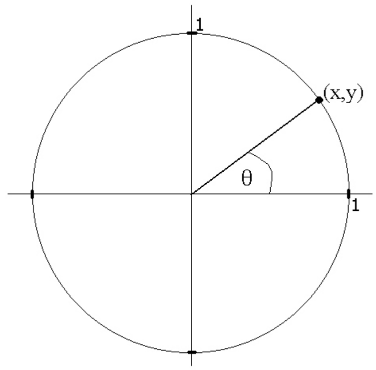
<u>An Introduction to Trigonometry</u>

P.Maidorn

I. Basic Concepts

The trigonometric functions are based on the unit circle, that is a circle with radius $r=1$. Since the circumference of a circle with radius r is $C=2\pi r$, the unit circle has circumference 2π.



For any point (x,y) on the unit circle, the associated angle θ can be measured in two different ways:

1. degree measure: in this case the circumference is divided into 360 equal parts, each part has measure one degree (written 1^o). A right angle, for example, is a 90^o angle. Positive angles are measured in the counter-clockwise direction.
2. radian measure: radian measure is defined as the actual length of the arc between the points (1,0) and (x,y). One entire revolution (i.e. 360^o) hence has a radian measure of 2π. A right angle (that is a quarter of one revolution) would have radian measure $\pi/2$. Note that the angle is simply denoted "$\pi/2$", not "$\pi/2$ radians".

One can easily convert between these two measures by keeping in mind that a 180^o angle (in degrees) is equivalent to a π angle (in radians). Note that angles in Calculus-related problems are usually denoted in radian measure, hence it is important to be comfortable with this measurement.

<u>Examples:</u>

1. A 270^o angle is 3/2 times a 180^o angle, hence in radian measure the angle would be denoted $3\pi/2$.
2. A $7\pi/5$ angle would simply have degree measure 7/5 times 180^o, i.e. 252^o.

Exercises:

Convert each angle to radians.

a) 120º b) 315º c) -420º

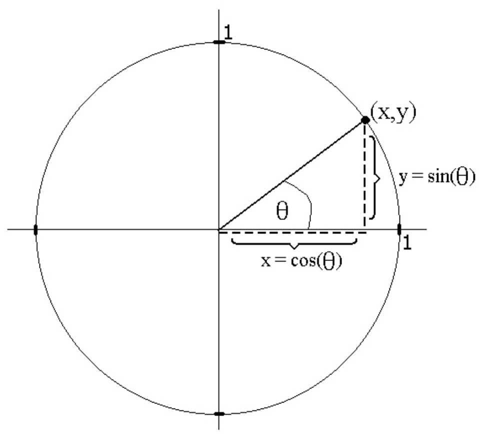
Convert each angle to degrees, to the nearest tenth of a degree.

d) -2π/3 e) 3π f) 4.52

Turning the above discussion around, each time we choose an angle θ, we find a unique point (x,y) on the unit circle. Hence both "x" and "y" can be considered functions of θ. Since these particular functions are of great importance to both pure and applied mathematics, they are given special names and symbols, and are called the trignometric functions.

Specifically:

The length "y" is called the sine of the angle θ, and is denoted by y=sin(θ).
The length "x" is called the cosine of the angle θ, and is denoted by x=cos(θ).

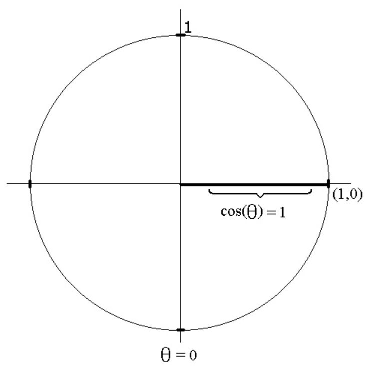


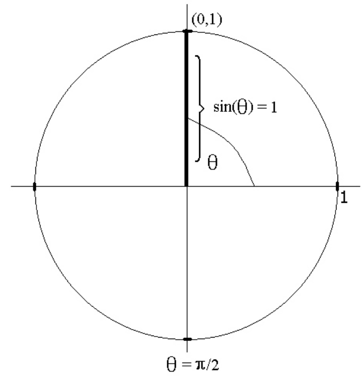
Other trigonometric functions can be calculated from the sine and cosine functions: the tangent of θ is defined as tan(θ)=sin(θ)/cos(θ) (or y/x), the secant of θ is defined as sec(θ)=1/cos(θ), the cosecant of θ is defined as csc(θ)=1/sin(θ), and the cotangent of θ is defined as cot(θ)=cos(θ)/sin(θ).

II. Calculating Trigonometric Functions of Special Angles

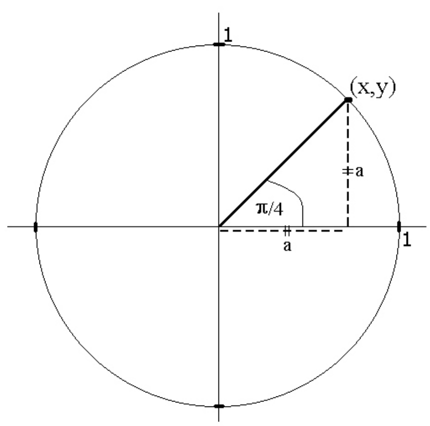
The first question that arises is how to calculate the sine or cosine of a given angle, that is how to find the lengths "x" (the cosine) and "y" (the sine) on the unit circle associated with a given angle.

We will begin to answer this question by looking at the angles 0º and 90º (or π/2). First, draw the unit circle, and on it indicate the angle θ=0 as well as the point (x,y) that is associated with that angle. If the angle is θ=0, the point (x,y) lies on the x-axis, that is x=1, and y=0 (remember that the radius of the circle is r=1). Hence cos(0)=1 and sin(0)=0. Similarly, the angle θ=π/2 is associated with the point (x,y)=(0,1). Therefore cos(π/2)=0 and sin(π/2)=1 (see diagrams).





Let's examine the angle θ=π/4 (or 45º) next.



Note that a right-angle triangle is formed, with a hypotenuse of length 1, and two adjacent sides of equal length, that is x=y. Let's denote that length "a". By the Pythagorean theorem, we have

$$a^2+a^2=1^2,$$

which we can solve for a=√½ or equivalently a=√2/2. Hence both "x" and "y" are equal to √2/2, and we have found that both sin(π/4)=√2/2 and cos(π/4)=√2/2.

One can also find trigonometric values for the angles θ=π/6 (or 30º) and θ=π/3 (or 60º). This set of angles is sometimes called the "special" angles, and their associated sine and cosine values are listed in the table below:

θ	**sin(θ)**	**cos(θ)**
0	0	1
π/6	½	√3/2
π/4	√2/2	√2/2
π/3	√3/2	½
π/2	1	0

We can immediately use these values to calculate other trigonometric functions of these special angles.

Examples:

1. Since tan(θ)=y/x, i.e. tan(θ)=sin(θ)/cos(θ), we simply divide sin(π/3) by cos(π/3) to find that tan(π/3) = √3.
2. Similarly csc(π/6) = 1 / sin(π/6), that is csc(π/6) = 2.

Note that not all trigonometric functions are defined for all angles. For example, the tangent of θ=π/2 does not exist, since here the denominator is equal to zero.

Exercises:

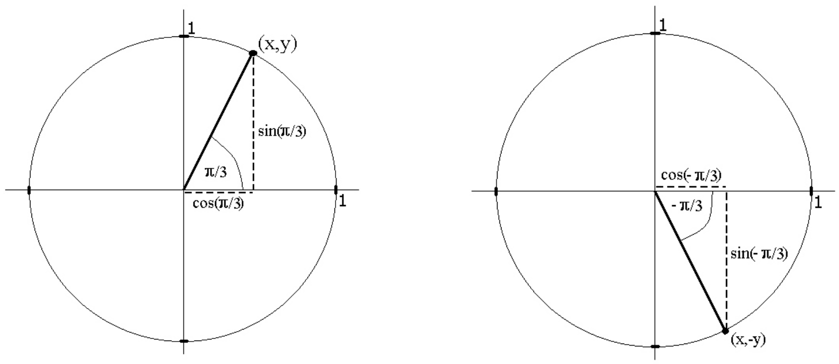
Calculate:

a) sec(π/3) b) csc(45º) c) cot(0)

III. Other Angles

Once you know how to find the trigonometric functions for the above special angles, it is important to learn how to extend your knowledge to any angle that is based on one of 0, π/6, π/4, π/3 or π/2, such as for example 2π/3, -π/6, 7π/4, -5π/2, and others.

Let's examine the angle θ=-π/3. Clearly, it is somehow related to the angle π/3. Draw a unit circle for both angles side-by-side, and indicate the sine and cosine on them:

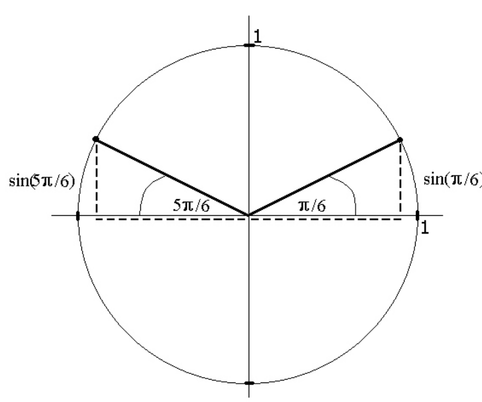


Clearly, the right-angle triangles that are formed are identical, except that they are mirror images of each other. The cosine ("x") in both cases is the same, hence we know that cos(-π/3) is identical to cos(π/3), that is cos(-π/3)=½. The sine ("y") is the same length, but has opposite sign (it is negative). Since sin(π/3)=√3/2, then sin(-π/3)=-√3/2.

In each of the examples below, proceed with the same method. It is imperative that you draw the unit circle each time until you become comfortable with these types of questions.

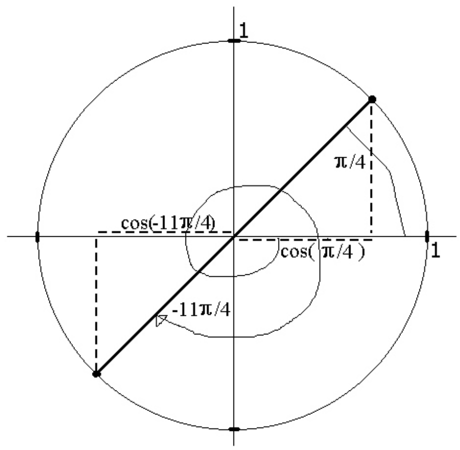
Examples:

1. Find sin(5π/6). The triangle formed by θ=5π/6 is identical to that formed by θ=π/6, except that it is reflected across the y-axis. The sine of 5π/6 is hence the same length, and has the same sign, as the sine of π/6. Therefore sin(5π/6)=½.



2. Find sec(-11π/4). To find the secant of an angle, remember to find the cosine first and take its reciprocal. The cosine of -11π/4 is related to the cosine of π/4. The angle -11π/4 is reached by completing one entire clockwise revolution (which equals -2π, or -8π/4) and then adding another -3π/4. Compare the angle -3π/4 to π/4. The cosine is the same length in each case, but has opposite sign. Since cos(π/4) = √2/2, cos(-11π/4) = -√2/2, and hence

sec(-11π/4) = -2/√2 = - √2.



Exercises:

Calculate

a) sin(-π/4) b) sin(5π/2) c) cos(11π/6)

d) cos(13π/4) e) cos(19π/6) f) sin(-510°)

IV. Using the Calculator

If you need to calculate the sine, cosine, or tangent of an angle other than the ones discussed above, you may need to use your calculator. Consult your calculator manual, if necessary, on how to use the trigonometric functions.

You should be aware of two facts:

1. In most cases, your calculator will not give you exact answers, but rather decimal approximations. For example, your calculator will tell you that the sine of a 45° angle is approximately .70710678, rather than giving you the exact answer √2/2.

2. You need to set your calculator to the appropriate angle measure, degrees or radians. Otherwise, your calculator might return the sine of the angle θ≈3.1415... degrees, when you enter "sin(π)" looking for the sine of θ=180°.

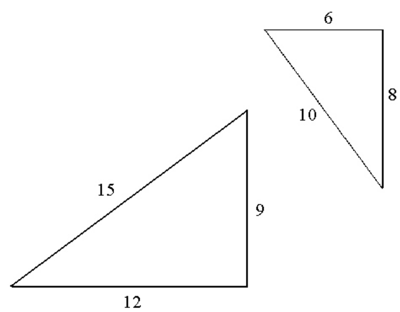
Exercises:

Use your calculator to compute, rounding to four decimal places:

a) $\sin(53.7^o)$ b) $\cos(11\pi/7)$ c) $\csc(-32^o)$

V. Right-Triangle Applications

In order to apply the trigonometric functions based on the unit circle to right triangles of any size, it is important to understand the concept of similar triangles. Two triangles are said to be similar if the ratio of any two sides of one triangle is the same as the ratio of the equivalent two sides in the other triangle. As a result, similar triangles have the same "shape", but might differ in size. For example, the sides in the triangles below have the same ratios to each other.



Similar Triangles

Consider the right triangle inscribed in the unit circle associated with an angle θ. We can calculate the length of the side adjacent to the angle θ (i.e. $\cos(\theta)$) and the length of the side opposite the angle θ (i.e. $\sin(\theta)$). Since the unit circle has radius one, the hypotenuse of these triangles is always equal to one.

If we were given a triangle with identical angle θ but with a hypotenuse twice the length, each of the other sides would be twice the length as well, as the triangles are similar. We can use this fact to now compute side-lengths of any right triangle, if the angle θ and one side-length are known. In general, we have

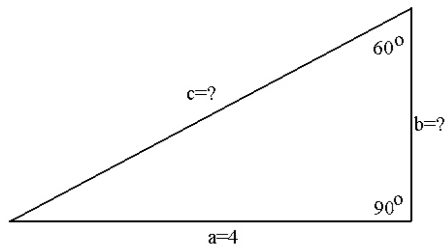
$\sin(\theta)$ = opposite / hypotenuse and $\cos(\theta)$ = adjacent / hypotenuse

and since $\tan(\theta)=\sin(\theta)/\cos(\theta)$, we have

$\tan(\theta)$ = opposite / adjacent.

Examples:

1. Find all missing sides and angles in the given triangle.



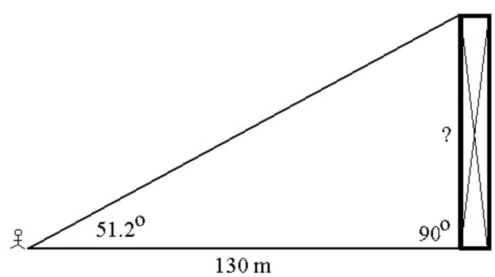
First, given that the sum of all angles in any triangle must equal 180° (or π), the missing angle measures 30°. We know the length of the opposite side of angle θ, hence we can use the sine to find the length of the hypotenuse. Since

$\sin(60^\circ) = \sqrt{3}/2$ and $\sin(\theta)$ = opposite / hypotenuse

we have $\sqrt{3}/2 = 4 / c$, which we can solve for $c = 8/\sqrt{3}$ or $c = 8\sqrt{3} / 3$.

Finally use, for example, the Pythagorean theorem to find that $b = 4\sqrt{3} / 3$.

2. To find the height of a building, a person walks to a spot 130m away from the base of the building and measures the angle between the base and the top of the building. The angle is found to be 51.2°. How tall is the building?



We know the length of the adjacent side of the right triangle that is formed, and wish to find the length of the opposite side. The quickest method to do so is to use the tangent, since $\tan \theta$ = opposite / adjacent. Using the calculator, $\tan(51.2^\circ) \approx 1.2437$. Hence the height of the building is

$$h = 130 \tan(51.2^\circ) \approx 161.7 \text{ metres.}$$

Exercises:

a) A ladder is leaning against the side of the building, forming an angle of 60º with the ground. If the foot of the ladder is 10m from the base of the building, how far up does the ladder reach, and how long is the ladder?

b) A person stationed on a 40m tall observation tower spots a bear in the distance. If the angle of depression (that is, the angle between the horizontal and the line of sight) is 30º, how far away is the bear, assuming that the land surrounding the tower is flat?

c) A 12m tall antenna sits on top of a building. A person is standing some distance away from the building. If the angle of elevation between the person and the top of the antenna is 60º, and the angle of elevation between the person and the top of the building is 45º, how tall is the building and how far away is the person standing?

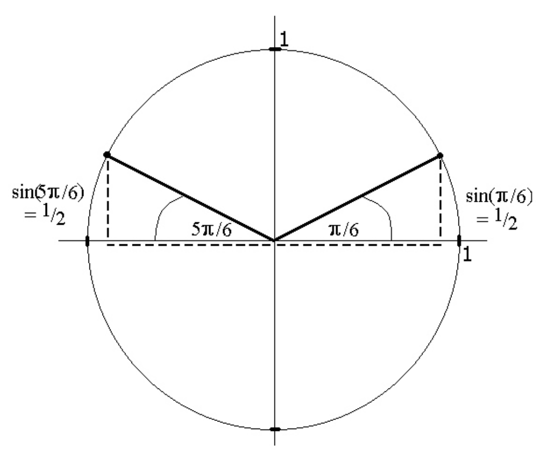
VI. Simple Trigonometric Equations

Given a value for the angle "x", we now know how to calculate sin(x), cos(x), etc. We need to proceed more carefully if however we wish to solve for the angle "x" given the value of sin(x) or cos(x).

Example: Find all values for "x" (in radian measure) such that sin(x)=½.

One such angle "x" can be immediately found from our table of special angles. Since sin(π/6)=½ , we know that one possible x-value is x=½. However, this is not the only possibility.

Consider the unit circle for $\theta_1=\pi/6$. Is it possible to find another angle such that the sine value (i.e. the height "y") is identical to that of $\theta_1=\pi/6$? The answer is 'yes'. Another such angle is on the other side of the y-axis. Since this angle is π/6 away from the angle π (180º), it measures $\theta_2 = \pi-\pi/6 = 5\pi/6$. Hence sin(5π/6) is also equal to ½ (check this on your calculator).



Are there any other such angles? By remembering that we can always add or subtract one complete revolution (2π) from an angle to end up in the same position, we can in fact generate infinitely many such angles. For example, $\theta_3 = \pi/6 + 2\pi = 13\pi/6$ is another such angle, as is $\theta_4 = 5\pi/6 - 4\pi = -19\pi/6$.

Hence all angles $\theta = \pi/6 + 2k\pi$ and $\theta = 5\pi/6 + 2k\pi$ for any $k \in I$ (that is k can be any integer ...-3,-2,-1,0,1,2,3,...) are solutions to the equation $\sin(x)=½$.

Exercises:

Find all "x" such that

a) $\cos(x) = \sqrt{3}/2$ b) $\cos(x) = -\sqrt{3}/2$ c) $\sin(x) = -1$

d) $\sin(x) = -½$ e) $\cos(x) = -½$ f) $\tan(x) = 1$

VII. Trigonometric Identities

There are several trigonometric identities, that is equations which are valid for any angle θ, which are used in the study of trigonometry.

From the unit circle, we have already seen that the cosine of an angle is identical to the cosine of the associated negative angle, that is

$$\cos(-\theta)=\cos(\theta) \quad \text{for any angle } \theta. \quad (1)$$

Similarly,

$$\sin(-\theta)=-\sin(\theta) \quad \text{for any angle } \theta. \quad (2)$$

For example, $\sin(\pi/6) = ½$ and $\sin(-\pi/6) = -½$.

Also from the unit circle, which has equation $x^2+y^2=1$, we can substitute $x=\cos(\theta)$ and $y=\sin(\theta)$ to obtain the identity

$$\sin^2(\theta)+\cos^2(\theta)=1, \quad \text{for any angle } \theta. \quad (3)$$

Another useful trigonometric identity concerns the sum of two angles A and B. We have:

$$\sin(A+B) = \sin(A)\cos(B) + \sin(B)\cos(A) \quad (4)$$

and

$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B) \quad (5)$$

for any angles A and B.

Note that you cannot simply "distribute" the sine through a sum. It is false to state that, for example, $\sin(A+B)=\sin(A) + \sin(B)$.

The above five identities can be used to derive many other useful identities, which then no longer need to be memorized.

Examples:

1. Dividing equation (3) by $\cos^2(\theta)$, we obtain the identity

$$1 + \tan^2(\theta) = \sec^2(\theta).$$

2. Identities involving the difference of two angles A-B can be obtained by combining equation (4) or (5) with equations (1) and (2). As an example,

$$\begin{aligned} \sin(A-B) &= \sin(A+(-B)) \\ &= \sin(A)\cos(-B) + \sin(-B)\cos(A) && \text{using (4)} \\ &= \sin(A)\cos(B) - \sin(B)\cos(A) && \text{using (1) and (2) on the left and right term respectively.} \end{aligned}$$

3. The double-angle formula $\cos(2\theta) = 2\cos^2(\theta) - 1$ can be derived using equation (5) and setting both $A=\theta$ and $B=\theta$:

$$\begin{aligned} \cos(2\theta) &= \cos(\theta + \theta) \\ &= \cos(\theta)\cos(\theta) - \sin(\theta)\sin(\theta) && \text{using (5)} \\ &= \cos^2(\theta) - \sin^2(\theta) \end{aligned}$$

Now, re-arrange (3) to obtain $\sin^2(\theta)=1-\cos^2(\theta)$ and substitute:

$$\begin{aligned} \cos(2\theta) &= \cos^2(\theta) - (1-\cos^2(\theta)) \\ &= 2\cos^2(\theta) - 1. \end{aligned}$$

4. Find all "x" such that $2\sin^2(x) + \cos(x) = 1$.

To solve this trigonometric equation, we will first need to simplify it using the trigonometric identities. Using (3) we can substitute $\sin^2(x)=1-\cos^2(x)$ and obtain an equation that only involves cosines:

$$2 - 2\cos^2(x) + \cos(x) = 1$$

or

$$2\cos^2(x) - \cos(x) - 1 = 0.$$

The above is a quadratic equation in cos(x), and can be factored

$$(2\cos(x)+1)(\cos(x) - 1) = 0.$$

As a result, we now need to find all "x" such that either

$2\cos(x) + 1 = 0$, i.e. $\cos(x)=-½$,

or $\cos(x)-1 = 0$, i.e. $\cos(x)=1$.

From the unit circle, $\cos(x)=1$ when $x= 0 + 2k\pi, k\in I$, i.e. $x=2k\pi, k\in I$

Further, $\cos(x)=-½$ when $x=2\pi/3+2k\pi$ or $x=4\pi/3+2k\pi$, $k\in I$.

The solution to the given equation is hence the set of all x-values listed above.

Exercises:

Derive the following equations using equations (1) through (5) only:

a) cos(A-B) = cos(A)cos(B) + sin(A)sin(B)

b) sin(2A) = 2sin(A)sin(B)

c) sin(A)sin(B) = -½ [cos(A+B) - cos(A-B)]

Solve the following equations for "x":

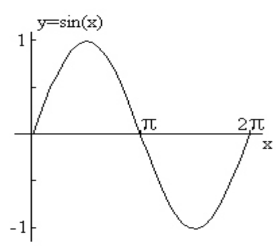
d) 1 - cos(2x) = 2sin(x)

e) $\sin^2(5x^3-2x^2+1) = 1 - \cos^2(5x^3-2x^2+1)$

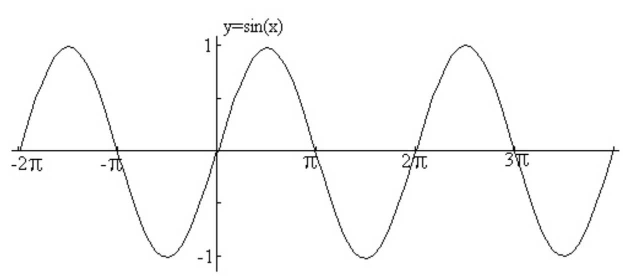
VIII. Graphing Trigonometric Functions

Recall that a function y=f(x) is a rule of correspondence between the independent variable ("x") and the dependent variable ("y"), such that each x-value is associated with one and only one y-value. Since each angle "x" produces only one value for sin(x), cos(x), etc., the relationships y = sin(x), y = cos(x), etc. are functions of "x".

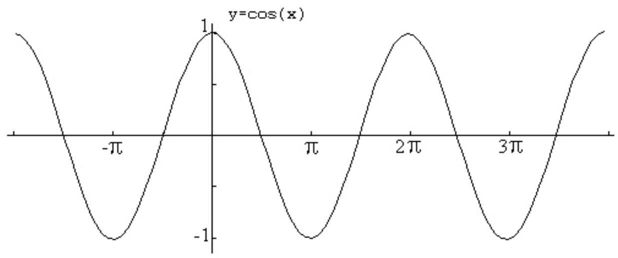
To obtain the graph of the function y = sin(x), trace how the height of the triangle inscribed in the unit circle changes as the angle "x" gradually moves from x=0 to x=2π. For example, the sine graph will start at (0,0) since the sine of zero is zero, obtain a maximum value at (π/2, 1), since the sine of π/2 equals one, then decrease towards (π,0).



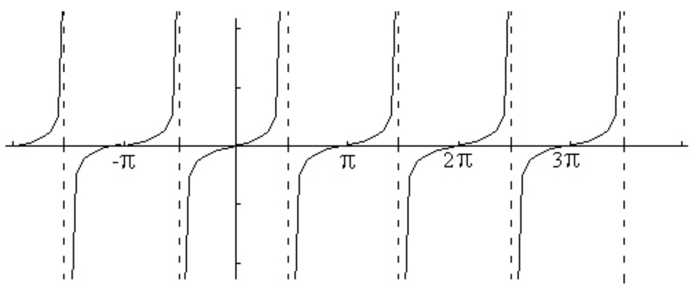
Since the graph of y = sin(x) will repeat as we complete more than one revolution, we can now obtain the complete graph of the real valued function y = sin(x).



The graph of y = cos(x) is obtained in a similar fashion.



To obtain the graph of y = tan(x), divide sin(x)/cos(x) as before. Note again that tan(x) does not exist for values of $x=\pi/2 + k\pi$, $k\in I$.

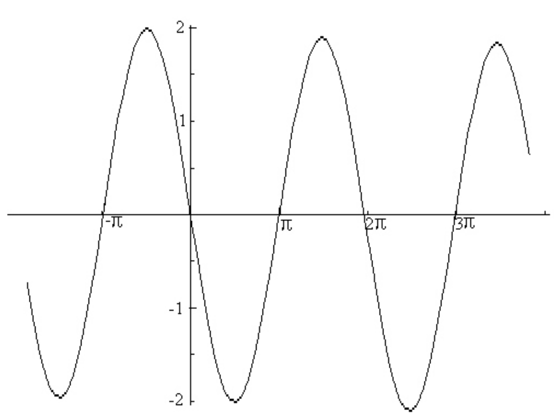


You can now use the techniques for shifting and scaling function graphs to obtain the graphs for any trigonometric function.

Examples:

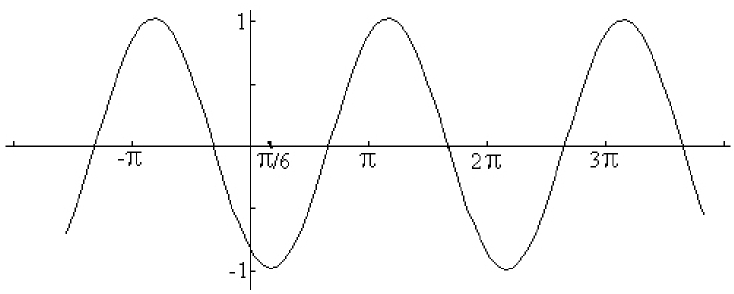
1. Graph $y = 2\sin(x+\pi)$.

This graph is identical to that of y=sin(x), except it is shifted to the left by π units, and scaled vertically by a factor of 2.



2. Graph y = - cos(x-π/6).

This graph is obtained by shifting the graph of y=cos(x) to the right by π/6 units, then flipping it across the x-axis.



<u>Exercises:</u>

Sketch graphs for the given functions:

a) y = cos(x+π/6) - ½

b) y = ½ sin(x-π/2) + 1

Answers to Exercises

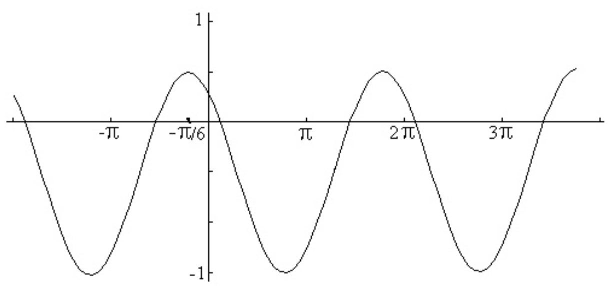
I.	a) $2\pi/3$	b) $7\pi/4$	c) $-7\pi/3$	d) -120°	e) 540°	f) 259.0°
II.	a) 2	b) $\sqrt{2}$	c) Does not exist			
III.	a) $-\sqrt{2}/2$	b) 1	c) $\sqrt{3}/2$	d) $-\sqrt{2}/2$	e) $-\sqrt{3}/2$	f) $-½$
IV.	a) .8059	b) .2225	c) -1.8871			

V. a) The ladder is 20m long and reaches up approximately 17.32m.
b) The bear is approximately 69.28m away.
c) The height of the building and the distance of the observer are both about 20.49m.

VI. a) $\pi/6 + 2k\pi$ and $11\pi/6 + 2k\pi$
b) $5\pi/6 + 2k\pi$ and $7\pi/6 + 2k\pi$
c) $3\pi/2 + 2k\pi$
d) $7\pi/6 + 2k\pi$ and $11\pi/6 + 2k\pi$
e) $2\pi/3 + 2k\pi$ and $4\pi/3 + 2k\pi$
f) $\pi/4 + k\pi$

VII. d) $k\pi$ and $\pi/2 + 2k\pi$
e) all x-values

VIII. a)



b)