Geometric origami

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Abstract In this short note we shall briefly describe a few flavours of contemporary geometric origami, from *kusudama* to tessellations and beyond. It will be an impressionistic and not technical presentation, just to give an idea of what can be done with geometric origami. However, we shall digress to present a way to trace a $d \times d$ square grid by folding only, and we shall touch upon the Kawasaki-Justin-Robertson theorem giving a necessary and sufficient condition for deciding when a sequence of folds produces a flat, 2-dimensional model — at least in theory...

1 Introduction

What is origami? The standard answer is "paper folding"; and this immediately brings to mind paper hats, maybe paper cranes or possibly (after having seen *Blade Runner* too many times) paper unicorns. But origami is much more than that. A better answer might be "the art of paper folding"; but after the impetuous development of origami in the last twenty-five years or so an even better answer is "the art and science of paper folding". Indeed, while the aim still is to realize beautiful objects (and thus it is an art), the techniques and methods required for planning the folds needed for reaching the desired goal have evolved so much to give rise to a (mostly mathematical) theory, up to the point that now congresses on the theory of origami are regularly held (see, e.g., [1] for the proceedings of a recent one).

A typical way (though not the only one) for creating a complex origami model starts with choosing the subject of the piece: an insect, a man reading a newspaper, a biplane, a heart with wings, a fractal... contemporary origami techniques allow the folding of practically any subject. Then the creator puts on the scientist hat and, using the theories and techniques available, plans (sometimes even using computer

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programs) the sequence of folds needed to get a suitable *base*, i.e., a still mostly abstract form close enough to the subject of the piece, with the right number and relative dimensions of flaps, appendices, points, etc. Then the artist hat comes on. First, there is the very important step of choosing the right paper: the color, the strength, the thinness are all elements that will greatly change the final appearance of the piece (and sometimes even the possibility of actually folding it). After having chosen the paper, the true folding starts; it might last a few minutes, or a few hours, or sometimes even days or months, depending on the difficulty of the model. The origamist follows the plan laid out in the preliminary theoretical step, and if everything goes well (which does not always happen: the paper might tear, or it might be too thick, or the theoretical folds turn out to be impossible to perform in practice) s/he gets to a good approximation of the desired aim: arms, legs and heads are all in the right place and of the right length, the wings (of insects, planes or hearts) are open, and so on. A last step remains, and the artist hat comes handy again: the modelling. The series of folds, nudges and delicate creasing transforming the approximation in a real work of art — and it is at this stage that one can tell apart an (even proficient) origami practitioner from a origami artist: the model folded by the first one is right, the same model folded by the second one is beautiful. The pictures in this article will give you a very vague idea of how beautiful a origami model can be; a good web site where one can find amazing pictures of incredible models is [2].

2 A very brief history of origami

Origami consists in obtaining beautiful objects folding pieces of paper. The piece of paper usually is a square; but there are exceptions. Usually the model is folded starting from one piece of paper, but again there are exceptions, and actually we shall later discuss a particular form of origami where models are composed by tenths (and sometimes hundreds) of pieces of paper all folded in the same way. But the rule that every contemporary origamist respects is: no cuts[1]. The paper should be folded with respect, and cutting it would be disrespectful.

While it is known that paper was invented in China around 100 BC, and that paper making arrived in Japan via Korea around 550 AD, in the Arab world around 750 AD and in Europe via the Arabs around 1000 AD, it is not known where and when recreational paper folding first started. There are references to the use of folded paper for ceremonial or official use in Japan in the Heian era (794–1185 AD), but apparently the earliest sure references to recreational paper folding dates from the 1600 AD, both in Japan and in Europe. The famous Japanese book *Senbazuru Orikata* (How to fold one thousand cranes) published in 1797 seems at present to be the oldest known book about recreational paper folding, but it is very plausible that other books or printed material on the subject were available before.

[1] Traditional models sometimes require cuts; but they were already rare fifty years ago, and have disappeared from contemporary origami. If you like to cut paper, you might try *kirigami*, the Japanese art of cutting the paper to get tridimensional models, like the ones in pop-up books.

The use of the word “origami” (from the Japanese words “oru”, meaning “to fold”, and “kami”, meaning “paper”) to refer to recreational (as opposed to ceremonial) paper folding is relatively recent, replacing at the end of the Nineteenth century the less specific word “orikata” (“folded shapes”). The word “origami” became commonly used in the Western world around the end of the 1950s, when modern origami books started to be published in English, but it should be remarked that Spanish countries refer to origami using the word “papiroflexia”, proposed by Vicente Solorzano Sagredo in Argentina in the 1930s.

The permanence of a different word in the Spanish world probably is due to the influence of a precursor of modern origami, the philosopher and poet Miguel Unamuno (1864-1936), that went beyond the traditional models introducing new techniques and devising his own style of folding, inspiring many followers in Spain and Argentina. But the man that actually gave birth to modern origami is Akira Yoshizawa (1911-2005), not only because he devoted his entire life to origami devising many new techniques and creating hundreds of new models, but because he invented writing in the 1950s. Until then, the instructions for making origami models were mostly passed on by oral tradition, accompanied by just a few sketches, and this greatly limited the diffusion and sharing of models. Yoshizawa instead created a standard set of symbols for recording the sequence of folds needed to reproduce a model. His system was promptly adopted in the West with just a few modifications due to Robert Harbin and Samuel Randlett in the 1960s, and it is now the standard in all origami publications around the world. The adoption of a standardized method of recording the folding of origami models allowed for a much easier sharing of models and techniques among origami artists, and possibly as a consequence many new talents appeared (Fred Rohm, Neal Elias, Patricia Crawford in the US, Adolfo Cerceda and Ligia Montoya in Argentina, Kunihiko Kasahara and Toshie Takahama in Japan, to name just a few), the number of available origami models passed from about one hundred to thousands in a few years, and a tradition of sharing ideas, techniques and instructions among origami folders with no limitations due to nationality or language (or race, gender, religion or anything else) took hold and it is still very much alive nowadays.

But even with writing one can go so far on intuition only; most of the models created in the 1960s and 1970s were relatively simple, stylised pieces, that can be folded in no more than 30-40 steps. To go beyond that one needs a more systematic approach; one needs a theory. After some initial innovations introduced in the 1980s by John Montroll in the US and Jun Maekawa in Japan (and others: for the sake of brevity I unfortunately have to leave out many worthwhile names), in the 1990s a breakthrough occurred. A new technique (the *circle/river packing* method) was independently developed by Robert J. Lang in the US and Toshiyuki Meguro in Japan, giving a general theoretical and practical framework for producing bases with the desired shape. At more or less the same time, the “bug wars” took place. A group of brilliant Japanese origamists (besides Maekawa and Meguro there were at least Issei Yoshino, Sejji Nishikawa, Fumiaki Kawahata and Satoshi Kamiya, founding members of the Tanteidan origami group which has become one of the more influential in the world of origami — “Tanteidan” means “Detective group”, by

Fig. 1 *Allomyrina dichotoma. Opus 655*. Model created and folded by Robert J. Lang. (From www.langorigami.com. Reproduced with permission).

the way) and one American (Robert J. Lang) started challenging each other to create insects and arachnids as realistic as possible, with many legs, wings, antennas... and at the same time as beautiful as possible. This allowed to show that the theory that was being developed could actually be used to create amazing origami models with an incredible amount of detail and harmony. A selection of the models created in the bug wars can be found in [3, 4, 5]; see also Fig. 1 for one example.

In the following years other powerful general theories and techniques were developed, and the net result of this theoretical revolution is that now it is possible to model in origami essentially anything, from a space station to a Kraken assaulting a full-rigged ship, from all kinds of animals to all kinds of human figures; see, e.g., Fig. 2 for a spectacular example. There are hundreds of active origami creators, and tens of thousands of available models (the site [6] lists more than fifty thousands models, considering only models published with instructions) ranging from one fold (the famous one-crease elephant [7]) to hundreds of folds.

The main ideas of modern origami design are described in the highly influential book *Origami design secrets* by Robert J. Lang, so successful that it recently had a second updated edition [7]. On the other hand, it is not that easy to find reliable information about the history of origami, at least in English; a starting point are the sites [8, 9] collecting notes by David Lister, one of the main Western origami historian.

3 Geometric origami

The vast majority of origami models are figurative: they represent something. But there is an increasing minority of models which are abstract — mostly geometrical in nature. There were some geometrical models in traditional origami (typically stars or decorations) but in the last fifteen years or so we have seen a strong increase of the interest in this kind of origami (and in particular in *kusudama* and tessellations;

Fig. 2 *Il Capitan*. Model created and folded by Eric Joisel. (From *Eric Joisel – The Magician of Origami*, M. Yamaguchi ed., Origami House, Tokyo, 2010. Reproduced with permission).

see below), possibly spurred by the mathematical approach to origami described in the previous section. In 2018 a new book by Robert J. Lang appeared [10], laying out the theory behind the design of geometric origami (mostly tessellations), and this probably will start a new wave of geometric models.

Geometric origami can be two-dimensional or three-dimensional. Typical subjects for two-dimensional geometric origami are stars (see, e.g., [11]) and spirals (see, e.g., [12]); typical subjects for three-dimensional geometric origami are polyhedra. Platonic solids have been folded by many authors, but a particular mention goes to John Montroll that not only designed all Platonic solids and many Archimedean and not Archimedean polyhedra (Fig. 3 is a 9-sided bipyramid, for instance), but he also published a book [13] explaining in details how to design origami polyhedra (and including instructions for more than 50 different models). He also published a book called *Origami and Math* [14], containing the instructions for his famous 8×8 chess board folded starting from a single square (see Fig. 4).

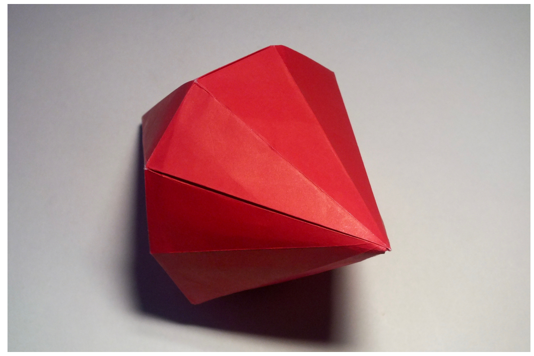

Fig. 3 *9-sided bipyramid.* Model created by John Montroll and folded by the author.

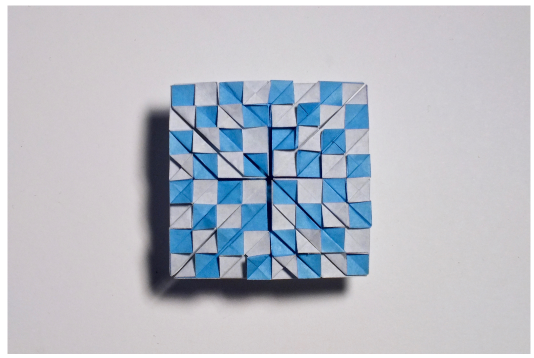

Fig. 4 8×8 *chess board.* Model created by John Montroll and folded by the author.

A different approach to polyhedra-like models is provided by another flavour of origami: *kusudama* (or *modular*[2]) origami. Kusudama origami models, instead of being made with a single piece of paper folded hundreds of times, are made by many (usually tenths, in a few cases hundreds) of small pieces of paper called *units*, usually all folded in the same way and joined together without glue inserting flaps into pockets created by the folding. The positions of the units often follow the geometry of a polyhedron, usually Archimedean or Platonic; the appearance and the beauty of the model mostly depend on the shape of the free (that is, not used for the joining) parts of the units and the color of the chosen paper — which in kusudama origami is even more important than usual. A good introductory book to kusudama origami is [15] by Ekaterina Lukasheva; Fig. 5 is one of her kusudama polyhedra.

Another flavour of geometric origami is *fractal origami.* A fractal origami model has a structure that can be repeated to smaller and larger scales, in principle *ab infinitum;* like a fractal, it is invariant under change of scale. Fractal origami are relatively rare; Fig. 6 shows a typical 3-D example by Jun Maekawa, and Fig. 7 a typical 2-D example by Shuzo Fujimoto. A somewhat atypical but exceptional example of modular 3-D fractal origami is the 4-level Menger sponge created by Serena Cicalò,

[2] Actually, modular origami is more general than kusudama, because a modular origami model can be formed with several different kinds of units and can be even figurative. Fig. 8 is an example of a (non-figurative) modular origami which is not kusudama.

Fig. 5 *Tiara variation*. Model created and folded by Ekaterina Pavlovic(Lukasheva) (From www.kusudama.me. Reproduced with permission).

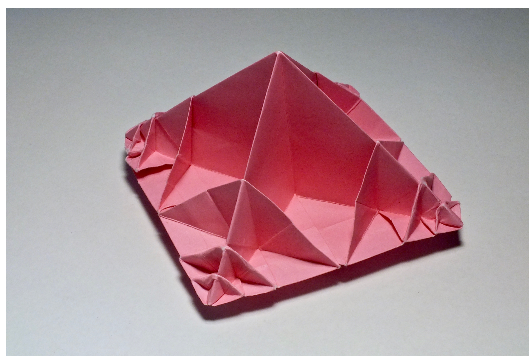

Fig. 6 *Pyramid*. Model created by Jun Maekawa and folded by the author.

larger than 1 meter and with a weight of about 25 kilograms, composed by hundred of thousands of different units of 9 different kinds; see Fig. 8 and [16].

4 A geometric interlude

In geometric origami it is often useful to fold a preliminary square grid on the paper (see, e.g., Fig. 16 where a 26×26 square grid is used). To fold a 2×2 square grid on a square of paper is easy: it suffices to fold in half the paper both in the horizontal and in the vertical directions. If on the paper we have two parallel lines, folding one line over the other produces a new line parallel to the previous two and exactly in the middle; using this trick one easily folds 4×4, 8×8 and in general $2^n \times 2^n$ square

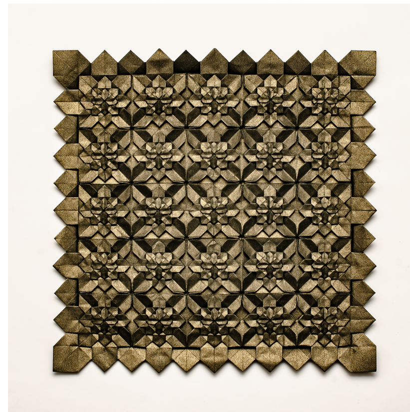

Fig. 7 *Hydrangea*. Model created by Shuzo Fujimoto and folded by Ed Sprake. (Reproduced with permission)

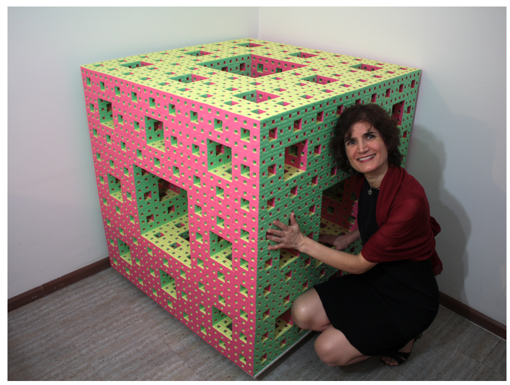

Fig. 8 *4-level Menger sponge*. Model created and folded by Serena Cicalò; photo by Francesco Finarolli. (From www.flickr.com/people/162102357@N06/ Reproduced with permission).

grids. What about $d \times d$ grids where d is not a power of 2? Taking measurements is (not precise and) not allowed; it should be achieved by folding only.

The answer (see, e.g., [13] where also other techniques are described) is surprisingly simple. Let us start by writing $d = 2^n + b$, where 2^n is the largest power of 2 less than d, and $0 \leq b < 2^n$. Assume for simplicity that the square of paper has length 1. Then:

(a) fold the diagonal from the upper left to the lower right corner;
(b) dividing by 2 horizontally enough times find the point distant $b/2^n$ from the lower right corner on the right side of the paper;
(c) trace the fold from the lower left corner to the point $b/2^n$ on the right side;
(d) fold vertically the paper across the intersection of the diagonal with the line created in the previous step;
(e) this vertical fold intersects the lower side in a point distant $2^n/d$ from the lower left corner; by dividing by 2 vertically one traces the lines whose distance from the vertical left side are $2^{n-1}/d$, $2^{n-2}/d$ and so on until one gets $1/d$;
(f) by repeately dividing by 2 one gets all the vertical lines of the $d \times d$ grid on the left of the line traced in step (d); to get the vertical lines on the right it suffices

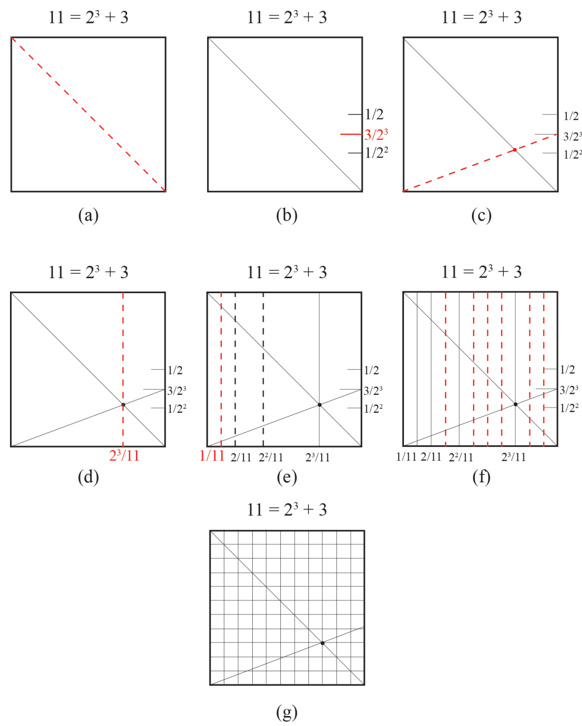

Fig. 9 How to fold an 11×11 grid.

to fold the right rectangle to the left along the $2^n/d$ vertical line and then fold it along the vertical lines already present in the left rectangle;

(g) to complete the grid it suffices to fold horizontal lines across the intersections between the diagonal and the vertical lines.

Fig. 9 shows the procedure for $d = 11 = 2^3 + 3$.

It is not difficult to prove that this procedure is exact. Choose a system of coordinates with origin in the lower left vertex of the square. The segment going from the origin to the point of coordinates $(1, b/2^n)$ identified in step (b) is parametrized by $(t, tb/2^n)$ for $0 \le t \le 1$. The diagonal folded in step (a) is parametrized by $(s, 1-s)$ for $0 \le s \le 1$. The intersection point should satisfy $(t, tb/2^n) = (s, 1-s)$, that is $t = s$ and

$$t\frac{b}{2^n} = 1 - t \quad \Longrightarrow \quad t = \frac{2^n}{b+2^n} = \frac{2^n}{d} \, .$$

Since t is the x-coordinate of the intersection point, the vertical line folded in (d) passes through the point $(2^n/d, 0)$, as claimed.

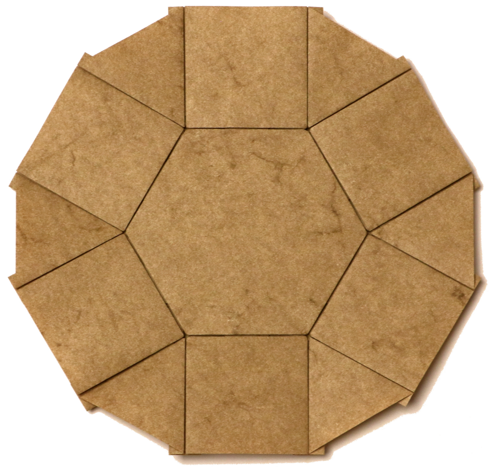

Fig. 10 *3.4.6.4 flagstone tessellation.* Model created and folded by Robert J. Lang. (From *Twists, Tilings, and Tessellations: Mathematical Methods for Geometric Origami,* CRC Press, Boca Raton, 2018. Reproduced with permission).

5 Tessellations

In the last twenty years or so another form of geometric origami has developed extensively: *tessellations.* The definition of origami tessellation in [10] is the following: an origami representation of a dissection of the plane into geometric patterns where the borders are formed by folded edges and/or variations in the number of layers. Usually, a tessellation is a periodic pattern that in principle can cover the entire plane, limited only by the dimension of the piece of paper (but there are also examples of aperiodic tessellations). Tessellations can be 2-dimensional (see, e.g., Fig. 10) or 3-dimensional (see, e.g., Fig. 11).

The study of tessellations in origami started in the 1970s with the work of Shuzo Fujimoto (Fig. 7 can be repeated to form a tessellation) and Yoshihide Momotani (see Fig. 13), but it developed in earnest in the 1990s thanks to the explorations conducted by Paulo Taborda Barreto and Chris K. Palmer. Nowadays there are many origami artists creating new beautiful tessellations (e.g., Alex Bateman, Joel Cooper, Ilan Garibi, Eric Gjerde, Ekaterina Lukasheva, Robin Scholz, and many others; among them I am pleased to mention the Italian origamist Alessandro Beber, who has an article in this volume with more about tessellations).

Tessellations differ from ordinary origami also in the way they are folded. Ordinary origami mostly uses an approach to folding that I will call *local*: any given fold affects only the paper close to it and can be done by itself, or at worst involving a handful of other folds to be performed together — and this property is important for the diagramming system introduced by Yoshizawa. On the other hand origami tessellations very often require *global* foldings: each fold affects the whole sheet of paper and all folds should be done at the same time for the model to form.

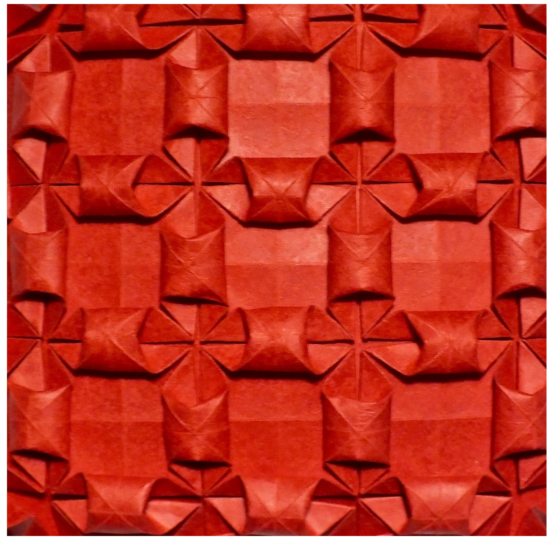

Fig. 11 *Red flower*. Model created by Ilan Garibi and folded by the author.

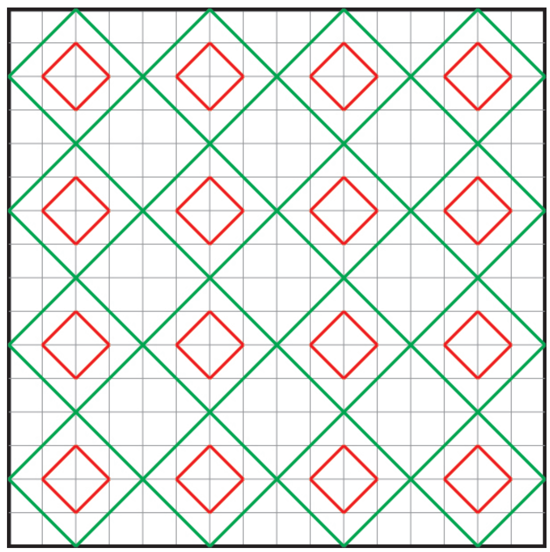

Fig. 12 Crease pattern for *Red flower* on a 16×16 grid.

As a consequence, instructions for tessellations are almost always presented not as a sequence of steps but as a *crease pattern*: a diagram showing in one picture all the creases needed to fold the model. Crease patterns are also used to present complex figurative origami, but in that case the crease pattern usually contains the most important creases only; the crease pattern of a tessellation contains instead all

Fig. 13 *Momotani's wall*. Model created by Yoshihide Momotani and folded by the author.

creases needed to fold the model. For instance, Fig. 12 contains the crease pattern[3] for the model shown in Fig. 11.

Besides being useful for folding origami, crease patterns are important because they show the underlying geometrical structure of the model, and they are suitable for a mathematical treatment of origami (see, e.g., [10]). For instance, a much studied problem, with several applications to tessellations, is the following: when a crease pattern can be folded so that the final model is flat (in short: when a crease pattern can be flat folded)?

It turns out that there is a complete theoretical answer to this question. A *vertex* in a crease pattern is a point where several creases meet. For the whole crease pattern to be flat foldable, at the very least it should be flat foldable near any of its vertices. There is a necessarily and sufficient condition for a vertex to be flat foldable:

Theorem 1 (Kawasaki-Justin-Robertson theorem). *An interior vertex of a crease pattern is flat foldable if and only if the alternate sum of the angles between the creases meeting at the vertex vanishes.*

This theorem has been first proved by S.A. Robertson in 1978 [17], and then it has been independently rediscovered by J. Justin [18] and T. Kawasaki [19] in the 1980s. Fig 13 is an example of a flat foldable tessellation; indeed its crease pattern (Fig. 14) satisfies the Kawasaki-Justin-Robertson condition at each internal vertex.

The proof of Theorem 1 is not difficult; it essentially boils down to the remark that if the vertex is flat-folded then if we walk a circle around the vertex when we cross a fold the paper changes orientation: if before the fold one side is up after the fold that side is down. This change of orientation can be recorded by changing the sign of the angle between two consecutive creases, because the change of orientation

[3] Green lines represent mountain folds and red lines represent valley folds. This is not Yoshizawa's convention but it makes the crease pattern more readable.

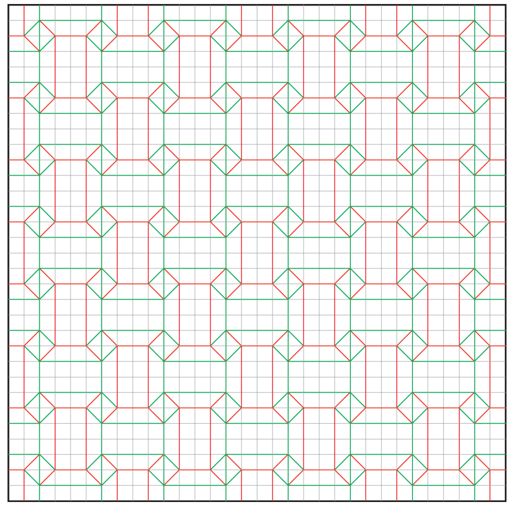

Fig. 14 Crease pattern for *Momotani's wall* on a 32×32 grid. It satisfies the Kawasaki-Justin-Robertson condition, and indeed the model is flat.

corresponds to a change in the direction of the walk. Since completing the walk around the vertex we come back to the starting point, necessarily the alternate sum of the angles should be zero.

Somewhat surprisingly, if all internal vertices are flat-foldable, that is if they satisfy the condition described by Kawasaki-Justin-Robertson theorem, then the whole crease pattern is *theoretically* flat foldable. This has been proved by Dacorogna, Marcellini and Paolini in 2008 (see [20, 21, 22]) by using techniques coming from the study of partial differential equations. However, I have emphasised "theoretically" because there are a few *caveat* to be considered.

First of all, a real piece of paper is finite, and so a real crease pattern contains also boundary vertices, points where the creases intersect the boundary of the paper. Flat foldable boundary vertices do not necessarily satisfy Kawasaki-Justin-Robertson theorem, and so they have to be treated differently (and indeed Dacorogna, Marcellini and Paolini spend some time discussing suitable boundary conditions).

But the main reason why this is only a theoretical result is that it does not take into account the fact that, during folding, the paper cannot self-intersect. The map produced by Dacorogna, Marcellini and Paolini might not be actually realisable because folding it might require for the paper to cross itself. Understanding for which crease patterns this does not happen is a global problem, very delicate, very complex and not yet completely solved (see, e.g., [10] for a discussion).

The third reason why this is only theoretical is that actual paper has thickness, and elasticity, and memory, and thus it does not always behave like an abstract plane figure. From an artistic point of view, this can be an advantage, as shown for instance in a sub-class of tessellations called *corrugations,* where the pattern is obtained without overlapping paper but playing with 3D effects created by not flattening the folds; see, e.g., Fig. 15, whose crease pattern is in Fig. 16.

A good introductory book on tessellations and corrugations is [23].

Fig. 15 *Childhood*. Model created by Ilan Garibi and folded by the author.

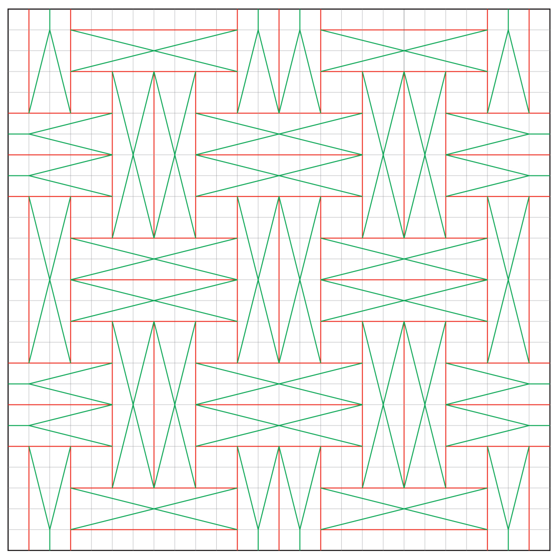

Fig. 16 Crease pattern for *Childhood* on a 26×26 grid. It does not satisfy the Kawasaki-Justin-Robertson condition, and indeed the model is not flat.

6 And more...

In this short note we have only scratched the surface of contemporary geometric origami. Putting together mathematics and art has led (and it is still leading) to the creation of models that are both artistically and mathematically beautiful and that could not even be imagined before. Figg. 17 and 18 are just two particularly striking examples among many of what can be obtained putting together techniques coming

Fig. 17 *Double diagonal shift vase 2*. Model created and folded by Rebecca Gieseking (From www.flickr.com/people/rgieseking/ Reproduced with permission).

Fig. 18 *Oread*. Model created and folded by Joel Cooper (From www.flickr.com/people/origamijoel/ Reproduced with permission).

from geometric origami with techniques coming from figurative origami. As already mentioned, much more information, both mathematical and otherwise, can be found in [7, 10]; I hope that this short introduction will entice you toward this beautiful subject. Art is just a fold away…

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