Understanding algebra

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UNDERSTANDING ALGEBRA

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FORWARD

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James W. Brennan
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Boise, Idaho

Chapter 1: The Numbers of Arithmetic

The Real Number System

The real number system evolved over time by expanding the notion of what we mean by the word "number." At first, "number" meant something you could count, like how many sheep a farmer owns. These are called the *natural numbers*, or sometimes the *counting numbers*.

Natural Numbers

or "Counting Numbers"

1, 2, 3, 4, 5, . . .

- The use of three dots at the end of the list is a common mathematical notation to indicate that the list keeps going forever.

At some point, the idea of "zero" came to be considered as a number. If the farmer does not have any sheep, then the number of sheep that the farmer owns is zero. We call the set of natural numbers plus the number zero the *whole numbers*.

Whole Numbers

Natural Numbers together with "zero"

0, 1, 2, 3, 4, 5, . . .

About the Number Zero

What is zero? Is it a number? How can the number of nothing be a number? Is zero nothing, or is it something?

Well, before this starts to sound like a Zen koan, let's look at how we use the numeral "0." Arab and Indian scholars were the first to use zero to develop the place-value number system that we use today. When we write a number, we use only the ten numerals 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9. These numerals can stand for ones, tens, hundreds, or whatever depending on their position in the number. In order for this to work, we have to have a way to mark an empty place in a number, or the place values won't come out right. This is what the numeral "0" does. Think of it as an empty container, signifying that that place is empty. For example, the number 302 has 3 hundreds, no tens, and 2 ones.

So is zero a number? Well, that is a matter of definition, but in mathematics we tend to call it a duck if it acts like a duck, or at least if it's behavior is for the most part duck-like. The number zero obeys *most* of the same rules of arithmetic that ordinary numbers do, so we call it a number. It is a rather special number, though, because it doesn't quite obey all the same laws as other numbers—you can't divide by zero, for example.

Note for math purists: In the strict axiomatic field development of the real numbers, both 0 and 1 are singled out for special treatment. Zero is the *additive identity*, because adding zero to a number does not change the number. Similarly, 1 is the *multiplicative identity* because multiplying a number by 1 does not change it.

Even more abstract than zero is the idea of negative numbers. If, in addition to not having any sheep, the farmer owes someone 3 sheep, you could say that the number of sheep that the farmer owns is negative 3. It took longer for the idea of negative numbers to be accepted, but eventually they came to be seen as something we could call "numbers." The expanded set of numbers that we get by including negative versions of the counting numbers is called the *integers*.

INTEGERS

Whole numbers plus negatives

. . . –4, –3, –2, –1, 0, 1, 2, 3, 4, . . .

About Negative Numbers

How can you have less than zero? Well, do you have a checking account? Having less than zero means that you have to add some to it just to get it up to zero. And if you take more out of it, it will be even further less than zero, meaning that you will have to add even more just to get it up to zero.

The strict mathematical definition goes something like this:

> For every real number n, there exists its *opposite*, denoted $-n$, such that the sum of n and $-n$ is zero, or
>
> $n + (-n) = 0$
>
> Note that the negative sign in front of a number is part of the symbol for that number: The symbol "–3" is one object—it stands for "negative three," the name of the number that is three units less than zero.
>
> The number zero is its own opposite, and zero is considered to be neither negative nor positive.
>
> Read the discussion of subtraction for more about the meanings of the symbol "–."

The next generalization that we can make is to include the idea of fractions. While it is unlikely that a farmer owns a fractional number of sheep, many other things in real life are measured in fractions, like a half-cup of sugar. If we add fractions to the set of integers, we get the set of *rational numbers*.

RATIONAL NUMBERS

All numbers of the form $\frac{a}{b}$, where a and b are integers (but b cannot be zero)

Rational numbers include what we usually call *fractions*

- Notice that the word "rational" contains the word "ratio," which should remind you of fractions.

The bottom of the fraction is called the *denominator*. Think of it as the *denomination*—it tells you what size fraction we are talking about: fourths, fifths, etc.

The top of the fraction is called the *numerator*. It tells you the *number*. That is, how many fourths, fifths, or whatever.

- **RESTRICTION**: The denominator cannot be zero! (But the numerator can)

  If the numerator is zero, then the whole fraction is just equal to zero. If I have zero thirds or zero fourths, than I don't have anything. However, it makes no sense at all to talk about a fraction measured in "zeroths." How can you divide something up into pieces of zero size?

- Fractions can be numbers smaller than 1, like 1/2 or 3/4 (called *proper fractions*), or they can be numbers bigger than 1 (called *improper fractions*), like two-and-a-half, which we could also write as 5/2

All integers can also be thought of as rational numbers, with a denominator of 1:

$$3 = \frac{3}{1}$$

This means that all the previous sets of numbers (natural numbers, whole numbers, and integers) are subsets of the rational numbers.

Now it might seem as though the set of rational numbers would cover every possible case, but that is not so. There are numbers that cannot be expressed as a fraction, and these numbers are called *irrational* because they are not rational.

IRRATIONAL NUMBERS

- Cannot be expressed as a ratio of integers.
- As decimals they never repeat or terminate (rationals always do one or the other)

Examples:

$\frac{3}{4} = 0.75$	Rational (terminates)
$\frac{2}{3} = 0.66666\overline{6}$	Rational (repeats)
$\frac{5}{11} = 0.454545\overline{45}$	Rational (repeats)
$\frac{5}{7} = 0.714285\overline{714285}$	Rational (repeats)
$\sqrt{2} = 1.41421356\ldots$	Irrational (never repeats or terminates)
$p = 3.14159265\ldots$	Irrational (never repeats or terminates)

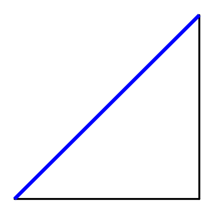
More on Irrational Numbers

It might seem that the rational numbers would cover any possible number. After all, if I measure a length with a ruler, it is going to come out to some fraction—maybe 2 and 3/4 inches. Suppose I then measure it with more precision. I will get something like 2 and 5/8 inches, or maybe 2 and 23/32 inches. It seems that however close I look it is going to be *some* fraction. However, this is not always the case.

Imagine a line segment exactly one unit long:

Now draw another line one unit long, perpendicular to the first one, like this:

Now draw the diagonal connecting the two ends:

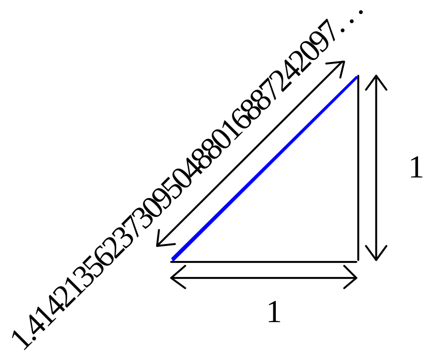


Congratulations! You have just drawn a length that cannot be measured by any rational number. According to the Pythagorean Theorem, the length of this diagonal is the square root of 2; that is, the number which when multiplied by itself gives 2.

According to my calculator,

$$\sqrt{2} = 1.41421356237$$

But my calculator only stops at eleven decimal places because it can not display any more. This number actually goes on forever past the decimal point, without the pattern ever terminating or repeating.



This is because if the pattern ever stopped or repeated, you could write the number as a fraction—and it can be proven that the square root of 2 can never be written as

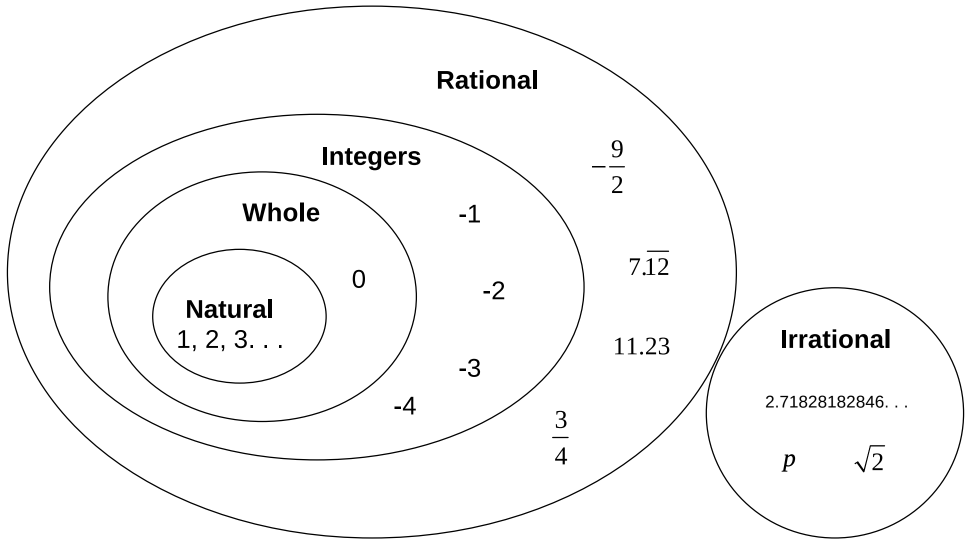
$$\sqrt{2} = \frac{a}{b}$$

for *any* choice of integers for *a* and *b*. The proof of this was considered quite shocking when it was first demonstrated by the followers of Pythagoras 26 centuries ago.

THE REAL NUMBERS

- Rationals + Irrationals
- All points on the number line
- Or all possible distances on the number line

When we put the irrational numbers together with the rational numbers, we finally have the complete set of real numbers. Any number that represents an amount of something, such as a weight, a volume, or the distance between two points, will always be a real number. The following diagram illustrates the relationships of the sets that make up the real numbers.



AN ORDERED SET

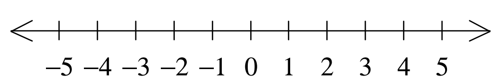
The real numbers have the property that they are *ordered*, which means that given any two different numbers we can always say that one is greater or less than the other. A more formal way of saying this is:

For any two real numbers *a* and *b*, one and only one of the following three statements is true:

1. *a* is less than *b*, (expressed as $a < b$)
2. *a* is equal to *b*, (expressed as $a = b$)
3. *a* is greater than *b*, (expressed as $a > b$)

THE NUMBER LINE

The ordered nature of the real numbers lets us arrange them along a line (imagine that the line is made up of an infinite number of points all packed so closely together that they form a solid line). The points are ordered so that points to the right are greater than points to the left:



- Every real number corresponds to a distance on the number line, starting at the center (zero).
- Negative numbers represent distances to the left of zero, and positive numbers are distances to the right.
- The arrows on the end indicate that it keeps going forever in both directions.

ABSOLUTE VALUE

When we want to talk about how "large" a number is without regard as to whether it is positive or negative, we use the *absolute value* function. The absolute value of a number is the distance from that number to the origin (zero) on the number line. That distance is always given as a non-negative number.

In short:

- If a number is positive (or zero), the absolute value function does nothing to it: $|4| = 4$
- If a number is negative, the absolute value function makes it positive: $|-4| = 4$

WARNING: If there is arithmetic to do inside the absolute value sign, you must do it before taking the absolute value—the absolute value function acts on the **result** of whatever is inside it. For example, a common error is

$$|5+(-2)| = 5+2 = 7 \quad \textbf{(WRONG)}$$

The mistake here is in assuming that the absolute value makes everything inside it positive. This is not true. It only makes the *result* positive. The correct result is

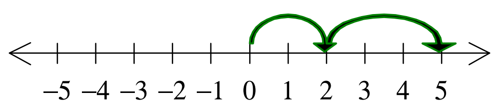
$$|5+(-2)| = |3| = 3$$

ADDITION AND SUBTRACTION OF REAL NUMBERS

All the basic operations of arithmetic can be defined in terms of addition, so we will take it as understood that you have a concept of what addition means, at least when we are talking about positive numbers.

ADDITION ON THE NUMBER LINE

A positive number represents a distance to the right on the number line, starting from zero (zero is also called the *origin* since it is the starting point). When we add another positive number, we visualize it as taking another step to the right by that amount. For example, we all know that 2 + 3 = 5. On the number line we would imagine that we start at zero, take two steps to the right, and then take three more steps to the right, which causes us to land on positive 5.



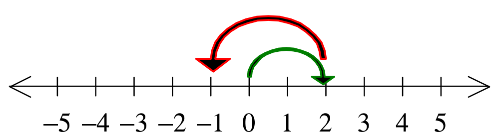
ADDITION OF NEGATIVE NUMBERS

What does it mean to add negative numbers? We view a negative number as a displacement to the left on the number line, so we follow the same procedure as before but when we add a negative number we take that many steps to the left instead of to the right.

Examples:

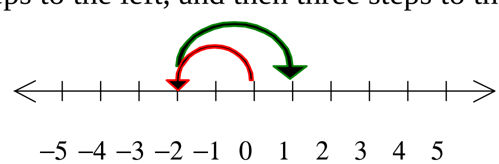
2 + (–3) = –1

First we move two steps to the right, and then three steps to the left:



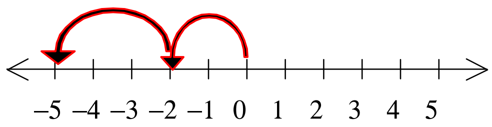
(–2) + 3 = 1

We move two steps to the left, and then three steps to the right:



(–2) + (–3) = –5

Two steps to the left, and then three more steps to the left:



From these examples, we can make the following observations:

1. If we add two positive numbers together, the result will be positive
2. If we add two negative numbers together, the result will be negative
3. If we add a positive and a negative number together, the result could be positive or negative, depending on which number represents the biggest step.

Subtraction

There are two ways to define subtraction: by a related addition statement, or as adding the opposite.

Subtraction as Related Addition

$a – b = c$ if and only if $a = b + c$

Subtraction as Adding the Opposite

For every real number b there exists its opposite $–b$, and we can define subtraction as adding the opposite:

$a – b = a + (–b)$

- In algebra it usually best to always think of subtraction as adding the opposite. That way, we never really do subtraction, because both addition and subtraction are just seen as addition.

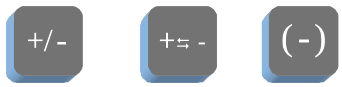
Distinction Between Subtraction and Negation

The symbol "–" means two different things in math. If it is between two numbers it means subtraction, but if it is in front of one number it means the opposite (or negative) of that number.

Subtraction is *binary* (acts on two numbers), but negation is *unary* (acts on only one number).

Calculators have two different keys to perform these functions. The key with a plain minus sign is only for subtraction:

Negation is performed by a key that looks like one of these:



Remember that subtraction can always be thought of as adding the opposite. In fact, we could get along just fine without ever using subtraction. If, for some reason, the subtraction key fell off of your calculator, you could still do subtraction by pressing the negation key and the addition key.

SUBTRACTION ON THE NUMBER LINE

Addition of a positive number moves to the right, and adding a negative moves to the left.

Subtraction is just the opposite: Subtraction of a positive number moves to the left, and subtracting a negative moves to the right.

- Notice that subtracting a negative is the same thing as adding a positive.

MULTIPLICATION AND DIVISION

MULTIPLICATION AS REPEATED ADDITION

We think of a multiplication statement like “2×3” as meaning “Add two threes together”, or

$$3 + 3$$

and “4×9” as “add 4 nines together”, or

$$9 + 9 + 9 + 9.$$

In general, $a \times b$ means to add b’s together such that the number of b’s is equal to a:

$$a \times b = b + b + b + \ldots + b \ (a \text{ times})$$

MULTIPLICATION WITH SIGNED NUMBERS

We can apply this same rule to make sense out of what we mean by a positive number times a negative number. For example,

$$3 \times (-4)$$

just means to take 3 of the number “negative four” and add them together:

$$3 \times (-4) = (-4) + (-4) + (-4) = -12$$

Unfortunately, this scheme breaks down when we try to multiply a negative number times a number. It doesn’t make sense to try to write down a number a negative number of times. There are two ways to look at this problem.

One way is to use the fact that multiplication obeys the *commutative law*, which means that the order of multiplication does not matter:

$$a \times b = b \times a.$$

This lets us write a negative times a positive as a positive times a negative and proceed as before:

$$(-3) \times 4 = 4 \times (-3) = (-3) + (-3) + (-3) + (-3) = -12$$

However, we are still in trouble when it comes to multiplying a negative times a negative. A better way to look at this problem is to see that multiplication obeys a consistent pattern. If we look at a multiplication table for positive numbers and then extend it to include negative numbers, the results in the table should continue to change in the same pattern.

For example, consider the following multiplication table:

a	b	$a \times b$
3	2	6
2	2	4
1	2	2
0	2	0

The numbers in the last column are decreasing by 2 each time, so if we let the values for *a* continue into the negative numbers we should keep decreasing the product by 2:

a	b	$a \times b$
3	2	6
2	2	4
1	2	2
0	2	0
–1	2	–2
–2	2	–4
–3	2	–6

We can make a bigger multiplication table that shows many different possibilities. By keeping the step sizes the same in each row and column, even as we extend into the negative numbers, we see that the following sign rules hold for multiplication:

SIGN RULES FOR MULTIPLICATION

$$(+)(+) = (+)$$

$$(-)(-) = (+)$$

$$(-)(+) = (-)$$

$$(+)(-) = (-)$$

Multiplication Table

Notice how the step size in each row or column remains consistent, regardless of whether we are multiplying positive or negative numbers.

	-5	-4	-3	-2	-1	0	1	2	3	4	5
-5	25	20	15	10	5	0	–5	–10	–15	–20	–25
-4	20	16	12	8	4	0	–4	–8	–12	–16	–20
-3	15	12	9	6	3	0	–3	–6	–9	–12	–15
-2	10	8	6	4	2	0	–2	–4	–6	–8	–10
-1	5	4	3	2	1	0	–1	–2	–3	–4	–5
0	0	0	0	0	0	0	0	0	0	0	0
1	–5	–4	–3	–2	–1	0	1	2	3	4	5
2	–10	–8	–6	–4	–2	0	2	4	6	8	10
3	–15	–12	–9	–6	–3	0	3	6	9	12	15
4	–20	–16	–12	–8	–4	0	4	8	12	16	20
5	–25	–20	–15	–10	–5	0	5	10	15	20	25

For math purists, here's the real reason:

The Real Reason

It should be obvious that the presentation of the rules of arithmetic given here is just a collection of motivational arguments, not a formal development. The formal development of the real number system starts with the *field axioms*. The field axioms are postulated, and then all the other properties follow from them. The field axioms are

1. The associative and commutative laws for addition and multiplication
2. The existence of the additive and multiplicative identities (0 and 1)

3. The existence of the additive inverse (opposites, or negatives) and the multiplicative inverse (the reciprocal)

4. The distributive law

All of these are essential, but the distributive law is particularly important because it is what distinguishes the behavior of multiplication from addition. Namely, multiplication distributes over addition but not vice-versa.

The rules of arithmetic like “a negative times a negative gives a positive” are what they are because that is the only way the field axioms would still hold. For example, the distributive law requires that

$$-2(3 - 2) = (-2)(3) + (-2)(-2)$$

We can evaluate the left side of this equation by following the order of operations, which says to do what is in parentheses first, so

$$-2(3 - 2) = -2(1) = -2.$$

Now for the distributive law to be true, the right side must also be equal to −2, so

$$(-2)(3) + (-2)(-2) = -2$$

If we use our sign rules for multiplication then it works out the way it should:

$$(-2)(3) + (-2)(-2) = -6 + 4 = -2$$

Notation for Multiplication

We are used to using the symbol “×” to represent multiplication in arithmetic, but in algebra we prefer to avoid that symbol because we like to use the letter “*x*” to represent a variable, and the two symbols can be easily confused. So instead, we adopt the following notation for multiplication:

1. Multiplication is implied if two quantities are written side-by-side with no other symbol between them.

   Example: ab means $a \times b$.

2. If a symbol is needed to prevent confusion, we use a dot.

   Example: If we need to show 3 times 5, we cannot just write them next to each other or it would look like the number thirty-five, so we write $3 \cdot 5$.

   - We can also use parentheses to separate factors. 3 times 5 could be written as 3(5) or (3)5 or (3)(5).

Division

There are two ways to think of division: as implying a related multiplication, or as multiplying by the reciprocal.

Division as Related Multiplication

The statement “$12 \div 3 = 4$” is true only because $3 \times 4 = 12$. A division problem is really asking the question “What number can I multiply the divisor by to get the dividend?” and so every division equation implies an equivalent multiplication equation. In general:

$$a \div b = c \text{ if and only if } a = b \times c$$

This also shows why you cannot divide by zero. If we asked “What is six divided by zero?” we would mean “What number times zero is equal to six?” But any number times zero gives zero, so there is no answer to this question.

Multiplicative Inverse (The Reciprocal)

For every real number a (except zero) there exists a real number denoted by $\frac{1}{a}$, such that

$$a\left(\frac{1}{a}\right) = 1$$

- The number $\frac{1}{a}$ is called the *reciprocal* or *multiplicative inverse* of a.

- Note that the reciprocal of $\frac{1}{a}$ is a. The reciprocal of the reciprocal gives you back what you started with.

This allows us to define division as multiplication by the reciprocal:

$$a \div b = a\left(\frac{1}{b}\right)$$

This is usually the most convenient way to think of division when you are doing algebra. In fact, this is very much like the situation with subtraction. Remember that we can do away with subtraction entirely by replacing it with adding the opposite. Similarly, if the division key falls off of your calculator, you can still perform division by pressing the reciprocal key and the multiplication key. The reciprocal key on calculators looks like one of these:

NOTATION FOR DIVISION

Instead of using the symbol "÷" to represent division, we prefer to write it using the fraction notation:

$$a \div b = \frac{a}{b}$$

SIGN RULES FOR DIVISION

Because division can always be written as a multiplication by the reciprocal, it obeys the same sign rules as multiplication.

If a positive is divided by a negative, or a negative divided by a positive, the result is negative:

$$-\frac{a}{b} = \frac{-a}{b} = \frac{a}{-b}$$

but if both numbers are the same sign, the result is positive:

$$\frac{-a}{-b} = \frac{a}{b}$$

Positive Integer Exponents

Meaning

$3^2 = 3 \times 3$

$3^3 = 3 \times 3 \times 3$

In general $x^n = x \cdot x \cdot x \cdot . . . \cdot x$ (n factors of x)

x is the *base*, and n is the *exponent* (or *power*)

Rules

Product of Different Powers: $a^m a^n = a^{m+n}$

- IMPORTANT: all the numbers must have the same bases (the same 'a')

Example: $(4^2)(4^3) = 4^5$

This is easy to see if you write out the exponents:

$(4^2)(4^3) = (4 \cdot 4) \cdot (4 \cdot 4 \cdot 4) = 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^5$

WARNING: Do not attempt to use this rule for addition:

$4^2 + 4^3$ is **NOT** 4^5. In fact there is no way to simplify $x^n + x^m$ if n and m are different powers.

Power Raised to a Power: $\left(a^m\right)^n = a^{mn}$

Example: $(4^2)^3 = 4^6$

This is also easy to see if you expand the exponents:

$(4^2)^3 = (4^2)(4^2)(4^2)$

$= (4 \cdot 4)\ (4 \cdot 4)\ (4 \cdot 4)$

$= 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4$

$= 4^6$

There are more rules for combining numbers with exponents, but this is enough for now.

ORDER OF OPERATIONS

When we encounter an expression such as $3 + 15 \div 3 + 5 \times 2^{2+3}$, it makes quite a difference how we choose which operations to perform first. We need a set of rules that would guide anyone to one unique value for this kind of expression. Some of these rules are simply based on convention, while others are forced on us by mathematical logic. In the chapter on the Properties of Real Numbers, you will see how the distributive law is consistent with these rules. The universally agreed-upon order in which to evaluate a mathematical expression is as follows:

1. Parentheses from Inside Out

By "parentheses" we mean anything that acts as a grouping symbol, including anything inside symbols such as [ ], { }, | | and $\sqrt{}$. Any expression in the numerator or denominator of a fraction or in an exponent is also considered grouped and should be simplified before carrying out further operations.

- If there are nested parentheses (parentheses inside parentheses), you work from the innermost parentheses outward.

2. Exponents

Also other special functions such as log, sin, cos, etc.

3. Multiplication and Division, left to right

The left-to-right order does not matter if only multiplication is involved, but it matters for division.

4. Addition and Subtraction, left to right

The left-to-right order does not matter if only addition is involved, but it matters for subtraction.

Example: Going back to our original example, $3 + 15 \div 3 + 5 \times 2^{2+3}$

Given: $3 + 15 \div 3 + 5 \times 2^{2+3}$

The exponent is an implied grouping, so the 2 + 3 must be evaluated first: $= 3 + 15 \div 3 + 5 \times 2^{5}$

Now the exponent is carried out: $= 3 + 15 \div 3 + 5 \times$ **32**

Now the multiplication and division, left to right, using $15 \div 3 = 5$ and $5 \times 32 = 160$: $= 3 +$ **5** $+$ **160**

Now the addition, left to right: $= 168$

Calculator Note: Most modern calculators "know" the order of operations, and you can enter expressions pretty much as they are written. Some older calculators will carry out each operation as soon as its key is pushed, which can result in the operations being carried out in the wrong order. Try some examples if you are not sure how your calculator behaves.

For example, if you enter

$$3 + 4 \times 5 =$$

The correct answer should be 23, because the multiplication should be performed before the addition, giving 3 + 20. But if your calculator carries out the "3 + 4" before getting to the "× 5", it will show a result of 35 because it will see it as 7×5.

Calculator Note: Use the parenthesis keys to force grouping. If you are evaluating an expression such as

$$\frac{4}{3+5}$$

the denominator needs to be simplified before doing the division. If you enter it into your calculator as $4 \div 3 + 5$, it will evaluate the "$4 \div 3$" first, and then add 5 to the result, given the incorrect answer of 6.3333. To make it perform the addition first, use parentheses:

$$4 \div (3 + 5) = 0.5$$

In fact, it is a good idea to always use the parentheses keys for the denominator of a fraction. It never hurts, and it can be essential.

Going back to our example problem above, the "2 + 3" in the exponent is an implied grouping, and you would also need to use parentheses to input it into your calculator. To enter the example expression in you calculator, the button sequence would be

$3 + 15 \div 3 + 5 \times 2$ ^ $(2 + 3) =$

(on some calculators the exponent button is labeled "^", while on others it is labeled "y^x" or "x^y")

FRACTIONS

Fractions, also called rational numbers, are numbers of the form $\frac{a}{b}$, where *a* and *b* are integers (but *b* cannot be zero).

The bottom number is called the *denominator*. Think of it as the denomination: it tells you what size units you are talking about—fourths, fifths, or whatever.

The top number is the *numerator*. It tells you how many of those units you have. For example, if I have 3 quarters in my pocket, then I have three-fourths of a dollar. The denomination is quarters (fourths), and I have three of them: 3/4.

IMPROPER FRACTIONS

Ordinarily we think of fractions as being between zero and one, like 3/4 or 2/3. These are called *proper fractions*. In these fractions, the numerator is smaller than the denominator—but there is no reason why we can not have a numerator bigger than the denominator. Such fractions are called *improper*.

What does an improper fraction like 5/4 mean? Well, if we have 5 quarters of something then we have more than one whole of that something. In fact, we have one whole plus one more quarter (if you have 5 quarters in change, you have a dollar and a quarter).

MIXED NUMBER NOTATION

One way of expressing the improper fraction 5/4 is as the *mixed number* $1\frac{1}{4}$, which is read as “one and one-fourth.” This notation is potentially confusing and is not advised in algebra.

One cause of confusion is that in algebra we use the convention that multiplication is implied when two quantities are written next to each other with no symbols in between. However, the mixed number notation implies addition, not multiplication. For example, $1\frac{1}{4}$ means 1 *plus* one-quarter.

It is possible to do arithmetic with mixed numbers by treating the whole number parts and the fractional parts separately, but it is generally more convenient in algebra to always write improper fractions. When you encounter a problem with mixed numbers, the first thing you should do is convert them to improper fractions.

Conversion

Mixed Number to Improper Fraction

Then add

$$2\frac{3}{4} = \frac{11}{4}$$

First Multiply

A. Multiply the integer part with the bottom of the fraction part.

B. Add the result to the top of the fraction.

The general formula is

$$a\tfrac{b}{c} = \frac{ac+b}{c}$$

Improper Fraction to Mixed Number

1. Do the division to get the integer part

2. Put the remainder over the old denominator to get the fractional part.

Multiplying, Reducing, and Dividing Fractions

Equivalent Fractions

Equivalent fractions are fractions that have the same value, for example

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8} \text{ etc.}$$

Although all these fractions are written differently, they all represent the same quantity. You can measure a half-cup of sugar or two quarter-cups of sugar, or even four eighth-cups of sugar, and you will still have the same amount of sugar.

- Multiply by a form of One

A fraction can be converted into an equivalent fraction by multiplying it by a form of 1. The number 1 can be represented as a fraction because any number divided by itself is equal to 1 (remember that the fraction notation means the same thing as division). In other words,

$$1 = \frac{1}{1} = \frac{2}{2} = \frac{3}{3} = \frac{4}{4} \quad \text{etc.}$$

Now if you multiply a number by 1 it does not change its value, so if we multiply a fraction by another fraction that is equal to 1, we will not be changing the value of the original fraction. For example,

$$\begin{aligned} \frac{2}{3} &= \frac{2}{3} \cdot 1 \\ &= \frac{2}{3} \cdot \frac{2}{2} \\ &= \frac{4}{6} \end{aligned}$$

In this case, 2/3 represents exactly the same quantity as 4/6, because all we did was to multiply 2/3 by the number 1, represented as the fraction 2/2.

Multiplying the numerator and denominator by the same number to produce an equivalent fraction is called *building up* the fraction.

Reduced Form

Numerator and Denominator Have No Common Factors

Procedure:

1. Write out prime factorization of Numerator and Denominator
2. Cancel all common factors

This procedure is just the opposite of building up a fraction by multiplying it by a fraction equivalent to 1.

Prime Factors

A number is *prime* if it has no whole number factors other than 1 times itself, that is, the number cannot be written as a product of two whole numbers (except 1 times itself).

Example: 6 is not prime because it can be written as 2×3

Example: 7 is prime because the only way to write it as a product of whole numbers is 1×7

- The first few prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, . . .
- There are an infinite number of prime numbers (the list goes on forever).

Any non-prime number can be decomposed into a product of prime numbers

Example: $4 = 2 \times 2$

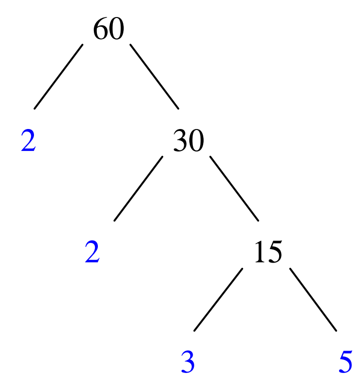
Example: $12 = 2 \times 2 \times 3$

The Branching Method

This method works well for larger numbers that might have many factors. All you need to do is think of any two numbers that multiply to give your original number, and write them below it. Continue this process for each number until each branch ends in a prime number. The factors of the original number are the prime numbers on the ends of all the branches.

Example: Factor the number 60

$60 = 2 \times 2 \times 3 \times 5$



Notes

1. Notice how I started with the smallest numbers: first 2's, then 3's, and so on. This is not required but it keeps the result nicely in order.
2. If a number is even, then it is divisible by 2.
3. If the digits of a number add up to a number divisible by 3, then the number is divisible by 3. In this example 15 gives 1 + 5 = 6, which is divisible by 3, and therefore 15 is divisible by 3.
4. If a number ends in 0 or 5, then it is divisible by 5.
5. Large numbers with large prime factors are notoriously hard to factor—it is mainly just a matter of trial and error. The public-key encryption system for sending secure computer data uses very large numbers that need to be

factored in order to break the code. The code is essentially unbreakable because it would take an enormous amount of computer time to try every possible prime factor.

MULTIPLYING FRACTIONS

Multiply Numerators and Denominators

Example:

$$\frac{2}{3}\cdot\frac{3}{4}=\frac{6}{12}$$

And reduce result if needed

$$\frac{6}{12}=\frac{1}{2}$$

Canceling common factors first makes multiplication easier

If you don't reduce the factors before multiplying, the answer will have to be reduced.

Example:

$$\frac{2}{3}\cdot\frac{3}{4}$$

$$\frac{\cancel{2}}{\cancel{3}}\cdot\frac{\cancel{3}}{\cancel{2}\cdot 2}=\frac{1}{2}$$

Remember that canceling always leaves a "1" behind, because you are really dividing the numerator and the denominator by the same number.

ADDING AND SUBTRACTING FRACTIONS

- Add Numerators when Denominators Are the Same

$$\frac{1}{5}+\frac{2}{5}=\frac{3}{5}$$

- If the denominators are not the same, *make* them the same by building up the fractions so that they both have a common denominator.
- Any common denominator will work, but the answer will have to be reduced if it is not the Least Common Denominator.

- The product of all the denominators is always a common denominator (but not necessarily the *Least* Common Denominator).

LEAST COMMON DENOMINATOR (LCD)

By Inspection

The LCD is smallest number that is evenly divisible by all the denominators. Another way of saying this is that it is the least common multiple of both denominators. That means that it is the smallest number that you can get to by multiplying either denominator by whole numbers. For small denominators, you can guess this number pretty quickly by just going through the multiples of each denominator until you find a match. For large numbers, though, these multiples exceed the multiplication table that you learned, and it is helpful to have a systematic method for finding the LCD. This method will also become important when we look at fractions made up of algebraic expressions instead of just plain numbers.

In General

The LCD is the product of all the prime factors of all the denominators, each factor taken the greatest number of times that it appears in any single denominator.

- **Example:**

$$\frac{1}{12}+\frac{4}{15}$$

Factor the denominators:

$$12 = 2\times 2\times 3$$

$$15 = 3\times 5$$

Assemble LCD:

$$2\times 2\times 3\times 5 = 60$$

Note that the three only appears once, because it is only needed once to make *either* the 12 or the 15:

15

$$2\times 2\times 3\times 5 = 60$$

12

Now that you have found the LCD, multiply each fraction (top and bottom) by whatever is needed to build up the denominator to the LCD:

$$\begin{aligned}
&\frac{1}{12}+\frac{4}{15}\\
&=\frac{1}{12}\left(\frac{5}{5}\right)+\frac{4}{15}\left(\frac{4}{4}\right)\\
&=\frac{5}{60}+\frac{16}{60}
\end{aligned}$$

Then add the numerators and reduce if needed (using the LCD does not guarantee that you won't have to reduce):

$$\frac{5}{60}+\frac{16}{60}=\frac{21}{60}=\frac{7}{20}$$

DECIMALS

DECIMALS ARE REALLY JUST FRACTIONS

Decimal notation is just a shorthand way of expressing certain fractions, namely those fractions with denominators that are powers of 10. For example, consider the number

2.345

Because of the place-values of the decimal digits, this really means

$$2.345=\frac{2}{1}+\frac{3}{10}+\frac{4}{100}+\frac{5}{1000}$$

CONVERTING DECIMALS TO FRACTIONS

Because all the denominators are powers of 10, it is very easy to add these fractions by finding a common denominator. In this example, the common denominator is 1000, and we get

$$\begin{aligned}
2.345&=\frac{2}{1}+\frac{3}{10}+\frac{4}{100}+\frac{5}{1000}\\
&=\frac{2000}{1000}+\frac{300}{1000}+\frac{40}{1000}+\frac{5}{1000}\\
&=\frac{2345}{1000}
\end{aligned}$$

This suggests a general rule for converting a decimal number to its fraction form:

- **Put all the digits over the denominator that corresponds to the place value of the last decimal place.**

In the number 2.345, the last decimal place value is the thousandths place, so we put the digits 2345 over the denominator 1000.

Of course we would usually want to reduce the resulting fraction to its simplest form. In this case

$$\frac{2345}{1000} = \frac{469}{200}$$

REPEATING FRACTIONS

The only time this method does not work is for repeating fractions. We know that 1/3 = 0.3333333... but how could we go from 0.3333333... back to 1/3? There is no 'last decimal place' because the decimals repeat forever. Fortunately, there is a simple trick for this:

- **Put the repeating digit over a denominator of 9.**

So we see that in the case of 0.3333333...., the repeating digit is 3, and we make the fraction 3/9, which reduces to 1/3.

If there is a group of more than one digit that repeats,

- **Put the repeating group of digits over as many 9's as there are digits.**

For example, in the fraction

$$0.151515\overline{15}$$

we see that the group of digits '15' repeats, so we put '15' over a denominator of '99' to get

$$0.151515\overline{15} = \frac{15}{99} = \frac{5}{33}$$

- One warning: This only works for the repeating fraction part of a number. If you have a number like 2.33333..., you should just work with the decimal part and rejoin it with the whole part after you have converted it to a fraction.

- Irrational numbers like π or $\sqrt{2}$ have non-repeating decimals, and so they cannot be written as fractions. You can, however, round them off at some point and produce an approximate fraction for them.

Why This Trick Works

If you have learned enough algebra to follow these steps, it is not difficult to see why this works. Let's start with the simple example 0.3333333... Since we already know that the answer is 1/3, we can concentrate on the procedure, not the answer.

Let

$$x = 0.3333333...$$

Multiply both sides by 10 to get

$$10x = 3.33333...$$

Notice that the decimal part (.33333…) is still the same. In fact, it is the very thing that we called x in the first place, so we can say that

$$10x = 3 + x$$

Now solve for x:

$$\begin{aligned} 10x &= 3 + x \\ 10x - x &= 3 \\ 9x &= 3 \\ x &= \frac{3}{9} = \frac{1}{3} \end{aligned}$$

This method will also work for repeating fractions that contain a group of repeating digits, but you have to multiply by a higher power of 10 in order to make the decimal portion stay the same as it was before. For example, suppose we had 0.345345345....

Let

$$x = 0.345345345....$$

Multiply both sides by 1000 to get

$$1000x = 345.345345345....$$

Notice that the decimal part is still the same. In fact, it is still the thing that we called x in the first place, so we can say that

$$1000x = 345 + x$$

Now solve for x:

$$1000x = 345 + x$$
$$1000x - x = 345$$
$$999x = 345$$
$$x = \frac{345}{999} = \frac{115}{333}$$

Converting Fractions to Decimals

We know the decimal equivalents for some common fractions without having to think about it: 1/2 = 0.5, 3/4 = 0.75, etc. But how do we arrive at these numbers? Remember that the fraction bar means the same thing as division.

- **To convert a fraction to a decimal, do the division.**

For example,

$$\frac{5}{7} = 5 \div 7 = 0.7142857\ldots$$

You can do the division with a calculator or by hand with long division.

Rounding

- Look only one digit to the right
- '5's or higher round up (there is some dispute about this rule, but it is good enough for most purposes)

Rounding 5's

Why round fives up? The number 3.5 is *exactly* halfway between 3.0 and 4.0, so it makes just as much sense to round it down as it does to round it up.

Most of the time there is no harm in using the 'always round fives up' rule. This is the rule that the United States Internal Revenue Service advises you to use on your taxes, and who is going to argue with them?

Sometimes, though, it can cause problems. Suppose you are adding a very large number of values that have all been rounded by this rule. The sum that you get will be a little bit bigger than it ought to be. This can be a very serious problem in computer programs. When thousands or even millions of additions are being performed, the accumulated roundoff error can be quite large.

One way of dealing with this problem is the *even-odd rule*. This rule says that:

- If the five is the last significant digit and the round-off digit (the one to the left of the 5) is odd, round up.
- If the five is the last significant digit and the round-off digit is even, don't round up.

Actually, you could reverse *even* and *odd* in this rule. All that matters is that about half the time you will be rounding up on a 5, and half the time down.

The reason it matters that the five is the last significant digit is because if there are any other non-zero digits past the five then you *must* round up, because the part that you are chopping off is more than 50% of the roundoff place-value. For example, suppose you want to round 3.351 to the nearest tenth. The decimal part represents the fraction 351/1000, which is 1/1000 closer to 400/1000 than it is to 300/100. Therefore you would always round this up to 3.4.

Example: Round 11.3826 to the nearest hundredth.

Solution: The hundredths place is where the '8' is. We look one digit to the right and see a '2', so we do not round up, leaving us with 11.38.

Example: Round 11.3826 to the nearest thousandth.

Solution: The thousandths place is where the '2' is. We look one place to the right and see a '6', so we round the '2' up, getting 11.383.

Trouble with '9's.

If the digit you are rounding up is a '9', then rounding it up will make it a '10', which is too big for one place. What happens is that the extra '1' gets added to the place to the left.

Example: Round 3.49721 to the nearest hundredth.

Solution: The hundredth place has a '9' in it. One step to the right is a '7', so we have to round up. This makes the '9' into a '10', but we really can't write the new number as 3.4(10). Instead, the extra '1' moves one place to the left and is added to the '4', giving us 3.50.

- It can happen that the place to the left contains *another* '9', in which case the extra '1' will cause *it* to become a '10', which pushes the '1' still further on to the next place to the left.

Example: Round 75.69996217 to the nearest ten-thousandth

Solution: The ten-thousandths place is the last '9' in the number, and the place to its right is a '6', which means we round up. This makes the '9' into a '10', like this:

75.699(10)

But of course we cannot put a '10' in that place, so the '1' moves to the left and gets added to the '9' there, making it into a '10':

75.69(10)0

This leaves us with the same problem, a '10' in one decimal place, so the extra '1' moves one more step to the left, turning *that* '9' into a '10':

75.6(10)00

Well, we still have the same problem, so we move the '1' yet another step to the left, where it adds on to the '6', finally leaving us with an acceptable answer:

75.7000

In general,

- The extra '1' migrates to the left until it finds a resting place.

This means that the '1' moves to the left until it can be added to a digit less than '9', or until it falls off the end as a new digit out in front.

Example: Round 999.96 to the nearest tenth.

The digit to the right of the tenths place is a '6', so we have to round up. But when we round up the '9' it becomes a '10', forcing the one to be added to the left. Unfortunately, we find another '9' there and the process is repeated for each of the '9's until we reach the leading '9', which becomes a '10' resulting in 1000.0

ARITHMETIC WITH DECIMALS

Although calculators have made it much easier to do arithmetic with decimal numbers, it is nice to know that you can still do it without a calculator.

Addition and Subtraction

To add or subtract decimal numbers, you use the familiar column method that you learned back in grade school. To use this method, the place values of the two numbers must be lined up. This means that the decimal points must be lined up, and you can fill in with zeros if one number has more decimal places than the other.

Example:

5.46 + 11.2

Becomes:

$$\begin{array}{r} 5.46 \\ \underline{11.20} \\ 16.66 \end{array}$$

Multiplication

To multiply two decimal numbers, you can use the column method just as you would with whole numbers. You ignore the decimal points as you carry out the multiplication, and then you put the decimal point in the result at the correct place. The product will have the number of decimal places as the total number of decimal places in the factors. In the following example, the first factor has 2 decimal places and the second factor has 1 decimal place, so the product must have 3 decimal places:

$$\begin{array}{r} 3.74 \\ \times 2.3 \\ \hline 1122 \\ +7480 \\ \hline 8.602 \end{array}$$

Division

You can divide decimal numbers using the familiar (?) technique of long division. This can be awkward, though, because it is hard to guess at products of decimals (long division, you may recall, is basically a guess-and-check technique). It can be made easier by multiplying both the dividend and divisor by '10's to make the divisor a whole number. This will not change the result of the division, because division is the same thing as fractions, and multiplying both the numerator and denominator of a fraction by the same number will not change the value of the fraction. For example, consider

$$12.24 \div 3.2$$

This is the same as the fraction $\frac{12.24}{3.2}$, which is equivalent to the fraction $\frac{122.4}{32}$, obtained by multiplying the numerator and denominator by 10. Thus

$$12.24 \div 3.2 = 122.4 \div 32,$$

which can be attacked with long division:

```
        3.825
  32)122.400
      96
      ----
      26.4
      25.6
        .80
        .64
       ----
        .16
        .16
       ----
       0
```

PERCENTS

Percent means "per hundred", so $x\% = \frac{x}{100}$, or x hundredths.

- A percent is just a fraction

However, not just any fraction, it is a fraction with a denominator of 100. When we write the percent, we are just writing the numerator of the fraction. The denominator of 100 is expressed by the percent symbol "%." Remembering that the percent symbol means "over one-hundred" can prevent a lot of confusion.

"%" means "/100"

TO CONVERT A PERCENT TO A DECIMAL

- Divide the percentage by 100 (or move the decimal point two places to the **left**).

Since $x\% = \frac{x}{100}$, the decimal equivalent is just the percentage divided by 100. But dividing by 100 just causes the decimal point to shift two places to the left:

$$75\% = \frac{75}{100} = 0.75$$

$$325\% = \frac{325}{100} = 3.25$$

TO CONVERT A DECIMAL TO A PERCENT

- Multiply the decimal number by 100 (or move the decimal point two places to the **right**).

Since $x\% = \frac{x}{100}$, it is also true that 100 $x\%$ = x. Another way to look at is to consider that in order to convert a number into a percent, you have to express it in hundredths. Recall that the hundredths place is the second place to the right of the decimal, so this is the digit that gives the units digit of the percent. Of course, all this means is that you move the decimal point two places to the right.

WARNING: If you just remember these rules as "move the decimal two places to the left" and "move the decimal two places to the right," you are very likely to get them confused. If you accidentally move the decimal in the wrong direction it will end up four places off from where it should be, which means that your answer will be either ten-thousand times too big or ten-thousand times too small. This is generally not an acceptable range of error. It is much better to remember these rules by simply remembering the meaning of the percent sign, namely that "%" means "/100." If you just write the problem that way, you should be able to see what you need to do in order to solve it.

Converting between percents and their decimal equivalents is so simple that it is usually best to express all percents in decimal form when you are working percent problems. The decimal numbers are what you will need to put in your calculator, and you can always express the result as a percent if you need to.

Calculator note: Some calculators have a percent key that essentially just divides by 100, but it can do other useful things that might save you a few keystrokes. For instance, if you need to add 5% to a number (perhaps to include the sales tax on a purchase), on most calculators you can enter the original number and then press " + 5 % = ". Just make sure you understand what it does before you blindly trust it. What it is doing in this example is multiplying the original number by 0.05 and then adding the result onto the original number. You should be able to work any percent problem without using this key, but once you understand what is going on it can be a convenient short-cut.

TO CONVERT A PERCENT TO A FRACTION

- Put the percentage over a denominator of 100 and reduce

Writing a percent as a fraction is very simple if you remember that the percent is the numerator of a fraction with a denominator equal to 100.

Examples:

$$75\% = \frac{75}{100} = \frac{3}{4}$$

$$325\% = \frac{325}{100} = \frac{13}{4} = 3\tfrac{1}{4}$$

$$0.2\% = \frac{0.2}{100} = \frac{2}{1000} = \frac{1}{500}$$

In this last example, the first fraction has a decimal in it, which is not a proper way to represent a fraction. To clear the decimal, just multiply both the numerator and the denominator by 10 to produce an equivalent fraction written with whole numbers.

To Convert a Fraction to a Percent

- Divide the numerator by the denominator and multiply by 100

To write a fraction as a percent you need to convert the fraction into hundredths. Sometimes this is easy to do without a calculator. For example, if you saw the fraction

$$\frac{13}{50},$$

you should notice that doubling the numerator and the denominator will produce an equivalent fraction that has a denominator of 100. Then the numerator will be the percent that you are seeking:

$$\frac{13}{50} = \frac{26}{100} = 26\%$$

With other fractions, though, it is not always so easy. It is not at all obvious how to convert a fraction like 5/7 into something over 100. In this case, the best thing to do is to convert the fraction into its decimal form, and then convert the decimal into a percent. To convert the fraction to a decimal, remember that the fraction bar indicates division:

$$\frac{5}{7} = 5 \div 7 \cong 0.7142857 \cong 71.4\%$$

The "approximately equal to" sign ($\cong$) is used because the decimal parts have been rounded off. Because it is understood that approximate numbers are rounded, we will not continue to use the approximately equal sign. It is more conventional to just use the standard equal sign with approximate numbers, even though it is not entirely accurate.

Working Percent Problems

In percent problems, just as in fraction problems, the word "of" implies multiplication:

"*x* percent *of* a number" means "*x*% *times* a number"

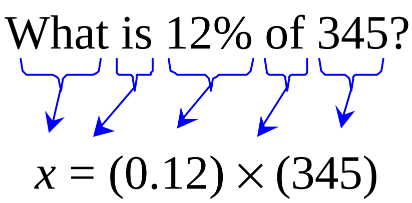
Example: What is 12% of 345?

12% is 12/100, which we can express in decimal form as 0.12. 12% *of* 345 means 12% *times* 345, or

$$\frac{12}{100}(345) = 0.12(345) = 41.4$$

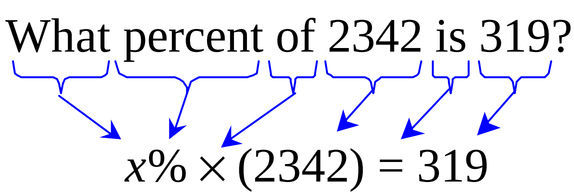
Notice how it is easier to just move the decimal over two places instead of explicitly dividing by 100.

We solve a problem like this by translating the question into mathematical symbols, using *x* to stand for the unknown "what" and that the "of" means "times":



Example: What percent of 2342 is 319?

Once again we translate this into mathematical symbols:



Solving this equation involves a little bit of algebra. To isolate the *x*% on one side of the equation we must divide both sides by 2342:

$$x\% = \frac{319}{2342}$$

The calculator tells use that

$x\% = 0.1362$

Now the right-hand side of this equation is the decimal equivalent that is equal to *x*%, which means that $x = 13.62$, or

319 is 13.62% of 2342

If that last step confused you, remember that the percent symbol means "over 100", so the equation

$x\% = 0.1362$

really says

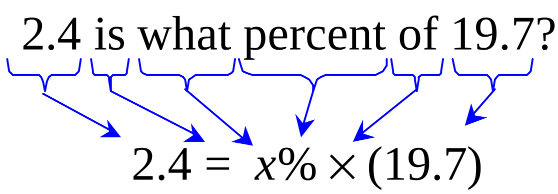
$$\frac{x}{100} = 0.1362$$

or

$$x = 100(0.1362)$$
$$x = 13.62$$

Example: 2.4 is what percent of 19.7?

Translating into math symbols:



Solving for x:

$$2.4 = x\%\ (19.7)$$

$$\frac{2.4}{19.7} = x\%$$

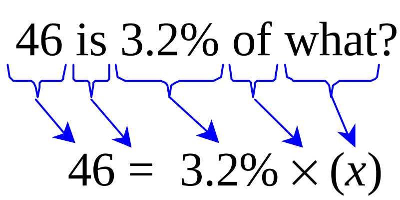
$$x\% = 0.1218$$

$$x = 12.18$$

So we can say that 2.4 is 12% of 19.7 (rounding to 2 significant figures)

Example: 46 is 3.2% of what?

Translating into math symbols:



Solving for x:

$$46 = 3.2\%\ (x)$$

$$46 = 0.032x$$

$$\frac{46}{0.032} = x$$

$x = 1437.5$

Therefore, we can say that 46 is 3.2% of 1400 (rounding to 2 significant figures). Notice that in the second step the percentage (3.2%) is converted into its decimal form (0.032).

PROPERTIES OF REAL NUMBERS

The following table lists the defining properties of the real numbers (technically called the *field axioms*). These laws define how the things we call numbers should behave.

ADDITION	MULTIPLICATION
Commutative For all real a, b $a + b = b + a$	**Commutative** For all real a, b $ab = ba$
Associative For all real a, b, c $a + (b + c) = (a + b) + c$	**Associative** For all real a, b, c $(ab)c = a(bc)$
Identity There exists a real number 0 such that for every real a $a + 0 = a$	**Identity** There exists a real number 1 such that for every real a $a \times 1 = a$
Additive Inverse (Opposite) For every real number a there exist a real number, denoted $(-a)$, such that $a + (-a) = 0$	**Multiplicative Inverse (Reciprocal)** For every real number a except 0 there exist a real number, denoted $\frac{1}{a}$, such that $a \times \frac{1}{a} = 1$
DISTRIBUTIVE LAW For all real a, b, c $a(b + c) = ab + ac$, and $(a + b)c = ac + bc$	

The commutative and associative laws do not hold for subtraction or division:

$a - b$ is not equal to $b - a$

$a \div b$ is not equal to $b \div a$

$a - (b - c)$ is not equal to $(a - b) - c$

$a \div (b \div c)$ is not equal to $(a \div b) \div c$

Try some examples with numbers and you will see that they do not work.

What these laws mean is that order and grouping don't matter for addition and multiplication, but they certainly do matter for subtraction and division. In this way, addition and multiplication are “cleaner” than subtraction and division. This will become important when we start talking about algebraic expressions. Often what we will want to do with an algebraic expression will involve rearranging it somehow. If the operations are all addition and multiplication, we don't have to worry so much that we might be changing the value of an expression by rearranging its terms or factors. Fortunately, we can always think of subtraction as an addition problem (adding the opposite), and we can always think of division as multiplication (multiplying by the reciprocal).

You may have noticed that the commutative and associative laws read exactly the same way for addition and multiplication, as if there was no difference between them other than notation. The law that makes them behave differently is the distributive law, because multiplication distributes over addition, not vice-versa. The distributive law is extremely important, and it is impossible to understand algebra without being thoroughly familiar with this law.

Example: $2(3 + 4)$

According to the order of operations rules, we should evaluate this expression by first doing the addition inside the parentheses, giving us

$$2(3 + 4) = 2(7) = 14$$

But we can also look at this problem with the distributive law, and of course still get the same answer. The distributive law says that

$$2(3 + 4) = 2 \times 3 + 2 \times 4 = 6 + 8 = 14$$

Chapter 2: Introduction to Algebra

ALGEBRAIC EXPRESSIONS

First we need to define some vocabulary. These words are very important—you don't want to confuse *terms* with *factors*, or you will not understand the discussions that follow.

VARIABLES

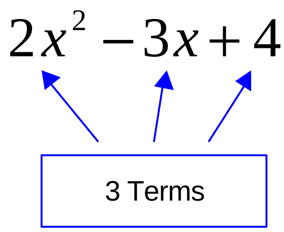
- Letters represent an unknown or generic real number.
- A variable could be any number. All you know is that it is *some* number.
- Sometimes with restrictions, such as a member of a certain set, or the set of values that makes an equation true.
- Often a letter from the end of the alphabet: x, y, z
- Or a letter that stands for a physical quantity: d for distance, t for time, etc.

CONSTANTS

- Fixed values, like 2 or 7.
- Can also be represented by letters: a, b, c, p , e, k

TERMS

Terms are parts of the expression that are added or subtracted. They will always be separated by + or –.

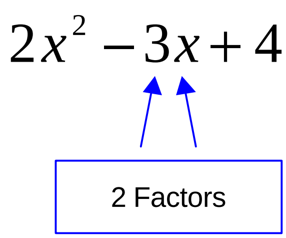


FACTORS

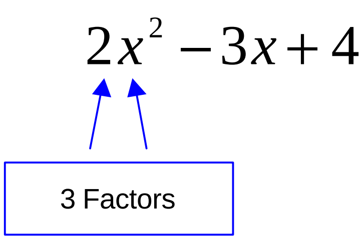
Factors are quantities that are multiplied together.

COEFFICIENTS

Coefficients are constant factors that multiply a variable or powers of a variable. For example, the middle term has 2 factors, namely –3 and x. We say that the coefficient of x is –3.

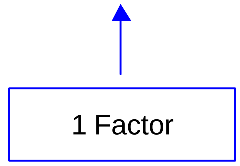


The first term has three factors, namely 2 and two factors of x. We say that 2 is the coefficient of x^2.



The last term is a factor all by itself because it is not multiplying anything (except a 1, which is a factor of everything).

$$2x^2 - 3x + 4$$



Simplifying Algebraic Expressions

By "simplifying" an algebraic expression, we mean writing it in the most compact or efficient manner, without changing the value of the expression. This mainly involves *collecting like terms*, which means that we add together anything that can be added together. The rule here is that only *like* terms can be added together.

Like (or Similar) Terms

Like terms are those terms which contain the same powers of same variables. They can have different coefficients, but that is the only difference.

Examples:

$3x$, x, and $-2x$ are like terms.

$2x^2$, $-5x^2$, and $\frac{1}{2}x^2$ are like terms.

xy^2, $3y^2x$, and $3xy^2$ are like terms.

xy^2 and x^2y are **NOT** like terms, because the same variable is not raised to the same power.

Combining Like Terms

Combining like terms is permitted because of the distributive law. For example,

$3x^2 + 5x^2 = (3 + 5)x^2 = 8x^2$

What happened here is that the distributive law was used in reverse—we "undistributed" a common factor of x^2 from each term. The way to think about this operation is that if you have three x-squareds, and then you get five more x-squareds, you will then have eight x-squareds.

Example: $x^2 + 2x + 3x^2 + 2 + 4x + 7$

Starting with the highest power of x, we see that there are four x-squareds in all ($1x^2 + 3x^2$). Then we collect the first powers of x, and see that there are six of them ($2x + 4x$). The only thing left is the constants $2 + 7 = 9$. Putting this all together we get

$$x^2 + 2x + 3x^2 + 2 + 4x + 7$$

$$= 4x^2 + 6x + 9$$

Parentheses

- Parentheses must be multiplied out before collecting like terms

You cannot combine things in parentheses (or other grouping symbols) with things outside the parentheses. Think of parentheses as opaque—the stuff inside the parentheses can't "see" the stuff outside the parentheses. If there is some factor multiplying the parentheses, then the only way to get rid of the parentheses is to multiply using the distributive law.

Example: $3x + 2(x - 4)$

$$= 3x + 2x - 8$$

$$= 5x - 8$$

Minus Signs: Subtraction and Negatives

Subtraction can be replaced by adding the opposite

$$3x - 2 = 3x + (-2)$$

Negative signs in front of parentheses

A special case is when a minus sign appears in front of parentheses. At first glance, it looks as though there is no factor multiplying the parentheses, and you may be tempted to just remove the parentheses. What you need to remember is that the minus sign indicating subtraction should always be thought of as adding the opposite. This means that you want to add the opposite of the entire thing inside the parentheses, and so you have to change the sign of each term in the parentheses. Another way of looking at it is to imagine an implied factor of one in front of the parentheses. Then the minus sign makes that factor into a negative one, which can be multiplied by the distributive law:

$$3x - (2 - x)$$

$$= 3x + (-1)[2 + (-x)]$$

$$= 3x + (-1)(2) + (-1)(-x)$$

$$= 3x - 2 + x$$

$$= 4x - 2$$

However, if there is only a plus sign in front of the parentheses, then you can simply erase the parentheses:

$3x + (2 – x)$

$= 3x + 2 – x$

A comment about subtraction and minus signs

Although you can always explicitly replace subtraction with adding the opposite, as in this previous example, it is often tedious and inconvenient to do so. Once you get used to *thinking* that way, it is no longer necessary to actually write it that way. It is helpful to always think of minus signs as being "stuck" to the term directly to their right. That way, as you rearrange terms, collect like terms, and clear parentheses, the "adding the opposite" business will be taken care of because the minus signs will go with whatever was to their right. If what is immediately to the right of a minus sign happens to be a parenthesis, and then the minus sign attacks every term inside the parentheses.

SOLUTIONS OF ALGEBRAIC EQUATION

Up until now, we have just been talking about manipulating algebraic expressions. Now it is time to talk about *equations*. An expression is just a statement like

$2x + 3$

This expression might be equal to any number, depending on the choice of x. For example, if $x = 3$ then the value of this expression is 9. But if we are writing an equation, then we are making a statement about its value. We might say

$2x + 3 = 7$

A mathematical equation is either true or false. This equation, $2x + 3 = 7$, might be true or it might be false, depending on the value chosen for x. We call such equations *conditional*, because their truth depends on choosing the correct value for x. If I choose $x = 3$, then the equation is clearly false because $2(3) + 3 = 9$, not 7. In fact, it is only true if I choose $x = 2$. Any other value for x produces a false equation. We say that $x = 2$ is the *solution* of this equation.

SOLUTIONS

- The solution of an equation is the value(s) of the variable(s) that make the equation a true statement.

An equation like $2x + 3 = 7$ is a simple type called a linear equation in one variable. These will always have one solution, no solutions, or an infinite number of solutions. There are other types of equations, however, that can have several solutions. For example, the equation

$$x^2 = 9$$

is satisfied by both $x = 3$ and $x = -3$, and so it has two solutions.

One Solution

This is the normal case, as in our example where the equation $2x + 3 = 7$ had exactly one solution, namely $x = 2$. The other two cases, no solution and an infinite number of solutions, are the oddball cases that you don't expect to run into very often. Nevertheless, it is important to know that they can happen in case you do encounter one of these situations.

Infinite Number of Solutions

Consider the equation

$$x = x$$

This equation is obviously true for every possible value of x. This is, of course, a ridiculously simple example, but it makes the point. Equations that have this property are called *identities*. Some examples of identities would be

$$2x = x + x$$

$$3 = 3$$

$$(x - 2)(x + 2) = x^2 - 4$$

All of these equations are true for any value of x. The second example, 3 = 3, is interesting because it does not even contain an x, so obviously its truthfulness cannot depend on the value of x. When you are attempting to solve an equation algebraically and you end up with an obvious identity (like 3 = 3), then you know that the original equation must also be an identity, and therefore it has an infinite number of solutions.

No Solutions

Now consider the equation

$$x + 4 = x + 3$$

There is no possible value for x that could make this true. If you take a number and add 4 to it, it will never be the same as if you take the same number and add 3 to it. Such an equation is called a *contradiction*, because it cannot ever be true.

If you are attempting to solve such an equation algebraically, you will eventually end up with an extremely obvious contradiction such as 1 = 2. This indicates that the original equation is a contradiction, and has no solution.

In summary,

- An *identity* is always true, no matter what x is
- A *contradiction* is never true for any value of x
- A *conditional equation* is true for some values of x

ADDITION PRINCIPLE

EQUIVALENT EQUATIONS

The basic approach to finding the solution to equations is to change the equation into simpler equations, but in such a way that the solution set of the new equation is the same as the solution set of the original equation. When two equations have the same solution set, we say that they are equivalent.

What we want to do when we solve an equation is to produce an equivalent equation that tells us the solution directly. Going back to our previous example,

$$2x + 3 = 7,$$

we can say that the equation

$$x = 2$$

is an equivalent equation, because they both have the same solution, namely $x = 2$. We need to have some way to convert an equation like $2x + 3 = 7$ into an equivalent equation like $x = 2$ that tells us the solution. We solve equations by using methods that rearrange the equation in a manner that does not change the solution set, with a goal of getting the variable by itself on one side of the equal sign. Then the solution is just the number that appears on the other side of the equal sign.

The methods of changing an equation without changing its solution set are based on the idea that if you change both sides of an equation in the same way, then the equality is preserved. Think of an equation as a balance—whatever complicated expression might appear on either side of the equation, they are really just numbers. The equal sign is just saying that the value of the expression on the left side is the same number as the value on the right side. Therefore, no matter how horrible the equation may seem, it is really just saying something like 3 = 3.

THE ADDITION PRINCIPLE

- Adding (or subtracting) the same number to both sides of an equation does not change its solution set.

Think of the balance analogy—if both sides of the equation are equal, then increasing both sides by the same amount will change the value of each side, but they will still be equal. For example, if

$3 = 3,$

then

$3 + 2 = 3 + 2.$

Consequently, if

$6 + x = 8$

for some value of x (which in this case is $x = 2$), then we can add any number to both sides of the equation and $x = 2$ will still be the solution. If we wanted to, we could add a 3 to both sides of the equation, producing the equation

$9 + x = 11.$

As you can see, $x = 2$ is still the solution. Of course, this new equation is no simpler than the one we started with, and this maneuver did not help us solve the equation.

If we want to solve the equation

$6 + x = 8,$

the idea is to get x by itself on one side, and so we want to get rid of the 6 that is on the left side. We can do this by subtracting a 6 from both sides of the equation (which of course can be thought of as adding a negative six):

$6 - 6 + x = 8 - 6$

or

$x = 2$

You can think of this operation as moving the 6 from one side of the equation to the other, which causes it to change sign.

- The addition principle is useful in solving equations because it allows us to move whole terms from one side of the equal sign to the other. While this is a convenient way to think of it, you should remember that you are not really “moving” the term from one side to the other—you are really adding (or subtracting) the term on both sides of the equation.

NOTATION NOTES

In the previous example, we wrote the –6 in-line with the rest of the equation. This is analogous to writing an arithmetic subtraction problem in one line, as in

$234 - 56 = 178.$

You probably also learned to write subtraction and addition problems in a column format, like

$$\begin{array}{r} 234 \\ -\ 56 \\ \hline 178 \end{array}$$

We can also use a similar notation for the addition method with algebraic equations.

Given the equation

$x + 3 = 2,$

we want to subtract a 3 from both sides in order to isolate the variable. In column format this would look like

$$\begin{array}{r} x+3=2 \\ -3=-3 \\ \hline x=-1 \end{array}$$

Here the numbers in the second row are negative 3's, so we are *adding* the two rows together to produce the bottom row.

The advantage of the column notation is that it makes the operation easier to see and reduces the chances for an error. The disadvantage is that it takes more space, but that is a relatively minor disadvantage. Which notation you prefer to use is not important, as long as you can follow what you are doing and it makes sense to you.

MULTIPLICATION PRINCIPLE

Multiplying (or dividing) the same non-zero number to both sides of an equation does not change its solution set.

Example:

$$6\times 2=12$$
$$3\times 6\times 2=3\times 12$$

so if $6x = 12$, then $18x = 36$ for the same value of x (which in this case is $x = 2$).

The way we use the multiplication principle to solve equations is that it allows us to isolate the variable by getting rid of a factor that is multiplying the variable.

Example: $2x = 6$

To get rid of the 2 that is multiplying the *x*, we can divide both sides of the equation by 2, or, equivalently, multiply by its reciprocal (one-half).

Either divide both sides by 2:

$$\begin{aligned} 2x &= 6 \\ \frac{2x}{2} &= \frac{6}{2} \\ x &= 3 \end{aligned}$$

or multiply both sides by a half:

$$\begin{aligned} 2x &= 6 \\ \left(\tfrac{1}{2}\right)2x &= \left(\tfrac{1}{2}\right)6 \\ x &= 3 \end{aligned}$$

- Whether you prefer to think of it as dividing by the number or multiplying by its reciprocal is not important, although when the coefficient is a fraction it is easier to multiply by the reciprocal:

Example: $\frac{4}{5}x = 8$

Multiply both sides by the reciprocal of the coefficient, or $\frac{5}{4}$

$$\frac{5}{4} \cdot \frac{4}{5}x = \frac{5}{4} \cdot \frac{8}{1}$$

After multiplying and reducing the fractions, we get the result

$x = 10$.

Note: When working with fractions it is convenient to express whole numbers like the 8 in the example above as a fraction with a denominator of 1.

USING THE PRINCIPLES TOGETHER

Suppose you were given an equation like

$2x - 3 = 5$.

You will need to use the addition principle to move the –3, and the multiplication principle to remove the coefficient 2. Which one should you use first? Strictly speaking, it does not matter—you will eventually get the right answer. In practice, however, it is usually simpler to use the addition principle first, and then the multiplication principle. The reason for this is that if we divide by 2 first we will turn everything into fractions:

Given: $2x - 3 = 5$

Suppose we first divide both sides by 2:

$$\frac{2x-3}{2} = \frac{5}{2}$$
$$\frac{2x}{2} - \frac{3}{2} = \frac{5}{2}$$
$$x - \frac{3}{2} = \frac{5}{2}$$

Now there is nothing wrong with doing arithmetic with fractions, but it is not as easy as working with whole numbers. In this example we would have to add 3/2 to both sides of the equation to isolate the *x*. To avoid fractions as much as possible, it is more convenient to use the addition principle first:

Given: $2x - 3 = 5$

Add 3 to both sides:

$$\begin{array}{r} 2x - 3 = 5 \\ 3 = 3 \\ \hline 2x = 8 \end{array}$$

At this point all we need to do is divide both sides by 2 to get the result $x = 4$.

Chapter 3: Word Problems

PROBLEM SOLVING STRATEGIES

UNDERSTAND

1. Read the problem carefully.
2. Make sure you understand the situation that is described.
3. Make sure you understand what information is provided, and what the question is asking.
4. For many problems, drawing a clearly labeled picture is very helpful.

PLAN

1. First focus on the objective. What do you need to know in order to answer the question?
2. Then look at the given information. How can you use that information to get what you need to know to answer the question?
3. If you do not see a clear logical path leading from the given information to the solution, just try *something*. Look at the given information and think about what you can find from it, even if it is not what the question is asking for. Often you will find another piece of information that you can then use to answer the question.

WRITE EQUATIONS

You need to express mathematically the logical connections between the given information and the answer you are seeking. This involves:

1. Assigning variable names to the unknown quantities. The letter x is always popular, but it is a good idea to use something that reminds you what it represents, such as d for distance or t for time. The trickiest part of assigning variables is that you want to use a minimum number of different variables (just one if possible). If you know how two quantities are related, then you can express them both with just one variable. For example, if Jim is two years older than John is, you might let x stand for John's age and $(x + 2)$ stand for Jim's age.

2. Translate English into Math. Mathematics is a language, one that is particularly well suited to describing logical relationships. English, on the other hand, is much less precise. The next page is a table of English phrases and their corresponding mathematical meanings, but don't take it too literally. The meaning of English words has to be taken in context.

SOLVE

Now you just have to solve the equation(s) for the unknown(s). Remember to answer the question that the problem asks.

CHECK

Think about your answer. Does your answer come out in the correct units? Is it reasonable? If you made a mistake somewhere, chances are your answer will not just be a little bit off, but will be completely ridiculous.

WORDS FOR OPERATIONS

Note: The English language is notoriously imprecise, and these suggested translations should be taken only as a guide, not as absolutes.

Subtraction	**minus**	"a number minus 2"	$x - 2$
	difference between	"the difference between a number and 8"	$x - 8$
	from	"2 from a number"	$n - 2$
	less	"a number less 3"	$n - 3$
	less than	"3 less than a number"	$y - 3$
	fewer than	"2 fewer than a number"	$y - 2$
	decreased by	"a number decreased by 2"	$x - 2$
	take away	"a number take away 2"	$x - 2$
Addition	**plus**	"a number plus 2"	$x + 2$
	and	"3 and a number"	$3 + n$
	added to	"8 added to a number"	$x + 8$
	greater than	"3 greater than a number"	$n + 3$
	more than	"3 more than a number"	$y + 3$
	increased by	"a number increased by 2"	$y + 2$
	total	"the total length"	$l_1 + l_2$
	sum of	"The sum of length and width"	$l + w$
Multiplication	**times**	"5 times a number"	$5n$
	product	"The product of 3 and a number"	$3y$
	at	"3 at 1.59"	3×1.59
	double, triple, etc.	"double a number"	$2x$
	twice	"twice a number"	$2y$
	of (fractions of)	"three-fourths of a number"	$\left(\frac{3}{4}\right)y$
Division	**quotient of**	"The quotient of 5 and a number"	$\frac{5}{n}$
	Half of	"half of a number"	$\frac{n}{2}$
	goes into	"a number goes into 6 twice"	$\frac{6}{n} = 2$
	per	"The price is $8 per 50"	$P = \frac{8}{50}$
Equals	**Is, is the same as, gives, will be, was, is equivalent to**		

General Word Problems

General Strategy

Recall the general strategy for setting up word problems. Refer to the Problem Solving Strategies page for more detail.

1. Read the problem carefully: Determine what is known, what is unknown, and what question is being asked.
2. Represent unknown quantities in terms of a variable.
3. Use diagrams where appropriate.
4. Find formulas or mathematical relationships between the knowns and the unknowns.
5. Solve the equations for the unknowns.
6. Check answers to see if they are reasonable.

Number/Geometry Problems

Example: Find a number such that 5 more than one-half the number is three times the number.

Let x be the unknown number.

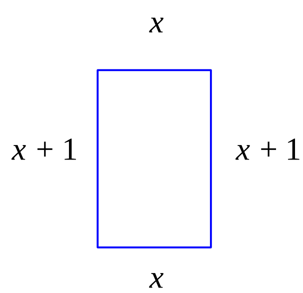
Translating into math: $5 + x/2 = 3x$

Solving:
(First multiply by 2 to clear the fraction)

$$5 + x/2 = 3x$$
$$10 + x = 6x$$
$$10 = 5x$$
$$x = 2$$

Example: If the perimeter of a rectangle is 10 inches, and one side is one inch longer than the other, how long are the sides?

Let one side be x and the other side be $x + 1$.



Then the given condition may be expressed as

$$x + x + (x + 1) + (x + 1) = 10$$

Solving:

$$4x + 2 = 10$$
$$4x = 8$$
$$x = 2$$

so the sides have length **2 and 3**.

RATE-TIME PROBLEMS

- Rate = Quantity/Time

or

- Quantity = Rate × Time

Example: A fast employee can assemble 7 radios in an hour, and another slower employee can only assemble 5 radios per hour. If both employees work together, how long will it take to assemble 26 radios?

The two together will build 7 + 5 = 12 radios in an hour, so their combined rate is 12 radios/hr.

Using Time = Quantity/Rate,

$$time = \frac{26 \text{ radios}}{12 \text{ radios/hr}} = 2\tfrac{1}{6} \text{ hr},$$

or **2 hours 10 minutes**

Example: you are driving along at 55 mph when you are passed by a car doing 85 mph. How long will it take for the car that passed you to be one mile ahead of you?

We know the two rates, and we know that the difference between the two distances traveled will be one mile, but we don't know the actual distances. Let D be the distance in miles that you travel in time t, and $D + 1$ be the distance in miles that the other car traveled in time t. Using the rate equation in the form *distance = speed · time* for each car we can write

$D = 55\,t$, and $D + 1 = 85\,t$

Substituting the first equation into the second,

$55t + 1 = 85t$
$-30t = -1$
$t = 1/30$ hr (or **2 minutes**)

MIXTURE PROBLEMS

Example: How much of a 10% vinegar solution should be added to 2 cups of a 30% vinegar solution to make a 20% solution?

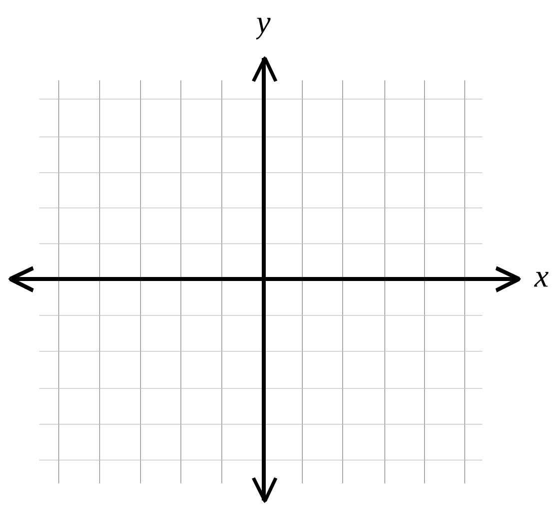
Let x be the unknown amount of 10% solution. Write an equation for the amount of vinegar in each mixture:

(amount of vinegar in first solution) + (amount of vinegar in second solution) = (amount of vinegar in total solution)

$0.1x + 0.3(2) = 0.2(x + 2)$
$0.1x + 0.6 = 0.2x + 0.4$
$-0.1x = -0.2$
$x =$ **2 cups**

Chapter 4: Graphing and Straight Lines

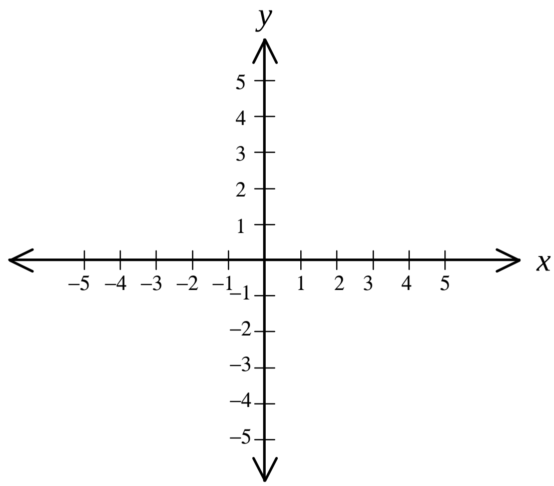
RECTANGULAR COORDINATES



The rectangular coordinate system is also known as the *Cartesian* coordinate system after Rene Descartes, who popularized its use in analytic geometry. The rectangular coordinate system is based on a grid, and every point on the plane can be identified by unique x and y coordinates, just as any point on the Earth can be identified by giving its latitude and longitude.

AXES

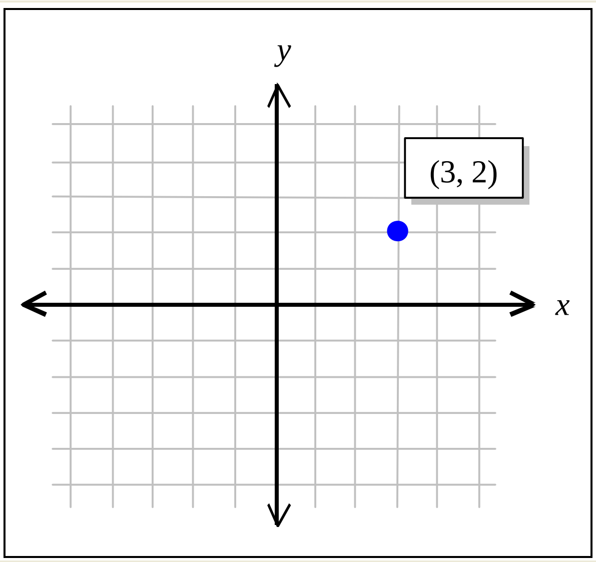
Locations on the grid are measured relative to a fixed point, called the *origin*, and are measured according to the distance along a pair of axes. The x and y axes are just like the number line, with positive distances to the right and negative to the left in the case of the x axis, and positive distances measured upwards and negative down for the y axis. Any displacement away from the origin can be constructed by moving a specified distance in the x direction and then another distance in the y direction. Think of it as if you were giving directions to someone by saying something like "go three blocks East and then 2 blocks North."

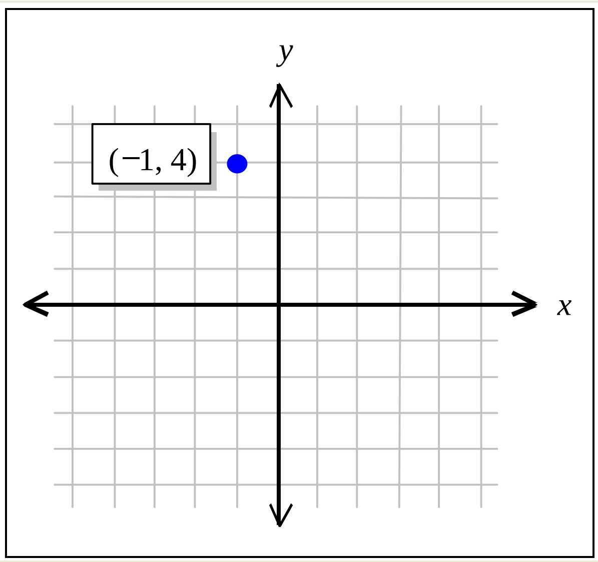


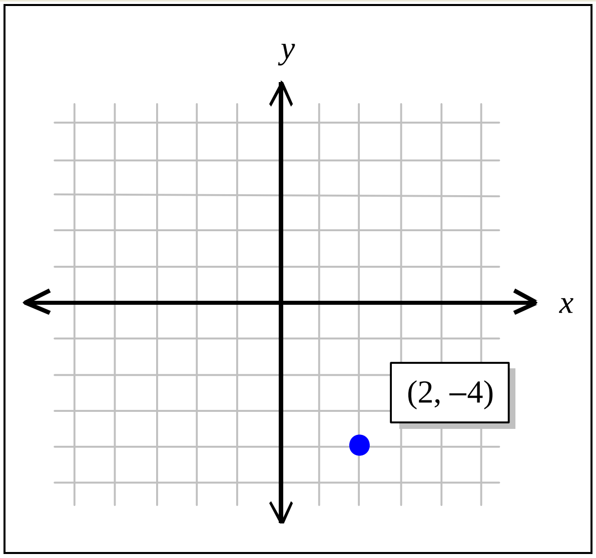
COORDINATES, GRAPHING POINTS

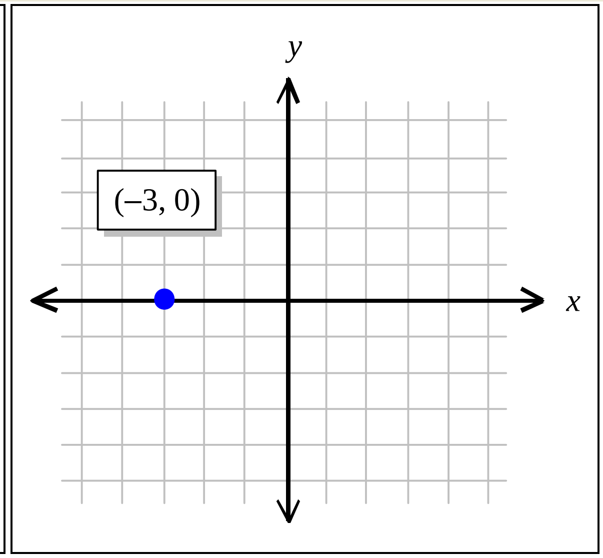
We specify the location of a point by first giving its *x* coordinate (the left or right displacement from the origin), and then the *y* coordinate (the up or down displacement from the origin). Thus, every point on the plane can be identified by a pair of numbers (*x*, *y*), called its *coordinates*.

Examples:



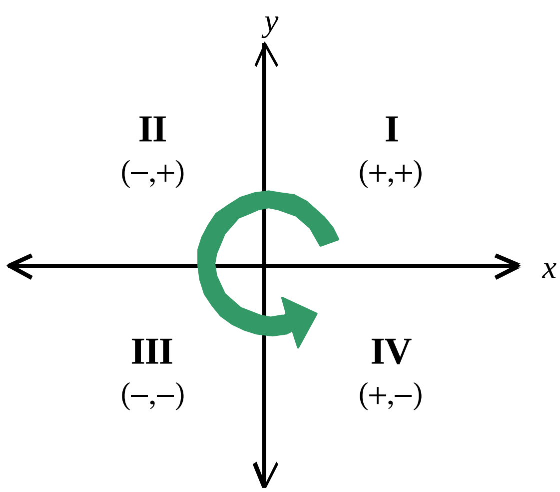






QUADRANTS

Sometimes we just want to know what general part of the graph we are talking about. The axes naturally divide the plane up into quarters. We call these *quadrants*, and number them from one to four. Notice that the numbering begins in the upper right quadrant and continues around in the counter-clockwise direction. Notice also that each quadrant can be identified by the unique combination of positive and negative signs for the coordinates of a point in that quadrant.



GRAPHING FUNCTIONS

Consider an equation such as

$$y = 2x - 1$$

We say that y is a *function* of x because if you choose any value for x, this formula will give you a unique value of y. For example, if we choose $x = 3$ then the formula gives us

$$y = 2(3) - 1$$

or

$$y = 5$$

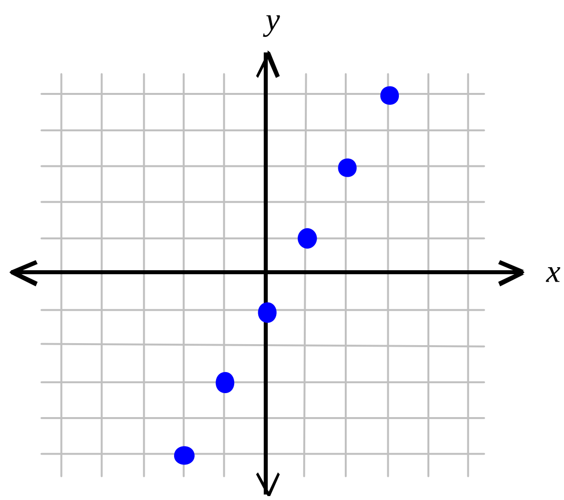
Thus we can say that the value $y = 5$ is generated by the choice of $x = 3$. Had we chosen a different value for x, we would have obtained a different value for y. In fact, we can choose a whole bunch of different values for x and get a y value for each one. This is best shown in a table:

x (Input)	***x*** ➔ **FORMULA** ➔ ***y***	***y*** (Output)
–2	2(–2) – 1 = –5	–5
–1	2(–1) – 1 = –3	–3
0	2(0) – 1 = –1	–1
1	2(1) – 1 = 1	1
2	2(2) – 1 = 3	3
3	2(3) – 1 = 5	5

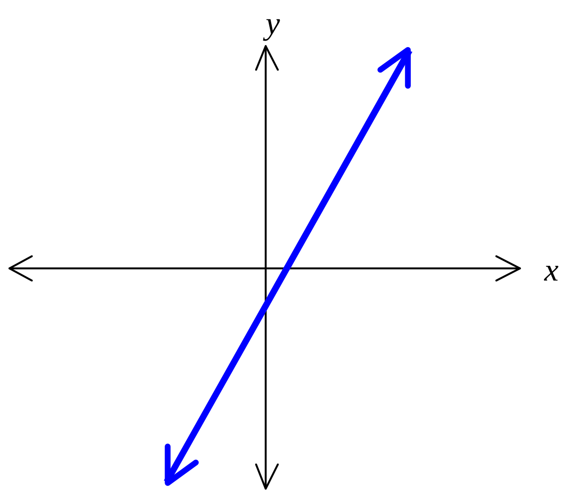
This relationship between x and its corresponding y values produces a collection of pairs of points (x, y), namely

$$(-2, -5)$$
$$(-1, -3)$$
$$(0, -1)$$
$$(1, 1)$$
$$(2, 3)$$
$$(3, 5)$$

Since each of these pairs of numbers can be the coordinates of a point on the plane, it is natural to ask what this collection of ordered pairs would look like if we graphed them. The result is something like this:

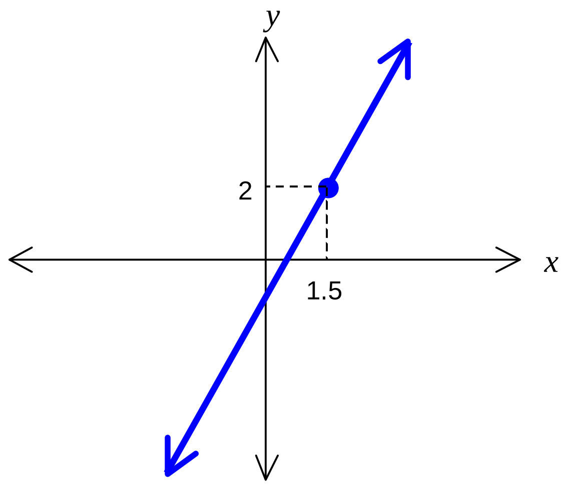


The points seem to fall in a straight line. Now, our choices for x were quite arbitrary. We could just as well have picked other values, including non-integer values. Suppose we picked many more values for x, like 2.7, 3.14, etc. and added them to our graph. Eventually the points would be so crowded together that they would form a solid line:



The arrows on the ends of the line indicate that it goes on forever, because there is no limit to what numbers we could choose for x. We say that this line is the *graph* of the function $y = 2x - 1$.

If you pick any point on this line and read off its x and y coordinates, they will satisfy the equation $y = 2x - 1$. For example, the point (1.5, 2) is on the line:



and the coordinates $x = 1.5$, $y = 2$ satisfy the equation $y = 2x - 1$:

$$2 = 2(1.5) - 1$$

- Note: This graph turned out to be a straight line only because of the particular function that we used as an example. There are many other functions whose graphs turn out to be various curves

Straight Lines

Linear Equations in Two Variables

The equation $y = 2x - 1$ that we used as an example for graphing functions produced a graph that was a straight line. This was no accident. This equation is one example of a general class of equations that we call *linear equations in two variables*. The two variables are usually (but of course don't have to be) x and y. The equations are called *linear* because their graphs are straight lines. Linear equations are easy to recognize because they obey the following rules:

1. The variables (usually x and y) appear only to the first power
2. The variables may be multiplied only by real number constants
3. Any real number term may be added (or subtracted, of course)

4. Nothing else is permitted!

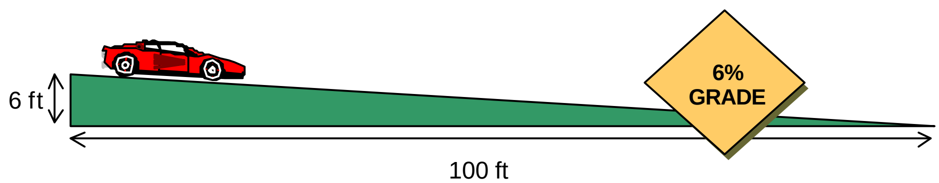
- This means that any equation containing things like x^2, y^2, $1/x$, xy, square roots, or any other function of x or y is not linear, and it's graph will not be a straight line.

Describing Lines

Just as there are an infinite number of equations that satisfy the above conditions, there are also an infinite number of straight lines that we can draw on a graph. To describe a particular line we need to specify two distinct pieces of information concerning that line. A specific straight line can be determined by specifying two distinct points that the line passes through, or it can be determined by giving one point that it passes through and somehow describing how "tilted" the line is.

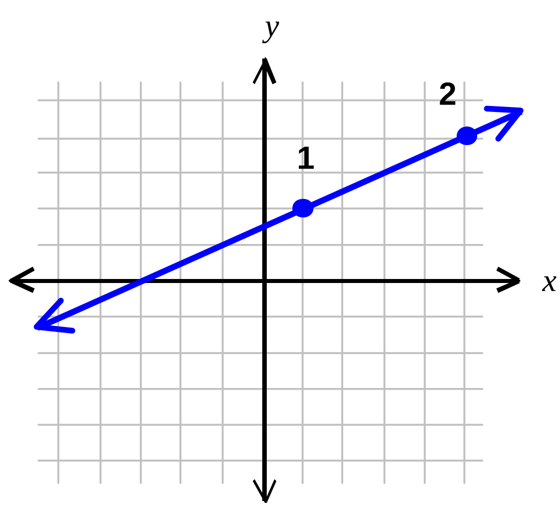
Slope

The *slope* of a line is a measure of how "tilted" the line is. A highway sign might say something like "6% grade ahead." What does this mean, other than that you hope your brakes work? What it means is that the ratio of your drop in altitude to your horizontal distance is 6%, or 6/100. In other words, if you move 100 feet forward, you will drop 6 feet; if you move 200 feet forward, you will drop 12 feet, and so on.

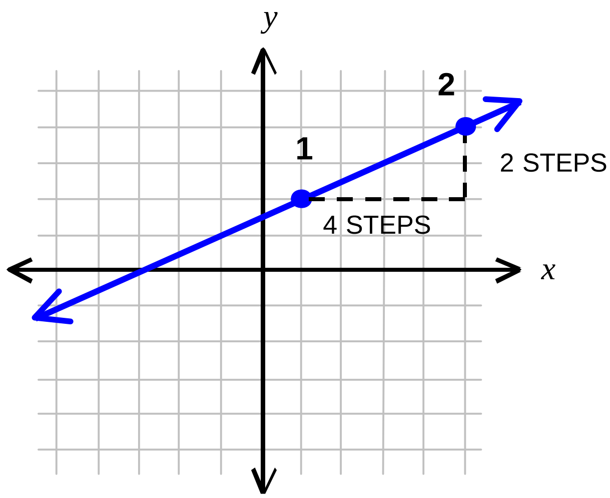


We measure the slope of lines in much the same way, although we do not convert the result to a percent.

Suppose we have a graph of an unknown straight line. Pick any two different points on the line and label them point 1 and point 2:



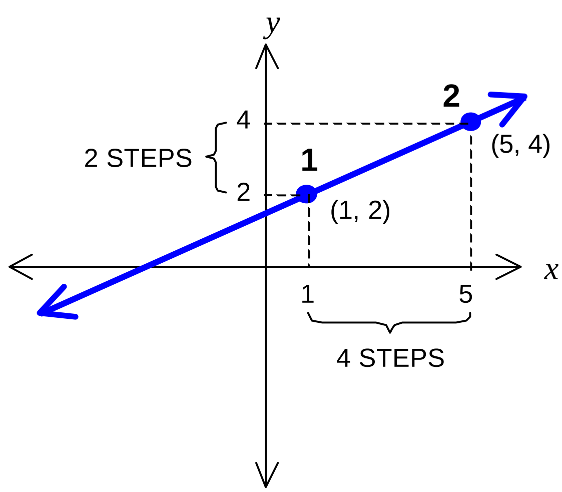
In moving from point 1 to point 2, we cover 4 steps horizontally (the x direction) and 2 steps vertically (the y direction):



Therefore, the ratio of the change in altitude to the change in horizontal distance is 2 to 4. Expressing it as a fraction and reducing, we say that the slope of this line is

$$\frac{2}{4} = \frac{1}{2}$$

To formalize this procedure a bit, we need to think about the two points in terms of their x and y coordinates.



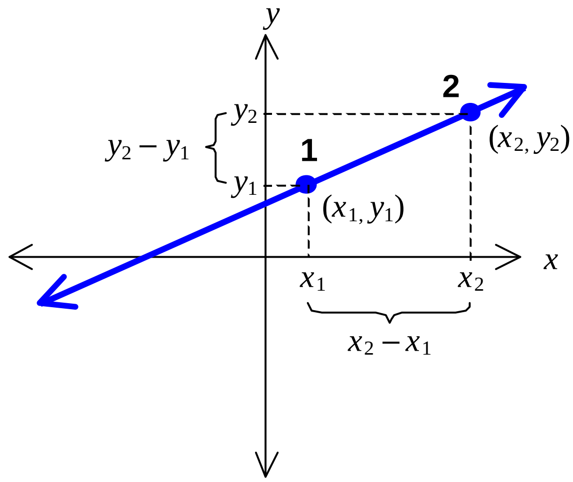
Now you should be able to see that the horizontal displacement is the difference between the x coordinates of the two points, or

$$4 = 5 - 1,$$

and the vertical displacement is the difference between the y coordinates, or

$$2 = 4 - 2.$$

In general, if we say that the coordinates of point 1 are (x_1, y_1) and the coordinates of point 2 are (x_2, y_2),

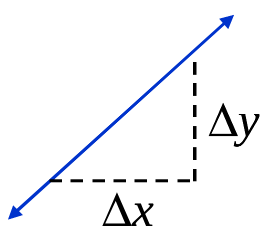


then we can define the slope *m* as follows:

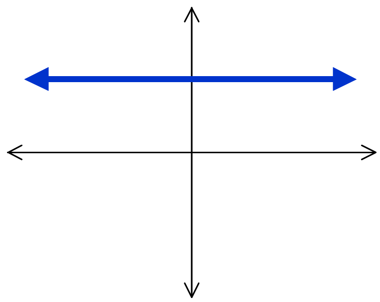
$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1}$$

where (x_1, y_1) and (x_2, y_2) are any two distinct points on the line.

- It is customary (in the US) to use the letter *m* to represent slope. No one seems to knows why.
- It makes *no difference* which two points are used for point 1 and point 2. If they were switched, both the numerator and the denominator of the fraction would be changed to the opposite sign, giving exactly the same result.
- Many people find it useful to remember this formula as “slope is rise over run.”
- Another common notation is $m = \frac{\Delta y}{\Delta x}$, where the Greek letter delta (Δ) means “the change in.” The slope is a *ratio* of how much *y* changes per change in *x*:



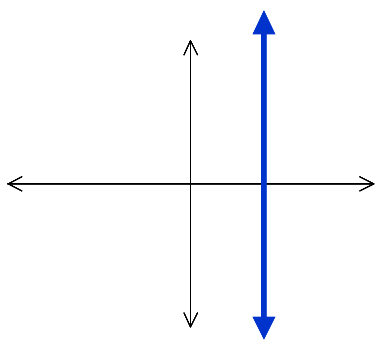
HORIZONTAL LINES



A horizontal line has zero slope, because there is no change in y as x increases. Thus, any two points will have the same y coordinates, and since $y_1 = y_2$,

$$m = \frac{rise}{run} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0}{x_2 - x_1} = 0 .$$

VERTICAL LINES



A vertical line presents a different problem. If you look at the formula

$$m = \frac{y_2 - y_1}{x_2 - x_1},$$

you see that there is a problem with the denominator. It is not possible to get two different values for x_1 and x_2, because if x changes then you are not on the vertical line anymore. Any two points on a vertical line will have the *same* x coordinates, and so $x_2 - x_1 = 0$. Since the denominator of a fraction cannot be zero, we have to say that **a vertical line has undefined slope**. Do not confuse this with the case of the horizontal line, which has a well-defined slope that just happens to equal zero.

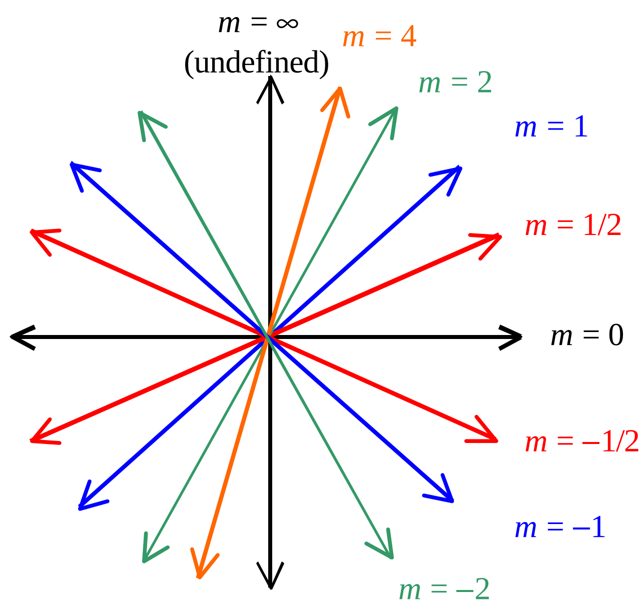
POSITIVE AND NEGATIVE SLOPE

The x coordinate increases to the right, so moving from left to right is motion in the positive x direction. Suppose that you are going uphill as you move in the positive x direction. Then both your x and y coordinates are increasing, so the ratio of rise over run will be positive—you will have a positive increase in y for a positive increase in x. On the other hand, if you are going downhill as you move from left to right, then the ratio of rise over run will be negative because you *lose* height for a given positive increase in x. The thing to remember is:

As you go from left to right,

- Uphill = Positive Slope
- Downhill = Negative Slope

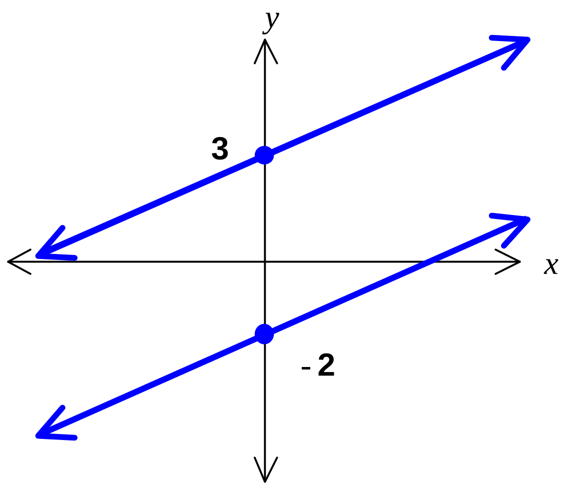
And of course, no change in height means that the line has zero slope.



Some Slopes

INTERCEPTS

Two lines can have the same slope and be in different places on the graph. This means that in addition to describing the slope of a line we need some way to specify exactly where the line is on the graph. This can be accomplished by specifying one particular point that the line passes through. Although any point will do, it is conventional to specify the point where the line crosses the y-axis. This point is called the *y-intercept*, and is usually denoted by the letter b. This is useful because every line except vertical lines will eventually cross the y-axis at some point, and we have to handle vertical lines as a special case anyway because we cannot define a slope for them.



Same Slopes, Different *y*-Intercepts

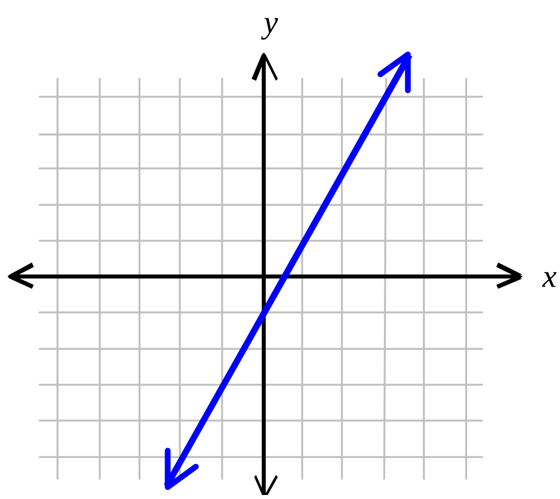
EQUATIONS

The equation of a line gives the mathematical relationship between the *x* and *y* coordinates of any point on the line.

Let's return to the example we used in graphing functions. The equation

$$y = 2x - 1$$

produces the following graph:



This line evidently has a slope of 2 and a *y* intercept equal to –1. The numbers 2 and –1 also appear in the equation—the coefficient of *x* is 2, and the additive constant is –1. This is not a coincidence, but is due to the standard form in which the equation was written.

Standard Form (Slope-Intercept Form)

If a linear equation in two unknowns is written in the form

$$y = mx + b$$

where m and b are any two real numbers, then the graph will be a straight line with a slope of m and a y intercept equal to b.

Point-Slope Form

As mentioned earlier, a line is fully described by giving its slope and one distinct point that the line passes through. While this point is customarily the y intercept, it does not need to be. If you want to describe a line with a given slope m that passes through a given point (x_1, y_1), the formula is

$$y - y_1 = m(x - x_1)$$

To help remember this formula, think of solving it for m:

$$m = \frac{y - y_1}{x - x_1}$$

Since the point (x, y) is an arbitrary point on the line and the point (x_1, y_1) is another point on the line, this is nothing more than the definition of slope for that line.

Two-Point Form

Another way to completely specify a line is to give two different points that the line passes through. If you are given that the line passes through the points (x_1, y_1) and (x_2, y_2), the formula is

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

This formula is also easy to remember if you notice that it is just the same as the point-slope form with the slope *m* replaced by the definition of slope, $m = \frac{y_2 - y_1}{x_2 - x_1}$

Chapter 5: Systems of Linear Equations

THE SOLUTIONS OF A SYSTEM OF EQUATIONS

A system of equations refers to a number of equations with an equal number of variables. We will only look at the case of two linear equations in two unknowns. The situation gets much more complex as the number of unknowns increases, and larger systems are commonly attacked with the aid of a computer.

A system of two linear equations in two unknowns might look like

$$\begin{cases} 2x + 4y = 3 \\ x - 3y = 1 \end{cases}$$

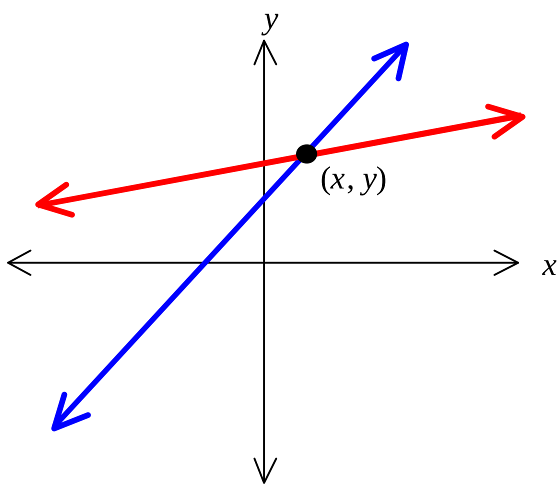
This is the standard form for writing equations when they are part of a system of equations: the variables go in the same order on the left side, and the constant term is on the right. The bracket on the left is meant to indicate that the two equations are intended to be solved simultaneously, but it is not always used.

When we talk about the *solution* of this system of equations, we mean the values of the variables that make both equations true at the same time. There may be many pairs of x and y that make the first equation true, and many pairs of x and y that make the second equation true, but we are looking for an x and y that would work in *both* equations. In the following pages we will look at algebraic methods for finding this solution, if it exists.

Because these are linear equations, their graphs will be straight lines. This can help us visualize the situation graphically. There are three possibilities:

1. INDEPENDENT EQUATIONS

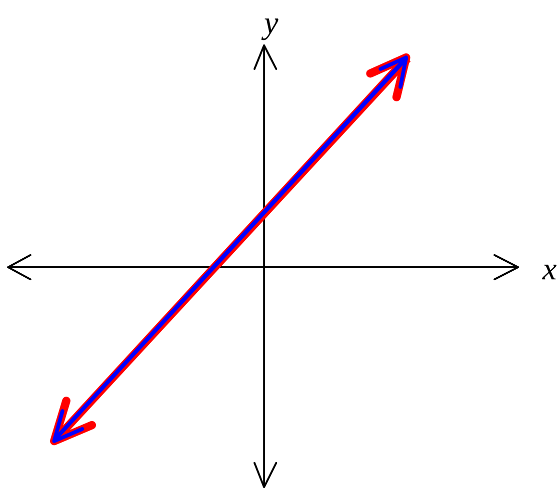
- ***Lines intersect***
- ***One solution***



In this case the two equations describe lines that intersect at one particular point. Clearly this point is on both lines, and therefore its coordinates (x, y) will satisfy the equation of either line. Thus the pair (x, y) is the one and only solution to the system of equations.

2. DEPENDENT EQUATIONS

- ***Equations describe the same line***
- ***Infinite number of solutions***



Sometimes two equations might look different but actually describe the same line. For example, in

$$\begin{cases} 2x+3y=1 \\ 4x+6y=2 \end{cases}$$

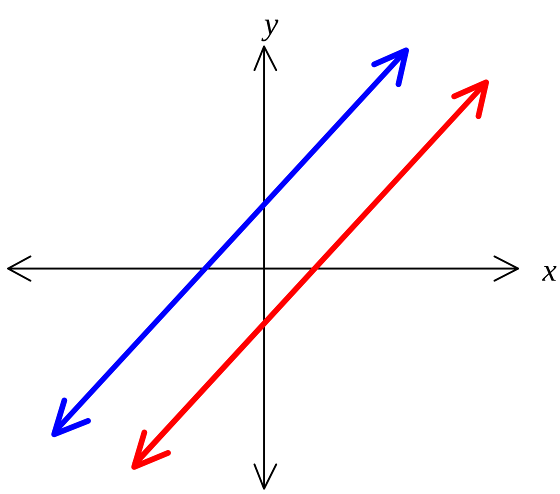
The second equation is just two times the first equation, so they are actually equivalent and would both be equations of the same line. Because the two equations describe the same line, they have *all* their points in common; hence there are an infinite number of solutions to the system.

- ***Attempting to solve gives an identity***

If you try to solve a dependent system by algebraic methods, you will eventually run into an equation that is an *identity*. An identity is an equation that is always true, independent of the value(s) of any variable(s). For example, you might get an equation that looks like $x = x$, or 3 = 3. This would tell you that the system is a dependent system, and you could stop right there because you will never find a unique solution.

3. Inconsistent Equations

- ***Lines do not intersect (Parallel Lines; have the same slope)***
- ***No solutions***



If two lines happen to have the same slope, but are not identically the same line, then they will never intersect. There is no pair (x, y) that could satisfy both equations, because there is no point (x, y) that is simultaneously on both lines. Thus these equations are said to be *inconsistent*, and there is no solution. The fact that they both have the same slope may not be obvious from the equations, because they are not written in one of the standard forms for straight lines. The slope is not readily evident in the form we use for writing systems of equations.

(But if you think about it you will see that the slope is the negative of the coefficient of x divided by the coefficient of y).

- ***Attempting to solve gives a false statement***

By attempting to solve such a system of equations algebraically, you are operating on a false assumption—namely that a solution exists. This will eventually lead you to a *contradiction*: a statement that is obviously false, regardless of the value(s) of the variable(s). At some point in your work you would get an obviously false equation like $3 = 4$. This would tell you that the system of equations is inconsistent, and there is no solution.

Solution by *Graphing*

For more complex systems, such as those that contain non-linear equations, finding a solution by algebraic methods can be very difficult or even impossible. Using a graphing calculator (or a computer), you can graph the equations and actually see where they intersect. The calculator can then give you the coordinates of the intersection point. The only drawback to this method is that the solution is only an approximation, whereas the algebraic method gives the exact solution. In most practical situations, though, the precision of the calculator is sufficient. For more demanding scientific and engineering applications there are computer methods that can find approximate solutions to very high precision.

ADDITION METHOD

The whole problem with solving a system of equations is that you cannot solve an equation that has two unknowns in it. You need an equation with only one variable so that you can isolate the variable on one side of the equation. Both methods that we will look at are techniques for eliminating one of the variables to give you an equation in just one unknown, which you can then solve by the usual methods.

The first method of solving systems of linear equations is the addition method, in which the two equations are added together to eliminate one of the variables.

Adding the equations means that we add the left sides of the two equations together, and we add the right sides together. This is legal because of the Addition Principle, which says that we can add the same amount to both sides of an equation. Since the left and right sides of any equation are equal to each other, we are indeed adding the same amount to both sides of an equation.

Consider this simple example:

Example:

$$\begin{cases} 3x+2y=4 \\ 2x-2y=1 \end{cases}$$

If we add these equations together, the terms containing y will add up to zero ($2y$ plus $-2y$), and we will get

$$\begin{aligned} 3x+2y&=4 \\ \underline{2x-2y}&\underline{=1} \\ 5x+\ \ 0&=5 \end{aligned}$$

or

$5x = 5$
$\boldsymbol{x = 1}$

However, we are not finished yet—we know x, but we still don't know y. We can solve for y by substituting the now known value for x into either of our original equations. This will produce an equation that can be solved for y:

$$\begin{aligned} 3x+2y&=4 \\ 3(1)+2y&=4 \\ 3+2y&=4 \\ 2y&=1 \\ y&=\frac{1}{2} \end{aligned}$$

Now that we know both x and y, we can say that the solution to the system is the pair (1, 1/2).

This last example was easy to see because of the fortunate presence of both a positive and a negative $2y$. One is not always this lucky. Consider

Example:

$$\begin{cases} x+2y=3 \\ 3x+4y=2 \end{cases}$$

Now there is nothing so obvious, but there is still something we can do. If we multiply the first equation by -3, we get

$$\begin{cases} -3x-6y=-9 \\ 3x+4y=2 \end{cases}$$

(Don't forget to multiply every term in the equation, on both sides of the equal sign). Now if we add them together the terms containing *x* will cancel:

$$\begin{array}{r} -3x-6y=-9 \\ \underline{3x+4y=2} \\ -2y=-7 \end{array}$$

or

$$y=\frac{7}{2}$$

As in the previous example, now that we know *y* we can solve for *x* by substituting into either original equation. The first equation looks like the easiest to solve for *x*, so we will use it:

$$\begin{aligned} &x+2y=3 \\ &x+2\left(\frac{7}{2}\right)=3 \\ &x+7=3 \\ &x=-4 \end{aligned}$$

And so the solution point is (–4, 7/2).

Now let's look at an even less obvious example:

Example:

$$\begin{cases} 5x-2y=6 \\ 2x+3y=10 \end{cases}$$

Here there is nothing particularly attractive about going after either the *x* or the *y*. In either case, *both* equations will have to be multiplied by some factor to arrive at a common coefficient. This is very much like the situation you face trying to find a least common denominator for adding fractions, except that here we call it a Least Common Multiple (LCM). As a general rule, it is easiest to eliminate the variable with the smallest LCM. In this case that would be the *y*, because the LCM of 2 and 3 is 6. If we wanted to eliminate the *x* we would have to use an LCM of 10 (5 times 2). So, we choose to make the coefficients of *y* into plus and minus 6. To do this, the first equation must be multiplied by 3, and the second equation by 2:

$$\begin{aligned} &(3)5x-(3)2y=(3)6 \\ &(2)2x+(2)3y=(2)10 \end{aligned}$$

or

$$15x - 6y = 18$$
$$4x + 6y = 20$$

Now adding these two together will eliminate the terms containing *y*:

$$\begin{array}{l} 15x - 6y = 18 \\ \underline{\ \ 4x + 6y = 20} \\ 19x \ \ = 38 \end{array}$$

or

$$x = 2$$

We still need to substitute this value into one of the original equation to solve for *y*:

$$\begin{aligned} &2x + 3y = 10 \\ &2(2) + 3y = 10 \\ &4 + 3y = 10 \\ &3y = 6 \\ &y = 2 \end{aligned}$$

Thus the solution is the point (2, 2).

SUBSTITUTION METHOD

When we used the Addition Method to solve a system of equations, we still had to do a substitution to solve for the remaining variable. With the substitution method, we solve one of the equations for one variable in terms of the other, and then substitute that into the other equation. This makes more sense with an example:

Example:

$2y + x = 3$ (1)

$4y – 3x = 1$ (2)

Equation 1 looks like it would be easy to solve for *x*, so we take it and isolate *x*:

$2y + x = 3$

$$x = 3 - 2y \qquad (3)$$

Now we can use this result and substitute 3 – 2*y* in for *x* in equation 2:

$$\begin{aligned} 4y - 3x &= 1 \\ 4y - 3(3 - 2y) &= 1 \\ 4y - 9 + 6y &= 1 \\ 10y - 9 &= 1 \\ 10y &= 10 \\ y &= 1 \end{aligned}$$

Now that we have *y*, we still need to substitute back in to get *x*. We could substitute back into any of the previous equations, but notice that equation 3 is already conveniently solved for *x*:

$$\begin{aligned} x &= 3 - 2y \\ x &= 3 - 2(1) \\ x &= 3 - 2 \\ x &= 1 \end{aligned}$$

And so the solution is (1, 1).

As a rule, the substitution method is easier and quicker than the addition method when one of the equations is very simple and can readily be solved for one of the variables.

Chapter 6: Polynomials

POLYNOMIALS

Definition: A polynomial is an algebraic expression that is a sum of terms, where each term contains only variables with whole number exponents and integer coefficients.

Example: The following expressions are all considered polynomials:

$x^2 + 2x - 7$

$x^4 - 7x^3$

x

When we write a polynomial we follow the convention that says we write the terms in order of descending powers, from left to right.

The following are NOT polynomials:

$\frac{1}{x}$

$\sqrt{x^3 - 4}$

$x^2 + 3x + 2x^{-2}$

A polynomial can have any number of terms ("poly" means "many"). We have special names for polynomials that have one, two, or three terms:

MONOMIAL

A monomial has one term ("mono" means "one"). The following are monomials:

x

$3x^4$

$2x^3$

Binomial

A binomial has two terms:

$x + 1$

$5x^2 - 3x$

Trinomial

A trinomial has three terms:

$x^4 + 2x^3 - 3x$

$2x^2 - 4x + 1$

Degree of a Term

The *degree* of an individual term in a polynomial is the sum of powers of all the variables in that term. We only have to use the plurals in this definition because of the possibility that there may be more than one variable. In practice, you will most often see polynomials that have only one variable (traditionally denoted by the letter 'x'). In that case, the degree will simply be the power of the variable.

Examples:

$2x^3$ Degree = 3

$3x^4$ Degree = 4

x Degree = 1

$3x^2y^5$ Degree = 7 (because 2 + 5 = 7)

37 Degree = 0

Why is the last example, which is just a plain number, considered to be of degree zero? It is because of the fact that $x^0 = 1$, and everything has a factor of 1. So we can say that 37 is the coefficient of x^0.

Degree of a Polynomial

The degree of the entire polynomial is the degree of the highest-degree term that it contains, so

$x^2 + 2x - 7$ is a second-degree trinomial, and $x^4 - 7x^3$ is a fourth-degree binomial.

Addition and Subtraction of Polynomials

Adding (or subtracting) polynomials is really just an exercise in collecting like terms. For example, if we want to add the polynomial

$2x^2 + 4x - 3$

to the polynomial

$6x + 4$,

we would just put them together and collect like terms:

$$\begin{aligned}(2x^2 + 4x - 3) + (6x - 4) &= 2x^2 + 4x - 3 + 6x - 4 \\ &= 2x^2 + 10x - 7\end{aligned}$$

Notice that the parentheses in the first line are only there to distinguish the two polynomials—they don't really do anything. If the operation was subtraction instead of addition, we would have to change the signs of all the terms in the second polynomial:

$$\begin{aligned}(2x^2 + 4x - 3) - (6x - 4) &= 2x^2 + 4x - 3 - 6x + 4 \\ &= 2x^2 - 2x - 1\end{aligned}$$

Although this is basically just a bookkeeping activity, it can get a little messy when there are many terms. One way to help keep things straight is to use the column format for addition, keeping like terms lined up in columns:

$$\begin{array}{r} 2x^2 + 4x - 3 \\ +6x - 4 \\ \hline 2x^2 + 10x - 7 \end{array}$$

This method is particularly helpful in the case of subtraction, because it is too easy to make a mistake distributing the minus sign when you write it all in one row.

MULTIPLICATION OF POLYNOMIALS

- The general rule is that **each term in the first factor has to multiply each term in the other factor**
- The number of products you get has to be the number of terms in the first factor times the number of terms in the second factor. For example, a binomial times a binomial gives four products, while a binomial times a trinomial gives six products.
- Be very careful and methodical to avoid missing any terms
- After the multiplication is complete you can try to collect like terms to simplify the result

EXAMPLE: PRODUCT OF A BINOMIAL AND A TRINOMIAL

$(x + 2)(x^2 - 2x + 3)$

There are six possible products. We can start with the x and multiply it by all three terms in the other factor, and then do the same with the 2. It would look like this:

$(x + 2)(x^2 - 2x + 3)$

$= (x)x^2 - (x)2x + (x)3 + (2)x^2 - (2)2x + (2)3$

$= x^3 - 2x^2 + 3x + 2x^2 - 4x + 6$

$= x^3 - x + 6$

This method can get hard to keep track of when there are many terms. There is, however, a more systematic method based on the stacked method of multiplying numbers:

Stack the factors, keeping like degree terms lined up vertically:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline \end{array}$$

Multiply the 2 and the 3:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline 6 \end{array}$$

Multiply the 2 and the $-2x$:

$$\begin{array}{r} x^2 - 2x + 3 \\ x + 2 \\ \hline -4x + 6 \end{array}$$

Multiply the 2 and the x^2:

$$\begin{array}{r} x^2-2x+3 \\ x+2 \\ \hline 2x^2-4x+6 \end{array}$$

Now multiply the x by each term above it, and write the results down underneath, keeping like degree terms lined up vertically:

$$\begin{array}{r} x^2-2x+3 \\ x+2 \\ \hline 2x^2-4x+6 \\ 3x \end{array}$$

$$\begin{array}{r} x^2-2x+3 \\ x+2 \\ \hline 2x^2-4x+6 \\ -2x^2+3x \end{array}$$

$$\begin{array}{r} x^2-2x+3 \\ x+2 \\ \hline 2x^2-4x+6 \\ x^3-2x^2+3x \end{array}$$

Then you just add up the like terms that are conveniently stacked above one another:

$$\begin{array}{r} x^2-2x+3 \\ x+2 \\ \hline 2x^2-4x+6 \\ x^3-2x^2+3x \\ \hline x^3+0-x+6 \end{array}$$

This stacked method is much safer, because you are far less likely to accidentally overlook one of the products, but it does take up more space on the paper.

PRODUCT OF A MONOMIAL AND A BINOMIAL: DISTRIBUTIVE LAW

If one of the polynomial factors is just a monomial, than the multiplication involves nothing more than distributing the monomial and simplifying the products of monomials.

Example:

$ab(2a + 1) = ab(2a) + ab(1) = 2a^2b + ab$

PRODUCT OF TWO BINOMIALS: FOIL (FIRST-OUTER-INNER-LAST)

Because the situation of a binomial times a binomial is so common, it helps to use a quick mnemonic device to help remember all the products. This is called the FOIL method.

Example:

$$(x + 2)(x + 3)$$

F O I L

$$x^2 + 3x + 2x + 6$$

1. The F stands for *first*, which means the x in the first factor times the x in the second factor
2. The O stands for *outer*, which means the x in the first factor times the 3 in the second factor
3. The I stands for *inner*, which means the 2 in the first factor times the x in the second factor
4. The L stands for *last*, which means the 2 in the first factor times the 3 in the second factor

- Of course you would then combine the $3x + 2x$ into a $5x$, because they are like terms, so the final result is $(x + 2)(x + 3) = x^2 + 5x + 6$

SPECIAL PRODUCTS OF BINOMIALS

Some products occur so frequently in algebra that it is advantageous to be able to recognize them by sight. This will be particularly useful when we talk about factoring.

In the following examples the special products of binomials are multiplied out using the FOIL method, and then simplified

DIFFERENCE OF TWO SQUARES

F O I L

$$(a + b)(a - b) = a^2 - ab + ab - b^2$$
$$= a^2 - b^2$$

SQUARING A BINOMIAL

F O I L

$$(a + b)^2 = (a + b)(a + b) = a^2 + ab + ab + b^2$$
$$= a^2 + 2ab + b^2$$

$$\begin{aligned}(a-b)^2 = (a-b)(a-b) &= \overset{F}{a^2} \overset{O}{-ab} \overset{I}{-ab} \overset{L}{+b^2} \\ &= a^2 - 2ab + b^2\end{aligned}$$

What you should to be able to recognize by sight are these three formulas:

$$(a+b)^2 = a^2 + 2ab + b^2$$

$$(a-b)^2 = a^2 - 2ab + b^2$$

$$(a+b)(a-b) = a^2 - b^2$$

You should be able to recognize these products both ways. That is, if you see the left side you should think of the right side, and if you see the right side you should think of the left side.

FACTORING POLYNOMIALS

Factoring a polynomial is the opposite process of multiplying polynomials. Recall that when we factor a number, we are looking for prime factors that multiply together to give the number; for example

$6 = 2 \times 3$, or $12 = 2 \times 2 \times 3$.

When we factor a polynomial, we are looking for simpler polynomials that can be multiplied together to give us the polynomial that we started with. You might want to review multiplying polynomials if you are not completely clear on how that works.

- When we factor a polynomial, we are usually only interested in breaking it down into polynomials that have **integer** coefficients and constants.

SIMPLEST CASE: REMOVING COMMON FACTORS

The simplest type of factoring is when there is a factor common to every term. In that case, you can factor out that common factor. What you are doing is using the distributive law in reverse—you are sort of un-distributing the factor.

Recall that the distributive law says

$a(b + c) = ab + ac$.

Thinking about it in reverse means that if you see $ab + ac$, you can write it as $a(b + c)$.

Example: $2x^2 + 4x$

Notice that each term has a factor of $2x$, so we can rewrite it as:

$$2x^2 + 4x = 2x(x + 2)$$

Removing More Complicated Factors

The common factor can be anything, even a complicated expression. Consider the following example:

$$3x^2p + 7p$$

These two terms have a common factor of p, which we can factor out:

$$3x^2p + 7p = p(3x^2 + 7)$$

Now suppose the p was in parentheses. This would not change anything, and we would have

$$3x^2(p) + 7(p) = (p)(3x^2 + 7).$$

What if what was in the parentheses was more than just a single p? Let's use "###" to mean "some expression", as long as all the expressions denoted by "###" are identical. Then we could say

$$3x^2(\#\#\#) + 7(\#\#\#) = (\#\#\#)(3x^2 + 7).$$

Example: $5x(x - 1) - 2(x - 1)$

Here the common factor is the binomial $(x - 1)$, and we can factor it out:

$$5x(x - 1) - 2(x - 1) = (x - 1)(5x - 2)$$

This technique will be used soon when we discuss the grouping method for factoring second-degree trinomials.

Difference of Two Squares

If you see something of the form $a^2 - b^2$, you should remember the formula

$$(a + b)(a - b) = a^2 - b^2$$

Example: $x^2 - 4 = (x - 2)(x + 2)$

- This only holds for a **difference** of two squares. There is no way to factor a **sum** of two squares such as $a^2 + b^2$ into factors with real numbers.

TRINOMIALS (QUADRATIC)

A quadratic trinomial has the form

$$\boldsymbol{ax^2 + bx + c},$$

where the coefficients a, b, and c, are real numbers (when discussing factoring we will only use integers, but in general they could be any real number). We are interested here in factoring quadratic trinomials with integer coefficients into factors that have integer coefficients. Not all such quadratic polynomials can be factored with integer coefficients. Therefore, when we say a quadratic can be factored, we mean that we can write the factors with only integer coefficients. Otherwise, it is like a prime number in that it cannot be broken down any further.

If a quadratic can be factored, it will be the product of two first-degree binomials, except for very simple cases that just involve monomials. For example x^2 by itself is a quadratic expression where the coefficient a is equal to 1, and b and c are zero. Obviously, x^2 factors into $(x)(x)$, but this is not a very interesting case.

A slightly more complicated case occurs when only the coefficient c is zero. Then you get something that looks like

$$2x^2 + 3x$$

This can be factored very simply by factoring out ('undistributing') the common factor of x:

$$2x^2 + 3x = x(2x + 3)$$

The most general case is when all three terms are present, as in

$$x^2 + 5x + 6$$

We look at two cases of this type. The easiest to factor are the ones where the coefficient of x^2 (which we are calling 'a') is equal to 1, as in the above example. If a is not 1 then things get a little bit more complicated, so we will begin by looking at $a = 1$ examples.

Coefficient of x^2 is 1

Since the trinomial comes from multiplying two first-degree binomials, let's review what happens when we multiply binomials using the FOIL method. Remember that to do factoring we will have to think about this process in reverse (you could say we want to "de-FOIL" the trinomial).

Suppose we are given

$$(x + 2)(x + 3)$$

Using the FOIL method, we get

$(x + 2)(x + 3) = x^2 + 3x + 2x + 6$

Then, collecting like terms gives

$(x + 2)(x + 3) = x^2 + 5x + 6$

Now look at this and think about where the terms in the trinomial came from. Obviously the x^2 came from x times x (the F in FOIL). The interesting part is what happens with the other parts, the '+ 2' and the '+ 3'. The last term in the trinomial, the 6 in this case, came from **multiplying** the 2 and the 3 (the L in FOIL). Where did the $5x$ in the middle come from? We got the $5x$ by **adding** the $2x$ and the $3x$ when we collected like terms (the O + I in FOIL). We can state this as a rule:

- If the coefficient of x^2 is one, then to factor the quadratic you need to find two numbers that:

    1. Multiply to give the constant term (which we call c)

    2. Add to give the coefficient of x (which we call b)

The best way to go about finding these numbers is to start by considering all the possible pairs of factors of the constant term c, and then seeing if any pair of factors adds up to b. If no pair of factors of c adds up to b, you can conclude that the trinomial can't be factored.

This rule works even if there are minus signs in the quadratic expression (assuming that you remember how to add and multiply with positive and negative numbers).

SPECIAL CASE: PERFECT SQUARE TRINOMIAL

Recall from special products of binomials that

$$(a+b)^2 = a^2 + 2ab + b^2$$

and

$$(a-b)^2 = a^2 - 2ab + b^2$$

The trinomials on the right are called *perfect squares* because they are the squares of a single binomial, rather than the product of two different binomials. A quadratic trinomial can also have this form:

$(x + 3)^2 = (x + 3)(x + 3) = x^2 + 6x + 9$

Notice that just as before the coefficient of x is the **sum** 3 + 3, and the constant term is the **product** 3×3. One can also say that

1. The coefficient of x is twice the number 3
2. The constant term is the number three squared

In general, if a quadratic trinomial is a perfect square, then

- The coefficient of x is twice the square root of the constant term

Or to put it another way,

- **The constant term is the square of half the coefficient of *x***

In symbolic form we can express this as

$$(x+a)^2 = x^2 + 2ax + a^2$$

It is helpful to be able to recognize perfect square trinomials. We will see them again when we talk about solving quadratic equations.

Coefficient of x^2 is not 1

A quadratic is more difficult to factor when the coefficient of the squared term is not 1, because that coefficient is mixed in with the other products from FOILing the two binomials. There are two methods for attacking these: either you can use a systematic guess-and-check method, or a method called factoring by grouping. We will first look at the guess-and-check method (which we could call factoring by *groping*).

If you need to factor a trinomial such as

$$2x^2 + x - 3,$$

you have to think about what combinations could give the $2x^2$ as well as the other two terms. In this example the $2x^2$ must come from $(x)(2x)$, and the constant term might come from either $(-1)(3)$ or $(1)(-3)$. The hard part is figuring out which combination will give the correct middle term. This gets messy because all those coefficients will be mixed in with the middle term when you FOIL the binomials. To see what is going on, let's see what happens when we FOIL the following binomials:

$$\begin{aligned}(x-1)(2x+3) &= 2x^2 + 3x - 2x - 3 \\ &= 2x^2 + x - 3\end{aligned}$$

What happened? There are several significant things to notice:

1. The leading term in the trinomial (the $2x^2$) is just the product of the leading terms in the binomials.

2. The constant term in the trinomial (the -3) is the product of the constant terms in the binomials (so far this is the same as in the case where the coefficient of x^2 is 1)

3. The middle term in the trinomial (the x) is the sum of the outer and inner products, which involves *all* the constants and coefficients in the binomials, in a messy way that is not always obvious by inspection.

Because 1 and 2 are relatively simple and 3 is complicated, it makes sense to think of the possible candidates that would satisfy conditions 1 and 2, and then test them in every possible combination by multiplying the resulting binomials to see if you get the correct middle term. This seems tedious, and indeed it can be if the numbers you are working with have a lot of factors, but in practice you usually only have to try a few combinations before you see what will work. As a demonstration, let's see how we would attack the example by this method.

Given

$$2x^2 + x - 3$$

We make a list of the possible factors of $2x^2$: The only choice is $(2x)(x)$.

Then we make a list of the possible factors of the constant term −3: it is either (1)(−3) or (−1)(3). (Notice that since we need a negative number, one factor must be negative and the other positive, so we have to try it both ways).

The possible factors of the trinomial are the binomials that we can make out of these possible factors, taken in every possible order. From these possibilities, we see that the candidate binomials are:

$(2x + 1)(x - 3)$
$(x + 1)(2x - 3)$
$(2x + 3)(x - 1)$
$(x + 3)(2x - 1)$

If we start multiplying these out, we will find that the third one works, and then we are finished. All you really need to check is to see if the sum of the outer and inner multiplications will give you the correct middle term, since we already know that we will get the correct first and last terms.

In short, the method is:

1. List all the possible ways to get the coefficient of x^2 (which we call a) by multiplying two numbers

2. List all the possible ways to get the constant term (which we call c) by multiplying two numbers

3. Try all possible combinations of these to see which ones give the correct middle term

   - Don't forget that the number itself times 1 is always a possibility for the pair of factors

   - If the number (a or c) is negative, remember to try the plus and minus signs both ways

Another method for factoring these kinds of quadratic trinomials is called factoring by grouping. Factoring by grouping can be a bit more tedious, and is often not worth the trouble if you can find the correct factors by some quick trial and error. However, it works quite well when the factors are not immediately obvious, such as when you have a very large number of candidate factors. When this happens, it is the trial and error method becomes tedious.

FACTORING A QUADRATIC TRINOMIAL BY GROUPING

Factoring by grouping is best demonstrated with a few examples.

Example:

Given: $5x^2 + 11x + 2$

Find the product *ac*: $(5)(2) = 10$

Think of two factors of 10 that add up to 11: 1 and 10

Re-write the original trinomial, but write the middle term $11x$ as the sum of $1x$ and $10x$: $5x^2 + 1x + 10x + 2$

Group the two pairs of terms: $(5x^2 + 1x) + (10x + 2)$

Remove common factors from each group: $x(5x + 1) + 2(5x + 1)$

Notice that the two quantities in parentheses are now identical. That means we can factor out a common factor of $(5x + 1)$: $(5x + 1)(x + 2)$

Example:

Given: $4x^2 + 7x - 15$

Find the product *ac*: $(4)(-15) = -60$

Think of two factors of -60 that add up to 7: -5 and 12

Write the $7x$ as the sum of $-5x$ and $12x$: $4x^2 - 5x + 12x - 15$

Group the two pairs of terms: $(4x^2 - 5x) + (12x - 15)$

Remove common factors from each group: $x(4x - 5) + 3(4x - 5)$

Notice that the two quantities in parentheses are now identical. That means we can factor out a common factor of $(4x - 5)$: $(4x - 5)(x + 3)$

WARNING: There is (as always) a potential pitfall with minus signs.

Example:

Given: $3x^2 - 23x + 14$

Find the product *ac*: $(3)(14) = 42$

Think of two factors of 42 that add up to –23. Since they multiply to give a positive number and add to give a negative number, they must both be negative: -2 and -21

Write the $-23x$ as the sum of $-2x$ and $-21x$: $3x^2 - 2x - 21x + 14$

Group the two pairs of terms: $(3x^2 - 2x) - (21x + 14)$

Now something is not right. Can you see what it is? Look at the last term, the 14. In the original trinomial it was positive, but now that it is in parentheses, the minus sign in front of the parentheses applies to it, too. If we cleared the parentheses we would have –14 , which is not equal to what we started with. In order to fix this problem we have to change the sign of the 14 inside the parentheses:

Correct: $(3x^2 - 2x) - (21x - 14)$

Now remove common factors from each group: $x(3x - 2) + 7(3x - 2)$

Factor out the common factor of $(3x - 2)$: $(3x - 2)(x - 7)$

- In short, whenever you end up with a minus sign in front of the parentheses after grouping, you must switch the sign of the last term.

The Procedure

Given a general quadratic trinomial

$ax^2 + bx + c$

1. Find the product ac.

2. Find two numbers h and k such that

$hk = ac$

(h and k are factors of the product of the coefficient of x^2 and the constant term)

AND

$h + k = b$

(h and k add to give the coefficient of x)

- If you can't find two numbers that add up to b after considering all the factors of ac, then the original trinomial cannot be factored.

3. Rewrite the quadratic as

$ax^2 + hx + kx + c$

4. Group the two pairs of terms that have common factors.

$(ax^2 + hx) + (kx + c)$

5. Factor out any common factors from both groups

6. Factor out the common binomial that will appear. Because of the way you chose h and k, you will always left with two identical expressions in parentheses, as in the examples above).

Why this works (for those who need to see the gory details)

Suppose the quadratic trinomial in question came from multiplying two arbitrary binomials:

$(px + n)(qx + m)$

If we multiply this out we will get

$pqx^2 + pmx + qnx + nm$

or

$pqx^2 + (pm + qn)x + nm$

Notice that the coefficient of x consists of a sum of two terms, pm and qn. These are the two numbers we called h and k above.

$pm = h$

$qn = k$

Now we see that the two numbers h and k add up to the coefficient of x, which we called b:

$h + k = b$

Obviously they are factors of their own product $pmqn$, but we notice that $pq = a$, and $mn = c$, so

$(pm)(qn) = (pq)(nm)$

which is equivalent to

$hk = ac$

Chapter 7: Rational Expressions

RATIONAL EXPRESSIONS

- A *rational expression* is a ratio of polynomials:

Examples:

$$\frac{x^2-1}{2-x}$$

$$\frac{1}{x}$$

$$\frac{x^3+2x^2-x+5}{x^2+3x-1}$$

EXCLUDED VALUES

Whenever an expression containing variables is present in the denominator of a fraction, you should be alert to the possibility that certain values of the variables might make the denominator equal to zero, which is forbidden. This means that when we are talking about rational expressions we can no longer say that the variable represents “any real number.” Certain values may have to be excluded. For example, in the expression

$$\frac{2x-1}{3x},$$

we cannot allow the value $x = 0$ so we would parenthetically add the comment ($x \neq 0$), and for

$$\frac{2x}{x-3}$$

we would say $x \neq 3$. In the case of

$$\frac{x+3}{1-x^2},$$

we would exclude both $x = 1$ and $x = -1$, since either choice would make the denominator zero.

We don't care if the **numerator** is zero. If the numerator is zero, that just makes the whole rational expression zero (assuming, of course, that the denominator is not zero), just as with common fractions. Recall that 0/4 = 0, but 4/0 is undefined.

It is important to keep this in mind as you work with rational expressions, because it can happen that you are trying to solve an equation and you get one of the "forbidden" values as a solution. You would have to discard that solution as being unacceptable. You can also get some crazy results if you don't pay attention to the possibility that the denominator might be zero for certain values of the variable. For example, consider the celebrated proof that 0 = 1.

Proof that 1 = 0

And other nonsense

Can you identify the flaw in this argument?

Let $x = 1$. Then

Given:	$x = 1$
Multiply both sides by x:	$x^2 = x$
Subtract x from both sides:	$x^2 - x = 0$
Factor out an x:	$x(x - 1) = 0$
Divide both sides by $(x - 1)$:	$x = 0$
But $x = 1$, so substitute 1 for x to get:	$1 = 0$

This is a very simple variant of this classic "proof". Once you see the trick*, you can construct more elaborate versions that do a better job of concealing the error, and you can vary it to "prove" other nonsense such as 1 = 2.

For example:

Let $x = 1$. Then

Given:	$x = 1$
Multiply both sides by –1:	$-x = -1$
Add x^2 to both sides:	$x^2 - x = x^2 - 1$

Factor both sides:	$x(x-1) = (x-1)(x+1)$
Divide both sides by $(x-1)$:	$x = (x+1)$
Substituting 1 back in for x gives the result:	$1 = 2$

*The trick: We are dividing by zero because if $x = 1$ then $(x-1) = 0$. Thus, all of these "proofs" are invalid because they use an illegal step.

SIMPLIFYING RATIONAL EXPRESSIONS

CANCELING LIKE FACTORS

When we reduce a common fraction such as

$$\frac{4}{6} \rightarrow \frac{2}{3},$$

we do so by noticing that there is a factor common to both the numerator and the denominator (a factor of 2 in this example), which we can divide out of both the numerator and the denominator.

$$\frac{4}{6} = \frac{2 \cdot 2}{2 \cdot 3} = \frac{\cancel{2} \cdot 2}{\cancel{2} \cdot 3} = \frac{2}{3}$$

We use exactly the same procedure to reduce rational expressions. Consider the rational expression

$$\frac{4x^2}{6x}$$

The numerator and denominator both have a common factor of $2x$, which we can cancel out:

$$\frac{4x^2}{6x} = \frac{2x \cdot 2x}{2x \cdot 3} = \frac{\cancel{2x} \cdot 2x}{\cancel{2x} \cdot 3} = \frac{2x}{3}$$

Polynomial / Monomial

Each term in the numerator must have a factor that cancels a common factor in the denominator.

$$\frac{4x+6}{2y}=\frac{2x+3}{y},$$

because all terms have a common factor of 2, but

$$\frac{2x+1}{2}$$

cannot be reduced because the 2 is not a common factor of the entire numerator.

WARNING You can only cancel a factor of the entire numerator with a factor of the entire denominator

> However, as an alternative, a fraction with more than one term in the numerator can be split up into separate fractions with each term over the same denominator; then each separate fraction can be reduced if possible:
>
> $$\begin{aligned}\frac{2x+1}{2}&=\frac{2x}{2}+\frac{1}{2}\\&=\frac{x}{1}+\frac{1}{2}\\&=x+\frac{1}{2}\end{aligned}$$
>
> - Think of this as the reverse of adding fractions over a common denominator.
>
> Sometimes this is a useful thing to do, depending on the circumstances. You end up with simpler fractions, but the price you pay is that you have more fractions than you started with.

- Polynomials must be factored first. You can't cancel factors unless you can see the factors:

Example:

$$\frac{x^2+2x-8}{x-2} = \frac{(x-2)(x+4)}{(x-2)}$$

$$= \frac{\cancel{(x-2)}(x+4)}{\cancel{(x-2)}}$$

$$= \frac{(x+4)}{1}$$

$$= x+4$$

- Notice how canceling (x – 2) from the denominator left behind a factor of 1

MULTIPLICATION AND DIVISION

Same rules as for rational numbers!

Multiplication

- Both the numerators and the denominators multiply together
- Common factors may be cancelled before multiplying

Example:

Given Equation: $\frac{x^2+3x+2}{x-2} \cdot \frac{x^2-4}{x+1}$

First factor all the expressions: (I also put the denominators in parentheses because then it is easier to see them as distinct factors) $= \frac{(x+1)(x+2)}{(x-2)} \cdot \frac{(x-2)(x+2)}{(x+1)}$

Now cancel common factors—any factor on the top can cancel with any factor on the bottom: $= \frac{\cancel{(x+1)}(x+2)}{(x-2)} \cdot \frac{(x-2)(x+2)}{\cancel{(x+1)}}$

$$= \frac{(x+2)}{\cancel{(x-2)}} \cdot \frac{\cancel{(x-2)}(x+2)}{1}$$

$$=\frac{(x+2)}{1}\cdot\frac{(x+2)}{1}$$

Now just multiply what's left. $=(x+2)(x+2)$

You usually do not have to multiply out the factors, just leave them as shown. $=(x+2)^2$

Division

- Multiply by the reciprocal of the divisor
- Invert the second fraction, then proceed with multiplication as above
- Do not attempt to cancel factors before it is written as a multiplication

ADDITION AND SUBTRACTION

Same procedure as for rational numbers!

- Only the numerators can add together, once all the denominators are the same

Finding the LCD

- The LCD is built up of all the factors of the individual denominators, each factor included the most number of times it appears in an individual denominator.
- The product of all the denominators is always a common denominator, but not necessarily the LCD (the final answer may have to be reduced).

Example:

Given equation: $\frac{x-1}{x^2-1}+\frac{2x}{x^2-2x+1}$

Factor both denominators: $\frac{x-1}{(x+1)(x-1)}+\frac{2x}{(x-1)(x-1)}$

Assemble the LCD: $LCD=(x+1)(x-1)(x-1)$

Note that the LCD contains both

denominators

$$LCD = \underbrace{(x+1)(x-1)}_{x^2-1}(x-1)$$

$$LCD = (x+1)\underbrace{(x-1)(x-1)}_{x^2-2x+1}$$

Build up the fractions so that they both have the LCD for a denominator:

(keep both denominators in factored form to make it easier to see what factors they need to look like the LCD)

$$\frac{x-1}{(x+1)(x-1)} + \frac{2x}{(x-1)(x-1)}$$

$$= \frac{(x-1)(x-1)}{(x+1)(x-1)(x-1)} + \frac{2x(x+1)}{(x+1)(x-1)(x-1)}$$

Now that they are over the same denominator, you can add the numerators:

$$= \frac{(x-1)(x-1)+2x(x+1)}{(x+1)(x-1)(x-1)}$$

And simplify:

$$= \frac{x^2-2x+1+2x^2+2x}{(x+1)(x-1)(x-1)}$$

$$= \frac{3x^2+1}{(x+1)(x-1)(x-1)}$$

Chapter 8: Exponents and Roots

EXPONENTS

DEFINITION

In x^n, x is the *base*, and n is the *exponent* (or *power*)

We defined positive integer powers by

$$x^n = x \cdot x \cdot x \cdot \ldots \cdot x \text{ (} n \text{ factors of } x\text{)}$$

PROPERTIES

The above definition can be extended by requiring other powers (for example negative integers) to behave just like the positive integer powers. For example, we know that

$$x^n x^m = x^{n+m}$$

for positive integer powers, because we can write out the multiplication.

Example:

$$x^2 x^5 = (x \cdot x)(x \cdot x \cdot x \cdot x \cdot x) = x \cdot x \cdot x \cdot x \cdot x \cdot x \cdot x = x^7$$

We now require that this rule hold even if n and m are not positive integers, although this means that we can no longer write out the multiplication as above (How do you multiply something by itself a negative number of times? Or a fractional number of times?).

We can find several new properties of exponents by considering the rule for dividing powers:

$$\frac{x^m}{x^n} = x^{m-n}$$

(We will assume without always mentioning it that $x \neq 0$, so that we are not committing the sin of dividing by zero). This rule is quite reasonable when m and n are positive integers and $m > n$. For example:

$$\frac{x^5}{x^2} = \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x} = \frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}} = \frac{x \cdot x \cdot x}{1} = x^3$$

where indeed 5 – 2 = 3.

However, in other cases it leads to situation where we have to define new properties for exponents. First, suppose that $m < n$. We can simplify it by canceling like factors as before:

$$\frac{x^2}{x^5} = \frac{x \cdot x}{x \cdot x \cdot x \cdot x \cdot x} = \frac{1}{x \cdot x \cdot x} = \frac{1}{x^3}$$

But following our rule would give

$$\frac{x^2}{x^5} = x^{2-5} = x^{-3}$$

In order for these two results to be consistent, it must be true that

$$\frac{1}{x^3} = x^{-3}$$

or, in general,

$$x^{-n} = \frac{1}{x^n}$$

- Notice that a minus sign in the exponent does not make the result negative—instead, it makes it the *reciprocal* of the result with the positive exponent.

This rule actually makes sense if you think about it. A positive exponent means to multiply by the number that many times, so a negative exponent must mean to "un-multiply" that many times. But "un-multiplying" is what we usually call division, so raising a number to a negative power means to divide by it that many times. That is exactly what is accomplished by putting the number in the denominator of a fraction.

Now suppose that $n = m$. The fraction becomes

$$\frac{x^n}{x^n},$$

which is obviously equal to 1. But following our rule gives

$$\frac{x^n}{x^n} = x^{n-n} = x^0$$

Again, in order to remain consistent we have to say that these two results are equal, and so we define

$$x^0 = 1$$

for all values of x (except $x = 0$, because 0^0 is undefined).

This rule looks kind of funny, because people generally expect that if there are zero factors of x, then you have nothing at all, and so it seems that x^0 should equal zero instead of one. One way to reconcile this is to remember that every number has a factor of 1 (it's standard equipment installed at the factory). So every factor x can be thought of as $1x$, and you see that if you factor out the x it always leaves behind a 1. Therefore, even if you have no factors of x, you still have that omnipresent factor of 1.

Summary of Exponent Rules

The following properties hold for all real numbers *x*, *y*, *n*, and *m*, with these exceptions:

1. 0^0 is undefined
2. Dividing by zero is undefined
3. Raising negative numbers to fractional powers can be undefined

$$x^1 = x \qquad (x^n)^m = x^{nm}$$

$$x^0 = 1 \qquad x^{-n} = \frac{1}{x^n}$$

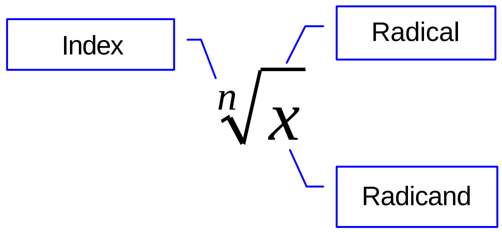
Note that the bases must be the same ⇨ $$x^n x^m = x^{n+m} \qquad \frac{x^m}{x^n} = x^{m-n}$$

$$(xy)^n = x^n y^n \qquad \left(\frac{x}{y}\right)^n = \frac{x^n}{y^n}$$

Roots

Definition

Roots are the inverse of exponents. An *n*th root "undoes" raising a number to the *n*th power. (The correct terminology for this types of relationship is *inverse function*, but powers and roots can only be strictly classified as inverse functions if we account for some ambiguities associated with plus or minus signs, so we will not worry about this yet). The most common example is the *square root*, which "undoes" the act of squaring. For example, take 3 and square it to get 9. Now take the square root of 9 and get 3 again. It is also possible to have roots related to powers other than the square. The cube root, for example, is the inverse of raising to the power of 3. The cube root of 8 is 2 because $2^3 = 8$. In general, the *n*th root of a number is written:



$\sqrt[n]{x} = y$ if and only if $y^n = x$

$\sqrt[3]{64} = 4$ because $4^3 = 64$

We leave the index off the square root symbol only because it is the most common one. It is understood that if no index is shown, then the index is 2.

$\sqrt{x} = y$ if and only if $y^2 = x$
$\sqrt{16} = 4$ because $4^2 = 16$

Square Roots

The square root is the inverse function of squaring (strictly speaking only for positive numbers, because sign information can be lost)

Principal Root

- Every positive number has two square roots, one positive and one negative

Example: 2 is a square root of 4 because $2 \times 2 = 4$, but –2 is also a square root of 4 because $(-2) \times (-2) = 4$

To avoid confusion between the two we **define** the symbol $\sqrt{}$ (this symbol is called a *radical*) to mean the **principal** or **positive** square root.

The convention is:

For any positive number *x*,

$\sqrt{x}$ is the positive root, and
$-\sqrt{x}$ is the negative root.

If you mean the negative root, use a minus sign in front of the radical.

Example:

$$\sqrt{25} = 5$$
$$-\sqrt{25} = -5$$

Properties

$\left(\sqrt{x}\right)^2 = x$ for all non-negative numbers x

$\sqrt{x^2} = x$ for all non-negative numbers x

However, if x happens to be negative, then squaring it will produce a positive number, which will have a positive square root, so the most general statement is

$\sqrt{x^2} = |x|$ for all real numbers x.

- You don't need the absolute value sign if you already know that x is positive. For example, $\sqrt{4} = 2$, and saying anything about the absolute value of 2 would be superfluous. You only need the absolute value signs when you are taking the square root of a square of a *variable*, which may be positive or negative.
- The square root of a negative number is undefined, because anything times itself will give a positive (or zero) result.

$\sqrt{-4} = \textit{undefined}$ (your calculator will probably say ERROR)

- Note: Zero has only one square root (itself). Zero is considered neither positive nor negative.

WARNING: Do not attempt to do something like the distributive law with radicals: $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$ (**WRONG)** or $\sqrt{a^2 + b^2} \neq a + b$ (**WRONG)**. This is a violation of the order of operations. The radical operates on the *result* of everything inside of it, not individual terms. Try it with numbers to see:

$\sqrt{9+16} = \sqrt{25} = 5$ (**CORRECT**)

But if we (incorrectly) do the square roots first, we get

$\sqrt{9+16} = \sqrt{9} + \sqrt{16} = 3 + 4 = 7$ (**WRONG**)

However, radicals <u>do</u> "distribute" over products:

$$\sqrt{ab} = \sqrt{a}\sqrt{b}$$

and

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$$

provided that both *a* and *b* are non-negative (otherwise you would have the square root of a negative number, which is not allowed).

PERFECT SQUARES

Some numbers are perfect squares, that is, their square roots are whole numbers:

$\sqrt{0} = 0$, $\sqrt{1} = 1$, $\sqrt{4} = 2$, $\sqrt{9} = 3$, $\sqrt{16} = 4$, $\sqrt{25} = 5$, $\sqrt{36} = 6$ etc.

It turns out that all other whole numbers have irrational square roots:

$\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, $\sqrt{6}$, $\sqrt{7}$, $\sqrt{8}$, $\sqrt{10}$, $\sqrt{11}$, $\sqrt{12}$ etc. are all irrational numbers.

For example, $\sqrt{2} = 1.41421356237309504880168872420 97\ldots$

- The square root of any whole number is either a whole number or an irrational number

SIMPLIFYING RADICAL EXPRESSIONS

$\sqrt{x^2} = |x|$ for all real numbers

$\sqrt{xy} = \sqrt{x}\sqrt{y}$ if both *x* and *y* are non-negative, and

$\sqrt{\frac{x}{y}} = \frac{\sqrt{x}}{\sqrt{y}}$ if both *x* and *y* are non-negative, and *y* is not zero

WARNING: Never cancel something inside a radical with something outside of it:

$\frac{\sqrt{3x}}{3} \neq \sqrt{x}$ **WRONG!** If you did this you would be canceling a 3 with $\sqrt{3}$, and they are certainly not the same number.

The general plan for reducing the radicand is to remove any perfect powers. We are only considering square roots here, so what we are looking for is any factor that is a perfect square. In the following examples we will assume that *x* is positive.

Example: $\sqrt{16x} = \sqrt{16}\sqrt{x} = 4\sqrt{x}$

In this case the 16 was recognized as a perfect square and removed from the radical, causing it to become its square root, 4.

Example: $\sqrt{x^3} = \sqrt{x^2 x} = \sqrt{x^2}\sqrt{x} = x\sqrt{x}$

Although x^3 is not a perfect square, it has a factor of x^2, which is the square of x.

Example: $\sqrt{x^5} = \sqrt{x^4 x} = \sqrt{x^4}\sqrt{x} = x^2\sqrt{x}$

Here the perfect square factor is x^4, which is the square of x^2. This little trick will work for any odd power. You can write it as a single factor times an even power, and then you can take the square root of the even power, which results in the power being halved.

Example: $\sqrt{8x^5} = \sqrt{4 \cdot 2 \cdot x^4 \cdot x} = \sqrt{4}\sqrt{x^4}\sqrt{2x} = 2x^2\sqrt{2x}$

In this example we could take out a 4 and a factor of x^2, leaving behind a 2 and one factor of x.

- The basic idea is to factor out anything that is “square-rootable” and then go ahead and square root it.

Rationalizing the Denominator

One of the “rules” for simplifying radicals is that you should never leave a radical in the denominator of a fraction. The reason for this rule is unclear (it appears to be a holdover from the days of slide rules), but it is nevertheless a rule that you will be expected to know in future math classes. The way to get rid of a square root is to multiply it by itself, which of course will give you whatever it was the square root of. To keep things legal, you must multiply the numerator by whatever you multiply the denominator, and so we have the rule:

If the Denominator is Just a Single Radical

- ***Multiply the numerator and denominator by the denominator***

Example:

$$\frac{3}{\sqrt{x-1}}$$

$$= \frac{3}{\sqrt{x-1}}\left(\frac{\sqrt{x-1}}{\sqrt{x-1}}\right)$$

$$= \frac{3\sqrt{x-1}}{x-1}$$

- Note: If you are dealing with an *n*th root instead of a square root, then you need *n* factors of that root in order to make it go away. For instance, if it is a cube root ($n = 3$), then you need to multiply by two more factors of that root to give a total of three cube-root factors.

IF THE DENOMINATOR CONTAINS TWO TERMS

If the denominator contains a square root plus some other terms, a special trick does the job. It makes use of the difference of two squares formula:

$$(a + b)(a - b) = a^2 - b^2$$

Suppose that your denominator looked like $a + b$, where b was a square root and a represents all the other terms. If you multiply it by $a - b$, then you will end up with $a^2 - b^2$, which contains the square of your square root, which means no more square roots. It is called the *conjugate* when you replace the plus with a minus (or vice-versa). An example would help.

Example:

Given: $\dfrac{x}{2+\sqrt{x}}$

Multiply numerator and denominator by the conjugate of the denominator: $\dfrac{x}{\left(2+\sqrt{x}\right)}\,\dfrac{\left(2-\sqrt{x}\right)}{\left(2-\sqrt{x}\right)}$

Multiply out: $\dfrac{2x - x\sqrt{x}}{4 - x}$

Chapter 9: Quadratic Equations

QUADRATIC EQUATIONS

DEFINITION

$ax^2 + bx + c = 0$

a, b, c are constants (generally integers)

ROOTS

Synonyms: Solutions or Zeros

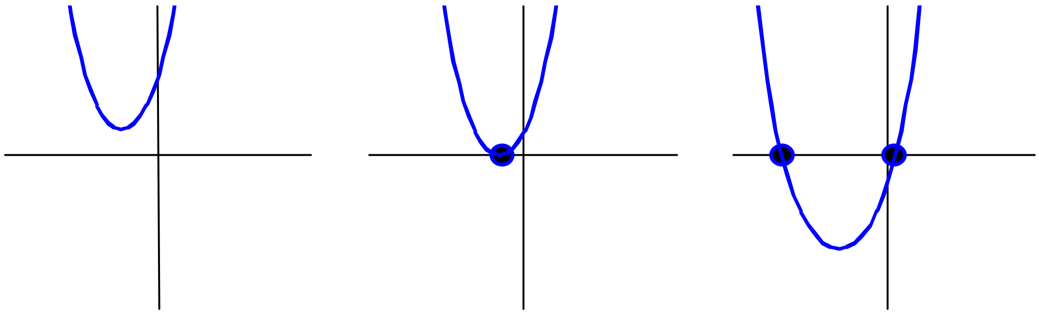
- Can have 0, 1, or 2 real roots

Consider the graph of quadratic equations. The quadratic equation looks like $ax^2 + bx + c = 0$, but if we take the quadratic *expression* on the left and set it equal to y, we will have a function:

$$y = ax^2 + bx + c$$

When we graph y vs. x, we find that we get a curve called a *parabola*. The specific values of a, b, and c control where the curve is relative to the origin (left, right, up, or down), and how rapidly it spreads out. Also, if a is negative then the parabola will be upside-down. What does this have to do with finding the solutions to our original quadratic equation? Well, whenever $y = 0$ then the equation $y = ax^2 + bx + c$ is the same as our original equation.

Graphically, y is zero whenever the curve crosses the x-axis. Thus, the solutions to the original quadratic equation ($ax^2 + bx + c = 0$) are the values of x where the function ($y = ax^2 + bx + c$) crosses the x-axis. From the figures below, you can see that it can cross the x-axis once, twice, or not at all.



No Solutions One Solution Two Solutions

> Actually, if you have a graphing calculator this technique can be used to find solutions to *any* equation, not just quadratics. All you need to do is
>
> 1. Move all the terms to one side, so that it is equal to zero
> 2. Set the resulting expression equal to y (in place of zero)
> 3. Enter the function into your calculator and graph it
> 4. Look for places where the graph crosses the x-axis
>
> Your graphing calculator most likely has a function that will automatically find these intercepts and give you the x-values with great precision. Of course, no matter how many decimal places you have it is still just an approximation of the exact solution. In real life, though, a close approximation is often good enough.

SOLVING BY SQUARE ROOTS

NO FIRST-DEGREE TERM

If the quadratic has no linear, or first-degree term (i.e. $b = 0$), then it can be solved by isolating the x^2 and taking square roots of both sides:

$$ax^2 + c = 0$$

$$ax^2 = -c$$

$$x^2 = \frac{-c}{a}$$

$$x = \pm\sqrt{-\frac{c}{a}}$$

- You need both the positive and negative roots because $\sqrt{x^2} = |x|$, so x could be either positive or negative.
- This is only going to give a real solution if either a or c is negative (but not both)

SOLVING BY FACTORING

Solving a quadratic (or any kind of equation) by factoring it makes use of a principle known as the zero-product rule.

Zero Product Rule

If $ab = 0$ then either $a = 0$ or $b = 0$ (or both).

In other words, if the product of two things is zero then one of those two things must be zero, because the only way to multiply something and get zero is to multiply it by zero.

Thus, if you can factor an expression that is equal to zero, then you can set each factor equal to zero and solve it for the unknown.

- The expression *must* be set equal to zero to use this principle
- You can always make any equation equal to zero by moving all the terms to one side.

Example:

Given:	$x^2 - x = 6$
Move all terms to one side (we added a –6 to both sides):	$x^2 - x - 6 = 0$
Factor:	$(x - 3)(x + 2) = 0$
Set each factor equal to zero and solve:	$(x - 3) = 0$ **OR** $(x + 2) = 0$
Solutions:	$x = 3$ **OR** $x = -2$

No Constant Term

If a quadratic equation has no constant term (i.e. $c = 0$) then it can easily be solved by factoring out the common x from the remaining two terms:

$$ax^2 + bx = 0$$
$$x(ax + b) = 0$$

Then, using the zero-product rule, you set each factor equal to zero and solve to get the two solutions:

$x = 0$ **or** $ax + b = 0$

$x = 0$ **or** $x = -b/a$

WARNING: Do not divide out the common factor of x or you will lose the $x = 0$ solution. Keep all the factors and use the zero-product rule to get the solutions.

Trinomials

When a quadratic has all three terms, you can still solve it with the zero-product rule if you are able to factor the trinomial.

- Remember, not all trinomial quadratics *can* be factored with integer constants

If it can be factored, then it can be written as a product of two binomials. The zero-product rule can then be used to set each of these factors equal to zero, resulting in two equations that are both simple linear equations that can be solved for x. See the above example for the zero-product rule to see how this works.

A more thorough discussion of factoring trinomials may be found in the chapter on polynomials, but here is a quick review:

Tips for Factoring Trinomials

1. Clear fractions (by multiplying through by the common denominator)
2. Remove common factors if possible
3. If the coefficient of the x^2 term is 1, then

   $x^2 + bx + c = (x + n)(x + m)$, where n and m

   i. Multiply to give c

   ii. Add to give b
4. If the coefficient of the x^2 term is not 1, then use either

   a. Guess-and Check

      i. List the factors of the coefficient of the x^2 term

   ii. List the factors of the constant term
   iii. Test all the possible binomials you can make from these factors
b. Factoring by Grouping
   i. Find the product *ac*
   ii. Find two factors of *ac* that add to give *b*
   iii. Split the middle term into the sum of two terms, using these two factors
   iv. Group the terms into pairs
   v. Factor out the common binomial

COMPLETING THE SQUARE

The technique of completing the square is presented here primarily to justify the quadratic formula, which will be presented next. However, the technique does have applications besides being used to derive the quadratic formula. In analytic geometry, for example, completing the square is used to put the equations of conic sections into standard form.
Before considering the technique of completing the square, we must define a perfect square trinomial.

Perfect Square Trinomial

What happens when you square a binomial? You can use the FOIL technique to verify that

$$(x+a)^2 = (x+a)(x+a) = x^2 + 2ax + a^2$$

- Note that the coefficient of the middle term ($2a$) is twice the square root of the constant term (a^2)
- Thus the constant term is the square of half the coefficient of x
- Important: These observations only hold true if the coefficient of x is 1.

This means that any trinomial that satisfies this condition is a perfect square. For example,

$$x^2 + 8x + 16$$

is a perfect square, because half the coefficient of x (which in this case is 4) happens to be the square root of the constant term (16). That means that

$$x^2 + 8x + 16 = (x + 4)^2$$

Multiply out the binomial (x + 4) times itself and you will see that this works.

The technique of completing the square is to take a trinomial that is not a perfect square, and make it into one by inserting the correct constant term (which is the square of half the coefficient of *x*). Of course, inserting a new constant term has to be done in an algebraically legal manner, which means that the same thing needs to be done to both sides of the equation. This is best demonstrated with an example.

Example:

Given Equation:	$x^2 + 6x - 2 = 0$
Move original constant to other side:	$x^2 + 6x = 2$
Add new constant to both sides (the square of half the coefficient of *x*):	$x^2 + 6x + 9 = 2 + 9$
Write left side as perfect square:	$(x+3)^2 = 11$
Square root both sides (remember to use plus-or-minus):	$x + 3 = \pm\sqrt{11}$
Solve for *x*:	$x = -3 \pm \sqrt{11}$

Notes

- Completing the square finds all real roots. Factoring can only find integer or rational roots.
- When you write it as a binomial squared, the constant in the binomial will be half of the coefficient of *x*.

If the Coefficient of x^2 is Not 1

First divide through by the coefficient, then proceed with completing the square.

Example:

Given Equation:	$2x^2 + 3x - 2 = 0$
Divide through by coefficient of x^2: (in this case a 2)	$\frac{1}{2}\left(2x^2 + 3x - 2 = 0\right)$ $x^2 + \frac{3}{2}x - 1 = 0$
Move constant to other side:	$x^2 + \frac{3}{2}x = 1$
Add new constant term: (the square of half the coefficient of *x*, in this case 9/16):	$x^2 + \frac{3}{2}x + \frac{9}{16} = 1 + \frac{9}{16}$
Write as a binomial squared: (the constant in the binomial is half the coefficient of *x*)	$\left(x + \frac{3}{4}\right)^2 = \frac{25}{16}$
Square root both sides: (remember to use plus-or-minus)	$x + \frac{3}{4} = \pm\frac{5}{4}$
Solve for *x*:	$x = \frac{-3 \pm 5}{4}$

Thus $x = ½$ or $x = -2$

The Quadratic Formula

The solutions to a quadratic equation can be found directly from the quadratic formula.

The equation

$$ax^2 + bx + c = 0$$

has solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The advantage of using the formula is that it always works. The disadvantage is that it can be more time-consuming than some of the methods previously discussed. As a general rule you should look at a quadratic and see if it can be solved by taking square roots; if not, then if it can be easily factored; and finally use the quadratic formula if there is no easier way.

- Notice the plus-or-minus symbol (±) in the formula. This is how you get the two different solutions—one using the plus sign, and one with the minus.
- Make sure the equation is written in standard form before reading off *a*, *b*, and *c*.
- Most importantly, make sure the quadratic expression is equal to zero.

The Discriminant

The formula requires you to take the square root of the expression $b^2 - 4ac$, which is called the *discriminant* because it determines the nature of the solutions. For example, you can't take the square root of a negative number, so if the discriminant is negative then there are no solutions.

If $b^2 - 4ac > 0$	There are two distinct real roots
If $b^2 - 4ac = 0$	There is one real root
If $b^2 - 4ac < 0$	There are no real roots

Deriving the Quadratic Formula

The quadratic formula can be derived by using the technique of completing the square on the general quadratic formula:

Given: $ax^2 + bx + c = 0$

Divide through by *a*: $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$

Move the constant term to the right side: $x^2 + \frac{b}{a}x = -\frac{c}{a}$

Add the square of one-half the coefficient of *x* to both sides: $x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$

Factor the left side (which is now a perfect square), and rearrange the right side: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a}$

Get the right side over a common denominator: $\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$

Take the square root of both sides (remembering to use plus-or-minus): $x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$

Solve for *x*: $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

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