounded measurable function on a set of finite measure.

Let 'f' be a bounded real -valued function defined on a set of finite measure E. We define the lower and upper Lebesgue integral respectively, of 'f' over E to be $\sup\left\{\int_E \phi : \phi \; simple\; and\; \phi \leq f \; on\; E\right\}$ and $\inf\left\{\int_E \Psi : \Psi \; simple\; and\; f \leq \Psi \; on\; E\right\}$.

Since 'f' is bounded by the monotonicity property of the integral for simple functions, the lower and upper integral are finite and the lower integral $\leq$ the upper integral.

6.3 DEFINITION

A bounded function 'f' on a domain E of finite measure is said to be Lebesgue integrable over E if its upper and lower Lebesgue integrals over E are equal. The common value of the upper and lower integrals is called the Lebesgue integrals or simply the integral, of 'f' over E and is denoted by $\int_E f$.

Example 2 : Show that a non negative bounded measurable function on a set E of finite measure is integrable E of finite measure is integrable over E.

Solution : Let 'f' be a bounded measurable function defined on E. where $m(E) < \infty$.

$\therefore$By simple Approximation Lemma

For $n \in \mathbb{N}, \exists$simple function ϕ_n and Ψ_n such that $\phi_n \le f \le \Psi_n$ and $0 \le \Psi_n - \phi_n < \frac{1}{n}$.

$$\therefore \int_E \Psi_n - \int_E \phi_n = \int_E \Psi_n - \phi_n < \int_E \frac{1}{n} = \frac{1}{n} m(E)$$

But, $\sup\left\{\int \phi; \phi \; simple, \phi \le f\right\} \ge \int_E \phi_n$ and

$$\inf\left\{\int \Psi; \Psi \; simple, \; f \le \Psi\right\} \le \Psi_n$$

$$0 \le \inf\left\{\int_E \Psi; \Psi \; simple, \Psi \ge f\right\} - \sup\left\{\int_E \phi; \phi \; simple, \phi \le f\right\}$$

$$\le \int_E \Psi_n - \int_E \phi_n < \frac{1}{n} m(E)$$

This is true for any $n \in \mathbb{N}$ and $m(E) < \infty$

$$\therefore \inf\left\{\int_E \Psi; \Psi \; simple, \Psi \ge f\right\}$$

$$= \sup\left\{\int_E \phi; \phi \; simple, \phi \le f\right\}$$

$\Rightarrow f$ is Lebesgue integrable over E.

Example :

Let 'f' be a bounded measurable function on a set E of finite measure. Show that if $\int_E f = 0$ then $f = 0$ a.e.

Solution : Suppose $\int_E f = 0$ and $f \ge 0$

tst $f = 0$ a.e.

Let $E_n = \left\{ x \in E; f(x) > \frac{1}{n} \right\}$ then $\frac{1}{n}\chi_{E_n}(x) < f(x)$.

By monotonicity,

$$\int \frac{1}{n}\chi_{E_n}(x) < \int_E f = 0$$

$$\Rightarrow \frac{1}{n} m(E_n) < 0$$

$$\Rightarrow m(E_n) = 0$$

But $E_0 = \{x \in E; f(x) > 0\} = \bigcup_{n=1}^{\infty} E_n$

$\therefore m(E_0)$

$\Rightarrow f = 0$ a.e. over E.

6.3.1 Properties of integral of bounded function :

Theorem : Let ‘f’ and ‘g’ be bounded measurable functions defined on a set of finite measure E then

1) Linearity : for any $'\alpha'$ and β

$$\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$$

Proof : Let f, g be bounded functions, $\alpha, \beta \in \mathbb{R}$

tst $\int_E \alpha f + \beta g = \alpha \int_E f + \beta \int_E g$

It is enough tst $\int_E \alpha f = \alpha \int_E f$ and $\int_E f + g = \int_E f + \int_E g$

If $\alpha = 0$ then $\alpha f = 0$

$$\Rightarrow \int_E \alpha f = 0 = \alpha \int_E f$$

Suppose $\alpha \neq 0$

$\therefore f$ is bounded $\Rightarrow \alpha f$ is bounded $\Rightarrow \alpha f$ is lebesgue integrable.

Let $\alpha > 0$

$\therefore \int_E \alpha f =$ upper lebesgue integrable of $'\alpha f'$

$= \inf\left\{\int \Psi : \Psi \text{ is simple} \,\&\, \Psi \geq \alpha - f\right\}$

$= \inf\left\{\alpha \int_E \left(\Psi/\alpha\right) : \Psi \text{ simple} \,\&\, \Psi/\alpha \geq f\right\}$

$= \alpha \inf\left\{\int_E \left(\Psi/\alpha\right) : \Psi/\alpha \text{ simple}, \Psi/\alpha \geq f\right\}$

$= \alpha \inf\left\{\int_E \phi : \phi \text{ simple}, \phi \geq f\right\}$

$= \alpha \int_E f$

Let $\alpha < 0$

Similarly for lower Lebesgue integral of αf

$\therefore \int_E \alpha f = \alpha \int_E f$

We now show that $\int_E f + g = \int_E f + \int_E g$

Let Ψ_1 and Ψ_2 be simple functions on E such that, $f \leq \Psi_1$ and $g \leq \Psi_2$ then Ψ_1 + Ψ_2 is a simple function and $f + g \leq \Psi_1 + \Psi_2$

$\therefore f$ and g are bounded $\Rightarrow f + g$ is bounded.

$\Rightarrow f + g$ is Lebesgue integrable

$\therefore$By definition

$\int_E f + g = \inf\left\{\int \Psi ; f + g \leq \Psi, \Psi \text{ is simple}\right\}$

$\leq \int_E \Psi_1 + \Psi_2 = \int_E \Psi_1 + \int_E \Psi_2$

This is true for any Ψ_1, Ψ_2 simple with $f \leq \Psi$, and $g \leq \Psi_2$

$\Rightarrow \int f + g$ is lower bound of

$\left\{\int_E \Psi_1 + \int_E \Psi_2 ; \Psi_1 \geq f_1, \Psi_2 \geq g, \Psi_1, \Psi_2 \text{ simple}\right\}$

$\Rightarrow \int_E f + g \leq \inf\left\{\int_E \Psi_1 + \int_E \Psi_2 ; \Psi_1 \geq f, \Psi_2 \geq g, \Psi_1, \Psi_2 \text{ simple}\right\}$

$\leq \inf\left\{\int \Psi_1 ; \Psi_1 \geq f, \Psi_1 \text{ simple}\right\} + \inf\left\{\int \Psi_2 ; \Psi_2 \geq g, \Psi_2 \text{ simple}\right\}$

$\leq \int_E f + \int_E g$

$\therefore \int_E f + g \leq \int_E f + \int_E g$

For the reverse inequality

Let ϕ_1 and ϕ_2 be simple function for which $\phi_1 \le f \; \& \phi_2 \le g$ on E then $\phi_1 + \phi_2 \le f + g$ and $\phi_1 + \phi_2$ is simple

$$\therefore \int_E f + g = \sup\left\{\int_E \phi; f + g \ge \phi, \phi \; simple\right\}$$

$$\ge \int_E \phi_1 + \phi_2$$

$$\ge \int_E \phi_1 + \int_E \phi_2$$

This is true for any ϕ_1, ϕ_2 simple with $f \ge \phi_1 \; \& \; g \ge \phi_2$

$\Rightarrow \int_E f + g$ is upper bound of

$$\left\{\int_E \phi_1 + \int_E \phi_2; \phi_1 \le f, \phi_2 \le g, \phi_1, \phi_2 \; simple\right\}$$

$$\Rightarrow \int_E f + g \ge \sup\left\{\int_E \phi_1 + \int_E \phi_2; \phi_1 \le f, \phi_2 \le g, \phi_1, \phi_2 \; simple\right\}$$

$$\ge \sup\left\{\int \phi_1; \phi_1 \le f, \phi_1 \; simple\right\} + \sup\left\{\int_E \phi_2; \phi_2 \le g, \phi_2 simple\right\}$$

$$\le \int_E f + \int_E g$$

$$\therefore \int_E f + g \ge \int_E f + \int_E g$$

$$\therefore \int_E f + g = \int_E f + \int_E g$$

2) Monotonicity : If $f \le g$ on E, then $\int_E f \le \int_E g$

Proof

Suppose f and g are bounded mesurable function on a set E of finite measurable function and $f \le g$

tst $\int_E f \le \int_E g$

Let $h = f - g \ge 0$

$\Rightarrow h$ is non-negative bounded function.

$\therefore$By linearity

$$\int_E g - \int_E f = \int_E g - f = \int_E h$$

$\therefore h$ is bounded & $h \ge 0$

$\Rightarrow h \geq \Psi$ where $\Psi = 0$ simple function

But $\int_E h = \sup\left\{\int_E \Psi; simple, \Psi \leq h\right\}$

$\Rightarrow \int_E h \geq \int_E \Psi = 0 * m(E) = 0$

$\therefore \int_E g - \int_E f = \int_E h \geq 0$

$\therefore \int_E g \geq \int_E f$

3) Additivity : For any two disjoint subsets, $A, B \subseteq E$ with finite measure.

$$\int_{A \cup B} f = \int_A f + \int_B f$$

Proof :

Let 'f' be bounded measurable function on a set E of finite measure and A,B disjoint subsets of E.

tst $\int_{A \cup B} f = \int_A f + \int_B f$

$\because f$ is bounded measure.

$\Rightarrow f\chi_{A \cup B}, f\chi_A, \chi_B$ are bounded measurable functions.

$$\therefore \int_{A \cup B} f = \int_E \chi_{A \cup B} = \int_E f(\chi_A + \chi_B)$$

$$= \int_E f\chi_A + f\chi_B$$

$$= \int_E f\chi_A + \int_E f\chi_B$$

$$\int_{A \cup B} f = \int_A f + \int_B f$$

4) Triangle inequality : Let f be a bounded measurable function on a set of finite measure E, Then $\left|\int_E f\right| \leq \left|\int_E |f|\right|$.

Proof :

Let f be bounded measurable function on a set E of finite measurable

$\Rightarrow |f|$ is measurable and bounded on E.

Note that

$\therefore -|f| \leq f \leq |f|$

$\therefore$ By monotonicity and linearity

$$-\int_E |f| \le \int_E f \le \int_E |f|$$

$$\Rightarrow \left|\int_E f\right| \le \int_E |f|$$

Example :

Let $\{f_n\}$ be a sequence of bounded measurable functions on a set of finite measure E. Show that if $f_n \to f$ uniformly on E, then $\lim_{n\to\infty}\int_E f_n = \int_E f$

Solution : Let $\{f_n\}$ be a sequence of bounded measurable function on a set E of finite and $f_n \to f$ uniformly on E

tst $\lim_{n\to\infty}\int_E f_n = \int_E f$

i.e. $\int_E f_n = \int_E f$

$\because f_n \to f$ uniformly on E

$\Rightarrow$ for a given $\in > 0$, $\exists_{n0} \in \mathbb{N}$

$\forall x \in E, \left|f_n(x) - f(x)\right| < \in/m(E) \; \forall n \ge n_0$

i.e. $|f_n - f| < \dfrac{\in}{m(E)} \forall n \ge n_0$ on E

For $n \ge n_0$

Now $\left|\int f_n - \int f\right| = \left|\int_E f_n - f\right|$

$$\le \int_E |f_n - f|$$

$$< \int \frac{\in}{m(E)}$$

$$< \frac{\in}{m(E)} \cdot m(E) = \in$$

By definition

$\therefore \lim_{n\to\infty}\int f_n = \int_E f$.

Example 5 :

Show by an example that the pointwise convergence alone is not sufficient to the passage of the limit under the integral sign.

Solution : Example

Let $f = 0$, function on $E = [0,1]$

Let $\phi_k = K\,\chi\left[0, \frac{1}{k}\right] \to 0$ as $k \to \infty$

$\therefore \phi_k \to f$ pointwise

$$\int_E \phi_k = K.m\left(\left[0, \frac{1}{k}\right]\right)$$

$$= K.\frac{1}{k} = 1$$

$$\int_E f = 0$$

$$\therefore \int_E \phi_k \not\to \int_E f$$

Example 6 :

Let f be a bounded measurable function on a set of finite measure E. Assume g is bounded and $f - g$ a.e. on E,

Show that $\int_E f = \int_E g$

6.4 THE GENERAL LEBESGUE INTEGRAL

For an extended real-valued function 'f' on E, the positive part f^+ and the negative part f^- of f defined by

$f^+(x) = ma \times \{f(x), 0\}$ and

$f^+(x) = ma \times \{-f(x), 0\}\ \forall x \subset E$

Then f^+ and f^- are non-negative functions on f

$f = f^+ - f^-$ on E and $|f| = f^+ + f^-$ on E

Thus f is measurable iff f^+ and f^- are measurable.

Example 7 :

Let f be a measurable function on E, show that f^+ and f^- are integrable over E iff $|f|$ is integrable over E.

Ans. Suppose f^+ and f^- are integrable

$$\Rightarrow \int_E f^+ < \infty \ \& \int_E f^- < \infty$$

But $|f| = f^+ + f^-$

$$\Rightarrow \int_E |f| = \int_E f^+ + f = \int_E f^+ + \int_E f < \infty$$

$\therefore |f|$ is integrable

Conversely, suppose $|f|$ is integrable

$$\Rightarrow \int_E |f| < \infty$$

But $f^+ \leq |f|$ & $f^- \leq |f|$

$$\Rightarrow \int_E f^+ \leq \int_E |f| < \infty \Rightarrow f^+ \text{ is integrable}$$

Similarly f^- is integrable.

Definition :

A measurable function f on E is said to be integrable over E if $|f|$ is integrable over E i.e. $\int_E |f| < \infty$. If ‘f’ is integrable over E, then we define the integral of ‘f’ over E by $\int_E f = \int_E f^+ - \int_E f^-$

Example :

Let ‘f’ be integrable over E. Show that f is finite a.e. on E and $\int_E f = \int_{E/E_0} f$ where $E_0 \subseteq E$ and $m(E_0) = 0$

Solution :

‘f’ is integrable on E

$\Rightarrow |f|$ is integrable

$$\Rightarrow \int_E |f| < \infty$$

Note that $|f|$ is non negative integrable function.

We now show that $|f|$ is finite a.e. on E.

Note that $\{x \in E; |f(x)| = \infty\}$

$= \bigcap \{x \in E; f(x) > x\}$

$\Rightarrow \{x \in E; |f(x)| = \infty\} \subseteq \{x \in E; f(x) > n\} \forall n$

But by chebychev's Lemma ………………………. (*)

$m(\{x \in E; |f(x) > n|\}) < \frac{1}{n}\int |f| \forall_n$

$\therefore |f|$ is integrable, $\int_E |f|$ is finite

i.e. $\int |f| < \infty$

$\Rightarrow m(\{x \in E; |f(n)| < n\}) = 0$

$\Rightarrow m(\{x \in E; |f(n)| = \infty\}) = 0$

$\Rightarrow |f(x)|$ is finite a.e. on E

$\therefore f \leq |f|$, we get

f is finite a.e. on E

Let $E_0 \subseteq E$ s.t. $m(E_0) = 0$

$\therefore$By definition

$\int_E f = \int_E f^+ - \int_E f^-$

$= \int_{E|E_0} f^+ - \int_{E|E_0} f^-$ ($\because f^+$ & f^- are non-negative integrable functions)

$= \int_{E/E_0} (f^+ - f^-) = \int_{E/E_0} f$

Example 9:

Define $f(x) = \frac{1}{x^{2/3}} \quad 0 < x < 1$

$= 0 \qquad x = 0$

Show that f is Lebesgue integrable on $[0,1]$ and $\int_0^1 \frac{1}{x^{2/3}} dx = 3$. Find also $f(x,2)$

Solution :

$\frac{1}{x^{2/3}} \to \infty$ as $x \to 0$

So f is unbounded in $[0,1]$ its Lebesgue integrability define

$f(x,n) = \frac{1}{x^{2/3}}$ if $\frac{1}{x^{3/2}} \leq x \leq 1$

$= n$ if $O < x < 1/n^{3/2}$

$= 0$ if $x = 0$

Now $\int_0^1 f(x,n)\,dx = \int_0^{1|_{n^{3/2}}} f(x,n)\,dx + \int_{1|_{n^{3/2}}}^1 f(x,n)\,dx$

$$= \int_0^{1|_{n^{3/2}}} n dx + \int_{1|_{n^{3/2}}}^1 \frac{1}{x^{2|3}}\,dx$$

$$= \frac{1}{\sqrt{n}} + 3\left[1 - \left(\frac{1}{n^{3/2}}\right)^{1/3}\right] = 3 - \frac{2}{\sqrt{n}}\ \forall n$$

by definition of the Lebesgue integral of on bounded functions

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty}\int_0^1 f(x,n)\,dx$$

$$= \lim_{n\to\infty}\left(3 - \frac{2}{\sqrt{n}}\right)$$

$$= 3$$

Lebesgue integrable define for $n = 2$

$$f(x,2) = \frac{1}{x^{2/3}} \text{ if } \frac{1}{z^{2/3}} \le x \le 1$$

$$= 2 \quad \text{if } 0 < x < \frac{1}{z^{2/3}}$$

$$= 0 \quad \text{if } x = 0$$

6.5 SUMMARY

In this chapter we have learned about

- Introduction concept of Lebesgue integral.
- Lebesgue integral of complex valued Measurable functions
- Lebesgue integral at a simple function.
- Lebesgue integral on bounded Measurable function general Lebesgue integral

6.6 UNIT END EXERCISE

1. Show that for a finite family $\{f_k\}_{n=1}$ of measurable functions with common domain E, the functions $Max\{f_1....f_n\}$ and $Min\{f_1....f_n\}$ also are measurable.
2. Show that the sum and product of two simple functions are simple.

3. For every non-negative and measurable function f on $[0,1]$ then show that $\int_{[0,1]} f\, dm = \inf \int_{[0,1]} \phi\, dm$.
4. Prove that a measurable function $f(x) L^1[0,1]$ if and only if $\sum_{n=1}^{\infty} 2^n m\left\{x \in [0,1]; |f(x)| \geq 2^n\right\} < \infty$
5. If $f \in L^1[0,1]$ find $\lim_{k\to\infty} \int_0^1 K \log\left(1 + \frac{|f(x)|^2}{K^2}\right) dx$
6. Let f be a Lebesgue integrable function on X use the positive and negative part of f to prove that $\left|\int_x f\, dx\right| \leq \int_x |f|\, dx$.
7. Let f be a non-negative measurable function on X and suppose that $f \leq M$ for some constant M prove that $\int_E f\, dx \leq \int_x |f|\, dx$ for
8. Calculate Lebesgue integral for the function $f(x) = \begin{cases} 1\ \textit{where x is rational} \\ 2\ \textit{where x is irrational} \end{cases}$
9. Evaluate $\int_0^5 f(x)\, dx$ if

   $$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & \{1 \leq x \leq 2\} \cup \{3 \leq x < 4\} \\ 2 & \{2 \leq x < 3\} \cup \{4 \leq x < 5\} \end{cases}$$

   by using Riemann and Lebesgue definition of the integral.
10. Show that if f is a non-negative measurable function then $f = 0$ a.e. on a set A iff $\int_A f\, dx = 0$
11. If $f(x) = 1/x$ if $0 < x < 1$

    $= 9$

    then f is not Lebesgue integrable in $[0,1]$
12. Let F be a non-negative measurable function on χ and suppose that $f \leq M$ for some constant M. Prove that $\int_E f\, d\mu \leq m\,\mu(E)$ for any measurable $E \subseteq \chi$.

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7

CONVERGENCE THEOREMS

Unit Structure :

7.1 INTRODUCTION

In this section we analyze the dynamics of integrability in the case when sequences of measurable functions are considered. Roughly speaking a "convergence theorem" states that integrability is preserved under taking limits. In other words, if one has a sequence $\left(f_n\right)_{n=1}^{\infty}$ of integrable functions, and if 'f' is some kind of a limit of the $f_n{}'s$ then we would like to conclude that 'f' itself is integrable, as well as the equality $\int f = \lim_{n\to\infty} \int f_n$ such results are employed in two instances.

i) When we want to prove that some function 'f' is integrable. In this case we would look for a sequence $\left(f_n\right)_{n=1}^{\infty}$, of integrable approximation for f.

ii) When we want to construct and integrable function in this case, we will produce first the approximates and then we will examine the existence of the limit.

The first convergence result, which is some how primote, but very useful in the following.

7.2 MEASURABLE FUNCTIONS

Theorem :

Let (X, A, μ) be a finite measure space, let $G(C-(o,\infty))$ and let $f_n : X \to [0,9], n \geq 1$ be a sequence of measurable functions satisfying.

1) $f_1 \geq f_2 \geq .... \geq 0$
2) $\lim_{n\to\infty} f_n(x) =, \forall x \in X$ Then one has the equality $\lim_{n\to\infty} \int_A f_n \, dx = 0$.

Proof :

Let for each $\in > 0$ and each integer $n \geq 1$, the set $A_K^C = \{x \in X_e; f_n(x) \geq \in\}$ obviously, we have $A_n^\in \in A, \forall \in > 0, n \geq 1$we are going to use the following case.

Claim I :

For every $\in > 0$, one has the equality $\lim_{n\to\infty} \mu(A_n^\in) = 0$.

Fix $\in > 0$, Let us first observe that (a) we have the inclusion $A_1^C \supset A_2^C \supset$ …………………………………….. (II)

Second using (b) we clearly have the equality $\bigcap_{k=1}^{\infty} A_k^\in = \phi$.

Since μ is finite using continuity property we have

$$\lim_{n\to\infty} \mu(A_n^\in) = \mu\left(\bigcap_{n=1}^{\infty} A_n^\in\right) = \mu(\phi) = 0$$

Claim II :

For every $\in > 0$, and every integer $n \geq 1$, one has the inequality $0 \leq \int_X f_n du \leq a\mu(A_n^\in) + \in \mu(x)$.

Fix $\in$ and n and let us consider the elementary functions.

$h_n^\in = ax_{A_n^\in} + \in x_{A_n^{-\in}}$ where $B_n^\in = X / A^E$ obviously, since $\mu(x) < \infty$ the function $h_n^\in$ is elementary integrable. By construction we clearly have $0 \leq f_n \leq h_n^\in$, so using the properties of integration, we get

$$0 \le \int_X f_n dx \le \int_X h_n^{\in} dx = a\mu\left(A_n^{\in}\right) + \in \mu\left(B^{\in}\right)$$
$$\le a\mu\left(A^{\in}\right) + \in \mu\left(X\right)$$

Using claim I & III it follows immediately that
$0 \le \lim_{n\to\infty} \inf \int_X f_n d\mu \le \lim_{n\to\infty} \sup \int_X f_n d\mu \le \in \mu\left(X\right)$
Since the last inequality hold for arbitary $\in > 0$, we get $\lim_{n\to\infty} \int_X f_n du = 0$

7.3 LEBESGUE THEOREM ON BOUNDED CONVERGENCE

Statement :

Let $\{f_n\}$be a sequence of functions measurable on a measurable subset $A \subseteq [a,b]$such that $\lim_{n\to\infty} f_n(x) = f(x)$then if there exists a constant M such that $|f_n(x)| \le M$ for all 'n' and for all 'x', we have $\lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A f(x)\, dx$.

Proof :

$\therefore \lim_{n\to\infty} f_n(x) = f(x)$ and $|f_n(x)| \le M$
$\Rightarrow |f(x)| \le M$
The function 'f' is bounded and measurable
Hence Lebesgue integrable.
Now we shall show that
$\lim_{n\to\infty} \int_A |f_n(x) - f(x)|\, dx = 0$
For a given $\in > 0$, we define a partition A into disjoint measurable sets A_k's as follows :
$A_k = \{x : |f_{k-1} - f| \ge \in, |f_n - f| < \in, \forall_n \ge k\}\ K = 1, 2, 3, ...$
In particular,
$A_1 = \{x : |f_n - f| < \in; n = 1, 2, 3, ...\}$
$A_2 = \{x : |f_1 - f| \ge \in; |f_n - f| n < \in; n = 2, 3, 4, ...\}$

Clearly,

$$A = \bigcup_{K=1}^{\infty} A_k = \left(\bigcup_{K=1}^{\infty} A_k\right) \cup \left(\bigcup_{K=n+1}^{\infty} A_k\right)$$

$$= P_n \cup Q_n$$

$$m_A = m(P_n \cup Q_n) = mP_n + mQ_n$$

Now $\int_A |f_n - f|\, dx = \int_{P_n} |f_n - f|\, dx + \int_{Q_n} |f_n - f|\, dx$ ………… (1)

For each 'n', we have

$|f_n - f| < \in$ on P_n and $|f_n - f| \leq |f_n| + |f| \leq 2m$ on Q_n

Thus, $\int_A |f_n - f|\, dx < \in mP_n + 2M\ mQ_n$

As $n \to \infty$, $\lim_{n\to\infty} mP_n = mA$ and $\lim_{n\to\infty} mQ_n = 0$

Thus, $\int_A |f_n - f|\, dx < \in mA$

$\in$ being an arbitrary value

$$\therefore \lim_{n\to\infty} \int_A f_n(x)\, dx = \int_A f(x)\, dx$$

Example 1 :

Verify Bounded Convergence.

Theorem for the sequence of functions

$$f_n = \frac{1}{\left(1 + x/n\right)^n}; O \leq x \leq 1, n \in \mathbb{N}.$$

$$|f_n(x)| = \left|\frac{1}{\left(1 + x/n\right)^n}\right| \leq 1 \forall n \text{ and } \forall x$$

Each f_n being bounded and measurable, the limit function.

$$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{\left(1 + x/n\right)^2} = \frac{1}{e^x}$$

It is also bounded and measurable. Now

$$\int_0^1 \frac{dx}{\left(1 + x/n\right)^n} = n \left.\frac{\left(1 + x/n\right)^{(-n+1)}}{(-n+1)}\right|_0^1$$

$$= \frac{n}{(n-1)} \left(1 - \frac{1 + 1/n}{\left(1 + 1/n\right)^n}\right)$$

$$\therefore \lim_{n\to\infty}\int_0^1 \frac{dx}{\left(1+x/n\right)^n}$$

$$=\lim_{n\to\infty}\frac{n}{n-1}\left(1-\frac{1+1/n}{\left(1+1/n\right)^n}\right)$$

$$=\lim_{n\to\infty}\frac{n}{(n-1)}\left(1-\frac{(n+1)/n}{\left(\frac{n+1}{n}\right)^n}\right)$$

$$=1-1/e$$

$$=\frac{e-1}{e}$$

Similarly,

$$\int_0^1 \lim_{n\to\infty}\frac{1}{\left(1+\frac{x}{n}\right)^n}dx=\int_0^1\frac{1}{e^x}dx=\int_0^1 e^{-x}\,dx$$

$$=\left[-e^{-x}\right]_0^1=\left(1-\frac{1}{e}\right)=\frac{e-1}{e}$$

Hence Bounded convergence theorem is verified.

7.4 LIMITS OF MEASURABLE FUNCTIONS

If $f_n:\mathbb{R}\to[-\infty,\infty](n;1,2,....)$ is an finite sequence of functions then we say that $f:\mathbb{R}\to[-\infty,\infty]$ is the pointwise limit of the sequence $(f_n)_n$ if we have $f(x)=\lim_{n\to\infty} f_n(x)$ for each $x\in\mathbb{R}$.

For any sequence $f_n:\mathbb{R}\to[-\infty,\infty]$ we can define $\limsup_{n\to\infty} f_n$ as the function with value at 'x' given by

$$\limsup_{n\to\infty} f_n(x)=\lim_{n\to\infty}\left(\sup_{k\geq n} f_k(x)\right)$$

Something that always makes sense because $\sup_{k\geq n} f_k(x)$ decreases n increases or atleast does not get any bigger as n increase. Suppose that $\{f_n\}$ is a sequence of real number. Let A be the set of numbers such that $f_n\to f$ for some subsequence f_{n_k} of f_n.

$\therefore f$ is called a limit point of f_n, so A is the set of all limit points of $\{f_n\}$. Then supremum and infimum of A are denoted by the following $\lim_{n\to\infty}\inf f_n = \inf A, \lim_{n\to\infty}\sup f_n = \sup A$.

7.5 FATOU'S LEMMA

Statement :

If $\{f_n\}$ is a sequence of non-negative measurable functions, then for any measurable set E.

$$\lim_{n\to\infty}\inf \int_E f_n\,dx \geq \int_E \left(\lim_{n\to\infty}\inf f_n\right) dx$$

Proof : We write $f(x) = \lim_{n\to\infty}\inf f(x)$

We recall that for any x, $\lim\inf f_n(x) = \inf\inf f_n$ where Ex is the set of all limit points of $f_n(x)$.

$\therefore f_n \to f$ pointwise convergence on E

$\Rightarrow f_n \to f$ pointwise on $E/E, m(E_1) = 0$

$\therefore f_n \not\to f$ pointwise on E_1

$\because E_1 \subseteq E$ and $m(E_1) = 0$

We may assume $f_n \to f$ pointwise on E

f_n's are non-negative measurable and $f_n \to f$

$\Rightarrow f$ is non-negative and measurable.

Now to show that $\int_E f \leq \lim_{n\to\infty}\inf \int_E f_n$

Let h be a bounded measurable function of finite support such that $0 < h < f$

$\Rightarrow m(E_0) < \infty$ where $E_0 = \{x \in E; h(x) \neq 0\}$

$\because h$ is bounded choose M such that $h(x) \leq M$ on E for $n \in \mathbb{N}$ Define $h_n = \min\{h, f_n\}$.

Clearly $h_n \geq 0$ is measurable bounded function and $h_n \leq M$. We can now show that $h_n \to a$ pointwise on E_0.

For $x \in E_0$ $h(x) \leq f(x)$

Case I :

$h(x) < f(x)$

$\Rightarrow f(x) - h(x) > 0$

$\because f_n \to f$ pointwise on E for $0 < \in < f(x) - h(x)$

$\exists\, n_0 \in \mathbb{N}$ such that $\left| f_n(x) - f(x) \right| < \in \ \forall_n \ge n_0$

$\Rightarrow f(x) - \in < f_n(x) < f(x) + \in$

$\therefore h(x) < f(x) - \in < f_n(x) \, \forall_n \ge n_0$

$\therefore h_n(x) = \min(h, f_n) = h(x) \, \forall n \ge n_0$

$\Rightarrow h_n \to h$ pointwise on E_0

Case II :

$h(x) = f(x)$

Then $h_n(x) = f_n(x)$ on $f(x) \forall n$

$\because f_n \to f$ pointwise on E_0

$\Rightarrow h_n \to f = h$ pointwise E_0

By bounded convergence Theorem

For the bounded sequence $\{h_n\}$ restricted to E_0

We have $\lim_{n\to\infty} \int_{E_0} h_n - \int_{E_0} h$

$\therefore \lim_{n\to\infty} \int_E h_n = \lim_{n\to\infty} \int_{E_0} h_n = \int_{E_0} h = \int_E h$

$[\because h_n = 0, on\ E/E_0 ; h = 0\ on\ E/E_0]$

$\int_E h = \lim_{n\to\infty} \int_E h_n = \lim\inf \int_E h_n \le \lim\inf \int_E f_n$

This is true for any bounded measurable function with finite support such that $0 \le h \le f$

$\therefore$ By definition of $\int_E f$

$\therefore \int_E f \le \lim_{h\to\infty}\inf \int_E f_n$

7.6 LEBESGUE INTEGRAL OF NON-NEGATIVE MEASURABLE FUNCTION

Definition :

Let f be a measurable function defined on E. The support of 'f' is defined as $\sup(f) = \{x \in E; f(x) \ne 0\}$.

Definition :

A measurable function f on E is said to vanish outside a set of finite measure if $\exists$ a subset E_0 of E for which $m(E_0)<\infty$ & $f=0$ on E/E_0. It is convenient to say that a function that vanishes outside a set of finite measure has finite support.

$\therefore$We have defined the integral of a bounded measurable function 'f' over a set of finite measure E. But $m(E)=\infty$ and f is bounded and measurable on E with finite. Support we can define its integral over E by $\int_E f=\int_{E_0} f$ where $m(E_0)<\infty$ and $f=0$ on E/E_0.

Definition :

For a non-negative measurable function f on E we define integral of 'f' over E by $\int_E = \sup\left\{\int_E h : h\right.$ bounded; measurable of finite support and $\left. 0\le h< f \; on\, E\right\}$.

Chebychev's Inequality :

Statement :

Let f be a non-negative measurable function on $E\subseteq\mathbb{R}$ then for any $\lambda>0$.

$$m\{x\in E; f(x)\ge\lambda\}\le\frac{1}{\lambda}\int_E f$$

Proof :

Let $E_\lambda=\{x\in E: f(x)\ge\lambda\}$

Case I :

$m(\lambda_n)=\infty$ for each $n\in\mathbb{N}$ define $E_\lambda^n=E_\lambda\bigcap[-n,n]$. Then $\Psi_n=\lambda\chi_{E_\lambda^n}$.

Then Ψ_n is bounded measurable function

$\therefore \lambda_m(E_\lambda^n)=\int_E \Psi_n$ and $\Psi_n\le f$

Note that $E_\lambda^n\le E_\lambda^{n+1}$ and $\bigcup_{n=1}^{\infty}E_\lambda^n=E_\lambda$

$\therefore$By continuity of measure.

$$\infty=\lambda_m(E_\lambda)=\lim_{n\to\infty}\lambda_m(E_\lambda^n)$$

$$=\lim_{n\to\infty}\int_E\Psi_n$$

$\because \Psi_n$ is bounded on E and $\Psi_n\le f$

$\therefore$ by definition $\int_E f$, we get

$$\int_E \Psi_n \leq \int_E f$$

$$\infty = \lambda_m (E_\lambda) \leq \int_E f$$

Both side $= \infty$

$$\therefore m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

Case II : $m(E_\lambda) \leq \infty$

Define $h = \lambda \chi_{E_\lambda}$ then h is bounded measurable function $h \leq f$

$\therefore$ by definition of $\int_E f$, we get $\lambda m(E_\lambda) = \int h \leq \int_E f$

$$m(E_\lambda) \leq \frac{1}{\lambda} \int_E f$$

$$\therefore m\{x \in E; f(x) \geq \lambda\} \leq \frac{1}{\lambda} \int_E f$$

7.7 THE MONOTONE CONVERGENCE THEOREM

Statement : Let $\{f_n\}$ be an increasing sequence on non-negative measurable functions on A. If $\lim_{n\to\infty} f_n(x) = f(x)$ then $\lim_{n\to\infty} \int_A f_n = \int_A f$.

Proof :

Let $\{f_n\}$ be an increasing sequence of non-negative measurable functions and $\lim_{n\to\infty} f_n = f(x)$ i.e. it is convergent at pointwise to f on A.

Now to show that $\lim_{n\to\infty} \int_A f_n = \int_A f$.

$\because f_n \to f$ pointwise on A and $f_n \leq f_{n+1} \; \forall \, n \in \mathbb{N}$

$\Rightarrow f_n \leq f \; \forall_n$ on A

$\Rightarrow \int_A f_n \leq \int_A f$ on A

$$\Rightarrow \sup \int_A f_n \leq \int_A f$$

$$\lim_{n\to\infty} \sup \int_A f_n \leq \int_A f \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \text{(I)}$$

By the Fatou's lemma

$$\int_A f \le \liminf_{n\to\infty} \int_A f_n \text{ .... (II)}$$

From I & II we get

$$\int_A f = \liminf_{n\to\infty} \int_A f_n = \limsup_{n\to\infty} \int_A f_n$$

$$\therefore \lim_{n\to\infty} \int_A f_n = \int_A f$$

7.8 DOMINATED CONVERGENCE THEOREM

(Generalisation of Bounded Convergence Theorem)

Statement : Let $\{f_n\}$ be a sequence of measurable function on E. Suppose there is a function 'g' that is integrable over E and dominates $\{f_n\}$ on E in the sense that $|f_n| \le g$ on E for all n. If $f_n \to f$ pointwise almost everywhere on E,then f is integrable over E and $\lim_{n\to\infty} \int_E f_n = \int_E f$.

Proof :

$\because |f_n| \le g \ \forall_n$ on E and $f_n \to f$ pointwise on E.

$\Rightarrow |f| \le g \le |g|$

$\Rightarrow \int |f| \le \int |g| < \infty$

$\Rightarrow f$ is measurable

$\because |f_n| \le g$ and $|f| \le g \Rightarrow g - f_n \ge 0$ and $g - f_n \to g - f$ pointwise

$\therefore$By Fatou's lemma

$$\int g - f \le \liminf \int g - f_n$$

$$\le \liminf \int_E \delta - \int_E f_n$$

$$\le \int_E \delta - \limsup \int_E f_n$$

$$\therefore \limsup \int_E f_n \le \int_E f \text{ .... (I)}$$

Similarly $g + f_n \ge 0$ & $g + f_n \to g + f$ pointwise on E.

$\therefore$ By Fatou's lemma,

$$\int_E g + f \le \liminf \int_E g + f_n$$

$$\int_E g + \int f \le \int_E g + \liminf \int_E f_n$$

$\int_E f \le \liminf \int_E f_n$ .... (III)

From I & II we get

$\liminf \int_E f_n = \limsup \int_E f_n - \int_E f$

$\therefore \lim_{n\to\infty} \int_E f_n = \int_E f$

Example 2 :

Check the convergence of

$$f_n(x) = 1/n ; |x| \le n$$
$$= 0 \;\; ; |x| > n$$

Solution : Let $f_n(x) = 1/n \;\; ; |x| \le n$

$$= 0 \;\;\; ; |x| > n$$

Then $f_n(x) \to 0$ uniformly on $\mathbb{R}$ but $\int_{-\infty}^{\infty} f_n dx = 2; \;\; n = 1, 2, 3, ...$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{n} = 0$ where $|x| \le n$

$= 0$ when $|x| > n$

$\therefore \lim_{n\to\infty} f_n(x) = 0$ uniformly on the whole real time.

Now, $\left| f_{2m}(x) - f_m(x) \right| = \left| \frac{1}{2m} - \frac{1}{m} \right| = \left| \frac{1}{2m} \right| < \in$

Whenever $M > \frac{1}{2 \in}$

Now $\int_{-\infty}^{\infty} f_n(x)\, dx = \int_{-\infty}^{-n} 0\, dx + \int_{-n}^{n} 1/n\, dx + \int_{n}^{\infty} 0\, dx = 2$.

This emplies that uniform converges of $\{f_n(x)\}$ is not enough for $\lim_{n\to\infty} \int f_n = \int \lim_{n\to\infty} f_n$

This equality is Lebesgue integration.

In general, is only due to dominated convergence of the sequence $\{f_n(x)\}$.

$\therefore$However on the set of finite measure uniformly convergent sequence of bounded function are bounded convergent.

7.9 LEBESGUE INTEGRAL OF COMPLEX VALUED FUNCTIONS

If f is a complex valued function on $E \subseteq \mathbb{R}^n$ we may write as $f(x) = u(x) + i\vartheta(x)$ where $u \,\&\, v$ are real functions called the real and imaginary part of f.

A complex valued measurable function, $f : u + iv$ on E is said to be integrable if $\int_E |f(x)| = \int_E \sqrt{u(x^2) + v(x^2)} < \infty$ and the integral of 'f' is given by $\int_E f = \int_E u + i\int_E v$

Theorem :

Show that a complex valued function is integrable if and only if both of its real and imaginary parts are integrable.

Proof :

Suppose $f : u + iv$ is integrable

$\Rightarrow \int |f| < \infty$

$\Rightarrow \int \sqrt{u^2 + v^2} < \infty$

$u \le |u| = \sqrt{u^2} \le \sqrt{u^2 + v^2}$

$\Rightarrow \int |u| \le \int \sqrt{u^2 + v^2} < \infty$

$\Rightarrow u$ is integrable

Similarly v is integrable

Conversely

Suppose $u \,\&\, v$ are integrable

$\Rightarrow \int |u| < \infty$ and $\int |V| < \infty$

By Minkowski's inequality

$|f| = \sqrt{u^2 + v^2} \le \sqrt{u^2} + \sqrt{v^2} = |u| + |v|$

$\Rightarrow \int |f| \le \int |u| + \int |v| < \infty$

$\therefore f$ is integrable.

Definition :

A measurable function $f : E \to C, E \subseteq \mathbb{R}^n$ is said to be an L^1 function if $\int_E |f| < \infty$.

Note : $L^1(\mathbb{R}^n) =$ {set of all complex valued function on $\mathbb{R}^n$ }

Definition : A family G of integrable function is dense in $L^1(\mathbb{R}^n)$ if for any $f \in \bigcup$ and $\in > 0 \exists g \in G$ so that $\int_E |f-g| < \in$

Example 3:

Show that the continuous function of compact support is dense in $L^1(\mathbb{R}^n)$.

Solution :

To show that : The continuous function of compact support is dense in $L^1(\mathbb{R}^n)$.

i.e. tst for any $f \in L^1$ and $\in > 0$.

$\exists$ a continuous function 'g' on $\mathbb{R}^n$ with compact support such that $\|f-g\|, < \in$ i.e. $\int |f-g| < \in$.

Let $f \in L^1(\mathbb{R}^n)$

We may assume 'f' is real valued becaue we may approximate its real and imaginary part independently.

In this can we write $f = f^+ - f^-$.

Where $f^+ \geq 0$ and $f^- \geq 0$

$\therefore$ It is enough to show the result $f \geq 0$.

$\therefore$ $f \geq 0$ can be approximated by integrable simple functions.

It is enough to show that the result for an integrable simple functions.

$\therefore A_n$ integrable simple functions is a Linear combination of characteristic function.

It is enough to show for $f = \chi_E$ where E is a measurable set of finite measurable.

Let $\in > 0$

$\therefore$E is measurable $\exists$ a compact set k and an open set Ω of $\mathbb{R}^n$ such that $K \subseteq E \subseteq \Omega$ and $m(\Omega | k) < \in$

By Urysohn's Lemma

$\exists$ a continuous function $g:\Omega \to k$ such that $g \equiv 0$ on $\Omega | k$ & $g \equiv 1$ on K

$\therefore$ g is continuous function with compact support
$\therefore |g - f| = |g - \chi_E| = 1 \quad E | k$ and $|g - \chi_E| = 0$ on outside $E | k$
$\therefore \int_{\mathbb{R}^n} |g - f| = \int_{E/k} 1 = m(E | k) \le m(\Omega | k) < \in$
$\therefore \exists$ continuous function of compact support such that $|g - f| < \in$.
$\therefore$ Continuous function of compact support is dense in $L^1(\mathbb{R}^n)$.

Example 4 :

Let $f \in L^1(\mathbb{R}^n)$ show that $\left|\int f\right| \le \int |f|$

Solution : Let $f \in L^1$ to show that $\left|\int f\right| \le \int |f|$
Let $z = \int f$
If $z = 0$ then clearly $\int |f| \ge 0 = z = |z| = \left|\int f\right|$
$\therefore \left|\int f\right| \le \int |f|$
If $z \ne 0$
Define $\alpha = \dfrac{\overline{z}}{|z|}$
$\therefore |\alpha| = 1$ and $\alpha z = |z|$
$\therefore \left|\int f\right| = |z| = \alpha z = \alpha \int f = \int \alpha f$

Let $\alpha f = u + iv$

By definition
$\int \alpha f = \int u + i \int v$
$\therefore \left|\int f\right| = \int u + i \int v$
$\therefore \left|\int f\right| \in \mathbb{R} \Rightarrow \int v = 0$
$\therefore \left|\int f\right| = \int u$ ………………………………………….. (I)

$u \le |u| \le |\alpha f| = |\alpha||f| = |f|$

By Monotonicity property
$\int u \le \int |f|$ …………………………………………….. (II)

By (I) and (II)

$\therefore \left|\int f\right| \leq \int |f|$ proved

Example 5 :

Show that $L^1\left(\mathbb{R}^n\right)$ is complete in its metric.

Solution :

Let $\{f_n\}$ be a Cauchy sequence in $L^1\left(\mathbb{R}^n\right)$ for $\in > 0, \exists\, n_0 \in \mathbb{N}$ such that $\|f_m - f_n\|_1 < \in \forall n, n \geq n_0$

$\therefore$ for each $k \in \mathbb{N}$

We can choose n_k such that for $m, n \geq n_k$ $\|f_m - f_n\|_1 < \frac{1}{2^k}$ and $n_k < n_{k+1}$

then the sequence f_{n_k} has the property that $\left\|f_{n_{k+1}} - f_{n_k}\right\| < \frac{1}{2^k}$.

Construct the series

$$f(n) = f_{n_1}(x) + f_{n_2}(x) - f_{n_1}(x) + f_{n_3}(x) - f_{n_2}(x) + ...$$

$$= f_{n_1}(x) + \sum_{K=1}^{\infty}\left(f_{n_{k+1}}(x) - f_{n_k}(x)\right)$$

and $g(x) = \left|f_{n_1}(x)\right| + \sum_{K=1}^{\infty}\left|f_{n_{k+1}}(x) - f_{n_k}(x)\right|$

Let $S_k(g)$ denote the k^{th} partial sum of the series g then.

$$S_k(g) = \left|f_{n_1}(x)\right| + \sum_{i=1}^{k+1}\left|f_{n_{i+1}}(x) - f_{n_i}(x)\right|$$

Then $\{S_k(g)\}$ is a sequence of non-negative function converges pointwise to g.

$S_k(g) \leq S_{k+1}(g) \forall n$

$\therefore$ By Monotone Convergence Theorem g is integrable and $\lim_{n\to\infty} \int S_k(g) = \int g$

Note that $|f| \leq g$

$\Rightarrow \int (f) \leq \int g < \infty$ ($\because$ g is integrable)

$\Rightarrow f$ is integrable

$\Rightarrow f$ is $L^1(\mathbb{R}^n)$

Let $S_k(f)$ denote the k^{th} partial sum of the series of f, then

$$S_k(f) = f_{n_1}(x) + \sum_{i=1}^{K-1}\left(f_{n_{i+1}}(x) - f_{n_i}(x)\right)$$

$$= f_{n_k}(x)$$

$\because S_k(f) \to f$ pointwise

$\Rightarrow f_{n_k} \to f$ pointwise

Now we show that $\Rightarrow f_{n_k} \to f$ in $L^1(\mathbb{R}^n)$

Note that $\left|f - f_{n_k}\right| \leq g \; \forall \, k$

By Dominated convergence Theorem

$\lim\limits_{n\to\infty} \int \left|f - f_{n_k}\right| = 0$

$\therefore \lim\limits_{n\to\infty} \int \left\|f - f_{n_k}\right\|_1 = 0$

$\therefore f_{n_k} \to f$ in $L^1(\mathbb{R}^n)$

$\because f_n$ is Cauchy and has convergent subsequence f_{n_k} converges of f.

We get $f_n \to f$

$\therefore$ Every Cauchy sequence in L^1 is convergent.

$\therefore L^1$ is complete in its metric. Proved

7.10 REVIEW

In this chapter we have learnt following points.

- Limits of Measurable function
- Bounded convergence theorem of measurable function
- Monotone convergence theorem of measurable function.
- Fatou's lemma of measurable function
- Dominated convergence Theorem
- Complex valued measurable function
- Compactness of $L^1(\mathbb{R}^n)$

7.12 UNIT END EXERCISE

1. show by an example that the inequality in Fatou's lemma may be a strict inequality.

Example : Consider a sequence of function $\left(f_n\right)_{n\in\mathbb{N}}$ defined on $[0,1]$ by $f_n(x)=\dfrac{nx}{1+n^2x^2}\ x\in[0,1]$.

i) Show that $\left(f_n\right)$ is uniformly bounded on $[0,1]$ and evaluate $\lim\limits_{n\to\infty}\int\limits_{[0,1]}\dfrac{nx}{1+n^2x^2}dx$

ii) Show that $\left(f_n\right)$ doesnot converge uniformly on $[0,1]$

Solution :

1) For all $n\in\mathbb{N}$ for all $x\in[0,1]$ we have $1+n^2x^2\geq 2nx\geq 0$ and $1+n^2x^2>0$

Hence $0\leq f_n(x)=\dfrac{nx}{1+n^2x^2}\leq\dfrac{1}{2}$

Thus $f(x)$ is uniformly bounded on $[0,1]$

Since each f_n is continuous on $[0,1]$

$\therefore f$ is Riemann integrable on $[0,1]$

In this case Lebesgue integral and Riemann integral on $[0,1]$. Consider

$$\int\limits_{[0,1]}\frac{nx}{1+n^2x^2}dx=\int_0^1\frac{nx}{1+n^2x^2}dx$$

Put $1+n^2x^2=t$

$$=\frac{1}{2x}\int_0^{1+n^2}1/t\;dt$$

$$\int\limits_{[0,1]}\frac{nx}{1+n^2x^2}dx=\frac{1}{2n}\log\left(1+n^2\right)=\frac{\log\left(1+n^2\right)}{2n}$$

Using L^1 Hospitalrule we get

$$\log_{n\to\infty}\frac{\log\left(1+n^2\right)}{2n}=0$$

Hence $\lim\limits_{n\to\infty}\int\limits_{[0,1]}\dfrac{nx}{1+n^2x^2}dx=0$

ii) For each $x\in[0,1]\Rightarrow\lim\limits_{n\to\infty}\int\limits_{[0,1]}\dfrac{nx}{1+n^2x^2}=0$

Hence $f_n\to f$ pointwise on $[0,1]$

Now to show that f_n does not converges to $f=0$ uniformly on $[0,1]$.

We find a sequence (x_n) in $[0,1]$.

Such that $x_n \to 0$ and $f_x(x_n) \not\to f(0)=0$ as $n\to\infty$, taking $x_n=\frac{1}{n}$ then $f_n(x)= 1/2$.

Thus $\lim_{n\to\infty} f_n(x_n)=\frac{1}{2}\neq f(0)=0$

Example 2 :

Evaluate $\lim_{n\to\infty}\int_0^n \left(\frac{1+x}{n}\right)^n e^{-2x}dx$

Solution : We know that

$$\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x \text{ and } \left(1+\frac{x}{n}\right)^n \le \left(1+\frac{x}{n+1}\right)^{n+1}.$$

Also we have $\left(1+\frac{x}{n}\right)^n \le e^x$

$\therefore \left(1+\frac{x}{n}\right)^n e^{-2x} \le e^{-x}$

$\therefore$ by Dominated convergence then to the function $\left(1+\frac{x}{n}\right)^n e^x$ with the dominating function e^{-x}

$$\therefore \lim_{n\to\infty}\int_0^n \left(1+\frac{x}{n}\right)^n e^{-2x}dx$$

$$=\lim_{n\to\infty}\int_0^\infty 1_{[0,1]}x\left(1+\frac{x}{n}\right)^n e^{-2x}dx$$

$$=\int_0^\infty \lim_{n\to\infty} 1_{[0,1]}x\left(1+\frac{x}{n}\right)^n e^{-2x}dx$$

$$=\int_0^\infty e^{-x}dx$$

$$=1$$

2) Show by an example that monotone convergence theorem does not hold for a decreasing sequence of functions.

3) Let $f_n(x) := \frac{x}{n^2}; 0<x<n$

= 0 ; otherwise

Evaluate $\lim_{n\to\infty}\int_0^n f_n(x)\,dx$ and $\int_0^n \lim_{n\to\infty} f_n(x)\,dx$ are these equal?

4) $g(x)=0 \quad 0\le x\le \frac{1}{2}$

$\quad = 1 \quad \frac{1}{2}\le x\le 1$

$f_{2k}(x)=g(x), 0\le x\le 1$

$f_{2k+1}(x)=g(1-x), 0\le x\le 1$

To show that $\liminf_{n\to\infty}\int_0 f_n(x)\,dx > \int_0 \liminf_{n\to\infty} f_n(x)\,dx$

5) If $f_n; X\to[0,\infty]$ is measurable for $n=1,2,....$ and $f(x)=\sum_{n=1}^{\infty} f_n(x)(x\in X)$ then show that $\int_X f\,dr=\sum_{n=1}^{\infty}\int_X f_n\,dr$.

6) Use the dominated convergence theorem to find $\lim_{n\to\infty}\int_1^{\infty} f_n(x)\,dx$ where $f_n(x)=\frac{\sqrt{x}}{1+nx^3}$.

7) If $a_n\le b_n$ for all n, then show that $\liminf_{n\to\infty} a_n \le \liminf_{n\to\infty} b_n$.

8) State and prove bounded convergence theorem of measurable function.

9) Use convergence theorem to show that $f(t)=\int_{[0,\infty]} e^{-x}\cos(\pi t)\,d\,u(x)$ is continuous.

10) Use the dominated, convergence theorem to prove that $\lim_{n\to\infty} n\int_0^1 \sqrt{x}e^{n^2x^2}\,dx=0$

11) Use the dominated convergence theorem to show that

$$\lim_{n\to\infty}\int_R \left(1+\frac{x^2}{n^2}\right)^{-\left(\frac{n+1}{2}\right)} dx=\int_R e^{-\frac{x^2}{2}}\,dx$$

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