Fluid Mechanics 1

034013

Exercise Booklet

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Foreword and Acknowledgments

Fluid Mechanics is an important and fundamental branch of Physics. Its governing equations and similar phenomena can be seen in various branches and disciplines of the Physical and Engineering world. Understanding these interactions provide a more accurate and general description of Nature, amongst which allow us to tame the ocean and the skies to transfer ourselves safely and conveniently around the globe. However, that is not the only reason we study Fluid Mechanics. As an undergraduate student, I immediately fell in love with the field of Fluid Mechanics due its richness of phenomena, simplicity and mathematical elegance. Since then I have taken numerous courses in the broad field of Fluid Mechanics and my PhD focuses on the flow of fluid through nanochannels with the fluid being driven by an electric force. Much of these phenomena will not be covered in this course, however the underlying principles remain the same and hopefully some of you will continue on a similar path as I have and enhance your knowledge in this field.

I would like to thank numerous people who have contributed to this booklet. Foremost I would like to thank Mr. Oriel Shoshani and Mr. Lior Atia for providing me with their class notes which were the base of this compilation and for numerous fruitful discussions on teaching Fluid Mechanics. Additionally, I would like to thank Prof. Moran Bercovici, Prof. Gilad Yossifon, Prof. Shimon Haber and Prof. Amir Gat, Mr. Nimrod Kruger, and Mr. Shai Elbaz whom I have had the pleasure to teach with and learn from their experience. Their input and ideas have been invaluable!

This is still an initial work and is not yet complete, thus I hope that I have not forgotten any contributor of importance in my list of thank and references (to be found at the end of this booklet) .

Please note that sometimes links to Wikipedia [1] will be embedded. Wikipedia is a great website to learn new and basic concepts. However Wikipedia is not without flaws. There are numerous errors and typos in varying entries. It will be your responsibility to judge and verify through the long source of references what is indeed correct.

Please note that not all questions will be solved in class. We are attempting to build a comprehensive booklet that will provide additional exercises for students to practice and learn from.

Good luck to us all!

Yoav

Table of contents by classes – will be updated frequently:

"On the first day of class, Scott recalled, "in the hall, there were 183 new freshman and a bowling ball hanging from the three-story ceiling to just above the floor. Feynman walked in, and without a word, grabbed the ball and backed against the wall with the ball touching his nose. He let go, and the ball swung slowly 60 feet across the room and back--stopping naturally, just short of crushing his face. Then he took the ball again, stepped forward, and said: **'I wanted to show you that I believe in what I'm going to teach you over the next two years.'**" ,Michael Scott

1. Ordinary Differential Equations, Differential Operators and Newtonian fluids

Ordinary Differential Equations

In the last two years of your studies you have taken many basic courses in Mathematics, from Calculus to Partial Differential equations and more. In all of these courses you learned Lemmas and rules and many techniques on how to solve various problems. However all these problems were mathematical in their essence and were disconnected from the physical world. In this course you will learn how to apply much of what you have learned to solve real physical problems. However as the semester has only started and we have not had time to derive anything we are not just ready to solve real problems. However, we will rehearse some of these techniques and ideas so that when the real problems come along the semester you will be well prepared.

Question 1

A parachutist with mass . $m = 80[kg]$. is undergoing free fall (see image). The drag force applied on him is $F_D = kv^2$, where v is the velocity measured relative to the air. The constant $k = 0.27\left[N\,s^2/m^2\right]$ is given. Find:

a. The terminal velocity, v_t

b. The distance traveled, h, until $v = 0.95v_t$ is achieved.

Solution:

Given details:

1. Mass: $m = 80[kg]$
2. Free fall: $F = mg$ with $g = 9.81\left[m/s^2\right]$.
3. Drag force: $F = kV^2$ with $k = 0.27\left[Ns^2/m^2\right]$.

a. The terminal velocity is the velocity achieved when a body under two opposing forces (such as gravity and drag) is no longer accelerating. From Newton's second law

$$m\frac{dv}{dt} = mg - kv^2 \ . \qquad (1.1)$$

Assuming no acceleration yields

$$v_t = \sqrt{\frac{mg}{k}} \ . \qquad (1.2)$$

Inserting the given values yields

$$v_t = \sqrt{\frac{mg}{k}} = \sqrt{\frac{80[kg]9.81\left[ms^{-2}\right]}{0.27\left[Ns^2m^{-2}\right]}} = \sqrt{2906.7\frac{[kg]\left[ms^{-2}\right]}{\left[kgms^{-2}s^2m^{-2}\right]}} = 53.91\left[\frac{m}{s}\right] \ . \qquad (1.3)$$

In varying units:

$$v_t = 53.91\left[\frac{m}{s}\right] = 53.91\left[\frac{m}{s}\right]\frac{1}{1000}\left[\frac{km}{m}\right]3600\left[\frac{s}{hr}\right] = 194\left[\frac{km}{hr}\right] \ . \tag{1.4}$$

b. We now change the gravity expression in Eq. (1.1) with $mg = kv_t^{\,2}$

$$m\frac{dv}{dt} = k\left(v_t^{\,2} - v^2\right) \ . \tag{1.5}$$

Using the chain rule

$$m\frac{dv}{dz}\underbrace{\frac{dz}{dt}}_{v} = k\left(v_t^{\,2} - v^2\right) \ . \tag{1.6}$$

After some algebraic manipulations, we get

$$dz = \frac{m}{k}\frac{vdv}{\left(v_t^{\,2} - v^2\right)} \ . \tag{1.7}$$

We now introduce the change of variables $\left(v = uv_t, dv = v_t du\right)$ into Eq. (1.7)

$$dz = \frac{m}{k}\frac{uv_t^{\,2}du}{\left(v_t^{\,2} - u^2 v_t^{\,2}\right)} = \frac{m}{k}\frac{udu}{\left(1-u^2\right)} \ . \tag{1.8}$$

Integration of Eq. (1.8) yields the final solution.

$$h = \int_0^h dz = \frac{m}{k}\int_0^{0.95}\frac{udu}{\left(1-u^2\right)} = -\frac{m}{2k}\ln\left(1-u^2\right)\Big|_0^{0.95} = 345[m] \ . \tag{1.9}$$

Question 2
Solving the following ordinary differential equations (ODE's) with the given boundary conditions (BC)

a. $\begin{cases} \partial_{yy}u(y) = A \\ u(y=0) = u(y=h) = 0 \end{cases}$

b. $\begin{cases} \partial_{yy}u(y) - V\partial_y u(y) = B, V > 0 \\ u(y=0) = u(y=h) = 0 \end{cases}$

c. $\begin{cases} r^2\partial_{rr}u(r) + r\partial_r u(r) - u(r) = 0 \\ u(r=a) = U_1, u(r=b) = U_2 \end{cases}$

Note that all these equations and BCs will be derived later on during the semester and will be used to solve real physical problems.

Solution:

a. The solution of problem (a) is straightforward. Integrating twice gives

$$u(y) = \frac{Ay^2}{2} + c_1 y + c_2 \tag{1.10}$$

Finding the constants c_1, c_2 is a simple task. Inserting the BC one gets

$$\begin{cases} u(y=0)=c_2=0 \\ u(y=h)=\frac{Ah^2}{2}+c_1h=0 \Rightarrow c_1=-\frac{Ah}{2} \end{cases} \cdot \tag{1.11}$$

The solution is

$$u(y)=\frac{A}{2}\left(y^2-hy\right) \tag{1.12}$$

b. The solution is no longer straightforward. Yet this problem is relatively simple to solve.

The equation that we would like to solve is linear and thus we can solve the homogeneous and particular solutions separately. Furthermore, since the coefficients are constant, it is easy to guess that the solution has the following form

$$u_h(y)=ce^{sy},\partial_y u_h(y)=sce^{sy}=su_h(y),\partial_{yy}u_h(y)=s^2ce^{sy}=s^2u_h(y). \tag{1.13}$$

where s is an eigenvalue and c is an undetermined constant. Inserting this into the homogeneous equation gives

$$\partial_{yy}u_h(y)-V\partial_y u_h(y)=0:\left(s^2-Vs\right)\cdot u_h(y)=0 \ . \tag{1.14}$$

Hence the solution is simply $\left(s_1=V,s_2=0\right)$. It can be seen that the second solution is simply a constant. We now tackle the particular/private solution. We guess the simplest form of the solution that will satisfy the equation. The solution $u_p(y)=C$ does not satisfy the equation and is already included in the homogeneous solution. The second simplest solution is

$$u_p(y)=Qy \ . \tag{1.15}$$

The constant term does not need to be included. Inserting Eq. (1.15) into the governing equation gives

$$\partial_{yy}u_p(y)-V\partial_y u_p(y)=B:0-VQ=B\Rightarrow Q=-B/V \ . \tag{1.16}$$

Hence the solution has the form

$$u(y)=u_h(y)+u_p(y)=c_1e^{Vy}+c_2-\frac{B}{V}y \tag{1.17}$$

Solving for the BC gives

$$u(y)=\frac{Bh}{V}\frac{e^{Vy}-1}{e^{hV}-1}-\frac{B}{V}y \tag{1.18}$$

c. Similar to problem (b), the equation is linear but it appears that the coefficients are non-longer constants.

This kind of equation is called an Euler Differential Equation [1]. Solving these equations is done in a similar manner to problem (b) except that the homogeneous solution now has the following form

$$u_h(r)=cr^s,\partial_r u_h(r)=scr^{s-1},\partial_{rr}u_h(r)=s(s-1)cr^{s-2} \ . \tag{1.19}$$

Inserting this into the governing equation gives

$$r^2\partial_{rr}u(r)+r\partial_r u(r)-u(r)=0:\left[s(s-1)+s-1\right]cr^s=0 \ . \tag{1.20}$$

The solutions of the quadratic equation for s are $s_{1,2}=\pm 1$. Hence the solution has the form

[1] See your ODE /מד"ר notebooks or see Ref. [2]

$$u(r) = c_1 r + \frac{c_2}{r} \ . \tag{1.21}$$

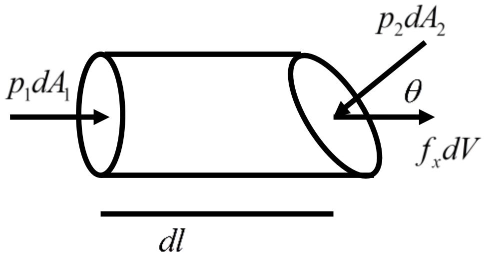
Solving for the BC's gives

$$c_1 = \frac{aU_1 - bU_2}{a^2 - b^2}, c_2 = ab\frac{U_2 a - bU_1}{a^2 - b^2} \ . \tag{1.22}$$

Differential Operators (forthcoming/next semester)

Pressure

Show that the pressure at a point in an inviscid fluid is independent of direction.



Solution

Let us look upon an infinitesimal cylinder that has different surfaces at it edges. The force being applied on the left surface is simply

$$F_{left} = p_1 dA_1 \ , \tag{1.23}$$

while the force on right surface is

$$\vec{F}_{right} = -p_2 d\vec{A}_2 \ , \tag{1.24}$$

however, only the horizontal component is needed and thus

$$F_{right} = -p_2 dA_2 \cos\theta \ . \tag{1.25}$$

We can additionally assume that this body is also under an applied force

$$F_{applied} = f_x dV \ , \tag{1.26}$$

where f_x has is the force per unit volume with the appropriate units of $\left[N/m^3\right]$. Newton's second law gives

$$\sum F = ma : p_1 dA_1 - p_2 dA_2 \cos\theta + f_x dV = ma = \rho a dV \ . \tag{1.27}$$

Where the third equality merely states that mass equals the density times the volume. We now need to simplify Eq. (1.27). This is done by accounting for two geometric factors

$$dA_1 = dA_2 \cos\theta, dV = dAdl \tag{1.28}$$

We see that initially all volumetric terms are multiplied by the area and an additionally small infinitesimal length. Hence, both these terms are negligible. Also, by projecting the area onto the horizontal direction, and assuming that cylinder has parallel generators, then the areas are equal. Thus Eq. (1.27) is reduced to $(p_1 - p_2) dA_1 = 0$□ .

Newtonian fluids

The resultant force from shear is given by

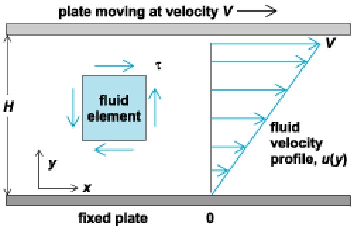
$$F = \tau A \ , \tag{1.29}$$

where A is the area where the shear stress, τ , is being applied. This stress is given by the following relation.

$$\tau = \mu \frac{\partial u}{\partial y} = \mu \frac{\Delta u}{\Delta y} \ , \tag{1.30}$$

with μ is the viscosity of the fluid. Note that Eq. (1.30) has a similar form to the stress-strain relation you have previously seen in Solid Mechanics and Theory of Elasticity

$$\tau_{xy} = G\gamma_{xy} \quad . \tag{1.31}$$



In the theory of elasticity, the stress is linearly proportional to the strain/displacement. In Fluid Mechanics, the strain is proportional to the strain rate. Thus the viscosity plays the role of the Young/Shear Modulus. In fact, sometimes, it is termed the dynamic Young Modulus. In a flow with a linear profile

$$\tau = \mu \frac{V}{H} \tag{1.32}$$

Question 1
The velocity profile of fluid in between two plates is given by (see Question 2.a from the previous subsection)

$$u = u_{\max}\left[1 - \left(\frac{y}{h/2}\right)^2\right] , \tag{1.33}$$

where the origin of the coordinate system is now located in the midsection of the channel. Calculate the force being applied on an area of length L_x in the direction of the flow and length L_z into the plane.

Solution

The velocity field given by Eq. (1.33) is known as a Poiseuille flow field. This flow field will be encountered numerous times throughout the course. The shear stress can easily be calculated.

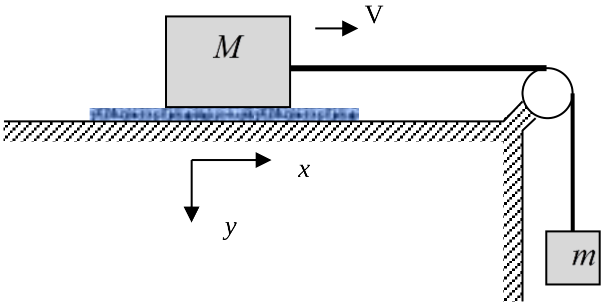
$$\tau = \mu \frac{\partial}{\partial y}\left(u_{\max}\left(1 - \left(\frac{2y}{h}\right)^2\right)\right) = -8\frac{\mu u_{\max}}{h^2} y \quad . \tag{1.34}$$

We now note that the shear stress at the plane $y = \pm h/2$ have different signs. How can this be? The field is symmetric and thus we expect the stresses to be in the same direction. This is explained in the following manner. Stresses are like forces- they have directions. We can see that on the upper surface of a fluid element, the stress is defined positive in a direction reverse to that on the lower surface (see how definitions in the above figure). So in fact, we have received that the shear stress at the top and bottom are the same in magnitude and direction. To find the force on the wall we simply multiply by the area and by a minus sign

$$F = -\tau A = 4\mu u_{\max} \frac{L_x L_z}{h} \quad . \tag{1.35}$$

Question 2
Mass M is sliding on a thin layer of oil and moving at a constant velocity V. Mass M is connected by a wire to mass m (see figure). At time, $t = 0$, mass m is allowed to fall freely. Assuming that the contact area of mass M with the oil is A and the thickness of the oil layer is h, find:

a. An expression for the viscous force applied on M as a function of the velocity, $v(t)$.
b. Derive the governing equation for the dynamics of the system.



c. Solve this equation and plot the solution.

Solution:

Given: The initial velocity: $v(t=0)=V$.

Assumptions:

1. Newtonian fluid.
2. Linear profile velocity within the fluid with a velocity $v(t)$ at $y=h$.

a. These two assumptions allow us to write

$$\tau=\mu v(t)/h \ . \tag{1.36}$$

Thus the viscous force is given by

$$F_D=\tau A=\frac{\mu A}{h}v(t) \ . \tag{1.37}$$

b. The equations of motion (EOM) for each of the bodies is written separately

$$\begin{cases} M\dfrac{dv}{dt}=T-F_D \\ m\dfrac{dv}{dt}=mg-T \end{cases} , \tag{1.38}$$

Addition of both equations yields

$$(M+m)\frac{dv}{dt}=mg-F_D=mg-\frac{\mu A}{h}v(t) \ . \tag{1.39}$$

Dividing Eq. (1.39) by $(M+m)$, one gets

$$\frac{dv}{dt}=\beta-\alpha v(t) \ , \tag{1.40}$$

with $\alpha=\dfrac{\mu A}{h(M+m)}, \beta=\dfrac{mg}{(M+m)}$.

c. We will solve Eq. (1.40) in two parts. First we will start with the homogeneous part. We shall guess a solution of the form

$$v_h(t)=c\exp(rt), v_h{}'(t)=rc\exp(rt)=rv_h(t) \ . \tag{1.41}$$

Inserting relation Eq. (1.41) into Eq. (1.40) yields

$$c\exp(rt)r=-\alpha c\exp(rt)\Rightarrow r=-\alpha \ . \tag{1.42}$$

To find the particular/private solution, we shall guess the simplest form of a solution that satisfies Eq. (1.40). This guess will be a constant. Thus,

$$v_p=\frac{\beta}{\alpha}=\gamma \ . \tag{1.43}$$

The solution thus has the form

$$v(t) = ce^{-\alpha t} + \gamma \ . \tag{1.44}$$

To find the constant c we shall use the initial condition $v(t=0)=V$

$$V = ce^{-\alpha 0} + \gamma \Rightarrow c = V - \gamma \ . \tag{1.45}$$

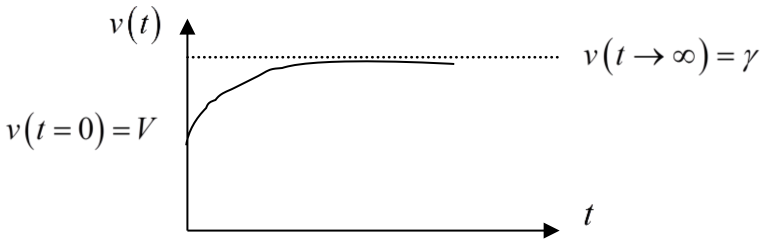
Finally,

$$v(t) = (V-\gamma)e^{-\alpha t} + \gamma \ . \tag{1.46}$$

It is easy to see that the maximal velocity that can be achieved in steady state is

$$V_{\max} = \gamma = \frac{\beta}{\alpha} = \frac{mgh}{\mu A} \tag{1.47}$$

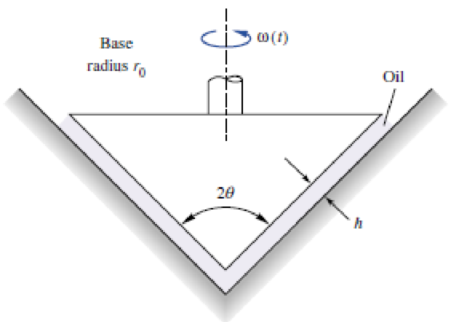
We plot the solution



It is interesting to note that the ancient Egyptians and the more modern Chinese used similar methods to transport very heavy stones tens or even hundreds of kilometers. Essentially they made sure that between the ground and the stone a thin film of ice/water existed (obviously ice was not to be found in Egypt) and were able to move the stone with an effectively smaller coefficient of friction.

Question 3
A conical pointed shaft turns in a conical bearing with a angular velocity ω. The gap between the shaft and bearing is filled with an oil of viscosity μ and the distance between these two is h. Obtain an expression for the shear stress that acts on the surface of the conical shaft. Calculate the viscous torque that acts on the shaft.

Solution



We can assume that the azimuthal velocity behaves linearly with radial distance and we know the velocity at the surface is $u = \omega r$. Thus the shear stress can be calculated

$$\tau = \mu \frac{\Delta u}{\Delta y} = \frac{\mu \omega r}{h} \ . \tag{1.48}$$

We see that the stress is dependent on the radial coordinate, thus the torque will must be integrated on the entire surface of the cone. It will now be useful to find a number of geometric relations:

$$dr = ds \sin\theta \Rightarrow ds = dr / \sin\theta \ , \tag{1.49}$$

$$dA = 2\pi r ds = 2\pi r dr / \sin\theta \ , \tag{1.50}$$

where the first describes the relation between the a differential length of the cone surface and differential radii. The second is the perimeter. The torque applied on a differential area located at radius r is

$$dT = r\tau dA = r\left(\frac{\mu\omega r}{h}\right)\left(\frac{2\pi r}{\sin\theta}\right)dr \quad . \tag{1.51}$$

To calculate the torque we integrate over the entire surface of the cone.

$$T = \int dT = \int_0^{r_0} r\left(\frac{\mu\omega r}{h}\right)\left(\frac{2\pi r}{\sin\theta}\right)dr = 2\pi\frac{\mu\omega}{h\sin\theta}\int_0^{r_0} r^3 dr = \frac{\pi}{2}\frac{\mu\omega r_0^{\,4}}{h\sin\theta} \quad . \tag{1.52}$$

Feynman solution algorithm:

"1. Write down the problem.

2. Think very hard.

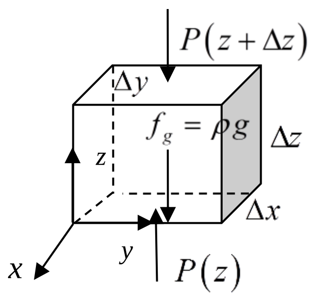
3. Write down the answer."

2. Hydrostatics

Hydrostatics is of utmost importance in any problem that includes within it the equations of Fluid Mechanics. Usually by assuming hydrostatic equilibrium in a system, many important conclusions on the governing physics of the system can be drawn. For example, the physics within a star include dozen of equations from varying fields (quantum mechanics, statistical physics and thermodynamics, electro-magnetism and more) but a simple model shows that a star doesn't collapse upon itself simply because gravity is counter-balanced by an outward pressure gradient. How this pressure gradient is formed is a completely different issue. We will see that most of the problems encountered in this chapter are mathematically simple but are rich in physics. Look for simplicity and you will find it!

Hydrostatic equilibrium

We make a distinction between two kinds of forces- surface forces and volumes forces, where the former will act on the surface of our body/control surface (C.S.) , for example pressure and drag, while the latter forces act upon the entire body/control volume (C.V.), for example gravity.



Note the direction of the coordinate system and that gravity is pointed in the negative .. direction.

The forces on the top and bottom surfaces are accordingly

$$F_{down} = p(x,y,z)\Delta x\Delta y \ . \tag{2.1}$$

$$F_{up} = -p(x,y,z+\Delta z)\Delta x\Delta y \ . \tag{2.2}$$

Gravity also applies a force on the C.V.

$$F_g = -\rho g\Delta x\Delta y\Delta z \tag{2.3}$$

Naturally, Eq. (2.2) is expanded using a Taylor series

$$F_{up} = -p(x,y,z+\Delta z)\Delta x\Delta y = -\left(p(x,y,z)+\partial_z p(x,y,z)\Delta z + O\left(\Delta z^2\right)\right)\Delta x\Delta y \tag{2.4}$$

Remembering that the small cube is in equilibrium and inserting Eqs. (2.1)-(2.4) we have

$$\sum\vec{F} = 0 : F_{down} + F_{up} + F_g = \cancel{p(z)}\Delta x\Delta y - \left(\cancel{p(z)} + \frac{\partial p(z)}{\partial z}\Delta z\right)\Delta x\Delta y - \rho g\Delta x\Delta y\Delta z = 0 \tag{2.5}$$

This yields

$$\left(-\frac{\partial p(z)}{\partial z} - \rho g\right)\Delta x\Delta y\Delta z = 0 \ . \tag{2.6}$$

Since the volume is arbitrary and non-zero, the final result shows that gravity is counter-balanced by a pressure gradient

$$-\frac{\partial p(z)}{\partial z} = \rho g \tag{2.7}$$

In similar manner to the above derivation we can generalize Eq. (2.7) to have a vector form

$$-\vec{\nabla} p = \rho \vec{g} \tag{2.8}$$

Furthermore, for a system undergoing acceleration this can also be generalized

$$-\vec{\nabla} p - \rho \vec{g} = \rho \vec{a} \tag{2.9}$$

In the case of constant density and gravity, Eq. (2.7) can be integrated to yield the result

$$p_2 - p_1 = -\rho g (z_2 - z_1) \tag{2.10}$$

The two important points that should be reflected upon are

1. If z_1 is deeper than z_2 then the pressure p_1 is larger than the pressure p_2 .
2. Pressure is energy (per unit volume)!

$$[Pa] = \left[\frac{N}{m^2}\right] = \left[\frac{Nm}{m^3}\right] = \left[\frac{J}{m^3}\right], \tag{2.11}$$

Also, remember the first law of thermodynamics

$$dE = \delta Q - \delta W = Tds - pdV \ , \tag{2.12}$$

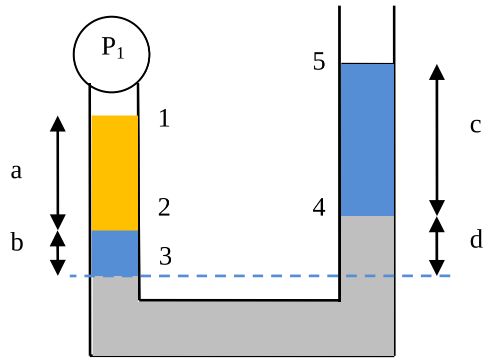
where pdV was the work being done. Eq. (2.10) can be rewritten

$$p_2 + \rho g z_2 = p_1 + \rho g z_1 \tag{2.13}$$

to have a form of energy equals energy which illustrates conservation of energy. This point will be expanded upon later in the course.

Question 1

Consider a manometer, which is an instrument used to measure pressure, which uses three different liquids for measurement. Water is colored in blue, mercury in silver/grey and oil is yellow/gold. Assuming that the gage pressure is $p_1 = 10[kPa]$, solve for the height d of the mercury on the unconstrained side.



Solution:

Given: The following geometrical and mechanical properties of the system are given, as well as the gage pressure

$$\begin{gathered} a = 0.05[m], b = 0.03[m], c = 0.07[m], p_1 - p_5 = 10[kPa] \\ \rho_w = 10^3\left[kg/m^3\right], \rho_{oil} = 0.88 \cdot 10^3\left[kg/m^3\right], \rho_{Hg} = 13.6 \cdot 10^3\left[kg/m^3\right] \end{gathered} \tag{2.14}$$

Assumptions:

1. The system is in hydrostatic equilibrium.
2. All fluids are incompressible – constant density.
3. Gravity is constant.

The pressure differences between two adjoining points are

$$\begin{cases} p_2 - p_1 = -\rho_{oil} g\left(z_2 - z_1\right) = \rho_{oil} ga \\ p_3 - p_2 = -\rho_w g\left(z_3 - z_2\right) = \rho_w gb \\ p_3 - p_4 = -\rho_{Hg} g\left(z_3 - z_4\right) = \rho_{Hg} gd \\ p_4 - p_5 = -\rho_w g\left(z_4 - z_5\right) = \rho_w gc \end{cases} \tag{2.15}$$

The solution of which is slightly tedious but straightforward

$$d = \frac{1}{\rho_{Hg}}\left(\frac{p_1 - p_5}{g} + \rho_{oil} a + \rho_w \left(b - c\right)\right) \tag{2.16}$$

Inserting the numbers given by Eq. (2.14) yield

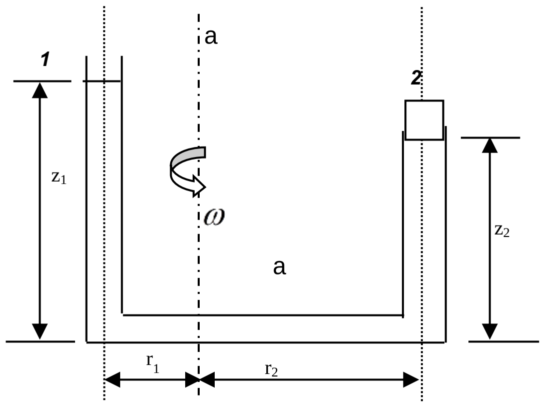
$$d = 0.075\cancel{2476}\left[m\right] \tag{2.17}$$

It is important to remember that when you actually calculate the solution, keep only the leading term.

Question 2 (Past midterm)

A U shaped pipe whose geometry is given in the figure has a diameter d. At one of its end a cork blocks the movement of water outward, while the other end is free. The density of the fluid is also given, ρ. Find:

a. The angular velocity ω about axis $a-a$ that will release the cork, if the required force for removal is F.
b. After a sufficiently long time from release, find the final heights of the fluid.



Solution

Given: Geometric lengths r_1, r_2, z_1, z_2, d, Density ρ, Atmospheric pressure P_a

Assumptions: Hydrostatic equilibrium.

The governing hydrostatic equations are

$$\begin{cases} \partial_z p = -\rho g \\ \partial_r p = \rho \omega^2 r \end{cases} \tag{2.18}$$

The solution of which is (see Question 2 (2012 Spring midterm) in the following section for a full derivation)

$$p\left(r,z\right) = \frac{\rho\omega^2 r^2}{2} - \rho gz + c \tag{2.19}$$

The two unknowns ω and c can be solved by requiring that

$$p\left(r_1, z_1\right) = P_a,\ p\left(r_2, z_2\right) = \frac{4F}{\pi d^2} \tag{2.20}$$

Inserting these BC, one gets

$$\begin{cases} P_a = \frac{1}{2}\rho\omega^2 r_1^{\ 2} - \rho g z_1 + c \\ \frac{4F}{\pi d^2} = \frac{1}{2}\rho\omega^2 r_2^{\ 2} - \rho g z_2 + c \end{cases} \tag{2.21}$$

It is now evident, that we can solve solely for the angular velocity

$$\omega = \sqrt{2\frac{P_a - \frac{4F}{\pi d^2} + \rho g\left(z_1 - z_2\right)}{\rho\left(r_1^{\ 2} - r_2^{\ 2}\right)}} \tag{2.22}$$

Up until now, the cork blocked the flow of water outside. Once the cork is removed, water can now flow outwards, the dynamics of which do not interest us. We are interested in what happens after a sufficiently long time when the system has reached a new equilibrium. The required BC are

$$p\left(r_1, \tilde{z}_1\right) = P_a, p\left(r_2, z_2\right) = P_a \ , \tag{2.23}$$

leading to the equations

$$\begin{cases} P_a = \frac{1}{2}\rho\omega^2 r_1^{\ 2} - \rho g \tilde{z}_1 + c' \\ P_a = \frac{1}{2}\rho\omega^2 r_2^{\ 2} - \rho g z_2 + c' \end{cases} \tag{2.24}$$

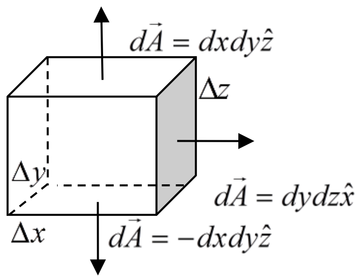
Note that constant c' is not equal to the constant c given in Eq. (2.19) because thus far we have released energy from our system (water spilling). Once more we note that we need to solve solely for $\tilde{z}_1$

$$\tilde{z}_1 = \frac{\omega^2}{2g}\left(r_1^{\ 2} - r_2^{\ 2}\right) + z_2 \ . \tag{2.25}$$

Expectedly, due to the loss of water, the height of column 1 has decreased. One last note is needed- this was a midterm question, it should be solved elegantly yet as short as possible. Put some thought as to what is important and what is not. Those who solved for c and c' wasted crucial test time on pointless mathematics for no apparent reason! This doesn't contribute by any means to the solution. **Think before you solve!**

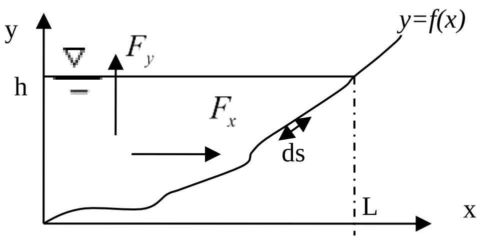
Immersed surfaces

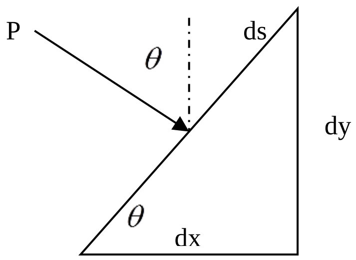
As we will see throughout the course, many of our equations will be dependent on the area and its direction. Remember that in general area is a vector where the normal to the surface defines the direction of the area, while its size is dependent on the geometry. As discussed in class, pressure is a scalar function so that it operates in all directions equally, however the force being applied on the surface of a body is dependent on the area.



$$d\vec{F} = -pd\vec{A} = -p\hat{n}dA \tag{2.26}$$

Let us look at a 2D surface described by the contour $y = f\left(x\right)$





It is obvious that $d\vec{A} = bd\vec{s}$ where $d\vec{s}$ is a differential element on the contour. This can also be written in a vector form

$$d\vec{s} = -dy\hat{x} + dx\hat{y} \ . \tag{2.27}$$

Inserting this into Eq. (2.26) gives

$$\begin{cases} dF_x = Pbdy \\ dF_y = -Pbdx \end{cases}, \tag{2.28}$$

so that the total force applied on the surface is

$$\begin{aligned} F_x &= \int dF_x = b\int_0^h Pdy \\ F_y &= \int dF_y = -b\int_0^L Pdx \end{aligned} \ . \tag{2.29}$$

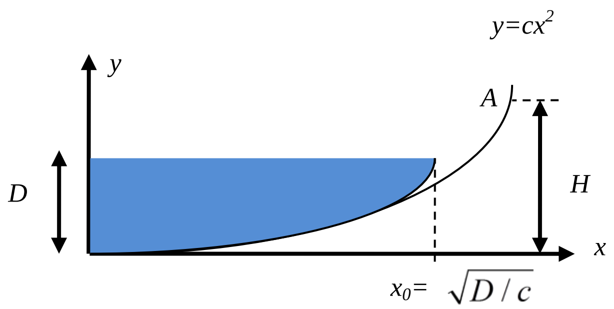
An alternative way to derive Eq. (2.28) is take the projection of the total force

$$dF_x = dF\sin\theta = dF\frac{dy}{ds} = \frac{dF}{ds}\frac{b}{b}dy = Pbdy \ . \tag{2.30}$$

A last note, if we were to look at the same geometry located at the negative x side then Eq. (2.27) would be modified: $d\vec{s} = dy\hat{x} + dx\hat{y}$.

Question 1

A hanging body of parabolic shape is immersed in water up to height D. The shape is 2D as shown and has a depth of b into the page. Calculate:



a. The hydrostatic pressure.
b. The applied force.
c. The applied moment.

Solution:

Assumption: 1. Hydrostatic equilibrium

2. The air pressure is atmospheric- $p(y = D) = P_a$.

Integration of the hydrostatic equation gives the pressure up to a constant

$$\frac{dp}{dy} = -\rho g \Rightarrow p = -\rho gy + P_0 \tag{2.31}$$

This constant is found by requiring the pressure by atmospheric at the surface

$$p = \rho g(D - y) + P_a \quad (2.32)$$

Due to the assumption of atmospheric pressure throughout, when calculating the forces, the atmospheric term can be ignored as it cancels out. Integration of the forces gives

$$\begin{cases} F_y = \int dF_y = -\rho g \int_0^{\sqrt{D/c}} (D-y)bdx = -\rho gb \int_0^{\sqrt{D/c}} \left(D - cx^2\right)dx = -\frac{2}{3}\frac{\rho gbD^{3/2}}{c^{1/2}} \\ F_x = \int dF_x = \rho g \int_0^D (D-y)bdy = \frac{1}{2}\rho gD^2 b \end{cases}, \quad (2.33)$$

unsurprisingly, the forces are in the expected direction, with F_y pushing downward and F_x pushing to the right.

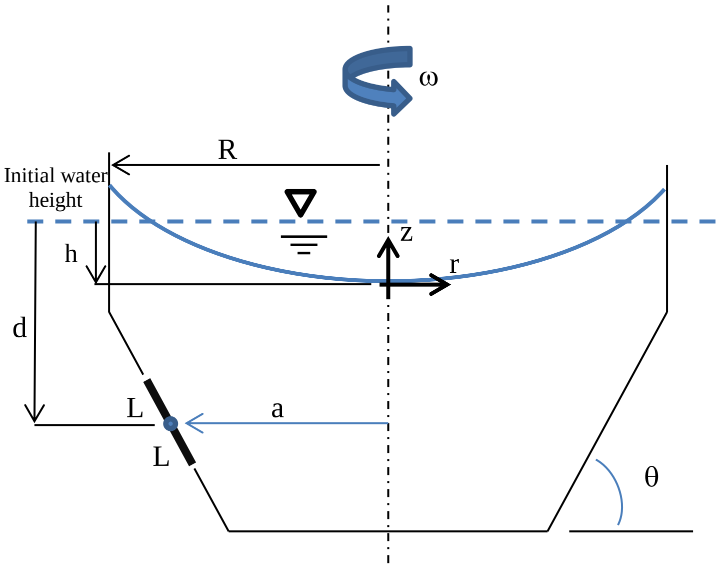
We calculate the applied moments

$$\begin{aligned} M_{z,1} &= \bar{x}F_y = \int xdF_y = -\int \rho gb(D-y)xdx = -\int_0^{\sqrt{D/c}} \rho gb\left(D - cx^2\right)xdx = -\frac{1}{4}\frac{\rho gbD^2}{c} \\ M_{z,2} &= F_x\bar{y} = -\int ydF_x = -\int \rho gby(D-y)dy = -\int_0^D \rho gby(D-y)dy = -\frac{1}{6}\rho gD^3 b \end{aligned}. \quad (2.34)$$

It can be seen that both moments are trying to flip the body over. Thus to be in static equilibrium, an external force needs to be applied.

Question 2 (2012 Spring midterm)

A cylindrical tank is filled with water. Within the tank a door of length $2L$ is installed. The door can rotate about its axis which is located at radius a and depth d relative to the initial height of the water. Assume that the radius of the tank is sufficiently large so that the door can be assumed flat. The tank rotates with a constant angular velocity ω about its axis. Assuming that the fluid moves as a solid body calculate



a. The pressure distribution within the tank.
b. What is the distance h between the initial and final heights of the water on the axis of symmetry. Assume that water level always remains with the region of constant cylinder radius.
c. What is the angular velocity ω so that the net moment on the door is zero.

Solution:

Assumptions:

1. The door is flat.
2. The surface of the water remains within the cylinder of constant radius.
3. The coordinate system is defined at the bottom of the water surface during rotation.
4. Fluid moving as a solid body is equivalent to assuming hydrostatic equilibrium.

The first step is to calculate the hydrostatic pressure. We start off with the hydrostatic equation

$$\vec{\nabla}p = \left(\partial_r p, \frac{1}{r}\partial_\theta p, \partial_z p\right) = \rho\left(-g\hat{z} - a_r\hat{r}\right) = \rho\left(-g\hat{z} + \omega^2 r \cdot \hat{r}\right) \quad . \tag{2.35}$$

This is written explicitly

$$\partial_r p = \rho\omega^2 r \tag{2.36}$$

$$\partial_z p = -\rho g \tag{2.37}$$

Integration of Eq. (2.36) yields the pressure up to a function in z

$$\frac{\partial p}{\partial r} = \rho\omega^2 r \Rightarrow p(r,z) = \frac{\rho\omega^2 r^2}{2} + f(z) \quad . \tag{2.38}$$

Taking this derivative and inserting it in Eq. (2.37) gives

$$f'(z) = -\rho g \Rightarrow f(z) = -\rho g z + c \quad . \tag{2.39}$$

The constant is found by requiring $p(0,0) = p_{atm}$ and it is readily seen that $c = p_{atm}$ thus the pressure is

$$p(r,z) = \frac{\rho\omega^2 r^2}{2} - \rho g z + p_{atm} \tag{2.40}$$

The surface of the water is an isobar of $p(r,z) = p_{atm}$ so that the surface of the water is described by

$$z(r) = \frac{\omega^2 r^2}{2g} \quad . \tag{2.41}$$

Requiring that the volume of water before and after rotation are the same

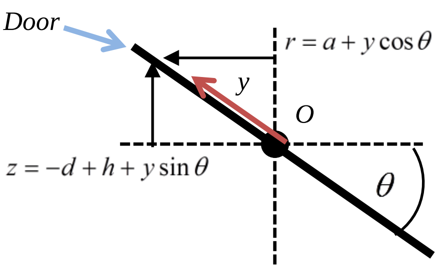
$$\pi R^2 h = \int_0^R z(r) \cdot 2\pi r dr = \frac{\pi\omega^2}{g}\int_0^R r^3 \cdot dr = \frac{\pi\omega^2 R^4}{4g} \quad , \tag{2.42}$$

gives

$$h = \frac{\omega^2 R^2}{4g} . \tag{2.43}$$

The pressure outside of the door is also atmospheric so that it will be beneficial to work with the gage pressure $p_{gage} = p - p_{atm}$.

Point O is the axis of the door, where a new coordinate y is defined along the door. The translation of this coordinate system is given in the figure.



The moment on the door about point O is given by

$$\sum M_O = \int_{-L}^{L} p_{gage}(r(y), z(y))\, y \underbrace{1}_{\substack{witdh \\ unit}} dy \quad , \tag{2.44}$$

this is calculated explicitly by inserting the pressure

$$\int_{-L}^{L}\left(\frac{1}{2}\rho\omega^2\left(a+y\cos\theta\right)^2-\rho g\left(-d+h+y\sin\theta\right)\right)ydy=\frac{2}{3}\left(\rho\omega^2 a\cos\left(\theta\right)-\rho g\sin\left(\theta\right)\right)L^3 . \quad (2.45)$$

This moment will zero when the angular velocity equals to

$$\omega=\sqrt{\frac{g}{a}\tan\theta} . \quad (2.46)$$

Buoyancy

Archimedes' principle

"Any object, wholly or partially immersed in a fluid, is buoyed up by a force equal to the weight of the fluid displaced by the object."

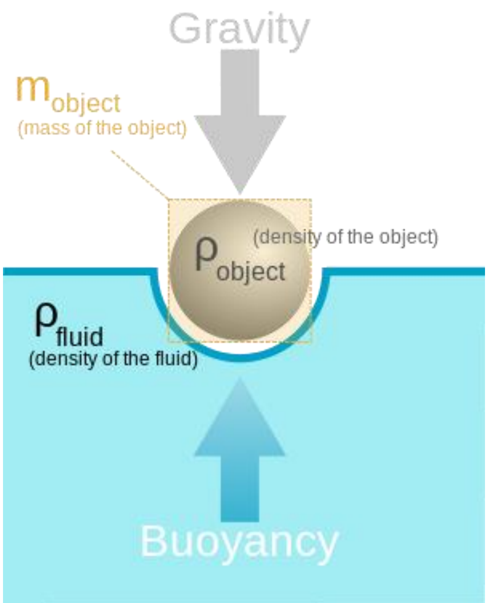
Simply put, an upward force exerted by a fluid opposes the weight of an immersed object. This force is termed buoyancy.

The force is given by the following equation

$$F_{buoy}=\rho_f V_{disp} g , \quad (2.47)$$

where ρ_f is the density of the immersing fluid, V_{disp} is the volume of the displace fluid and not necessarily the volume of the entire body and g is gravity in a constant field. Note that Eq. (2.47) would be modified in the case gravity was non-constant.

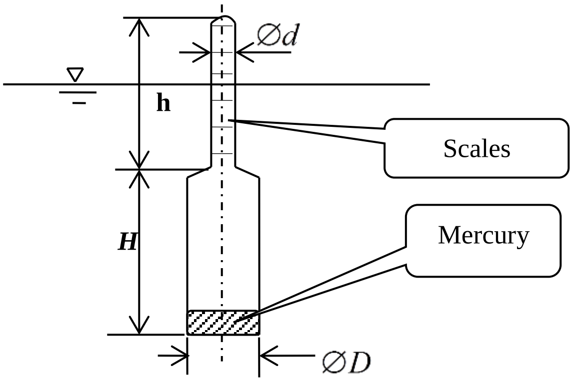
The figure to the right was taken from Ref. [3].



Question 1

A hydrometer whose dimensions are given in the figure is used to measure the density of varying fluids. The mass of the apparatus, not including the mass of the mercury at the bottom, is m_{body}. If the hydrometer is immersed in water whose density is ρ_w, then half of the upper cylinder is immersed in water (the scale $h/2$ is at the water level). If a lighter fluid of density $\eta\rho_w$, with $\eta<1$ is used, the entire body is immersed, yet buoyant. Find

a. The mass m_{Hg} of the mercury at the bottom of the hydrometer and the height H of the lower cylinder.
b. What is the range of fluid densities that this hydrometer can be used for?



Solution

A hydrometer is used to measure fluids of varying densities. Assuming that the weight of the body and mercury are known, then by balancing out gravity and buoyancy the density can be measured. Thus the denser the fluid, the less the body will penetrate in the fluid (i.e. the displaced volume will be smaller)

Given: Geometric dimensions: d, D, h

Body mass: m_{body}

The gravitational force applied on the body is

$$F_g = \left(m_{body} + m_{Hg}\right) g \quad , \tag{2.48}$$

which is balanced out by buoyancy

$$F_{bouy} = \rho_f g V_{disp} \, . \tag{2.49}$$

Requiring equilibrium gives

$$\left(m_{body} + m_{Hg}\right) g = \rho_f g V_{disp} \tag{2.50}$$

We know that when the fluid is water, the density of the fluid and the displaced volume are given by

$$\rho_f = \rho_w, V_1 = \frac{\pi}{4}\left(D^2 H + \frac{h}{2} d^2\right) \; , \tag{2.51}$$

while for the lighter fluid these quantities are

$$\rho_f = \eta \rho_w, V_1 = \frac{\pi}{4}\left(D^2 H + h d^2\right) \; . \tag{2.52}$$

Inserting both Eqs. (2.51) and (2.52) into Eq. (2.50) gives

$$\begin{cases} \rho_w g V_1 = \left(m_{body} + m_{Hg}\right) g \\ \eta \rho_w g V_2 = \left(m_{body} + m_{Hg}\right) g \end{cases} . \tag{2.53}$$

Solving for the m_{Hg} and H yields

$$m_{Hg} = \frac{\pi \eta}{8(1-\eta)} d^2 h \rho_w - m_{body}, H = \left(\frac{d}{D}\right)^2 \frac{h}{2}\left(\frac{1-2\eta}{\eta - 1}\right) \tag{2.54}$$

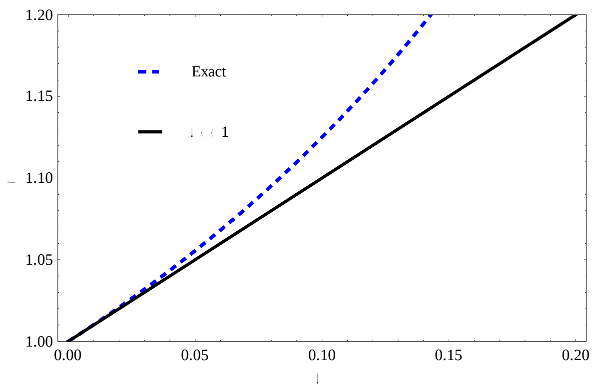
We are now asked to find the highest density the hydrometer can measure- where only the lower cylinder is immersed. Prior to solving this formally, let's ask what the expected answer should be? For example, had $\eta = 0.95$, we would expect that due to symmetry, the highest density would be approximately $1.05\rho_w$. Let us verify this prediction. For case of the highest density we have

$$\rho_f = \alpha \rho_w, V_3 = \frac{\pi}{4} H D^2 \; . \tag{2.55}$$

Inserting this into Eq. (2.50) gives

$$\alpha \rho_w g \frac{\pi}{4} H D^2 = \left(m_{body} + m_{Hg}\right) g = \left(\frac{\pi \eta}{8(1-\eta)} d^2 h \rho_w\right) g \; . \tag{2.56}$$

After some algebraic manipulation (and inserting Eq. (2.54)) one gets



$$\alpha = \frac{\frac{\eta}{(1-\eta)} d^2 h}{2HD^2} = \frac{\frac{\eta}{(1-\eta)} d^2 h}{2\left(\frac{d}{D}\right)^2 \frac{h}{2}\left(\frac{1-2\eta}{\eta-1}\right) D^2} = \frac{\eta}{2\eta-1} \; . \quad (2.57)$$

Assuming that hydrometer measure densities of fluids that are similar to the density of water, we can write $\eta = 1-\Delta$ with $\Delta \ll 1$. Inserting this into Eq. (2.57) and Taylor expanding gives

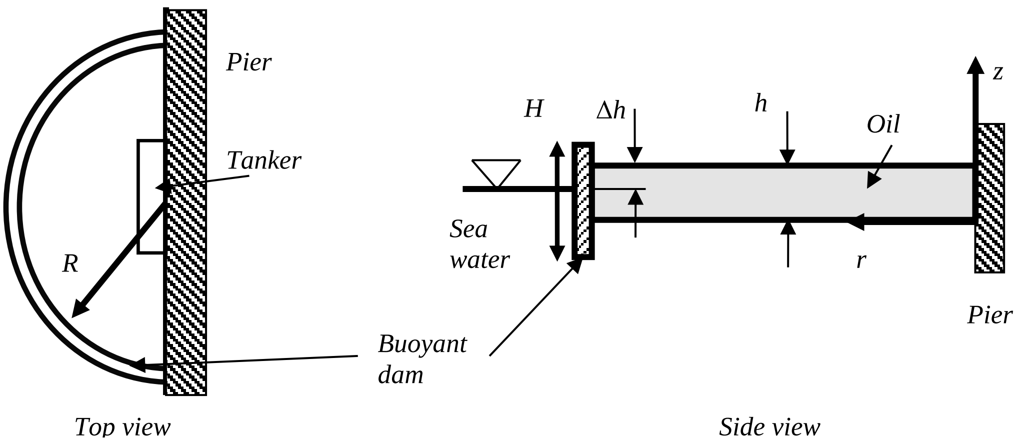
$$\alpha = \frac{\eta}{2\eta-1} = \frac{1-\Delta}{1-2\Delta} = (1-\Delta)\left(1+2\Delta+O\left(\Delta^2\right)\right) = 1+\Delta+O\left(\Delta^2\right). \quad (2.58)$$

This is the result we predicted based upon the simple assumption of symmetry.

Question 2 (Past midterm)

An oil tanker docked at a pier is encircled by a buoyant damn to contain the oil within it in case of a leakage(see top view). The dam dimensions must account for leakage both from below and above. The radius of the dam is given, R, as well as the densities of water and oil, ρ_w and ρ_o, respectively. Assuming that a volume V of oil has been spilled, calculate:

a. The final height of the spilled oil layer.
b. The height difference Δh between the sea water level and oil level.
c. The applied horizontal force per unit length of the dam.
d. The applied force at one of the two connecting points between the damn and pier.
e. Assuming that the height of the dam is $H = h + 2h_2$, find the buoyant dam density such the dam will protrude equally from the top and bottom of the oil.



Solution

Assumptions: The system has reached equilibrium, so that hydrostatics principles can be used.

a. The height of the oil is easily calculated

$$h = \frac{2V}{\pi R^2} \quad (2.59)$$

b. Both the water and oil are governed by the hydrostatic equation thus the pressures are given by

$$\partial_z P = -\rho g \Rightarrow P_{oil/w} = -\rho_{oil/w} g z + A_{oil/w} \quad (2.60)$$

Requiring that the pressures at the free surfaces are zero (why can we assume this?) gives

$$\begin{cases} P_w\left(z=h-\Delta h\right)=0 \Rightarrow P_w=\rho_w g\left(h-\Delta h-z\right) \\ P_{oil}=\left(z=h\right)=0 \Rightarrow P_{oil}=\rho_{oil} g\left(h-z\right) \end{cases} \tag{2.61}$$

These pressures are equal at $z=0$ giving

$$\rho_{oil} gh=\rho_w g\left(h-\Delta h\right) \Rightarrow \Delta h=\left(1-\frac{\rho_{oil}}{\rho_w}\right)h \tag{2.62}$$

Two alternate derivations for the solution of this problem exist. One assumes that the potential energy of each of the surfaces is equal, in fact this exactly what is written in the first term of Eq. (2.62) . The second derivation which was pointed out to me by students during class is probably the most elegant and is left as an exercise for you – use the consideration of buoyancy to describe the floating oil in equilibrium.

c. This question focuses on immersed surfaces where the horizontal force (per unit length) is given by

$$f_x=\frac{F_x}{b}=\int_0^H Pdy \quad , \tag{2.63}$$

so that the force being applied by the oil is

$$f_{oil}=\int_0^h \rho_{oil} g\left(h-z\right)dz=\frac{\rho_{oil} gh^2}{2} \quad , \tag{2.64}$$

and the force being applied by the water is

$$f_w=\int_0^{h-\Delta h} \rho_w g\left(h-\Delta h-z\right)dz=\frac{\rho_w g\left(h-\Delta h\right)^2}{2} \quad . \tag{2.65}$$

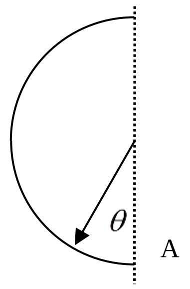
Inserting Eq. (2.62) into Eq. (2.65) and subtracting Eq. (2.65) from Eq. (2.64) gives the force per unit length

$$q=\frac{\rho_{oil} gh^2}{2}\left(1-\frac{\rho_{oil}}{\rho_w}\right). \tag{2.66}$$

This force is positive in the radial direction – the oil is pushing outwards.

d. To calculate the reactions at the contact points we must write Eq. (2.66) in a vector form

$$\begin{aligned} q_x &= q\sin\theta \\ q_y &= q\cos\theta \end{aligned} \tag{2.67}$$



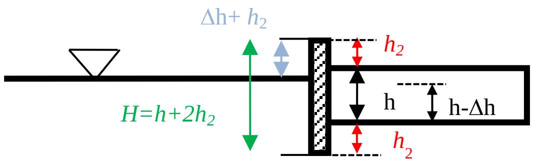
For reasons of symmetry the force in the $\hat{y}$ direction must cancel out while the force in the $\hat{x}$ is readily calculated

$$.\ Q=\int_0^\pi q_x Rd\theta=\int_0^\pi q\sin\theta Rd\theta=2qR \quad . \tag{2.68}$$

Thus the reaction forces at point A are

$$F_{A,x}=Q/2=qR, F_{A,y}=0 \tag{2.69}$$

e. We are now told that the dam protrudes h_2 above and below the water. This indicates the amount of fluid/volume displaced by the damn.



Note that the solution contains the thickness of the dam. Since this thickness is not given in the question, this is an indication that this length scale must cancel off.

$$V_{disp} = \underbrace{\pi R}_{length} \underbrace{L}_{thickness} \underbrace{\left(h + 2h_2 - \Delta h - h_2\right)}_{height} = \underbrace{\pi R}_{length} \underbrace{L}_{thickness} \underbrace{\left(h + h_2 - \Delta h\right)}_{height} \; . \tag{2.70}$$

The density used to calculate the buoyancy force $\left(F_{buoy} = \rho_w g V_{disp}\right)$ is that of water- because it is the water that is responsible for the floatation of the dam. The gravitational force is simply

$$F_g = -\rho_{body} g V_{body} = -\rho_{body} g \underbrace{\pi R}_{length} \underbrace{L}_{thickness} \underbrace{\left(h + 2h_2\right)}_{height} \; . \tag{2.71}$$

Requiring equilibrium (and noting that L does indeed cancel out), we find the density of the body

$$\rho_{body} = \rho_w \frac{V_{disp}}{V_{body}} = \rho_w \frac{\left(h + h_2 - \Delta h\right)}{\left(h + 2h_2\right)} \tag{2.72}$$

“It can scarcely be denied that the supreme goal of all theory is to make the irreducible basic elements as simple and as few as possible without having to surrender the adequate representation of a single datum of experience.” , Albert Einstein

Simply put: “Everything should be made as simple as possible, but no simpler.”

3. Integral Equations of Motion

Conceptually, nothing is easier than to understand “what goes in minus what goes out equals to that which changes in time”. In this chapter, we will discuss conservation and balance laws for mass, momentum and energy. We will see that much knowledge can be gained by looking at integral quantities without knowledge of local quantities. This method usually gives very good estimates on how average quantities behave at the boundaries of a system without knowing exactly what is occurring within the system.

Conservation of mass

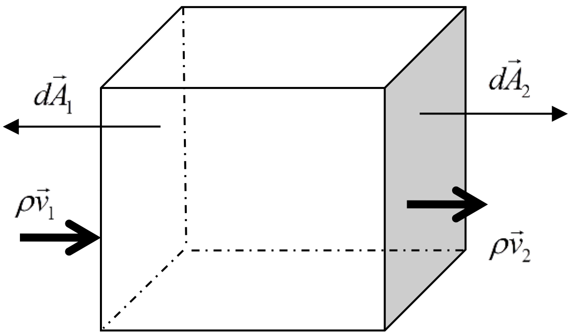
The flux entering through a surface minus the flux leaving equals to that which accumulates in the volume encompassed by the surface. This is best understood by looking at the following equation

$$\frac{\partial}{\partial t}\iiint_{C.V.}\rho dV = -\oiint_{C.S.}\rho\vec{q}\cdot d\vec{A} \ , \tag{3.1}$$

where

$$\vec{q} = \vec{u} - \vec{u}_{C.S.} \ , \tag{3.2}$$

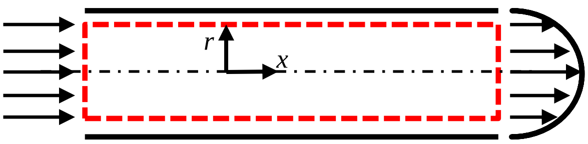
is the relative fluid velocity to that of the control surface (which can change shape and translate in time). The left hand side (LHS) of the equation describes the accumulation of mass within the control volume. For example, if $\rho = \rho(x, y, z, t)$, then integration over the entire volume gives us the mass trapped within, while the time derivative shows us how the mass within changes. The right hand side of the equation describes the mass flux in and out of the system. The minus is there simply because the contribution of entering flux is positive. In the figure we see that the scalar product between the surface and mass flux in the entrance is negative while in the exit it is positive. Also, it can be seen that both expressions, have units of mass per unit time.



One last comment, for the sake of brevity, in future exercises and chapters, the volume and area will most likely be marked as V and A , respectively. We leave it to the reader to understand the connotation.

Question 1

An incompressible fluid flows through a long circular channel of radius R .Given that the entrance velocity is uniform, U , and that the exit velocity has a parabolic profile $u = u_{\max}\left(1 - r^2 / R^2\right)$. Find the maximal velocity of the exit profile.



<u>Solution:</u>

Assumption: The problem is time independent.

Control volume: We choose a non-changing and non-moving control volume.

The governing equation is

$$-\oint\!\!\oint_A \rho\vec{u}\cdot d\vec{A} = -\left(\iint_{left}\rho\vec{u}\cdot d\vec{A} + \iint_{right}\rho\vec{u}\cdot d\vec{A}\right) = 0 \quad . \tag{3.3}$$

The velocity and surface at the left (indicated by 1 in the figure is)

$$\vec{v} = U\hat{x}, d\vec{A} = -rdrd\theta\hat{x} \quad , \tag{3.4}$$

thus the flux is

$$-\iint_{A_1} U\rho rdrd\theta\left(\hat{x}\cdot\hat{x}\right) = U\rho\int_0^R\int_0^{2\pi} rdrd\theta = \pi R^2 U\rho \quad . \tag{3.5}$$

At the right surface, the velocity, surface and flux are

$$\vec{v} = u_{\max}\left(1-\left(\frac{r}{R}\right)^2\right)\hat{r}, d\vec{A} = Rd\theta dx\hat{r} \quad , \tag{3.6}$$

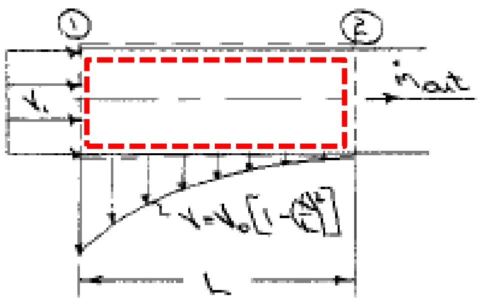
$$-\int_0^R\int_0^{2\pi}\rho u_{\max}\left(1-\left(\frac{r}{R}\right)^2\right)rdrd\theta\left(\hat{x}\cdot\hat{x}\right) = -\pi u_{\max}\frac{R^2}{2}\rho \quad . \tag{3.7}$$

Thus,

$$-\oint\!\!\oint_A \rho\vec{v}\cdot d\vec{A} = 0 : 2\pi R^2 U\rho - 2\pi u_{\max}\frac{R^2}{2}\rho = 0 \Rightarrow u_{\max} = 2U \quad . \tag{3.8}$$

Question 2

An incompressible fluid of density, ρ, flows through a porous pipe of radius, R. The flow enters the pipe with a uniform velocity, V_1, and leaks out radially (axis-symmetrically) with the profile $V = V_0\left[1-\left(x/L\right)^2\right]$. Calculate the mass outflow at $x = L$.



Solution

Assumption: The problem is time independent.

Control volume: We choose a non-changing and non-moving control volume.

The governing equation is

$$-\oint\!\!\oint_A \rho\vec{u}\cdot d\vec{A} = -\left(\iint_{left}\rho\vec{u}\cdot d\vec{A} + \iint_{right}\rho\vec{u}\cdot d\vec{A} + \iint_{bottom}\rho\vec{u}\cdot d\vec{A}\right) = 0 \quad . \tag{3.9}$$

The velocity and surface at the left (indicated by 1 in the figure is)

$$\vec{u} = V_1\hat{x}, d\vec{A} = -rdrd\theta\hat{x} \quad , \tag{3.10}$$

thus the flux is

$$-\iint_{A_1} -V_1\rho rdrd\theta\left(\hat{x}\cdot\hat{x}\right) = V_1\rho\int_0^R\int_0^{2\pi} rdrd\theta = \pi R^2 V_1\rho \quad . \tag{3.11}$$

At the bottom surface, or more exact, at the circumference of the pipe, the velocity, surface and flux are

$$\vec{u} = V_0\left(1-\left(\frac{x}{L}\right)^2\right)\hat{r}, d\vec{A} = Rd\theta dx\hat{r} \ , \quad (3.12)$$

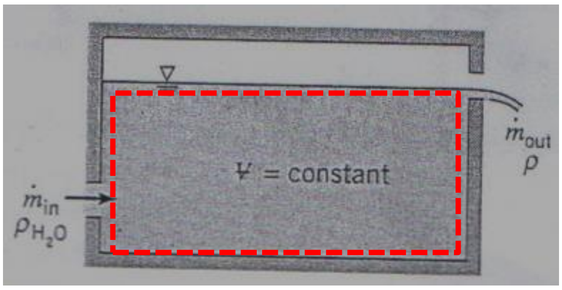
$$-\iint_{A_2} V_0\left(1-\left(\frac{x}{L}\right)^2\right)\rho Rd\theta dx\hat{r}\left(\hat{r}\cdot\hat{r}\right) = -\int_0^L\int_0^{2\pi} V_0\left(1-\left(\frac{x}{L}\right)^2\right)\rho Rd\theta dz = -\frac{4}{3}\pi RLV_0\rho \ . \quad (3.13)$$

Inserting Eq. (3.11) and (3.13) in to Eq. (3.9) gives

$$\pi R^2\rho V_1 - \frac{4}{3}\pi RLV_0\rho - \dot{m}_{out} = 0 \Rightarrow \dot{m}_{out} = \pi R^2\rho V_1 - \frac{4}{3}\pi RL\rho V_0 \ . \quad (3.14)$$

Question 3

A tank initially holds within it salt water with a density $\rho_i > \rho_{H_2O}$. At $t=0$ water is pumped through an entrance located at the left, with a volumetric flux Q. Assuming that the volume of the fluid inside, V, remains constant:



a. Write the equation governing the dynamics of the density.
b. Solve for the time it takes the fluid to reach density ρ_f .

Solution

Note, the information of volumetric flux Q contains information about both the velocity and area size at the entrance.

Assumption: Assume that the mixing time of the fresh water and salt water is instantaneous $\rho(x,y,z,t)=\rho(t)$.

Control volume: We choose a non-changing and non-moving control volume.

That the volume of the fluid inside does not change indicates that volumetric flux at the exit is the same as the entrance. Hence,

$$\dot{m}_{in} - \dot{m}_{out} = \left(\rho_{H_2O} - \rho\right)Q = -\left(\rho - \rho_{H_2O}\right)Q \ . \quad (3.15)$$

The accumulation term is

$$\frac{\partial}{\partial t}\iiint_V \rho(t)dV = \frac{\partial}{\partial t}\rho(t)V = V\frac{\partial\rho(t)}{\partial t} \ . \quad (3.16)$$

Demanding equality gives

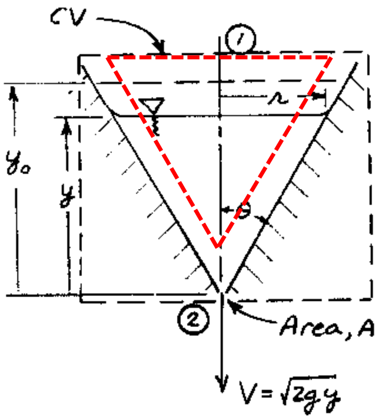
$$V\frac{\partial\rho(t)}{\partial t} = V\frac{d\rho(t)}{dt} = -\left(\rho - \rho_{H_2O}\right)Q \ . \quad (3.17)$$

Solving for the time gives

$$dt = -\frac{V}{Q}\frac{d\rho}{\rho-\rho_{H_2O}} \Rightarrow t = -\frac{V}{Q}\log\left(\rho-\rho_{H_2O}\right)\Big|_{\rho_i}^{\rho_f} = -\frac{V}{Q}\log\left(\frac{\rho_f-\rho_{H_2O}}{\rho_i-\rho_{H_2O}}\right) \ . \quad (3.18)$$

Question 4

An incompressible fluid is emptying out from a small aperture at the bottom of a conic tank. Assuming that the initial height of the fluid was y_0, the angle of the cone is , θ, the radius of the aperture is , R, and the exit velocity is given by $\sqrt{2gy}$, find the time when the tank is empty.



Solution

Control volume: We will choose two different control volumes. One that is constant in space and time while the other will follow the volume of the fluid. Both will yield the same result.

We will solve first for the constant volume. It is obvious that the flux at the upper surface is zero because the velocity there is zero. Hence the exiting flux is responsible for the fluid depleting from the tank. At the aperture the velocity and surface are given by

$$\vec{u} = -\sqrt{2gy}\,\hat{y}, d\vec{A} = -rdrd\theta\hat{y} \qquad (3.19)$$

Thus the flux is

$$-\iint_A \left(-\sqrt{2gy}\right)\rho_f\left(-rdrd\theta\right)\left(\hat{y}\cdot\hat{y}\right) = -\int_0^R\int_0^{2\pi}\sqrt{2gy}\,\rho_f rdrd\theta = -\pi R^2\sqrt{2gy}\,\rho_f \qquad (3.20)$$

The volume term is the following

$$\frac{\partial}{\partial t}\iiint_V \rho dV = \frac{\partial}{\partial t}\left(\iiint_{V_f}\rho_f dV + \iiint_{V_{air}}\rho_{air} dV\right) \qquad (3.21)$$

Assuming that $\rho_{air} \ll \rho_f$ the second term can be neglected. Hence, the volume term is

$$\frac{\partial}{\partial t}\iiint_{V_f}\rho_f dV = \frac{\partial}{\partial t}\rho_f\iiint_{V_f} dV = \frac{\partial}{\partial t}\rho V_f \;, \qquad (3.22)$$

The volume of a cone (of radius, R, and height, H) is given by

$$V_{cone} = \frac{\pi R^2 H}{3} \;, \qquad (3.23)$$

hence,

$$\frac{\partial}{\partial t}\rho_f V_f = \frac{\partial}{\partial t}\rho_f\left(\pi r^2\frac{y}{3}\right) \;. \qquad (3.24)$$

The radius and height are related by geometry

$$r = y\tan\theta \;. \qquad (3.25)$$

Substitution gives

$$\frac{\partial}{\partial t}\rho_f\left(\pi r^2\frac{y}{3}\right) = \frac{\partial}{\partial t}\rho_f\left[\pi\left(y\tan\theta\right)^2\frac{y}{3}\right] = \frac{\partial}{\partial t}\rho_f\left(\frac{\pi}{3}y^3\tan^2\theta\right) = \rho_f\pi y^2\tan^2\theta\frac{\partial y}{\partial t} \;. \qquad (3.26)$$

Requiring equality between Eq. (3.20) and Eq. (3.26) yields

$$-\pi R_0^{\ 2}\sqrt{2gy}\rho_f = \rho_f \pi y^2 \tan^2\theta \frac{dy}{dt} \ . \tag{3.27}$$

The solution of this equation is

$$dt = -\frac{\tan^2\theta}{R_0^{\ 2}\sqrt{2g}} y^{3/2}dy \Rightarrow T\left(y=0\right) = -\int_{y_0}^{0} \frac{\tan^2\theta}{R_0^{\ 2}\sqrt{2g}} y^{3/2}dy = \frac{2\tan^2\theta}{5R_0^{\ 2}\sqrt{2g}} y_0^{\ 5/2} \ . \tag{3.28}$$

Alternately, we now choose a CV that follows the fluid. The flux at the bottom aperture remains the same. The relative velocity between the fluid and control volume is zero so that the flux is still zero at the top. However, the volumetric change no longer needs to account for the air. Thus, Eq. (3.21) is simplified

$$\frac{\partial}{\partial t}\iiint_V \rho dV = \frac{\partial}{\partial t}\iiint_{V_f} \rho_f dV \ . \tag{3.29}$$

Of course it is now obvious that the rest of the solution remains unchanged.

Integral equations of momentum balance

Newton's second law not only applies to point particles but can also be written in integral form.

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{u}dV + \oint\!\!\!\oint_A \rho\vec{u}\left(\vec{q}\cdot d\vec{A}\right) = -\oint\!\!\!\oint_A pd\vec{A} + \vec{F}_S + \vec{F}_V \tag{3.30}$$

where $\vec{q} = \vec{u} - \vec{u}_{C.S.}$ and is given by Eq. (3.2). The first term on the LHS represents the momentum accumulating within the CV, while the second term on the LHS is simply the fluxes through CS. These are balanced by forces that are applied both on the surface (surfaces forces) and volume (body forces). Due to its importance, the pressure, which is a surface force, is written explicitly. To conclude, the accumulation of momentum within a control volume plus the flux of momentum through the control surface are balanced out by external forces.

One should look at the form of the second term on the LHS. Not only is it non-linear, but one should also remember that this equation is a vector equation. Thus one can expect interesting behavior. For example, for a general velocity vector (do not confuse the vector $\vec{u}$ with velocity component u) and flux through a non-moving surface given by

$$\vec{u} = \left(u,v,w\right), d\vec{A} = udydz\hat{x} \ , \tag{3.31}$$

the flux will be

$$\vec{u}\left(\vec{u}\cdot d\vec{A}\right) = \vec{u}\left(udydz\right) \ . \tag{3.32}$$

Eq. (3.32) illustrates that due to the convection of momentum in one direction, one can expect a change in the momentum in the transverse directions.

Conservation of energy

Bernoulli's equation is manifestation of conservation of energy when all the applied forces within the system are conservative (viscosity is a non-conservative force). The energy upon a streamline is constant

$$E_{line} = P_1 + \rho\frac{v_1^{\ 2}}{2} + \rho g z_1 = P_2 + \rho\frac{v_2^{\ 2}}{2} + \rho g z_2 \ , \tag{3.33}$$

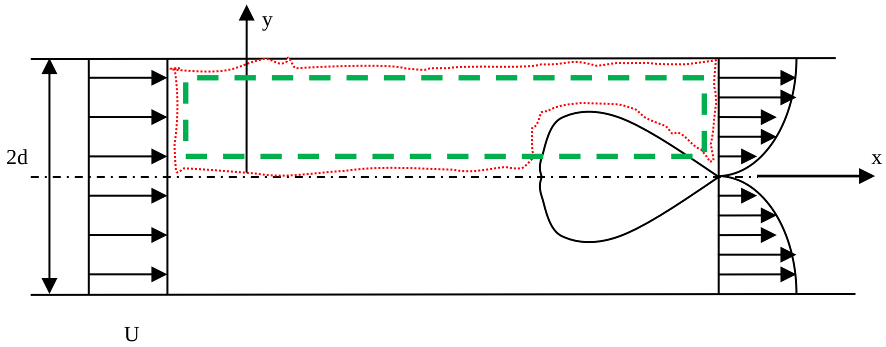
where points 1 and 2 are on the same stream line. The conditions for using the above equations are

1. Steady flow.
2. Incompressible flow.
3. Ideal flow.
4. Inviscid flow.

Further along the course you will learn that there are additional conditions that will constrain the energy to be constant in space and not just upon a certain streamline.

Question 1 (From 2005/6 Final)

An incompressible fluid, of density ρ, enters a 2D channel with a uniform and steady velocity profile, U. The channel height is $2d$. At the center of the channel is symmetric body. Assume:



I. At the exit (section) the pressure is constant and the velocity has a parabolic profile.
II. The velocity is zero on the symmetry axis.
III. At $y=\pm d$ the velocity derivative is zero $\partial_y u = 0$.

Find:

a. The exit velocity profile as a function of U and d.
b. The drag force applied on the body.

<u>Solution</u>

Control volume: Due to symmetry we look only at the upper half of the channel. In fact we are looking at a control volume per unit depth, so that in fact $C.V. \to C.S.$ and $C.S. \to \mathrm{C.L.}$

Both the red (dotted) and green (dashed) CV are the natural choices but the question is which is more convenient. For mass conservation, obviously, they will both produce the give the same result and one is not more advantageous over the other. However, when calculating the momentum in $\hat{x}$ direction, the red CV requires knowledge of the pressure on the surface of the body, which is unknown, while the green doesn't. In fact, had we known the pressure distribution on the body, our problem would have been solved. We will use the green CV for the remainder of this exercise.

The exit velocity profile is given by

$$u(y) = Ay^2 + By + C \,. \tag{3.34}$$

Applying the two BC (given by assumptions II and III)

$$u(0) = 0, \partial_y u(d) = 0 \,, \tag{3.35}$$

yields $(B = -2Ad, C = 0)$, thus Eq. (3.34) is written as

$$u(y) = Ay(y - 2d) \,. \tag{3.36}$$

To find the final unknown constant we require that the entering flux is equal to exiting flux (per unit depth)

$$\oint_{C.L.} \rho \vec{u} \cdot d\vec{l} = -\int_0^d U dy + \int_0^d u(y) dy = 0 \Rightarrow A = -\frac{3U}{2d^2} \,. \tag{3.37}$$

Thus the velocity profile is

$$u(y) = \frac{3U}{2d^2} y(2d - y) \,. \tag{3.38}$$

To calculate the drag force, we must use the momentum balance equation. However, to be able to use this equation we need to know the pressure difference between entrance and exit. In fact we should return to assumption III. The meaning that $\partial_y u(d)=0$, is that the flow is not experiencing shear stresses (i.e. energy is not lost to viscous induced heat). Thus, we can assume that energy is conserved on the streamline $y=d$. Hence we can apply Bernoulli's theorem

$$p_{entrance}+\rho\frac{U^2}{2}=p_{exit}+\rho\frac{u(y=d)^2}{2}\Rightarrow\Delta p=p_{entrance}-p_{exit}=\frac{\rho}{2}\left[\left(\frac{3U}{2}\right)^2-U^2\right]=\frac{5\rho U^2}{8} \tag{3.39}$$

Eq. (3.30) is written for this problem

$$\int_0^d \rho U\hat{x}\left[U\hat{x}\cdot(-\hat{x})dy\right]+\int_0^d \rho u\hat{x}\left(u\hat{x}\cdot\hat{x}dy\right)=\int_0^d\left(p_{entrance}-p_{exit}\right)dy\hat{x}+\frac{D}{2}\ , \tag{3.40}$$

where D is the total drag force on the fluid and the factor 2 accounts for half the geometry. We now account for assumption I and insert the velocities and pressure into Eq. (3.40)

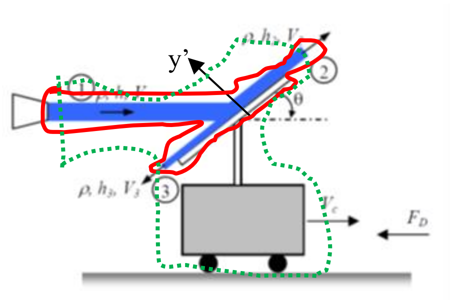
$$\frac{\rho U^2 d}{5}=\frac{5}{8}\rho U^2 d+\frac{D}{2}\Rightarrow D=-\frac{17}{20}\rho U^2 d \tag{3.41}$$

Hence the total drag force on the body is minus that on the fluid

$$D_{body}=-D_{flow}=\frac{17}{20}\rho U^2 d \tag{3.42}$$

Question 2 (Winter 2012 midterm)
A jet of water, with density ρ, thickness h, and velocity V , impacts a board tilted by angle θ that is on top of a wheeled cart. The incoming jet splits into two smaller jets. Assuming that the fluid is ideal, neglecting viscous effects between the water and board, and neglecting the effect of gravity, find

a. The velocities (V_2, V_3) when the cart is at rest.
b. The thickness of the jet (h_2, h_3) when the cart is at rest.
c. The same as questions a. and b. when the cart is moving with a constant velocity V_c .
d. Find the terminal velocity of the kart, V_c, if the drag experienced by the kart is $F=kV_c^2$. Neglect the friction between the cart and the ground.
e. Based upon the results of the previous question, at what angle θ , will the cart move at a maximal and minimal velocity. What are these velocities?



Solution

Assumptions:

1. Steady flow.
2. Incompressible flow.
3. Ideal fluid.
4. Inviscid flow.
5. Uniform velocities at all sections.
6. Atmospheric pressure throughout.
7. Gravity is negligible.

Control volumes: We will use the red volume when the cart is at rest and when it is moving but when we are uninterested of the drag it is experiencing. The green volume will used when we are interested in the applied drag.

Given all the above assumptions, it is readily seen that energy is conserved thus

$$P_{atm} + \frac{\rho V^2}{2} + \rho g h = P_{atm} + \frac{\rho {V_{2,3}}^2}{2} + \rho g h \Rightarrow V_{2,3} = V \ . \tag{3.43}$$

To find the thickness, we will require conservation of mass and balance of tangential momentum

$$\rho V h = \rho V_2 h_2 + \rho V_3 h_3 = \rho V \left(h_2 + h_3\right) \ , \tag{3.44}$$

$$\rho V^2 h \cos\theta = \rho {V_2}^2 h_2 - \rho {V_3}^2 h_3 = \rho V^2 \left(h_2 - h_3\right) \ . \tag{3.45}$$

The solution of which is

$$h_2 = \frac{h}{2}\left(1+\cos\theta\right), h_3 = \frac{h}{2}\left(1-\cos\theta\right) \ . \tag{3.46}$$

When the cart is moving at a constant velocity V_c, the relative entrance velocity is now $V - V_c$ but this is true also for the exit velocities, thus in Eqs. (3.44) and (3.45), all $V \rightarrow V - V_c$ but these cancel out and the solution (Eq. (3.46)) remains unchanged.

We shall now use the green CV which accounts also for the drag applied on the cart. The momentum in the direction parallel to the surface we can write the momentum equation

$$-\left(V - V_c\right)^2 h + \left(V - V_c\right)^2 h_2 \cos\theta - \left(V - V_c\right)^2 h_3 \cos\theta = -\frac{k}{\rho} {V_c}^2 \ . \tag{3.47}$$

Inserting Eq. (3.46) and after some algebraic manipulations we have

$$\left(\frac{V_c}{V - V_c}\right)^2 = \frac{\rho h}{k}\sin^2\theta \Rightarrow \frac{V_c}{V} = \frac{\sin\theta}{\left(k/\rho h\right)^{1/2} + \sin\theta} \ . \tag{3.48}$$

It is easy to see that the velocity is maximal at $\theta = \pi/2$ and minimal at $\theta = 0$

$$V_c\left(\theta = \pi/2\right) = \frac{V}{\left(k/\rho h\right)^{1/2} + 1}, V_c\left(\theta = 0\right) = 0 \ , \tag{3.49}$$

where the former is when the board is perpendicular to the ground and the latter is parallel.

An alternate derivation of the solution is based upon the observation that the total amount of momentum applied in the perpendicular direction, marked by y', is

$$F_{y'} = \rho\left(V - V_c\right)^2 h \sin\theta \tag{3.50}$$

Taking the projection in the $\hat{x}$ direction (and accounting for a minus sign) gives

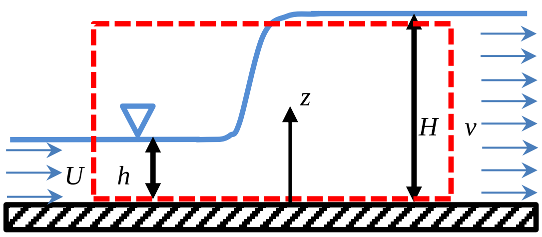
$$-\rho\left(V - V_c\right)^2 h \sin\theta^2 = -k V_c^2 \tag{3.51}$$

The projection in the $\hat{y}$ direction balances out

$$\rho\left(V - V_c\right)^2 \sin\theta\cos\theta = \rho\left(V - V_c\right)^2 \left(h_2\cos\theta - h_3\cos\theta\right) \tag{3.52}$$

Question 3

Incompressible fluid flowing at high velocities in a wide, horizontal open channel can experience a hydraulic jump. Assuming uniform velocities at the



entrance and exits, and assuming hydrostatic pressure distributions, find

a. The height and velocity of the fluid at the exit.
b. Calculate the rate of energy dissipation in the system.

Solution

The most familiar example of a hydraulic jump is when you open your kitchen sink and you see that near the jet the fluid is thin but moving at high velocity and suddenly at a constant radius the fluid jumps to a higher height and slows down.

Control volume: We choose a CV which doesn't require knowledge of what is occurring at the jump (which in the mathematical limit is a discontinuity).

Assumption: We will use gage pressure.

Conservation of mass gives a simple equation

$$Uh = vH \tag{3.53}$$

It is easy to see that due to hydrostatic equilibrium the pressure distribution are given by

$$p_{entrance} = \rho g\left(h - z\right), p_{exit} = \rho g\left(H - z\right) \tag{3.54}$$

Thus the momentum equation in the $\hat{x}$ direction is

$$-\rho hU^2 + \rho v^2 H = \int_0^h p_{entrance}\left(z\right)dz - \int_0^H p_{exit}\left(z\right)dz \tag{3.55}$$

Inserting both Eq. (3.53) and (3.54) into Eq. (3.55) yields (after dividing by ρ)

$$-hU^2 + \frac{h^2U^2}{H} = \frac{g\left(h^2 - H^2\right)}{2} \quad , \tag{3.56}$$

It can be seen that the solution $H = h$ is one of the roots of this equation, thus after algebraic manipulation one gets

$$\left(h - H\right)\left(gH\left(h + H\right) - 2hU^2\right) = 0 \tag{3.57}$$

How to get this equation? Simply write

$$\left(h - H\right)\left(AH^2 + BH + C\right) \tag{3.58}$$

Expand this expression and that in Eq. (3.57) and compare polynomial coefficients. The trivial solution of $H = h$ is of little interest as this mean the jump is non-existent. Thus we solve the quadratic equation given in Eq. (3.57), where only one solution is physical

$$H = -\frac{h}{2} \pm \frac{h}{2}\sqrt{1 + \frac{8U^2}{hg}} \quad . \tag{3.59}$$

Thus the velocity is at the exit

$$v = \frac{Uh}{H} = U\left(-\frac{1}{2} + \frac{1}{2}\sqrt{1 + \frac{8U^2}{hg}}\right)^{-1} \tag{3.60}$$

The expression in the square root is called the Froude number and is a ratio of kinetic energy and gravitational potential energy

$$F_r = \frac{U}{\sqrt{gh}} \ . \tag{3.61}$$

Rewriting the height and velocity in term of the Froude number gives

$$H = \frac{h}{2}\left(\sqrt{1+8F_r^{\,2}}-1\right), v = 2U\left(\sqrt{1+8F_r^{\,2}}-1\right)^{-1} \tag{3.62}$$

The integral equation for balance of energy is given by

$$\partial_t \iiint_V \rho e dV = -\oiint_S \rho e \vec{u}\cdot d\vec{A} \tag{3.63}$$

The first term is the time derivative of total change to the system energy, $\dot{E}_{sys}$. The energy density is given by

$$e = \frac{u^2}{2}+\frac{p}{\rho}+gz \tag{3.64}$$

The RHS of Eq. (3.63) gives

$$\int_0^h \left[\frac{u^2}{2}+g(h-z)+gz\right]U\rho dz - \int_0^H \left[\frac{v^2}{2}+g(H-z)+gz\right]\rho v dz \tag{3.65}$$

Evaluation gives

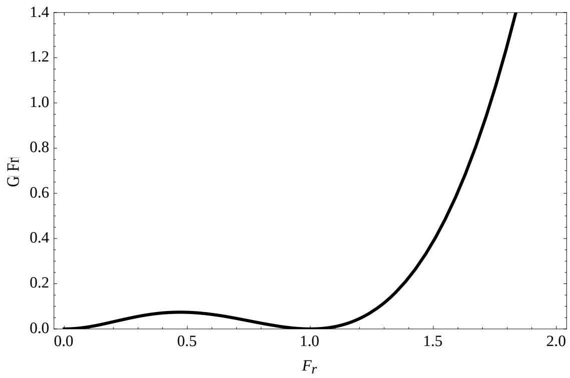
$$\dot{E}_{sys} = \frac{\rho}{2}\left(hU^3 - Hv^3\right)+\rho g\left(h^2U - H^2 v\right) \ . \tag{3.66}$$

Inserting Eq. (3.53) into Eq. (3.66) gives

$$\dot{E}_{sys} = \frac{U\rho h(H-h)}{2H^2}\left[(h+H)U^2 - 2gH^2\right] \tag{3.67}$$

Additional simplification and inserting the solution for H gives

$$\frac{\dot{E}_{sys}}{\frac{U\rho hg(H-h)}{2H^2}} = (h+H)hF^2 - 2H^2 = h^2\left(\sqrt{1+8F_r^{\,2}}\frac{F_r^{\,2}}{2}+\sqrt{1+8F_r^{\,2}}-\frac{7F_r^{\,2}}{2}-1\right) = h^2 G(F_r) \tag{3.68}$$



Exploring $G(F_r)$ shows this function is semi-positive definite, meaning that energy is constantly being transported into the system. However, we are in steady state so we expect that energy will not accumulate in the system. This means we should have added an additional heat sink term in to Eq. (3.63) which is responsible for dissipating heat within the system and this occurs at the vicinity of the discontinuity.

Accelerating systems

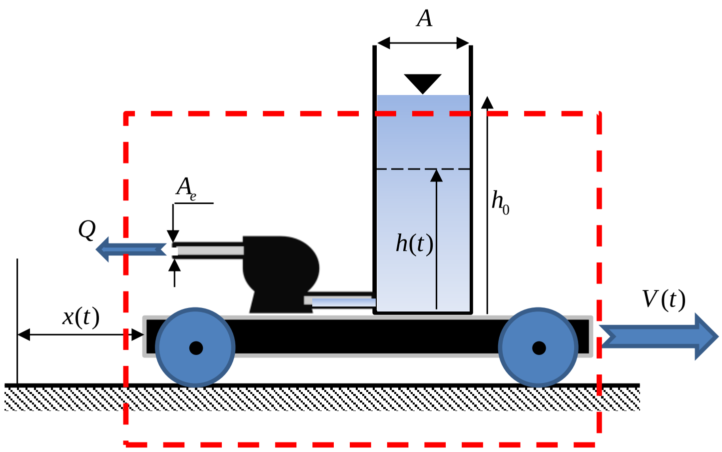
If a system is accelerating and a non-inertial frame is being used, then Eq. (3.30) needs to be modified. This done by adding a term which accounts for the total acceleration within the system

$$\frac{\partial}{\partial t}\iiint_V \rho\vec{u}dV + \oiint_A \rho\vec{u}\left(\vec{q}\cdot d\vec{A}\right)+\iiint_V \rho\vec{a}dV = -\oiint_A pd\vec{A}+\vec{F}_S+\vec{F}_V \tag{3.69}$$

Question 1

A large water tank is fitted onto a cart that is at rest and has an initial mass, M_0. The initial height of the water is h_0 and the area of the tank is A. At $t=0$, the water is pumped out at a constant volumetric flux rate Q through an aperture with an area A_e. The friction coefficient of the cart and the surface is μ. Assuming that the change of inertia within the tank can be neglected, solve the following problem using a non-inertial frame (NIF) and a inertial frame (IF)

a. The minimal flux that will allow the cart to start moving at $t=0$.
b. Solve for the acceleration, velocity and displacement as a function of time (including after the cart is emptied completely).



Solution

Control volume: Our control volume will be one that covers the entire cart. Our CV will move with the velocity of the cart, so that depending on our frame of reference we will need to account for $\vec{q}=\vec{v}-\vec{v}_{C.S.}$ accordingly.

Assumptions

1. Atmospheric pressure[2].

Let us begin with the NIF

Mass conservation gives

$$\rho\partial_t V=\rho A\partial_t h=-\rho Q \ . \tag{3.70}$$

Solution of which gives the both the height as a function of time and the time until the cart is emptied

$$h(t)=h_0-\frac{Qt}{A}, t_f=\frac{h_0 A}{Q} \ . \tag{3.71}$$

Obviously, the mass of the cart and tank is then simply

$$M(t)=\begin{cases} M_0+\rho Ah(t)=M_0+\rho h_0 A-\rho Qt & ,t<t_f \\ M_0 & ,t\geq t_f \end{cases} \ . \tag{3.72}$$

The equations governing momentum balance are:

$$\hat{x}:\frac{\partial}{\partial t}\iiint_V \rho u dV+\oiint_A \rho u\left(\vec{q}\cdot d\vec{A}\right)+\iiint_V \rho a_x dV=-\mu N \ , \tag{3.73}$$

$$\hat{y}:\frac{\partial}{\partial t}\iiint_V \rho v dV+\oiint_A \rho v\left(\vec{q}\cdot d\vec{A}\right)+\iiint_V \rho a_y dV=N-M(t)g \ . \tag{3.74}$$

In Eq. (3.74) the first term on the LHS falls because of the given assumption of negligible vertical inertia. The second term is zero because the velocity v on the control surface is identically zero. The third term is dropped because the system is not accelerating in this direction. Unsurprisingly, the equation gives the relation between the normal force and gravity. On the other hand, Eq. (3.73) should be inspected thoroughly. The first term is dropped because the momentum is not accumulating within the CV. Since the CV is moving with the cart, the relative velocity of the fluid and the CV is the velocity of the fluid/cart, thus

[2] The closed surface integral of a constant is zero.

$$\oint\!\!\oint_A \rho u\left(\vec{q}\cdot d\vec{A}\right)=\iint_{A_e}\rho\left(-V_e\hat{x}\right)\left[V_e\left(-\hat{x}\right)\cdot dA_e\left(-\hat{x}\right)\right]=-\rho V_e^2 A_e \ , \tag{3.75}$$

where $V_e = Q / A_e$. The third term is simply

$$\iiint_V \rho a_x dV = a_x \iiint_V \rho dV = M(t) a_x \ . \tag{3.76}$$

Inserting Eqs. (3.74)-(3.76) into Eq. (3.73) gives

$$-\rho A_e V_e^2 + M(t)\dot{u}(t) = -\mu M(t) g \ . \tag{3.77}$$

The critical flux occurs when the acceleration if zero. Inserting $V_e = Q / A_e$ and Eq. (3.72) gives

$$\rho Q^2 / A_e \mu = \mu\left(M_0 + \rho h_0 A - \rho Q \not{t}\right) \Rightarrow Q = \sqrt{\frac{\mu g A_e}{\rho}\left(M_0 + \rho h_0 A\right)} \ . \tag{3.78}$$

Eq. (3.77) is rewritten (remember that when $Q = 0$, there no longer is contributing accelerating force but friction still exists)

$$\dot{u}(t) = -\mu g + \frac{\rho A_e}{M(t)} V_e^2 = -\mu g + \frac{\rho Q^2}{A_e M(t)} = \begin{cases} -\mu g + \dfrac{\rho Q^2}{A_e\left(M_0 + \rho h_0 A - \rho Q t\right)}, & t < t_f \\ -\mu g, & t \geq t_f \end{cases} \ . \tag{3.79}$$

Integration gives the velocity of the cart

$$u(t) = \int \dot{u}(t) dt = \begin{cases} \dfrac{Q}{A_e}\ln\left(\dfrac{M_0 + \rho h_0 A}{M_0 + \rho h_0 A - \rho Q t}\right), & t < t_f \\ u\left(t = t_f\right) - \mu g t, & t \geq t_f \end{cases} \ . \tag{3.80}$$

The displacement is found by an additional (implicit) integration

$$x(t) = \int u(t) dt \ . \tag{3.81}$$

We shall now solve the problem in an inertial reference frame. Thus, we only need to modify Eq. (3.73). We note the following

1. The velocity of the CV is $u(t)$.
2. The absolute velocity of the fluid at the exit aperture is $u(t) - V_e$.
3. The relative velocity is now $q = u(t) - V_e - u(t) = -V_e$.

Obviously, Eqs. (3.70)-(3.72) and Eq. (3.74) remain unchanged. Thus the first term in the momentum equation is

$$\partial_t \iiint_V \rho u_x dV = \partial_t \iiint_V \rho u(t) dV = \partial_t\left[M(t)u(t)\right] = \dot{M}(t)u(t) + M(t)\dot{u}(t) \ . \tag{3.82}$$

The flux term is

$$\oint\!\!\oint_A \rho u\left(\vec{q}\cdot d\vec{A}\right) = \iint_{A_e}\rho\left[\left(-V_e + u(t)\right)\hat{x}\right]\left[-V_e\hat{x}\cdot dA_e\left(-\hat{x}\right)\right] = -\rho V_e^2 A_e + \rho V_e A_e u(t) \ . \tag{3.83}$$

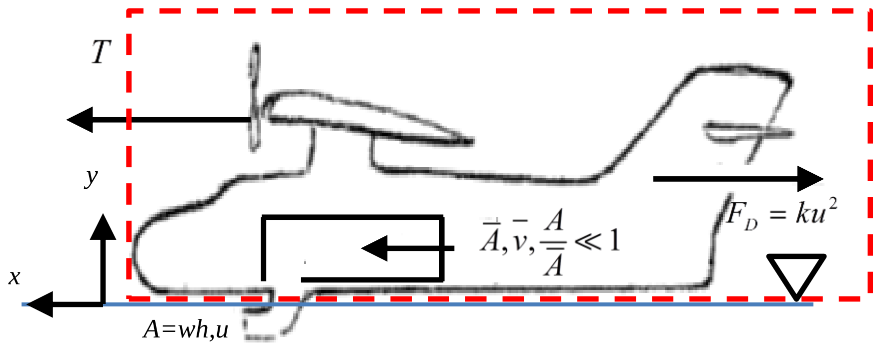
We can thus write

$$\dot{M}(t)u(t)+M(t)\dot{u}(t)-\rho V_e^2 A_e+\rho V_e A_e u(t)=-\mu M(t)g \tag{3.84}$$

However we now remember that $Q=V_e A_e$ and insert the relation for $\dot{M}(t)$ from Eq. (3.72) and we see that the first and fourth terms cancel out. Thus the governing EOM, unsurprisingly, remains unchanged.

Question 2

A fire-fighting plane scoops water from the surface of a lake to fill an internal tank. The mass of the (empty) plane is M_0, while the volume of the tank is V. The initial velocity of the plane prior to scooping is u_0. The area of the entrance aperture is A. The depth of penetration of the aperture into the water is negligible. A drag force $F_D=ku^2$ is applied on the entire plane. Find:



a. The time required to fill the tank while the plane is flying at the initial velocity.
b. The additional thrust needed to fly at the initial velocity.

Assuming that the engines shut down when the plane makes contact with the lake, calculate

c. The rate of change of the mass of the plane dM/dt as a function of the instantaneous velocity $u(t)$
d. The de/acceleration of the plane as a function of the mass, $M(t)$, and velocity, $u(t)$.
e. Once the tank has completely filled, how will the acceleration of the plane change?
f. Solve problems d. and e. by using the chain rule $\frac{du}{dt}=\frac{du}{dM}\frac{dM}{dt}$.

Solution

Control volume: The entire plane.

Assumptions:

1. Atmospheric pressure.
2. Incompressible fluid.

Conservation of mass gives

$$\rho\partial_t V-\rho u_0 A=0 \ . \tag{3.85}$$

Thus the time for the tank to be complete filled is

$$\int_0^V dV=\int_0^{t_{full}} u_0 A dt \Rightarrow t_{full}=\frac{V}{u_0 A} \ . \tag{3.86}$$

Prior to water scooping, the thrust and drag were at equilibrium, so that $T_0=ku_0^2$. Since the plane remains at the same velocity, the drag remains unchanged. However, the thrust can now be divided into two $T=T_0+\Delta T$, where the second term is the required thrust required to overcome the force of the incoming flux. One comment should be made regarding the momentum equation in the $\hat{y}$ direction. The is no incoming flux in this direction which means we need only to look at the accumulation term. Due to the fact that the entrance aperture area is small compared to the area of the tank itself, the velocity within the tank will also be small. Thus, we will neglect this term, and we are left we gravity is balanced by the lift. This accumulation term can also be neglected in the $\hat{x}$ in this direction. The momentum equation in the $\hat{x}$ is

$$\iint_A \rho\left[u_0\left(-\hat{x}\right)\right]\left[u_0\left(-\hat{x}\right)\cdot dA\hat{x}\right]=T_0+\Delta T-ku_0^{\ 2}\Rightarrow \Delta T=\rho u_0^{\ 2}A \quad . \tag{3.87}$$

The engines are now shut off, so that all the equations need to be slightly modified. It is readily seen that the mass rate of the plane is

$$\dot{M}\left(t\right)=\rho Au\left(t\right) \quad . \tag{3.88}$$

We know use the momentum equations that account for the fact that we are no longer in an IF

$$\iint_A \rho\left[u\left(t\right)\left(-\hat{x}\right)\right]\left[u\left(t\right)\left(-\hat{x}\right)\cdot dA\hat{x}\right]+\iiint_V \rho a_x dV=-ku\left(t\right)^2 \quad . \tag{3.89}$$

Integration of space independent values yield simple results

$$\rho u\left(t\right)^2 A+M\left(t\right)a_x\left(t\right)=-ku\left(t\right)^2\Rightarrow a_x=\frac{du}{dt}=-\frac{\rho A+k}{M\left(t\right)}u\left(t\right)^2=-\frac{\alpha}{M\left(t\right)}u\left(t\right)^2 \quad . \tag{3.90}$$

It can be seen that once the tank has been filled, $M\left(t\right)=M_0+\rho V$ $M\left(t\right)=M_0+\rho V$, so the above equation is simplified, however the existence of the drag force will result in the continued deceleration of the plane.

Inserting the chain rule gives

$$\frac{du}{dt}=\frac{du}{dM}\dot{M}\left(t\right)=-\frac{\alpha}{M\left(t\right)}u\left(t\right)^2 \quad . \tag{3.91}$$

Inserting Eq. (3.88)

$$\frac{du}{dM}\rho Au\left(t\right)=-\frac{\alpha}{M\left(t\right)}u\left(t\right)^2\Rightarrow\frac{du}{u\left(t\right)}=-\frac{\alpha}{\rho A}\frac{dM}{M\left(t\right)}=-\gamma\frac{dM}{M\left(t\right)} \quad . \tag{3.92}$$

Integration gives

$$\ln\left[\frac{u\left(t\right)}{u_0}\right]=-\gamma\ln\left[\frac{M\left(t\right)}{M_0}\right]\Rightarrow u\left(t\right)=u_0\left[\frac{M_0}{M\left(t\right)}\right]^{\gamma} \quad . \tag{3.93}$$

This can now be inserted back into Eq. (3.88)

$$\dot{M}\left(t\right)=\rho Au\left(t\right)=\rho Au_0M_0^{\ \gamma}M\left(t\right)^{-\gamma}=\beta M\left(t\right)^{-\gamma}\Rightarrow M\left(t\right)^{\gamma}dM=\beta dt\Rightarrow M\left(t\right)=\left(M_0^{\ \gamma+1}+\beta\left(\gamma+1\right)t\right)^{1/(\gamma+1)} \quad . \tag{3.94}$$

Inserting a number of typical values for these planes gives interesting results. See if you can discover them all.

$$M_0=8{,}000\left[kg\right],u_0=60\left[m/s\right],A=0.01\left[m^2\right],k=1.5\left[\frac{Ns^2}{m^2}\right],\rho=1{,}000\left[\frac{kg}{m^3}\right] \quad .$$

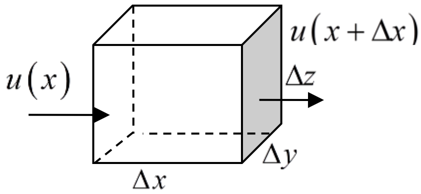
4. Kinematics (forthcoming/next semester)

“Truth and clarity are complementary.” Neils Bohr

5. Navier-Stokes Equations

In chapter 3 we started off by discussing the meaning of the integral equations of motion. Why? Because conceptually they are easier to understand and provide most of the physical intuition required to deal with the mathematically more difficult differential equations that describe the local changes. In the following chapter we shall discuss how these differential equations determine the dynamics of the system and what are the required BCs.

Introduction to the governing differential equations



Divergence theorem/ Gauss's theorem/Ostrogradsky's theorem
In the previous section we showed that the integral equation governing conservation of mass was dependent on the flux exiting and entering a surface. This was proportional to the integral over the surface $\rho\vec{u}\cdot d\vec{A}$

$$\partial_t \iiint_V \rho dV = -\oiint_S \rho\vec{u}\cdot d\vec{A} \qquad (5.1)$$

Let us now look upon a differential cube as shown in the figure. The fluxes in all three directions are given by

$$\begin{cases} \hat{x}:\left[\rho(x)u(x)-\rho(x+\Delta x)u(x+\Delta x)\right]\Delta y\Delta z \\ \hat{y}:\left[\rho(y)v(y)-\rho(y+\Delta y)v(y+\Delta y)\right]\Delta x\Delta z \\ \hat{z}:\left[\rho(z)w(z)-\rho(z+\Delta z)w(z+\Delta z)\right]\Delta x\Delta y \end{cases} , \qquad (5.2)$$

where for the sake of brevity instead of writing all spatial coordinates for the density and velocity $\rho(x+\Delta x,y,z,t)$ only the relevant coordinate has been specified $\rho(x+\Delta x)$. Each of these terms can be Taylor expanded to yield

$$\begin{cases} \hat{x}:-\partial_x(\rho u)\Delta x\Delta y\Delta z \\ \hat{y}:-\partial_y(\rho v)\Delta x\Delta y\Delta z \\ \hat{z}:-\partial_z(\rho w)\Delta x\Delta y\Delta z \end{cases} . \qquad (5.3)$$

Summation of all these terms yields

$$-\vec{\nabla}\cdot(\rho\vec{u})dV \quad , \qquad (5.4)$$

since this relation hold locally at every point it also holds globally, thus integration over the volume gives

$$-\iiint_V \vec{\nabla}\cdot(\rho\vec{u})dV = -\oiint_S \rho\vec{u}\cdot d\vec{A} \ . \qquad (5.5)$$

This is the proof Gauss’s theorem. Inserting Eq. (5.5) into Eq. (5.1) yields

$$\iiint_V \left[\partial_t\rho+\vec{\nabla}\cdot(\rho\vec{u})\right]dV = 0 \ . \qquad (5.6)$$

Since the integral is zero for any volume, then of course, the integrand must be zero indefinitely

$$\partial_t\rho+\vec{\nabla}\cdot(\rho\vec{u}) = 0 \ . \qquad (5.7)$$

Eq. (5.7) is differential mass conservation equation or mass continuity equation. In the professional jargon it is often termed the continuity equation. This equation can be in many branches of physics, including heat transfer, electric currents and even quantum mechanics. This equation is very generic. This equation can be rewritten in number of ways

$$\partial_t \rho + \partial_x \rho u + \partial_y \rho v + \partial_z \rho w = \partial_t \rho + u\partial_x \rho + v\partial_y \rho + w\partial_z \rho + \rho\partial_x u + \rho\partial_y v + \rho\partial_z w = \frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\vec{u} = 0 \ . \tag{5.8}$$

The definition of an incompressible fluid is when $D\rho / Dt = 0$. Thus the continuity equation reduces to

$$\vec{\nabla}\cdot\vec{u} = 0 \ . \tag{5.9}$$

Navier-Stokes equations

Now, instead of looking at the mass density, we can look at the momentum density, such that $\rho \to \rho\vec{u}$, of course the resulting equation is a vector equation

$$\rho\frac{D\vec{u}}{Dt} = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} + \vec{f} \ . \tag{5.10}$$

Eq. (5.10) can be written in a more explicit manner

$$\rho\left(\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right) = -\vec{\nabla}p + \mu\vec{\nabla}^2\vec{u} + \vec{f} \ . \tag{5.11}$$

The second term in the brackets is non-linear. It should also remind you of the non-linear term in Eq. (3.30). See our discussion there as to the physical meaning of this term. Eq. (5.11) is now written explicitly in a Cartesian coordinate system (with gravity in the $\hat{z}$ direction)

$$\begin{aligned}
\hat{x} &: \rho\left(\partial_t u + u\partial_x u + v\partial_y u + w\partial_z u\right) = -\partial_x p + \mu\left(\partial_{xx}u + \partial_{yy}u + \partial_{zz}u\right) + f_x \\
\hat{y} &: \rho\left(\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v\right) = -\partial_y p + \mu\left(\partial_{xx}v + \partial_{yy}v + \partial_{zz}v\right) + f_y \\
\hat{z} &: \rho\left(\partial_t w + u\partial_x w + v\partial_y w + w\partial_z w\right) = -\partial_z p + \mu\left(\partial_{xx}w + \partial_{yy}w + \partial_{zz}w\right) - \rho g + f_z
\end{aligned} \ . \tag{5.12}$$

The LHS represents the change of momentum (including the convective terms). We see that the convection in perpendicular directions to the direction of interest can lead to loss of momentum in that direction. The change in momentum is balanced out by additional forces – pressure, viscosity, gravity and any other force we would like to add. Both pressure and viscous drag have been converted from surface forces to body forces. In fact all these forces are now body forces.

For the special case, $\mu = 0$, the Navier Stokes (NS) equation reduces to the Euler Equation (which we will discuss in the next chapter).

Boundary Conditions

To solve these (continuity and NS) equations, we need to supply appropriate BCs. There are three kinds of BCs: 1)Dirichlet BC where the value of the field is known at the boundary. 2)Neumann BC where the derivatives of the field are known at the boundary. 3) A combination of these two are known as Mixed/Robin type BC.

The most commonly used BC is when a fluid is moving adjacent to a wall. Obviously, unless the wall is porous, flow cannot penetrate it and the fluid at the wall must move with the velocity of the wall $\vec{u}_{fluid} = \vec{u}_{wall}$, explicitly

$$\vec{u}_{fluid,\perp} = 0, \vec{u}_{fluid,\|} = \vec{u}_{wall,\|} \ . \tag{5.13}$$

When the wall is at rest the second condition is called the no-slip condition, otherwise it is known as the slip condition. Naturally we can extend this discussion to two layers of fluid. At the interface of two immiscible (non-mixing fluids) we require $\vec{u}_A = \vec{u}_B$

$$\vec{u}_A\cdot\hat{n} = \vec{u}_B\cdot\hat{n} = 0, \vec{u}_A\cdot\hat{t} = \vec{u}_B\cdot\hat{t} \ . \tag{5.14}$$

Additionally for mechanical equilibrium, we require that stresses in each fluid are equal $\vec{\tau}_A = \vec{\tau}_B$, explicitly

$$\vec{\tau}_A \cdot \hat{t} = \vec{\tau}_B \cdot \hat{t} \quad : \tau_{xy,A} = \tau_{xy,B} \ , \tag{5.15}$$

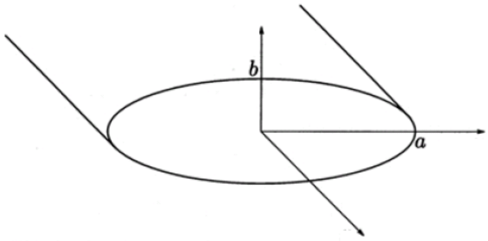
$$\vec{\tau}_A \cdot \hat{n} = \vec{\tau}_B \cdot \hat{n} \quad : p_A + \sigma_{nn,A} = p_B + \sigma_{nn,B} \ , \tag{5.16}$$

where the former is an example for the 2D case and the latter is demonstration of the Robin type BC when capillary forces are negligible.

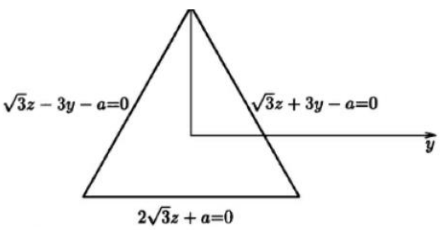
Question 1

Using only the no-slip condition find the velocity field in a long channel

a. in between two parallel plates with a distance $2H$ in between them.
b. with a circular cross section.
c. with an elliptical cross section.
d. Use the solution found in (c) and show that when
   i. $a = b = R$ the solution reduces to solution of (b)
   ii. $a = H, b \gg H$ the solution reduces to solution of (a)
e. Similarly derive a solution for channel with a equilateral triangular cross section.



Elliptical cross section



Solution

Assumptions

1. Steady state $\partial_t = 0$.
2. Incompressible fluid, $\rho = const$.
3. 1D flow $(v = w = 0)$.
4. Developed flow $\partial_x \vec{u} = \vec{0}$

The continuity equation is vastly reduced under these assumptions

$$\underbrace{\partial_t \rho}_{1} + \partial_x \rho u + \underbrace{\partial_y \rho v}_{3} + \underbrace{\partial_z \rho w}_{3} = 0 \Rightarrow \partial_x u = 0 \Rightarrow u = u(y,z) \ . \tag{5.17}$$

Before proceeding to the momentum equations, we note the following, the velocity field is solely in the $\hat{x}$ direction, however Eq. (5.17) shows that this velocity is dependent solely on the $\hat{y}$ and the $\hat{z}$ coordinates. It can thus be expected that the term $(\vec{u} \cdot \vec{\nabla})\vec{u} = 0$. This will reduce the complexity of the NS equation as the non-linear term cancels and the governing equation is now ***linear***. The equation in the $\hat{x}$ direction is

$$\rho\left(\underbrace{\partial_t u}_{1} + \underbrace{u\partial_x u}_{above} + \underbrace{v\partial_y u}_{3} + \underbrace{w\partial_z u}_{3}\right) = -\partial_x p + \mu\left(\underbrace{\partial_{xx} u}_{above} + \partial_{yy} u + \partial_{zz} u\right) \ . \tag{5.18}$$

As expected, the term $(\vec{u} \cdot \vec{\nabla})\vec{u}$ indeed cancels out. The equation governing the dynamics is

$$-\partial_x p + \mu\left(\partial_{yy} u + \partial_{zz} u\right) = 0 \ . \tag{5.19}$$

For case (a) where the channels are parallel, we can additionally assume that $\partial_z = 0$, thus

$$\partial_{yy} u = \frac{1}{\mu} \partial_x p \quad . \tag{5.20}$$

The LHS is a function of y while the RHS is a function of x , thus both sides must be equal to a constant. From straightforward integration we find the velocity

$$u(y) = \frac{\partial_x p}{2\mu} y^2 + cy + d \quad . \tag{5.21}$$

We find the constants of integration by using the BCs $u(y = \pm H) = 0$

$$u = \frac{\partial_x p}{2\mu}\left(y^2 - H^2\right). \tag{5.22}$$

However, there exists an alternate way to solve this problem which will prove to be useful to solve for the more complicated geometries. The geometry of the two parallel plates can be described by the relation $y^2 = H^2$. Additionally, we know that the velocity on these surfaces is zero. If one were to guess that the solution has the form

$$u = \alpha\left(y^2 - H^2\right) \quad , \tag{5.23}$$

then the requirements that velocity is zero at the walls is automatically met and all we need to do is find α . Substituting Eq. (5.23) into Eq. (5.20) gives that

$$\alpha = \frac{\partial_x p}{2\mu} \quad , \tag{5.24}$$

which is exactly the solution given in Eq. (5.22). This velocity field is called Poiseuille. Let us now return to the circular cylinder. We know that $z^2 + y^2 = R^2$ so that guessing that the velocity has the following form

$$u = \alpha\left(z^2 + y^2 - R^2\right) \quad . \tag{5.25}$$

makes sense. Substitution into Eq. (5.19) gives

$$\alpha = \frac{\partial_x p}{4\mu} \quad , \tag{5.26}$$

$$u = \frac{\partial_x p}{4\mu}\left(z^2 + y^2 - R^2\right) = \frac{\partial_x p}{4\mu}\left(r^2 - R^2\right) \quad . \tag{5.27}$$

This field is also known as Hagen-Poiseuille flow. An ellipse is given by the curve $\frac{z^2}{b^2} + \frac{y^2}{a^2} = 1$ $\frac{z^2}{b^2} + \frac{y^2}{a^2} = 1$. Thus we guess the velocity has the form

$$u = \alpha\left(\frac{z^2}{b^2} + \frac{y^2}{a^2} - 1\right) \quad . \tag{5.28}$$

The final solution is

$$\alpha = \frac{\partial_x p}{2\mu} \frac{a^2 b^2}{a^2 + b^2} \quad , \tag{5.29}$$

$$u = \frac{\partial_x p}{2\mu}\frac{a^2b^2}{a^2+b^2}\left(\frac{z^2}{b^2}+\frac{y^2}{a^2}-1\right). \tag{5.30}$$

Substitution of $a = b = R$ gives

$$u = \frac{\partial_x p}{2\mu}\frac{R^2R^2}{R^2+R^2}\left(\frac{z^2}{R^2}+\frac{y^2}{R^2}-1\right) = \frac{\partial_x p}{4\mu}\left(z^2+y^2-R^2\right) \ , \tag{5.31}$$

which is the solution given by Eq. (5.27). When $a = H$ and $b \gg H$ it is fair to assume that except at the edges where $z \sim b$ then $z \ll b$ so that

$$u = \frac{\partial_x p}{2\mu}\frac{H^2b^2}{H^2+b^2}\left(\cancel{\frac{z^2}{b^2}}+\frac{y^2}{H^2}-1\right) = \frac{\partial_x p}{2\mu}\left(y^2-H^2\right) \ , \tag{5.32}$$

which is the solution given in Eq. (5.22).

Let us define the axis to sit at the center of the triangle. We now describe each of the curves. The bottom most curve is a located at a distance of a third of the triangle height $\sqrt{3}a/2$ $\sqrt{3}a/2$

$$2\sqrt{3}z + a = 0 \ . \tag{5.33}$$

The two remaining curves are found in a similar manner. We know that the slope of the curves are $m = \pm\tan(\pi/3)$ while they pass through the point $\left(y = 0, z = \frac{2a}{3}\sin\frac{\pi}{3} = \frac{2a}{3}\frac{\sqrt{3}}{2} = \frac{a}{\sqrt{3}}\right)$. These curves are given by

$$\sqrt{3}z \pm 3y - a = 0 \ . \tag{5.34}$$

Multiplication of all three curves gives us the natural guess for the flow field

$$u = \alpha\left(2\sqrt{3}z+a\right)\left(\sqrt{3}z+3y-a\right)\left(\sqrt{3}z-3y-a\right) = \alpha\left(6\sqrt{3}z^3-9az^2-18\sqrt{3}y^2z-9ay^2+a^3\right) \ . \tag{5.35}$$

To find α we need to take the second derivatives and we have

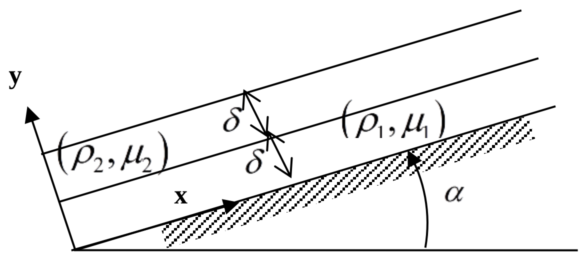
$$\partial_{zz}u = \alpha\left(-18a+36\sqrt{3}z\right), \partial_{yy}u = \partial_{zz}u = \alpha\left(-18a-36\sqrt{3}z\right) \ . \tag{5.36}$$

Substitution of this into Eq. (5.19) will yield the final solution

$$\alpha = -\frac{\partial_x p}{36a\mu} . \tag{5.37}$$

Question 2

Find the velocity distribution of two fluids with densities and viscosities (ρ_1, μ_1) and (ρ_2, μ_2) placed upon a sloped surface with angle α . The thickness of each layer of fluid is δ .



Solution

Assumptions

1. Steady state $\partial_t = 0$.
2. Incompressible fluid, $\rho = const$.
3. 2D flow $(w = 0, \partial_z = 0)$.

4. Developed flow $\partial_x \vec{u} = \vec{0}$.
5. The pressure is atmospheric and doesn't change in the horizontal direction $\hat{x}$.

The continuity equation gives

$$\underset{1}{\partial_t \rho} + \underset{2,4}{\partial_x \rho u} + \partial_y \rho v + \underset{2,3}{\partial_z \rho w} = 0 \Rightarrow \partial_y v = 0 \Rightarrow v = const \quad , \tag{5.38}$$

yet we know that the velocity perpendicular to the surface is $v(y=0)=0$ so that the constant is zero everywhere. This result will be marked as assumption (6). The momentum equations are

$$\rho\left(\underset{1}{\partial_t u} + \underset{4}{u\partial_x u} + \underset{6}{v\partial_y u} + \underset{3}{w\partial_z u}\right) = -\underset{5}{\partial_x p} + \mu\left(\underset{4}{\partial_{xx} u} + \partial_{yy} u + \underset{3}{\partial_{zz} u}\right) - \rho g \sin\alpha \quad , \tag{5.39}$$

$$\rho\underbrace{\left(\partial_t v + u\partial_x v + v\partial_y v + w\partial_z v\right)}_{6} = -\partial_y p + \mu\underbrace{\left(\partial_{xx} v + \partial_{yy} v + \partial_{zz} v\right)}_{6} - \rho g \cos\alpha \quad . \tag{5.40}$$

What remains of Eq. (5.40) expresses hydrostatic equilibrium in the vertical direction. While Eq. (5.39) governs the dynamics of the two fluids in the horizontal direction

$$\partial_{yy} u_i - \beta_i = 0 \quad , \tag{5.41}$$

with $\beta_i = \rho_i g \sin\alpha / \mu_i$ for $i = 1,2$. Direct integration gives

$$u_i = \frac{\beta_i}{2} y^2 + c_i y + d_i \quad . \tag{5.42}$$

To find the constant of integration we now define the appropriate BCs. No slip at the surface $u_1(y=0)=0$

$$d_1 = 0 \quad . \tag{5.43}$$

Slip velocity between the two layers requires that $u_1(\delta) = u_2(\delta)$

$$\frac{\beta_1}{2}\delta^2 + c_1\delta = \frac{\beta_2}{2}\delta^2 + c_2\delta + d_2 \, . \tag{5.44}$$

Mechanical equilibrium requires that the shear stresses be equal, $\tau_{xy}^{(1)}(\delta) = \tau_{xy}^{(2)}(\delta)$,

$$\mu_1 \partial_y u_1(\delta) = \mu_2 \partial_y u_2(\delta) \quad : \mu_1(\beta_1\delta + c_1) = \mu_2(\beta_2\delta + c_2) \, . \tag{5.45}$$

And finally, the top of the second layer is a free surface so that it is shear-less, $\tau_{xy}^{(2)}(y = 2\delta) = 0$,

$$2\beta_2\delta + c_2 = 0 \quad . \tag{5.46}$$

Solutions of Eqs. (5.43)-(5.46) yields

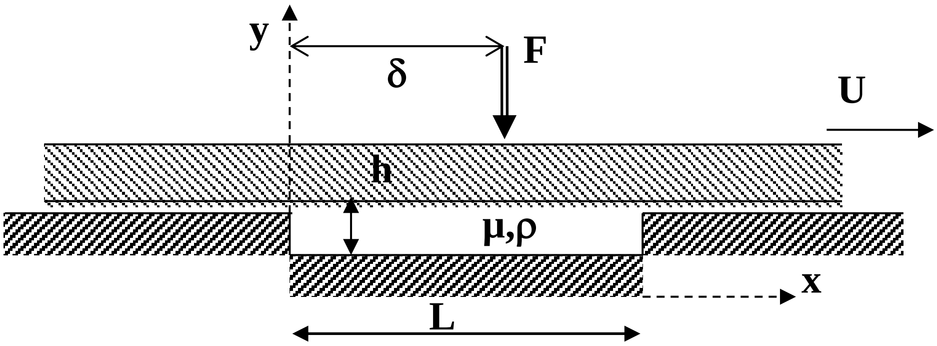
$$c_1 = -\delta\left(\beta_1 + \beta_2\frac{\mu_2}{\mu_1}\right), c_2 = -2\beta_2\delta, d_1 = 0, d_2 = \frac{\delta^2}{2}\left(3\beta_2 - \beta_1 - \beta_2\frac{\mu_2}{\mu_1}\right) \quad . \tag{5.47}$$

So that we can now write the flow field for both layers

$$\left\{u_1 = \frac{\beta_1}{2}y^2 - \delta\left(\beta_1 + \beta_2\frac{\mu_2}{\mu_1}\right)y, v_1 = 0\right\}, \left\{u_2 = \frac{\beta_1}{2}y^2 - 2\beta_2\delta y + \frac{\delta^2}{2}\left(3\beta_2 - \beta_1 - \beta_2\frac{\mu_2}{\mu_1}\right), v_2 = 0\right\} . \tag{5.48}$$

Question 3

Consider an infinitely long surface moving at constant velocity U relative to a fixed surface beneath it. The lower surface has a long slit of length L and height . An incompressible fluid of constant density ρ and viscosity μ resides within this volume. Due to the movement of this inner plate, an internal pressure field is created.

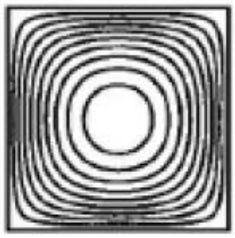


To ensure that the fluid cannot escape from in between the plates , an external force F is applied on the upper plate. Assuming that the slit is very long $\left(L \gg h\right)$, it is possible to ignore edge effects. Find:

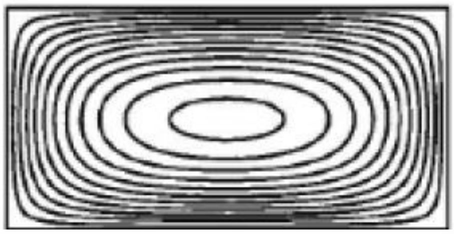
a. The velocity field within the slit.
b. The flow rate for a vertical section.
c. The pressure distribution within the fluid
d. Find the force F as a function of the fluid velocity and geometry.
e. Where force F should be applied.

<u>Solution</u>

For a square geometry we would expect the solution to look like



While for a rectangular geometry we expect this solution to be streched.



For a rectanglar geonetry where $L \gg h$, one can expect that this stretching will be further enhanced. Ignoring the edges, it appears that the center of the slit has an almost 1D behavior.

Assumptions:

1. Steady state $\partial_t = 0$.
2. Incompressible fluid, $\rho = const$.
3. 2D flow $\left(w = 0, \partial_z = 0\right)$.
4. Developed flow $\partial_x \vec{u} = 0$.
5. No additional body force terms.

The continuity equation gives (result will be marked as assumption 6)

$$\underbrace{\partial_x u}_{4} + \partial_y v + \underbrace{\partial_z w}_{3} = 0 \Rightarrow v\left(y\right) = const \quad , \tag{5.49}$$

yet we know that the velocity perpendicular to the surface, $v\left(y=0\right)=0$ thus the constant is zero everywhere. The momentum equation in the horizontal direction is (the vertical equation is trivial – why?)

$$\rho\left(\underset{1}{\partial_t u}+\underset{4}{u\partial_x u}+\underset{6}{v\partial_y u}+\underset{3}{w\partial_z u}\right)=-\partial_x p+\mu\left(\underset{4}{\partial_{xx}u}+\partial_{yy}u+\underset{3}{\partial_{zz}u}\right)\Rightarrow\partial_{yy}u=\frac{\partial_x p}{\mu} \quad . \tag{5.50}$$

Direct integration gives

$$u\left(y\right)=\frac{\partial_x p}{2\mu}y^2+cy+d \quad . \tag{5.51}$$

Using the BCs $u\left(y=0\right)=0,u\left(y=h\right)=U$, gives $d=0,c=\frac{U}{h}-\frac{h}{2\mu}\partial_x p$. The flow rate is given by

$$Q=\int_0^h u\left(y\right)dy=-\frac{h^3\partial_x p}{12\mu}+\frac{hU}{2} \quad . \tag{5.52}$$

However the total flux at any section must be zero so that

$$\partial_x p=\frac{6\mu U}{h^2}\Rightarrow p\left(x\right)=\frac{6\mu U}{h^2}+P_0 \quad . \tag{5.53}$$

The overall force applied by the fluid on the upper plate is

$$F=\int_0^L p\left(x\right)dx=\frac{3\mu UL^2}{h^2}+P_0 L \quad . \tag{5.54}$$

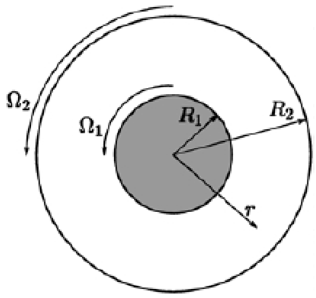
We calculate the moment about the origin to find the total moment applied by this force and the location of the counterforce F

$$\delta=\frac{\int_0^L p\left(x\right)dx}{F}=\frac{L\left(h^2P_0+4LU\mu\right)}{2h^2P_0+6LU\mu} \quad . \tag{5.55}$$

Question 4

Calculate the steady state velocity profile of two rotating concentered cylinders. Consider the extreme cases of

a. $R_1=0$.
b. Ω_1 .
c. $R_2\rightarrow\infty$.



Solution

Assumptions:

1. Steady state $\partial_t=0$.
2. Incompressible fluid, $\rho=const$.
3. 2D flow $\left(w=v_z=0,\partial_z=0\right)$.
4. Axisymmetric flow $\partial_\theta=0$.
5. No additional body force terms and pressure build up.

The continuity equation gives (result will be marked as assumption 6)

$$\vec{\nabla}\cdot\vec{u}=\frac{1}{r}\partial_r v_r r+\underbrace{\frac{1}{r}\partial_\theta v_\theta}_{4}+\underbrace{\partial_z v_z}_{3}=\frac{1}{r}\partial_r v_r r=0\Rightarrow v(r)=\frac{c}{r}\quad , \tag{5.56}$$

yet we know that the velocity perpendicular to the surface, $v_r\left(r=R_{1,2}\right)=0$ thus the constant is zero everywhere. The momentum equation in the tangential direction (the remaining equations are trivial)

$$\rho\left(\underbrace{\partial_t v_\theta}_{1}+\underbrace{v_r\partial_r v_\theta}_{6}+\underbrace{\frac{v_\theta}{r}\partial_\theta v_\theta}_{4}+\underbrace{v_z\partial_z v_\theta}_{3}+\underbrace{\frac{v_r v_\theta}{r}}_{6}\right)=\mu\left[\frac{1}{r}\partial_r\left(\frac{\partial_r r v_\theta}{r}\right)+\underbrace{\frac{1}{r^2}\partial_{\theta\theta}v_\theta}_{4}+\underbrace{\partial_{zz}v_\theta}_{3}+\underbrace{\frac{2}{r^2}\partial_\theta v_r}_{4,6}\right], \tag{5.57}$$

leaving us with (for more details on the solution process see Ordinary Differential Equations-Question 2)

$$\frac{1}{r}\partial_r\left(\frac{\partial_r r v_\theta}{r}\right)=0\Rightarrow v_\theta=c_1 r+\frac{c_2}{r}\quad , \tag{5.58}$$

The BCs are $v_\theta\left(r=R_1\right)=\Omega_1 R_1, v_\theta\left(r=R_2\right)=\Omega_2 R_2$. Yielding

$$c_1=\frac{R_1^{\,2}\Omega_1-R_2^{\,2}\Omega_2}{R_1^{\,2}-R_2^{\,2}}, c_2=\frac{R_1^{\,2}R_2^{\,2}\left(\Omega_2-\Omega_1\right)}{R_1^{\,2}-R_2^{\,2}}\quad , \tag{5.59}$$

When $R_1=0$

$$c_1=\Omega_2, c_2=0\Rightarrow v_\theta=\Omega_2 r\quad . \tag{5.60}$$

When $\Omega_1=0$

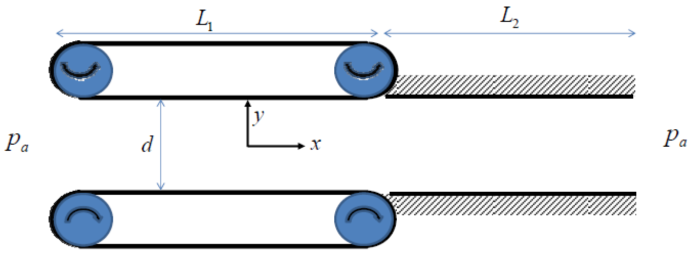
$$c_1=-\frac{R_2^{\,2}\Omega_2}{R_1^{\,2}-R_2^{\,2}}, c_2=\frac{R_1^{\,2}R_2^{\,2}\Omega_2}{R_1^{\,2}-R_2^{\,2}} \tag{5.61}$$

When $R_2\to\infty$, to ensure that the solution doesn't diverge the constant $c_1=0$. This equivalent to requiring that the rotation velocity $\Omega_2=0$

$$c_1=\frac{R_1^{\,2}\Omega_1-R_2^{\,2}\Omega_2}{R_1^{\,2}-R_2^{\,2}}\approx\frac{R_2^{\,2}\Omega_2}{R_2^{\,2}}\approx\Omega_2=0, c_2=\frac{R_1^{\,2}R_2^{\,2}\left(\Omega_2-\Omega_1\right)}{R_1^{\,2}-R_2^{\,2}}=\Omega_1 R_1^{\,2}\Rightarrow v_\theta=\frac{\Omega_1 R_1^{\,2}}{r}\,. \tag{5.62}$$

Question 5

Four identical wheels with radius R rotating with a rotational velocity ω cause two (top and bottom) long belts to circulate. Due to this circulation, fluid is sucked into between the plates. Assume that edge effects are negligible ($L_{1,2}\gg d$).



a. What is the velocity distribution between the plates.
b. Two stationary plates are added at the exit of the conveyer belt section. What is the velocity distribution for each section?
c. A plug is inserted at the exit to the stationary plates. What is the new steady state velocity distribution?

<u>Solution</u>

From Question 1-3 we have seen that in the case of steady state, fully developed 2D flow the governing equation is

$$\partial_{yy}u = \frac{\partial_x p}{\mu} . \tag{5.63}$$

However, we know that at both the entrance and exit the pressure is atmospheric thus the pressure gradient is zero. This leads to a linear velocity profile

$$u = cy + e \ . \tag{5.64}$$

Inserting the BCs $u(y = \pm d/2) = \omega R$ yields a constant velocity profile

$$u = \omega R \ . \tag{5.65}$$

Upon addition of the two stationary surfaces and inspection we find that the governing equation given by (5.63) remains unchanged and is correct in each region separately. The BCs in the second region are $u(y = \pm d/2) = 0$. Inspection of the previous assumption that the pressure gradient is zero no longer holds. While the pressure at the center is not known, due to the assumption of fully developed flow, we can assume that the pressure gradient in each region is constant $\partial_x p_i$. Using the BC for each region we find the velocity distributions

$$u_1 = \frac{\partial_x p_1}{2\mu}\left(y^2 - \frac{d^2}{4}\right) + R\omega \ , \tag{5.66}$$

$$u_2 = \frac{\partial_x p_2}{2\mu}\left(y^2 - \frac{d^2}{4}\right) \ , \tag{5.67}$$

We can now write the pressure gradients explicitly

$$\partial_x p_1 = \frac{p_{12} - p_a}{L_1}, \partial_x p_2 = \frac{p_a - p_{12}}{L_2} \tag{5.68}$$

where p_{12} is the pressure at the intermediate point of the two regions. To find this last constant which is unknown we require that the total flux in each section is equal

$$\int_{-d/2}^{d/2} u_1 dy = \int_{-d/2}^{d/2} u_2 dy \tag{5.69}$$

Inserting the velocity profiles Eqs. (5.66)-(5.67) and the relations in Eq. (5.68) into Eq. (5.69) gives

$$p_{12} = p_a + \frac{12R\omega\mu}{d^2}\frac{L_1 L_2}{L_1 + L_2} \tag{5.70}$$

When the plug is inserted a number of things occur:

I. The net flow is zero.
II. The velocity in region 2 must be zero and the pressure there is constant.
III. Pressure is buildup at p_{12} .

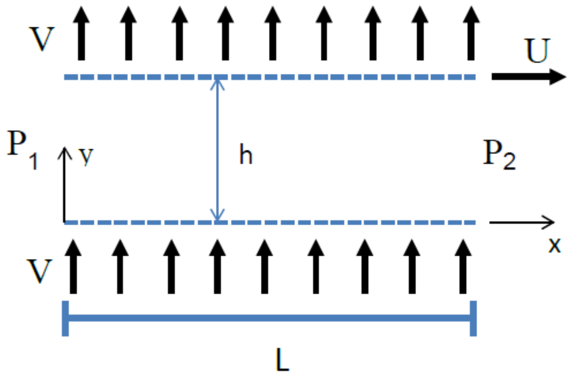
Inserting Eq. (5.66) and the first term of Eq. (5.68) into the LHS of (5.69) will give the relation required to find p_{12}

$$\int_{-d/2}^{d/2}\left[\frac{p_{12} - p_a}{2\mu L_1}\left(y^2 - \frac{d^2}{4}\right) + R\omega\right]dy = 0 \Rightarrow p_{12} = p_a + 12\mu\omega\frac{R}{d}\frac{L_1}{d} \tag{5.71}$$

We see that due to the ratio L_1/d one can expect the pressure build up to be relatively large.

Question 6

Between two long and porous plates, a fluid of density ρ and viscosity μ flows steadily. The flow is fully developed and flows due to a pressure difference Δp. The velocity of the upper plate is U and fluid is injected a constant velocity V.



a. Define all the required BC and parameters needed to solve this problem.

b. Write the required assumptions and show that the final governing equation is given by

$$V\frac{du}{dy}=-\frac{1}{\rho}\frac{\Delta p}{L}+\nu\frac{d^2u}{dy^2}.$$

c. Solve the equation of motions.

d. Show that when $V\to 0$ the solution reduces to the standard Poiseuille flow.

$$u(y)=U\frac{y}{h}-\frac{h^2}{2\mu}\frac{\Delta p}{L}\left(\frac{y}{h}-\frac{y^2}{h^2}\right).$$

e. For the case $V=0$, find the velocity U that the plate at $y=h$ feels no shear.

f. Show that when $V\gg \nu/h,(h\Delta p)/(\rho UL)$, for all practical purposes the velocity will be zero everywhere except in the region $y=h$.

<u>Solution</u>

The BCs are

$$\begin{aligned} y=0&:u=0,v=V \\ y=h&:u=U,v=V \end{aligned}, \tag{5.72}$$

and the pressure gradient is given by

$$\partial_x p=\frac{\Delta p}{L}=\frac{p_1-p_2}{L}=const. \tag{5.73}$$

Assumptions:

1. Constant density and viscosity.
2. Fully developed flow, $\partial_x\vec{u}=0$.
3. 2D flow $w=0,\partial_z=0$.
4. Steady flow $\partial_t=0$.
5. No additional body forces, $\vec{f}=0$.
6. Pressure gradient only in the horizontal direction and is given by Eq. (5.73).

The continuity equation gives

$$\underbrace{\partial_x u}_{2}+\partial_y v+\underbrace{\partial_z w}_{3}=0\Rightarrow\partial_y v=0\Rightarrow \underbrace{v=V}_{7}=const, \tag{5.74}$$

where we have used the permeable surface velocity BC and marked it as assumption 7. The horizontal and vertical NS equations, respectively, give

$$\rho\left(\underset{4}{\partial_t u}+\underset{2}{u\partial_x u}+v\partial_y u+\underset{3}{w\partial_z u}\right)=-\partial_x p+\mu\left(\underset{2}{\partial_{xx}u}+\partial_{yy}u+\underset{3}{\partial_{zz}u}\right)+\underset{5}{f_x}\ , \tag{5.75}$$

$$\rho\left(\underset{4}{\partial_t v}+\underset{2}{u\partial_x v}+\underset{7}{v\partial_y v}+\underset{3}{w\partial_z v}\right)=\underset{6}{-\partial_y p}+\mu\left(\underset{2}{\partial_{xx}v}+\underset{7}{\partial_{yy}v}+\underset{3}{\partial_{zz}v}\right)+\underset{5}{f_y}\ . \tag{5.76}$$

From Eq. (5.75) we are left with

$$\nu\frac{d^2u}{dy^2}-V\frac{du}{dy}=\frac{1}{\rho}\frac{\Delta p}{L}\ , \tag{5.77}$$

Where $\nu=\mu/\rho$. We solve this second order ODE (following the solution in Question 2 of Section 1) in the following manner. For the homogeneous solution, we assume a solution of the form $u=ce^{\lambda y}$ and receive the characteristic equation

$$\nu\lambda^2-V\lambda=0\Rightarrow\lambda_1=0,\lambda_2=V/\nu\ . \tag{5.78}$$

The particular solution is found by guessing a polynomial of order 1, $u=Ay$. Insertion into Eq. (5.77) gives

$$\cancel{\nu\frac{d^2u}{dy^2}}-VA=\frac{1}{\rho}\frac{\Delta p}{L}\Rightarrow A=-\frac{\Delta p}{V\rho L}\ . \tag{5.79}$$

Hence the general solution is given by

$$u=c_1e^{Vy/\nu}+c_2-\frac{\Delta p}{\rho VL}y\ . \tag{5.80}$$

Using the horizontal BCs given in Eq. (5.72) yields

$$\begin{cases}u=-\dfrac{\Delta p}{\rho VL}y+U\left(1+\dfrac{h\Delta p}{\rho LVU}\right)\left(\dfrac{1-e^{Vy/\nu}}{1-e^{Vh/\nu}}\right)\\ v=V\end{cases} \tag{5.81}$$

Inserting $V\to 0$ is non-trivial as the expression is singular in this term. Thus we must either take the formal limit or conduct a Taylor expansion of the expression

$$\begin{aligned}u(y)&=-\frac{\Delta p}{\rho VL}y+U\left(1+\frac{h\Delta p}{\rho LVU}\right)\left(\frac{Vy/\upsilon+(Vy/\upsilon)^2/2+...}{Vh/\upsilon+(Vh/\upsilon)^2/2+...}\right)\\ &\approx-\frac{\Delta p}{\rho VL}y+U\left(1+\frac{h\Delta p}{\rho LVU}\right)\left(\frac{2\upsilon Vy+(Vy)^2}{2\upsilon Vh+(Vh)^2}\right)\\ &=\frac{1}{2\upsilon Vh+\underset{negligable}{(Vh)^2}}\left\{-\frac{\Delta p}{\rho VL}y\left(2\upsilon Vh+(Vh)^2\right)+U\left(1+\frac{h\Delta p}{\rho LVU}\right)\left(2\upsilon Vy+(Vy)^2\right)\right\}\\ &\approx\frac{1}{2\upsilon Vh}\left\{-\cancel{\frac{2\upsilon h\Delta py}{\rho L}}-\underbrace{\frac{Vh^2\Delta py}{\rho L}}_{O(V)}+\underbrace{2\upsilon VUy}_{O(V)}+\cancel{\frac{2\upsilon h\Delta py}{\rho L}}+\underbrace{V^2Uy^2}_{O(V^2)-negligable}+\underbrace{\frac{Vh\Delta py^2}{\rho L}}_{O(V)}\right\}\\ &\approx\frac{1}{2\upsilon Vh}\left\{-\frac{Vh^3\Delta p}{\rho L}\frac{y}{h}+2\upsilon VUy+\frac{Vh^3\Delta p}{\rho L}\left(\frac{y}{h}\right)^2\right\}=U\frac{y}{h}-\frac{1}{2}\frac{h^2\Delta p}{\mu L}\left[\frac{y}{h}-\left(\frac{y}{h}\right)^2\right]\end{aligned} \tag{5.82}$$

Eq. (5.82) is simply a Taylor expansion up to order $O\left(V^2\right)$. After some algebraic manipulation and neglections the known solution is retrieved. The flow will be shear-less if

$$\frac{du}{dy}\left(y=h\right)=0 \Rightarrow U=-\frac{h\Delta p}{2\mu L} \ , \tag{5.83}$$

we see that the shear flow is the opposite direction of the pressure driven flow. When $V \gg \nu/h, \left(h\Delta p\right)/\left(\rho UL\right)$, we have the following relations

$$\begin{aligned} &Vh/\nu \to \infty \\ &\left(h\Delta p\right)/\left(\rho UVL\right) \to 0 \end{aligned} \tag{5.84}$$

We take the limit of the horizontal velocity given in (5.81)

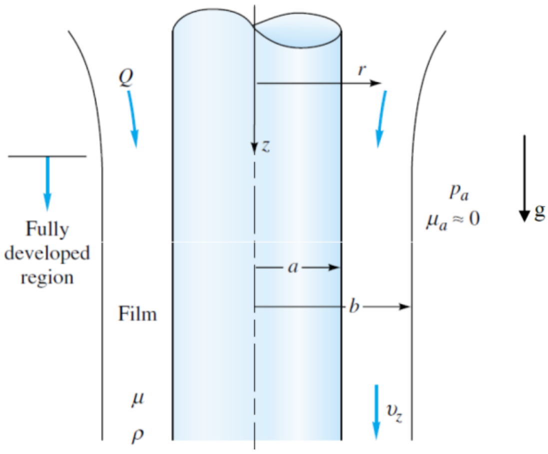
$$\lim_{\substack{Vh/\nu\to\infty \\ \frac{h\Delta p}{\rho UVL}\to 0}} \left(\frac{u}{U}\right) = \lim_{\substack{Vh/\nu\to\infty \\ \frac{h\Delta p}{\rho UVL}\to 0}} \left[-\underbrace{\frac{\Delta p}{\rho VLU}}_{\to 0} y + \left(1+\underbrace{\frac{h\Delta p}{\rho LVU}}_{\to 0}\right)\left(\frac{1-e^{Vy/\nu}}{1-e^{Vh/\nu}}\right)\right] = \lim_{Vh/\nu\to\infty}\left[\left(\frac{1-e^{Vy/\nu}}{1-e^{Vh/\nu}}\right)\right] = e^{V(y-h)/\nu} \ . \tag{5.85}$$

We can see that the velocity decays exponentially fast as the distance increases from $y=h$.

Question 7

A film of fluid drains downwards on a dormant rod of radius a. At a sufficiently long enough distance from the start of the drainage the flow can be considered fully developed. The radius of the film thickness (including the rod) is b. Assume that the atmosphere does not apply a shear force on the fluid $\left(\mu_{air}=0\right)$. Find:

a. The governing EOMs and BC.
b. Solve the above
c. Find the relation between the volumetric flow, Q, rate and the geometric parameters.
d. For the same Q, how will the radius b change if the rod is drawn downward with a velocity U?



Solution

Assumptions:

1. Steady state $\partial_t = 0$.
2. Axisymmetric flow $\partial_\theta = 0, v_\theta = 0$.
3. Fully developed flow $\partial_z \vec{u} = 0$.
4. Incompressible fluid, $\rho = const$.
5. No pressure gradients- atmospheric pressure.
6. Gravity is pointed in the positive z direction.

The only non-trivial term in the continuity equation is the radial term

$$\vec{\nabla}\cdot\vec{u} = \frac{1}{r}\partial_r v_r r + \underbrace{\frac{1}{r}\partial_\theta v_\theta}_{2} + \underbrace{\partial_z v_z}_{3} = \frac{1}{r}\partial_r v_r r = 0 \Rightarrow v\left(r\right) = \frac{c}{r} = 0 \ , \tag{5.86}$$

where the last equality is due the no-penetration condition. The axial momentum equation reduces to

$$\mu\left(\partial_{rr}v_z+\frac{1}{r}\partial_r v_z\right)+\rho g=0\,. \tag{5.87}$$

The solution to the homogeneous equation is found by guessing $v_z=r^s$. Inserting this gives us a characteristic equation

$$r^s s^2=0\ , \tag{5.88}$$

Thus $s=0$ is the solution with an algebraic multiplicity of 2. Thus the solution is

$$v_z=c_1\ln r+c_2\ . \tag{5.89}$$

The particular solution is found by guessing a polynomial of order 2, $v_{z,p}=Ar^2+Br$. Inserting this into Eq. (5.87) gives

$$A=-\rho g/4\mu=-\beta/4, B=0\ . \tag{5.90}$$

The velocity is given by

$$v_z=c_1\ln r+c_2-\frac{\beta r^2}{4}\ , \tag{5.91}$$

with the BC

$$v_z\left(r=a\right)=0,\partial_r v_z\left(r=b\right)=0\,. \tag{5.92}$$

Yielding a final velocity

$$v=\frac{b^2\beta}{2}\ln\frac{r}{a}+\frac{a^2\beta}{4}-\frac{\beta r^2}{4}\ . \tag{5.93}$$

The volumetric flow rate is

$$Q=\int_a^b v_z\left(r\right)2\pi r dr=\frac{\pi\beta b^4}{2}\left(\ln\frac{b}{a}-\frac{3}{8}\right)+\frac{\pi\beta a^2b^2}{2}-\frac{\pi\beta a^4}{8}\ . \tag{5.94}$$

Let us denote this solution as Q_b . Now if we were to pull the rod at a constant velocity U downward, then the BC are modified

$$v_z\left(r=a\right)=U,\partial_r v_z\left(r=\tilde{b}\right)=0\ , \tag{5.95}$$

where $\tilde{b}$ is the new radius of the film. Solution of Eqs. (5.91) and (5.95) yield

$$v=\frac{b^2\beta}{2}\ln\frac{r}{a}+\frac{a^2\beta}{4}-\frac{\beta r^2}{4}+U \tag{5.96}$$

This is exactly the solution given in Eq. (5.93) up to a constant. Thus the flux as given by (5.94) changes by the velocity multiplied by the area thus the flux is

$$Q_{\tilde{b}}+\pi U\left(\tilde{b}^2-a^2\right)\ . \tag{5.97}$$

Requiring that the fluxes are equal means that

$$Q_b = Q_{\tilde{b}} + \pi U\left(\tilde{b}^2 - a^2\right) \quad , \qquad (5.98)$$

as the second term on the RHS is positive, then $Q_b > Q_{\tilde{b}}$, hence $\tilde{b} < b$.

"One can state, without exaggeration, that the observation of and the search for similarities and differences are the basis of all human knowledge.", Alfred Nobel

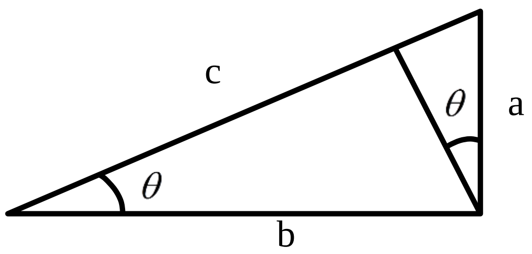
6. Similarity and non-dimensional analysis

Before starting our discussion on similarity and non-dimensional analysis, let's start off with a simple exercise – we shall prove Pythagoras theorem based on geometric similarity arguments.

Question 1

Show that for a right-angled triangle with hypoteneuse length c and shorter sides of lengths a and b that

$$c^2 = a^2 + b^2 . \tag{6.1}$$



Solution

From elementary school we know that the area of the triangle is exactly $ab/2$. Let's assume that we don't know trigonometry as well as let's forget this well-known fact for a second. We can assume that the area of the triangle is dependent solely on c and θ

$$S_c = f(c,\theta) . \tag{6.2}$$

Due to the fact that the area must be dependent on the length squared, where c has dimensions length and θ is dimensionless, this can be written as

$$S_c = c^2 g(\theta) . \tag{6.3}$$

The function $g(\theta)$ is currently unknown and is arbitrary (for now). We can divide the larger triangle into two smaller right-angled triangles, and based on the above arguments, the area of each smaller triangle is

$$S_a = a^2 g(\theta), S_b = b^2 g(\theta), \tag{6.4}$$

where once more $g(\theta)$ appears. It is obvious that this value is the same for all the triangles as they all have the same angle. It is also obvious that the area of the both smaller triangles equals to the area of the larger triangle

$$a^2 g(\theta) + b^2 g(\theta) = c^2 g(\theta) . \tag{6.5}$$

So long as $g(\theta) \neq 0$ for all $\theta \neq 0$ (zero area) then

$$c^2 = a^2 + b^2 \therefore \tag{6.6}$$

This is just one out of 370 known proofs of this theorem. Go ahead and discover more for yourself.

For two systems to be similar to each other, they must not only be geometrically similar as we have just seen above, but they must also be kinematically similar as well as dynamically similar. The meaning of kinematically similar is that both systems undergo the similar times rates such as the same streamlines, whereas dynamically similar means that the ratio of all forces in the system are the same. When all the constraints are held, the systems are said to be similar, and measurements (forces, moments, fluxes, and etc') on one system can be transformed to the other.

Buckingham π Theorem

1. Write the n parameters of the problem (density, length scales, viscosity, and etc').
2. Write the m dimensions of the problem (length, time, mass, temperature, and etc').

3. Write the parameters in terms of their dimensions.
4. Choose m parameters that span the dimension space.
5. Express the remaining $p=n-m$ parameters as a functions of the above chosen m parameters: $\pi_i = X_1^{a_1} X_2^{a_2} \cdots X_m^{a_m} X_i, i=1,\dots,p$ and receive non-dimensional parameters.
6. Interpret the physical meaning of these non-dimensional parameters.

Question 2
Use dimensionless analysis to find the maximal velocity for a fully developed flow in between infinite plates of length $\tilde{L}$ that are separated by a distance of $2h$. The pressure difference is given, Δp , as well as the viscosity of the fluid, μ .

Solution

The only fore-knowledge we bring in this problem is that velocity profile is parabolic so that the second derivative is constant

$$u(y) = Ay^2 + By + C \quad . \tag{6.7}$$

However, due to symmetry, $B=0$, and due to the no-slip condition, $A=-C=u_{\max}$

$$u(y) = u_{\max}\left[\left(\frac{y}{h}\right)^2 - 1\right] , \tag{6.8}$$

where y is normalized so that the parameter multiplying the parentheses will have the dimensions of velocity.

The parameters of the problem are $\tilde{L}, h, u_{\max}, \Delta p, \mu$.

The dimensions are L,T,M , length, time, and mass, accordingly. The dimensions of the parameters are

$$\tilde{L}[m \sim L], h[m \sim L], u_{\max}\left[\frac{m}{s} \sim \frac{L}{T}\right], \Delta p\left[Pa = \frac{N}{m^2} = \frac{kgm}{m^2 s^2} \sim \frac{M}{LT^2}\right], \mu\left[Pa \cdot s \sim \frac{M}{LT}\right] . \tag{6.9}$$

Let us assume that the maximal velocity is a function of the remaining parameters

$$u_{\max} = \tilde{L}^a h^b \Delta p^c \mu^d \quad . \tag{6.10}$$

Then based on dimension analysis

$$\frac{L}{T} = L^a L^b \left(\frac{M}{LT^2}\right)^c \left(\frac{M}{LT}\right)^d \tag{6.11}$$

We now require that the dimensions are equal for both sides.

$$M: 0 = \mathrm{c} + \mathrm{d} \ , \tag{6.12}$$

$$T: -1 = -2c - d \ , \tag{6.13}$$

$$L: 1 = a + b - c - d \ . \tag{6.14}$$

It is quite easy to see that Eqs. (6.12)-(6.13) have the simple equation $c=1, d=-1$. Thus Eq. (6.14) reduces to

$$1 = a + b \ , \tag{6.15}$$

where it is evident that the remaining parameter cannot be extracted without additional previous knowledge or assumptions. Strictly speaking, for dimensional analysis purposes, any choice of one parameter will lead to a correct solution, for example choosing $b=0$ gives $, a=1$ and will lead to

$$u_{\max} = \frac{h\Delta p}{\mu} \ . \quad (6.16)$$

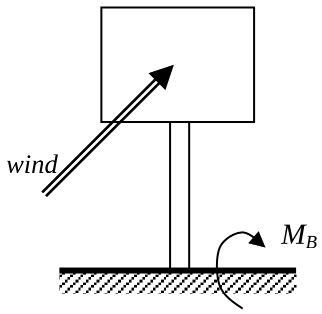
It now remains for us to choose a normalization that includes physical understanding of the problem. We expect, and have seen this numerous times this semester, that the pressure gradient should decay with the length of the plates, thus a better guess would be $b=1$, leading to $a=2$ and

$$u_{\max} = \frac{\Delta p}{L}\frac{h^2}{\mu} \quad (6.17)$$

This is solution we have received numerous times up to a factor of $1/2$, yet we have not solved the problem in an explicit manner! This would be a good time to ask yourselves why we have not included the density of the fluid in this solution.

Question 3

To design a large posted sign, the affect of strong winds $\left(100\left[km/h\right]\right)$ on the bending moments at the bottom of the signs need to be considered. A small-scale model, with a geometric ratio of $1:20$, will be tested in a wind-tunnel. The pressure in the channel is substantially higher than atmospheric pressure so that the density in the channel is 8 times larger than the density at standard atmospheric conditions. The viscosity of the air remains unchanged.



a. In order to be able to use similarity what should the velocity of the air be within the tunnel.
b. In the channel, a bending moment of $M_B = 35\left[N\cdot m\right]$ is measured for the model. What is the corresponding moment for the real sign.

<u>Solution</u>

The parameters of the problems are $M_B, \rho, \mu, \mathrm{u}, \tilde{\mathrm{L}}$ so that $n=5$.

The dimensions of the problems are M, L, T so that $m=3$. Let us write the dimensions of each parameter as function of the dimensions

	$\tilde{L}$	M_B	ρ	μ	u
M	0	1	1	1	0
L	1	2	-3	-1	1
T	0	-2	0	-1	-1

The number of dimensionless parameters will be $p = n - m = 2$. We will use $\rho, u, \tilde{L}$ to span the dimensional space. The dimensional parameter are

$$\pi_1 = \rho^a u^b \tilde{L}^c \mu, \pi_2 = \rho^{\tilde{a}} u^{\tilde{b}} \tilde{L}^{\tilde{c}} M_B \ . \quad (6.18)$$

To find all the coefficients $a, b..., \tilde{c}$ we shall compare that all the π_i are dimensionless. Let us start with the first one

$$M^0 L^0 T^0 = \pi_1 = \rho^a u^b \tilde{L}^c \mu = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b L^c \left(\frac{M}{LT}\right) , \quad (6.19)$$

requiring equality gives

$$\begin{cases} M: 0 = a+1 \\ L: 0 = -3a+b+c-1 \\ T: 0 = -b-1 \end{cases} , \quad (6.20)$$

the solution of which is trivial $a=b=c=-1$, thus we have

$$\pi_1 = \frac{\mu}{\tilde{L}u\rho} = \frac{1}{\text{Re}} \ , \tag{6.21}$$

where Re is the Reynolds number. The second dimensionless number (dropping all the tildas)

$$M^0 L^0 T^0 = \pi_2 = \rho^a u^b \tilde{L}^c M_B = \left(\frac{M}{L^3}\right)^a \left(\frac{L}{T}\right)^b L^c \left(\frac{ML^2}{T^2}\right) , \tag{6.22}$$

requiring equality gives

$$\begin{cases} M: 0 = a+1 \\ L: 0 = -3a+b+c+2 \\ T: 0 = -b-2 \end{cases} \tag{6.23}$$

The solution of which is also trivial $(a=-1, b=-2, c=-3)$, thus we have

$$\pi_2 = \frac{M_B}{\rho u^2 \tilde{L}^3} \ . \tag{6.24}$$

We would like to compare between the velocities between the model (index m) and the real sign (index p for prototype). Requiring that both are undergoing the same time rate, means the Reynolds number is equal

$$(\pi_1)_m = (\pi_1)_p : \frac{\mu_m}{\tilde{L}_m u_m \rho_m} = \frac{\mu_p}{\tilde{L}_p u_p \rho_p} \Rightarrow u_m = u_p \left(\frac{\mu_m}{\mu_p}\right)\left(\frac{\tilde{L}_p}{\tilde{L}_m}\right)\left(\frac{\rho_p}{\rho_m}\right) = 1 \cdot 20 \cdot \frac{1}{8} = 2.5 u_p = 250 [km/h] \ . \tag{6.25}$$

Additionally, requiring dynamical similarity gives

$$(\pi_2)_m = (\pi_2)_p : \frac{M_{B,m}}{\tilde{L}_m{}^3 u_m{}^2 \rho_m} = \frac{M_{B,p}}{\tilde{L}_p{}^3 u_p{}^2 \rho_p} \Rightarrow M_{B,p} = M_{B,m} \left(\frac{u_p}{u_m}\right)^2 \left(\frac{\tilde{L}_p}{\tilde{L}_m}\right)^3 \left(\frac{\rho_p}{\rho_m}\right) = \frac{1}{2.5^2} \cdot 20^3 \cdot \frac{1}{8} \cdot 35 = 5.6 [kN \cdot m] . \tag{6.26}$$

For further thought- had we also been interested in the generated force F , what would have been the ratio of this force between the model and prototype? The answer is surprisingly simple. We would get an additional non-dimensional number $\pi_3 = F / \left(\rho u^2 \tilde{L}^2\right)$. Requiring equality, we would get

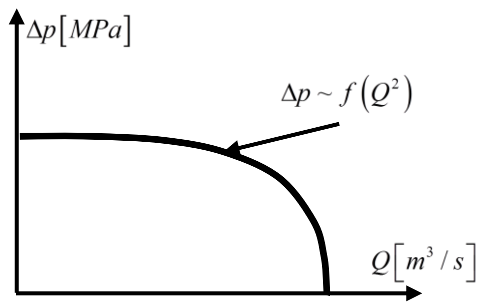
$$\frac{F_m}{F_p} = \left(\frac{\rho_m}{\rho_p}\right)\left(\frac{u_m}{u_p}\right)^2 \left(\frac{\tilde{L}_m}{\tilde{L}_p}\right)^2 = 8 \cdot \left(\frac{5}{2}\right)^2 \cdot 20^{-2} = \frac{1}{8} \ , \tag{6.27}$$

or through an alternative means

$$\frac{F_m}{F_p} = \frac{M_{B,m} / \tilde{L}_m}{M_{B,p} / \tilde{L}_p} = \frac{M_{B,m}}{M_{B,p}} \frac{\tilde{L}_p}{\tilde{L}_m} = \frac{35}{5600} \cdot 20 = \frac{1}{8}$$

Question 4

The pressure drop Δp in a centrifugal pump depends on its diameter, D , rotational velocity ω and volumetric flow rate Q of a fluid with density ρ and viscosity μ .



a. Write the general non-dimensional relation between the pressure drop and flow rate.
b. Assuming that the viscosity of the fluid is negligible, and the relation between the pressure

drop and flow rate is parabolic (with a maximal pressure drop with zero flow), write the explicit relation between these two. What is the number of unknown parameters in this problem.

c. Two sets of experiments were conducted for a fluid of density $\rho = 10^3 \left[kg/m^3 \right]$ rotating with an angular velocity $\omega = 200 \left[rad/\sec \right]$ in a pipe of diameter $D = 0.12 \left[m \right]$. Uses these experiments to determine these unknowns

	Exp. 1	Exp. 2
$Q\left[m^3/s \right]$	0.05	0.15
$\Delta p\left[MPa \right]$	0.7	0.2

d. What will be the pressure drop of the diameter and angular velocities are increased threefold and the density is decreased to $\rho = 800 \left[kg/m^3 \right]$?

<u>Solution</u>

The number of parameters in this problem is $n = 6$: $\Delta p, \rho, \mu, \omega, D, Q$.

The number of dimensions is $m = 3$: M, L, T .

	D	ρ	μ	Δp	ω	Q
M	0	1	1	1	0	0
L	1	-3	-1	-1	0	3
T	0	0	-1	-2	-1	-1

The number of dimensionless parameters will be $p = n - m = 3$. We will use ρ, ω, D to span the dimensional space. The dimensional parameter are

$$\pi_1 = \rho^a \omega^b D^c \mu, \pi_2 = \rho^{\tilde{a}} \omega^{\tilde{b}} D^{\tilde{c}} \Delta p, \pi_3 = \rho^{\bar{a}} \omega^{\bar{b}} D^{\bar{c}} Q \quad . \tag{6.28}$$

For the first dimensionless number we have

$$M^0 L^0 T^0 = \pi_1 = \rho^a \omega^b D^c \mu = \left(\frac{M}{L^3} \right)^a T^{-b} L^c \frac{M}{LT} \tag{6.29}$$

requiring equality gives

$$\begin{cases} M : 0 = a + 1 \\ L : 0 = -3a + c - 1 \\ T : 0 = -b - 1 \end{cases} , \tag{6.30}$$

the solution of which is trivial $\left(a = -1, b = -1, c = -2 \right)$, thus we have

$$\pi_1 = \frac{\mu}{D^2 \omega \rho} = \frac{1}{\text{Re}_\omega} \quad , \tag{6.31}$$

where this time we are looking at Reynolds number caused due to angular velocity. The second dimensionless number (dropping all the tildas and overbars)

$$M^0 L^0 T^0 = \pi_2 = \rho^a \omega^b D^c \Delta p = \left(\frac{M}{L^3} \right)^a T^{-b} L^c \frac{M}{LT^2} \quad , \tag{6.32}$$

requiring equality gives

$$\begin{cases} M: 0 = a+1 \\ L: 0 = -3a+c-1 \\ T: 0 = -b-2 \end{cases} \tag{6.33}$$

the solution of which is trivial $(a=-1, b=c=-2)$, thus we have

$$\pi_2 = \frac{\Delta p}{D^2\omega^2\rho} \ , \tag{6.34}$$

this is called the dimensional pressure coefficient. The third dimensionless number

$$M^0L^0T^0 = \pi_3 = \rho^a\omega^b D^c Q = \left(\frac{M}{L^3}\right)^a T^{-b}L^c\frac{L^3}{T} \ , \tag{6.35}$$

requiring equality gives

$$\begin{cases} M: 0 = a \\ L: 0 = -3a+c+3 \\ T: 0 = -b-1 \end{cases} \tag{6.36}$$

The solution of which is also trivial $(a=0, b=-1, c=-3)$, thus we have

$$\pi_3 = \frac{Q}{D^3\omega} \ . \tag{6.37}$$

From Buckingham π theorem, we know that $\pi_2 = f(\pi_1, \pi_3)$ where the exact structure of the function f is unknown.

Assuming that the effects of viscosity are negligible then $f(\pi_1, \pi_3) = f(\pi_3)$. If we know that the relation is parabolic we can write the following two relations

$$\Delta p = \tilde{A}Q^2 + \tilde{B} , \tag{6.38}$$

or

$$\pi_2 = A\pi_3^{\ 2} + B . \tag{6.39}$$

The linear contribution has been removed because it can be seen in the figure that the derivative at $Q=0$ is zero. We see that we have two unknowns in either Eq. (6.38) or (6.39). From the table we have enough data points to find either set of these two unknowns. However, if we find the first set, we will not be able to say anything regarding a similar system with additional work.

	Exp. 1	Exp. 2
$Q\left[m^3/s\right]$	0.05	0.15
$\pi_3 = Q/\left(D^3\omega\right)$	0.144	0.434
$\Delta p[MPa]$	0.7	0.2
$\pi_2 = \Delta p/\left(\rho\omega^2 D^2\right)$	1.21	0.347

From this table and Eq. (6.39) we have

$$\begin{cases} 1.21 = 0.144^2 A + B \\ 0.347 = 0.434^2 A + B \end{cases} \tag{6.40}$$

The solution of which is

$$A = -5.148, B = 1.316 \ . \tag{6.41}$$

Since we now the non-dimension relation, we can look at different systems. Inserting Eqs. (6.34), (6.41), and (6.37) into Eq. (6.39) gives

$$\frac{\Delta p}{\rho\omega^2 D^2} = A\left(\frac{Q}{\omega D^3}\right)^2 + B \Rightarrow \Delta p = A\frac{\rho}{D^4}Q^2 + B\rho\omega^2 D^2 \tag{6.42}$$

Inserting the values given in subsection *d*. gives

$$\Delta p = -0.24Q^2 + 0.61 \tag{6.43}$$

"The shortest path between two truths in the real domain passes through the complex domain.", Jacques Hadamard

7. Potential Flow

For the case of an incompressible, invisicid, irrotational and ideal fluid, the NS equation given by Eq. (5.11)

$$\rho\left(\frac{\partial \vec{u}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right)=-\vec{\nabla}p+\cancel{\mu\vec{\nabla}^2\vec{u}}+\vec{f} \ , \tag{7.1}$$

reduces to

$$\rho\left(\frac{\partial \vec{u}}{\partial t}+\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u}\right)=-\vec{\nabla}p+\vec{f} \ . \tag{7.2}$$

This a result of the viscous forces being negligible compared to the inertia forces, or more exactly $\mathrm{Re}\to\infty$. Eq. (7.2) is called the Euler equation. To solve this equation, we supplement the continuity equation

$$\vec{\nabla}\cdot\vec{u}=0 \ . \tag{7.3}$$

If we define the velocity to be the gradient of a scalar function

$$\vec{u}=\vec{\nabla}\phi \ , \tag{7.4}$$

then we get a number of interesting results. Note that unlike electrostatics or gravity, the potential is **<u>not</u>** defined with a minus sign. First, we shall insert Eq. (7.4) into Eq. (7.3)

$$\vec{\nabla}\cdot\vec{u}=\partial_x u+\partial_y v=\partial_x\partial_x\phi+\partial_y\partial_y\phi=\partial_{xx}\phi+\partial_{yy}\phi=\vec{\nabla}^2\phi=0 \ , \tag{7.5}$$

where we have received Laplace's equation. The two important points to realize are that by solving Eq. (7.5) we know the dynamics of the velocity field everywhere and we no longer need to solve the non-linear equation given by Eq. (7.2). Also, Laplace's equation is linear, so that if have a number of solutions to the equation, then a superposition of all these solutions is also a solution. Also, due to the fact that velocity is a gradient of a scalar function, then it can be shown that this field is conservative

$$\zeta_z=\left(\vec{\nabla}\times\vec{U}\right)_z=\partial_x v-\partial_y u=\partial_x\partial_y\phi-\partial_y\partial_x\phi=0 \ . \tag{7.6}$$

We can now write the relationship between the stream function and the potential function in Cartesian and cylindrical coordinates

$$u_x=\partial_x\phi=\partial_y\psi,\ \ v_y=\partial_y\phi=-\partial_x\psi \ , \tag{7.7}$$

$$v_r=\partial_r\phi=\frac{\partial_\theta\psi}{r},\ \ v_\theta=\frac{\partial_\theta\phi}{r}=-\partial_r\psi \ , \tag{7.8}$$

where the relation between these two coordinates systems are given by

$$\begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases} \Leftrightarrow \begin{cases} r=\sqrt{x^2+y^2} \\ \theta=\arctan\left(y/x\right) \end{cases} . \tag{7.9}$$

Based on Eqs. (7.7)-(7.8) it is evident that if know either the stream function or the potential function, we can find the other one. Also, we know that the stream function describe the motion of the fluid while, the potential function describe contour lines, that that derivative between two adjacent contours gives the velocity. Attached is a table of a number flows that are potential and uphold Eqs. (7.5)-(7.6).

		$\phi(x,y)$	$\psi(x,y)$	$\phi(r,\theta)$	$\psi(r,\theta)$
Uniform flow	$u = A$	Ax	Ay	$Ar\cos\theta$	$Ar\sin\theta$
Uniform Flow	$v = B$	By	$-Bx$	$Br\sin\theta$	$-Br\cos\theta$
Rotating flow	$v_\theta = \omega / r$			$\omega\theta$	$-\omega\log r$
Source	$v_r = Q/(2\pi r)$			$Q\log(r)/2\pi$	$Q\theta/2\pi$
Sink	$v_r = -Q/(2\pi r)$			$-Q\log(r)/2\pi$	$-Q\theta/2\pi$
Doublet				$q\cos\theta/r$	$q\sin\theta/r$

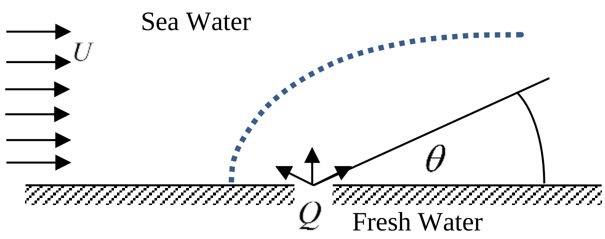
One last comment is required. Note that both the source and sink are divided by 2π. This is true if the sink is located within the domain of the problem. Let's assume that all we know is $v_r = A/r$, but we also know that the total flux at any given radius is Q , thus

$$Q = \oint \vec{v}\cdot d\vec{l} = \oint v_r \cdot r d\theta = \oint \frac{A}{r} r d\theta = \oint A d\theta = 2\pi A \Rightarrow A = \frac{Q}{2\pi} \tag{7.10}$$

However, if the source is located on the wall, then obviously the contour integration will not be from $[0,2\pi]$ but rather $[0,\pi]$ or dependent on the geometry.

Question 1

Fresh water is injected from a narrow slot into sea water that at infinity moves at a uniform velocity U. The volumetric flow rate of the fresh water is Q. Assume that the difference in densities is negligible. Additionally, assume that the flow is ideal and no forces are exerted within the system. Find:



a. The point of maximal pressure and the maximal pressure.
b. An expression for the streamline dividing the fresh and sea water.
c. The criteria when a point located on $\theta = \pi/2$ is located in the fresh water area.
d. The maximal distance from the wall that the fresh water will penetrate into the sea water.

Solution

The potential function of the flow is given by

$$\phi(r,\theta) = \phi_{uniform} + \phi_{source} = Ur\cos\theta + \frac{Q}{\pi}\ln r \ . \tag{7.11}$$

The velocities are given by

$$v_r = \partial_r \phi = \partial_r\left(Ur\cos\theta + \frac{Q}{\pi}\ln r\right) = U\cos\theta + \frac{Q}{\pi r} \ , \tag{7.12}$$

$$v_\theta = \frac{\partial_\theta \phi}{r} = \frac{1}{r}\partial_\theta\left(Ur\cos\theta + \frac{Q}{\pi}\ln r\right) = -U\sin\theta \ . \tag{7.13}$$

As we have already seen in the first part of the course, based upon the Bernoulli principle, when the velocity decreases, the pressure increases. Thus we need to find the point where the velocity is the lowest. Investigation of the flow fields shows us the azimuthal velocity is zero at $\theta = 0,\pi$. In contrast, the radial velocity is positive at $\theta = 0$ but can change signs when $\theta = \pi$. This change of sign occurs at the stagnation point when $v_r = v_\theta = 0$

$$r = R_0 = \frac{Q}{\pi U} \ . \tag{7.14}$$

Since we have assumed that the flow is ideal we can write the Bernoulli principle (conservation of energy) and we can assume that we know that ambient pressure

$$p\left(r = R_0, \theta = \pi\right) = p_\infty + \frac{\rho}{2}\left(v_{r,\infty}{}^2 + v_{\theta,\infty}{}^2\right) = p_\infty + \frac{\rho U^2}{2} . \tag{7.15}$$

To find the stream function we integrate one of the velocity equations

$$\frac{\partial_\theta \psi}{r} = v_r \Rightarrow \psi = \int v_r r d\theta = Ur\sin\theta + \frac{Q\theta}{\pi} + f\left(r\right) \ . \tag{7.16}$$

The function $f\left(r\right)$ is found be taking the r derivative and comparing this azimuthal velocity

$$\partial_r \psi = \partial_r \left(Ur\sin\theta + \frac{Q\theta}{\pi} + f\left(r\right)\right) = U\sin\theta + f\,'\left(r\right) = -v_\theta = U\sin\theta \Rightarrow f\,'\left(r\right) = 0 \Rightarrow f\left(r\right) = const = 0 \ . \tag{7.17}$$

The constant is arbitrary and so we set it to zero. We will now conduct a small sanity check. We have stated and shown on numerous occasions that streamlines cannot cross each other and that the difference between two streamlines is the volumetric flow rate. Let us ask what the is the value of the stream function when $\theta = 0, \pi$

$$\psi\left(r,0\right) = Ur\sin 0 + \frac{Q}{\pi}0 = 0 \ , \tag{7.18}$$

$$\psi\left(r,\pi\right) = Ur\sin\pi + \frac{Q\pi}{\pi} = Q \ . \tag{7.19}$$

Hence these two streamlines indeed have the expected flux of Q and it is the latter one that divides between the fresh and salted water (Eq. (7.19) also describes the value of the stream function at the stagnation point). The value on at this point and every point on the dividing line is Q thus

$$Ur\sin\theta + \frac{Q\theta}{\pi} = Q \Rightarrow r\left(\theta\right) = Q\frac{1-\frac{\theta}{\pi}}{U\sin\theta} \ . \tag{7.20}$$

By substituting $\theta = \pi/2$ into Eq. (7.20) we get

$$r_{divide} = \frac{Q}{2U} \ . \tag{7.21}$$

Thus, if the radial coordinate of point *A* is larger than the above equation it will be in the sea water, otherwise it will be in the fresh water. To find the maximal horizontal distance from the wall, we remember that

$$y = r\sin\theta \ . \tag{7.22}$$

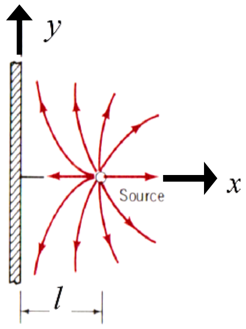
Inserting Eq. (7.20) into Eq. (7.22) yields

$$y = r\sin\theta = Q\frac{1-\frac{\theta}{\pi}}{U\sin\theta}\sin\theta = \frac{Q}{U}\left(1-\frac{\theta}{\pi}\right) \ . \tag{7.23}$$

It is quite easy to see that

$$y_{\max}\left(\theta = 0\right) = \frac{Q}{U} \ , \tag{7.24}$$

and this occurs at infinity.



Question 2 (Final Winter 2011)

A source is located at distance l from a thin and infinite wall as shown in the figure. Assuming that the velocity field is irrotational, find:

a. The potential function and show that this velocity obeys the appropriate BC.
b. The velocity distribution on the wall.
c. Assuming that ambient pressure far away from the wall (and on the other side of the wall) is p_a , find the pressure distribution.
d. Find the force that the source is exerting on the wall.

<u>Solution</u>

We have seen that the potential function of a source is

$$\phi = \frac{Q \ln r}{2\pi} = \frac{Q}{2\pi} \ln\left[\left(x^2 + y^2\right)^{1/2}\right] = \frac{Q}{4\pi} \ln\left(x^2 + y^2\right) . \tag{7.25}$$

Thus the potential function of a source translated to point $(l, 0)$ is simply

$$\phi = \frac{Q}{4\pi} \ln\left[(x - l)^2 + y^2\right] . \tag{7.26}$$

However, such a potential field would not uphold the no penetration velocity $u(x = 0, y) = 0$ as required. However, if we were to add an additional source located at $(-l, 0)$, then we would be able to obey this constraint.

$$\phi(x, y) = \frac{Q}{4\pi}\left\{\log\left[(x - l)^2 + y^2\right] + \log\left[(x + l)^2 + y^2\right]\right\} , \tag{7.27}$$

The horizontal velocity is given by

$$u(x, y) = \partial_x \phi(x, y) = \frac{Q}{2\pi}\left(\frac{x - l}{(x - l)^2 + y^2} + \frac{x + l}{(x + l)^2 + y^2}\right) , \tag{7.28}$$

where it is easy to see that $u(0, y) = 0$. The vertical velocity is given by

$$v(x, y) = \partial_y \phi(x, y) = \frac{Q}{2\pi} y\left(\frac{1}{(x - l)^2 + y^2} + \frac{1}{(x + l)^2 + y^2}\right) , \tag{7.29}$$

while the velocity on the wall is no longer zero (slip is allowed!)

$$v(0, y) = \frac{Qy}{\pi\left(y^2 + l^2\right)} . \tag{7.30}$$

The field is irrotational so we can use Bernoulli's principle throughout the field

$$p_\infty = p(x = 0, y) + \frac{\rho v(x = 0, y)^2}{2} \Rightarrow p(x = 0, y) = p_\infty - \frac{Q^2 \rho}{2\pi^2}\left(\frac{y}{y^2 + l^2}\right)^2 \tag{7.31}$$

The force exerted on the wall is given by

$$F = \int_{-\infty}^{\infty} \left[p_a - p(y)\right] dy = \int_{-\infty}^{\infty} \frac{Q^2 \rho}{2\pi^2}\left(\frac{y}{y^2 + l^2}\right)^2 dy , \tag{7.32}$$

where the first term is the force being applied from the left side of the wall. We transform $y = z \cdot l$

$$F = \int_{-\infty}^{\infty} \frac{Q^2 \rho}{2\pi^2} \left(\frac{zl}{z^2 l^2 + l^2} \right)^2 l dz = \frac{Q^2 \rho}{2\pi^2 l} \int_{-\infty}^{\infty} \frac{z^2}{\left(z^2+1\right)^2} dz \ . \tag{7.33}$$

Eq. (7.33) is integrated by parts

$$\int uv' dz = u(z) v(z) \Big|_{z_{low}}^{z_{high}} - \int u' v dz \ . \tag{7.34}$$

We define

$$\left| \begin{matrix} u = z & u' = 1 \\ v' = \dfrac{z}{\left(z^2+1\right)^2} & v = -\dfrac{1}{2\left(z^2+1\right)} \end{matrix} \right| , \tag{7.35}$$

so that the integration becomes trivial

$$\int_{-\infty}^{\infty} \frac{z^2}{\left(z^2+1\right)^2} dz = \cancel{-\frac{z}{2\left(z^2+1\right)}\Bigg|_{-\infty}^{\infty}} + \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{\left(z^2+1\right)} dz = \frac{1}{2} \arctan(z) \Big|_{-\infty}^{\infty} = \frac{\pi}{2} \ . \tag{7.36}$$

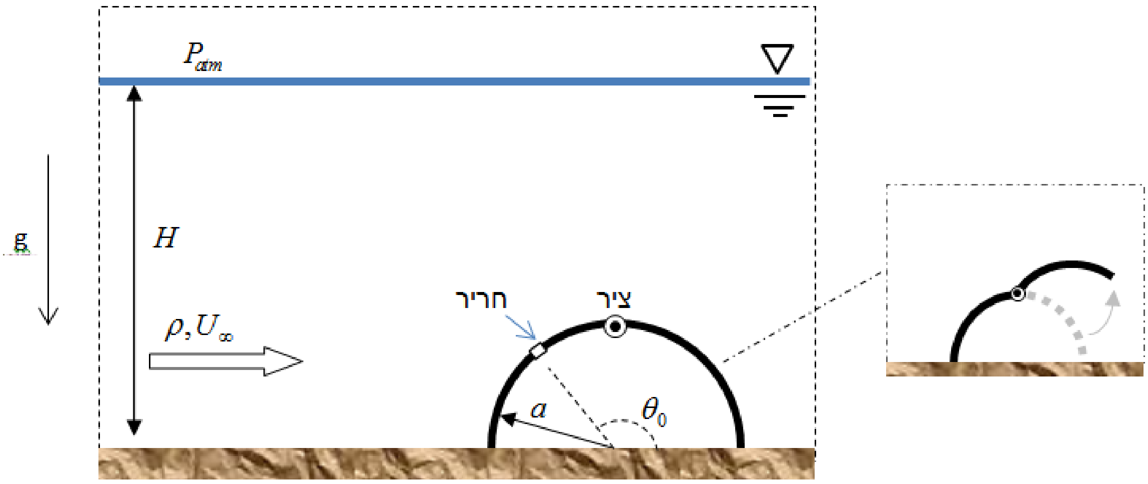
Inserting this into Eq. (7.33) yields

$$F = \frac{Q^2 \rho}{4\pi l} \ . \tag{7.37}$$

The most important to note from this solution is the wall/board is pulled towards the source and is not repelled from it as one might intuitively expect! While the source is expelling material onto the wall and pushing it to the left, the pressure drop occurring at the wall is by far more substantial.

Next time you are on a fast moving train and you pass by another train, see what happens? And explain to yourself why!

Question 3 (Final Spring 2011-2012)
At the bottom of a pool of depth H, half of a thin cylindrical shell is placed. The radius of the shell is a is much smaller than the length (into the page) b , $b \gg a$. The far field velocity is U_∞ . The shell is comprised of two parts: the left component is fixed to the ground, while the right part is free to move around a hinge located the top (see figure). The right part of the door has a mass <u>per unit area</u> σ . At the affixed part, at an angle of θ_0 an orifice is placed. Assume that the orifice is small so that it does not affect the velocity field. Also assume that the shell is completely filled with water.



a. When can this problem be solved using potential theorem? Write down the condition/s using the given problem parameters. Explain <u>briefly</u> the physical meaning of each condition.

b. Under these conditions, and assuming steady state, what is the mass per unit area , $\sigma\left[kg/m^3\right]$, required so that door will continue to stay shut?

c. At what angle should the orifice be placed to minimize the weight of the door? Give a short explaination (3-4 sentences). Note that this question can be solved without the full solution of question b.

<u>Solution</u>

The two conditions required to solve this problem using potential flow are

I. The flow is irrotational $\vec{\nabla}\times\vec{u}=0$.

II. The Reynolds number, $\mathrm{Re}=U_\infty \rho a/\mu \gg 1$, is very large, thus the effects of viscosity are negligible.

You have seen in class that the potential field for such a flow is simply the superposition of a uniform flow and a doublet (when the origin of the axis is defined at the center of the cylinder)

$$\phi=\phi_{doublet}+\phi_{uniform}=Ur\left(1+\frac{a^2}{r^2}\right)\cos\theta \; . \qquad (7.38)$$

The velocities are derived

$$v_r=\partial_r\phi=U\cos\theta\left(1-\frac{a^2}{r^2}\right), \qquad (7.39)$$

$$v_\theta=\frac{1}{r}\partial_\theta\phi=-U\sin\theta\left(1+\frac{a^2}{r^2}\right). \qquad (7.40)$$

It is easy to that the radial velocity on the surface is zero $v_r(r=a)=0$, while the azimuthal velocity is $v_\theta(r=a)=-2U\sin\theta$. Using Bernoulli's theorem, on the cylinder, we have

$$P_{atm}+\rho gH+\frac{\rho U^2}{2}=P_{out}+\rho ga\sin\theta+\frac{\rho}{2}{v_\theta}^2 \; . \qquad (7.41)$$

Thus the pressure is easily calculated

$$P_{out}(\theta)=\rho g(H-a\sin\theta)+\frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)-P_{atm} \ . \tag{7.42}$$

For the sake of convenience we shall now denote P_{out} as the gage pressure, and thus we get rid of the term P_{atm} .

Relative to the surface, the inner pressure is given by

$$P_{in}(\theta)=P_{ref}-\rho ga\sin\theta \ . \tag{7.43}$$

The orifice doesn't allow substantial flow through it, however the hydrostatic pressure in the shell must be equal to the hydrostatic pressure outside of the shell at $\theta=\theta_0$. Thus we find the reference pressure

$$P_{out}(\theta_0)=P_{ref}-\rho ga\sin\theta_0 \Rightarrow P_{ref}=P_{out}(\theta_0)+\rho ga\sin\theta_0 \Rightarrow P_{in}(\theta)=P_{out}(\theta_0)+\rho ga\left(\sin\theta_0-\sin\theta\right) \tag{7.44}$$

The difference in the outer and inner pressure is found by subtracting Eq. (7.43) from Eq. (7.42).

$$\begin{aligned}\Delta P=P_{out}-P_{in}&=\rho g(H-a\sin\theta)+\frac{1}{2}\rho U^2\left(1-4\sin^2\theta\right)-P_{out}(\theta_0)-\rho ga\left(\sin\theta_0-\sin\theta\right)\\ &=2\rho U^2\left(\sin^2\theta_0-\sin^2\theta\right)\end{aligned}, \tag{7.45}$$

The moment about the axis of door due to the pressure difference is

$$\begin{aligned}M_{axis}&=\int\Delta P(\theta)d(\theta)ds=\int\Delta P(\theta)(a\cos\theta)(bad\theta)=a^2b\int_0^{\pi/2}\Delta P(\theta)\cos\theta d\theta=\\ &2\rho U^2a^2b\int_0^{\pi/2}\left(\sin^2\theta_0-\sin^2\theta\right)\cos\theta d\theta=2\rho U^2a^2b\left[\sin^2\theta_0\sin\theta-\frac{\sin^3\theta}{3}\right]_0^{\pi/2}=2\rho U^2a^2b\left(\sin^2\theta_0-\frac{1}{3}\right)\end{aligned} \tag{7.46}$$

The moment about the axis due the weight of the door is

$$M_g=\int_s(\sigma bdsg)(a\cos\theta)=\int_0^{\pi/2}(\sigma bagd\theta)(a\cos\theta)=a^2b\sigma g\sin\theta\Big|_0^{\pi/2}=a^2b\sigma g \tag{7.47}$$

The door will remain closed so long as the total moment will be large than zero

$$M=M_P+M_g=2\rho U^2a^2b\left[\sin^2\theta_0-\frac{1}{3}\right]+a^2b\sigma g>0 \ , \tag{7.48}$$

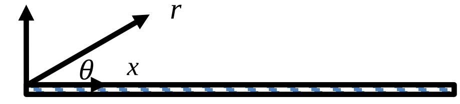
or more appropriately

$$\sigma>\frac{2\rho U^2}{g}\left(\frac{1}{3}-\sin^2\theta_0\right) \ . \tag{7.49}$$

The flow outside of the shell results in a decrease in the hydrostatic pressure. Due to the presence of the orifice, this will also cause a decrease in the pressure within the shell. Hence, the lower the pressure is within the shell, the lower the weight of the door is needed. The maximal velocity occurs at the top of the shell $(\theta=\pi/2)$, and this is the optimal place for the orifice.

Question 4 (Final Spring 2012-2013)

For an incompressible flow around a semi-infinite and thin plate, the radial velocity $v_r=\left(A\cos(\theta/2)\right)/\left(2r^{1/2}\right)$ is given. We know that $A>0$, $r>0$ and $2\pi\geq\theta\geq0$. Find :



a. The azimuthal velocity $v_\theta(r,\theta)$.

b. The stream function $\psi(r,\theta)$.
c. Plot the streamlines of the stream function.
d. Given that $\psi(r=0,\theta=0)=0$, calculate the vorticity $\zeta_z(r,\theta)$ when $r \neq 0$.
e. Calculate the drag force on the section $x \in [0,L]$.
f. Calculate the flux between the corner located at $x=0, y=0$ and the point located at $x=-A^2, y=0$.

<u>Solution</u>

The velocity field must obey the continuity equation

$$\frac{1}{r}\partial_r(rv_r)+\frac{1}{r}\partial_\theta v_\theta=0 \ , \tag{7.50}$$

which yields

$$v_\theta=-\int \partial_r(rv_r)d\theta+f(r) \ , \tag{7.51}$$

inserting the radial velocity component we have

$$v_\theta=-\frac{A}{2}r^{-1/2}\sin\frac{\theta}{2}+f(r) \ . \tag{7.52}$$

Due to the no-penetration BC, we expect that the azimuthal velocity will be zero on the surface $\theta=0,2\pi$ so we require that

$$f(r)=0 \ . \tag{7.53}$$

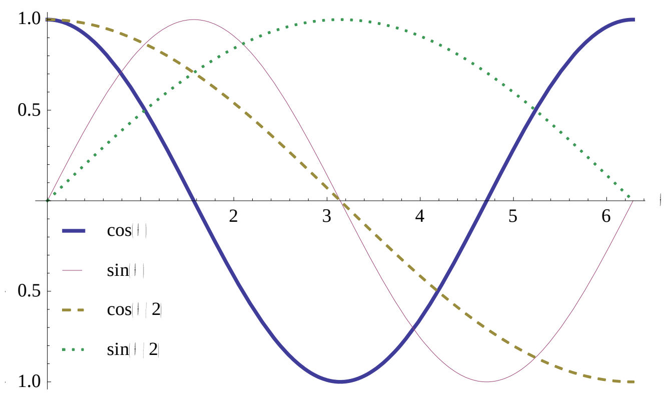
To find the stream function we merely integrate the radial velocity component

$$\psi=\int v_r d\theta+g(r)=Ar^{1/2}\sin\frac{\theta}{2}+g(r) \ . \tag{7.54}$$

Once more, due to the no-penetration BC, the function $g(r)$ cannot be a function of the radius and at the very most be a constant. Due to the fact that $\psi(r=0,\theta=0)=0$, we set this constant to be zero. To plot the streamlines, we must first analyze the velocity field

$$v_r=\frac{A}{2}r^{-1/2}\cos\frac{\theta}{2}, v_\theta=-\frac{A}{2}r^{-1/2}\sin\frac{\theta}{2} \ . \tag{7.55}$$

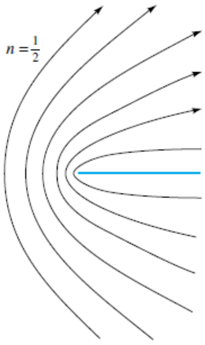
It is simple to see that the azimuthal velocity component $\sim\sin(\theta/2)$ is always negative in the domain $\theta\in[0,2\pi]$. This is an indication that throughout the domain, the velocity is clockwise. In contrast, we see that the radial velocity $\sim\cos(\theta/2)$ changes sign in this domain. Moreover, this velocity component is positive in the domain $\theta\in[0,\pi]$ and negative in $\theta\in[\pi,2\pi]$. Hence, in the latter region, the field converges to the corner/origin while in the former regions it diverges from the corner.



This gives the following picture.

We calculate the vorticity using the following relation

$$\zeta_z = \vec{\nabla}^2\psi = \partial_{rr}\psi + \frac{1}{r}\partial_r\psi + \frac{1}{r^2}\partial_{\theta\theta}\psi = \frac{A}{r^{3/2}}\sin\frac{\theta}{2}\left(-\frac{1}{4}+\frac{1}{2}-\frac{1}{4}\right) = 0\,. \quad (7.56)$$



Since the vorticity is zero everywhere in the domain, then the flow field is potential. It can be shown that the this potential is given by

$$\phi = Ar^{1/2}\cos\frac{\theta}{2} \quad . \qquad (7.57)$$

Furthermore, because this flow field is zero then the drag force and any other force on this flow must be zero. Calculating the flow rate gives

$$Q = \Delta\psi = \psi\left(r = A^2, \theta = \pi\right) - \psi\left(r = 0, \theta = 0\right) = A^2 \qquad (7.58)$$

Ask yourself what would happen if you would have gotten the same answer with a minus sign.

8. References

Mathematics

An excellent book that provides most of the required mathematical skill for solving ODEs and PDEs can be found in Ref [2].

Hydrodynamics

Many of the exercises in the booklet are based or taken from the introductory-level books of Fox and McDonalds [4]and White[5]. These books are very good for undergraduate level studies as they introduce most of the required concepts by explaining them through simple examples, exact calculations and numerous diagrams per example. However, they do not dwell into the deeper meanings and provide in-depth interpretations of the almost infinite phenomena related to fluid mechanics.

For a broader perspective, there are number of excellent books that provide amazing physical insight, solutions to fundamental problems encountered on a day to day basis and the relationship to other branches of physics. In my opinion, the book by Currie [6] is an excellent book that is both mathematically rigorous and provides excellent insights and probably best suites the required level of this course. Additionally, the all-encompassing book by Milne-Thompson[7] is an excellent book and is a favorite among fluid dynamists.

The literature in hydrodynamics is rich and infinite however most of the books are graduate level text book that at this point might prove to be too difficult for you to study from. However for the sake of completeness and because these books have inspired much of the physical interpretation in this booklet, a full list will be provided. Truly, one of the finest book written in Hydrodynamics is that by Landau and Lifshitz [8]. Till this day, Landau is considered one of the greatest physicists of all times. This graduate level text book epitomizes Lawerance Bragg's saying :" The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them." No one excelled in this better than Landau who provided unique physical insight on just about every important phenomena with simple, and often too brief explanations. I recommend that you look through this book at some point in your career. An excellent book on Low-Reynolds Hydrodynamics is that written by Happel and Brenner[9] which surveys the field of (Low Reynolds) Stokes flow. This book is rigorous and provides full mathematical derivation of many important theorems. An additional book that I am fond of is a book [10] written by a former teacher, Gregory Falkovich, who is one of the greatest teachers I have had. This books, likes his classes, are full of insights on various topics in Fluid Mechanics and their relation to additional branches of physics. During one of his classes he stated one of my favorite insights on how to be a scientist (not the exact words but almost exact): "As physicists, it is our duty/responsibility to create analogs from branch of physics to the other and increase our understanding. Mathematicians have a higher way of thought, they create analogs to analogs."

[1] 2013, "Main Page," Wikipedia, the free encyclopedia.
[2] Boyce W. E., and DiPrima R. C., 2009, Elementary differential equations, John Wiley & Sons, Hoboken, N.J.
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[6] Currie I. G., 1974, Fundamental mechanics of fluids, McGraw-Hill.
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[8] Landau L. D., and Lifshitz E. M., 1959, Fluid mechanics.
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[10] Falkovich G., 2011, Fluid mechanics: a short course for physicists, Cambridge University Press, Cambridge.